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THE ADS/CFT CORRESPONDENCE IN A NON-SUPERSYMMETRIC SETTING

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Declaration

I declare that this thesis is my own, unaided work. It is being submitted for the Degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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(Signature of candidate)

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Abstract

We find giant graviton solutions in Frolov's three parameter generalization of the Lunin-Maldacena background. This class of backgrounds provide a non-supersymmetric example of the gauge theory/gravity correspondence that can be tested quantitatively, as recently shown by Frolov, Roiban and Tseytlin. The giant graviton solutions are shown to have the same energy spectrum as in the undeformed case, and to be perturbatively stable. The spectrum of fluctuations about the semiclassical solutions are shown to depend upon the deformation parameters. We find striking quantitative agreement between the gauge theory and gravity descriptions of open strings attached to the giant, at the level of the semiclassical Hamiltonians.

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Chapter 1

Introduction

The AdS/CFT correspondence[1] provides a new approach to the study of non-Abelian gauge theories. It is a conjectured correspondence between gauge theories at large N and string theories. The most well understood example of the correspondence is between $\mathcal{N} = 4$ 3+1-dimensional SYM and type IIB string theory on $AdS_5 \times S^5$. In this case the gauge theory possesses both conformal invariance and supersymmetry, features which are not part of any field theory known to describe nature. One may hope that ultimately the correspondence may even be used to understand non-perturbative aspects of QCD, which is at the time of writing, a formidable problem. If this hope is ever to be realized, we must gain an understanding of the gauge theory/gravity correspondence in situations with no supersymmetry or conformal symmetry. Recently, a significant step in this direction was achieved by Lunin and Maldacena[2], who identified the gravitational dual of β deformed $\mathcal{N} = 4$ super Yang-Mills theory. The dual gravitational theory has an AdS_5 times a deformed S^5 geometry. Since the AdS_5 factor is not deformed, the field theory is still conformally invariant. However, it only has $\mathcal{N} = 1$ supersymmetry. This deformation was further generalized by

Frolov[3] who gave a background determined by three parameters, that in general, preserves no supersymmetry. The gauge theory/gravity correspondence for this background was explored in detail by Frolov, Roiban and Tseytlin[4]. These authors went on to show a quantitative agreement between the semi-classical energies of strings with large angular momentum and the 1-loop anomalous dimensions of the corresponding gauge theory operators. This is a significant result. The gauge theory/gravity correspondence is a strong weak coupling duality in the 't Hooft coupling. At weak coupling computations in the field theory are straight forward; the dual gravitational theory however, has a highly curved geometry. At strong coupling computations in the field theory are not (in general) under control; in this case curvature corrections in the dual gravitational theory can be neglected. The correspondence is usually explored by computing "nearly protected quantities." These can be computed at weak coupling in the field theory. Since they are nearly protected, they can reliably be extrapolated to the strong coupling regime where comparison with the dual gravity theory is possible. Typically, one appeals to the supersymmetry of the problem to find these nearly protected quantities. The agreement of [4] is striking because it provides an example of quantitative agreement between the gravity and field theory descriptions, in a setting without any supersymmetry. It is important to see how far this quantitative agreement in non-supersymmetric settings can be extended. This is the primary motivation for our work.

Giant gravitons[5],[6],[7] provide a very natural framework for the study of non-perturbative effects in the string theory, in supersymmetric examples of the gauge theory/gravity correspondence. Since giant gravitons are BPS objects, they lead to effects that are protected and hence may be extrapolated between strong and weak coupling. Moreover, they have a simple description in terms of a string worldsheet theory - to leading order they simply determine the boundary conditions for strings with no other affect on the worldsheet sigma model. A lot is also known about giant gravitons in the dual field theory. Operators dual to giant gravitons have been studied in both the $U(N)$[8] and the $SU(N)$[9] gauge theories. These half BPS states also have a simple description in terms of free fermions for a one matrix model[10] which has recently been connected to a description which accounts for the full back reaction of the geometry in the supergravity limit[11]. A tantalizing attempt to go beyond one matrix dynamics has appeared in[12]. Further, the technology needed to study strings attached to giant gravitons is well developed[13],[14]. Given the recent progress in constructing non-supersymmetric examples of the gauge theory/gravity correspondence, it seems natural to ask if there are giant gravitons solutions in these new geometries. We will construct giant gravitons for the deformed background with $\tilde{\gamma}_1 = 0$ and $\tilde{\gamma}_2 = \tilde{\gamma}_3 = \tilde{\gamma}$.

A particularly efficient way to organize and sum the Feynman diagrams of the field theory, is through the use of a spin chain[15]. In this approach, one identifies the dilatation operator of the field theory with the Hamiltonian of the spin chain. Constructing operators with a definite scaling dimension as well as the spectrum of scaling dimensions becomes the problem of diagonalizing the spin chain Hamiltonian. This approach has been extremely powerful because it allows one to identify and match the integrability of the gauge theory dilatation operator[16] with that of the world sheet sigma model[17]. Understanding the field theory beyond the one-loop approximation involves studying spin chains with varying number of sites[18]. In this article we would like to use the spin chain approach to study operators dual to open strings attached to giant gravitons. For non-maximal giants this again corresponds to studying a spin chain with a variable number of sites. A very convenient approach to these problems has been developed in[19]. The idea is to map the spin chain into a dual boson model on a lattice. For the boson model, the number of sites is fixed; the variable number of sites in the original spin chain is reflected in the fact that the number of bosons in the dual boson model is not conserved. In this article we will construct the boson lattice model which describes open strings attached to giant gravitons in the deformed background.

Apart from the three parameter deformed backgrounds studied in this article, there have been many other interesting developments following Lunin and Maldacena's work. In[20] energies of semiclassical string states in the Lunin Maldacena background were matched to the anomalous dimensions of a class of gauge theory scalar operators. The spin chain for the twisted $\mathcal{N} = 4$ super Yang-Mills has been studied in [21]. The logic employed by Lunin and Maldacena to obtain the gravitational theory dual to the deformed field theory has been extended in a number of ways. Recently, instead of deforming the $\mathcal{N} = 4$ super Yang-Mills theory, deformations of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ theories have been considered[22]. Further, deformations of eleven dimensional geometries of the form $\mathrm{AdS}_4 \times \mathrm{Y}_7$ with Y_7 a seven dimensional Sasaki-Einstein[23],[24] or weak G_2 or tri-Sasakian[24] space have been considered. The pp-wave limit of the Lunin Maldacena background, and the relation to BMN[25] operators in the dual field theory has been considered in [26]. Recent studies of the β-deformed field theory include[27]. Semiclassical strings were studied in [28]. Finally, in [29], interesting instabilities in the general three parameter backgrounds have been discovered.

This thesis is organised as follows:

- Chapters 2-4 provides a cursory review of the AdS/CFT correspondence, giant gravitons, and deformation of string and field theories, highlighting features important for the rest of the work in the thesis.

- Chapters 5-7 present the work published in [30] which formed the basis of this MSc. degree - specifying and testing a giant graviton ansatz and showing that the correspondence holds in the non-supersymmetric case.

Chapter 2

The AdS/CFT correspondence

In this chapter we review Maldecena's original duality conjecture [1], after briefly examining the Large N 't Hooft limit and its relevance to string theories. Many more details can be found in the review listed at [1]. Note that although one particular duality is described here, many more have since been explored. A common feature of these dualities is supersymmetry in the corresponding gauge theory. It should be noted that on one side of this duality is a theory of gauged particle interactions in flat space, with no mention of gravity and in fact the apparent impossibility of incorporating gravity. On the other hand we have the string theory - an intrinsically gravitational theory in that consistency requirements imply the existence in the theory of the graviton, and force Einstein's field equations upon spacetime. The goal of this thesis is to present our work [30] testing a duality also in the case of no supersymmetry.

2.1 Large N gauge theories and string theories

String theory was originally envisioned as a description of the strong interaction, which exhibits some stringy behaviour, like confinement. Then an $SU(3)$ gauge theory, QCD, was discovered. This theory exhibits asymptotic freedom - at high energies the coupling constant decreases. At low energy there are interesting physical effects such as confinement and chiral symmetry breaking which are mathematically inaccessible in the full theory. Many attempts have been made to describe QCD at low energy in terms of flux tubes (which behave like strings). Such a description seems natural given, for example, that if ψ is a quark field, then $\bar{\psi}\psi$ is not gauge invariant whereas $\bar{\psi}(x_1)\int_{x_1}^{x_2} A_\mu dx^\mu \psi(x_2)$ is. 't Hooft pointed out that $SU(N)$ gauge theories simplify at large N, where an expansion is admitted in $\frac{1}{N}$. We argue that this expansion indicates a possible relationship between string theories and large N gauge theories.

Suppose we have a general gauge theory given by (the fields have been rescaled by $\Phi \to \frac{1}{g_{\rm YM}}\Phi$)

$$\mathcal{L} \sim \frac{1}{g_{\rm YM}^2}\left[\mathrm{Tr}(d\Phi_i d\Phi_i) + a^{ijk}\mathrm{Tr}(\Phi_i\Phi_j\Phi_k) + b^{ijkl}\mathrm{Tr}(\Phi_i\Phi_j\Phi_k\Phi_l)\right].$$

The coefficient $\frac{1}{g_{\rm YM}^2} = \frac{N}{\lambda}$ diverges in the large N limit. This should not be thought of as a classical limit however, since the number of components for each field also goes to infinity. For a general gauge theory containing matter fields in the adjoint representation, the Feynman diagrams for correlation functions of arbitrary products of gauge-invariant fields are drawn in a double-line notion. The following discussion is for $U(N)$ and $SU(N)$ gauge theories, but the conclusion is valid for general gauge theories, up to the anticipated string theory dual.

The double lines of a diagram can be viewed as the simplicial decomposition of some surface. The closed single line loops define the oriented surfaces of the decomposition. We add a point at infinity so that these decompositions correspond to closed oriented surfaces. From the Lagrangian above it can be inferred that each vertex comes with a factor $\sim \frac{N}{\lambda}$ and each propagator with a factor $\sim \frac{\lambda}{N}$. For each closed loop in the diagram (representing a sum over one set of indices) there is a factor of N. So a diagram of V vertices, E propagators (edges) and F loops (faces) contributes

$$N^{V-E+F}\lambda^{E-V} = N^{2-2g}\lambda^{E-V},$$

where g is the genus of the surface. The relation follows since for compact oriented surfaces the Euler character $V - E + F = 2 - 2g$. For any diagram the perturbative expansion is

$$\sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda),$$

where f_g is any polynomial in λ. The expansion has the same form as that obtained from oriented closed string theory, with a coupling $\sim \frac{1}{N}$. In the latter case the perturbative expansion is over surfaces of increasing genus. This is compelling evidence to suggest that field theories and string theories are related, at least perturbatively. In the remainder of the chapter we present an example of such a correspondence.

2.2 5-dimensional anti-de Sitter space, AdS_5

Manifolds are commonly defined as surfaces embedded in simpler spaces of higher dimensionality. To get AdS_5, one begins in a space with metric

$$ds^2 = -(dY^{-1})^2 - (dX^0)^2 + (dX^1)^2 + (dX^2)^2 + (dX^3)^2 + (dX^4)^2.$$

The surface in this space given by

$$(Y^{-1})^2 + (X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2 - (X^4)^2 = R^2$$

is AdS_5. In this form the $SO(2,4)$ isometry of AdS_5 is manifest. Switching to Poincaré coordinates yields the metric

$$ds^2 = \frac{R^2}{r^2}(-dt^2 + d\vec{x}^2 + dr^2).$$

The Poincaré coordinates are defined by the change of variables

$$\begin{aligned}
X^{-1} &= \frac{1}{2r}(\Lambda^2 + \vec{x}^2 + r^2 + t^2) \\
X^0 &= \Lambda\frac{t}{r} \\
X^1 &= -\frac{1}{2r}(-\Lambda^2 + \vec{x}^2 + r^2 - t^2) \\
X^2 &= \Lambda\frac{x^1}{r} \\
X^3 &= \Lambda\frac{x^2}{r} \\
X^4 &= \Lambda\frac{x^3}{r}.
\end{aligned}$$

The singularity at $r = 0$ is only a coordinate singularity and it is possible to define globally valid coordinates on the space.

2.3 D-branes, open strings and gauge theories

Dp-branes are objects in string theory which arise from assigning Dirichlet boundary conditions to string end points. These boundary conditions only apply to spatial coordinates since string end points should be able to move in time. In d spatial dimensions, suppose that some end points satisfy Dirichlet boundary conditions in $d-p$ coordinates. They are thus free to move in p spatial dimensions. The surface on which these end points move is called a Dp-brane.

It was shown in [31] that such objects are BPS states. Stable branes in IIA theory are even-dimensional, while in IIB theory they are odd-dimensional. Such states have a charge in the perturbative regime which may be reliably extrapolated to the non-perturbative regime. Because of the strong/weak coupling nature of the AdS/CFT correspondence (see section 2.5) this is a singularly important feature for probing the relationship between gravity and gauge theories. BPS objects have charge equal to their rest mass, which puts them in a short supersymmetry multiplet. Were this condition to change, the states would exist in a larger multiplet. It is not possible to find a continuous operation which takes a state from the short to long multiplet. Moving from a perturbative regime (small coupling) to a non-perturbative regime (large coupling) is a continuous operation on the coupling constant. So for these objects, we can be certain that the mass equals charge condition holds in the non-perturbative regime (otherwise they would have been discontinuously transformed).

How does this translate into a charge which is the same in both perturbative and non-perturbative regime? On the gravity side the relevant charge is angular momentum, which is obviously unchanged by increasing the string coupling. So we are guaranteed that the charge (and hence the mass) is exactly the same in both energy regimes on the gravity side. In the gauge theory we can explicitly calculate the scaling dimension of the candidate dual operator. (The scaling dimension of a gauge theory operator is matched to the mass of a gravity state, see 2.5). So one can see directly whether the scaling dimension depends on the coupling.

Although it may seem strange to assign ontological status to sets of boundary conditions it turns out that this was just the route by which D-branes were discovered. It is known that D-branes are physical objects intrinsic to string theory - among other things, they are the sources of RR charge required for internal consistency.

We now turn to a duality in the interpretation of a string theory Lagrangian. One may view this Lagrangian as both specifying the dynamics of (extended) objects in spacetime or as defining a conformal field theory on the parameter space of the coordinates.

Imagine string theory in some $d+1$-dimensional spacetime. Moving through spacetime, a string traces out a 2-dimensional surface called the worldsheet. This surface is parameterized by two coordinates σ, τ. The spacetime coordinates of the string are described by functions $X^\mu(\sigma,\tau)$, $\mu = 0,\ldots,d$. The action for a string is proportional to the area of the worldsheet in spacetime. This is analogous to the case of a relativistic point particle, whose action is proportional to the length of its worldline.

Equivalently, we can think of the coordinates X^μ as fields living on the $1+1$-dimensional parameter space, and thus we have an interpretation of the system as some kind of field theory. One symmetry of the Polyakov action is Weyl rescaling invariance, $g(x) \to \Lambda(x)g(x)$. That is to say, what we call the area of the worldsheet does not depend upon scaling of the worldsheet metric. This symmetry of the action has profound consequences. Firstly, it imposes a constraint on the metric so that it satisfies the (suitably generalised) Einstein field equations. When combined with the natural emergence of (quantum) field theories, this makes it look promising that string theory can shed some light on the question of quantum gravity. Secondly, enforcing the symmetry gives rise to a conformal field theory on the $1+1$ dimensional parameter space.

D-branes similarly give rise to a worldvolume field theory. Below we shall be interested in a system with N coincident D3-branes. The embedding of these branes into spacetime again has an interpretation as fields existing on the worldvolume of the branes. Quantising the string theory one finds that the massless modes of open strings ending on the N branes give rise to $\mathcal{N} = 4$ $U(N)$ super Yang-Mills theory. It can also be seen that these open string states are describing excitations of the D-branes. The conclusion is that the worldvolume CFT on N coincident D3-branes is the SYM theory of open strings ending on these branes. For the work presented later in this thesis we will be interested primarily in transverse fluctuations of the branes, which are described by the six scalar fields in the adjoint representation of $U(N)$.

2.4 Two equivalent descriptions of a system of D-branes

The system of N cooincident D-branes introduced above can be described in two different ways. Many calculations have been done for corresponding objects in these theories leading to faith in their exact equivalence. The AdS/CFT conjecture rests on this equivalence. Firstly, the descriptions are organised into systems which decouple in the low energy limit, and then these limits are compared, and a correspondence drawn between subsystems.

2.4.1 Branes in flat space

Here we begin with type IIB string theory in flat, 10-dimensional Minkowski space, M_{10}. Consider N parallel, coincident D3-branes at the origin. This theory admits two kinds of perturbative excitations. There are closed string modes propagating throughout empty space, and open strings ending on the D-branes, which describe the excitations of the branes. In order to describe interactions between the two types of excitations, it is neccessary to use an effective Lagrangian, wherein all but the massless modes have been integrated out. Schematically, one obtains an action

$$S = S_{\text{space}} + S_{\text{brane}} + S_{\text{interaction}}.$$

For closed strings the effective action is IIB supergravity with some corrections. For the open strings one has the expected 3+1-dimensional SYM gauge theory and some corrections. Finally, there is an interaction term between the brane modes and closed string modes.

For the strings, unexcited states are massless, while the excited states have mass $\sim \frac{1}{l_s}$. Thus at an energies smaller than $\frac{1}{l_s}$ only massless modes can be excited. The low energy picture is obtained by taking $l_s \to 0$ while keeping all dimensionless parameters fixed. In this limit, the corrections mentioned above, as well as the interaction term, vanish, leaving the action

$$S = S_{M_{10}}^{\text{IIB sugra}} + S_{M_4}^{U(N)\ \text{SYM}}.$$

2.4.2 D-branes in supergravity

An alternate description is given by solving supergravity for the heavy stack of N D3-branes, which produce a curved classical metric along with some background fields. This solution is

$$ds^2 = f^{-\frac{1}{2}}(-dt^2 + d\vec{x}^2) + f^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2), \tag{2.1}$$

with

$$f = 1 + \frac{R^4}{r^4},\ R^4 \equiv 4\pi g_s {\alpha'}^2 N.$$

R is the radius of curvature of the geometry. The non-constant g_{tt} component implies a redshift of the energy E_p as measured by an observer at r compared with the energy E measured at infinity:

$$E = f^{-\frac{1}{4}} E_p.$$

So the energy of an object brought closer to $r = 0$ would appear to have a lower energy as measured by an observer at infinity. In other words, the brane at $r = 0$ sets up a gravitational potentional well which makes it harder for excitations near $r = 0$ to escape to the asymtoptic region. This is true of any excitation in the near-horizon region, not just the massless modes. In the limit $r \to 0$ we expect these excitations to decouple completely from low energy excitations in the bulk of the spacetime. These latter bulk excitations in the low energy limit have wavelengths much larger than R, the characteristic gravitational size of the brane, and are described by IIB supergravity in M_{10}. The flat space emerges because really long wavelength modes are insensitive to the curvature which is localised near the brane. We have established that all excitations in the near horizon region contribute to the low energy effective theory, so we must ask what spacetime looks like to these excitations. For $r << R$, we get $f \approx \frac{R^4}{r^4}$ and equation 2.1 becomes (this is somewhat schematic - more details in [1])

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + d\vec{x}^2) + R^2 \frac{dr^2}{r^2} + R^2 d\Omega_5^2$$

which is the geometry of $AdS_5 \times S^5$, as described above. In total we have a low energy action like

$$S = S_{M_{10}}^{\text{IIB sugra}} + S_{AdS_5 \times S^5}.$$

2.5 Comparison and consequences

Both descriptions above yielded decoupled theories in the low energy limit. Both contained IIB supergravity in flat space. It is natural then to identify the remaining pieces. This is the conjecture that $\mathcal{N} = 4$ $U(N)$ super Yang-Mills theory in 3+1 dimensions is dual to IIB superstring theory on $AdS_5 \times S^5$ [1].

There is one particular feature of this duality which makes it both potentially very powerful and quite tricky to test. First let us note that the various parameters are related as follows:

$$g_{YM}^2 N \sim g_s N \sim \frac{R^4}{l_s^4}.$$

Only perturbative calculations are generally accessible to present mathematical technology, and these are generally valid only for certain parameter ranges. In the gauge theory, perturbation theory is valid for $g_{YM}^2 N \ll 1$. On the other hand, perturbation theory in the gravity description relies on a large radius of curvature compared to the string length, ie $\frac{R^4}{l_s^4} \gg 1$. Refering to the relation above, it is clear that these are inverse conditions, and so the perturbative regimes of the two theories are perfectly incompatible. This has two immediate consequences. Firstly, if we are confident in the validity of the conjecture, then well understood perturbative calculations in the one theory can be used to make statements about the otherwise inaccessible non-perturbative regime of the other. Secondly, to test the conjecture for validity one has to find objects on each side with some properties which are protected from corrections when extrapolating the perturbative result to the non-perturbative regime.

As a final observation, notice that the conformal SYM theory has the symmetry group $SO(2,4) \times SO(6)$. $SO(2,4)$ is the conformal group in 3+1 dimensions and the $SO(6)$ describes the R-symmetry of the six real scalar fields. We have already seen that $SO(2,4)$ is the isometry group of AdS_5, and there is on the string theory side the additional isometry group of the S^5, namely $SO(6)$. Hence we see that the two theories enjoy the same symmetry groups. This is a general feature of dualities between string theories and gauge theories. It is natural to suppose that charges in the two theory should be compared based on which symmetries they correspond to, and this turns out to be the case. For example, $SO(2,4)$ decomposes into $SO(2)\times SO(4)$. In the gauge theory, the charge under $SO(2)$ is the scaling dimension, Δ. In the gravity theory it is the energy of the state. A common test of the AdS/CFT correspondence is to check that these values match for protected states and their dual description.

Chapter 3

Giant gravitons

Giant gravitons were first described in [5]. These are spherical D3-branes orbiting in the sphere of the $AdS_5 \times S^5$ spacetime. Their discovery provided a mechanism to explain the stringy exclusion principle, which places a limit on the angular momentum of states propagating on the sphere, equal to N. On the string theory side, for $AdS_5 \times S^5$, the giant couples to a Chern-Simons potential which causes it to expand, while the kinetic (Dirac-Born-Infeld) term favours shrinking the giant. A set of stable angular momentum states can be found (details below) for which the brane exists with a certain radius. The maximum size of the brane is the radius of the S^5, which state has the maximum angular momentum, N.

This also fixed a divergence problem with the traditional Kaluza-Klein gravitons of high angular momentum - more exciting details below.

Below we present a description of this expansion process by first reviewing some analogies and then sketching the calculations of [5].

3.1 Exotic dipole action

Consider a hydrogen atom in an electric field, $\vec{E}$, which is polarized by the field. The positive nucleus and the expected position of the negative electron align with the field and are separated by $\vec{d}$. We can define the polarizability of the atom, α as

$$q \cdot d = \alpha E, \tag{3.1}$$

since $\vec{d}$ and $\vec{E}$ are parallel. Note that q is the sum of the absolute values of the charges.

Now replace our atom with an electric dipole such that the two point charges have equal mass (but a very small mass so the argument below neglects any contributions from momentum/kinetic energy to the angular momentum of the dipole). Let's imagine that this dipole lives on a 2-sphere, orbiting on the equator. It will be convenient for our puny human brains to imagine this sphere embedded in $\mathcal{R}^3$. Place a magnetic monopole at the centre of this sphere so that the charges orbiting the equator experience a force in opposite directions - they will move so that one charge orbits in the top hemisphere, and the other in the bottom hemisphere, at equal angles off the equator. This is possible because the force experienced by the particles in the magnetic field falls off as they move away from the equator. It can be shown (full details in [5]) that for a particular field strength, the angle off the equator is determined by the angular momentum of the orbiting dipole. We will present details for the case of the inflating giant; the present example serves to highlight conceptual critical points.

Continuing the analysis for the dipole, one finds that the maximum angular momentum (equal to the magnetic flux on the sphere) occurs when the dipole has expanded so that one charge is at each pole (where it is stationary). How odd. Well, the magnetic potential couples to the motion of the centre of mass of the dipole (go read [5]). By the symmetry of the arrangement the centre of mass is never going to leave the equator. In particular, when the charges are at the poles, the centre of mass will still be on the *equator*, happily orbiting away and contributing angular momentum through its coupling to the magnetic field. (Remember that these are vanishingly light charges - we are neglecting any contribution to angular momentum through mv-type terms). How well-defined is the centre of mass of a dipole with charges on opposite poles of a sphere? I imagine that if I lived in a 2-sphere and had a rod which I stretched out until it was half the circumference long, it would still have a well-defined centre of mass. Similarly the position of the centre of mass of a dipole stretched to half the circumference would lie on the geodesic along which it was stretched.

3.2 Polarizability of the giant graviton

The giant is an S^3 whose centre of mass orbits a ring in S^5 parameterized by ϕ. The radius of the giant is r. The radius of the S^5, and hence the radius of the centre of mass orbit is R.

Now we will wonder in what sense a homogeneous 3-brane can be understood to possess a dipole charge and thereby justify the elaborate analogy above. A point particle moving along a worldline parameterized by dx^μ couples to a 1-form potential A_μ with a term like

$$S = \cdots + e \int A_\mu dx^\mu,$$

where e is the charge/coupling constant. The sum over indices implicity includes the appropriate spacetime metric elements and hence this object is invariant under the spacetime isometries. Moving in the opposite direction would make the charge look negative. Sane conventions in this case can give us a well defined charge for the particle. Not so for our 3-brane.

Generalising in a natural way, the D3-brane couples to a 4-form potential. The potential is related to a 5-form field strength through

$$F_5 = dC_4. \tag{3.2}$$

This is analogous to the 2-form field strength in electromagnetism, which is the exterior derivative of the vector potential. The 5-form field strength is given in equation 3.4. Looking at equations 3.2 and 3.4 we see there are several ways to write C_4 which would satisfy 3.2. We are interested in the coupling of the potential to the brane worldvolume which selects the set of indices we want C_4 to carry, namely t and Ω_3, which parameterizes the D3-brane. So we write

$$C_4 = BR\frac{r^4}{4}d\phi d\Omega_3 = BR\dot{\phi}\frac{r^4}{4}dtd\Omega_3. \tag{3.3}$$

Now consider a small D3-brane and pick some point on its worldvolume. Suppose $d\psi$ is one of the angles parameterizing $d\Omega_3$ in the equation above, so that the volume element is $\sim d\psi$. Circling the worldvolume in the direction of $d\psi$ until we reach the antipode, we find that the sign of $d\psi$ has changed, while all other directions on the worldvolume are the same, hence the "direction of the parameterization" has changed by a minus sign. As described above with the point particle, this means that antipodal points on the brane appear to carry opposite charge under the 4-form potential, and hence expand in different directions.

Let's try capture the polarizability of a giant graviton. Analogously to equation 3.1, we have

$$2RQ = \alpha F_5.$$

F_5 is the 5-form field strength on the sphere, which has a constant flux B over the sphere. Quantization of flux requires

$$\Omega_5 B R^5 = 2\pi N,$$

where $\Omega_5 = \pi^3$ is the volume of a unit 5-sphere. In total we have

$$\begin{aligned} F_5 &= B \times \text{volume element on } S^5 \\ &= BRr^3 dr d\phi d\Omega_3. \end{aligned} \tag{3.4}$$

The radius of the S^5 part of the spacetime geometry is given by

$$R = (4\pi g_s N)^{\frac{1}{4}} l_s,$$

in terms of the string coupling and string length. Q is the charge of the giant graviton, which is equal to its mass since it is a D-brane and hence BPS. To calculate the mass of a giant graviton with radius r we multiply the tension (constant for all D3-branes)

$$T_{D_3} = \frac{1}{(2\pi)^3 l_s^4 g_s}$$

by the volume of the brane

$$r^3 \Omega_3 = 2\pi^2 r^3.$$

This gives us

$$m = Q = \frac{N r^3}{R^4}. \tag{3.5}$$

Putting it all together yields the D3-brane polarizability

$$\alpha_{\mathrm{D3}} = \pi^2 R,$$

which is constant for a given S^5 geometry.

3.3 Expansion in the S^5

The bosonic Lagrangian for the D3-brane is

$$\mathcal{L} = \mathcal{L}_{DBI} + \mathcal{L}_{CS}.$$

The Dirac-Born-Infeld term gives the kinetic energy of the brane. Derived in [32], the expression is

$$\mathcal{L}_{DBI} = \sqrt{-\det G_{\mu\nu}},$$

where $G^{\mu\nu}$ is the induced metric on the D3-brane worldvolume. The Chern-Simons term is the coupling of the brane to the 4-form potential 3.3. The total Lagrangian is

$$\mathcal{L} = -T_{D_3}\Omega_3 r^3 \sqrt{1-(R^2-r^2)\dot{\phi}^2} + \dot{\phi} N \frac{r^4}{R^4}.$$

Clearly the angular momentum conjugate to ϕ is conserved. In terms of $\dot{\phi}$:

$$L = \frac{m\dot{\phi}(R^2-r^2)}{\sqrt{1-\dot{\phi}^2(R^2-r^2)}} + N\frac{r^4}{R^4},$$

where m is the mass, as determined in 3.5. Notice that the maximum angular momentum N occurs for $r = R$ and $\dot{\phi}R = 1$. The second condition says that the centre of mass of the maximal giant (which, remember, is still orbiting at a radius R, even though the brane itself is centred in its orbital plane) moves at the speed of light.

In terms of L, the energy is

$$E = \sqrt{m^2 + \frac{(L - Nr^4/R^4)^2}{R^2 - r^2}}.$$

Varying with respect to r at fixed L we find a stable minimum at

$$r^2 = \frac{L}{N} R^2$$

with energy

$$E = \frac{L}{R},$$

for large L, which is where the semiclassical treatment presented above it valid. The traditional point-like graviton orbiting on the sphere with angular momentum $\sim N$ is a singular solution from the perspective of the gravitational field, suffering from uncontrolled quantum corrections due to concentrating such a large amount of energy at one point. The corresponding giant graviton has smoothed out this divergence by expanding into a 3-sphere.

Chapter 4

Deformed string backgrounds and field theories

4.1 B-fields in string theory

We saw in chapter 3 that the 3+1-dimensional D3-brane couples to a 4-form potential. Similarly the 1+1-dimensional string worldsheet couples to a 2-form potential called the NS-NS B field. Choosing an Euclidean metric for the string worldsheet Σ, the string Lagrangian is [33]

$$\frac{1}{4\pi\alpha'}g_{ij}\partial_a x^i\partial^a x^j - \frac{i}{2}B_{ij}\epsilon^{ab}\partial_a x^i\partial_b x^j.$$

For coordinates x^i along the Dp-branes we find boundary conditions

$$g_{ij}\partial_n x^j + 2\pi\alpha' B_{ij}\partial_t x^j|_{\partial\Sigma} = 0,$$

where ∂_t is a tangential derivative along the worldsheet boundary $\partial\Sigma$, and ∂_n is a derivative normal to $\partial\Sigma$. For $B = 0$ these are Neumann boundary conditions, while for $B \to \infty$ they are Dirichlet boundary conditions. The B field gives rise to a non-commutativity term

$$\theta^{ij} = -(2\pi\alpha')^2 \left(\frac{1}{g + 2\pi\alpha' B} B \frac{1}{g - 2\pi\alpha' B} \right)^{ij},$$

where

$$[x^i(\tau), x^j(\tau)] = i\theta^{ij}.$$

This effect is captured by replacing the ordinary product of fields with the star product

$$f(x) * g(x) = e^{\frac{i}{2}\theta^{ij}\frac{\partial}{\partial\xi^i}\frac{\partial}{\partial\zeta^i}} f(x+\xi) g(x+\zeta)|_{\xi=\zeta=0}. \tag{4.1}$$

In conclusion, the effect of a non-zero B field on the string CFT is to replace usual products with the star product above.

4.2 Marginal deformations of field theories

Generally speaking, in quantum field theory, coupling constants are not constant. They change as we flow to different energy scales. Suppose we have some energy scale μ, and a momentum cutoff Λ. If g_i is the set of coupling constants for some theory, then the effect of changing energy scale is determined by the β functions

$$\mu \frac{\partial g_i}{\partial \mu} = \beta_i(g_k, \mu).$$

For example, maximally supersymmetric YM theory is conformally invariant - that is to say it has a scale invariance. At all energy scales the dynamics are the same because the couplings do not change with energy scale. This is because all of the β functions of the theory vanish.

Working very schematically, the fields are defined by

$$\phi^a(x) = \int_{p^2<\Lambda^2} (A(p)e^{-ip\cdot x} + A^*(p)e^{ip\cdot x})dp.$$

The momentum cut off avoids UV divergences in momentum integrals. We can split up the fields into two pieces, the high-momentum sum and the low-momentum sum. Then

$$\phi^a(x) = \phi^a_<(x) + \phi^a_>(x),$$

where

$$\begin{aligned}\phi^a_< &= \int_{p^2<\mu^2} (A(p)e^{-ip\cdot x} + A^*(p)e^{ip\cdot x})dp,\\ \phi^a_> &= \int_{\mu^2<p^2<\Lambda^2} (A(p)e^{-ip\cdot x} + A^*(p)e^{ip\cdot x})dp.\end{aligned}$$

Correlation functions can be computed using the path integral approach. Imagine a set of operators $\mathcal{O}_i$ which can be expressed in terms of only $\phi_<$. Then it is possible to completely eliminate $\phi_>$ from the path integral as follows

$$\begin{aligned}\langle \mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n\rangle &= \int \mathcal{D}\phi^a \mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n e^{iS}\\ &= \int \mathcal{D}\phi^a_< \mathcal{D}\phi^a_> \mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n e^{iS}\\ &= \int \mathcal{D}\phi^a_< \mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n e^{iS_{\text{eff}}}.\end{aligned}$$

In the last line we have integrated out the high-momentum degrees of freedom to obtain an effective low-energy action which locally describes the theory at an energy scale less that μ. The effective action in the $\mu \to 0$ limit is called infrared effective action. It is sometimes possible to determine such an action exactly, although in many cases the above discussion is severely complicated by the fact that the low energy degrees of freedom may be completely different to the original degrees of freedom. For example, QCD at high energy is described by quark and gluon degrees of freedom, but at low energy by hadron and meson degrees of freedom.

This discussion provides a framework by which to classify deformations of field theories in terms of how their couplings behave as the high-momentum degrees of freedom are integrated out. Deforming a theory amounts to adding a new term to the Lagrangian, which carries a coupling coefficient. If the coupling gets smaller as $\phi_>$ is integrated out the deformation is called irrelevant. If it grows we call the deformation relevant. The case we shall be interested in is where the coupling remains constant as $\phi_>$ is integrated out - these are called marginal deformations.

Consider a supersymmetric field theory with three adjoint left chiral superfields Φ_i and superpotential

$$f = a\mathrm{Tr}(\Phi_1\Phi_2\Phi_3) + b\mathrm{Tr}(\Phi_1\Phi_3\Phi_2) + c\mathrm{Tr}(\Phi_1^2 + \Phi_2^2 + \Phi_3^2).$$

Upon integrating out the high-momentum modes, the new fields, denoted Φ_i, create a particle with probability < 1. They must therefore be renormalised. The renormalised fields are called canonically normalised fields,

$$\Phi_i^{\mathrm{cn}} \equiv \sqrt{Z_n(\mu)}\Phi_i,$$

and they create a single particle state with probability 1. The anomalous dimensions of the fields are given by

$$\gamma_i = \frac{d\log Z_n}{d\log\mu}.$$

The superpotential f above has a Z_3 symmetry acting as $\Phi_1 \to \Phi_2$, $\Phi_2 \to \Phi_3$, $\Phi_3 \to \Phi_1$, and hence we must have $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$. Usual $\mathcal{N} = 4$ SYM is obtained for $a = -b = g$ and $c = 0$. The β deformed $\mathcal{N} = 1$ SYM theory obtained in [2] is obtained for

$$f = e^{i\pi\gamma}\mathrm{Tr}(\Phi_1\Phi_2\Phi_3) + e^{-i\pi\gamma}\mathrm{Tr}(\Phi_1\Phi_3\Phi_2).$$

In both cases it can be shown that the β functions vanish, and so the field theories are conformally invariant.

Finally, it should be noted that the $\mathcal{N} = 1$ theory preserves a $U(1) \times U(1)$ non-R-symmetry which will be discussed in the section 4.4. If Q_i is the charge of a chiral superfield under the $U(1) \times U(1)$, then the deformation described above can be seen as arising from a redefinition of the usual field product to

$$\Phi_i(x) * \Phi_j(x) = e^{\frac{i}{2}\theta^{ij}(Q_i^1 Q_j^2 - Q_i^2 Q_j^1)}\Phi_i(x)\Phi_j(x).$$

4.3 A solution generating transformation

10-dimensional type IIA string theory contains a single scalar field which can be reinterpreted as a dimension to yield 11-dimensional M-theory. IIA can then be recovered by compactifying along one of the M-theory directions, say ψ_1. Any angular momentum along ψ_1 will look like a conserved charge from the 10-dimensional theory, which would label some particular state.

This idea was exploited in [2] to develop a technique for generating new supergravity solutions from old ones. Suppose we have a IIA supergravity solution, which we lift to M-theory. A requirement for this technique is that the M-theory solution contain a geometric T^2 symmetry, for reasons which will be elucidated momentarily. Suppose this symmetry is realised in the ψ_1, ψ_2 coordinates. An $SL(2,R)$ symmetry transformation can be applied to the solution in M-theory, changing the momentum of the solution about the aforementioned coordinates. Now a Kaluza-Klein reduction along the ψ_1 coordinate yields a new IIA solution - the conserved charge which labels the state has changed.

Of course, it is possible in principle to find more general symmetry transformations and thus produce more general new backgrounds. However, one constraint is that the new backgrounds must be non-singular. When transforming a 2-torus in the background possible singularities arise from the torus shrinking to an S^1 or a point. We are guaranteed by the action of $SL(2,R)$ that for values of the deformation parameter which could lead to such cases, the new background is identical to the original background. Therefore, this set of transformations guarantees that a non-singular original background produces a non-singular new background. The most general transformation found in [2], which possess this crucial property, is $SL(3,R)$.

The solutions in which we are interested contain giant gravitons. These are D3-branes and hence stable only in IIB theory. Happily, there is a duality connecting IIA and IIB theories. This is called T-duality - for a background with a compact dimension, exchanging winding number of a state about the compact dimension with its momentum around the compact dimension yields IIB from IIA and vice versa.

Applying this method to the original $AdS_5 \times S^5$ IIB solution yields a new solution

$$ds^2 = R^2 \left[ds^2_{AdS_5} + \Sigma_i (d\mu_i^2 + G\mu_i^2 d\phi_i^2) + \hat{\gamma}^2 \mu_1^2 \mu_2^2 \mu_3^2 (\Sigma_i d\phi_i^2) \right].$$

In the above expression, μ_i, ϕ_i are three pairs of radii and angles which replace the six Cartesian coordinates parameterizing S^5. They therefore satisfy

$$\mu_1^2 + \mu_2^2 + \mu_3^2 = 1.$$

We also have

$$G^{-1} = 1 + \hat{\gamma}^2(\mu_1^2\mu_2^2 + \mu_2^2\mu_3^2 + \mu_1^2\mu_3^2)$$
$$\hat{\gamma} = R^2\gamma,\ \ R^4 \equiv 4\pi e^{\phi_0}N.$$

If ϕ_0 is the original dilaton field, the new dilaton is given by

$$e^{2\phi} = e^{2\phi_0}G.$$

The background field content includes a 2-form B potential, coupling to fundamental strings

$$B = \hat{\gamma}R^2G(\mu_1^2\mu_2^2 d\phi_1 d\phi_2 + \mu_2^2\mu_3^2\phi_2 d\phi_3 + \mu_1^2\mu_3^2\phi_3 d\phi_1), \tag{4.2}$$

a C potential coupling to D-strings (D1-branes)

$$C = -3\gamma(16\pi N)\omega_1 d\psi,\ \ d\omega_1 = \cos\alpha\sin^3\alpha\sin\theta\cos\theta d\alpha d\theta,$$

and also a 5-form field strength which couples to D3-branes

$$F_5 = (16\pi N)(\omega_{AdS_5} + G\omega_{S^5}),\ \ \omega_{S^5} = d\omega_1 d\phi_1 d\phi_2 d\phi_3,\ \omega_{AdS_5} = d\omega_4.$$

Here ω_5 is the volume element on a unit S^5 (notation has been chosen to match [2]).

4.4 The conjectured dual gauge theory

It has already been mentioned that the symmetries of the field theory and the isometries of the gravity theory background must agree. In order to apply the method for generating new solutions, the background must contain a 2-torus, on which the action of $SL(2, R)$ is a symmetry. That is, shifts on the two angles parameterizing the torus are a symmetry of the theory. Thus we expect the dual field theory to contain a $U(1) \times U(1)$ non-R-symmetry, under which the fields Φ_i have charges $Q_i^1 \times Q_i^2$.

Having seen that the new supergravity solution contains a B field, we anticipate a new field product along the lines of 4.1. There is one difference though. In the present case, from 4.2 we see that the B flux is transverse to the N branes which originally set up the $AdS_5 \times S^5$ spacetime and which can be thought of as the boundary M_4. So the new product replaces momenta transverse to the brane with the $U(1) \times U(1)$ charges of the fields:

$$f(x) * g(x) = e^{\frac{i}{2}\theta^{ij}(Q_f^1 Q_g^2 - Q_f^2 Q_g^1)} f(x) g(x).$$

This does not give rise to a non-commutative field theory; rather it is a marginal deformation of a normal field theory, which picks some extra phases in the Lagrangian, as in section 4.2.

Now we have some hints for the dual gauge theory to this deformed gravity theory. Since the background has $\mathcal{N} = 1$ supersymmetry we expect the same in the field theory. The presence of the undeformed AdS_5 space indicates that the gauge theory should be conformally invariant, ie under the AdS_5 isometry group $SO(2, 4)$. Finally we expect the gauge theory to have a marginal β deformation.

4.5 Generalisation of the deformation

In [3] it was shown that the deformation outlined above can be obtained in a different way for real γ deformations. Start with a IIB background that has at least two cyclic variables, say ψ_1 and ψ_2. Perform a T-duality transformation on ψ_1, followed by a shift in the coordinate ψ_2,

$$\psi_2 \to \psi_2 + \hat{\gamma}\psi_1,$$

where $\hat{\gamma}$ is any constant. Finally, another T-duality transformation about ψ_1 recovers a new IIB background identical with the one described in section 4.3. This solution generating technique is called a TsT-transformation.

Examining the relationship between the original cyclic variables and the TsT-transformed variables yields an interesting map between string solutions in the γ-deformed background and the original background. Using the notation of [3] we denote the original angle variables of S^5 by $\tilde{\tilde{\phi}}_i$ and conjugate momenta by $p_i \equiv \partial_0 \tilde{\tilde{\phi}}_i$, and the TsT-transformed variables by ϕ_i and p_i. We further define $\gamma = \frac{\hat{\gamma}}{\sqrt{\lambda}}$. Then the variables are related by

$$\begin{aligned}
\tilde{\tilde{\phi}}'_1 &= \phi'_1 + \gamma(p_2 - p_3), \\
\tilde{\tilde{\phi}}'_2 &= \phi'_1 + \gamma(p_3 - p_1), \\
\tilde{\tilde{\phi}}'_3 &= \phi'_1 + \gamma(p_1 - p_2),
\end{aligned}$$

where $'$ denotes differentiation with respect to σ. Integrating over σ yields twisted boundary conditions for the angles $\tilde{\tilde{\phi}}_i$. The equations of motions for strings in the two backgrounds have the same form. This implies that, if ϕ_i solve the equations of motion in the γ-deformed background, then $\tilde{\tilde{\phi}}_i$ solve the equations of motion in the original $AdS_5 \times S^5$, with twisted boundary conditions.

Another interesting feature of the TsT-transformation is that it admits a natural generalisation in which each of the angular variables is deformed by a different constant γ_i. The deformed background is obtained by a chain of three TsT-transformations, starting with each angle variable of S^5 in turn. In the case $\gamma_1 = \gamma_2 = \gamma_3$, the deformed background is the same as that obtained by a single TsT-transformation. In the general case, the supersymmetry of the theory is completely broken, so we expect to find a non-supersymmetric dual field theory. As already discussed, this is a very important direction of research for the AdS/CFT correspondence. The remainder of this thesis deals with the general γ_i-deformed background. Details of the background will be presented in the following sections.

Chapter 5

Giant graviton solutions in the deformed background

In this chapter we will obtain giant graviton solutions in the deformed background. The chapter is based on work published in [30], which was the focus of this MSc. degree.

The giant graviton solutions we consider are D3 branes that have blown up in the deformed sphere part of the geometry. Our ansatz for the giant, made at the level of the action, assumes that it has a constant radius and a constant angular velocity, in analogy with chapter 3. This ansatz will be justified in chapter 6 where we will argue that our solution does indeed extremize the action.

5.1 D3-brane action

To write down the action for the D3-brane, we need the metric and dilaton of the background (to write down the Dirac-Born-Infeld term in the action), the NS-NS two form potential and the RR two and four form potentials (to write down the Chern-Simons terms in the action). For the giant gravitons described in chapter 3 only the 4-form potential was non-zero, and the dilaton was constant on the brane worldvolume, due to greater symmetry of the background. The AdS_5 and the deformed sphere spaces are orthogonal to each other

$$ds^2 = ds^2_{AdS_5} + ds^2_{S^5_{def}}.$$

We will use the following spacetime coordinates:

(1) For AdS_5 use $(t, \alpha_1, \alpha_2, \alpha_3, \alpha_4)$. In terms of these coordinates, the metric is

$$ds^2_{AdS_5} = -\left(1 + \sum_{k=1}^{4} \alpha_k^2\right) dt^2 + R^2 \left(\delta_{ij} + \frac{\alpha_i \alpha_j}{1 + \sum_{k=1}^{4} \alpha_k^2}\right) d\alpha_i d\alpha_j.$$

These coordinates are useful when studying small fluctuations of the giant graviton, since the make the $SO(4)$ subgroup of the $SO(2,4)$ isometry of AdS_5 manifest.

(2) For the deformed five sphere, use $(\alpha, \theta, \phi_1, \phi_2, \phi_3)$. In terms of these coordinates, the metric is

$$ds^2_{S^5_{def}} = R^2 \left(d\alpha^2 + \sin^2 \alpha d\theta^2 + G \sum_{i=1}^{3} \rho_i^2 d\phi_i^2\right) + R^2 G \rho_1^2 \rho_2^2 \rho_3^2 \left(\sum_{i=1}^{3} \tilde{\gamma}_i d\phi_i\right)^2,$$

$$\rho_1 = \cos\alpha, \qquad \rho_2 = \sin\alpha \cos\theta, \qquad \rho_3 = \sin\alpha \sin\theta,$$

$$G^{-1} = 1 + \tilde{\gamma}_1^2 \rho_2^2 \rho_3^2 + \tilde{\gamma}_2^2 \rho_1^2 \rho_3^2 + \tilde{\gamma}_3^2 \rho_2^2 \rho_1^2.$$

$\tilde{\gamma}_i$ is a rescaled γ_i from section 4.5, ie $\tilde{\gamma}_i = R^2 \gamma_i$. In terms of the dilaton ϕ_0 of the undeformed background, the dilaton is

$$e^{\phi} = \sqrt{G} e^{\phi_0}.$$

The dilaton of the undeformed background satisfies $R^4 e^{-\phi_0} = 4\pi N l_s^4$. The five form field strength of the background is

$$F_5 = 4R^4 e^{-\phi_0} \left(\omega_{AdS_5} + G\omega_{S^5}\right), \qquad \omega_{S^5} = \cos\alpha \sin^3\alpha \sin\theta \cos\theta d\alpha d\theta d\phi_1 d\phi_2 d\phi_3.$$

Finally, the RR two form potential is

$$C_2 = -4R^2 e^{-\phi_0} \omega_1 d\left(\sum_{i=1}^{3} \tilde{\gamma}_i \phi_i\right), \qquad d\omega_1 = \cos\alpha \sin^3\alpha \sin\theta \cos\theta d\alpha d\theta,$$

and NS-NS two form potential is

$$B = R^2 G \omega_2, \qquad \omega_2 = \tilde{\gamma}_3 \rho_1^2 \rho_2^2 d\phi_1 d\phi_2 + \tilde{\gamma}_1 \rho_2^2 \rho_3^2 d\phi_2 d\phi_3 + \tilde{\gamma}_2 \rho_3^2 \rho_1^2 d\phi_3 d\phi_1.$$

This notation deviates slightly from that of chapter 4, and is chosen to coincide with that of [4], where it was suggested that two types of BPS solution exist in the deformed background, organised in terms of their momenta on the deformed S^5. The first type is of the form $(J, 0, 0)$, $(0, J, 0)$ or $(0, 0, J)$, and the second of the type (J_1, J_2, J_3) where $J_i \propto \tilde{\gamma}_i$. Testing the correspondence in this non-supersymmetric problem will build up evidence that the AdS/CFT correspondence is a feature of dynamics, rather than an artefact of supersymmetry.

To write down the D3-brane action

$$S = -\frac{1}{(2\pi)^3 l_s^4} \int d^4y \, e^{-\phi} \sqrt{|\det(G+B)|} + \int C_4 - \int C_2 \wedge B,$$

we will use static gauge

$$y^0 = t, \quad y^1 = \theta, \quad y^2 = \phi_2, \quad y^3 = \phi_3.$$

Note the relative sign between the C_4 and C_2 terms, see [34] which corrects and error in [30].

5.2 Giant graviton ansatz

Our ansatz for the giant graviton is $\alpha = \alpha_0$, $\phi_1 = \omega_0 t$ where α_0 and ω_0 are constants, independent of y^μ. It is now a simple matter to integrate the Lagrangian density over y^1, y^2 and y^3 to obtain the Lagrangian

$$L = -m\sqrt{1 - a\dot{\phi}_1^2} + b\dot{\phi}_1,$$

where

$$m = 2\pi^2 r_0^3 \frac{e^{-\phi_0}}{(2\pi)^3 l_s^4} = N\frac{r_0^3}{R^4}, \qquad a = R^2 - r_0^2, \qquad b = N\frac{r^4}{R^4}.$$

$r_0 \equiv R\sin\alpha_0$ is the radius of the giant, T_3 is the D3 brane tension and R is the radius of curvature of the AdS space and the radius of the (undeformed) sphere. The deformations in the worldvolume metric and and B potential cancelled exactly, as did the deformations in the C_4 and C_2 terms. So this ansatz represents a giant graviton insensitive to the deformation $\tilde{\gamma}_i$. Solving for $\dot{\phi}_1$ in terms of the angular momentum

$$\mathcal{M} = \frac{\partial L}{\partial \dot{\phi}_1},$$

we obtain

$$\dot{\phi}_1 = \pm \frac{\mathcal{M} - b}{\sqrt{a\big[\mathcal{M} - b\big]^2 + m^2 a^2}}. \tag{5.1}$$

The energy of the giant graviton is now easily computed

$$E = \dot{\phi}_1 \mathcal{M} - L = \sqrt{m^2 + \frac{\big[\mathcal{M} - b\big]^2}{a}}.$$

We determine α_0 by minimizing the energy at fixed $\mathcal{M}$. The energy has two degenerate minima, one at $r = 0$ (corresponding to the point graviton) and the the other at $r = R\sqrt{\frac{\mathcal{M}}{N}}$ (corresponding to the expanded giant). The energy at these minima is $E = \frac{\mathcal{M}}{R}$, with $\dot{\phi}_1 = \omega_0 = \frac{1}{R}$. These results are identical with the undeformed case.

Chapter 6

Fluctuations

We have no guarantee that our ansatz of the previous chapter in fact minimizes the action. In this section we check that this is indeed the case and further, we study the spectrum of certain vibration modes of the giant. There are a number of interesting questions that can be answered using the vibration spectrum of giant gravitons. If our giants belong to a family of solutions that all have the same energy and angular momentum, there will be modes with zero energy. Secondly, if our giant graviton solution is (perturbatively) unstable, there will be tachyonic vibration mode(s). The excitations we consider correspond to motions of the branes in spacetime. Consequently, we do not consider the possibility of exciting fermionic modes or gauge fields that live on the giant graviton's worldvolume. Our results show that the giant graviton is perturbatively stable. This chapter follows closely [34].

6.1 The first order expansion

Obviously, if the ansatz does represent a valid solution to the specified action, the first order variation around the ansatz should vanish. Our ansatz for the giant graviton is

$$\alpha_i = \epsilon\delta\alpha_i, \qquad i = 1,2,3,4,$$

$$r = r_0 + \epsilon\delta r,$$

$$\phi_1 = \omega_0 t + \epsilon\delta\phi_1.$$

r_0 and ω_0 are constants, independent of y^μ. We expand the action to second order in ϵ schematically as

$$S = \int dy^0 dy^1 dy^2 dy^3 d\{\mathcal{L}_0 + \varepsilon\mathcal{L}_1 + \varepsilon^2\mathcal{L}_2 + \cdots\}$$

Despite their names, we have not yet given any reason to identify r_0 and ω_0 with the constants appearing in our ansatz of section 5.2. We now plug this ansatz into the action and expand in ϵ. If the linear order in ϵ contribution to the action vanishes, for $\omega = \dot{\phi}_1$ computed using (5.1) and for the value of α_0 that minimizes the energy, we know that the giants of section 5.2 minimize the action and that they are indeed classical solutions. The quadratic in ϵ contribution to the action can be used to learn about the energies of vibration modes of the giant.

Plugging this ansatz into the action and expanding, the term linear in ϵ is

$$\begin{aligned}
\mathcal{L}_1 \quad &= \quad -T_3\, e^{-\phi_0}\, \sin\theta\cos\theta \\
r_0^2 \quad &\left\{ \left[\frac{4r_0^2\omega_0^2 + 3(1 - R^2\omega_0^2)}{\sqrt{1-(R^2-r_0^2)\omega_0^2}} - 4r_0\omega_0 \right] \delta r + \right. \\
- \quad &\left. \left[\frac{(R^2-r_0^2)r_0\omega_0}{\sqrt{1-(R^2-r_0^2)\omega_0^2}} + r_0^2 \right] \frac{\partial\delta\phi_1}{\partial t} \right\}
\end{aligned}$$

The coefficient of δr above vanishes for $\omega_0 = \frac{1}{R}$ and the second vanishes because the variation is done with fixed end points and the coefficient of $\frac{\partial \delta\phi_1}{\partial t}$ is constant. This confirms that the giant gravitons written down in section 5.2 are indeed solutions to the equations of motion following from the D3 brane action.

6.2 Fluctuations - the expansion to second order

Fluctuations about the solution are found at order ϵ^2. Performing the expansion we find

$$\begin{aligned}
\mathcal{L}_2 &= T_3\, e^{-\phi_0}\, r_0^2\, \sin\theta\cos\theta \\
\Bigg\{\Bigg[&- \frac{R^3}{2(R^2-r_0^2)}\,\frac{\partial^2\delta r}{\partial t^2} + \frac{R}{2(R^2-r_0^2)}\,\Delta_{S^3}\,\delta r + \\
&+ \frac{1}{2R}\left(\hat\gamma_3^2\frac{\partial^2\delta r}{\partial\phi_2^2} + \hat\gamma_2^2\frac{\partial^2\delta r}{\partial\phi_3^2} - 2\hat\gamma_2\hat\gamma_3\frac{\partial^2\delta r}{\partial\phi_2\partial\phi_3}\right)\Bigg]\delta r + \\
\Bigg[&- \frac{R^3(R^2-r_0^2)}{2r_0^2}\,\frac{\partial^2\delta\phi_1}{\partial t^2} + \frac{R(R^2-r_0^2)}{2r_0^2}\,\Delta_{S^3}\,\delta\phi_1\Bigg]\delta\phi_1 + \\
&+ \frac{2R^2}{r_0}\,\frac{\partial\delta\phi_1}{\partial t}\,\delta r + \\
\Bigg[&- \frac{R^3}{2}\frac{\partial^2\delta\alpha_k}{\partial t^2} + \frac{R}{2}\,\Delta_{S^3}\,\delta\alpha_k - \frac{R}{2}\,\delta\alpha_k \\
&+ \frac{R^2-r_0^2}{2R}\left(\hat\gamma_3^2\frac{\partial^2\delta\alpha_k}{\partial\phi_2^2} + \hat\gamma_2^2\frac{\partial^2\delta\alpha_k}{\partial\phi_3^2} - 2\hat\gamma_2\hat\gamma_3\frac{\partial^2\delta\alpha_k}{\partial\phi_2\partial\phi_3}\right)\Bigg]\delta\alpha_k\Bigg\}.
\end{aligned}$$

The sum over k is implicit and Δ_{S^3} is the Laplacian on the unit 3-sphere. From the form of $\mathcal{L}_2$ we see that the $\delta\alpha_k$ modes decouple from the δr and $\delta\phi_1$ modes.

The original $SO(4)$ worldvolume symmetry that we had in the undeformed case has been broken to $U(1)\times U(1)$. These two $U(1)$ symmetries correspond to translations of ϕ_2 and ϕ_3. It is possible to choose spherical harmonics with definite $U(1)\times U(1)$ quantum numbers (m_1, m_2). In particular we have

$$\begin{aligned}\Delta_{S^3}\, \mathcal{Y}_s^{m_2,m_3}(\theta,\phi_2,\phi_3) &= -Q_s^2 \mathcal{Y}_s^{m_2,m_3}(\theta,\phi_2,\phi_3)\\ \frac{\partial}{\partial\phi_{2,3}}\, \mathcal{Y}_s^{m_2,m_3}(\theta,\phi_2,\phi_3) &= i m_{2,3} \mathcal{Y}_s^{m_2,m_3}(\theta,\phi_2,\phi_3)\end{aligned}$$

For spherical harmonics on S^3, $Q_s^2 = s(s+2)$. We expand the perturbations as

$$\begin{aligned}\delta r(t,\theta,\phi_2,\phi_3) &= A_r\, e^{-i\omega t}\, \mathcal{Y}_s^{m_2,m_3}(\theta,\phi_2,\phi_3)\\ \delta\phi_1(t,\theta,\phi_2,\phi_3) &= A_{\phi_1}\, e^{-i\omega t}\, \mathcal{Y}_s^{m_2,m_3}(\theta,\phi_2,\phi_3)\\ \delta\alpha_k(t,\theta,\phi_2,\phi_3) &= A_{\alpha_k}\, e^{-i\omega t}\, \mathcal{Y}_s^{m_2,m_3}(\theta,\phi_2,\phi_3)\end{aligned}$$

It is useful to define the positive quantity

$$\hat{\Gamma}^2 = \left(1 - \frac{r_0^2}{R^2}\right)(\hat{\gamma}_3 m_2 - \hat{\gamma}_2 m_3)^2$$

which contains the entire dependence of the fluctuations on the deformation parameters and radius of the giant. It is now straight forward to obtain the spectrum for fluctuations α_k in the AdS spacetime:

$$\omega_k^2 = \frac{1}{R^2}\left(1 + Q_s^2 + \hat{\Gamma}^2\right).$$

Note that this is a positive quantity so the vibration frequency is real and hence the graviton is perturbatively stable with respect to these modes.

For the coupled δr and $\delta\phi_1$ modes we need to solve

$$\begin{bmatrix} \frac{R}{R^2-r_0^2}\left(\omega^2 R^2 - Q_s^2 - \hat{\Gamma}^2\right) & -2i\omega\frac{R^2}{r_0} \\ 2i\omega\frac{R^2}{r_0} & \frac{R(R^2-r_0^2)}{r_0^2}\left(\omega^2 R^2 - Q_s^2\right)\end{bmatrix}\begin{bmatrix} A_r \\ A_{\phi_1}\end{bmatrix} = 0.$$

Since the matrix is not invertible we can set the determinant to zero and obtain the spectrum

$$\omega_{\pm}^2 = \frac{1}{R^2}\left[2 + Q_s^2 + \frac{\hat{\Gamma}^2}{2} \pm 2\sqrt{1 + Q_s^2 + \frac{\hat{\Gamma}^2}{2}\left(1 + \frac{\hat{\Gamma}^2}{8}\right)}\right],$$

which is again a positive quantity signalling perturbative stability of the giant.

The term $\hat{\Gamma}^2$ depends on both the deformation parameters and the radius r_0 of the giant. This size-dependence is absent in the undeformed case.

Chapter 7

Open Strings

We now compute a semiclassical spin chain action for the dual giant graviton operators. We then examine the action derived in the same limit from the open string sigma model. This model is an expansion of the Polyakov action for open strings to first order. At this order we know that open string dynamics are determined by D-branes which impose boundary conditions, and therefore this action is appropriate for the gravity description of our system in the semiclassical regime.

7.1 Large L spin chain

The background studied in chapter 4 is conjectured[3] to be dual to the field theory with scalar potential

$$V = \mathrm{Tr} \sum_{n>m=1}^{3} |e^{-i\pi\alpha_{mn}} \Phi_m \Phi_n - e^{i\pi\alpha_{mn}} \Phi_n \Phi_m|^2 + \mathrm{Tr} \sum_{n=1}^{3} \left[\Phi_n, \bar{\Phi}_n\right]^2,$$

where

$$\alpha_{mn} = -\epsilon_{mni}\gamma_i.$$

Below we will give a precise relation between the parameters γ_i of the gauge theory and the parameters $\tilde{\gamma}_i$ of the gravity background. Our giant graviton solutions correspond to branes orbiting with angular momentum along the ϕ_1 direction. The $\mathcal{R}$ charge of Φ_1 corresponds to the angular momentum $\mathcal{M}$ of section 2. Thus, a giant graviton with angular momentum $\mathcal{M}$ should be dual to an operator built out of $\mathcal{M}$ Φ_1 fields. From now on we use Z to denote Φ_1 and X, Y to denote Φ_2, Φ_3. For simplicity we set $\gamma_1 = 0$ and $\gamma_2 = \gamma_3 = \gamma$ so that

$$\begin{aligned} V &= \mathrm{Tr}\left[|e^{i\pi\gamma}ZY - e^{-i\pi\gamma}YZ|^2 + |e^{i\pi\gamma}XZ - e^{-i\pi\gamma}ZX|^2 + |YX - XY|^2 \right. \\ &+ \left. \left[X,\bar{X}\right]^2 + \left[Y,\bar{Y}\right]^2 + \left[Z,\bar{Z}\right]^2\right]. \end{aligned}$$

This is sufficient to break all supersymmetry in the problem.

We would like to determine the spin chain of this deformed $\mathcal{N} = 4$ super Yang-Mills theory relevant for the dual description of open strings attached to giants. The spin chain for the deformed $\mathcal{N} = 4$ super Yang-Mills theory was found in [39]; describing the open strings amounts to determining what boundary conditions must be imposed on this spin chain. In the undeformed theory with gauge group $U(N)$, operators dual to sphere giants are given by Schur polynomials of the totally antisymmetric representations[8], which are labeled by Young diagrams with a single column. The cut off on the number of rows of the Young diagram perfectly matches the cut off on angular momentum arising because the sphere giant fills the S^5 of the $\mathrm{AdS}_5 \times \mathrm{S}^5$ geometry. For maximal giants, the Schur polynomials are determinant like operators. Attaching a string to the maximal giant gives an operator of the form

$$\mathcal{O} = \epsilon^{j_1 \cdots j_N}_{i_1 \cdots i_N} Z^{i_1}_{j_1} \cdots Z^{i_{N-1}}_{j_{N-1}} (M_1 M_2 \cdots M_n)^{i_N}_{j_N}.$$

The open string is given by the product $(M_1 M_2 \cdots M_n)^{i_N}_{j_N}$. The M_i could in principle be fermions, covariant derivatives of Higgs fields or Higgs fields themselves. To describe excitations of the string involving only coordinates from the S^5, we would restrict the M_i to be Higgs fields. We will restrict ourselves even further and require that the M_i are Z or Y. A spin chain description can then be constructed by identifying $(M_1 M_2 \cdots M_n)^{i_N}_{j_N}$ with a spin chain that has n-sites. If $M_i = Z$ the ith spin is spin up; if $M_i = Y$ the ith spin is spin down. It is not possible for Z's to hop off and onto the string attached to a maximal giant; as soon as $M_1 = Z$ or $M_n = Z$ the operator factorizes into a closed string plus a maximal giant graviton. This implies the boundary constraint $M_1 \neq Z \neq M_n$. However, for non-maximal giants, Z's can hop between the graviton and the open string. In this case, the number of sites in the spin chain is dynamical. If however, one identifies the spaces between the Y's as lattice sites and the Z's as bosons which occupy sites in this lattice, the number of sites is again conserved[19]. For the undeformed theory this leads to the Hamiltonian[19]

$$H = 2\lambda\alpha^2 + 2\lambda \sum_{l=1}^{L} \hat{a}_l^\dagger \hat{a}_l - \lambda \sum_{l=1}^{L-1} \left(\hat{a}_l^\dagger \hat{a}_{l+1} + \hat{a}_l \hat{a}_{l+1}^\dagger \right) + \lambda\alpha(\hat{a}_1 + \hat{a}_1^\dagger) + \lambda\alpha(\hat{a}_L + \hat{a}_L^\dagger).$$

The operators in the above Hamiltonian are Cuntz oscillators[19]

$$a_i a_i^\dagger = I, \qquad a_i^\dagger a_i = I - |0\rangle\langle 0|.$$

For a giant with angular momentum p/R, the parameter

$$\alpha = \sqrt{1 - \frac{p}{N}},$$

measures how far from a maximal giant we are.

Due to the deformation, hopping is now accompanied by an extra phase. To see how this comes about, note that the deformation replaces

$$[Z,Y] \to ZYe^{i\pi\gamma} - YZe^{-i\pi\gamma},$$

$$[Z,Y]\,[Z,Y]^{\dagger} \to ZY\bar{Y}\bar{Z} + YZ\bar{Z}\bar{Y} - ZY\bar{Z}\bar{Y}e^{2\pi i\gamma} - YZ\bar{Y}\bar{Z}e^{-i2\pi\gamma}.$$

It is straight forward to see what interactions in the spin chain Hamiltonian these terms induce (the overbraces indicate Wick contractions)

$$\mathrm{Tr}(YZ\,\overbrace{\bar{Z}\,\overbrace{\bar{Y})\mathrm{Tr}(Y}\,Z}\,.....) \to \mathrm{Tr}(YZ...) \leftrightarrow a_l^{\dagger}a_l$$

$$\mathrm{Tr}(ZY\,\overbrace{\bar{Y}\,\overbrace{\bar{Z})\mathrm{Tr}(Z}\,Y}\,...) \to \mathrm{Tr}(ZY...) \leftrightarrow a_l^{\dagger}a_l$$

$$\mathrm{Tr}(ZY\,\overbrace{\bar{Z}\,\overbrace{\bar{Y}e^{i2\pi\gamma})\mathrm{Tr}(Y}\,Z}\,....) \to e^{i2\pi\gamma}\mathrm{Tr}(ZY....) \leftrightarrow e^{i2\pi\gamma}a_l^{\dagger}a_{l+1}$$

$$\mathrm{Tr}(YZ\,\overbrace{\bar{Y}\,\overbrace{\bar{Z}e^{-i2\pi\gamma})\mathrm{Tr}(Z}\,Y}\,...) \to e^{-i2\pi\gamma}\mathrm{Tr}(YZ...) \leftrightarrow e^{-i2\pi\gamma}a_l a_{l+1}^{\dagger}.$$

To hop onto the spin chain, we are hopping from the "zeroth site", which is the Schur polynomial/giant graviton, and onto the first site of the string. The term which does this has an $e^{-i2\pi\gamma}$ coefficient. Another way to hop onto the spin chain is to hop from the $L+1$th site into the Lth site. The term which does this has an $e^{i2\pi\gamma}$ coefficient. It is straight forward to argue for the phases when we hop off of the giant graviton. From the above discussion we see that the deformation modifies this Hamiltonian to

$$
\begin{aligned}
H &= 2\lambda\alpha^2 + 2\lambda\sum_{l=1}^{L}\hat{a}_l^\dagger\hat{a}_l - \lambda\sum_{l=1}^{L-1}\left(\hat{a}_l^\dagger\hat{a}_{l+1}e^{i2\pi\gamma} + \hat{a}_l\hat{a}_{l+1}^\dagger e^{-i2\pi\gamma}\right) \\
&+ \lambda\alpha(\hat{a}_1 e^{i2\pi\gamma} + \hat{a}_1^\dagger e^{-i2\pi\gamma}) + \lambda\alpha(\hat{a}_L e^{-i2\pi\gamma} + \hat{a}_L^\dagger e^{i2\pi\gamma}).
\end{aligned}
$$

In the above derivation of the deformed Hamiltonian we have considered only the terms which look like F-terms. For this to be valid, it is necessary that the self energy, vector exchange and terms which look like D-terms, continue to cancel as they did in the supersymmetric theory. It has been argued[4] that this is indeed the case, using the similarity between the β deformation[40] and non-commutative theories[41].

The semiclassical limit, in which the action derived from coherent states should provide a good approximation to the dynamics, is obtained by taking

$$
L \sim \sqrt{N} \to \infty, \qquad \lambda \to \infty,
$$

holding $\frac{\lambda}{L^2}$, $L\gamma$ and α fixed. To obtain the low energy effective action, we will use the coherent states

$$
|z\rangle = \sqrt{1-|z|^2}\sum_{n=0}^{\infty} z^n|n\rangle,
$$

with parameter

$$
z_l = r_l e^{i\phi_l},
$$

for the lth site. The coherent state action is given as usual by

$$
S = \int dt\left(i\langle Z|\frac{\partial}{\partial t}|Z\rangle - \langle Z|H|Z\rangle\right).
$$

In the above expression the coherent state $|Z\rangle$ is written as a product over all sites

$$|Z\rangle = \prod_l |z_l\rangle.$$

As an illustration of the manipulations which follow, we describe the evaluation of the first term in the action. It is straight forward to see that

$$\frac{\partial}{\partial t}|z_l\rangle = -\frac{r_l \dot{r}_l}{\sqrt{1-r_l^2}} \sum_{n=0}^{\infty} r_l^n e^{in\phi_l}|n\rangle + \sqrt{1-r_l^2}\sum_{n=0}^{\infty} n\left(\frac{\dot{r}_l}{r_l} + i\dot{\phi}_l\right) r_l^n e^{in\phi_l}|n\rangle,$$

$$\langle z_m|\frac{\partial}{\partial t}|z_l\rangle = i\frac{r_l^2\dot{\phi}_l}{1-r_l^2}\delta_{lm}.$$

Thus,

$$\langle Z|\frac{\partial}{\partial t}|Z\rangle = i\sum_{l=1}^{L}\frac{r_l^2\dot{\phi}_l}{1-r_l^2}.$$

In the large L limit, to leading order in L we have

$$\langle Z|\frac{\partial}{\partial t}|Z\rangle = iL\int_0^1 \frac{r(\sigma)^2\dot{\phi}(\sigma)}{1-r(\sigma)^2}d\sigma.$$

To compute $\langle z|H|z\rangle$ use the fact that the coherent state $|z\rangle$ is an eigenstate of a and the coherent state $\langle z|$ is an eigenstate of $a^\dagger$

$$a|z\rangle = z|z\rangle, \qquad \langle z|a^\dagger = \langle z|\bar{z}.$$

Then (every site is in a coherent state i.e. for the spin chain, $|z\rangle$ obviously means $\prod_l |z_l\rangle$ where the product is over all sites)

$$
\begin{aligned}
\langle z|H|z\rangle &= 2\lambda\alpha^2 + 2\lambda\sum_{l=1}^{L}\bar{z}_l z_l - \lambda\sum_{l=1}^{L-1}\left(\bar{z}_l z_{l+1}e^{i2\pi\gamma} + z_l\bar{z}_{l+1}e^{-i2\pi\gamma}\right) \\
&\quad + \lambda\alpha(z_1 e^{i2\pi\gamma} + \bar{z}_1 e^{-i2\pi\gamma}) + \lambda\alpha(z_L e^{-i2\pi\gamma} + \bar{z}_L e^{i2\pi\gamma}) \\
&= 2\lambda\alpha^2 + \lambda\sum_{l=1}^{L-1}\bar{z}_l(z_l - e^{i2\pi\gamma}z_{l+1}) + \lambda\sum_{l=1}^{L-1}z_l(\bar{z}_l - \bar{z}_{l+1}e^{-i2\pi\gamma}) + 2\lambda\bar{z}_L z_L \\
&\quad + \lambda\alpha(z_1 e^{i2\pi\gamma} + \bar{z}_1 e^{-i2\pi\gamma}) + \lambda\alpha(z_L e^{-i2\pi\gamma} + \bar{z}_L e^{i2\pi\gamma}).
\end{aligned}
\tag{7.1}
$$

As we mentioned above, we are interested in the semi-classical limit $L \to \infty$ in which we may replace sums by integrals. The basic tool that we use is the Euler-Maclaurin formula which states

$$
S = I + \sum_{k=1}^{p}\frac{B_{2k}}{(2k)!}(f^{(2k-1)}(n) - f^{(2k-1)}(0)) + R,
$$

where B_{2k} are the Bernoulli numbers,

$$
S = \frac{f(0)+f(n)}{2} + \sum_{k=1}^{n-1}f(k),
$$

$$
I = \int_0^n f(x)dx,
$$

$$
|R| \leq \frac{2}{(2\pi)^p}\int_0^n |f^{(2p+1)}(x)|dx.
$$

Rescaling $l = \sigma L$ and dropping $\mathcal{O}(\frac{1}{L^2})$ yields

$$
\begin{aligned}
\lambda \sum_{l=1}^{L-1} \bar{z}_l (z_l - e^{i2\pi\gamma} z_{l+1}) &= \lambda(1 - e^{i2\pi\gamma}) L \int_0^1 \bar{z}(\sigma) z(\sigma) d\sigma \\
&- \lambda e^{i2\pi\gamma} L \int_0^1 \bar{z}(\sigma) \left(\frac{1}{L} \frac{\partial z}{\partial \sigma} + \frac{1}{2! L^2} \frac{\partial^2 z}{\partial \sigma^2} \right) d\sigma \\
&- \frac{\lambda}{2}(1 - e^{i2\pi\gamma}) \bar{z}(0) z(0) + \frac{\lambda}{2} e^{i2\pi\gamma} \bar{z}(0) \left(\frac{1}{L} \frac{\partial z}{\partial \sigma} \right) |_{\sigma=0} \\
&- \frac{\lambda}{2}(1 - e^{i2\pi\gamma}) \bar{z}(1) z(1) + \frac{\lambda}{2} e^{i2\pi\gamma} \bar{z}(1) \left(\frac{1}{L} \frac{\partial z}{\partial \sigma} \right) |_{\sigma=1} \\
&+ \frac{\lambda B_2}{2L}(1 - e^{i2\pi\gamma}) \frac{\partial}{\partial \sigma} \left(\bar{z}(\sigma) z(\sigma) \right) \Big|_{\sigma=1} - \frac{\lambda B_2}{2L}(1 - e^{i2\pi\gamma}) \frac{\partial}{\partial \sigma} \left(\bar{z}(\sigma) z(\sigma) \right) \Big|_{\sigma=0}.
\end{aligned}
$$

Complex conjugation gives the other sum in equation 7.1. We finally drop all terms beyond $\mathcal{O}(\frac{1}{L})$ in the bulk expressions and work to $\mathcal{O}(1)$ in the boundary expressions.

A straight forward computation along these lines gives

$$
\begin{aligned}
S &= -\int dt \left[L \int_0^1 \frac{r^2 \dot{\phi}}{1 - r^2} d\sigma + 2\lambda\alpha^2 + \frac{\lambda}{L} \int_0^1 \left((r')^2 + r^2 (\phi' + 2\pi\tilde{\gamma})^2 \right) d\sigma \right. \\
&+ \left. \lambda \bar{z}(1) z(1) + \lambda \bar{z}(0) z(0) + \lambda\alpha(z(0) + \bar{z}(0)) + \lambda\alpha(z(1) + \bar{z}(1)) \right].
\end{aligned}
$$

We identify

$$\tilde{\gamma} = L\gamma.$$

Write this action in terms of γ and rescale $\sigma \to \frac{\sigma}{\pi}$. Clearly, the deformation replaces

$$\phi' \to \phi' + 2L\gamma.$$

7.2 The string sigma model

Lets now consider the description of the open strings using the dual sigma model. The undeformed case has been studied in[19]. The work [19] uses a coordinate system in which the brane is static, a gauge in which p_{ϕ_2} is homogeneously distributed along the string, $p_{\phi_2} = 2\mathcal{J}$ and $\tau = t$. After taking a low energy limit, the string sigma model action is

$$-\sqrt{\lambda_{YM}}\int dt \int_0^{2\pi} \frac{d\sigma}{2\pi}\left[\frac{r^2\dot{\phi}_1}{1-r^2} + \frac{\lambda_{YM}}{8\pi^2\mathcal{J}^2}(r'^2 + r^2\phi_1'^2)\right],$$

in perfect agreement with the undeformed result from the field theory[19], after identifying $L = \mathcal{J}$ and $\lambda_{YM} = 8\pi^2\lambda$.

The background studied in section 2 can be obtained by performing a sequence of TsT transformations[3]. A TsT transformation exploits a two torus, with coordinates (ϕ_1, ϕ_2) say, in the geometry. A TsT transformation begins with a T-duality with respect to ϕ_1, then a shift $\phi_2 \to \phi_2 + \gamma\phi_1$ and finally a second T-duality along ϕ_1. In the $AdS_5 \times S^5$ background there are three natural tori (ϕ_1, ϕ_2) (ϕ_2, ϕ_3) and (ϕ_3, ϕ_1). This allows three independent TsT transformations giving the three parameter deformation of section 2. See [3] for details. The TsT transformation has a particularly simple action on the string sigma model, something which was exploited in[3] to obtain the Lax pair for the bosonic part of the sigma model. To obtain the sigma model for the deformed theory, we simply need to shift[3]

$$\phi_i' \to \phi_i' - \epsilon_{ijk}\gamma_j p_k.$$

For the above action, we only need to consider ϕ_1'

$$\phi_1' \to \phi_1' - \epsilon_{1jk}\gamma_j p_k.$$

Next, since we set $X = 0$ we know that $p_3 = 0$. Thus,

$$\phi_1' \to \phi_1' - \epsilon_{1j2}\gamma_j p_2 = \phi_1' - \epsilon_{132}\gamma_3 p_2 = \phi_1' + \gamma_3 p_2.$$

Now, we have set $\gamma_3 = \gamma_2 = \gamma$ and in our gauge $p_2 = 2\mathcal{J}$, so that

$$\phi_1' \to \phi_1' + 2\gamma\mathcal{J} = \phi_1' + 2\gamma L.$$

This is in complete agreement with the spin chain result.

Chapter 8

Conclusion

In this thesis we have presented giant graviton solutions in the general γ_i-deformed background. We have also considered the spectrum of small fluctuations about these giants. The spectrum depends on the radius of the giant in contrast to the undeformed case where the spectrum is independent of the size of the giant[7]. For small deformations, we have argued that the giant graviton is perturbatively stable. We have also considered the semiclassical dynamics of open strings attached to these giants. We find that there is perfect quantitative agreement between the gauge theory and the string theory, at the level of the semicalssical Hamiltonians. Indeed, the deformation in the gauge theory exactly reproduces the TsT transformation relating the deformed and undeformed sigma models.

The comparison in this paper provides further quantitative agreement following from AdS/CFT duality in a non-supersymmetric case. This is encouraging because it indicates that the mechanism for the correspondence lies in the dynamics of these theories, rather than in symmetry artefacts.

There are a number of directions in which the present work can be extended.

It would be interesting to look for giant gravitons in the general three parameter deformed background. One could also consider giants which have expanded into the AdS_5 space; the giant will be the same as the solution presented in[6]; the deformation should however modify the small fluctuation spectrum[7]. Since publication of this work, these cases have been studied in [34]. Further, the open string fluctuations we have considered are certainly not the most general fluctuations that can be considered. It would be interesting to extend our results to see if the agreement we have found continues to hold for more general open string configurations.

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