Repair cost-limit replacement policies with minimal repairs

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A dissertation submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, in fulfilment of the requirements for the degree of Master of Science.

Johannesburg, 2008

Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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Abstract

This dissertation examines a number of maintenance models that are implemented to reduce incidences of system failures or to return a failed system to the operating state. In particular, the repair cost-limit (RCL) replacement models are investigated. On failure the system is replaced by an equivalent new one if the random repair costs C exceeds a given repair cost limit c, or at age τ by a preventive replacement if in the time interval $(0,\tau)$ no failure induced a replacement of the system. Otherwise, after failure a minimal repair is carried out. After any replacement, the time is returned to zero.

For instance, in everyday life, one has to make such decisions once in a while: repair or replace a car, a television set or a refrigerator, depending on the cost of a repair. If the cost of repair is high, then it is more economical to replace the item by a new one.

This dissertation considers the combination of the RCL replacement policy with the age replacement policy. An expression for the expected long run cost per unit time (maintenance cost rate) is derived. The optimum values of c and τ are obtained.

Dedication

Dedicated to my lovely daughters, Lesedi and Boitumelo Mohlala.

Acknowledgments

I would like to thank the school of Statistics and Actuarial Science for providing me with the opportunity to undertake this research, in particular my supervisor, Professor F.E. Beichelt, for his patience, guidance and thoughtful comments as well as for arranging NRF funding and engaging me in many interesting discussions.

Symbols and Abbreviations

X system lifetime, a continuous random variable

$F(t), \bar{F}(t)$ Distribution function, Survival function of X

$\bar{F}(t) = 1 - F(t)$

$f(t)$ Probability density of X : $f(t) = F'(t)$

$\lambda(t)$ Failure rate of X : $\lambda(t) = f(t)/\bar{F}(t)$

$\Lambda(t)$ Integrated failure rate : $\Lambda(t) = \int_0^t \lambda(\mathrm{x})dx$

X_t Residual lifetime of a system after time t given that it has survived interval $[0,t]$

$F_t(x)$ Distribution function of X_t

$\bar{F}_t(x)$ Survival function of X_t : $\bar{F}_t(x) = 1 - F_t(x)$

c_m, c_p, c_r Mean cost of a minimal repair, a preventive replacement(pm), an unscheduled replacement after a type 2 failure, $c_m < c_p < c_r$

$o(h)$ Landau's order symbol, i.e $o(h)$ is any function with property $\lim_{h \to 0} o(h)/h = 0.$

$\{N(t), t \geq 0\}$ counting process

$\Gamma(x)$ $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$, Gamma function, $x > 0$

$C(t)$ Cumulative maintenance cost in $[0,t]$

K maintenance cost rate (= long - run total maintenance cost per unit time)

$H(t)$	renewal function
$Y \text{ or } L, G(t)$	replacement cycle length, distribution function of $Y \text{ or } L$
$p(t), \bar{p}(t) = 1 - p(t)$	probability of a type 2 (type 1) failure
$c(t)$	repair cost limit at failure time t
$R(x)$	distribution function of the random repair cost
c_0	optimal constant repair cost limit
τ_0	optimal constant repair time limit
$\mu(t)$	mean residual lifetime of the system
$F^{(n)}(t)$	n - fold convolution power of $F(t)$
X_n	time to n$^{\text{th}}$ failure
T_0	economic life of the system
t_0	optimal replacement time
n_0	optimal number of failures before replacement
c	constant repair cost limit
μ	average estimated repair cost
c_s	cost of an inspection, a constant
$A_k(t)$	probability of k type 1 failure and no type 2 failure in $(0,t]$, $A_0(t) = \bar{F}(t)$
T	repair time limit

$Q(t)$	distribution function of major repair time
c_R	constant repair cost
c_S	constant shortage cost
c_D	constant ordering cost

Contents

1. Introduction

Among a great number of reliability problems, repair-limit replacement problems are important for the economical decision of whether to repair or replace a failed system.

From time to time, figures are being published reflecting the costliness of maintenance actions. While estimates of actual rand costs vary, they invariably illustrate the magnitude of maintenance expenditures. These high expenditures clearly indicate the continuous need for the development of efficient maintenance techniques.

The past five decades have seen continuing interest in maintenance policies for systems that are subject to stochastic failures. Maintenance policies are implemented to reduce incidences of system failures or to return a failed system to the operating state.

There're two types of maintenance, namely, preventive and corrective maintenance. In preventive maintenance, inspections are carried out, worn-out parts replaced, adjustments done to keep the system in proper condition, and even complete system replacements are carried out. Corrective maintenance involves unscheduled maintenance performed on a failed system in order to restore it to an operating condition by replacing or repairing the failed component(s) that caused the system failure. The type of maintenance applied depends on the costs involved and on safety implications of failures.

Repair cost limits have long been used, their values often being based on the principle that no more should be spend on the repair of an item than the item itself is worth. As such, policies based on repair cost limits have been regarded as an efficient way to decide on whether to repair or replace.

This dissertation will focus mainly on a replacement model that depends on both repair costs (assumed to be independent and identically distributed random variables) and age of a single unit system. The system is maintained according to the failure type. Type 1 failure causes only a minor damage and is removed by a minor repair. After a minor repair the unit is as ‘good as old’. Type 2 failure requires a unit to be replaced.

The objectives of this dissertation are:

i. Presentation of the theoretical foundations of the replacement policies with minimal repairs at failure, in particular discussion of the failure rate concepts, nonhomogeneous Poisson processes (used here to model the stochastic behaviour of failures under minimal repair), two failure type modes and renewal reward processes.

ii. To combine the repair cost limit replacement policies with other standard replacement policies found in the literature.

iii. The numerical examples of the resulting policies will be considered and the maintenance cost rate will be calculated.

iv. Some problems regarding the gap between theoretical results and practical applications will be highlighted.

Aim (i) will be covered in Chapters 2 and 3 in this dissertation, whilst Chapter 4 will deal comprehensively with methods and literature studies pertaining to the research.

In Chapter 5, we consider the combination of repair cost limit replacement policy with the well-known age replacement policy in order to come up with a policy which is more cost-efficient.

Chapter 6 will deal with the optimization of the combined policy and presents also a numerical example.

Basic Assumptions:

1) Maintenance actions are restricted to minimal repairs and replacements
2) After a replacement, a system is 'as good as new'
3) All maintenance actions take only negligible time
4) The system is aging (its failure rate is increasing).

Definition 1:

Minimal Repair: This is a repair which restores the capability of the system to continue its work, but does not affect its failure rate.

2. Theoretical Foundations

2.1 Residual Lifetime, System Aging and Failure Rate

Extensive discussion or exposition of the principles of this chapter can be found in most textbooks in the field of Reliability and Maintenance theory. For the development of this section, we refer to Barlow and Proschan [1965], Beichelt [2006], Grosh [1989], Kapur et al. [1999] and Nachlas [2005].

Let X be a random lifetime of a system with distribution function $F(t)$. The lifetime of a system is the time span from its start-up time point to its failure, where 'failure' is assumed to be an instantaneous event which need not be equivalent to the end of the system's useful life. In the engineering context, $F(t)$ is called *failure probability* of the system and $\bar{F}(t) = 1 - F(t)$ is called *survival probability* with regard to the interval $[0,t]$, because $F(t)$ and $\bar{F}(t)$ are the respective probabilities that the system does or does not fail in $[0,t]$.

Let $F_t(x)$ be the distribution function of the residual lifetime X_t of the system, which has already worked t time units without failing. This conditional failure probability is given by

$$F_t(x) = P\left(X_t \le x\right) = P(X - t \le x | X > t) \tag{2.1}$$

In probability theory, the conditional probability of A on condition B is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\therefore F_t(x) = \frac{P(X - t \le x \cap X > t)}{P(X > t)} = \frac{P(t < X \le t + x)}{P(X > t)}$$

$$= \frac{F(t + x) - F(t)}{\overline{F}(t)}; \qquad x \ge 0, \;\; t \ge 0$$

The corresponding conditional survival probability $\overline{F}_t(x) = 1 - F_t(x)$ is given by

$$\overline{F}_t(x) = \frac{\overline{F}(t + x)}{\overline{F}(t)}; \qquad x \ge 0, \;\; t \ge 0 \tag{2.2}$$

Hence, the mean residual lifetime $\mu(t) = E(X_t)$ of the system is

$$\mu(t) = \frac{1}{\overline{F}(t)} \int_t^{\infty} \overline{F}(x)\,dx \tag{2.3}$$

The engineering background of the conditional failure probability motivates the following definition:

Definition 2: A system is aging in the interval $[t_1, t_2]$, $t_1 < t_2$, if for an arbitrary but fixed *x*, the conditional failure probability $F_t(x)$ is increasing with increasing *t*, $t_1 \le t \le t_2$.

The aging behaviour of the system can be modelled in several ways; one of them is the failure rate. Provided the existence of the density function $f(t) = F'(t)$, the failure rate $\lambda(t)$ is defined as:

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{P\left[t + \Delta t \geq X \middle| X > t\right]}{\Delta t}$$

Thus, $\lambda(t)$ reflects the system' tendency to fail at time t given that it has survived up to time t.

By definition,

$$P\left[t + \Delta t \geq X \middle| X > t\right] = \frac{P\left[t + \Delta t \geq X \cap X > t\right]}{P\left[X > t\right]},$$

so that,

$$P\left[t + \Delta t \geq X \cap X > t\right] = P(t + \Delta t \geq X > t) = F(t + \Delta t) - F(t).$$

Since,

$$P\left[X > t\right] = 1 - F(t),$$

$$\therefore \lambda(t) = \frac{1}{1 - F(t)} \lim_{\Delta t \to 0} \frac{F(t + \Delta t) - F(t)}{\Delta t}$$

$$= \frac{F'(t)}{1 - F(t)}$$

$$= \frac{f(t)}{\overline{F}(t)} \tag{2.4}$$

Integration on both sides of this relationship yields:

$$F(t) = 1 - \exp\left(-\int_0^t \lambda(x)dx\right)$$

$$= 1 - e^{-\Lambda(t)}, \qquad t \geq 0.$$

The function $\Lambda(t)$ is called the integrated failure rate of the system.

$$\Lambda(t) = \int_0^t \lambda(x)dx \tag{2.5}$$

and,

$$F_t(x) = 1 - e^{-[\Lambda(t+x)-\Lambda(t)]},$$

$$\bar{F}_t(x) = e^{-[\Lambda(t+x)-\Lambda(t)]}$$

This representation of $\bar{F}_t(x)$ implies an important property of the failure rate:

A system ages in $[t_1, t_2]$, $t_1 < t_2$, if its failure rate $\lambda(t)$ is increasing in this interval.

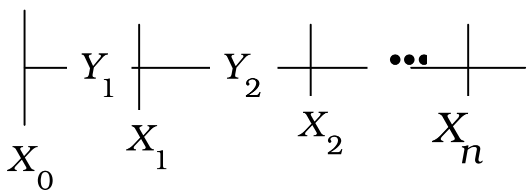
However, definition 2 is more general than this implication, since definition 2 does not require the existence of the density function.

2.2 Renewal Process and Instantaneous Repairs

Think about the state of the system following maintenance. Conceptually, we think in terms of a system that operates until it fails, at which time it is immediately replaced by a statistically equivalent new one that, in turn, operates until its failure.

In this context, the replacements of the failed systems are also called renewals. The sequence of the operating times generates a renewal process.

Definition 3: An ordinary renewal process is a sequence of independent and identically distributed nonnegative random variables, $\{Y_i; \quad i = 1,2,\ldots\}$.



In this context, Y_i is the time between the $(i-1)$th and the ith replacement.

X_n denotes the time point of the nth replacement. Hence, X_n is given by:

$$X_n = \sum_{i=1}^{n} Y_i, \quad n = 1,2,\ldots \tag{2.6}$$

The random variables $\{Y_i;\ \ i=1,2,\ldots\}$ have a well defined probability distribution function, which do not change over time as the process continues.

The renewal counting process $\{N(t),\ t\geq 0\}$ is defined as

$$N(t)=\begin{cases}\max\{n|X_n \leq t\} \\ 0 \quad \text{for } t < X_1\end{cases}. \tag{2.7}$$

Thus, $N(t)$ is the random number of renewals (replacements) that occur during interval (0, t]. Since $\{Y_i;\ \ i=1,2,\ldots\}$ is a sequence of identically distributed random variables with distribution function $F(t)$, the probability distribution function of X_n is the n-fold convolution power of $F(t)$:

$$F_{X_n}(t)=F^{(n)}(t) \tag{2.8}$$

given $F^{(0)} \equiv 1$.

The corresponding density function of X_n is

$$f_{X_n}(t)=f^{(n)}(t),\quad n=1,2,\ldots$$

Two questions need to be addressed using the above model.

a) Examining the time until a certain number of renewals can occur, and
b) The probabilities of the number of renewals occurring in a time interval.

A key relationship in this context is

$$F_{X_n}(t) = P[N(t) \geq n] = P[X_n \leq t] \tag{2.9}$$

To determine the probability distribution of $N(t)$, we use

$$P[N(t) = n] = P[N(t) \geq n] - P[N(t) \geq n+1]$$

$$= F_{X_n}(t) - F_{X_{n+1}}(t); \qquad F_{X_0}(t) = 1, \quad n = 0,1,\ldots$$

A special case of this result is,

$$P[N(t) = 0] = F_{X_0}(t) - F_{X_1}(t)$$

$$= 1 - F_{X_1}(t)$$

$$= \overline{F}_{X_1}(t).$$

The mean value

$$H(t) = E[N(t)],$$

of the random number $N(t)$ of renewals over a fixed time interval $(0,t]$ as a function of t is called renewal function. It can also be expressed as:

$$H(t) = \sum_{n=0}^{\infty} nP[N(t) = n] \tag{2.10}$$

The renewal function $H(t)$ is the trend function of the renewal counting process $\{N(t), t \geq 0\}$.

By using equations (2.8) and (2.9),

$$H(t) = \sum_{n=0}^{\infty} n\left[F_{X_n}(t) - F_{X_{n+1}}(t)\right]$$

$$= \sum_{n=1}^{\infty} F^{(n)}(t)$$

$$= F(t) + \sum_{n=2}^{\infty} \int_0^t F^{(n-1)}(t-x)\,dF(x)$$

$$= F(t) + \int_0^t \sum_{n=1}^{\infty} F^{(n)}(t-x)\,dF(x) \tag{2.11}$$

The Laplace transform of the renewal function yields a direct relationship between the renewal function and the underlying probability distribution of the cycle length $F(t)$.

In view of equation (2.10),

$$H(t) = \sum_{n=1}^{\infty} F^{(n)}(t) \tag{2.12}$$

An analytical solution for the renewal function usually does not exist. A number of numerical methods have been suggested for obtaining $H(t)$ [see example, Baxter et al, (1981) and Kao (1988)].

2.3 Nonhomogeneous Poisson Process (NHPP)

In section 2.2, we focused primarily on renewal process techniques for analyzing repairable systems. The times between failures $\{Y_i;\ i = 1,2,\ldots\}$ were independent and identically distributed observations from a single population. If the i.i.d assumptions do not hold in practice, the renewal process is not a suitable model.

In real life, the time to failure $\{X_i;\ i = 1,2,\ldots\}$ depends on many factors, e.g. basic system design, operating conditions, quality of the materials used to repair, competency of the technician (human factors) and so on.

Therefore, there is a genuine possibility of a non-renewal process in which $\{Y_i;\ i = 1,2,\ldots\}$ is independent and not identically distributed. As an example, think of large complex systems, such as cars, for which the replacement of a few of its many components does not appreciably change the 'age' of the system.

When system age following repair is nonzero, successive operating periods do not have a common distribution and the renewal model does not apply.

Definition 4: A counting process $\{N(t), t \geq 0\}$ satisfying $N(0) = 0$ is called a nonhomogeneous Poisson process (NHPP) with intensity $\lambda(t)$ if it has the following properties:

1) $\{N(t), t \geq 0\}$ has independent increments
2) for $h \to 0$,

$$P[N(t+h) - N(t) = 1] = \lambda(t)h + o(h)$$

   where $o(h)$ is the Landau order symbol (i.e. any function that approaches zero faster than h, when h approaches zero, i.e $\lim_{h \to 0} 0(h)/h = 0$.

3) $P[N(t+h) - N(t) \geq 2] = o(h)$

   This property means that in an interval of length h, the probability that two or more Poisson events occur tends to 0 as $h \to 0$.

   If the intensity function is constant, $\lambda(t) = \lambda$, then NHPP becomes the homogeneous Poisson process

From properties (2) and (3), it follows that as $h \to 0$,

$$P[N(t+h) - N(t) = 0] = 1 - \lambda(t)h + o(h)$$

a) Let us look at the probability distribution of the increments of a nonhomogeneous Poisson process (NHPP)

$$P_i(s,t) = P[N(t) - N(s) = i]; \qquad s < t, \quad i = 0,1,\ldots$$

In view of property 1) of definition 4,

$$P_0(s,t+h) = P[N(t+h) - N(s) = 0]$$

$$= P[N(t) - N(s) = 0].P[N(t+h) - N(t) = 0].$$

Hence, it follows that

$$\frac{P_0(s,t+h) - P_0(s,t)}{h} = -\lambda(t)P_0(s,t) + \frac{o(h)}{h}$$

$$P_n'(s,t) = -\lambda(t)P_n(s,t) + \lambda(t)P_{n-1}(s,t). \tag{2.13}$$

Letting $h \to 0$ yields a partial differential equation of the first order:

$$\frac{\partial}{\partial t}P_0(s,t) = -\lambda(t)P_0(s,t) \tag{2.14}$$

Since the probability of zero occurrences by time 0 is equal to one.

i.e $P_0(0,0) = 1$ or equivalently, $N(0) = 0$,

the solution to equation (2.14) is

$$P_0(s,t) = e^{-[\Lambda(t) - \Lambda(s)]}, \quad \text{where } \Lambda(y) = \int_0^y \lambda(x)dx \tag{2.15}$$

Inserting equation (2.15) into (2.13) for n=1 yields,

$$P_1'(s,t) = -\lambda(t)P_1(s,t) + \lambda(t)P_0(s,t)$$

$$P_1'(s,t) + \lambda(t)P_1(s,t) = \lambda(t)e^{-[\Lambda(t) - \Lambda(s)]}$$

$$P_1(s,t) = [\Lambda(t) - \Lambda(s)]e^{-[\Lambda(t) - \Lambda(s)]}$$

The expression for $P_1(s,t)$ can be substituted in equation (2.13) with $n = 2$, and so on. After few steps, the pattern emerges for the general form of the equation.

Therefore, the joint solution for equations (2.13) and (2.15) has the form:

$$P_n(s,t) = \frac{[\Lambda(t) - \Lambda(s)]^n}{n!} e^{-[\Lambda(t)-\Lambda(s)]}, \qquad n = 0,1,\ldots$$

of which equation (2.15) is just a special case.

Differentiation of (2.15) yields (2.14).

Thus, the increment of the NHPP in the interval $[s,t]$ has a Poisson distribution with mean value $[\Lambda(t) - \Lambda(s)]$.

In particular, the absolute state probabilities of the NHPP at time t,

$$P_n(t) = P_n(0,t) = P[N(t) = n],$$

are

$$P_n(t) = \frac{[\Lambda(t)]^n}{n!} e^{-\Lambda(t)}, \qquad t \geq 0, \qquad n = 0,1,\ldots \qquad (2.16)$$

Thus, $N(t)$ has Poisson distribution with mean value $\Lambda(t)$:

b) Let $F_{X_1}(t) = P(X_1 \leq t)$ be the distribution function and $f_{X_1}(t)$ the probability density of the random time X_1 to the occurrence of the first Poisson event.

Then,

$$P_0(t) = P_0(0,t) = P\left(X_1 > t\right) = 1 - F_{X_1}(t)$$

From (2.16)

$$P_0(t) = e^{-\Lambda(t)}, \quad t \geq 0$$

Hence,

$$F_{X_1}(t) = 1 - e^{-\Lambda(t)} \tag{2.17}$$

$$= 1 - e^{-\int_0^t \lambda(x)\,dx}, \qquad f_{X_1}(t) = \lambda(t)e^{-\int_0^t \lambda(x)\,dx}, \ \ t \geq 0$$

From these formulas, we can see that the intensity function $\lambda(t)$ of the NHPP $\{N(t), t \geq 0\}$ is identical to the failure rate of X_1.

And since $F_{X_n}(t) = P\left(X_n \leq t\right) = P\left[N(t) \geq n\right]$

Using (2.16), the time of the occurrence of the nth Poisson event has distribution function:

$$F_{X_n}(t) = \sum_{n=j}^{\infty} \frac{\left[\Lambda(t)\right]^n}{n!} e^{-\Lambda(t)}, \quad j \geq 1 \tag{2.18}$$

Differentiation with respect to t yields the probability density of X_n:

$$f_{X_n}(t) = \frac{\left[\Lambda(t)\right]^{n-1}}{(n-1)!} \lambda(t) e^{-\Lambda(t)} \qquad \mathrm{t} \geq 0, \ n = 0,1,\ldots$$

$$= \frac{\left[\Lambda(t)\right]^{n-1}}{(n-1)!} f_{X_1}(t) \tag{2.19}$$

The mean value of X_n is

$$E(X_n) = \int_0^\infty \left[1 - F_{X_n}(t)\right] dt$$

$$= \int_0^\infty e^{-\Lambda(t)} \left(\sum_{i=0}^{n-1} \frac{[\Lambda(t)]^i}{i!} \right) dt \tag{2.20}$$

The expected time between $(n-1)$th and nth Poisson events is

$$E(Y_n) = E\left[X_n - X_{n-1}\right]$$

$$= E(X_n) - E(X_{n-1}) \quad \text{becomes}$$

$$E(Y_n) = \frac{1}{(n-1)!} \int_0^\infty \Lambda(t)^{n-1} e^{-\Lambda(t)} dt \quad , n = 1,2,\ldots \tag{2.21}$$

By letting, $\lambda(t) = \lambda$ and $\Lambda(\mathrm{t})$ yields the corresponding HPP, in particular $E(Y_n) = \frac{1}{\lambda}$.

Conditional probability

$$P(X_2 \le t_2 | X_1 = t_1) = 1 - P_0(t_1, t_2)$$

$$= 1 - e^{-\left[\Lambda(t_2) - \Lambda(t_1)\right]}, \qquad t_1 < t_2$$

Differentiation with respect to t_2 yields the corresponding probability density

$$f_{X_2}(t_2 | t_1) = \lambda(t_2) e^{-\left[\Lambda(t_2) - \Lambda(t_1)\right]}, \qquad 0 \le t_1 < t_2$$

the joint p.d.f of (X_1, X_2) is

$$f\left(t_{1,}t_{2}\right)=\begin{cases}\lambda\left(t_{1}\right)F_{X_1}\left(t_{2}\right) & \text{for } t_{1}<t_{2}\\ 0 & \text{otherwise}\end{cases}$$

By induction, we obtain the joint p.d.f of $\left\{X_i,\ i=1,2,\ldots,n\right\}$

$$f\left(t_{1},t_{2},\ldots t_{n}\right)=\begin{cases}\lambda\left(t_{1}\right)\lambda\left(t_{2}\right),\ldots\lambda\left(t_{n-1}\right)f_{X_1}\left(t_{n}\right), & t_{1}<t_{2}<\ldots<t_{n}\\ 0 & \text{otherwise.}\end{cases} \tag{2.22}$$

3. Basic Maintenance Policies

3.1 Nonhomogeneous Poisson Process (NHPP) and Minimal Repair Policy

This section considers the nonhomogeneous Poisson process (NHPP) and a simple maintenance policy. NHPP is used here to model the stochastic behaviour of failures under minimal repairs. The discussion will provide formulas which are needed for analyzing policies in the forthcoming sections.

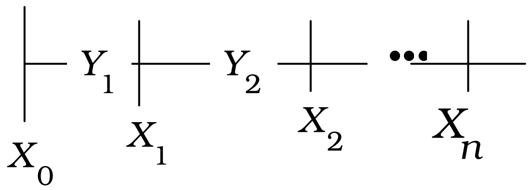
Let X be the random lifetime of a system to be maintained with the distribution function $F(t)$, the survival function $\bar{F}(t)$, the density function $f(t) = F'(t)$, its failure rate $\lambda(t) = f(t)/\bar{F}(t)$,and the hazard function (also known as the integrated failure rate)

$$\Lambda(t) = \int_0^t \lambda(x)dx$$

X is a nonnegative random variable, i.e. $F(0) = 0$. By definition, a minimal repair carried out after a system failure, restores the capability of the system to continue its work, but does not affect the failure rate of the system.

Minimal repair policy: Every system failure is removed by a minimal repair.

Let X_n represent the time to the n^{th} failure. Then



$Y_n = X_n - X_{n-1}$ is the time between the $(n-1)$ th and the n th system failure and $X_0 = 0$.

Suppose system start to work at $t = 0$. System failure happens at time $T = t$ and a minimal repair is conducted. Then, the failure rate of the repaired system is by definition $\lambda(t)$.

The repaired system has a residual lifetime X_t with distribution function (2.1):

$$P\left(X_t \le x\right) = F_t(x) = \frac{F(t+x) - F(t)}{\bar{F}(t)}$$

and the corresponding survival function is given as (2.2):

$$\bar{F}_t(x) = 1 - F(x) = \frac{\bar{F}(t+x)}{\bar{F}(t)}$$

Equations (2.1) and (2.2) are also applicable if t is the time point of any failure, for instance the nth failure.

Let $f_n(t)$ denote the probability density of X_n (time to the nth failure). From equation (2.19),

$$f_n(t) = \frac{[\Lambda(t)]^{n-1}}{(n-1)!} f(t) \qquad (3.1)$$

Since $E(X) = \int_0^\infty x f(x)\,dx$, it follows that

$$E(X_n) = \frac{1}{(n-1)!} \int_0^\infty t[\Lambda(t)]^{n-1} f(t)\,dt \qquad (3.2)$$

and by partial integration,

$$E(X_n) = E(X_{n+1}) - \int_0^\infty \frac{[\Lambda(t)]^n}{n!} \bar{F}(t)\,dt$$

Hence, if Y_n represents the time between $(n-1)$ th and n th failure, from equation (2.21)

$$E(Y_n) = \frac{1}{(n-1)!} \int_0^\infty [\Lambda(t)]^{n-1} e^{-\Lambda(t)}\,dt, \quad n = 1,2,\ldots$$

$$= \int_0^\infty \frac{[\Lambda(t)]^n}{n!} \bar{F}(t)\,dt, \qquad n = 0,1,\ldots \qquad (3.3)$$

Let $N(t)$ be the random number of failures corrected by minimal repairs in $[0,t]$.

Then, from equation (2.16)

$$P[N(t) = n] = P_n(0,t)$$

In particular, $N(t)$ has probability distribution

$$P[N(t) = n] = \frac{[\Lambda(t)]^n}{n!} e^{-\Lambda(t)}, \quad \mathrm{t} \geq 0;\ \mathrm{n} = 0,1,\ldots$$

So that

$$E[N(t)] = \Lambda(t) \qquad (3.4)$$

where

$$\Lambda(t) = \int_0^t \lambda(x)dx$$

$\{N(t), t \geq 0\}$ is an equivalent way to characterize the nonhomogeneous Poisson process (NHPP).

Therefore, the time points of subsequent system failures under the condition that each failure is removed by a minimal repair create a NHPP with intensity function $\Lambda(t)$.

If M_i denotes the cost of the ith minimal repair, then the total repair cost in $[0, t]$ are

$$M(t) = \sum_{i=1}^{N(t)} M_i \tag{3.5}$$

Assuming the (M_i) to be independent of each other and independent of $N(t)$, as well as identically distributed as *M*, the trend function of the stochastic process $\{M(t), t \geq 0\}$ is by Wald's identity :

$$E[M(t)] = E[M]E[N(t)]$$

Therefore, if $c_m = E[M]$ denotes the expected cost of minimal repair and $E[N(t)] = \Lambda(t)$,

$$E[M(t)] = c_m \Lambda(t) \tag{3.6}$$

The non-homogeneous Poisson point process $\{X_1, X_2, \ldots\}$ is an ingredient to a marked point process $\{(X_1, M_1), (X_2, M_2), \ldots\}$.

Let L_i, be the random length of the i^{th} replacement cycle (time between two neighbouring replacements) and C_i the total repair plus replacement cost in the i^{th} replacement cycle. Assume that $L_1, L_2, \ldots$ are independent and identically distributed as L. This assumption implies that the replaced system is statistically as good as new with respect to its lifetime.

The $C_1, C_2, \ldots$ are assumed to be independent and identically distributed as C and independent on the L_i.

$$C(t) = \sum_{i=1}^{N(t)} C_i$$

If $E(C_i)$ and $E(L_i)$ are finite, then with probability 1,

$$\frac{C(t)}{t} \to \frac{E(C)}{E(L)} \text{ as } t \to \infty$$

Proof :

$$\frac{C(t)}{t} = \frac{\sum_{i=1}^{N(t)} C_i}{N(t)} \times \frac{N(t)}{t}$$

but $\frac{\sum_{i=1}^{N(t)} C_i}{N(t)} \to E(C)$ by the strong law of large numbers and

$$\frac{N(t)}{t} \to \frac{1}{E(L)} \quad \text{as } t \to \infty$$

$$\therefore K = \frac{E(C)}{E(L)} \tag{3.7}$$

K is referred to as the (long run) maintenance cost rate. Thus, the maintenance cost rate is equal to the mean maintenance cost per cycle divided by the mean cycle length. The aim now, is to minimize K by proper choice of the parameters on which it depends, in particular on the optimal time between replacements and the repair cost limit.

3.2 General Failure Model and Maintenance Policy

Consider a new system which starts operating at time $t = 0$. Suppose that there are two types of failures which can happen.

Type 1 failures: Failures of this type may be interpreted as minor ones easily removed by minimal repairs.
Type 2 failures: Failures of this type may be complete system breakdowns which require replacement.

This failure model was first proposed and applied to maintenance modelling by [Beichelt, 1976], see also his monographs [Beichelt, 1979]. A couple of years later, the same results had been published by other specialists as well [see Beichelt, 1993] for a survey on this development.

A failure which occurs at system age t, is a type 2 failure with probability $p(t)$,with $0 < p(t) < 1$ and a type 1 failure with probability $\overline{p}(t) = 1 - p(t)$. It is assumed that type 1 and type 2 failures occur independently of each other.

Equivalently, the system following a repair is renewed ('as good as new') with probability $p(t)$ and minimally repaired ('as bad as old') with probability $\overline{p}(t) = 1 - p(t)$.

The result is that the stochastic process which represents the system behaviour is a mixture of the renewal process and the minimal repair NHPP. The key to the analysis is that the times at which a replacement is performed constitute a renewal process that is embedded in the more general point process produced by the system failures.

The points at which a replacement occurs are restart points, i.e. the system failure rate after any replacement is $\lambda(0)$.
Throughout this section, we assume instant replacement and instant repair.

The maintenance cost rate of this policy is given by [Beichelt, 1976] as:

$$K = \frac{\left(\int_0^\infty \Lambda(t)\,dG(t) - 1\right)c_m + c_r}{\int_0^\infty \overline{G}(t)\,dt} \tag{3.8}$$

where c_m, c_r are the mean cost of a minimal repair, a replacement, $(c_m < c_r)$ respectively.
Within this model, a replacement cycle is the time between two neighbouring type 2 failures.

Let $G(t)$ denote the distribution function of the time interval between replacements (random cycle length Y) and $\overline{G}(t) = 1 - G(t)$. Its failure rate is denoted as $\lambda_Y(t)$.

$$G(t) = P(Y \le t)$$

Let $F(t)$ be the system' lifetime distribution function. If only minimal repairs occurs during $[0,t]$, then at time t the system behaves as if it has age t . Its failure rate would be $\lambda(t)$.

If a failure occurs at time t, it will be a type 2 failure with probability $p(t)$. Further,

$$\lambda_Y(t) = p(t)\lambda(t) = p(t)\frac{f(t)}{\overline{F}(t)} \tag{3.9}$$

would be the conditional intensity of the occurrence of a replacement (type 2 failure).

The survival function corresponding to $\lambda_Y(t)$ will be

$$\overline{G}(t) = \exp\left\{-\int_0^t \lambda(x)p(x)\,dx\right\}$$

and,

$$G(t) = 1 - e^{-p(t)\Lambda(t)} \tag{3.10}$$

This is a general result applicable to all choices of the underlying lifetime distributions.

If $p(t) = 1$, $\overline{G}(t)$ is the survival function of the system, i.e.,

$$\overline{G}(t) = e^{-\Lambda(t)} = \overline{F}(t).$$

Let N be the random number of type 1 failures, to the occurrence of the first type 2 failures.

Then,

$$P(N=0)=\int_0^\infty p(t)\,dF(t) \tag{3.11}$$

where

$$f(t)=\lambda(t)e^{-\int_0^t \lambda(x)dx}, \quad t\ge 0$$

given by equation (2.17) and for $n\ge 1$, using equation (2.22)

$$P(N=n)=\int_0^\infty \int_0^{x_{n+1}} \dots \int_0^{x_3} \int_0^{x_2} \prod_{i=1}^{n} \overline{p}(x_i)\lambda(x_i)\,dx_i\, p(x_{n+1}) f(x_{n+1})\,dx_{n+1}$$

Note that for any integrable function $g(x)$,

$$\int_0^t \int_0^x \dots \int_0^{x_3} \int_0^{x_2} \prod_{i=1}^{n} g(x_i)\,dx_i = \frac{1}{n!}\left[\int_0^t g(x)\,dx\right]^n, \quad n\ge 2 \tag{3.12}$$

Therefore,

$$P(N=n)=\frac{1}{n!}\int_0^\infty \left(\int_0^t \overline{p}(x)\lambda(x)\,dx\right)^n p(t) f(t)\,dt, \; n=0,1,\dots \tag{3.13}$$

and

$$\begin{aligned} E(N) &= \sum_{n=1}^{\infty} nP(N=n) \\ &= \int_0^\infty \Lambda(t)\,dG(t)-1 \end{aligned} \tag{3.14}$$

The details of the derivation are given in the Appendix (A1).

In the special case when $p(t)=p, \quad p>0$:

$$\overline{G}(t)=1-G(t)=\left[\overline{F}(t)\right]^p \quad \text{and} \quad E(N)=\frac{1-p}{p}.$$

Then (3.8) simplifies to

$$K = \frac{\left(\frac{1-p}{p}\right)c_m + c_r}{\int_0^{\infty}\left[\bar{F}(t)\right]^p dt} \tag{3.15}$$

The above equations show that if $p(t) \equiv p$, the distribution on the time between replacements will be of the same class as the lifetime distribution $F(t)$, i.e. Increasing Failure Rate (IFR), Decreasing Failure Rate (DFR), and so on.

Note that, if $p(t)$ is increasing, the times between replacements tend to become shorter and if $p(t)$ is decreasing, the times tend to become longer.

4. Methods and Literature Survey

4.1 Basic Repair Cost Policies

Probably the first cost-based replacement policy for technical systems consists in replacing a system by a new one at the end of its economic lifetime. This will occur when the sum of the capital depreciation payments and the maintenance costs are at a minimum. This is referred to by Clapham [1957], as the ‘economic life’ of a system.

Clapham [1957] suggest that if the unit is kept for less than this time, greater expense will be incurred due to heavier depreciation charges, if it is kept for longer, the rising maintenance charges will cause extra loss costs.

However, a replacement policy only makes sense if the system suffers from wear out problems. Implementing a replacement policy while there is no wear out problems is economically not beneficial, because a replaced unit would have the same or a higher probability of failing compared to a unit that was not replaced.

Clapham found that there is a point at which it is on average cheaper to replace a system than to continue with repairs. He derived a replacement time that will minimize the total life cycle cost, while taking into account repair cost and the replacement cost. He applied his results to the underground coal haulage.

Derivation of the economic life:

Usually, system maintenance costs rise with age. Denoting again c_p as the replacement cost, c_m as the average repair cost and that the unit is used for T years.

Clapham [1957], showed that the economic life model take on a simple form when the increase in maintenance cost with age is linear. In addition, let $N(T)$ be the number of failures in $[0, \mathrm{t}]$.

The total system cost between replacements is:

$$TSC(T) = c_m E[N(T)] + c_p \tag{4.1}$$

so the average system cost is:

$$AvC(T) = \frac{c_m E[N(T)] + c_p}{T} \tag{4.2}$$

Knowing that,

$$E[N(t)] = \Lambda(t),$$

for $\lambda(\mathrm{t}) = at \qquad a > 0,$ and a constant

$$\Lambda(t) = a\frac{T^2}{2}$$

The instantaneous maintenance cost at time T is equal to:

$$IMC(T) = ac_m T \tag{4.3}$$

The long term average system cost is minimized when the system is replaced at time T_0, such that the instantaneous maintenance cost equals the average system cost.

The following equation holds at optimum replacement time T_0:

$$ac_m T_0 = \left(ac_m \frac{T_0^2}{2} + c_p \right) \Big/ T_0$$

$$T_0 = \left(\frac{c_p}{0.5ac_m} \right)^{\frac{1}{2}}$$

if $\quad \Lambda(t) = \lambda T^{\beta}$ (Weibull lifetime)

$$\therefore T_0 = \left(\frac{c_p}{\lambda(\beta - 1)c_m} \right)^{\frac{1}{\beta}} \tag{4.4}$$

The value of T_0 is called the ‘economic life’ of the unit and is the operating time when the average cost of operation per unit time is at the minimum.

The estimate for the optimum replacement time to minimize life cycle cost is given by:

$$\hat{T}_0 = \left(\frac{c_p}{\hat{\lambda}(\hat{\beta} - 1)c_m} \right)^{\frac{1}{\hat{\beta}}} \tag{4.5}$$

When the system fails, a decision has to be made to determine whether it is economical to replace the system, or to repair (replace) the failed components and reset the system to operation.

If a repair or replacement of the failed component restores function to the entire system but the failure rate of the system remains more or less the same (as it was just before failure), then the repair is called minimal repair (see definition 1, page 14).

For example, if the fuel pump of a car fails and is replaced, the overall failure rate of the car is essentially unchanged. Since the failure rate of systems increase with age, it would become too expensive or impossible to maintain operation by minimal repairs.

Some serious major failures will directly induce replacements of the system. The question becomes, when is it optimal to replace the entire system instead of performing minimal repairs? The original work on maintenance policy which includes replacements and minimal repairs was first introduced and discussed by Barlow and Hunter [1960], using a periodic replacement Model with minimal repairs between replacements.

The original objective of the minimal repair model was to find a replacement age t_0, which minimizes the long-run expected cost per unit time of replacement and minimal repairs.

Failures that occur before t_0 , are removed by minimal repairs. The system is replaced with an identical new system at time t_0. The long run expected cost per unit time using a replacement age t_0 for the basic model is given by:

$$K(t) = \frac{c_m E[N(t)] + c_p}{t} \tag{4.6}$$

Where $E[N(t)]$ represents the expected number of failures (minimal repairs) during the period $(0, t]$.

For Weibull system lifetime, the minimum cost per unit time is

$$K(t_0) = c_m \lambda\beta \left[\frac{c_p}{(\beta - 1)c_m} \right]^{\frac{(\beta-1)}{\beta}} \tag{4.7}$$

The basic minimal repair model developed by Barlow and Hunter has been generalized and modified by amongst other authors, Makabe and Morimura [1963], Muth [1977] and Cléroux et al. [1979], to fit more realistic situations.

Makabe and Morimura [1963] modified entirely the concept of age replacement under minimal repair introduced by Barlow and Hunter [1960]. Instead of finding a fixed time (age) to a replacement, they found the optimal number of minimal repairs before the system is replaced.

Policy:

The system is replaced at the time of n^{th} failure $(n = 1,2,\ldots)$ after its installation and undergoes only minimal repairs at failures between replacements.

Assuming $F(t)$ with finite mean μ and density $f(t)$,

$\overline{F}(t) = \exp[-\Lambda(t)]$, where $\overline{F}(t) = 1 - F(t)$.

Further assumed that the failure rate $\lambda(t)$, is continuous, monotone increasing and remains undisturbed by minimal repairs.

Then, if the times for repairs and replacement are negligible, then the expected cost rate is

$$K(n) = \frac{(n-1)c_m + c_p}{E(X_n)} \tag{4.8}$$

where $E(X_n)$ refers to equation (3.2)

An optimal n_0 is found to be the smallest integer n satisfying

$$\underset{(3.2)}{\underset{\downarrow}{E(X_n)}} - \left[n - 1 + c_p/c_m\right] \underset{(3.3)}{\underset{\downarrow}{E(Y_{n+1})}} \geq 0 \quad ; \quad n = 1,2,\ldots \tag{4.9}$$

If X is Weibull distributed with

$$F(t) = 1 - \exp(-\lambda t)^{\beta}, \qquad \beta > 1,\ t \geq 0$$

Under this assumption, condition (4.9) becomes

$$\beta n - \left[n - 1 + c_p/c_m\right] \geq 0 \ ; \quad n = 1,2,\ldots$$

Hence, if $c_p > c_m, \quad \beta > 1$

$$n_0 = \left\| \frac{1}{\beta - 1}\left(\frac{c_p}{c_m} - 1\right) \right\| + 1,$$

where $\|x\|$ is the largest integer being less than or equal to x. If $x < 0$, then $\|x\| = 0$.

Other cases,

$$n_0 = 1 \qquad \text{if, } \beta > 1,\ c_p \le c_m$$

$$n_0 = \infty \qquad \text{if, } \beta = 1,\ c_p \ge c_m$$

$$n_0 = 1 \qquad \text{if, } \beta = 1,\ c_p < c_m$$

$$n_0 = \infty \qquad \text{if, } \beta < 1$$

Later, Morimura [1970] extended the policy by Makabe and Morimura [1963] by introducing another policy variable T (accumulated operating time).

Policy:

Under this policy, all failures before the n^{th} failure are corrected by minimal repair. If the n^{th} failure occurs before an accumulated operating time T, it is corrected by minimal repair and the next failure induces a replacement.

If however, the n^{th} failure occurs after T, it induces replacement of the unit at time T. The policy decision variables are n and T. If the policy decision variable T is zero, this policy reduces to that of Makabe and Morimura [1963].

Muth [1977], developed a model in which the policy was to have minimal repair if a failure occurred before a fixed time t^* and to have system replacement at the first failure after t^*.The meaning of t^* changes from the fixed replacement time as used by Barlow and Hunter [1960] to a threshold value for replacement.

They used the mean residual life function of the system to prove that their model yields a lower long-run expected cost per unit time than the basic model by Barlow and Hunter.

Muth assumes that the mean residual life function of the system is strictly decreasing after some age t_0. Under this condition, called positive aging, the system deteriorates and eventually reaches a condition where it is no longer economically justifiable to repair the system after failure.

Policy:
At failure, the system undergoes minimal repair if the failure time $t < t^*$, and is replaced otherwise.

The cost function to minimize in Muth's model is given by

$$K(t) = \frac{c_m E[N(t)] + c_p}{t + \mu(t)} \tag{4.10}$$

where $\mu(t)$ is the mean residual life function of the system at time t $(i.e., \quad \mu(t) = E[\tau - t \mid \tau > t])$.

Muth suggests that this model equation above can be solved for the optimum t_0 using a simple calculus approach. He does not give a specific algorithm for the case when a calculus approach cannot be used.

A policy similar to Makabe and Morimura [1963] was also investigated by Park [1979], in which a unit is replaced at the n^{th} failure and minimal repairs are performed for the first (n-1) failures.

Park [1979] shows a closed-form solution of his model for the case when the failure distribution of the system is Weibull. He also compares the results obtained from this solution with the optimal age replacement policy as given by Barlow and Hunter. Park's policy yields lower long-run expected cost per unit time than a fixed time replacement policy.

The number of failures before replacement is less in Park's count policy than in the fixed time policy. However, all these results are shown numerically for the Weibull distribution and no mathematical proof is given.

The cost function for Park's model is

$$K(n) = \frac{(n-1)c_m + c_p}{E\{cycle\}} \tag{4.11}$$

where the value for $E\{cycle\}$ equivalent to $E(X_n)$ depends on the failure distribution of the system and on the number of failures n before replacement. Park gives a simple expression that has to be satisfied for n_0 to be an optimal solution to equation (4.11).

For Weibull distribution,

$$E\{X_n\} = \frac{\Gamma\left(n + \frac{1}{\beta}\right)}{\left[\lambda\Gamma(n)\right]} \tag{4.12}$$

and the cost function becomes

$$K(n) = \left[(n-1)c_m + c_p\right]\frac{\lambda\Gamma(n)}{\Gamma\left(n + \frac{1}{\beta}\right)} \tag{4.13}$$

Phelps [1981] takes the minimal repair models developed by Barlow and Hunter[1960], Muth [1977], and Park [1979] and compares them using an increasing failure rate distribution.

Using the long-run expected cost per unit time, Phelps shows that Muth's policy, replace at the first failure that occurs after time t, is the optimal among the three policies. Phelps also finds that Park's policy is better than that of Barlow and Hunter.

He suggested that, since the t_0 (optimum age) for Muth's model is difficult to obtain, a t_0 obtained from Park's model can be used and still make a good replacement policy.

4.2 Two Failure type Model

Nakagawa [1981] considered the system with two types of failures introduced by Beichelt [1976]. Under this policy, when a system fails, type 1 failure occurs with probability (*1-p*) and is removed by minimal repair; type 2 failure occurs with probability *p* and is removed by replacement.

Type 1 failure are minor failures which are easily restored to the same operating state as before failure by minimal repair, whereas type 2 failure induces a total breakdown of the system.

Policy:

The system is replaced at the times of type 2 failure or n^{th} type 1 failure, whichever occurs first.

The expected number of minimal repairs (i.e., type 1 failures) before replacement is

$$(n-1)\bar{p}^{n} + \sum_{k=1}^{n}(k-1)\bar{p}^{k-1}p$$

$$= \left(\bar{p} - \bar{p}^{n}\right)\Big/(1-\bar{p}) \qquad (4.14)$$

the expected cost rate is given by

$$K(n,p) = \frac{c_m\left[\left(\overline{p} - \overline{p}^n\right) \big/ (1 - \overline{p})\right] + c_p}{E[L]} \tag{4.15}$$

where, $\quad E[L] = \sum_{n=0}^{k-1} \overline{p}^n E(Y_n)$

and $\quad E(Y_n)$ is given by equation (3.3)

when $p = 0$,

$K(n,0)$ will be given as

$$K(n,0) = \frac{(n-1)c_m + c_p}{E(X_n)} \tag{4.16}$$

and the optimal policy by Makabe and Morimura [1963] is applied.

when $p = 1$,

$$K(n,1) = \frac{c_p}{\mu} \tag{4.17}$$

which is constant for all n, and hence the system is replaced only at the occurrence of type 2 failure.

4.3 Repair Cost Limit models

The repair cost limit replacement policy states that as soon as the repair cost exceeds a given limit, the system is replaced by a statistically equivalent new one; otherwise, a minimal repair is carried out.

Repair cost limit replacement policies were first proposed by Gardent and Nonat [1963]. Drinkwater and Hastings [1967] and Hastings [1969] considered the repair-limit problem for army vehicles.

Policy:
When a unit fails, it is first inspected and the repair cost is estimated. If the estimated cost exceeds a certain amount, then the unit is not repaired but it is replaced.

Drinkwater and Hastings [1967] further derived the repair limit value, at which the expected future cost per vehicle-year when the failed vehicle is repaired, is equal to the cost at which the failed vehicle is scrapped and a new one is substituted.

The repair limit is a limit on the amount of money which can be spent on the repair of a vehicle.
The values of the repair limits are dependent on the type and/or the age of the vehicle.

Vehicles needing repair work in excess of the appropriate repair limit are not repaired but scrapped and replaced. Repair costs were assumed to be a function of the age of the vehicle.

Repair decision:

When a vehicle fails: 2 alternative courses can be taken:

1. Repair the vehicle
2. Replace the vehicle (scrap the vehicle and substitute with a new one).

Consider a vehicle at age t which requires maintenance:

c_1 cost of a repair

$m(t)$ expected total cost of future repairs

$\mu(t)$ expected remaining life of the vehicle

θ expected future cost per vehicle - year

$c(t)$ repair limit at time t

Course 1

Repair the vehicle

$$\text{future cost} = \frac{c_1 + m(t)}{\mu(t)} \tag{4.18}$$

Course 2

Replace the vehicle

Let the expected future cost per vehicle-year be: θ

and ignoring the resale value for the moment, the decision will be to repair if

$$\frac{c_1 + m(t)}{\mu(t)} < \theta.$$

Otherwise, the decision will be to scrap the vehicle (i.e. replacement).

The limiting value of $c(t)$ occurs when

$$\frac{c(t)+m(t)}{\mu(t)}=\theta \qquad (4.19)$$

Hence, $c(t)=\theta\times\mu(t)-m(t)$, called the repair limit equation.

Drinkwater and Hastings [1967] also proposed three methods for optimizing the repair-limit policy as follows: by simulation and/or, by use of certain frequency distribution functions which are found to represent the repair cost data and by using dynamic programming methods.

Hastings [1969] further assumes year by year failure rates and costs formulating the problem as a Markov decision process. They also formulated and discussed cases of finite and infinite planning horizons on the replacement model and the issue of discounted and undiscounted costs.

Di Veróli [1974] investigated the problem of finding the optimal replacement policies for equipment subject to failures with randomly distributed repair costs, the degree of reliability of the equipment being considered as a state of a Markov process.

He then described the procedure to obtain an optimal discrete policy and further proposed algorithms to find optimal combined policies, both for preventive replacement and for replacement in case of failure by using repair-limit strategies.

Repair cost limit replacement policies are characterised as follows: When a system fails at age t, it is first inspected and the repair cost is estimated. If the estimated cost exceeds a certain prescribed limit c called the repair limit, then the system is not minimally repaired but it is replaced by an equivalent new one. Otherwise, a minimal repair is carried out.

The mathematical basis for the analysis of the repair cost limit replacement policies is the two failure type model.

Let C be the random repair cost of the system and its distribution function as $R(x) = P(C \leq x)$. For a given repair limit - c , two types of system failures are generated:

A system failure at time t is of type 1 (type 2) if the corresponding random repair costs $C \leq c$ $(C > c)$, respectively. The system will be replaced with probability

$$p = \overline{R}(c) = P(C > c)$$

and minimally repaired with probability

$$\overline{p} = R(c) = P(C \leq c)$$

Let c_r be the mean cost of a replacement after a type 2 failure with $0 < c < c_r$. It is further assumed that the cost of a minimal repair (c_m is constant) does not depend on the repair cost limit.

Given the failure type probabilities above, the random cycle length L has, according to (3.10), a distribution function

$$G(t) = 1 - e^{-\int_0^t \overline{R}(c)\lambda(x)dx}$$

and mean value

$$E[L] = \int_0^\infty e^{-\int_0^t \overline{R}(c)\lambda(x)dx} dt$$

From (3.14), the mean number of minimal repairs in a cycle is

$$E[N] = \int_0^\infty \Lambda(t)\lambda(t)\overline{R}(c)e^{-\int_0^t \overline{R}(c)\lambda(x)dx} dt - 1 \tag{4.20}$$

and by (3.7), the long run expected cost per unit time from repairs and replacement (maintenance cost rate) is

$$K = \frac{\left[\int_0^\infty \Lambda(t)\lambda(t)\overline{R}(c)e^{-\int_0^t \overline{R}(c)\lambda(x)dx} dt - 1\right]c_m + c_r}{\int_0^\infty e^{-\int_0^t \overline{R}(c)\lambda(x)dx} dt} \tag{4.21}$$

In the special case where $p = \overline{R}(c), \quad p > 0$,

$$E[L] = \int_0^\infty \left(\left[\overline{F}(t)\right]^p \right) dt,$$

and

$$E[N] = \frac{(1-p)}{p} = \frac{R(c)}{\overline{R}(c)}.$$

The details of the derivation are given in the Appendix (A2). Therefore,

$$K = \frac{\left[\frac{(1-p)}{p}\right] c_m + c_r}{\int_0^\infty \left(\left[\bar{F}(t) \right]^p \right) dt} \tag{4.22}$$

The two special cases- viz $c = 0$ and $c = \infty$ correspond to always replacement and always minimal repair, respectively.

By applying numerical methods, it is in principle easy to obtain a repair cost limit c being optimal with respect to K.

Using the basic minimal repair model by Barlow and Hunter [1960], Cléroux et al. [1979] investigated an optimal replacement policy by assuming that the minimal repair cost c_m in the basic model is a random variable denoted by C_m.

Policy:
At failure, if the random minimal repair cost C_m is greater than a constant $0 \leq \delta \leq 1$ multiplied by the fixed replacement cost c_p, the system is replaced. However, if $C_m \leq \delta c_p$, minimal repair is performed. The repair cost limit is denoted as δc_p.

They interpret the parameter δ as a given percentage of the cost c_p that a decision maker selects according to some experience. Under the assumption of increasing failure rate (IFR), Cléroux et al. [1979] give a simple algorithm to find the optimal replacement age t_0.

Their algorithm essentially calls for the solution of the derivative of equation (4.23) set equal to zero.

Cléroux et al. [1979]' cost function model is

$$K(t) = \frac{\frac{c^*}{1-p} G(t) + c_r G(t) + c_p \overline{G}(t)}{\int_0^{\tau} \overline{G}(t)\,dt}, \qquad 0 < c_p < c_r \tag{4.23}$$

where c^* represent the average repair cost given that a repair is made; p the probability of repairing at failure and assumed that $0 < p < 1$. However, they were not interested in deriving the optimal repair cost limits.

In a later work, Park [1983] considered an optimum minimal repair cost limit policy, in which the failed system is minimally repaired (or replaced by a new one) if the repair cost is estimated to be less (greater) than a constant cost limit, c. It must be pointed out at this stage that the structure of Park' model is the same as that presented by Beichelt [1976].

A Weibull distribution of the time to failure and a negative exponential distribution of estimated repair cost were assumed for analytical evaluation.

The average cost per unit time for repairs and replacement is

$$K(c) = \frac{c_r + c_m \frac{R(c)}{\overline{R}(c)}}{E[L]} \tag{4.24}$$

where the value of $E[L]$, depends on the failure distribution of the system. For the Weibull time to failure distribution,

$$E[L] = \frac{1}{\lambda}\Gamma\left(1 + \frac{1}{\beta}\right)\left[\overline{R}(c)\right]^{-\frac{1}{\beta}}, \qquad \lambda > 0, \quad \beta > 1$$

and the probability that the estimated repair cost will exceed a repair cost limt, c is

$$\overline{R}(c) = \exp\left(-c/\mu\right) = 1 - R(c) \tag{4.25}$$

where μ represent the average estimated repair cost.

More recently, Park [1985] considered a variant of his earlier repair cost limit replacement policy by assuming an exponentially declining repair cost limit, negative exponentially distributed repair cost and a Weibull time to failure distribution.

He presented a mathematical model to evaluate a cost limit replacement policy for a system that follows a general time to failure distribution. When the failed system requires repair, it is first inspected and the repair cost is estimated. Minimal repair is only then undertaken if the estimated cost is less than the exponentially declining repair cost limit.

They assumed that repair cost limit set at time t, declines exponentially as follows:

$$c(t) = c_r \exp(-\alpha t) \qquad (4.26)$$

where α, represents the exponential decline rate of the cost limit, and it then follows that

$$\overline{R}(c) = \exp\left[{-c_r \exp(-\alpha t)}\big/{\mu}\right] \qquad (4.27)$$

since the estimated repair cost follows a negative exponential distribution with average μ.

The long run average cost per unit time from repairs and replacement is

$$K(c) = \frac{c_r + c_m \dfrac{R(c)}{\overline{R}(c)}}{\int_0^\infty \overline{G}(t)dt} \qquad (4.28)$$

Under the assumption of IFR, Park [1985] gives a simple microcomputer program to evaluate the effects of α on the average cost. However, no formal optimization of the average cost was attempted.

Yun and Bai [1987], proposed a repair cost limit policy in which, when a system fails, the repair cost is estimated and the repair is undertaken if the estimated cost is less than a pre-determined limit , c and the repair is minimal.

Otherwise, the system is replaced. This policy by Yun and Bai, is a generalization from the one by Drinkwater and Hastings [1967].

The cost function for Yun and Bai's model is

$$K(c) = \frac{c_r \overline{R}(c) + c_m R(c)}{\overline{R}(c) \int_0^\infty e^{-\Lambda(t)\overline{R}(c)} dt} \qquad (4.29)$$

The structure of the function is similar to the one given by Park [1985]. The optimal c, was studied for the exponential repair cost distribution and Weibull distributed lifetime.

A numerical example was also given. In the numerical evaluation it was shown that the optimal repair cost limit is sensitive to c_r and p (which is the probability that the system after repair has the same failure rate as before failure), but insensitive to β (Weibull distribution's shape parameter).

Kapur et al.[1989] also combine the failure type model with repair cost limits, but in another way than done in this thesis. Replacement are performed when a type 2 failure occurs and/or when the repair cost exceeds the limit. Hence in this case the failure type does not depend on the corresponding repair cost.

Kapur et al. [1989] proposed three extended models that generalizes model due to Park [1983, 1987]. In each model the minimal repair cost is estimated by inspection.

The system begins to operate at time zero (*t*=0) and is replaced at the n^{th} failure or when the estimated repair cost exceeds the pre-determined repair limit. In Model *a*, and Model *b*, the failed system is replaced at the n^{th} type 1 failure; type 2 failure or when the repair cost due to type 1 failure exceeds the limit.

Type 1 failures are minimal failures and occur with constant probability and Model *c*, the failed system is replaced at n^{th} type1 failure, type 2 failure occurring before n^{th} type 1 failure or when the repair cost due to type 1 failure exceeds the cost limit.

The costs involved in the models presented in Kapur et al., involves, besides the basic minimal repair cost at failure c_m, and cost of scheduled replacement c_p, and the cost due to inspection c_s.

The long-run expected cost per unit time for each of the models *a, b,* and *c*, respectively, are

$$K_a(n,c) = \frac{c_p + \left(c_s + c_m R(c)\right)\sum_{n=0}^{k-2} R(c)^n}{\sum_{n=0}^{k-1} R(c)^n E\left(Y_n\right)}, \qquad k = 2,3,\ldots \tag{4.30}$$

where $R(x)$ is the cumulative distribution function of the minimal repair cost.

$$K_b(n,c) = \frac{c_p + \overline{p}\left(c_s + c_m R(c)\right)\sum_{n=0}^{k-2} \overline{p}R(c)^n}{\sum_{n=0}^{k-1} R(c)^n \int_0^\infty \frac{\overline{p}\Lambda(t)^n}{n!} \overline{F}(t)\,dt} \tag{4.31}$$

where $\overline{p}$ is the probability of type 1 failure.

If $\overline{p} = 1$, we get $K_a(n,c)$.

$$K_c(n,c) = \frac{c_p + \left(c_s + c_m R(c)\right)\sum_{n=0}^{k-2} R(c)^n \int_0^\infty A_k(t)\overline{p}(x)\,d\Lambda(x)}{\sum_{n=0}^{k-1} R(c)^n \int_0^\infty A_k(t)\,dt} \tag{4.32}$$

where $A_k(t)$ represents the probability of k type 1 failures and no type 2 failures in the interval $(0,t]$; $\overline{p}(t)$ is the probability of type 1 failure when the age of the system reaches t.

$$A_k(t) = \frac{\left(\int_0^t \overline{p}(x)\,d\Lambda(x)\right)^k}{k!}\exp\left[-\Lambda(t)\right], \qquad k = 0,1,2,\ldots \tag{4.33}$$

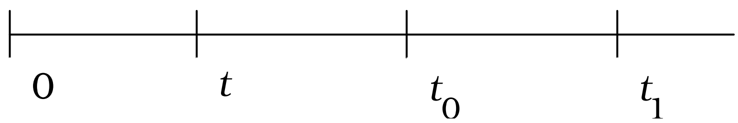
with $A_0(t) = e^{-\Lambda(t)} = \overline{F}(t)$

$A_k(t)$ was derived in Beichelt [1976]. An application of these models to computer science was also discussed.

Butani [1991] presents two modifications which generalize the model given by Muth [1977], by considering 2 types of failures as introduced by Beichelt [1976], e.g type 1 and type 2 failures, each occurring with probabilities $\overline{p}$ and p, respectively.

The policy by Muth[1977], does not consider the case when there is a major breakdown before the system attains age t_0. In such a case it may not be worthwhile undergoing minimal repairs and may be advisable to replace the system by a new one.
The modifications give alternative models that emphasize practical considerations.

In Model I, depicted graphically below:



System begins to operate at time zero (*t=0*).
System is replaced because of type 2 failure before t_0.
System is replaced on failure after age t_0.
The expected cost rate, K(t), for Butani's Model I is

$$K(t) = \frac{c_r + c_m\left[\int_0^t \overline{G}(x)\,d\Lambda(x) - G(t)\right]}{\int_0^t \overline{G}(x)\,dx + \overline{G}(t)\mu(t)} \tag{4.34}$$

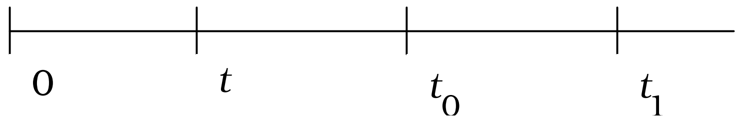
where $\int_0^t \overline{G}(x)\,d\Lambda(x) - G(t) = E\left(Z_t\right)$

z_t represents the random number of minimal repairs during the time interval $\left[0, \min\{Y, t\}\right]$,

$\mu(t)$ is again the mean residual life of the system at time *t*.

$\overline{G}(t) = \exp\left[-p\Lambda(t)\right]$

Model II



System begins to operate at time zero (*t=0*).

System is replaced on type 1 failure before time t_0, when the repair cost determined by inspection exceeds limit *c*.

System is replaced because of type 2 failure before t_0. System is replaced on failure after age t_0.

The expected cost rate, *K*(*t,c*) for Butani's Model II is

$$K(t,c) = \frac{c_r + \dfrac{\overline{p}\left[c_m R(c) + c_S\right]}{1 - \overline{p}R(c)}\left\{1 - e^{-H(t)}\right\}}{\int_0^t e^{-H(x)}dx + e^{-H(t)}\mu(t)} \tag{4.35}$$

where c_S represents inspection costs.

$H(t) = \left[1 - \overline{p}R(c)\right]\Lambda(t)$

Butani [1991] also looked at the special cases where $\bar{p}=1$, for Model II, the cost function becomes

$$K(t,c)=\frac{c_r+\left[c_m R(c)+c_s\right]G(t)/\bar{R}(c)}{\int_0^t \bar{G}(x)\,dx+\bar{G}(t)\,\mu(t)} \tag{4.36}$$

Beichelt and Krober [1992] considered maintenance policy with time-dependent repair cost limits. Since system aging implies an increase in the mean failure frequency and in the mean repair costs with increasing system age t.

They assumed for a given $c_1 > c_2$ that

repair cost limit, $c(t)$ be given by $c(t)=\begin{cases} c_1 & 0 \le t \le T \\ c_2 & t > T \end{cases}$

They also assumed that the cost of minimal repairs depends on the repair cost limit.

The probability to carry out a replacement at failure time t, is

$$P(C > c(t)) = p(t) = \begin{cases} \bar{R}(c_1), & 0 \le t \le T \\ \bar{R}(c_2), & t > T \end{cases}$$

The cycle length L has survival function

$$\bar{G}(t)=\exp\left(-\int_0^t p(x)\lambda(x)\,dx\right)=\begin{cases} \bar{F}(t)^{\bar{R}(c_1)}, & 0 \le t \le T \\ \dfrac{\bar{F}(T)^{\bar{R}(c_1)}}{\bar{F}(T)^{\bar{R}(c_2)}}\bar{F}(t)^{\bar{R}(c_2)}, & t > T \end{cases}$$

The maintenance cost rate is

$$K(c_1, c_2, T) = \frac{G(T)\left(\frac{\int_0^{c_1} \overline{R}(x)\,dx}{\overline{R}(c_1)} - c_1\right) + \overline{G}(T)\left(\frac{\int_0^{c_2} \overline{R}(x)\,dx}{\overline{R}(c_2)}\right) + c_r}{\int_0^{\infty} \overline{G}(t)\,dt} \qquad (4.37)$$

Beichelt and Mamabolo [2004], further investigated the influence of the continuous age-dependent repair cost limits on the maintenance cost rate. Contrary to Beichelt and Krober's model above, they assumed that the cost of minimal repair does not depend on the repair cost limit.

They considered the hyperbolic and linearly decreasing repair cost limits as follows:
For hyperbolical repair cost limit they assume

$$c(t) = \begin{cases} c_r, & 0 \le t < \frac{d}{c_r - c} \\ c + \frac{d}{t}, & \frac{d}{c_r - c} \le t < \infty \end{cases}$$

$$\text{with} \qquad 0 \le c < c_r$$

The system failure rate at age t implies a replacement with probability

$$p(t) = \begin{cases} 0 & , 0 \le t < \frac{d}{c_r - c} \\ \frac{c_r - c}{c_r} - \frac{d}{c_r t} & , \frac{d}{c_r - c} \le t < \infty \end{cases}$$

Otherwise, our basic notation and assumptions still apply. They also did the comparison to the constant repair cost limit policy. Under the assumptions that the repair cost distribution follows a uniform distribution

$$R(x) = P(C \le x) = \begin{cases} 1 - \left(\dfrac{c_r - x}{c_r}\right)^h & ,0 \le x \le c_r \\ 1 & ,c_r \le x < \infty \end{cases} \quad \text{with } h = 1$$

and the system lifetime has a Rayleigh distribution with

$$F(t) = 1 - e^{-\lambda t^2} \quad \text{for } \ t \ge 0,\ \lambda > 0,$$

the maintenance cost rate is

$$K(r,s) = \frac{1}{r + \sqrt{\dfrac{\pi s}{4\lambda}}}\left(1 - s + r\sqrt{\frac{\lambda\pi}{s}} + r^2\frac{\lambda}{s} + k\right)c_m \qquad (4.38)$$

where

$$r = \frac{d}{c_r}, \quad s = \frac{c_r - c}{c_r},$$

$$k = \frac{c_r}{c_m} > 1 \ \text{ by assumption.}$$

Under the same assumptions, they also assumed that repair cost limit function is given linearly by

$$c(t) = \begin{cases} c_r - dt & for \ \ 0 \le t < \dfrac{c_r}{d} \\ 0 & for \ \ \dfrac{c_r}{d} < t \end{cases}$$

A system failure at age t, implies a replacement with probability

$$p(t) = \begin{cases} \frac{t}{z} & ,0 \le t < z \\ 1 & z \le t \end{cases} \quad \text{with } z = \frac{c_r}{d}$$

Hence, the maintenance cost rate is

$$K(z) = \frac{c_m}{K_L} \lambda^{\frac{1}{3}} \left[K_N (\lambda z)^{\frac{1}{3}} + (k-1) z^{\frac{-1}{3}} \right] \tag{4.39}$$

where $K_L = 1.5^{\frac{1}{3}} \Gamma\left(\frac{4}{3}\right) \approx 1.02268$

and $K_N = 1.5^{\frac{2}{3}} \Gamma\left(\frac{5}{3}\right) \approx 1.18295$

$d = z c_r$

4.4 Repair time limit models.

Analogously to repair cost limit maintenance policies, Nakagawa and Osaki [1974], proposed a repair time limit policy in which a unit is replaced at failure: An alternative considered here is to repair the failed unit if the repair time is short and replace the unit if the repair time is long. This is achieved by stopping a repair if it is not completed within a specified time T (repair limit time), and the unit is then replaced by a new one, otherwise the repaired unit is put back into operation again.

It is assumed that failure of the unit is immediately detected and the failed unit is immediately repaired or replaced upon failure. It is also assumed that the unit is 'as good as new' upon repair or replacement.

The total expected maintenance cost rate up to time t is

$$K(T) = \frac{c_R \overline{R}(T) + \int_0^T \overline{R}(t)\, dc_R(t)}{\lambda + \int_0^T \overline{R}(t)\, dt}, \qquad \lambda > 0 \tag{4.40}$$

where λ represents the mean failure time and $c_R(t)$, the expected repair cost during $(0, t]$, also includes all costs incurred due to repair and system down cost during $(0, t]$.

It is evident from equation (4.40) that if the mean repair time $\mu = \int_0^\infty \overline{R}(t)\, dt$ is finite, then

$$K(0) = \lim_{T \to 0} K(T) = \frac{c_R}{\lambda}, \tag{4.41}$$

and

$$K(\infty) = \lim_{T \to \infty} K(T) = \frac{\int_0^\infty \overline{R}(t)\, dc_R(t)}{\lambda + \mu} \ . \tag{4.42}$$

$K(0)$ represent the expected costs per unit time with only replacement. $K(\infty)$ represent the expected costs with only repair maintenance.

Two special cases were also considered by the authors.

1. The repair cost is proportional to time

$c_R(t) = at^b \qquad a > 0,\ b \geq 0\ \&\ t \geq 0$

2. The repair cost grows exponential

$c_R(t) = a[\exp(bt) - 1] \qquad$ for $a > 0, b > 0.$

The optimum repair time T_0, for both scenarios were obtained analytically and numerical examples presented.

Nguyen and Murthy [1980] developed a model where the repair limit T, is a random variable (instead of being a constant repair limit) governed by some distribution $B(t)$ selected optimally with Nakagawa and Osaki [1974] assumptions and $c_R(0) \neq 0$.

Consider one cycle from the beginning of the operative unit to the completion of the repair (or replacement), then the expected cost of one cycle is

$$\int_0^\infty \left[c_R \overline{R}(t) + \int_0^t \overline{R}(x)\, dc_R(x) + c_R(0) \right] dB(t) \tag{4.43}$$

and the mean time of one cycle is

$$\int_0^\infty \left[\lambda + \int_0^t \overline{R}(x)\, dx \right] dB(t) \tag{4.44}$$

By using the renewal reward theorem Smith [1958], Ross [1970] the expected cost per unit time for an infinite time span if a repair limit policy with the random repair limit time T governed by distribution $B(t)$ is

$$K(T) = \int_0^\infty V(t)\,dB(t) \,/\, \int_0^\infty S(t)\,dB(t) \qquad (4.45)$$

where

$$V(T) = c_R \overline{R}(t) + \int_0^t \overline{R}(x)\,dc_R(x) + c_R(0)$$

$$S(T) = \lambda + \int_0^t \overline{R}(x)\,dx$$

They also concluded that with a suitable choice of $B(t)$, the deterministic policy by Nakagawa and Osaki [1974], is the optimal policy among the class of random repair limit policies.

Nguyen and Murthy [1981] obtained optimum repair limit replacement policy with imperfect repair (the repaired unit is not as good as new).

In their model, it is assumed that the mean life of a repaired unit is less than that of a new unit and the mean life of a repaired unit is independent of the number of times it has been repaired. Also they studied a repair limit replacement policy with two types of repairs- local and central repair.

In the local repair, the failed unit is repaired on site, while in the central repair the failed unit is brought to a central workshop for both repair and complete overhaul.

It is assumed that the local repair is imperfect whereas central repair is perfect, which may take a longer time to complete. They obtained the optimal solution by using the theory of semi-markov processes.

Koshimae et al., [1996] considered another repair time limit policy. Under this policy, when the original unit fails, the repair is started immediately. If the repair is completed in a pre-specified time limit T, $T \in [0, \infty)$, then the repaired unit is installed as soon as the repair is finished.

Otherwise, the failed unit is scrapped and the spare unit is ordered immediately and delivered after some lead time (amount of time between placing of an order and the receipt of the goods ordered). It is assumed that the unit once repaired, is presumed 'as good as new'.

The concept of estimated repair time is not included in the problem, but the lead time for the spare is considered. The policy decision variable is the repair time limit T.

Unit begins operation at $t = 0$.
Mean failure time for each unit is $\lambda(> 0)$. $R(t)$ assumed to be continuous and strictly increasing having an inverse function $R^{-1}(t)$. They also define the interval from start of operation $t = 0$, to the following start as a result of repair or replacement of the unit as one cycle.

Further costs included are as follows:

repair costs= c_R , shortage costs= c_S and ordering cost= c_D where it is assumed that $C_D > C_R L$, implying that ordering costs are larger than the repair costs during the interval $t \in [0, L]$ i.e., until the delivery of a new unit.

Expected repair costs are:

$$c_R \int_0^T \overline{R}(t)\,dt \tag{4.46}$$

where $\overline{R}(t) = 1 - R(t)$

Expected shortage costs are given by:

$$c_S \left\{ \int_0^T \overline{R}(t)\,dt + L\overline{R}(T) \right\} \tag{4.47}$$

Expected ordering costs are:

$$C_D \overline{R}(T) \tag{4.48}$$

Mean time of one cycle is given by:

$$E(T_0) = \lambda + \int_0^T \overline{R}(t)\,dt + L\overline{R}(T) \tag{4.49}$$

Then, the total expected cost per unit time is

$$K(T) = \lim_{t\to\infty} \frac{E\left[\text{total cost on } (0,t)\right]}{t}$$

$$= E_c(T) \Big/ \left\{\lambda + \int_0^T \overline{R}(t)\,dt + L\overline{R}(T)\right\} \qquad (4.50)$$

where

$$E_c(T) = \left(c_R + c_S\right)\int_0^T \overline{R}(t)\,dt + \left(c_S L + C_D\right)\overline{R}(T)$$

Special cases if

$T \to 0$ and $T \to \infty$,

$$K(0) = \frac{c_S L + c_D}{\lambda + L} \qquad (4.51)$$

and

$$K(\infty) = \frac{\left(c_R + c_S\right)\mu}{\lambda + \mu} \qquad (4.52)$$

respectively.

Dohi et al., [1996] considered a generalized repair time limit replacement problem with lead time and imperfect repair, which is subject to a time constraint and proposed a nonparametric solution procedure to estimate the optimal repair time limit. The optimal policies are given directly from the repair time data.

Rattihalli [2002] considered a model in which cost per unit time can be reduced by controlling both lifetime and repair time of a unit. This policy is called "AGE AND REPAIR LIMIT REPLACEMEMT POLICY."

The description of the model:

1. A single unit system starts working at time zero.
2. There are two types of failures.

The repair time is independent of the age of the system. If the repair time exceeds 'T' units the failed unit is replaced by a new unit. Otherwise the repaired unit which is as good as new is put into operation.

3. If the unit attains age 'τ' without type 2 failure it is exchanged by a new unit.
4. Units need maintenance while working.

Suppose that the state space is {N, N_E, R, N_C, N_R}.The interpretation of these states is

N: a new unit is in use;

N_E: a new unit which has been used as an exchange to a unit of age 'a' is working;

R: a unit under major repair;

N_C: a unit which was made as good as new after completion of major repair;

N_R: a new unit which was used as replacement to failed unit due to incomplete repair is in use.

Let

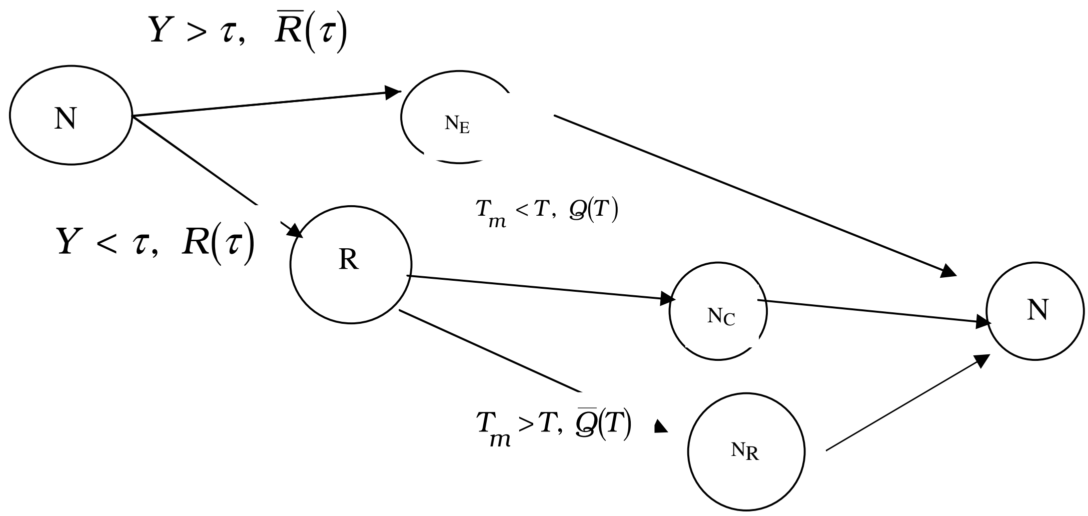
Y: time until type 2 failure of a new unit.

$\overline{R}(\tau)$: distribution function of Y.

T_m: Major repair time

$Q(T)$: distribution function of T_m.

STATE TRANSITION DIAGRAM OF THE SYSTEM:



The epochs at which the process enters state N can be regarded as regenerative epochs. Let $\{S_i : i \geq 0\}$ be a sequence of epochs of successive entrances to state N with $S_0=0$ and $L_i(\tau,T) = S_i - S_{i-1}$. Then the sequence $\{S_i : i \geq 0\}$ is a renewal process and $L_i(\tau,T)$ a cycle length.

$$L(\tau,T) = \min\{Y,\tau\} + \min\{T_m,T\}[Y \leq \tau]$$

The cost per cycle $C(\tau,T)$ is given by

$$C(\tau,T) = M\left(\min\{Y,\tau\} + C_E[Y > \tau]\right) + \left\{C\left(\min\left\{T_m,T\right\} + C_R\left[T_m > T\right]\right)\right\}[Y \le \tau]$$

Where M(t)= cost of maintenance and of minor repairs over the period (0,t).

C(t) = the total cost of repair over the interval (0, t).

C_E = the cost of exchanging a unit.

C_R = the cost of new unit for replacing a failed unit whose repair is not completed by time *T*.

The long run expected cost per unit time is given by

$$K(\tau,T) = \frac{E\{C(\tau,T)\}}{E\{L(\tau,T)\}}$$

For some specific models, optimum values of τ and T were obtained so as to minimize $K(\tau,T)$.

In what follows, we are only going to concentrate on the repair cost limit replacement policies, where our limit will be cost instead of time and our optimality criterion will be minimizing the maintenance cost rate under an infinite time horizon.

5. Combined Repair Cost Limit and Preventive Replacement Policy

5.1 Development of the combined policy

A disadvantage of the repair cost limit policy (RCL) is that the replacement or repair decision depends only on the cost of a single repair. Long lasting situations characterised by frequent repairs whose costs are below the corresponding limit do not directly influence the time of replacement. The combined model that follows can overcome one of the drawbacks of RCL model – less efficiency. However, the combined model also brings a new extra management effort- tracking the time of type 2 failure (renewal point).

Policy: On failure the system is replaced by an equivalent new one if the random repair cost C exceeds a given repair cost limit c, or at age τ (τ=time after its installation) by a preventive replacement if in an interval $(0,\tau)$, no failure induced a replacement of the system. After any replacement, the time is returned to zero. Otherwise, after failure a minimal repair is carried out.

This is equivalent to saying that: the system is maintained according to the failure type model (refer to section 3.2), whereas, if in a fixed interval $(0,\tau)$ no type 2 failure occurs, then the system is renewed at time τ by a preventive replacement. This policy arrangement combines the attributes of the repair cost limit and the "classical age replacement model".

Let X_n be the moment of the nth failure in a cycle, with $X_0 = 0$. The probability density of the random vector $(X_1, X_2, \ldots, X_n)$ is given by equation (2.22). Let S_n be the random number of minimal repairs in the interval $(0,t)$.

Then according to equation (2.16),

$$P(S_n = n) = P(S_n < t \cap S_{n+1} \geq t) = P_n(0,t)$$

$$= \frac{\Lambda(t)^n}{n!} e^{-\Lambda(t)}$$

$$= \frac{\Lambda(t)^n}{n!} \overline{F}(t) \qquad\qquad n = 0,1,\ldots.$$

Hence, $\{S_n; t \geq 0\}$ is a NHPP with trend function given as:

$$E[S_n] = \Lambda(t).$$

Let Y denote the time until first type 2 failure without preventive maintenance since the last replacement; that is, the time between successive replacement.

In what follows, we use facts presented in section 3.2.

Let Z_t be the random number of minimal repairs during the time interval $\{0, \min(Y,t)\}$ and

$$\overline{G}(t) = 1 - G(t) = P(Y \geq t)$$

$$= \exp\left(-\int_0^t p(x)\lambda(x)dx\right). \tag{5.1}$$

Then,

$$P\left(Z_t = n \middle| Y = t\right), \quad t = 0,1,2,\ldots.$$

is given as

$$\lim_{\Delta t \to 0} \frac{P\left(Z_t = n \cap t \leq Y \leq t + \Delta t\right)}{P(t \leq Y \leq t + \Delta t)} \tag{5.2}$$

Hence, for any continuous random variable Z_t , with condition $Y = t$ and $\Delta t > 0$, we obtain

$$\lim_{\Delta t \to 0} \frac{P\left(t \leq Y = X_{n+1} \leq t + \Delta t\right)}{G(t + \Delta t) - G(t)}$$

By using equation (2.22) and (3.11)

$$P(t \leq Y = X_{n+1} \leq t + \Delta t)$$

$$= \int_t^{t+\Delta t} \int_0^{x_{n+1}} \ldots \int_0^{x_3} \int_0^{x_2} \prod^{n} \overline{p}(x_i)\lambda(x_i)dx_i\, p\left(x_{n+1}\right) f\left(x_{n+1}\right) dx_{n+1}$$

$$= \frac{1}{n!} \int_t^{t+\Delta t} \left(\int_0^y \overline{p}(x)\lambda(x)dx\right)^n p(y) f(y) dy$$

Hence the limit (5.2) can be obtained after dividing the numerator and denominator by (Δt) (taking into account equation (5.1) as

$$P\left(Z_t = n \middle| Y = t\right) = \frac{1}{n!}\left(\int_0^t \overline{p}(x)\lambda(x)dx\right)^n \exp\left(-\int_0^t \overline{p}(x)\lambda(x)dx\right)$$

Thus, given that $Y = t$, the random variable Z_t is Poisson distributed with mean

$$E\left(Z_t \middle| Y = t\right) = \int_0^t \overline{p}(x)\lambda(x)dx \tag{5.3}$$

It must be noted at this stage that this conditional probability distribution of Z_t on condition that $Y = t$ and $Y \geq t$, are the same.

i.e.

$$E\left(Z_t \middle| Y = t\right) = E\left(Z_t \middle| Y \geq t\right) = \int_0^t \overline{p}(x)\lambda(x)dx \tag{5.4}$$

$$\begin{aligned} E\left(Z_t \middle| Y < t\right) &= \int_0^t E\left(Z_t \middle| Y = x\right) dG(x) / G(t) \\ &= \int_0^t \int_0^x \overline{p}(y)\lambda(y)\, dy\, dG(x) / G(t) \end{aligned} \tag{5.5}$$

The details of the derivation are given in the Appendix (A3).

Now the unconditional mean value of Z_t can be derived as

$$E(Z_t) = E(Z_t | Y < t)\ G(t) + E(Z_t | Y \geq t)\ \overline{G}(t)$$

$$= \int_0^t \overline{G}(x)\lambda(x)\ dx - G(t) \qquad (5.6)$$

The details of the derivation are given in the Appendix (A4).

Let c_m, c_r, and c_p with $0 < c_m < c_p < c_r$ denote the mean cost of a minimal repair, a replacement after a type 2 failure and a preventive maintenance (pm), respectively. Then the random length of a replacement cycle L_τ is:

$$L_\tau = \min(\tau, Y)$$

with mean value

$$E(L_\tau) = \int_0^\tau P(Y > t)\ dt = \int_0^\tau \overline{G}(t)\, dt \qquad (5.7)$$

Thus, in view of equation (5.6) and (5.7) the maintenance cost rate has the structure

$$K = \frac{c_m E(Z_t) + c_r G(\tau) + c_p \overline{G}(\tau)}{E(L_\tau)}$$

where

$$E(Z_t) = \int_0^\tau \overline{G}(t)\lambda(t)\ dt - G(\tau).$$

$$\therefore \qquad K = \frac{c_m\left[\int_0^\tau \overline{G}(t)\lambda(t)\, dt - G(\tau)\right] + c_r G(\tau) + c_p \overline{G}(\tau)}{\int_0^\tau \overline{G}(t)\ dt} \qquad (5.8)$$

In particular, for $p(t) \equiv p, \quad p > 0$

$$K = \frac{\left\{c_r + \left[\frac{(1-p)}{p}\right]c_m\right\}G(\tau) + c_p\overline{G}(\tau)}{\int_0^\tau \left(\left[\overline{F}(t)\right]^p\right)dt} \tag{5.9}$$

The details of the derivation are given in the Appendix (A5).
If we assume that the system lifetime has a Weibull distribution with distribution function

$$F(t) = P(X \le t) = 1 - e^{-\lambda t^\beta}, \quad t \ge 0,\ \beta > 1,\ \lambda > 0.$$

In case of constant repair cost limit c:

$$p = \overline{R}(c), \quad \overline{p} = R(c).$$

Then,

$$K(\tau,c) = \frac{\left\{c_r + \left[\frac{R(c)}{\overline{R}(c)}\right]c_m\right\}G(\tau) + c_p\overline{G}(\tau)}{\int_0^\tau e^{-\overline{R}(c)\lambda t^\beta}\,dt} \tag{5.10}$$

The maintenance cost rate depends on c and on τ.

5.2 Discussion of the Optimal Solution

Necessary conditions for the existence of a unique optimal policy.

Let $\delta = c_r - c_p + \frac{\bar{p}}{p} c_m$ and $\alpha = \delta + c_p$.

Then (5.9) becomes,

$$K(\tau, p) = \frac{\left[\alpha - \delta\left(\bar{G}(\tau)\right)\right]}{\int_0^\tau \bar{G}(t)\,dt} \tag{5.11}$$

For a fixed *p*, the $\tau_0 = \tau_0(p)$ that minimizes (5.11), is obtained by setting the partial derivative of $K(\tau, p)$ with respect to τ (while holding *p* constant) and equating the result to zero.

The optimal policy satisfies

$$\frac{\partial}{\partial \tau} K(\tau, p) = 0,$$

$$\frac{\partial}{\partial p} K(\tau, p) = 0.$$

From, $$\frac{\partial}{\partial \tau} K(\tau, p) = 0$$

$$\left(\int_0^\tau \bar{G}(t)\,dt\right)\left(\delta g(\tau)\right) - \bar{G}(\tau)\left[\alpha - \delta \bar{G}(\tau)\right] = 0$$

$$\left(\int_0^\tau \bar{G}(t)\,dt\right)\left(\delta g(\tau)\right) = \bar{G}(\tau)\left[\alpha - \delta \bar{G}(\tau)\right]$$

$$p\lambda(\tau)\int_0^\tau \bar{G}(t)\,dt = \frac{\alpha}{\delta} - \bar{G}(\tau)$$

$$\lambda(\tau)\int_0^{\tau}\overline{G}(t)\,dt + \frac{\overline{G}(\tau)}{p} = \frac{\alpha}{p\delta}\,. \tag{5.12}$$

If $\lambda(\tau)$ is continuous and strictly increasing to ∞, then the LHS (left hand side) of equation (5.12) is continuous and increasing. If $\tau \to \infty$, then LHS increases to ∞, whilst the RHS (right hand side of the equation) is constant. However, if $\tau = 0$, then LHS equals $\frac{1}{p}$ or $\frac{1}{\overline{R}(c)}$ and RHS is greater than $\frac{1}{p}$ since $\alpha > \delta$.

Therefore, if $\lambda(t)$ is continuous and strictly increasing to ∞, then the τ_0 which minimizes $K(\tau, p)$ for $0 < p < 1$ is unique and finite.

6. Optimization

6.1 Optimal Solution and Numerical Example.

In this section, we obtain the optimum replacement interval τ_0 for model equation (5.10), which minimizes the expected cost rate.

Equation (5.10) contains two variables τ and c. Therefore, its optimization is not straight forward. There are two situations with respect to c. Either the management decides in advance and puts a predetermined limit on the repair cost or does not make any decision about c.

If we assume the value of c is known, we can find an optimum τ_0 which minimizes the maintenance cost rate. In case the optimum c is also to be determined, the marginal analysis of equation (5.10) and (5.12) should be performed with various values of c.

The vector $\left(\tau_0; c_0\right)$ giving the lowest maintenance cost rate is the optimal solution.

As an example, we assume that:

A. The system lifetime follows a Weibull failure distribution with the survival function

$$F(t) = 1 - e^{-\left(\frac{t}{\theta}\right)^{\beta}}, \qquad t > 0,\ \beta > 1,\ \theta > 0,$$

where β is a shape parameter.

Its hazard function is:

$$\Lambda(t) = \left(\frac{t}{\theta}\right)^{\beta}, \qquad \beta > 1.$$

Differentiation yields the failure rate

$$\lambda(t) = \frac{\beta}{\theta}\left(\frac{t}{\theta}\right)^{\beta-1}.$$

Expected value of X is

$$E(X) = \theta\Gamma\left(1 + \frac{1}{\beta}\right)$$

where $\Gamma(x)$ is the gamma function :

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt.$$

The distribution is most popular due to its properties:

- Many distributions (including the truncated normal distribution) can be represented, at least roughly, by suitable choice of parameters.
- It has an increasing failure rate when $\beta > 1$. The lifetime distributions of many systems can be described adequately which, when in normal use, experience positive aging.
- when $\beta = 1$: Exponential distribution which is widely used to describe the failure characteristics of certain systems eg electronic appliances.
- when $\beta = 2$: Rayleigh distribution

B. The estimated repair cost follows a negative exponential distribution with average - μ , the probability that the repair cost will exceed a repair cost limit c is

$$\overline{R}(c) = \exp\left({}^{-c}\!/_{\mu} \right) = 1 - R(c)$$

C. The replacement cycle begins at $t = 0$, and ends at replacement.

Note that there are no theoretical justifications for our choice of $R(c)$- estimated repair cost distribution; as one can also consider log-normal or other forms of repair cost distributions. As the exponential distribution is relatively easy to handle algebraically, it was used in preference to the log-normal in the analytical approach which is introduced next.

Looking at equation (5.10), it has been shown that the average cost per unit time for repairs and replacement is

$$K(\tau,c) = \frac{\left\{c_r + \left[\dfrac{R(c)}{\overline{R}(c)}\right]c_m\right\}G(\tau) + c_p\overline{G}(\tau)}{\int_0^\tau \overline{G}(t)dt}$$

If the repair cost , follows exponential distribution with mean μ

$\therefore\ \overline{R}(c) = \exp(-c/\mu)$

$\int_0^\tau \overline{G}(t)dt = \int_0^\tau e^{-\overline{R}(c)\lambda t^\beta} dt$

For a fixed c, $\tau_0 = \tau_0(c)$ is computed by minimizing $K(\tau,c)$.

Taking into consideration that equation (5.12) satisfies the following:

$$\overline{R}(c)\lambda(\tau)\int_0^\tau \overline{G}(t)\ dt - G(\tau) = \frac{\overline{R}(c)c_p}{\left(c_r - c_p - c_m\right)\overline{R}(c) + c_m}$$

A sufficient condition for the existence of a unique solution of (5.12) is $\lambda(x) \to \infty$ for $x \to \infty$.

A unique solution exists if $0 \le c < c_r - c_p$ and $c_r + \dfrac{R(c)}{\overline{R}(c)}c_m > c_p$

Suppose,

$\mu = 25$, $c_r = R\,100$, $c_p = R\,67$

$c_m = R\,13$, $\lambda = 1$

$\beta = 2$ (Rayleigh distribution)

$\Lambda(t) = \lambda t^2 = t^2$

$\lambda(t) = 2\lambda t = 2t$

Using nonlinear integer programming software (See Appendix), the solution (τ_0, c_0) that minimize the maintenance cost rate (5.10) is

$$\tau_0 = [2.0802] \quad c_0 = [32{,}6639]$$

and the corresponding optimal cost rate is

$$K(\tau_0, c_0) = [76{,}31]$$

The results show that the optimal maintenance policy is to perform preventive replacement every 2,1 times at a cost of R67 and minimally repair the system upon failure at the cost of R13 if the repair cost limit of approximately R33 is not exceeded. Otherwise replace the failed system at a cost of R100.

Checking the effect of $\beta = 1$ and $\beta = 3$.

The example indicates that the optimal solution $K(\tau_0, c_0)$ is sensitive to the change in the shape parameter β.

This table shows some values for the numerical evaluation of $K(\tau_0, c_0)$ for

$$\left(\lambda = 1,\ \mu = R\,25,\ c_r = R\,100,\ c_m = R\,13,\ c_p = R\,67\right)$$

β	τ_0	c_0	$K(\tau_0, c_0)$
1	138,32	33	36,24
2	2,0802	32,6639	76,31
3	1,25	33	85,62

We now consider a constant cost limit, c and $\tau = \infty$.
Keeping the other parameters the same, the optimal solution is

$$\tau = \infty,\quad c_0 = [33]$$

and the corresponding optimal cost rate is

$$K(c_0, \infty) = [79{,}1201],$$

which coincides with results given by Park[1985]. Thus, at this time the optimal cost rate is somewhat increased compared to the combined model above.

For finite τ and $c = 0$ the optimal maintenance cost rate is

$$K(\tau_0, 0) = [112{,}593]$$

Since the present model is only for the evaluation of combined (τ, c) values, direct comparison to the optimal results from the other policies like equation (4.22) is unfair, except that the close conformity of the two results somewhat supports the validity of the present modelling approach.

6.2 Special Cases

Case where *p=0* (minimal repair policy)

This is the case where one always proceeds to a minimal repair at failure. The system is always replaced at age-τ. In case of failures between the periodic replacements a minimal repair is carried out [Barlow and Hunter, (1960), policy II].

All cycle lengths are equal to -τ. From equation (5.8), if p=0, it can be seen that

$$G(\tau) = 0$$

$$\overline{G}(\tau) = 1 \qquad \text{and } \int_0^{\tau} \overline{G}(t)\,dt = \tau$$

$$E[N(t)] = -\ln \overline{F}(\tau) = \int_0^{\tau} \lambda(t)\,dt = \Lambda(\tau)$$

Thus, the maintenance cost rate is given as

$$K(\tau) = \frac{c_p + c_m \Lambda(\tau)}{\tau} \tag{6.1}$$

Replacement interval $\tau = \tau_0$, which minimizes $K(\tau)$, satisfies condition

$$\tau\lambda(\tau) - \Lambda(\tau) = c_p / c_m \tag{6.2}$$

In case of Weibull distributed lifetime,

$$F(t) = 1 - e^{-(\lambda t)^{\beta}} \qquad \beta > 1$$

$$\lambda(\mathrm{t}) = \lambda \beta t^{\beta - 1}$$

$$\Lambda(t) = \lambda t^{\beta}$$

Optimal preventive maintenance interval is given by:

$$\tau_0 = \left[\frac{c_p}{(\beta - 1)\lambda c_m} \right]^{\frac{1}{\beta}} \tag{6.3}$$

If $\lambda(t) \to \infty$ as $t \to \infty$, there exists a unique solution $\tau = \tau_0$ of this equation.

Minimal maintenance cost rate will be

$$K(\tau_0) = c_m \lambda(\tau_0) \tag{6.4}$$

The details of the derivation are given in the Appendix (A6).

Case where *p=1* (age replacement policy)

Equation (5.8), becomes

$$K(\tau, 1) = \frac{c_r F(\tau) + c_p \overline{F}(\tau)}{\int_0^{\tau} \overline{F}(t)\, dt} \tag{6.5}$$

This is the case where no minimal repair is made. It is the so called classical age replacement policy. The system is replaced upon failure or at age τ by a preventive replacement whichever occurs first, where τ is constant.

After a replacement the system has the same lifetime distribution as the original one, i.e it is ‘as good as new’.

The age replacement problem has been discussed, formulated mathematically and solved in general by several authors, firstly by Senju [1957]and later by Barlow and Hunter [1960], Barlow and Proschan [1962, 1965], and Morse [1958].

Differentiating $K(\tau,1)$ with respect to τ and equating to zero yields the equation which the optimal τ_0, has to satisfy as:

$$\lambda(\tau_0)\int_0^{\tau}\overline{F}(t)\,dt - F(\tau_0) = \frac{c_p}{c_r - c_p} \tag{6.6}$$

from equation (6.6),

$$\left[c_r - c_p\right]\lambda(\tau)\int_0^{\tau}\overline{F}(t)\,dt - F(\tau)\left[c_r - c_p\right] = c_p$$

$$\left[c_r - c_p\right]\lambda(\tau)\int_0^{\tau}\overline{F}(t)\,dt + F(\tau)c_p - c_p = F(\tau)c_r$$

When finite τ_0 exists, we obtain by combining (6.5) and (6.6), the minimum expected cost per unit time as:

$$K\left(\tau_0\right) = \frac{\left[c_r - c_p\right]\lambda(\tau)\int_0^\tau \overline{F}(t)\,dt + c_p F(\tau) - c_p + c_p - c_p F(\tau)}{\int_0^\tau \overline{F}(t)\,dt}$$

$$\therefore K\left(\tau_0\right) = \left(c_r - c_p\right)\lambda\left(\tau_0\right) \qquad (6.7)$$

If no finite solution to equation (6.6) exists, then $\tau_0 = \infty$ is the optimal policy.

Solutions of equation (6.6) have been tabulated by Glasser [1967] for the classes of Weibull, Gamma and truncated normal distributed lifetimes. Tables are no longer up to date in view of computers.

7. Conclusions

In conclusion, it is perhaps important to note that maintenance policies have become more and more general because they include some previous policies as special cases. The optimal maintenance plans obtained from these general policies may result in some cost savings since the optimal maintenance schedules under them might be “globally” optimal (optimal in a larger range). However, as the policies become more and more complicated, these policies may also cause inconvenience for implementation in practice. What has hampered application amongst others are:

- lack of decision support systems for combining models and data via a friendly interface.
- the multitude of technical systems, and the many different ways in which they function and in which they deteriorate, render maintenance management problems highly complex. Taking into consideration that a model is, by definition, a simplification of reality (thus, it will only work if the simplification is relevant to the problem considered).
- A shortage of data.
- A lack of understanding with respect to mathematical models (unfamiliarity with knowledge of the stochastic processes).

Basically, each maintenance policy for a one-unit system either depends on counting/recording of the number of repairs, preventive maintenance (PM) time, and/or estimating repair cost. In practice, counting number of repairs and recording PM time, and estimating repair costs after failure are all possible ways.

The current research seems to intend to use two or more of them as policy decision variables in a single policy. Much of our research reflects these developments, in fact, there is no new theoretical research presented in this dissertation. The combination of these established maintenance policies as well as their numerical evaluation are regarded as our greatest contribution.

Lastly, some further research along these lines seems to be useful, since repair cost limit policies seems to be suitable for practical applications.

8. Appendix A: Derivation of formulas

(A1)

$$E[N] = \sum_{n=1}^{\infty} nP(N = n)$$

$$\textit{and} \qquad P(N = n) = \frac{1}{n!}\int_0^{\infty}\left(\int_0^t \overline{p}(x)\lambda(x)dx\right)^n p(t)f(t)dt, \quad n = 0,1,\ldots$$

$$\therefore \; E[N] = \sum_{n=1}^{\infty} n * \frac{1}{n!}\int_0^{\infty}\left(\int_0^t \overline{p}(x)\lambda(x)dx\right)^n p(t)f(t)dt$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n-1)!}\int_0^{\infty}\left(\int_0^t \overline{p}(x)\lambda(x)dx\right)^n p(t)f(t)dt$$

$$= \int_0^{\infty} \sum_{n=1}^{\infty}\left(\int_0^t \overline{p}(x)\lambda(x)dx\right)\frac{1}{(n-1)!}\left(\int_0^t \overline{p}(x)\lambda(x)dx\right)^{n-1} p(t)f(t)dt$$

$$= \int_0^{\infty}\left(\int_0^t \overline{p}(x)\lambda(x)dx\right)\sum_{n=0}^{\infty}\frac{1}{n!}\left(\int_0^t \overline{p}(x)\lambda(x)dx\right)^n p(t)f(t)dt$$

$$= \int_0^{\infty}\left(\int_0^t \overline{p}(x)\lambda(x)dx\right)e^{\int_0^t \overline{p}(x)\lambda(x)dx} p(t)f(t)dt$$

$$= \int_0^{\infty}\left(\int_0^t \overline{p}(x)\lambda(x)dx\right)e^{\int_0^t \overline{p}(x)\lambda(x)dx} p(t)\lambda(t)e^{-\Lambda(t)}dt$$

$$= \int_0^{\infty}\left(\int_0^t \overline{p}(x)\lambda(x)dx\right)e^{-\int_0^t p(x)\lambda(x)dx} p(t)d\Lambda(t)$$

$$F_{X_n}(t) = \sum_{n=j}^{\infty}\frac{\left[\Lambda(t)\right]^n}{n!}e^{-\Lambda(t)}, \quad j \geq 1$$

(A2)

$$E[N] = \overline{R}(c)\int_0^{\infty} \Lambda(t)\lambda(t)e^{-\overline{R}(c)\Lambda(t)}dt - 1$$

let $\quad x = \overline{R}(c)\Lambda(t), \qquad \Lambda(t) = \dfrac{x}{\overline{R}(c)}$

$$dx = \overline{R}(c)\lambda(t)dt$$

$$\therefore \qquad E[N] = \overline{R}(c)\int_0^{\infty} \frac{x}{\overline{R}(c)}\lambda(t)e^{-x}\,\frac{dx}{\overline{R}(c)\lambda(t)} - 1$$

$$= \frac{1}{\overline{R}(c)}\int_0^{\infty} xe^{-x}\,dx - 1$$

and $\quad \int_0^{\infty} xe^{-x}dx \quad$ by parts

$$u = x, \qquad du = dx$$

$$dv = e^{-x}dx, \qquad v = \int e^{-x}dx = -e^{-x}$$

$$I = uv - \int v\,du$$

$$= x\left(-e^{-x}\right) - \int -e^{-x}dx$$

$$= \left[-xe^{-x}\right]_0^{\infty} - \left[e^{-x}\right]_0^{\infty}$$

$$= 1$$

$$\therefore \qquad E[N] = \frac{1}{\overline{R}(c)} - 1$$

$$= \frac{R(c)}{\overline{R}(c)}$$

(A3)

$$E\left(Z_t \middle| Y < \tau\right) = \int_0^\tau E\left(Z_t \middle| Y = x\right) dG(t) \,/\, G(\tau)$$

$$= \frac{1}{G(\tau)} \sum_{n=1}^{\infty} n \int_0^\tau \frac{\left(\int_0^t \overline{p}(x)\lambda(x)\,dx\right)^n}{n!} p(t) f(t)\,dt$$

$$= \frac{1}{G(\tau)} \int_0^\tau \sum_{n=1}^{\infty} \left(\int_0^t \overline{p}(x)\lambda(x)\,dx\right) \frac{1}{(n-1)!} \left(\int_0^t \overline{p}(x)\lambda(x)\,dx\right)^{n-1} p(t) f(t)\,dt$$

$$= \frac{1}{G(\tau)} \int_0^\tau \left(\int_0^t \overline{p}(x)\lambda(x)\,dx\right) \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int_0^t \overline{p}(x)\lambda(x)\,dx\right)^{n} p(t) f(t)\,dt$$

$$= \frac{1}{G(\tau)} \int_0^\tau \left(\int_0^t \overline{p}(x)\lambda(x)\,dx\right) e^{\int_0^t \overline{p}(x)\lambda(x)\,dx} p(t) f(t)\,dt$$

since $\quad e^{\int_0^t \overline{p}(x)\lambda(x)\,dx} \; p(t) f(t)$

$$= e^{-\int_0^t p(x)\lambda(x)\,dx} p(t)\lambda(t) = dG(t)$$

$$\therefore E\left(Z_t \middle| Y < \tau\right) = \frac{1}{G(\tau)} \int_0^\tau \left(\int_0^t \overline{p}(x)\lambda(x)\,dx\right) dG(t)$$

$$E\left(Z_t\right) = E\left(Z_t \middle| Y < t\right) G(\tau) + E\left(Z_t \middle| Y \geq t\right) \overline{G}(\tau)$$

$$= \int_0^\tau \int_0^t \overline{p}(x)\lambda(x)\,dx\,dG(t) + \overline{G}(\tau) \int_0^\tau \overline{p}(x)\lambda(x)\,dx$$

From Fubini's Theorem, remember that,

$$\int_0^\tau \int_0^t f(x,y)\,dy\,dX$$

$$= \int_0^\tau \int_y^\tau f(x,y)\,dX\,dy$$

(A4)

$$\therefore\ E\left(Z_t\right) = \int_0^\tau \int_x^\tau \overline{p}(x)\lambda(x)g(t)\,dt\,dx + \overline{G}(\tau)\int_0^\tau \overline{p}(x)\lambda(x)\,dx$$

$$= \int_0^\tau \overline{p}(x)\lambda(x)\left[G(\tau) - G(x)\right]dx + \int_0^\tau \overline{p}(x)\lambda(x)\,dx - G(\tau)\int_0^\tau \overline{p}(x)\lambda(x)\,dx$$

$$= \int_0^\tau \overline{p}(x)\lambda(x)G(\tau)\,dx - \int_0^\tau \overline{p}(x)\lambda(x)G(x)\,dx + \int_0^\tau \overline{p}(x)\lambda(x)\,dx$$

$$- \ G(\tau)\int_0^\tau \overline{p}(x)\lambda(x)\,dx$$

$$= \int_0^\tau \overline{p}(x)\lambda(x)\,dx - \int_0^\tau \overline{p}(x)\lambda(x)G(x)\,dx$$

$$= \int_0^\tau \left[\overline{p}(x)\lambda(x)\left\{1 - G(x)\right\}dx\right]$$

$$= \int_0^\tau \overline{G}(x)\overline{p}(x)\lambda(x)\,dx$$

$$= \int_0^\tau \left(1 - p(x)\right)\lambda(x)\overline{G}(x)\,dx$$

$$= \int_0^\tau \overline{G}(x)\lambda(x)\,dx - \int_0^\tau \overline{G}(x)p(x)\lambda(x)\,dx$$

$$= \int_0^\tau \overline{G}(x)\lambda(x)\,dx - \int_0^\tau dG(x)$$

$$= \int_0^\tau \overline{G}(x)\lambda(x)\,dx - G(\tau)$$

(A5)

for $p(t) = p$,

$$\int_0^{\tau} \overline{G}(t)\lambda(t)\,dt - G(\tau)$$

$$= \int_0^{\tau} e^{-\int_0^{\tau} p\lambda(x)dx} \lambda(t)\,dt - \left(1 - e^{-p\int_0^{\tau} \lambda(t)\,dt}\right)$$

$$= -\frac{1}{p}\int_0^{\tau} e^{-p\Lambda(\tau)}\left(-p\lambda(t)\right)dt - 1 + e^{-p\Lambda(\tau)}$$

$$= \frac{1}{p}\left[-e^{-p\Lambda(\tau)} + 1\right] - 1 + e^{-p\Lambda(\tau)}$$

$$= -\frac{1}{p}e^{-p\Lambda(\tau)} + \frac{1}{p} - 1 + e^{-p\Lambda(\tau)}$$

$$= \overline{G}(\tau)\left[1 - \frac{1}{p}\right] + \frac{1}{p} - 1$$

$$= \left(1 - G(\tau)\right)\left(1 - \frac{1}{p}\right) + \frac{1}{p} - 1$$

$$= G(\tau)\left[\frac{1}{p} - 1\right]$$

$$= \frac{(1 - p)}{p} G(\tau)$$

(A6)

$$K(\tau) = \frac{c_p + c_m \Lambda(\tau)}{\tau} \qquad \text{..................(1)}$$

$$\frac{dK(\tau)}{d\tau} = \frac{-c_p}{\tau^2} - c_m \Lambda(\tau)\left(\tau^{-2}\right) + c_m \Lambda'(\tau)\tau^{-1}$$

$$= \frac{-c_p}{\tau^2} - \frac{c_m \Lambda(\tau)}{\tau^2} + \frac{c_m \Lambda'(\tau)}{\tau} = 0$$

$$= -c_p \tau^{-2} + \left[-c_m \Lambda(\tau)\tau^{-2} + c_m \Lambda'(\tau)\tau^{-1}\right]$$

$$\tau c_m \Lambda'(\tau) - c_m \Lambda(\tau) = c_p$$

$$c_m\left[\tau\Lambda'(\tau) - \Lambda(\tau)\right] = c_p$$

$$\tau \frac{d}{d\tau}\int_0^\tau \lambda(t)dt - \Lambda(\tau) = \frac{c_p}{c_m}$$

$$\tau\lambda(\tau) - \Lambda(\tau) = \frac{c_p}{c_m} \qquad \text{...........(2)}$$

Substitute (2) into (1)

from (2) $\quad c_m \Lambda\left(\tau^*\right) = c_m \tau\lambda\left(\tau^*\right) - c_p$

from (1) $\quad c_m \Lambda\left(\tau^*\right) = \tau K\left(\tau^*\right) - c_p$

$\therefore \quad c_m \tau\lambda\left(\tau^*\right) - c_p = \tau K\left(\tau^*\right) - c_p$

$$K\left(\tau^*\right) = c_m \lambda\left(\tau^*\right)$$

which is the function value when $\tau = \tau^*$

9. Appendix B: Computer code for numerical example.

```
a[t_, c_] = Exp[- Exp[- (c/u)] λ1 t^β];
β = 2;
A[τ_, c] = ∫_0^τ a[t, c] ⅆt
```

$$\frac{e^{\frac{c}{2u}}\sqrt{\pi}\ \mathrm{Erf}\left[e^{-\frac{c}{2u}}\tau\sqrt{\lambda_1}\right]}{2\sqrt{\lambda_1}}$$

```
K[τ_, c_] =
  1/A[τ, c]
   ((Exp[c/u] - 1) Cm (1 - a[τ, c]) + Cr (1 - a[τ, c]) +
     Cp a[τ, c] )
```

$$\left(2e^{-\frac{c}{2u}}\left((-1+e^{c/u})\left(1-e^{-e^{-\frac{c}{u}}\tau^2\lambda_1}\right)C_m+e^{-e^{-\frac{c}{u}}\tau^2\lambda_1}C_p+\left(1-e^{-e^{-\frac{c}{u}}\tau^2\lambda_1}\right)C_r\right)\sqrt{\lambda_1}\right)\Big/\left(\sqrt{\pi}\ \mathrm{Erf}\left[e^{-\frac{c}{2u}}\tau\sqrt{\lambda_1}\right]\right)$$

```
λ1 = 0.01; Cm = 13 ; Cr = 100 ; Cp = 67; u = 25;
NMinimize[{K[τ, c], τ > 0, Cr - Cp > c ≥ 0}, {τ, c}]
{7.63131, {c→33., τ→20.802}}
```

```
a[t_, c_] = Exp[-Exp[-(c/u)] λ1 t^β];
β = 1;
A[τ_, c] = ∫_0^τ a[t, c] dt
```

$$\frac{e^{c/u}\left(1-e^{-e^{-\frac{c}{u}}\tau\lambda_1}\right)}{\lambda_1}$$

```
K[τ_, c_] = 1/A[τ, c] ((Exp[c/u] - 1) Cm (1 - a[τ, c]) + Cr (1 - a[τ, c]) + Cp a[τ, c])
```

$$\frac{e^{-\frac{c}{u}}\left((-1+e^{c/u})\left(1-e^{-e^{-\frac{c}{u}}\tau\lambda_1}\right)C_m+e^{-e^{-\frac{c}{u}}\tau\lambda_1}C_p+\left(1-e^{-e^{-\frac{c}{u}}\tau\lambda_1}\right)C_r\right)\lambda_1}{1-e^{-e^{-\frac{c}{u}}\tau\lambda_1}}$$

```
λ1 = 1; Cm = 13 ; Cr = 100 ; Cp = 67; u = 25;
NMinimize[{K[τ, c], τ > 0, Cr - Cp > c ≥ 0}, {τ, c}]
```

{36.2408,{c→33.,τ→138.319}}

```
Clear[β];
a[t_, c_] = Exp[-Exp[-(c/u)] λ1 t^β];
β = 2;
A[τ_, c] = ∫_0^τ a[t, c] dt
```

$$\frac{1}{2}e^{c/50}\sqrt{\pi}\,\mathrm{Erf}[e^{-c/50}\tau]$$

```
K[τ_, c_] = 1/A[τ, c] ((Exp[c/u] - 1) Cm (1 - a[τ, c]) + Cr (1 - a[τ, c]) + Cp a[τ, c])
```

$$\frac{2e^{-c/50}\left(67e^{-e^{-c/25}\tau^2}+100\left(1-e^{-e^{-c/25}\tau^2}\right)+13\,(-1+e^{c/25})\left(1-e^{-e^{-c/25}\tau^2}\right)\right)}{\sqrt{\pi}\,\mathrm{Erf}[e^{-c/50}\tau]}$$

```
λ1 = 1; Cm = 13 ; Cr = 100 ; Cp = 67; u = 25;
NMinimize[{K[τ, c], τ > 0, Cr - Cp > c ≥ 0}, {τ, c}]
```

{76.3131,{c→33.,τ→2.0802}}

```
Clear[β]
a[t_, c_] = Exp[-Exp[-(c/u)] λ1 t^β];
β = 3;
A[τ_, c] = ∫_0^τ a[t, c] dt
```

$$\frac{\tau\left(3\,\mathrm{Gamma}\left[\frac{4}{3}\right]-\mathrm{Gamma}\left[\frac{1}{3},e^{-c/25}\tau^3\right]\right)}{3\,(e^{-c/25}\tau^3)^{1/3}}$$

```
K[τ_, c_] = 1/A[τ, c] ((Exp[c/u] - 1) Cm (1 - a[τ, c]) + Cr (1 - a[τ, c]) + Cp a[τ, c])
```

$$\frac{3\left(67e^{-e^{-c/25}\tau^3}+100\left(1-e^{-e^{-c/25}\tau^3}\right)+13\,(-1+e^{c/25})\left(1-e^{-e^{-c/25}\tau^3}\right)\right)(e^{-c/25}\tau^3)^{1/3}}{\tau\left(3\,\mathrm{Gamma}\left[\frac{4}{3}\right]-\mathrm{Gamma}\left[\frac{1}{3},e^{-c/25}\tau^3\right]\right)}$$

```
λ1 = 1; Cm = 13 ; Cr = 100 ; Cp = 67; u = 25;
NMinimize[{K[τ, c], τ > 0, Cr - Cp > c ≥ 0}, {τ, c}]
```

{85.6173,{c→33.,τ→1.24735}}

```
a[t_, c_] = Exp[- Exp[-(c/u)] λ1 t^β];
β = 2;
A[τ_, c] = ∫_0^τ a[t, c] dt
```

$$\frac{1}{2} e^{c/50} \sqrt{\pi}\,\mathrm{Erf}[e^{-c/50}\tau]$$

```
K[τ_, c_] = 1/A[τ, c] ((Exp[c/u] - 1) Cm (1 - a[τ, c]) + Cr (1 - a[τ, c]) + Cp a[τ, c])
```

$$\frac{2 e^{-c/50}\left(67 e^{-e^{-c/25}\tau^2} + 100\left(1 - e^{-e^{-c/25}\tau^2}\right) + 13\,(-1 + e^{c/25})\left(1 - e^{-e^{-c/25}\tau^2}\right)\right)}{\sqrt{\pi}\,\mathrm{Erf}[e^{-c/50}\tau]}$$

```
λ1 = 1; Cm = 13 ; Cr = 100 ; Cp = 67; u = 25;
NMinimize[{K[τ, 0], τ > 0, Cr - Cp > c ≥ 0}, {τ, c}]
{112.593,{c→0.0613369,τ→1.70596}}
```

```
Clear[β]
```

```
a[t_, c_] = Exp[- Exp[-(c/u)] λ1 t^β];
β = 2;
A[τ_, c] = ∫_0^τ a[t, c] dt
```

$$\frac{1}{2} e^{c/50} \sqrt{\pi}\,\mathrm{Erf}[e^{-c/50}\tau]$$

```
K[τ_, c_] = 1/A[τ, c] ((Exp[c/u] - 1) Cm (1 - a[τ, c]) + Cr (1 - a[τ, c]) + Cp a[τ, c])
```

$$\frac{2 e^{-c/50}\left(67 e^{-e^{-c/25}\tau^2} + 100\left(1 - e^{-e^{-c/25}\tau^2}\right) + 13\,(-1 + e^{c/25})\left(1 - e^{-e^{-c/25}\tau^2}\right)\right)}{\sqrt{\pi}\,\mathrm{Erf}[e^{-c/50}\tau]}$$

```
λ1 = 1; Cm = 13 ; Cr = 100 ; Cp = 67; u = 25;
NMinimize[{K[∞, c], τ > 0, Cr - Cp > c ≥ 0}, {τ, c}]
{79.1201,{c→33.,τ→1.12894}}
```

```
a[t_, c_] = Exp[- Exp[-(c/u)] λ1 t^β];
```

```
β = 1;
A[τ_, c] = ∫_0^τ a[t, c] ⅆt
```

$$\frac{e^{c/u}\left(1 - e^{-e^{-\frac{c}{u}}\tau\lambda_1}\right)}{\lambda_1}$$

```
K[τ_, c_] =
  1/A[τ, c]
  ((Exp[c/u] - 1) C_m (1 - a[τ, c]) + C_r (1 - a[τ, c]) +
    C_p a[τ, c] )
```

$$\frac{1}{1 - e^{-e^{-\frac{c}{u}}\tau\lambda_1}}\left(e^{-\frac{c}{u}}\left((-1 + e^{c/u})\left(1 - e^{-e^{-\frac{c}{u}}\tau\lambda_1}\right)C_m + e^{-e^{-\frac{c}{u}}\tau\lambda_1}C_p + \left(1 - e^{-e^{-\frac{c}{u}}\tau\lambda_1}\right)C_r\right)\lambda_1\right)$$

```
λ_1 = 1; C_m = 13 ; C_r = 100 ; C_p =  67; u = 25;
NMinimize[{K[∞, c], τ > 0, C_r - C_p > c ≥ 0}, {τ, c}]
```

{36.2408, {c→33., τ→1.12947}}

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