

# Forward and inverse spectral theory of Sturm-Liouville operators with transmission conditions

*A thesis presented for the degree of  
Doctor of Philosophy*

*by*

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## **Abstract**

Forward and inverse spectral problems concerning Sturm-Liouville operators without discontinuities have been studied extensively. By comparison, there has been limited work tackling the case where the eigenfunctions have discontinuities at interior points, a case which appears naturally in physical applications. We refer to such discontinuity conditions as transmission conditions. We consider Sturm-Liouville problems with transmission conditions rationally dependent on the spectral parameter. We show that our problem admits geometrically double eigenvalues, necessitating a new analysis. We develop the forward theory associated with this problem and also consider a related inverse problem. In particular, we prove a uniqueness result analogous to that of H. Hochstadt on the determination of the potential from two sequences of eigenvalues. In addition, we consider the problem of extending Sturm's oscillation theorem, regarding the number of zeroes of eigenfunctions, from the classical setting to discontinuous problems with general constant coefficient transmission conditions.

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# Declaration

I, Casey Ann Bartels, hereby declare the contents of this thesis to be my own work. This thesis is submitted for the degree of Doctor of Philosophy at the University of the Witwatersrand. This work has not been submitted to any other university, or for any other degree.

A handwritten signature in black ink, appearing to read 'Casey Ann Bartels', is written over a light grey rectangular background.

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Casey Ann Bartels

Signed on this the 30<sup>th</sup> day of May 2017 at Johannesburg, South Africa.

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# Chapter 1

## Introduction

Boundary value problems involving Sturm-Liouville equations have a rich and diverse history. Such problems appear in areas including quantum mechanics, nuclear physics, electronics, geophysics and other branches of the natural sciences. Even though the equations are only of second order, the spectral theory associated with such problems is deep, encouraging study purely for mathematical interest. Despite being almost 200 years old, Sturm-Liouville theory remains a highly active area of research, attracting interest from mathematicians, physicists and engineers alike.

In recent years there has been growing interest in problems involving Sturm-Liouville equations in which the eigenfunctions have discontinuities at interior points of the underlying interval. We refer to such discontinuity conditions as transmission conditions. In the physical context such problems are associated with a change in medium, and arise naturally in a varied assortment of physical transfer problems, including heat and mass transfer. Of particular interest are cases where the spectral parameter enters not only in the differential equation but also in the boundary conditions and/or transmission conditions.

Our aim is to extend some of the results of classical Sturm-Liouville theory to discontinuous problems of the above type. We consider Sturm-Liouville equations

$$\ell y := -(py')' + qy = \lambda r y, \quad (1.1)$$

on  $(-a, 0) \cup (0, b)$ ,  $a, b > 0$ . We impose separated boundary conditions

$$y(-a) \cos \alpha = (py')(-a) \sin \alpha, \quad \alpha \in [0, \pi), \quad (1.2)$$

$$y(b) \cos \beta = (py')(b) \sin \beta, \quad \beta \in (0, \pi], \quad (1.3)$$

and transmission conditions of the form

$$m_{11}(\lambda)y(0^-) + m_{12}(\lambda)(py')(0^-) + m_{13}(\lambda)y(0^+) + m_{14}(\lambda)(py')(0^+) = 0, \quad (1.4)$$

$$m_{21}(\lambda)y(0^-) + m_{22}(\lambda)(py')(0^-) + m_{23}(\lambda)y(0^+) + m_{24}(\lambda)(py')(0^+) = 0. \quad (1.5)$$

Here  $y(0^\pm) = \lim_{x \rightarrow 0^\pm} y(x)$ ,  $(py')(0^\pm) = \lim_{x \rightarrow 0^\pm} py'(x)$ . In particular, we are interested in two cases. We study Sturm-Liouville problems with transmission conditions rationally dependent on the spectral parameter  $\lambda$ . We also consider the case where  $m_{ij} \in \mathbb{R}$ ,  $i = 1, 2$ ,  $j = 1, 2, 3, 4$  with  $m_{14} = m_{23} = 0$  and  $\frac{m_{11}m_{22} - m_{12}m_{21}}{m_{13}m_{24}} > 0$ .

Precise definitions of the above two types of transmission conditions are given in Chapter 2. Here, we also present background for discontinuous Sturm-Liouville problems, in particular, focussing on transmission conditions of the form (1.4), (1.5) above. We give a brief overview of recent activity in the area. More detailed literature reviews are contained in subsequent chapters.

In Chapters 3 and 4 we study Sturm-Liouville problems with transmission conditions rationally dependent on the eigenparameter. So far, investigations into discontinuous Sturm-Liouville problems with eigenvalue dependent transmission conditions have been limited to the affine case, although transmission conditions with a polynomial dependence on the spectral parameter have been considered for the Dirac operator (see for example [41]). To the best of our knowledge, this is the first time spectral theory has been developed for discontinuous problems where the transmission conditions have a rational dependence on the spectral parameter. In addition to the added complexity caused by the discontinuity in the eigenfunctions, we show that our problem admits geometrically double eigenvalues. This necessitates a new analysis. In Chapter 3 we develop the “forward” theory associated with this problem. This work forms the foundation for a related inverse problem which is studied in Chapter 4. Here, we consider a uniqueness problem of determining the potential  $q$  from given spectral data. In particular, we extend the theory developed by H. Hochstadt in [39] to the case of discontinuous Sturm-Liouville equations of the type discussed above.

In Chapter 5 we develop oscillation theory for generalized Sturm-Liouville equations of the form (1.1) with constant coefficient transmission conditions. Very little work has been done to extend classical Sturmian oscillation theory to the case of discontinuous problems. Our aim is to adapt existing Prüfer methods to analyse transmission conditions of the type known commonly in the physics literature as “one-dimensional point interactions”. Essentially, transmission conditions of this type describe a linear relationship between the solution and its derivative on either side of the discontinuity. We consider general non-singular  $2 \times 2$  transfer matrices. We study the effect of the transfer on the oscillation counts of eigenfunctions, and consider the problem of indexing eigenvalues by the oscillation count of the associated eigenfunction.

Finally, we conclude the thesis with a discussion of future work in Chapter 6.



# Chapter 2

## Background

### 2.1 Discontinuous Sturm-Liouville eigenvalue problems

In the 1830's Charles Sturm and Joseph Liouville published a series of papers ([75], [76], [77]) on second order linear differential equations of the form

$$-\frac{d}{dx} \left( p(x) \frac{d}{dx} y(x) \right) + q(x)y(x) = \lambda r(x)y(x), \quad -a \leq x \leq b. \quad (2.1)$$

This work laid the foundation for what is known today as “forward” or “direct” spectral theory in differential equations. Before this time investigations into differential equations were mostly limited to finding analytic solutions to equations. Sturm and Liouville were among the first to seek properties of solutions directly from the equations, even when no analytic solution was possible. Due to the significance of their work, boundary value problems involving differential equations of the form (2.1) became known as Sturm-Liouville problems.

Sturm-Liouville problems in which the eigenfunctions have a discontinuity at an interior point arise naturally in a host of physical applications. Elementary examples include vibrating strings loaded in assorted configurations with point masses ([8], [79], [87]), as well as certain problems for heat transfer ([80], [87]). The inverse problem of reconstructing the material properties of a medium from external data is a problem of central importance in physics and engineering. Here the so called data consists usually of a combination of the natural frequencies of vibration (eigenvalues), vibrational amplitudes (norming constants) and positions of zero wave amplitude (nodal positions), all of which can be observed externally by disturbing the medium in some way. Because the assumption of a homogeneous medium is often an over-simplification, any change in medium results in a discontinuous inverse problem. Such problems occur for example in electromagnetism, where spectral data can be used to reconstruct the conductivity and permittivity profiles of a medium with discontinuities ([47], [48]).

Forward and inverse spectral theory for Sturm-Liouville equations with discontinuities has been gaining traction in recent years. However, studies seem to be largely limited to very specific types of discontinuity conditions. Common examples are simple jump discontinuities, whereby either the discontinuity in  $y$  is independent of the discontinuity in  $y'$ , or cases where  $y$  is continuous and the change in  $y'$  is assumed to be proportional to  $y$ . By comparison, more general constant coefficient transmission conditions involving both  $y$  and  $y'$  have received very little attention. In particular, there is much work to be done in extending classical oscillation theorems to discontinuous problems with minimally restrictive transmission conditions of constant coefficient type. Problems where the spectral parameter enters into the transmission conditions have been considered, although again, there is much room for growth. Discontinuous Sturm-Liouville problems

where the transmission conditions are dependent on the spectral parameter have thus far been limited to affine case ([2], [65], [66], [85]). In comparison, continuous problems have been studied where the boundary conditions have polynomial or rational dependence on the eigenparameter (see for example [11], [12], [28]), yielding interesting spectral structure.

Our aim in this thesis is to target some of these deficient areas. In particular, we are interested in studying the oscillatory properties of eigenfunctions corresponding to Sturm-Liouville equations with general constant coefficient transmission conditions. We also aim to develop theory for discontinuous Sturm-Liouville problems where the transmission conditions have a rational dependence on the spectral parameter. Exact definitions of these two particular types of transmission conditions are given in Sections 2.2 and 2.3 below.

## 2.2 Transmission conditions dependent on the spectral parameter

Boundary value problems where the spectral parameter appears not only in the differential equation but also in the boundary conditions and/or transmission conditions are of particular interest both mathematically and in physical applications (see [8], [79], [87]).

Transmission conditions of the form

$$\begin{bmatrix} y(0^+) \\ y'(0^+) \end{bmatrix} = \begin{bmatrix} c & 0 \\ h(\lambda) & c^{-1} \end{bmatrix} \begin{bmatrix} y(0^-) \\ y'(0^-) \end{bmatrix}, \quad (2.2)$$

where  $c \in \mathbb{R}^+$  and  $h$  is affine in  $\lambda$ , constitute the vast majority of cases of “eigenvalue dependent transmission conditions”. We refer the reader to [66], [86] and the references therein for examples. Recently, the discontinuity condition

$$\begin{bmatrix} y_1(0^+) \\ y_2(0^+) \end{bmatrix} = \begin{bmatrix} c & 0 \\ h(\lambda) & c^{-1} \end{bmatrix} \begin{bmatrix} y_1(0^-) \\ y_2(0^-) \end{bmatrix}, \quad (2.3)$$

with  $c \in \mathbb{R}^+$  and  $h$  a polynomial in  $\lambda$  was considered in [41] for the Dirac operator

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{dY}{dx} + \begin{bmatrix} p(x) & q(x) \\ q(x) & r(x) \end{bmatrix} Y = \lambda Y, \quad Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad (2.4)$$

with boundary conditions also polynomially dependent on the spectral parameter.<sup>1</sup>

To the best of our knowledge, this is the first time spectral theory has been presented for transmission conditions with rational dependence on the eigenparameter. In particular, our rationally-dependent transmission conditions take the form

$$y(0^+) = r(\lambda) [y'(0^+) - y'(0^-)], \quad (2.5)$$

$$y'(0^-) = s(\lambda) [y(0^+) - y(0^-)]. \quad (2.6)$$

Here

$$r(\lambda) = -\sum_{i=1}^N \frac{\beta_i^2}{\lambda - \gamma_i}, \quad s(\lambda) = \sum_{j=1}^M \frac{\alpha_j^2}{\lambda - \delta_j}, \quad (2.7)$$

---

<sup>1</sup>Note that for reasons of notational simplicity we will state all problems in literature reviews on the interval  $(-a, b)$ ,  $a, b > 0$  with points of discontinuity at  $x = 0$  (except in cases of multiple discontinuities). This is not necessarily the setting chosen by the authors, but is equivalent by a simple change of variables.

where

$$\gamma_1 < \gamma_2 < \cdots < \gamma_N, \quad (2.8)$$

$$\delta_1 < \delta_2 < \cdots < \delta_M, \quad (2.9)$$

and  $\beta_i, \alpha_j > 0$  for  $i = 1, \dots, N$  and  $j = 1, \dots, M$ . It is easy to check that  $s(\lambda)$ ,  $-r(\lambda)$ ,  $-\frac{1}{s(\lambda)}$  and  $\frac{1}{r(\lambda)}$  are Herglotz-Nevanlinna functions. Recall that a function  $f : \mathbb{C} \rightarrow \mathbb{C}$  is Herglotz-Nevanlinna if  $f(\bar{z}) = \overline{f(z)}$  and  $f$  maps the closed upper half plane to itself. Such functions have real, simple poles.

Note that we impose the following interpretation at zeroes and poles of  $r$  and  $s$ . If  $r(\lambda) = 0$  then (2.5) reduces to the Dirichlet condition  $y(0^+) = 0$  at  $\lambda$ , while if  $\lambda$  is a pole of  $r$  then (2.5) becomes  $y'(0^+) = y'(0^-)$ . Similarly, if  $s(\lambda) = 0$  then (2.6) reduces to the Neumann condition  $y'(0^-) = 0$  at  $\lambda$ , while if  $\lambda$  is a pole of  $s$  then (2.6) becomes  $y(0^-) = y(0^+)$ .

### 2.3 Transmission conditions with constant coefficients

Here we consider the particular case where the coefficients of the transmission conditions (1.4), (1.5) are real numbers with  $m_{14} = m_{23} = 0$  and  $[m_{11}m_{22} - m_{12}m_{21}] / m_{13}m_{24} > 0$ . These restrictions allow for (1.4), (1.5) to be written in the form

$$\begin{bmatrix} y(0^+) \\ (py')(0^+) \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} y(0^-) \\ (py')(0^-) \end{bmatrix}, \quad (2.10)$$

where  $T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$  is a real  $2 \times 2$  matrix with  $\det T > 0$ . Transmission conditions of this type yield self-adjoint problems (subject to suitable restrictions on the coefficients  $p$ ,  $q$  and  $r$ ). The case for  $\det T > 0$  has been discussed in the recent paper, [81], expanding on the usual theory which requires the transmission matrix to have determinant 1.

Discontinuity conditions of the form (2.10) are of central focus in the description of quantum mechanical systems. If  $r = p \equiv 1$  in (1.1) then  $\ell y = -y'' + qy$  is the one dimensional Schrödinger operator with potential  $q$ . In quantum mechanics discontinuity conditions of the form (2.10) are called point interactions. Point interaction models occur also in solid state physics, atomic and nuclear physics, in the description of certain electromagnetic, chemical and biological phenomena, as well as in the study of quantum chaotic systems ([3], [4]).

Of all one dimensional point interactions three types have a special significance in connection with the Schrödinger operator,  $-\frac{d^2}{dx^2} + q(x)$ , with potential  $q$  ([19]).

I The  $\delta$ -interaction or  $\delta$ -potential with intensity  $\epsilon$  is classified by the point transfer condition

$$\begin{bmatrix} y(0^+) \\ y'(0^+) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \epsilon & 1 \end{bmatrix} \begin{bmatrix} y(0^-) \\ y'(0^-) \end{bmatrix}. \quad (2.11)$$

II The  $\delta'$ -interaction with intensity  $\sigma$  has

$$\begin{bmatrix} y(0^+) \\ y'(0^+) \end{bmatrix} = \begin{bmatrix} 1 & \sigma \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y(0^-) \\ y'(0^-) \end{bmatrix}. \quad (2.12)$$

III The  $\delta'$ -potential with intensity  $\varsigma$  is characterized by

$$\begin{bmatrix} y(0^+) \\ y'(0^+) \end{bmatrix} = \begin{bmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{bmatrix} \begin{bmatrix} y(0^-) \\ y'(0^-) \end{bmatrix}, \quad (2.13)$$

$$\text{where } \theta = \frac{2+\varsigma}{2-\varsigma}.$$

Point interaction models of this type are studied in a variety of different settings. These include descriptions as singular perturbations of the negative Laplacian in suitable  $L^2$ -spaces ([4]); as self-adjoint extensions of the minimal operator  $-\frac{d^2}{dx^2}$ , defined on functions belonging to the class  $C_0^\infty((-\infty, 0) \cup (0, \infty))$  for point interactions at  $x = 0$  ([3], [4], [19]); and as definitions in terms of Dirichlet forms ([5]).

Schrödinger operators with point interactions have received a lot of attention in recent years in connection with nodal problems on graphs. Here, the oscillation counts of eigenfunctions correspond to so called nodal counts. Nodal counting theory has been developed for graph problems where the matching conditions are either standard Kirchhoff conditions or of so called  $\delta$ -type ([70], [71]). For graphs with a vertex of degree 2 at  $x = 0$ , these conditions correspond, respectively, to full continuity conditions  $y(0^-) = y(0^+)$ ,  $y'(0^-) = y'(0^+)$  (Kirchhoff), or  $\delta$ -interactions of type I above ( $\delta$ -type). Note that both conditions require the eigenfunctions to be continuous at the vertices. Our aim is to extend this theory to transmission conditions permitting discontinuities in both  $y$  and  $y'$ .

We will employ a novel parametrization of the transfer matrix,  $T$ , in order to the study oscillatory properties of the Sturm-Liouville problem (1.1)-(1.3) with general transmission conditions of the form (2.10). In particular, we make use of the Iwasawa decomposition of  $SL(2, \mathbb{R})$ , which gives each  $g \in SL(2, \mathbb{R})$  a unique representation in the form

$$g = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} \gamma & 0 \\ 0 & 1/\gamma \end{bmatrix} \begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix}. \quad (2.14)$$

Here  $\gamma \in \mathbb{R}^+$ ,  $\delta \in \mathbb{R}$  and we restrict  $\phi \in [-\pi, \pi)$ . In particular, writing  $T = g\sqrt{\det T}$  with  $g = (1/\sqrt{\det T})T \in SL(2, \mathbb{R})$  it can be shown that  $\phi$ ,  $\gamma$  and  $\delta$  are determined uniquely by the following formulae

$$\cos \phi = \frac{t_{11}}{\sqrt{t_{11}^2 + t_{21}^2}}, \quad \sin \phi = \frac{t_{21}}{\sqrt{t_{11}^2 + t_{21}^2}}, \quad (2.15)$$

$$\gamma = \sqrt{\frac{t_{11}^2 + t_{21}^2}{\det T}}, \quad \delta = \frac{t_{11}t_{12} + t_{21}t_{22}}{t_{11}^2 + t_{21}^2}. \quad (2.16)$$

We note that the three point interactions mentioned above have the following representations in the Iwasawa decomposition as follows:

$$\text{I } \cos \phi = \frac{1}{\sqrt{1+\epsilon^2}}, \sin \phi = \frac{\epsilon}{\sqrt{1+\epsilon^2}}, \gamma = \sqrt{1+\epsilon^2}, \delta = \frac{\epsilon}{1+\epsilon^2},$$

$$\text{II } \phi = 0, \gamma = 1, \delta = \sigma,$$

$$\text{III } \cos \phi = \text{sgn}(\theta), \gamma = |\theta|, \delta = 0.$$

As a sample result, we prove that for transfer matrices  $T$  satisfying the condition  $\tan \phi = \gamma^2 \delta$  (where  $\phi$ ,  $\gamma$  and  $\delta$  are as in equations (2.15)-(2.16) above) the  $n$ th eigenfunction has total oscillation count  $n - 1$  in  $(-a, b)$ . Note that the  $\delta$ -interaction in  $I$  above satisfies this condition. This result corresponds to known theory obtained for the nodal counts of quantum graphs with  $\delta$ -interactions (see for example R. Band [9]). However, the above condition is also satisfied by the  $\delta'$ -potential in III, yielding a new result. For a discussion of the oscillation counts permitted by  $\delta'$ -interactions (type II) see Chapter 5.

# Chapter 3

## Sturm-Liouville eigenvalue problems with transmission conditions Herglotz dependent on the eigenparameter

### 3.1 Introduction

Recently, there has been growing interest in spectral problems involving differential operators with discontinuity conditions. We refer to such conditions as transmission conditions (see also [24], [60], [61], [62], [73], [82]), although they appear under the guise of many names. These include point interactions in the physics literature, with important examples being the  $\delta$  and  $\delta'$  interactions from quantum mechanics (see for example [3], [19], [25] and the references therein); interface conditions ([46], [95], [96]); as well as matching conditions on graphs ([91], [94]). Also related to this particular class of problems are the more general multi-point conditions, containing both interior points of discontinuity and endpoints (see for example [45], [55], [59]). For an interesting exposition of transmission condition problems that arise naturally in applications we refer the reader to the book by A. N. Tikhonov and A. A. Samarskii, [79].

Direct and inverse problems for continuous Sturm-Liouville equations with eigenparameter dependent boundary conditions have been studied extensively (see [10], [11], [12], [21], [29], [30], [58], [72], [85] for a sample of the literature). Investigations into Sturm-Liouville equations with discontinuity conditions depending on the spectral parameter have been thus far only limited to the affine case (see [2], [65], [66], [85]). Although, this sometimes coupled with higher order  $\lambda$ -dependence in the boundary conditions. In [67], A. S. Ozkan studies Sturm-Liouville equations where the eigenparameter is rationally contained in the boundary conditions and an affine dependence in the transmission conditions.

We consider the equation

$$\ell y := -y'' + qy = \lambda y \quad (3.1)$$

on the intervals  $(-a, 0)$  and  $(0, b)$  with  $y|_{(-a, 0)} \in W^{2,2}(-a, 0)$  and  $y|_{(0, b)} \in W^{2,2}(0, b)$ , where  $a, b > 0$  and  $q \in L^2(-a, b)$  is a real-valued function. We impose boundary conditions

$$y(-a) \cos \alpha = y'(-a) \sin \alpha, \quad (3.2)$$

$$y(b) \cos \beta = y'(b) \sin \beta, \quad (3.3)$$

where  $\alpha \in [0, \pi)$  and  $\beta \in (0, \pi]$ , and transmission conditions

$$y(0^+) = r(\lambda) \Delta y' \quad (3.4)$$

$$y'(0^-) = s(\lambda)\Delta y. \quad (3.5)$$

Here

$$\begin{aligned} \Delta y &= y(0^+) - y(0^-), \\ \Delta y' &= y'(0^+) - y'(0^-), \\ r(\lambda) &= -\sum_{i=1}^N \frac{\beta_i^2}{\lambda - \gamma_i}, \end{aligned} \quad (3.6)$$

$$s(\lambda) = \sum_{j=1}^M \frac{\alpha_j^2}{\lambda - \delta_j}, \quad (3.7)$$

where

$$\gamma_1 < \gamma_2 < \cdots < \gamma_N, \quad (3.8)$$

$$\delta_1 < \delta_2 < \cdots < \delta_M, \quad (3.9)$$

and  $\beta_i, \alpha_j > 0$  for  $i = 1, \dots, N$ , and  $j = 1, \dots, M$ . Then  $s(\lambda)$ ,  $-r(\lambda)$ ,  $-\frac{1}{s(\lambda)}$  and  $\frac{1}{r(\lambda)}$  are Herglotz-Nevanlinna functions, and consequently have real, simple poles.

Note that  $r(\lambda) = 0$  reduces (3.4) at  $\lambda$  to the condition  $y(0^+) = 0$ , while if  $\lambda$  is a pole of  $r$  then (3.4) becomes  $\Delta y' = 0$ , i.e.  $y'(0^+) = y'(0^-)$ . Similarly, if  $s(\lambda) = 0$  then (3.5) at  $\lambda$  becomes  $y'(0^-) = 0$ , while if  $\lambda$  is a pole of  $s$  then (3.5) becomes  $\Delta y = 0$ , i.e.  $y(0^-) = y(0^+)$ .

The remainder of the chapter is structured as follows. Eigenvalue multiplicities are considered in Section 3.2. We show that the maximum geometric multiplicity of the eigenvalues of (3.1)-(3.5) is 2, and that geometrically double eigenvalues can occur only at zeroes of  $r(\lambda)$  or  $s(\lambda)$ . All eigenvalues not at zeroes of  $r(\lambda)$  or  $s(\lambda)$  are geometrically simple. Furthermore, we show how to construct potentials  $q$  for which (3.1)-(3.5) has precisely  $k$  double eigenvalues, where  $0 \leq k \leq N + M - 2$  and  $N$  and  $M$  are defined in (3.6) and (3.7). In Section 3.3 we formulate (3.1)-(3.5) as a self-adjoint operator eigenvalue problem with eigenvalues that agree up to multiplicity. We also determine the form of the corresponding eigenfunctions. In Section 3.4 we define the characteristic determinant of (3.1)-(3.5). In Section 3.5 we construct the Green's function and resolvent operator corresponding to the self-adjoint operator eigenvalue problem. Lastly, in Sections 3.6 and 3.7 we give asymptotic approximations for eigenvalues and solutions. From this asymptotic approximations for eigenfunctions can be found using the observations of Section 3.3.

The results contained in this chapter provide the foundation for an associated inverse problem discussed in Chapter 4.

## 3.2 Preliminaries

**Lemma 3.2.1.** *All eigenvalues of (3.1)-(3.5) not at zeroes of  $r(\lambda)$  or  $s(\lambda)$  are geometrically simple. In this case the transmission conditions (3.4)-(3.5) can be expressed as*

$$\begin{bmatrix} y(0^+) \\ y'(0^+) \end{bmatrix} = T \begin{bmatrix} y(0^-) \\ y'(0^-) \end{bmatrix}, \quad (3.10)$$

where

(i)  $T = I$  if  $\lambda$  is a pole of both  $r$  and  $s$ .

(ii)  $T = \begin{bmatrix} 1 & \frac{1}{s(\lambda)} \\ 0 & 1 \end{bmatrix}$  if  $s(\lambda) \in \mathbb{C} \setminus \{0\}$  and  $\lambda$  is a pole of  $r$ .

(iii)  $T = \begin{bmatrix} 1 & 0 \\ \frac{1}{r(\lambda)} & 1 \end{bmatrix}$  if  $r(\lambda) \in \mathbb{C} \setminus \{0\}$  and  $\lambda$  is a pole of  $s$ .

(iv)  $T = \begin{bmatrix} 1 & \frac{1}{s(\lambda)} \\ \frac{1}{r(\lambda)} & 1 + \frac{1}{r(\lambda)s(\lambda)} \end{bmatrix}$  if  $r(\lambda), s(\lambda) \in \mathbb{C} \setminus \{0\}$

*Proof.* As  $T$  is invertible the imposing of (3.2) restricts the solution space of (3.1) to one dimension.  $\square$

**Theorem 3.2.2.** *The maximum geometric multiplicity of an eigenvalue of (3.1)-(3.5) is 2 and such eigenvalues can only occur at zeroes of  $r(\lambda)$  or  $s(\lambda)$ . An eigenvalue  $\lambda$  has geometric multiplicity 2 if and only if  $r(\lambda) = 0$  or  $s(\lambda) = 0$ ,  $\lambda$  is an eigenvalue of (3.1) on  $(-a, 0)$  with boundary conditions (3.2) and  $y'(0^-) + s(\lambda)y(0^-) = 0$ , and  $\lambda$  is an eigenvalue of (3.1) on  $(0, b)$  with boundary conditions  $y(0^+) - r(\lambda)y'(0^+) = 0$  and (3.3).*

*Proof.* The conclusion that these are only instances in which non-simple eigenvalues are possible follows from Lemma 3.2.1. That the multiplicity is 2 in the given circumstances is directly evident.  $\square$

Note that in the above theorem, if  $\lambda$  is a pole of  $r$  then  $y(0^+) - r(\lambda)y'(0^+) = 0$  is taken to mean  $y'(0^+) = 0$ , while if  $\lambda$  is a pole of  $s$  then  $y'(0^-) + s(\lambda)y(0^-) = 0$  is taken to mean  $y(0^-) = 0$ .

**Note 3.2.3.** *If  $a_1 < b_1 < a_2 < b_2 < \dots < b_{m-1} < a_m$  and*

$$g(\lambda) = \frac{\prod_{j=1}^{m-1} (b_j - \lambda)}{\prod_{k=1}^m (a_k - \lambda)}$$

then

$$g(\lambda) = \sum_{r=1}^m \frac{c_r}{a_r - \lambda}$$

where

$$c_r = \frac{\prod_{j=1}^{m-1} (b_j - a_r)}{\prod_{k \neq r} (a_k - a_r)}$$

and  $c_r > 0$  for all  $r = 1, \dots, m$ .

**Theorem 3.2.4.** *For any  $N, M \in \mathbb{N}$  there are potentials  $q \in L^2(-\pi, \pi)$  and parameters  $\gamma_1 < \gamma_2 < \dots < \gamma_N$ ,  $\delta_1 < \delta_2 < \dots < \delta_M$ , and  $\beta_i, \alpha_j > 0$  for  $i = 1, \dots, N$ , and  $j = 1, \dots, M$*



such that (3.1)-(3.5) with  $a = b = \pi$  has precisely  $N + M - 2$  double eigenvalues (the maximum number possible).<sup>1</sup>

*Proof.* Assume that  $N \leq M$ . We take boundary conditions  $y(\pm\pi) = 0$  and set  $q(x) = 0$  for  $x \in [0, \pi]$  and

$$r(\lambda) = \frac{\prod_{j=1}^{N-1} (j^2 - \lambda)}{\prod_{k=1}^N \left( \left( k - \frac{1}{2} \right)^2 - \lambda \right)}.$$

Now  $1^2/4, 3^2/4, \dots, (2M - 3)^2/4$  are eigenvalues of (3.1) on  $[0, \pi]$  with boundary conditions  $y(\pi) = 0 = y'(0^+)$ , while  $1^2, 2^2, \dots, (N - 1)^2$  are eigenvalues of (3.1) on  $[0, \pi]$  with boundary conditions  $y(\pi) = 0 = y(0^+)$ . In particular,  $\lambda = 1^2, 2^2, \dots, (N - 1)^2$  are eigenvalues of (3.1) on  $[0, \pi]$  with boundary conditions  $y(\pi) = 0$  and  $y(0^+) = r(\lambda)y'(0^+)$  (when  $r(\lambda) = 0$ ).

Let  $a_j = (j - 1/2)^2$  for  $j = 1, \dots, N - 1$  and  $a_j = \mu_j$  for  $j = N, \dots, M - 1$ , where  $\lambda = \mu_N < \mu_{N+1} < \dots < \mu_{M-1}$  are eigenvalues of (3.1) on  $[0, \pi]$  with boundary conditions  $y(\pi) = 0$  and  $y(0^+) = r(\lambda)y'(0^+)$  with  $\lambda > (N - 1)^2$ . Define  $b_j = (j - 1)^2$  for  $j = 1, \dots, N$ , and  $b_j = (a_j + a_{j-1})/2$  for  $j = N + 1, \dots, M - 1$  and  $b_M = a_{M-1} + 1$ . Let

$$s(\lambda) = -\frac{\prod_{j=1}^{M-1} (a_j - \lambda)}{\prod_{k=1}^M (b_k - \lambda)}.$$

We now take  $q$  on  $[-\pi, 0)$  to be an  $L^2$  potential so that the eigenvalues of (3.1) on  $[-\pi, 0)$  with boundary condition  $y(-\pi) = 0$  and  $y(0^-) = 0$  contains the set

$$\{1^2, 2^2, \dots, (N - 1)^2, \mu_N, \dots, \mu_{M-1}\},$$

while the eigenvalues of (3.1) on  $[-\pi, 0)$  with boundary condition  $y(-\pi) = 0$  and  $y'(0^-) = 0$  contains the set

$$\{1^2/4, 3^2/4, \dots, (N - 1)^2/4\}.$$

This is possible via the Gelfand-Levitan theory of inverse spectral problems (see for example [27]). It is now easily verified that  $\lambda = (2j - 1)^2/4$  for  $j = 1, \dots, 2N - 2$ , as well as  $\mu_j$ , for  $j = N, \dots, M - 1$  are double eigenvalues of the transmission problem with  $q, r, s$  as constructed here with boundary conditions  $y(\pm\pi) = 0$ .  $\square$

We note that using similar methods to those of the above proof, it can be shown that any number of eigenvalues between 0 and  $N + M - 2$  can be constructed to be double. Due to notational opacity we will only present a proof of the other extreme case, that of no double eigenvalues.

**Theorem 3.2.5.** *For any  $N, M \in \mathbb{N}$  there are potentials  $q \in L^2(-\pi, \pi)$  and parameters  $\gamma_1 < \gamma_2 < \dots < \gamma_N$ ,  $\delta_1 < \delta_2 < \dots < \delta_M$ , and  $\beta_i, \alpha_j > 0$  for  $i = 1, \dots, N$  and  $j = 1, \dots, M$  such that (3.1)-(3.5) with  $a = b = \pi$  has no double eigenvalues.*

<sup>1</sup>Since the number of zeroes of  $r(\lambda)$  is  $N - 1$  and the number of zeroes of  $s(\lambda)$  is  $M - 1$ , the maximal number of  $N + M - 2$  double eigenvalues is achieved when the zeroes of  $r$  are disjoint from those of  $s$  and at each zero the conditions of Theorem 3.2.2 are satisfied.

*Proof.* For no double eigenvalues we require that:

- I. The boundary value problem consisting of the equation  $\ell y = \lambda y$  on  $(0, b)$ , with boundary conditions  $y(0^+) = 0$  and (3.3) does not have an eigenvalue at a root of  $r(\lambda) = 0$ ;
- II. The boundary value problem consisting of  $\ell y = \lambda y$  on  $(-a, 0)$ , with boundary conditions  $y'(0^-) = 0$  and (3.2) does not have an eigenvalue at a root of  $s(\lambda) = 0$ .

For  $a = \pi = b$ ,  $\alpha = \pi/2 = \beta$  and  $q = 0$ , the conditions I and II can be met by taking

$$r(\lambda) = \frac{\prod_{j=1}^{N-1} (j^2 - \lambda)}{\prod_{k=1}^N \left( \left( k - \frac{1}{2} \right)^2 - \lambda \right)}$$

and

$$s(\lambda) = -\frac{\prod_{j=1}^{M-1} \left( \left( j + \frac{1}{2} \right)^2 - \lambda \right)}{\prod_{k=1}^M (k^2 - \lambda)}.$$

□

### 3.3 Hilbert space setting

We now formulate (3.1) with boundary conditions (3.2)-(3.3) and transmission conditions (3.4)-(3.5) as a self-adjoint operator eigenvalue problem. Set

$$LY = \begin{bmatrix} \ell y \\ (\gamma_i y_i^1 + \beta_i \Delta y')_{i=1}^N \\ (\delta_j y_j^2 + \alpha_j \Delta y)_{j=1}^M \end{bmatrix}, \quad Y = \begin{bmatrix} y \\ (y_i^1)_{i=1}^N \\ (y_j^2)_{j=1}^M \end{bmatrix}, \quad (3.11)$$

with domain

$$\mathcal{D}(L) = \left\{ Y := \begin{bmatrix} y \\ (y_i^1)_{i=1}^N \\ (y_j^2)_{j=1}^M \end{bmatrix} \left| \begin{array}{l} y|_{(-a,0)} \in W^{2,2}(-a, 0), \\ y|_{(0,b)} \in W^{2,2}(0, b), \\ y \text{ obeys (3.2) and (3.3)} \\ -y(0^+) = \sum_{i=1}^N \beta_i y_i^1 \\ y'(0^-) = \sum_{j=1}^M \alpha_j y_j^2 \end{array} \right. \right\}, \quad (3.12)$$

where  $W^{2,2}$  is the Sobolev space. Note that for notational simplicity we will write  $(y_i^1)_{i=1}^N$  as  $(y_i^1)$ , and similarly  $(y_j^2)_{j=1}^M$  as  $(y_j^2)$ . These are still to be understood as vectors in  $\mathbb{C}^N$  and  $\mathbb{C}^M$  respectively.

**Theorem 3.3.1.** *The eigenvalue problems  $LY = \lambda Y$ , and (3.1) with boundary conditions (3.2)-(3.3) and transmission conditions (3.4)-(3.5) are equivalent in the sense that  $\lambda$  is an eigenvalue of  $LY = \lambda Y$  with eigenvector  $Y$  if and only if  $\lambda$  is an eigenvalue with eigenfunction  $y$  of (3.1) with boundary conditions (3.2)-(3.3) and transmission conditions (3.4)-(3.5). Here  $Y$  and  $y$  are related by*

$$Y = \begin{bmatrix} y \\ \left( \frac{\beta_i}{\lambda - \gamma_i} \Delta y' \right) \\ \left( \frac{\alpha_j}{\lambda - \delta_j} \Delta y \right) \end{bmatrix} \quad (3.13)$$

provided that  $\lambda \neq \gamma_i, \delta_j$  for all  $i = \overline{1, N}$  and  $j = \overline{1, M}$ . If  $\lambda = \gamma_p$  for some  $p \in \{1, \dots, N\}$  then

$$y_p^1 = -\frac{y(0^+)}{\beta_p} \quad \text{and} \quad y_k^1 = 0 \quad \forall k \neq p. \quad (3.14)$$

Whereas, if  $\lambda = \delta_\mu$  for some  $\mu \in \{1, \dots, M\}$  then

$$y_\mu^2 = \frac{y'(0^-)}{\alpha_\mu} \quad \text{and} \quad y_k^2 = 0 \quad \forall k \neq \mu. \quad (3.15)$$

The geometric multiplicity of  $\lambda$  as an eigenvalue of  $L$  is the same as the geometric multiplicity of  $\lambda$  as an eigenvalue of (3.1)-(3.5).

*Proof.* Suppose that  $LY = \lambda Y$ . Then  $\ell y = \lambda y$ , where  $y|_{(-a,0)} \in W^{2,2}(-a,0)$ ,  $y|_{(0,b)} \in W^{2,2}(0,b)$  and, moreover,

$$\begin{aligned} \gamma_i y_i^1 + \beta_i \Delta y' &= \lambda y_i^1 \implies \begin{cases} y_i^1 = \frac{\beta_i}{\lambda - \gamma_i} \Delta y' & \text{if } \lambda \neq \gamma_i, \\ \Delta y' = 0 & \text{if } \lambda = \gamma_i, \end{cases} \quad i = \overline{1, N}, \\ \delta_j y_j^2 + \alpha_j \Delta y &= \lambda y_j^2 \implies \begin{cases} y_j^2 = \frac{\alpha_j}{\lambda - \delta_j} \Delta y & \text{if } \lambda \neq \delta_j, \\ \Delta y = 0 & \text{if } \lambda = \delta_j, \end{cases} \quad j = \overline{1, M}, \end{aligned}$$

where

$$-y(0^+) = \sum_{i=1}^N \beta_i y_i^1, \quad y'(0^-) = \sum_{j=1}^M \alpha_j y_j^2.$$

Thus, we conclude that

$$y(0^+) = -\sum_{i=1}^N \frac{\beta_i^2}{\lambda - \gamma_i} \Delta y',$$

provided that  $\lambda \neq \gamma_i$  for all  $i = \overline{1, N}$ , whereas if  $\lambda = \gamma_p$  for some  $p \in \{1, \dots, N\}$  then

$$\Delta y' = 0, \quad y_p^1 = -\frac{y(0^+)}{\beta_p}, \quad y_k^1 = 0 \quad \forall k \neq p.$$

Also,

$$y'(0^-) = \sum_{j=1}^M \frac{\alpha_j^2}{\lambda - \delta_j} \Delta y,$$

if  $\lambda \neq \delta_j$  for all  $j = \overline{1, M}$ , else if  $\lambda = \delta_\mu$  for some  $\mu \in \{1, \dots, M\}$  then

$$\Delta y = 0, \quad y_\mu^2 = \frac{y'(0^-)}{\alpha_\mu}, \quad y_k^2 = 0 \quad \forall k \neq \mu.$$

Hence, the eigenvalues of  $L$  are eigenvalues of (3.1) with boundary conditions (3.2)-(3.3) and transmission conditions (3.4)-(3.5) with corresponding eigenfunction  $y = [Y]_0$ , the functional component of  $Y$ .

For the converse, suppose that  $\lambda$  is an eigenvalue, with corresponding eigenfunction  $y$ , of (3.1) with boundary conditions (3.2)-(3.3) and transmission conditions (3.4)-(3.5). Then  $\ell y = \lambda y$  with  $y|_{(-a,0)} \in W^{2,2}(-a,0)$  and  $y|_{(0,b)} \in W^{2,2}(0,b)$ . Define  $Y$  as given in (3.13)-(3.15). Now, if  $\lambda \neq \gamma_i$  for all  $i = \overline{1, N}$  then

$$\sum_{i=1}^N \beta_i y_i^1 = \sum_{i=1}^N \frac{\beta_i^2}{\lambda - \gamma_i} \Delta y' = -r(\lambda) \Delta y' = -y(0^+),$$

since  $y$  obeys (3.4). Whereas, if  $\lambda = \gamma_p$  for some  $p \in \{1, \dots, N\}$ , then

$$\sum_{i=1}^N \beta_i y_i^1 = \beta_p y_p^1 = -y(0^+),$$

by (3.14). Similarly, if  $\lambda \neq \delta_j$  for all  $j \in \overline{1, M}$  then

$$\sum_{j=1}^M \alpha_j y_j^2 = s(\lambda) \Delta y = y'(0^-),$$

since  $y$  obeys (3.5), while if  $\lambda = \delta_\mu$  for some  $\mu \in \{1, \dots, M\}$  then by (3.15),

$$\sum_{j=1}^m \alpha_j y_j^2 = \alpha_\mu y_\mu^2 = y'(0^-).$$

Next we consider the correspondence of geometric multiplicities. If  $\lambda$  is an eigenvalue of (3.1)-(3.5) with linearly independent eigenfunctions  $y^{[1]}, \dots, y^{[k]}$  then the vectors  $Y^{[1]}, \dots, Y^{[k]}$  as given by (3.13)-(3.15) are linearly independent eigenvectors of  $L$  with eigenvalue  $\lambda$ . Hence, the geometric multiplicity of  $\lambda$  as an eigenvalue of  $L$  is at least as large as the geometric multiplicity of  $\lambda$  as an eigenvalue of (3.1)-(3.5).

If  $Y^{[1]}, \dots, Y^{[k]}$  are linearly independent eigenvectors of  $L$  for the eigenvalue  $\lambda$  then it remains only to show that the corresponding functional components  $[Y^{[1]}]_0, \dots, [Y^{[k]}]_0$  are linearly independent eigenfunctions of (3.1)-(3.5) for the eigenvalue  $\lambda$ . That they are eigenfunctions of (3.1)-(3.5) for the eigenvalue  $\lambda$  follows from the first part of this theorem, it remains only to prove independence. Supposing that  $[Y^{[1]}]_0, \dots, [Y^{[k]}]_0$  are linearly dependent, there are  $\rho_1, \dots, \rho_k$ , not all zero, such that

$$0 = \sum_{n=0}^k \rho_n [Y^{[n]}]_0 = \left[ \sum_{n=0}^k \rho_n Y^{[n]} \right]_0.$$

The linear independence of  $Y^{[1]}, \dots, Y^{[k]}$  gives that

$$0 \neq \sum_{n=0}^k \rho_n Y^{[n]} := Y.$$

As  $Y := \begin{bmatrix} y \\ (y_i^1) \\ (y_j^2) \end{bmatrix} \in \mathcal{D}(L)$ , from  $y = 0$  it follows that

$$0 = \sum_{i=1}^N \beta_i y_i^1 \quad \text{and} \quad 0 = \sum_{j=1}^M \alpha_j y_j^2. \quad (3.16)$$

But  $LY = \lambda Y$ , and again  $y = 0$ . Hence,  $\lambda y_i^1 = \gamma_i y_i^1$  and  $\lambda y_j^2 = \delta_j y_j^2$  for all  $i = \overline{1, N}$  and  $j = \overline{1, M}$ . However, by (3.8) at most one of  $y_1^1, \dots, y_N^1$  is non-zero, so by (3.16) all are zero. Similarly, by (3.9) at most one of  $y_1^2, \dots, y_M^2$  is non-zero, so by (3.16) all are zero. Thus we get a contradiction to  $Y \neq 0$ .  $\square$

We conclude this section by showing that  $L$  is a self-adjoint operator.

**Theorem 3.3.2.** *The operator  $L$  is self-adjoint in  $\mathcal{H} = L^2(-a, b) \oplus \mathbb{C}^M \oplus \mathbb{C}^N$ .*

*Proof.* We begin by showing that  $\mathcal{D}(L)$  is dense in  $\mathcal{H}$ . Let  $F = \begin{bmatrix} f \\ \mathbf{f}^1 \\ \mathbf{f}^2 \end{bmatrix} \in \mathcal{H}$ , where  $\mathbf{f}^1 = (f_i^1)$ ,  $\mathbf{f}^2 = (f_j^2)$ . For  $m \geq 2$  let

$$w_m(x) = \begin{cases} \frac{(x+a)^m}{ma^{m-1}} \sum_{j=1}^M \alpha_j f_j^2, & x < 0 \\ -\frac{(b-x)^m}{b^m} \sum_{i=1}^N \beta_i f_i^1, & x > 0 \end{cases} \quad \text{then} \quad W_m = \begin{bmatrix} w_m \\ \mathbf{f}^1 \\ \mathbf{f}^2 \end{bmatrix} \in \mathcal{D}(L). \quad (3.17)$$

As  $q \in L^2(-a, b)$  it follows that  $(C_0^\infty(-a, 0) \oplus C_0^\infty(0, b)) \oplus \{\mathbf{0}\} \oplus \{\mathbf{0}\} \subset \mathcal{D}(L)$ . Here,  $C_0^\infty(-a, 0) \oplus C_0^\infty(0, b)$  is dense in  $L^2(-a, b)$  so there is a sequence  $\{g_n\} \subset C_0^\infty(-a, 0) \oplus C_0^\infty(0, b)$  with  $g_n \rightarrow f - w_m$  in norm. Here,

$$G_n := \begin{bmatrix} g_n \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \in \mathcal{D}(L)$$

and thus  $W_m + G_n \in \mathcal{D}(L)$ . Now,  $W_m + G_n \rightarrow F$  in norm as  $n \rightarrow \infty$  giving that  $\mathcal{D}(L)$  is dense in  $\mathcal{H}$ .

We now show that  $L$  is symmetric. As  $q \in L^2(-a, b)$ , we have that if  $f|_{(-a, 0)}$ ,  $f|'_{(-a, 0)}$ ,  $\ell f|_{(-a, 0)} \in L^2(-a, 0)$ , then  $f|_{(-a, 0)} \in C^1(-a, 0)$  with  $f|'_{(-a, 0)}$  absolutely continuous, and similarly for  $f|_{(0, b)}$ . It is thus possible to impose the conditions (3.2) and (3.3) on such a function  $f$ . Let  $F, G \in \mathcal{D}(L)$ , then the functional components  $f$  and  $g$  of  $F$  and  $G$  respectively obey

$$(\ell f, g) - (f, \ell g) = (-f' \bar{g} + f \bar{g}') (0^-) + (f' \bar{g} - f \bar{g}') (0^+),$$

where  $(f, g) := \int_{-a}^b f \bar{g} dx$ . Moreover, the vector components satisfy

$$\begin{aligned} & \langle (\gamma_i f_i^1 + \beta_i \Delta f'), (g_i^1) \rangle_N - \langle (f_i^1), (\gamma_i g_i^1 + \beta_i \Delta g') \rangle_N \\ &= \Delta f' \langle (\beta_i), (g_i^1) \rangle_N - \Delta \bar{g}' \langle (f_i^1), (\beta_i) \rangle_N \\ &= -\Delta f' \bar{g}(0^+) + \Delta \bar{g}' f(0^+), \end{aligned}$$

where  $\langle \cdot, \cdot \rangle_N$  is the Euclidean inner product in  $\mathbb{C}^N$ , and

$$\begin{aligned} & \langle (\delta_j f_j^2 + \alpha_j \Delta f), (g_j^2) \rangle_M - \langle (f_j^2), (\delta_j g_j^2 + \alpha_j \Delta g) \rangle_M \\ &= \Delta f \langle (\alpha_j), (g_j^2) \rangle_M - \Delta \bar{g} \langle (f_j^2), (\alpha_j) \rangle_M \\ &= \Delta f \bar{g}'(0^-) - \Delta \bar{g} f'(0^-), \end{aligned}$$

where  $\langle \cdot, \cdot \rangle_M$  is the Euclidean inner product in  $\mathbb{C}^M$ . Let

$$\langle F, G \rangle := (f, g) + \langle (f_i^1), (g_i^1) \rangle_N + \langle (f_j^2), (g_j^2) \rangle_M.$$

Then a direct computation gives

$$(f' \bar{g} - f \bar{g}')(0^-) - (f' \bar{g} - f \bar{g}')(0^+) = \Delta \bar{g}' f(0^+) - \Delta f' \bar{g}(0^+) + \Delta f \bar{g}'(0^-) - \Delta \bar{g} f'(0^-).$$

Thus  $\langle LF, G \rangle - \langle F, LG \rangle = 0$ . So  $L$  is symmetric, giving  $\mathcal{D}(L) \subset \mathcal{D}(L^*)$ .

To show that  $L$  is self-adjoint it remains only to verify that  $\mathcal{D}(L^*) \subset \mathcal{D}(L)$ . Let  $G \in \mathcal{D}(L^*)$  then  $\langle LF, G \rangle = \langle F, L^*G \rangle$  for all  $F \in \mathcal{D}(L)$ , and the map  $F \mapsto \langle F, L^*G \rangle$  defines a continuous linear functional on  $\mathcal{H}$ . Hence, the map  $F \mapsto \langle LF, G \rangle$  is a continuous linear functional on  $\mathcal{H}$  restricted to the dense subspace  $\mathcal{D}(L)$ . In particular, there is  $k \geq 0$  so that for all  $F \in (C_0^\infty(-a, 0) \oplus \{0\}) \oplus \{0\} \oplus \{0\}$  we have that

$$\left| \int_{-a}^0 f'' \left( -\bar{g} + \int_{-a}^x \int_{-a}^t q \bar{g} d\tau dt \right) dx \right| \leq k \|f\|_2, \quad (3.18)$$

for all  $f \in C_0^\infty(-a, 0)$ . Hence, see [1, Chapter 1 & 2],

$$g - \int_{-a}^x \int_{-a}^t q g d\tau dt \in H^2(-a, 0). \quad (3.19)$$

We note here that  $qg \in L^1(-a, 0)$ , giving that  $\int_{-a}^t qg d\tau \in L^2(-a, 0)$ . Hence,  $g \in H^1(-a, 0)$  and differentiating (3.19) gives

$$g' - \int_{-a}^x qg d\tau \in H^1(-a, 0). \quad (3.20)$$

Thus  $g''$  exists as a weak derivative and is in  $L^1(-a, 0)$ . Applying the above in (3.18) gives

$$\left| \int_{-a}^0 f (-\bar{g}'' + q\bar{g}) dx \right| \leq k \|f\|_2, \quad (3.21)$$

and hence  $\ell^*g = \ell g$  exists in  $L^2(-a, 0)$ .

Similarly, we obtain  $g, g', \ell^*g = \ell g$  exists in  $L^2(0, b)$ . Thus  $g \in H^2(-a, 0) \oplus H^2(0, b)$  with  $\ell^*g = \ell g \in L^2(-a, b)$ . Hence,

$$\int_{-a}^b f \bar{\ell g} dx = \int_{-a}^b (\ell f) \bar{g} dx = \langle LF, G \rangle = \langle F, L^*G \rangle = \int_{-a}^b f \overline{[L^*G]_0} dx$$

for all  $F \in (C_0^\infty(-a, 0) \oplus C_0^\infty(0, b)) \oplus \{0\} \oplus \{0\}$ , giving  $[L^*G]_0 = \ell g$ .

Now, for each  $\mathbf{f}^1 \in \mathbb{C}^N$  and  $\mathbf{f}^2 \in \mathbb{C}^M$  let  $W_n$  be as in (3.17), then we have that

$$\begin{aligned} & \int_{-a}^b \ell w_n \bar{g} dx + \langle (\gamma_i f_i^1 + \beta_i \Delta w_n'), [G]^1 \rangle_N + \langle (\delta_j f_j^2 + \alpha_j \Delta w_n), [G]^2 \rangle_M \\ &= \langle LW_n, G \rangle \\ &= \langle W_n, L^*G \rangle \\ &= \int_{-a}^b w_n \ell \bar{g} dx + \langle \mathbf{f}^1, [L^*G]^1 \rangle_N + \langle \mathbf{f}^2, [L^*G]^2 \rangle_M. \end{aligned}$$

Applying integration by parts to the pair of integrals in the above expression we have

$$\begin{aligned} & (w_n' \bar{g})(0^+) - (w_n' \bar{g})(0^-) + \langle (\gamma_i f_i^1 + \beta_i \Delta w_n'), [G]^1 \rangle_N + \langle (\delta_j f_j^2 + \alpha_j \Delta w_n), [G]^2 \rangle_M \\ &= (w_n \bar{g}') (0^+) - (w_n \bar{g}') (0^-) + \langle \mathbf{f}^1, [L^*G]^1 \rangle_N + \langle \mathbf{f}^2, [L^*G]^2 \rangle_M. \end{aligned}$$

Here

$$\begin{aligned} w_n(0^-) &= an^{-1} \langle \mathbf{f}^2, \boldsymbol{\alpha} \rangle_M, \quad w_n(0^+) = - \langle \mathbf{f}^1, \boldsymbol{\beta} \rangle_N, \\ w_n'(0^-) &= \langle \mathbf{f}^2, \boldsymbol{\alpha} \rangle_M, \quad w_n'(0^+) = nb^{-1} \langle \mathbf{f}^1, \boldsymbol{\beta} \rangle_N, \end{aligned}$$

where  $\boldsymbol{\alpha} = (\alpha_i)$  and  $\boldsymbol{\beta} = (b_j)$ . Thus we have

$$\begin{aligned} & nb^{-1} \langle \mathbf{f}^1, \boldsymbol{\beta} \rangle_N \bar{g}(0^+) - \langle \mathbf{f}^2, \boldsymbol{\alpha} \rangle_M \bar{g}(0^-) \\ &+ \langle \mathbf{f}^1, (\gamma_i [G]_i^1) \rangle_N + nb^{-1} \langle \mathbf{f}^1, \boldsymbol{\beta} \rangle_N \langle \boldsymbol{\beta}, [G]^1 \rangle_N - \langle \mathbf{f}^2, \boldsymbol{\alpha} \rangle_M \langle \boldsymbol{\beta}, [G]^1 \rangle_N \\ &+ \langle \mathbf{f}^2, (\delta_j [G]_j^2) \rangle_M - \langle \mathbf{f}^1, \boldsymbol{\beta} \rangle_N \langle \boldsymbol{\alpha}, [G]^2 \rangle_M - an^{-1} \langle \mathbf{f}^2, \boldsymbol{\alpha} \rangle_M \langle \boldsymbol{\alpha}, [G]^2 \rangle_M \\ &= - \langle \mathbf{f}^1, \boldsymbol{\beta} \rangle_N \bar{g}'(0^+) - an^{-1} \langle \mathbf{f}^2, \boldsymbol{\alpha} \rangle_M \bar{g}'(0^-) + \langle \mathbf{f}^1, [L^*G]^1 \rangle_N + \langle \mathbf{f}^2, [L^*G]^2 \rangle_M. \end{aligned}$$

Allowing  $\mathbf{f}^1$  and  $\mathbf{f}^2$  to vary throughout  $\mathbb{C}^N$  and  $\mathbb{C}^M$  respectively gives

$$nb^{-1} \boldsymbol{\beta} (g(0^+) + \langle [G]^1, \boldsymbol{\beta} \rangle_N) - \boldsymbol{\beta} (\langle [G]^2, \boldsymbol{\alpha} \rangle_M - g'(0^+)) = [L^*G]^1 - (\gamma_i [G]_i^1)$$

and

$$\boldsymbol{\alpha} (g(0^-) + \langle [G]^1, \boldsymbol{\beta} \rangle_N) + an^{-1} \boldsymbol{\alpha} (\langle [G]^2, \boldsymbol{\alpha} \rangle_M - g'(0^-)) = (\delta_j [G]_j^2) - [L^*G]^2.$$

Now allowing  $n$  to vary, we get

$$\begin{aligned} g(0^+) + \langle [G]^1, \boldsymbol{\beta} \rangle_N &= 0, \\ -\boldsymbol{\beta} (\langle [G]^2, \boldsymbol{\alpha} \rangle_M - g'(0^+)) &= [L^*G]^1 - (\gamma_i [G]_i^1), \\ \langle [G]^2, \boldsymbol{\alpha} \rangle_M - g'(0^-) &= 0, \\ \boldsymbol{\alpha} (g(0^-) + \langle [G]^1, \boldsymbol{\beta} \rangle_N) &= (\delta_j [G]_j^2) - [L^*G]^2. \end{aligned}$$

Thus

$$\begin{aligned} -g(0^+) &= \langle [G]^1, \boldsymbol{\beta} \rangle_N, \\ g'(0^-) &= \langle [G]^2, \boldsymbol{\alpha} \rangle_M, \\ [L^*G]^1 &= (\gamma_i [G]_i^1 + \beta_i \Delta g'), \\ [L^*G]^2 &= (\delta_j [G]_j^2 + \alpha_j \Delta g). \end{aligned}$$

Hence, it follows that  $G \in \mathcal{D}(L)$  and  $L^*G = LG$ . □

### 3.4 The characteristic determinant

Let  $u_-(x; \lambda)$  denote the solution of (3.1) on  $[-a, 0)$  satisfying the initial conditions

$$u_-(-a; \lambda) = \sin \alpha \quad \text{and} \quad u'_-(-a; \lambda) = \cos \alpha, \quad (3.22)$$

and  $v_+(x; \lambda)$  denote the solution of (3.1) on  $(0, b]$  satisfying the terminal conditions

$$v_+(b; \lambda) = \sin \beta \quad \text{and} \quad v'_+(b; \lambda) = \cos \beta. \quad (3.23)$$

We note that  $u_-(x; \lambda)$  and  $v_+(x; \lambda)$  can be extended to solutions  $u_+(x; \lambda)$  and  $v_-(x; \lambda)$  of (3.1) defined on  $(0, b]$  and  $[-a, 0)$  respectively by imposing the transmission conditions (3.4) and (3.5). At values of the eigenparameter not coinciding with a zero of  $r(\lambda)$  or  $s(\lambda)$  this is achieved simply by applying transfer matrix  $T$  defined in Lemma 3.2.1. That is, we define

$$\begin{bmatrix} u_+(0^+; \lambda) \\ u'_+(0^+; \lambda) \end{bmatrix} = T \begin{bmatrix} u_-(0^-; \lambda) \\ u'_-(0^-; \lambda) \end{bmatrix} \quad (3.24)$$

and

$$\begin{bmatrix} v_-(0^-; \lambda) \\ v'_-(0^-; \lambda) \end{bmatrix} = T^{-1} \begin{bmatrix} v_+(0^+; \lambda) \\ v'_+(0^+; \lambda) \end{bmatrix} \quad (3.25)$$

and write

$$\begin{aligned} u_+(x; \lambda) &= u_+(0^+; \lambda)w_1(x; \lambda) + u'_+(0^+; \lambda)w_2(x; \lambda) \quad \text{for } -a \leq x < 0 \\ v_-(x; \lambda) &= v_-(0^-; \lambda)w_1(x; \lambda) + v'_-(0^-; \lambda)w_2(x; \lambda) \quad \text{for } 0 < x \leq b, \end{aligned}$$

where  $w_1(x; \lambda)$  and  $w_2(x; \lambda)$  are solutions of (3.1) on  $[-a, 0) \cup (0, b]$  satisfying

$$w_1(0; \lambda) = 1, \quad w_2(0; \lambda) = 0 \quad (3.26)$$

$$w'_1(0; \lambda) = 0, \quad w'_2(0; \lambda) = 1. \quad (3.27)$$

At zeroes of  $r(\lambda)$  or  $s(\lambda)$  we extend  $u_+$  and  $v_-$  by continuity, defining

$$u_+(x; \lambda) = \lim_{\mu \rightarrow \lambda} [u_+(0^+; \mu)w_1(x; \mu) + u'_+(0^+; \mu)w_2(x; \mu)] \quad \text{for } -a \leq x < 0$$

$$v_-(x; \lambda) = \lim_{\mu \rightarrow \lambda} [v_-(0^-; \mu)w_1(x; \mu) + v'_-(0^-; \mu)w_2(x; \mu)] \quad \text{for } 0 < x \leq b$$

if the limits exist. Define

$$u(x; \lambda) = \begin{cases} u_-(x; \lambda) & \text{if } -a \leq x < 0 \\ u_+(x; \lambda) & \text{if } 0 < x \leq b \end{cases} \quad (3.28)$$

and

$$v(x; \lambda) = \begin{cases} v_-(x; \lambda) & \text{if } -a \leq x < 0 \\ v_+(x; \lambda) & \text{if } 0 < x \leq b \end{cases}. \quad (3.29)$$



Let

$$\begin{aligned}
\omega(\lambda) = & \left[ \sum_{j=1}^M \alpha_j^2 \prod_{k=1, k \neq j}^M (\lambda - \delta_k) \right] \left[ \sum_{i=1}^N \beta_i^2 \prod_{k=1, k \neq i}^N (\lambda - \gamma_k) \right] [u(0^-)v'(0^+) - u'(0^-)v(0^+)] \\
& + \prod_{i=1}^N (\lambda - \gamma_i) \left[ \sum_{j=1}^M \alpha_j^2 \prod_{k=1, k \neq j}^M (\lambda - \delta_k) \right] u(0^-)v(0^+) \\
& + \prod_{j=1}^M (\lambda - \delta_j) \left[ \sum_{i=1}^N \beta_i^2 \prod_{k=1, k \neq i}^N (\lambda - \gamma_k) \right] u'(0^-)v'(0^+) \\
& + \prod_{i=1}^N (\lambda - \gamma_i) \prod_{j=1}^M (\lambda - \delta_j) u'(0^-)v(0^+). \tag{3.30}
\end{aligned}$$

Here,  $\omega(\lambda)$  will be referred to as the characteristic determinant of (3.1)-(3.5). In the following theorem we show that  $\omega$  has the properties expected of the characteristic determinant.

**Theorem 3.4.1.** *The function  $\omega(\lambda)$  is entire, has zeroes at precisely the eigenvalues of (3.1) - (3.5) with the order of the zeroes of  $\omega(\lambda)$  coinciding with the geometric multiplicity of  $\lambda$  as an eigenvalue of (3.1) - (3.5), and hence of  $L$ .*

*Proof.* Let  $u_-, v_+$  be defined as in equations (3.22), (3.23). Then any function of the form

$$y(x; \lambda) = \begin{cases} C(\lambda)u_-(x; \lambda), & \text{if } -a \leq x < 0, \\ D(\lambda)v_+(x; \lambda), & \text{if } 0 < x \leq b, \end{cases} \tag{3.31}$$

is a solution of (3.1) satisfying the end boundary conditions (3.2) and (3.3). For any function  $y$  given by (3.31), above, we define the following forms,

$$U_1(y; \lambda) = \prod_{i=1}^N (\lambda - \gamma_i) y(0^+) + \sum_{i=1}^N \beta_i^2 \prod_{k=1, k \neq i}^N (\lambda - \gamma_k) \Delta y', \tag{3.32}$$

$$U_2(y; \lambda) = \prod_{j=1}^M (\lambda - \delta_j) y'(0^-) - \sum_{j=1}^M \alpha_j^2 \prod_{k=1, k \neq j}^M (\lambda - \delta_k) \Delta y. \tag{3.33}$$

Clearly,  $\lambda$  is an eigenvalue of (3.1) with boundary conditions (3.2)-(3.3) and transmission conditions (3.4)-(3.5) if and only if  $U_1(y; \lambda) = U_2(y; \lambda) = 0$ . That is, the eigenvalues of (3.1) - (3.5) coincide with the zeroes of

$$\omega(\lambda) = \det \begin{bmatrix} U_1(u_-; \lambda) & U_1(v_+; \lambda) \\ U_2(u_-; \lambda) & U_2(v_+; \lambda) \end{bmatrix}, \tag{3.34}$$

where by  $U_i(u_-; \lambda)$  (respectively  $U_i(v_+; \lambda)$ ) for  $i = 1, 2$  we mean  $U_i(y; \lambda)$  with  $C(\lambda) = 1$ ,  $D(\lambda) = 0$  (respectively  $C(\lambda) = 0$ ,  $D(\lambda) = 1$ ). By expanding the right hand side of (3.34) it is easy to check that this agrees with the right hand side of (3.30). It remains only to confirm that the order of  $\lambda$  as a zero of  $\omega$  coincides with the geometric multiplicity of  $\lambda$  as an eigenvalue of (3.1)-(3.5).

Suppose that  $\lambda$  is an eigenvalue of (3.1)-(3.5) with  $r(\lambda) = 0$  or  $s(\lambda) = 0$ . We give details only for the case of  $r(\lambda) = s(\lambda) = 0$ , as the arguments for the remaining cases are similar. In this case,

the transmission conditions reduce to  $y(0^+) = 0$  and  $y'(0^-) = 0$  at  $\lambda$ . Since  $r(\lambda) = s(\lambda) = 0$ , the main diagonal elements  $U_1(u_-; \lambda)$  and  $U_2(v_+; \lambda)$  are both automatically zero. Indeed

$$U_1(u_-; \lambda) = - \left[ \sum_{i=1}^N \beta_i^2 \prod_{k=1, k \neq i}^N (\lambda - \gamma_k) \right] u'_-(0^-), \quad (3.35)$$

$$U_2(v_+; \lambda) = - \left[ \sum_{j=1}^M \alpha_j^2 \prod_{k=1, k \neq j}^M (\lambda - \delta_k) \right] v_+(0^+), \quad (3.36)$$

both of which vanish as the terms in square brackets vanish. Next consider the off-diagonal elements of the matrix in (3.34).  $U_2(u_-; \lambda)$  and  $U_1(v_+; \lambda)$  are only zero if  $u'_-(0^-) = 0$  and  $v_+(0^+) = 0$ , respectively, since

$$U_1(v_+; \lambda) = \prod_{i=1}^N (\lambda - \gamma_i) v_+(0^+) + \left[ \sum_{i=1}^N \beta_i^2 \prod_{k=1, k \neq i}^N (\lambda - \gamma_k) \right] v'_+(0^+), \quad (3.37)$$

$$U_2(u_-; \lambda) = \prod_{j=1}^M (\lambda - \delta_j) u'_-(0^-) + \left[ \sum_{j=1}^M \alpha_j^2 \prod_{k=1, k \neq j}^M (\lambda - \delta_k) \right] u_-(0^-), \quad (3.38)$$

and the terms in square brackets vanish for  $r(\lambda) = s(\lambda) = 0$ . By P. Binding, P. Browne and B. A. Watson [11]  $U_1(v_+; \lambda)$  and  $U_2(u_-; \lambda)$  only have simple zeroes ( $U_2(u_-; \lambda) = 0$  can be viewed as the eigencondition for the boundary value problem:  $-y'' + qy = \lambda y$  on  $(-a, 0)$  with boundary conditions  $y(-a) \cos \alpha = y'(-a) \sin \alpha$  and  $y'(0) + s(\lambda)y(0) = 0$ , which is of the type considered in [11], and similarly for  $U_1(v_+; \lambda)$ ). For a geometrically double eigenvalue (refer to Theorem 3.2.2) we require both  $u'_-(0^-) = 0$  and  $v_+(0^+) = 0$  at  $\lambda$ , which gives  $U_2(u_-; \lambda) = U_1(v_+; \lambda) = 0$  and results in a zero of order 2 for  $\omega$ . On the other hand, a geometrically simple eigenvalue occurs when  $r(\lambda) = s(\lambda)$  if either:

- I.  $u'_-(0^-) \neq 0$  and  $v_+(0^+) = 0$  (so  $\lambda$  is not an eigenvalue of (3.1) on  $(-a, 0)$  with boundary conditions (3.2) and  $y'(0^-) = 0$ , but is an eigenvalue of (3.1) on  $(0, b)$  with boundary conditions  $y(0^+) = 0$  and (3.3)); or
- II.  $u'_-(0^-) = 0$  and  $v_+(0^+) \neq 0$  (so  $\lambda$  is an eigenvalue of (3.1) on  $(-a, 0)$  with boundary conditions (3.2) and  $y'(0^-) = 0$ , but is not an eigenvalue of (3.1) on  $(0, b)$  with boundary conditions  $y(0^+) = 0$  and (3.3)).

From (3.37)-(3.38),  $u'_-(0^-) \neq 0$  or  $v_+(0^+) \neq 0$  at  $\lambda$  gives  $U_2(u_-; \lambda) \neq 0$  or  $U_1(v_+; \lambda) \neq 0$  respectively, resulting in a zero of order 1 for  $\omega$ .

Finally, suppose that  $r(\lambda) \neq 0$  and  $s(\lambda) \neq 0$ , then we can rewrite (3.34) as

$$\omega(\lambda) = \left[ \sum_{i=1}^N \beta_i^2 \prod_{k=1, k \neq i}^N (\lambda - \gamma_k) \right] \left[ \sum_{j=1}^M \alpha_j^2 \prod_{k=1, k \neq j}^M (\lambda - \delta_k) \right] \det \begin{bmatrix} v_+(0^+) & u_+(0^+) \\ v'_+(0^+) & u'_+(0^+) \end{bmatrix}.$$

Here,  $\begin{bmatrix} u_+(0^+) \\ u'_+(0^+) \end{bmatrix} = T \begin{bmatrix} u_-(0^-) \\ u'_-(0^-) \end{bmatrix}$  at  $\lambda$  with  $T$  defined as in Lemma 3.2.1, and  $u_-$  is extended to a unique solution  $u_+(x; \lambda)$  defined for  $0 < x \leq b$  by the note at the beginning of this section.

Next, assume that  $\lambda$  is an eigenvalue of (3.1)-(3.5), then by Lemma 3.2.1  $\lambda$  is geometrically simple. Since the solution space is one dimensional,  $u_+(x; \lambda)$  and  $v_+(x, \lambda)$  are linearly dependent, hence there exists  $k \neq 0$  such that  $v_+ = ku_+$  at  $\lambda$ . We already know that  $\omega(\lambda) = 0$ , implying that

$$0 = \det \begin{bmatrix} v_+(0^+) & u_+(0^+) \\ v'_+(0^+) & u'_+(0^+) \end{bmatrix} = W[v, u] = \sin \beta u'(b; \lambda) - \cos \beta u(b; \lambda). \quad (3.39)$$

Now suppose that  $\dot{\omega}(\lambda) = 0$ , where the dot indicates differentiation with respect to  $\lambda$ . Differentiating the above expression for  $\omega$  and using (3.39), we deduce that

$$0 = \sin \beta u'_\lambda(b; \lambda) - \cos \beta u_\lambda(b; \lambda).$$

Hence

$$\begin{aligned} [uu'_\lambda - u'u_\lambda](b; \lambda) &= \frac{1}{k} [vu'_\lambda - v'u_\lambda](b; \lambda) \\ &= \frac{1}{k} [\sin \beta u'_\lambda(b; \lambda) - \cos \beta u_\lambda(b; \lambda)] = 0. \end{aligned}$$

Using the fact that  $-u'' + qu = \lambda u$  and  $-u''_\lambda + qu_\lambda = u + \lambda u_\lambda$  we obtain

$$u''u_\lambda - uu''_\lambda = u^2,$$

and integrating by parts gives

$$\int_{-a}^b u^2(t; \lambda) dt = [uu'_\lambda - u'u_\lambda](0^+; \lambda) - [uu'_\lambda - u'u_\lambda](0^-; \lambda) \quad (3.40)$$

(clearly  $[uu'_\lambda - u'u_\lambda](-a; \lambda) = 0$ ). We obtain a contradiction if the right hand side of (3.40) is less than or equal to 0, which is proven in Lemma 3.4.2 below. So  $\omega$  has a zero of order 1 at  $\lambda$ . □

**Lemma 3.4.2.** *Let  $\lambda$  be an eigenvalue of (3.1)-(3.5) with  $r(\lambda) \neq 0$  and  $s(\lambda) \neq 0$ . Then*

(i) *if  $\lambda$  is a pole of both  $r$  and  $s$ , say,  $\lambda = \gamma_n = \delta_m$  for some  $1 \leq n \leq N, 1 \leq m \leq M$ ,*

$$[uu'_\lambda - u'u_\lambda](0^+; \lambda) - [uu'_\lambda - u'u_\lambda](0^-; \lambda) = - \left[ \frac{u'(0^-; \lambda)}{\alpha_m} \right]^2 - \left[ \frac{u(0^+; \lambda)}{\beta_n} \right]^2;$$

(ii) *if  $s(\lambda) \in \mathbb{C} \setminus \{0\}$  and  $\lambda$  is a pole of  $r$ , say  $\lambda = \beta_n$  for some  $1 \leq n \leq N$ ,*

$$[uu'_\lambda - u'u_\lambda](0^+; \lambda) - [uu'_\lambda - u'u_\lambda](0^-; \lambda) = \dot{s}(\lambda) \left[ \frac{u'(0^-; \lambda)}{s(\lambda)} \right]^2 - \left[ \frac{u(0^+; \lambda)}{\beta_n} \right]^2;$$

(iii) *if  $r(\lambda) \in \mathbb{C} \setminus \{0\}$  and  $\lambda$  is a pole of  $s$ , say,  $\lambda = \alpha_m$  for some  $1 \leq m \leq M$ ,*

$$[uu'_\lambda - u'u_\lambda](0^+; \lambda) - [uu'_\lambda - u'u_\lambda](0^-; \lambda) = - \left[ \frac{u'(0^-; \lambda)}{\alpha_m} \right]^2 - \dot{r}(\lambda) \left[ \frac{u(0^+; \lambda)}{r(\lambda)} \right]^2;$$

(iv) *if  $r(\lambda), s(\lambda) \in \mathbb{C} \setminus \{0\}$ ,*

$$[uu'_\lambda - u'u_\lambda](0^+; \lambda) - [uu'_\lambda - u'u_\lambda](0^-; \lambda) = \dot{s}(\lambda) \left[ \frac{u'(0^-; \lambda)}{s(\lambda)} \right]^2 - \dot{r}(\lambda) \left[ \frac{u(0^+; \lambda)}{r(\lambda)} \right]^2.$$

*Proof.* Suppose that  $r(\mu), s(\mu) \in \mathbb{C} \setminus \{0\}$ . The transmission conditions for  $y = u(x; \mu)$  can be rewritten as follows

$$\frac{u(0^+; \mu)}{r(\mu)} - u'(0^+; \mu) = -u'(0^-; \mu), \quad (3.41)$$

$$u(0^+; \mu) = \frac{u'(0^-; \mu)}{s(\mu)} + u(0^-; \mu). \quad (3.42)$$

Differentiating with respect to  $\mu$  we obtain

$$\frac{u_\mu(0^+; \mu)}{r(\mu)} - \frac{\dot{r}(\mu)u(0^+; \mu)}{r^2(\mu)} - u'_\mu(0^+; \mu) = -u'_\mu(0^-; \mu), \quad (3.43)$$

$$u_\mu(0^+; \mu) = \frac{u'_\mu(0^-; \mu)}{s(\mu)} - \frac{\dot{s}(\mu)u'(0^-; \mu)}{s^2(\mu)} + u_\mu(0^-; \mu). \quad (3.44)$$

Multiplying corresponding sides of (3.41) and (3.44), and, similarly, multiplying (3.42) and (3.43), and subtracting the results we get:

$$\begin{aligned} & [uu'_\mu - u'u_\mu](0^+; \mu) - [uu'_\mu - u'u_\mu](0^-; \mu) \\ &= \frac{\dot{s}(\mu)}{s^2(\mu)} [u'(0^-; \mu)]^2 - \frac{\dot{r}(\mu)}{r^2(\mu)} [u(0^+; \mu)]^2. \end{aligned}$$

Taking limits as  $\mu \rightarrow \lambda$  yields the results stated in (i) – (iv).  $\square$

### 3.5 The Green's function and resolvent operator

Let  $u(x; \lambda)$  and  $v(x; \lambda)$  be defined by (3.28) and (3.29) as in the previous section. Let  $\psi(\lambda) = W[u, v]$  denote the Wronskian of  $u$  and  $v$ . Clearly  $W[u, v]$  is independent of  $x$  on  $[-a, 0)$  and  $(0, b]$ . Because of the nature of the transmission conditions it is also easy to check that the value of the Wronskian at  $0^-$  and  $0^+$  is equal.

**Theorem 3.5.1.** *The Green's function of (3.1)-(3.5) is given by*

$$G(x, t; \lambda) = \begin{cases} \frac{u(x; \lambda)v(t; \lambda)}{\psi(\lambda)}, & \text{if } x < t \text{ and } x, t \in [-a, 0) \cup (0, b], \\ \frac{u(t; \lambda)v(x; \lambda)}{\psi(\lambda)}, & \text{if } t < x \text{ and } x, t \in [-a, 0) \cup (0, b], \end{cases} \quad (3.45)$$

in the sense that if  $f \in L^2(-a, b)$  and  $\lambda$  is not an eigenvalue of (3.1)-(3.5) then

$$g(x; \lambda) = \int_{-a}^b G(x, t; \lambda) f(t) dt := \mathfrak{G}f \quad (3.46)$$

is a solution of  $(\lambda - \ell)g = f$  on  $[-a, 0)$  and  $(0, b]$ , and, moreover,  $g$  obeys the boundary conditions (3.2)-(3.3) and the transmission conditions (3.4)-(3.5).

*Proof.* From the above definition of  $G$  and  $g$ , we have

$$g(x; \lambda)\psi(\lambda) = u(x; \lambda) \int_x^b v(t; \lambda) f(t) dt + v(x; \lambda) \int_{-a}^x u(t; \lambda) f(t) dt. \quad (3.47)$$

Differentiating  $g$  with respect to  $x$  gives

$$g'(x; \lambda)\psi(\lambda) = u'(x; \lambda) \int_x^b v(t; \lambda)f(t) dt + v'(x; \lambda) \int_{-a}^x u(t; \lambda)f(t) dt, \quad (3.48)$$

and a further differentiation gives

$$\begin{aligned} g''(x; \lambda)\psi(\lambda) &= u''(x; \lambda) \int_x^b v(t; \lambda)f(t) dt + v''(x; \lambda) \int_{-a}^x u(t; \lambda)f(t) dt + \psi(\lambda)f(x) \\ &= (g(x) - \lambda)g'(x; \lambda)\psi(\lambda) + \psi(\lambda)f(x). \end{aligned}$$

So  $(\lambda - \ell)g = f$ . Further, for (3.47) and (3.48)

$$\begin{bmatrix} g(x; \lambda) \\ g'(x; \lambda) \end{bmatrix} \psi(\lambda) = \begin{bmatrix} u(x; \lambda) \\ u'(x; \lambda) \end{bmatrix} \int_x^b v(t; \lambda)f(t) dt + \begin{bmatrix} v(x; \lambda) \\ v'(x; \lambda) \end{bmatrix} \int_{-a}^x u(t; \lambda)f(t) dt,$$

from which it follows that

$$\begin{bmatrix} g(-a; \lambda) \\ g'(-a; \lambda) \end{bmatrix} \psi(\lambda) = \begin{bmatrix} u(-a; \lambda) \\ u'(-a; \lambda) \end{bmatrix} \int_{-a}^b v(t; \lambda)f(t) dt,$$

so (3.2) is obeyed as this condition is obeyed by  $u$ , and

$$\begin{bmatrix} g(b; \lambda) \\ g'(b; \lambda) \end{bmatrix} \psi(\lambda) = \begin{bmatrix} v(b; \lambda) \\ v'(b; \lambda) \end{bmatrix} \int_{-a}^b u(t; \lambda)f(t) dt,$$

so (3.3) is obeyed as this condition is obeyed by  $v$ . Further, if  $\lambda$  is not a zero of  $r$  or  $s$  then

$$\begin{bmatrix} g(0^\pm; \lambda) \\ g'(0^\pm; \lambda) \end{bmatrix} \psi = \begin{bmatrix} u(0^\pm; \lambda) \\ u'(0^\pm; \lambda) \end{bmatrix} \int_0^b v(t; \lambda)f(t) dt + \begin{bmatrix} v(0^\pm; \lambda) \\ v'(0^\pm; \lambda) \end{bmatrix} \int_{-a}^0 u(t; \lambda)f(t) dt,$$

so (3.4) and (3.5) are obeyed as these conditions are obeyed by  $u$  and  $v$ . If either  $r(\lambda) = 0$  or  $s(\lambda) = 0$  then more careful analysis is required. We present only the case of  $r(\lambda) = s(\lambda) = 0$  as remaining cases are similar. In this case, the transmission conditions (3.4) and (3.5) reduce to  $y(0^+; \lambda) = 0$  and  $y'(0^-; \lambda) = 0$  respectively. Then  $v(0^+) \neq 0$  and  $u'(0^-) \neq 0$  at  $\lambda$ , otherwise choosing either  $y(x; \lambda) = \chi_{(0,b]}v(x; \lambda)$  or  $y(x; \lambda) = \chi_{[-a,0)}u(x; \lambda)$  would give a solution of (3.1) obeying (3.2)-(3.5) at  $\lambda$ , contradicting  $\lambda$  not an eigenvalue of (3.1)-(3.5). Using (3.24) and (3.25) with  $\lambda$  replaced by  $\mu$ , and taking limits as  $\mu \rightarrow \lambda$  we find that

$$\begin{aligned} r(\mu)s(\mu)u(x; \mu) &\rightarrow u'(0^-; \lambda)w_2(x; \lambda), & 0 < x \leq b, \\ r(\mu)s(\mu)v(x; \mu) &\rightarrow v(0^+; \lambda)w_1(x; \lambda), & -a \leq x < 0, \end{aligned}$$

giving

$$\begin{aligned} r(\mu)s(\mu)\psi(\mu) &= r(\mu)s(\mu) [u(0^\pm; \mu)v'(0^\pm; \mu) - u'(0^\pm; \mu)v(0^\pm; \mu)] \\ &\rightarrow -u'(0^-; \lambda)v(0^+; \lambda) \end{aligned}$$

as  $\mu \rightarrow \lambda$ . Thus for  $-a \leq x < 0$ ,

$$\begin{aligned} g(x; \lambda) &= \lim_{\mu \rightarrow \lambda} \left[ u(x; \mu) \int_x^b \frac{r(\mu)s(\mu)v(t; \mu)}{r(\mu)s(\mu)\psi(\mu)} f(t) dt + \frac{r(\mu)s(\mu)v(x; \mu)}{r(\mu)s(\mu)\psi(\mu)} \int_{-a}^x u(t; \mu)f(t) dt \right] \\ &= -\frac{u(x; \lambda)}{u'(0^-; \lambda)} \int_x^0 w_1(t; \lambda)f(t) dt - w_1(x; \lambda) \int_{-a}^x \frac{u(t; \lambda)}{u'(0^-; \lambda)} f(t) dt. \end{aligned}$$

So  $g'(0^-; \lambda) = 0$ . Similarly, for  $0, x \leq b$ ,

$$g(x; \lambda) = -w_2(x; \lambda) \int_x^b \frac{v(t; \lambda)}{v(0^+; \lambda)} f(t) dt - \frac{v(x; \lambda)}{v(0^+; \lambda)} \int_0^x w_2(t; \lambda) f(t) dt,$$

giving  $g(0^+; \lambda) = 0$ . □

**Theorem 3.5.2.** *Let  $\lambda \in \mathbb{C}$  be different from the eigenvalues of (3.1)-(3.5) as well as the zeroes and poles of  $r$  and  $s$ . Then the resolvent to  $L$  at  $\lambda$  is given by  $(\lambda - L)^{-1}F = Y$ , where*

$$Y(x; \lambda) = \left[ \begin{array}{c} \int_{-a}^b G(x, t; \lambda) f(t) dt + A(\lambda) \chi_{[-a, 0)}(x) u(x; \lambda) + B(\lambda) \chi_{(0, b]}(x) v(x; \lambda) \\ \left( \frac{f_i^1 + \beta_i \Delta y'}{\lambda - \gamma_i} \right) \\ \left( \frac{f_j^2 + \alpha_j \Delta y}{\lambda - \delta_j} \right) \end{array} \right],$$

with

$$A(\lambda) = \frac{1}{\psi(\lambda)} \sum_{i=1}^N \left[ \frac{\beta_i}{\lambda - \gamma_i} \Delta v' \right] f_i^1 + \frac{1}{\psi(\lambda)} \sum_{j=1}^M \left[ \frac{\alpha_j}{\lambda - \delta_j} \Delta v \right] f_j^2, \quad (3.49)$$

$$B(\lambda) = \frac{1}{\psi(\lambda)} \sum_{i=1}^N \left[ \frac{\beta_i}{\lambda - \gamma_i} \Delta u' \right] f_i^1 + \frac{1}{\psi(\lambda)} \sum_{j=1}^M \left[ \frac{\alpha_j}{\lambda - \delta_j} \Delta u \right] f_j^2, \quad (3.50)$$

and  $G(x, t; \lambda)$  as in Theorem 3.5.1.

*Proof.* Let  $\lambda$  be different from all eigenvalues and zeroes and poles of  $r$  or  $s$ . Consider

$$(\lambda - L)Y = F. \quad (3.51)$$

For the  $L^2$  component, the general solution to the above equation is given by

$$y(x; \lambda) = g(x; \lambda) + A(\lambda) \chi_{[-a, 0)} u(x; \lambda) + B(\lambda) \chi_{(0, b]} v(x; \lambda), \quad (3.52)$$

for some  $A(\lambda), B(\lambda)$ . Further, we require that

$$\lambda y_i^1 - (\gamma_i y_i^1 + \beta_i \Delta y') = f_i^1, \quad i = \overline{1, N},$$

$$\lambda y_j^2 - (\delta_j y_j^2 + \alpha_j \Delta y) = f_j^2, \quad j = \overline{1, M}.$$

Since  $Y \in \mathcal{D}(L)$ ,

$$\begin{aligned} -B(\lambda)v(0^+; \lambda) - g(0^+; \lambda) &= -y(0^+) \\ &= \sum_{i=1}^N \beta_i y_i^1 \\ &= \sum_{i=1}^N \frac{\beta_i}{\lambda - \gamma_i} [f_i^1 + \beta_i \Delta y'] \\ &= \sum_{i=1}^N \frac{\beta_i}{\lambda - \gamma_i} f_i^1 - r(\lambda) [B(\lambda)v'(0^+; \lambda) - A(\lambda)u'(0^-; \lambda) + \Delta g'] \end{aligned}$$

and

$$\begin{aligned}
A(\lambda)u'(0^-; \lambda) + g'(0^-; \lambda) &= y'(0^-) \\
&= \sum_{j=1}^M \alpha_j y_j^2 \\
&= \sum_{j=1}^M \frac{\alpha_j}{\lambda - \delta_j} [f_j^2 + \alpha_j \Delta y] \\
&= \sum_{j=1}^M \frac{\alpha_j}{\lambda - \delta_j} f_j^2 + s(\lambda) [B(\lambda)v(0^+; \lambda) - A(\lambda)u(0^-; \lambda) + \Delta g].
\end{aligned}$$

So

$$\begin{aligned}
&\begin{bmatrix} -r(\lambda)u'(0^-; \lambda) & r(\lambda)v'(0^+; \lambda) - v(0^+; \lambda) \\ u'(0^-; \lambda) + s(\lambda)u(0^-; \lambda) & -s(\lambda)v(0^+; \lambda) \end{bmatrix} \begin{bmatrix} A(\lambda) \\ B(\lambda) \end{bmatrix} \\
&= \begin{bmatrix} \sum_{i=1}^N \frac{\beta_i}{\lambda - \gamma_i} f_i^1 + g(0^+; \lambda) - r(\lambda)\Delta g' \\ \sum_{j=1}^M \frac{\alpha_j}{\lambda - \delta_j} f_j^2 + s(\lambda)\Delta g - g'(0^-; \lambda) \end{bmatrix}.
\end{aligned}$$

The determinant of the above system is

$$-r(\lambda)s(\lambda) \det \left[ \begin{bmatrix} 1 & \frac{1}{s(\lambda)} \\ \frac{1}{r(\lambda)} & 1 + \frac{1}{r(\lambda)s(\lambda)} \end{bmatrix} \begin{bmatrix} u(0^-; \lambda) \\ u'(0^-; \lambda) \end{bmatrix} : \begin{bmatrix} v(0^+; \lambda) \\ v'(0^+; \lambda) \end{bmatrix} \right] = -r(\lambda)s(\lambda)\psi(\lambda),$$

giving

$$\begin{aligned}
A(\lambda)\psi(\lambda) &= \frac{v(0^+; \lambda)}{r(\lambda)} \left[ \sum_{i=1}^N \frac{\beta_i}{\lambda - \gamma_i} f_i^1 + g(0^+; \lambda) - r(\lambda)\Delta g' \right] \\
&+ \frac{1}{s(\lambda)} \left[ v'(0^+; \lambda) - \frac{v(0^+; \lambda)}{r(\lambda)} \right] \left[ \sum_{j=1}^M \frac{\alpha_j}{\lambda - \delta_j} f_j^2 + s(\lambda)\Delta g - g'(0^-; \lambda) \right]
\end{aligned}$$

and

$$\begin{aligned}
B(\lambda)\psi(\lambda) &= \frac{1}{r(\lambda)} \left[ u(0^-; \lambda) + \frac{u'(0^-; \lambda)}{s(\lambda)} \right] \left[ \sum_{i=1}^N \frac{\beta_i}{\lambda - \gamma_i} f_i^1 + g(0^+; \lambda) - r(\lambda)\Delta g' \right] \\
&+ \frac{u'(0^-; \lambda)}{s(\lambda)} \left[ \sum_{j=1}^M \frac{\alpha_j}{\lambda - \delta_j} f_j^2 + s(\lambda)\Delta g - g'(0^-; \lambda) \right].
\end{aligned}$$

But

$$\begin{aligned}
v'(0^-; \lambda) &= v'(0^+; \lambda) - \frac{v(0^+; \lambda)}{r(\lambda)}, \\
v(0^-; \lambda) &= v(0^+; \lambda) - \frac{1}{s(\lambda)} \left[ v'(0^+; \lambda) - \frac{v(0^+; \lambda)}{r(\lambda)} \right]
\end{aligned}$$

and

$$u(0^+; \lambda) = u(0^-; \lambda) + \frac{u'(0^-; \lambda)}{s(\lambda)},$$

$$u'(0^+; \lambda) = u'(0^-; \lambda) + \frac{1}{r(\lambda)} \left[ u(0^-; \lambda) + \frac{u'(0^-; \lambda)}{s(\lambda)} \right].$$

Moreover,  $g(0^+; \lambda) - r(\lambda)\Delta g' = 0$  and  $s(\lambda)\Delta g - g'(0^-; \lambda) = 0$ . Thus

$$A(\lambda) = \frac{1}{\psi(\lambda)} \sum_{i=1}^N \left[ \frac{\beta_i}{\lambda - \gamma_i} \Delta v' \right] f_i^1 + \frac{1}{\psi(\lambda)} \sum_{j=1}^M \left[ \frac{\alpha_j}{\lambda - \delta_j} \Delta v \right] f_j^2 \quad (3.53)$$

and

$$B(\lambda) = \frac{1}{\psi(\lambda)} \sum_{i=1}^N \left[ \frac{\beta_i}{\lambda - \gamma_i} \Delta u' \right] f_i^1 + \frac{1}{\psi(\lambda)} \sum_{j=1}^M \left[ \frac{\alpha_j}{\lambda - \delta_j} \Delta u \right] f_j^2, \quad (3.54)$$

giving  $(\lambda - L)^{-1}F = Y$  with

$$Y(x; \lambda) = \begin{bmatrix} \int_{-a}^b G(x, t; \lambda) f(t) dt + A(\lambda) \chi_{[-a, 0]} u(x; \lambda) + B(\lambda) \chi_{(0, b]} v(x; \lambda) \\ \left( \frac{f_i^1 + \beta_i \Delta y'}{\lambda - \gamma_i} \right) \\ \left( \frac{f_j^2 + \alpha_j \Delta y}{\lambda - \delta_j} \right) \end{bmatrix}$$

$$= \begin{bmatrix} \mathfrak{G}f + \frac{\chi_{[-a, 0]} u \Delta v' + \chi_{(0, b]} v \Delta u'}{\psi(\lambda)} \sum_{i=1}^N \frac{\beta_i f_i^1}{\lambda - \gamma_i} + \frac{\chi_{[-a, 0]} u \Delta v + \chi_{(0, b]} v \Delta u}{\psi(\lambda)} \sum_{j=1}^M \frac{\alpha_j f_j^2}{\lambda - \delta_j} \\ \left( \frac{f_i^1 + \beta_i \Delta y'}{\lambda - \gamma_i} \right) \\ \left( \frac{f_j^2 + \alpha_j \Delta y}{\lambda - \delta_j} \right) \end{bmatrix}. \quad (3.55)$$

Note, this solution can be extended to zeroes and poles of  $r$  or  $s$  (not coinciding with eigenvalues of (3.1)-(3.5)) by using (3.24) and (3.25) and taking appropriate limits.  $\square$

We will refer to the functional component,  $[Y]_0$ , of the resolvent operator  $Y = (\lambda - L)^{-1}F$  as the Green's operator corresponding to  $\ell$ . Note the relationship between the Green's function and the Green's operator. The Green's function is the kernel of the integration operator  $g(x; \lambda)$ , and  $g$  together with a finite summation gives us the Green's operator corresponding to  $\ell$ .

### 3.6 Eigenvalue asymptotics

The following result is a direct consequence of Theorem 3.7.1 (see the appendix of this chapter).

**Theorem 3.6.1.** *Let  $\eta = |\Im(\sqrt{\lambda})|$  and let  $u(x; \lambda)$ ,  $v(x; \lambda)$  and  $\omega(\lambda)$  be defined as in equations (3.28), (3.29) and (3.30) respectively. Let  $q_1(x)$  and  $q_2(x)$  be defined as in Theorem 3.7.1. Then as  $|\lambda| \rightarrow \infty$  the following asymptotics are valid.*



If  $\alpha = 0$  and  $\beta = \pi$ ,

$$\begin{aligned}
\omega(\lambda) &= \lambda^{N+M} \cos \sqrt{\lambda} a \frac{\sin \sqrt{\lambda} b}{\sqrt{\lambda}} \\
&+ \lambda^{N+M} \left[ q_1(0) \frac{\sin \sqrt{\lambda} a}{\sqrt{\lambda}} + \frac{1}{2} \int_{-a}^0 \frac{\sin \sqrt{\lambda}(2t+a)}{\sqrt{\lambda}} q(t) dt \right] \frac{\sin \sqrt{\lambda} b}{\sqrt{\lambda}} \\
&- \lambda^{N+M-1} \cos \sqrt{\lambda} a \left[ \left[ q_2(0) + \sum_{i=1}^N \beta_i^2 \right] \cos \sqrt{\lambda} b - \frac{1}{2} \int_0^b \cos \sqrt{\lambda}(2t-b) q(t) dt \right] \\
&+ O(\lambda^{N+M-3/2} e^{\eta(a+b)}). \tag{3.56}
\end{aligned}$$

If  $\alpha \neq 0$  and  $\beta = \pi$ ,

$$\begin{aligned}
\omega(\lambda) &= -\lambda^{N+M+1} \sin \alpha \frac{\sin \sqrt{\lambda} a}{\sqrt{\lambda}} \frac{\sin \sqrt{\lambda} b}{\sqrt{\lambda}} \\
&+ \lambda^{N+M} \sin \alpha \frac{\sin \sqrt{\lambda} a}{\sqrt{\lambda}} \left[ \left[ q_2(0) + \sum_{i=1}^N \beta_i^2 \right] \cos \sqrt{\lambda} b - \frac{1}{2} \int_0^b \cos \sqrt{\lambda}(2t-b) q(t) dt \right] \\
&+ \lambda^{N+M} \sin \alpha \left[ [q_1(0) + \cot \alpha] \cos \sqrt{\lambda} a + \frac{1}{2} \int_{-a}^0 \cos \sqrt{\lambda}(2t+a) q(t) dt \right] \frac{\sin \sqrt{\lambda} b}{\sqrt{\lambda}} \\
&+ O(\lambda^{N+M-1} e^{\eta(a+b)}). \tag{3.57}
\end{aligned}$$

If  $\alpha = 0$  and  $\beta \neq \pi$ ,

$$\begin{aligned}
\omega(\lambda) &= \lambda^{N+M} \sin \beta \cos \sqrt{\lambda} a \cos \sqrt{\lambda} b \\
&+ \lambda^{N+M} \sin \beta \cos \sqrt{\lambda} a \left[ \left[ q_2(0) - \cot \beta + \sum_{i=1}^N \beta_i^2 \right] \frac{\sin \sqrt{\lambda} b}{\sqrt{\lambda}} \right. \\
&\quad \left. + \frac{1}{2} \int_0^b \frac{\sin \sqrt{\lambda}(2t-b)}{\sqrt{\lambda}} q(t) dt \right] \\
&+ \lambda^{N+M} \sin \beta \left[ q_1(0) \frac{\sin \sqrt{\lambda} a}{\sqrt{\lambda}} + \frac{1}{2} \int_{-a}^0 \frac{\sin \sqrt{\lambda}(2t+a)}{\sqrt{\lambda}} q(t) dt \right] \cos \sqrt{\lambda} b \\
&+ O(\lambda^{N+M-1} e^{\eta(a+b)}). \tag{3.58}
\end{aligned}$$

If  $\alpha \neq 0$  and  $\beta \neq \pi$ ,

$$\begin{aligned}
\omega(\lambda) &= -\lambda^{N+M+1} \sin \alpha \sin \beta \frac{\sin \sqrt{\lambda} a}{\sqrt{\lambda}} \cos \sqrt{\lambda} b \\
&- \lambda^{N+M+1} \sin \alpha \sin \beta \frac{\sin \sqrt{\lambda} a}{\sqrt{\lambda}} \left[ \left[ q_2(0) - \cot \beta + \sum_{i=1}^N \beta_i^2 \right] \frac{\sin \sqrt{\lambda} b}{\sqrt{\lambda}} \right. \\
&\quad \left. + \frac{1}{2} \int_0^b \frac{\sin \sqrt{\lambda}(2t-b)}{\sqrt{\lambda}} q(t) dt \right] \\
&+ \lambda^{N+M} \sin \alpha \sin \beta \left[ [q_1(0) + \cot \alpha] \cos \sqrt{\lambda} a + \frac{1}{2} \int_{-a}^0 \cos \sqrt{\lambda}(2t+a) q(t) dt \right] \cos \sqrt{\lambda} b \\
&+ O(\lambda^{N+M-1/2} e^{\eta(a+b)}). \tag{3.59}
\end{aligned}$$

**Theorem 3.6.2.** Suppose that  $a$  and  $b$  in equations (3.2) and (3.3) are rationally related. That is,

$$\frac{a}{b} = \frac{c}{d}, \quad \text{for some } c, d \in \mathbb{N}, \quad \gcd(c, d) = 1.$$

Then there are constants  $\kappa, \kappa_1, \kappa_2, N_0 \in \mathbb{N}$  with  $N_0$  sufficiently large such that

$$\left\{ \sqrt{\lambda_n} : n \geq N_0 \right\} = \bigcup_{k=N_0}^{\infty} \Sigma^k,$$

where

$$\Sigma^k := \left\{ \sqrt{\lambda_n} : 0 \leq n - (k(c+d) + \kappa) \leq c+d-1 \right\}$$

and each  $\Sigma^k$  is the disjoint union<sup>2</sup>

$$\Sigma^k = \sigma_1^k \dot{\cup} \sigma_2^k$$

with

$$\begin{aligned} \sigma_1^k &= \left\{ s_n^1 : 0 \leq n - (kc + \kappa_1) \leq c-1 \right\}, \\ \sigma_2^k &= \left\{ s_n^2 : 0 \leq n - (kd + \kappa_2) \leq d-1 \right\}, \end{aligned}$$

and

$$\begin{aligned} s_n^1 &= \begin{cases} \frac{(n+1/2)\pi}{a} + \frac{q_1(0)}{n\pi} + \frac{1}{2n\pi} \int_{-a}^0 \cos \frac{(2n+1)\pi t}{a} q(t) dt + O\left(\frac{1}{n^2}\right), & \text{if } \alpha = 0, \\ \frac{n\pi}{a} + \frac{q_1(0) + \cot \alpha}{n\pi} + \frac{1}{2n\pi} \int_{-a}^0 \cos \frac{2n\pi t}{a} q(t) dt + O\left(\frac{1}{n^2}\right), & \text{if } \alpha \in (0, \pi), \end{cases} \\ s_n^2 &= \begin{cases} \frac{n\pi}{b} + \frac{q_2(0) + \sum_{i=1}^N \beta_i^2}{n\pi} + \frac{1}{2n\pi} \int_0^b \cos \frac{2n\pi t}{b} q(t) dt + O\left(\frac{1}{n^2}\right), & \text{if } \beta = \pi, \\ \frac{n+1/2\pi}{b} + \frac{q_2(0) + \sum_{i=1}^N \beta_i^2 - \cot \beta}{n\pi} + \frac{1}{2n\pi} \int_0^b \cos \frac{(2n+1)\pi t}{b} q(t) dt + O\left(\frac{1}{n^2}\right), & \text{if } \beta \in (0, \pi). \end{cases} \end{aligned}$$

Moreover, for each  $\sqrt{\lambda_n} \in \Sigma^k$  we have

$$\left| \sqrt{\lambda_n} - \sqrt{\lambda_{k(c+d)+\kappa+\lfloor \frac{c+d-1}{2} \rfloor}} \right| < \min \left\{ \frac{(c+1/2)\pi}{2a}, \frac{(d+1/2)\pi}{2b} \right\}.$$

*Proof.* We prove in detail only the case for  $\alpha = 0, \beta = \pi$  as the remaining proofs are similar. In this case, as  $|\Im(\sqrt{\lambda})| \rightarrow \infty$ ,

$$\omega(\lambda) = f(\lambda) + g(\lambda) + O(\lambda^{N+M-3/2} e^{\eta(a+b)}),$$

where

$$\begin{aligned} f(\lambda) &= \lambda^{N+M} \cos \sqrt{\lambda} a \frac{\sin \sqrt{\lambda} b}{\sqrt{\lambda}}, \\ g(\lambda) &= \lambda^{N+M} \left[ q_1(0) \frac{\sin \sqrt{\lambda} a}{\sqrt{\lambda}} + \frac{1}{2} \int_{-a}^0 \frac{\sin \sqrt{\lambda}(2t+a)}{\sqrt{\lambda}} q(t) dt \right] \frac{\sin \sqrt{\lambda} b}{\sqrt{\lambda}} \\ &\quad - \lambda^{N+M-1} \cos \sqrt{\lambda} a \left[ \left[ q_2(0) + \sum_{i=1}^N \beta_i^2 \right] \cos \sqrt{\lambda} b - \frac{1}{2} \int_0^b \cos \sqrt{\lambda}(2t-b) q(t) dt \right] \end{aligned}$$

<sup>2</sup>In set theory, disjoint union is different from the usual union operation in that it does not identify the common elements from different sets. An easy way to define the disjoint union of two sets  $A$  and  $B$  is to define  $A \dot{\cup} B = A \times \{0\} \cup B \times \{1\}$ . Thus each element of  $A \dot{\cup} B$  is identified with an element either of  $A$  or of  $B$  and is labelled using the notation of the set from which it comes.

Then, for sufficiently large  $k$ ,  $|g| < |f|$  for  $\lambda \in \Gamma_k = \Gamma_k^- \cup \Gamma_k^0 \cup \Gamma_k^+$ , where

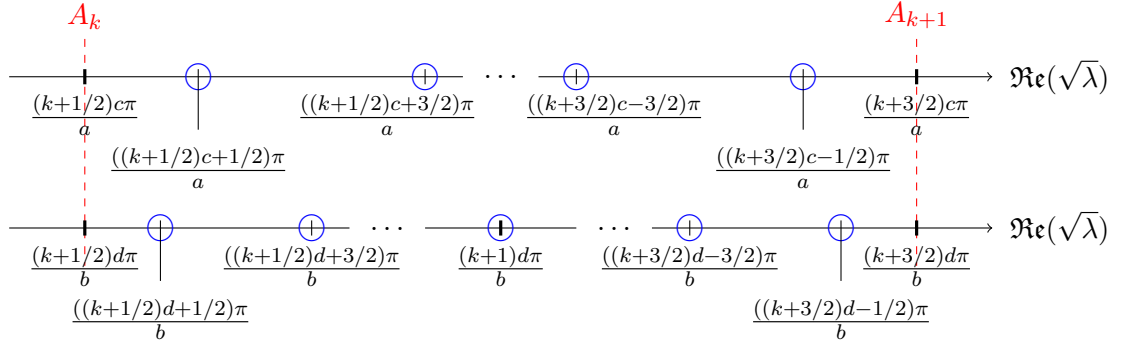
$$\begin{aligned}\Gamma_k^- &= \{(\zeta - iA_k)^2 : \zeta \in [0, A_k]\}, \\ \Gamma_k^0 &= \{(A_k + i\zeta)^2 : \zeta \in [-A_k, A_k]\}, \\ \Gamma_k^+ &= \{(\zeta + iA_k)^2 : \zeta \in [0, A_k]\},\end{aligned}$$

and

$$A_k = \begin{cases} \frac{(k+1/2)c\pi}{a} = \frac{(k+1/2)d\pi}{b}, & \text{if } c \text{ even, } d \text{ odd,} \\ \min \left\{ \frac{((k+1/2)c+3/4)\pi}{a}, \frac{((k+1/2)d+1/4)\pi}{b} \right\}, & \text{if } c \text{ odd, } d \text{ odd,} \\ \min \left\{ \frac{((k+1/2)c+1/4)\pi}{a}, \frac{((k+1/2)d+1/4)\pi}{b} \right\}, & \text{if } c \text{ odd, } d \text{ even.} \end{cases}$$

By Rouché's Theorem  $\omega(\lambda)$  and  $f(\lambda)$  have the same number of zeroes inside  $\Gamma_k$ .

(i) If  $c$  is even and  $d$  is odd,



then by Rouché's Theorem there are a total of

$$\left(kc + \frac{c-2}{2} + 1\right) + \left(kd + \frac{d-1}{2}\right) + M + N = k(c+d) + \frac{c+d-1}{2} + M + N$$

zeroes of  $\omega(\lambda)$  inside  $\Gamma_k$ . Moreover,  $\Gamma_{k+1}$  encloses an additional  $c+d$  zeroes. Let

$$\Sigma^k := \left\{ \sqrt{\lambda_n} : 0 \leq n - k(c+d) - \frac{c+d-1}{2} - M - N \leq c+d-1 \right\}$$

for  $k \in \mathbb{N}$ ,  $k \geq N_0$  with  $N_0$  sufficiently large. Taking small loops about each zero of  $f(\lambda)$  for  $\Re(\sqrt{\lambda})$  large we see that  $\Sigma^k$  can be decomposed into the disjoint union  $\sigma_1^k \dot{\cup} \sigma_2^k$  where

$$\begin{aligned}\sigma_1^k &= \{s_n^1 : 0 \leq n - (k+1/2)c \leq c-1\}, \\ \sigma_2^k &= \{s_n^2 : 0 \leq n - [(k+1/2)d+1/2] \leq d-1\}\end{aligned}$$

for each  $k \geq N_0$  and the  $s_n^1, s_n^2$  are given asymptotically by

$$\begin{aligned}s_n^1 &= \frac{(n+1/2)\pi}{a} + O\left(\frac{1}{n}\right), \\ s_n^2 &= \frac{n\pi}{b} + O\left(\frac{1}{n}\right).\end{aligned}$$

Substituting these estimates into equation (3.56) and bootstrapping we obtain

$$s_n^1 = \frac{(n+1/2)\pi}{a} + \frac{q_1(0)}{n\pi} + \frac{1}{2n\pi} \int_{-a}^0 \cos \frac{(2n+1)\pi t}{a} q(t) dt + O\left(\frac{1}{n^2}\right), \quad (3.60)$$

$$s_n^2 = \frac{n\pi}{b} + \frac{q_2(0) + \sum_{i=1}^N \beta_i^2}{n\pi} + \frac{1}{2n\pi} \int_0^b \cos \frac{2n\pi t}{b} q(t) dt + O\left(\frac{1}{n^2}\right). \quad (3.61)$$

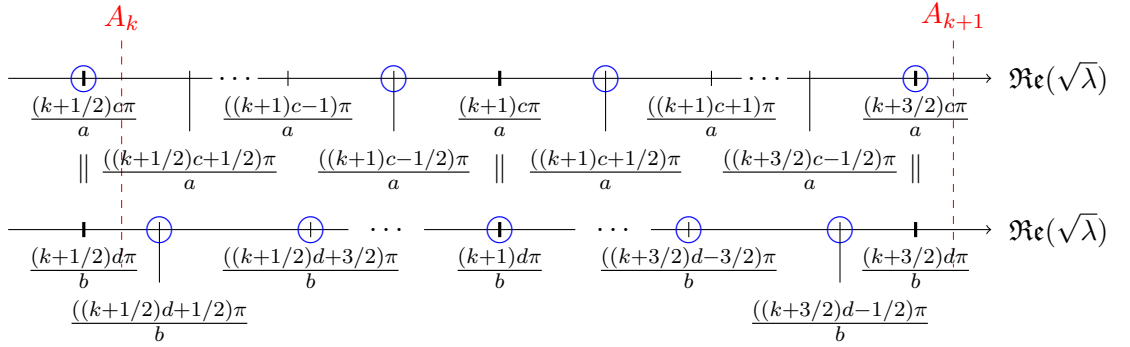
Finally, we observe that for  $\sqrt{\lambda_n} \in \Sigma^k$ , we have

$$\left| \sqrt{\lambda_n} - s_{(k+1)d}^2 \right| < \frac{d\pi}{2b} = \frac{1}{2} \left[ \frac{(k+3/2)d\pi}{b} - \frac{(k+1/2)d\pi}{b} \right] = \frac{A_{k+1} - A_k}{2}$$

where  $s_{(k+1)d}^2 = \sqrt{\lambda_{(k+1)(c+d)+M+N-1}}$  and as a result  $\sqrt{\lambda_{(k+1)(c+d)+M+N-1}}$  is given asymptotically by

$$\sqrt{\lambda_{(k+1)(c+d)+M+N-1}} = \frac{(k+1)d\pi}{b} + O\left(\frac{1}{k+1}\right).$$

(ii) If both  $c$  and  $d$  are odd,



then  $\Gamma_k$  encloses

$$\left( kc + \frac{c+1}{2} \right) + \left( kd + \frac{d-1}{2} \right) + M + N = \left( k + \frac{1}{2} \right) (c+d) + M + N$$

zeroes of  $\omega(\lambda)$  and clearly  $\Gamma_{k+1}$  encloses an additional  $c+d$  zeroes. Let

$$\Sigma^k = \left\{ \sqrt{\lambda_n} : 0 \leq n - (k+1/2)(c+d) - M - N \leq c+d-1 \right\}$$

for  $k \in \mathbb{N}$ ,  $k \geq N_0$ . Then  $\Sigma^k = \sigma_1^k \cup \sigma_2^k$  with

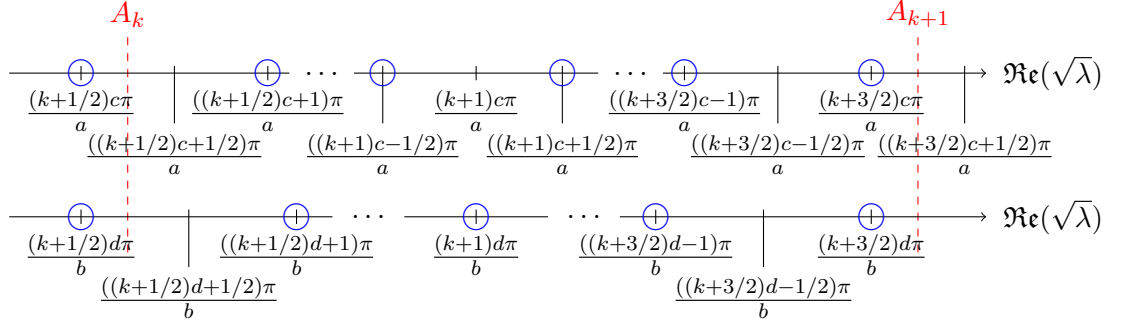
$$\begin{aligned} \sigma_1^k &= \left\{ s_n^1 : 0 \leq n - [(k+1/2)c + 1/2] \leq c-1 \right\}, \\ \sigma_2^k &= \left\{ s_n^2 : 0 \leq n - [(k+1/2)d + 1/2] \leq d-1 \right\} \end{aligned}$$

and  $s_n^1, s_n^2$  given asymptotically by (3.60), (3.61). For  $\sqrt{\lambda_n} \in \Sigma^k$ ,

$$\left| \sqrt{\lambda_n} - s_{(k+1)d}^2 \right| < \min \left\{ \frac{d+1/2\pi}{2b}, \frac{c+1/2\pi}{2a} \right\},$$

where  $s_{(k+1)d}^2 = \sqrt{\lambda_{(k+1)(c+d)+M+N-1}}$ .

(iii) If  $c$  is odd and  $d$  is even,



then  $\Gamma_k$  encloses

$$\left(kc + \frac{c+1}{2}\right) + \left(kd + \frac{d}{2}\right) + M + N = k(c+d) + \frac{c+d+1}{2} + M + N$$

zeroes of  $\omega(\lambda)$ . Let

$$\Sigma^k = \left\{ \sqrt{\lambda_n} : 0 \leq n - k(c+d) - \frac{c+d+1}{2} - M - N \leq c+d-1 \right\}$$

for  $k \geq N_0$  with  $N_0 \in \mathbb{N}$  sufficiently large. Then  $\Sigma^k = \sigma_1^k \cup \sigma_2^k$  where

$$\sigma_1^k = \left\{ s_n^1 : 0 \leq n - kc - \frac{c+1}{2} \leq c-1 \right\},$$

$$\sigma_2^k = \left\{ s_n^2 : 0 \leq n - kd - \frac{d}{2} - 1 \leq d-1 \right\}$$

and  $s_n^1$  and  $s_n^2$  are given asymptotically as above. Finally, for  $\sqrt{\lambda_n} \in \Sigma^k$  we have

$$\left| \sqrt{\lambda_n} - s_{(k+1)d}^2 \right| < \min \left\{ \frac{d+1/2\pi}{2b}, \frac{c+1/2\pi}{2a} \right\}$$

and

$$\left| \sqrt{\lambda_n} - s_{(k+1)c}^1 \right| < \frac{c\pi}{2a} = \frac{d\pi}{2b},$$

where  $s_{(k+1)d}^2 = \sqrt{\lambda_{(k+1)(c+d)+M+N-1}}$  and  $s_{(k+1)c}^1 = \sqrt{\lambda_{(k+1)(c+d)+M+N}}$

□

### 3.7 Appendix - Initial value solution asymptotics

**Theorem 3.7.1.** Let  $\eta = |\Im(\sqrt{\lambda})|$  and define

$$q_1(x) = \frac{1}{2} \int_{-a}^x q(t) dt, \quad q_2(x) = \frac{1}{2} \int_x^b q(t) dt. \quad (3.62)$$

Then, as  $|\lambda| \rightarrow \infty$ , the following asymptotics are valid.

If  $\alpha = 0$  in (3.2) then

$$\begin{aligned} u(x; \lambda) &= \frac{\sin \sqrt{\lambda}(x+a)}{\sqrt{\lambda}} - \frac{q_1(x)}{\lambda} \cos \sqrt{\lambda}(x+a) \\ &\quad + \frac{1}{2\lambda} \int_{-a}^x \cos \sqrt{\lambda}(x-a-2t)q(t)dt + O\left(\frac{e^{\eta(x+a)}}{\lambda^{3/2}}\right), \\ u'(x; \lambda) &= \cos \sqrt{\lambda}(x+a) + q_1(x) \frac{\sin \sqrt{\lambda}(x+a)}{\sqrt{\lambda}} \\ &\quad - \int_{-a}^x \frac{\sin \sqrt{\lambda}(x-a-2t)}{2\sqrt{\lambda}} q(t)dt + O\left(\frac{e^{\eta(x+a)}}{\lambda}\right). \end{aligned}$$

if  $-a \leq x < 0$ , else if  $0 < x \leq b$ ,

$$\begin{aligned} u(x; \lambda) &= \frac{-\lambda^2}{\sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2} \left[ \cos \sqrt{\lambda}a \frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}} - \frac{1}{\lambda} \sum_{i=1}^N \beta_i^2 \cos \sqrt{\lambda}a \cos \sqrt{\lambda}x \right. \\ &\quad + \frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}} \int_{-a}^0 \cos \sqrt{\lambda}t \frac{\sin \sqrt{\lambda}(a+t)}{\sqrt{\lambda}} q(t)dt \\ &\quad \left. + \cos \sqrt{\lambda}a \int_0^x \frac{\sin \sqrt{\lambda}t \sin \sqrt{\lambda}(x-t)}{\sqrt{\lambda}} q(t)dt + O\left(\frac{e^{\eta(x+a)}}{\lambda^{3/2}}\right) \right], \\ u'(x; \lambda) &= \frac{-\lambda^2}{\sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2} \left[ \cos \sqrt{\lambda}a \cos \sqrt{\lambda}x + \sum_{i=1}^N \beta_i^2 \cos \sqrt{\lambda}a \frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}} \right. \\ &\quad + \cos \sqrt{\lambda}x \int_{-a}^0 \cos \sqrt{\lambda}t \frac{\sin \sqrt{\lambda}(a+t)}{\sqrt{\lambda}} q(t)dt \\ &\quad \left. + \cos \sqrt{\lambda}a \int_0^x \frac{\sin \sqrt{\lambda}t}{\sqrt{\lambda}} \cos \sqrt{\lambda}(x-t)q(t)dt + O\left(\frac{e^{\eta(x+a)}}{\lambda}\right) \right]. \end{aligned}$$

Whereas if  $\alpha \in (0, \pi)$  then

$$\begin{aligned} u(x; \lambda) &= \sin \alpha \cos \sqrt{\lambda}(x+a) + (\sin \alpha q_1(x) + \cos \alpha) \frac{\sin \sqrt{\lambda}(x+a)}{\sqrt{\lambda}} \\ &\quad + \sin \alpha \int_{-a}^x \frac{\sin \sqrt{\lambda}(x-a-2t)}{2\sqrt{\lambda}} q(t)dt + O\left(\frac{e^{\eta(x+a)}}{\lambda}\right), \\ u'(x; \lambda) &= -\sqrt{\lambda} \sin \alpha \sin \sqrt{\lambda}(x+a) + (\sin \alpha q_1(x) + \cos \alpha) \cos \sqrt{\lambda}(x+a) \\ &\quad + \frac{1}{2} \sin \alpha \int_{-a}^x \cos \sqrt{\lambda}(x-a-2t)q(t)dt + O\left(\frac{e^{\eta(x+a)}}{\sqrt{\lambda}}\right) \end{aligned}$$

if  $-a \leq x < 0$ , else if  $0 < x \leq b$ ,

$$u(x; \lambda) = \frac{\lambda^3 \sin \alpha}{\sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2} \left[ \frac{\sin \sqrt{\lambda} a \sin \sqrt{\lambda} x}{\sqrt{\lambda} \sqrt{\lambda}} - \frac{1}{\lambda} \sum_{i=1}^N \beta_i^2 \frac{\sin \sqrt{\lambda} a}{\sqrt{\lambda}} \cos \sqrt{\lambda} x - \frac{\cot \alpha}{\lambda} \cos \sqrt{\lambda} a \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} - \frac{1}{\lambda} \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} \int_{-a}^0 \cos \sqrt{\lambda} t \cos \sqrt{\lambda} (a+t) q(t) dt + \frac{\sin \sqrt{\lambda} a}{\sqrt{\lambda}} \int_0^x \frac{\sin \sqrt{\lambda} t \sin \sqrt{\lambda} (x-t)}{\sqrt{\lambda} \sqrt{\lambda}} q(t) dt + O\left(\frac{e^{\eta(x+a)}}{\lambda^2}\right) \right],$$

$$u'(x; \lambda) = \frac{\lambda^3 \sin \alpha}{\sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2} \left[ \frac{\sin \sqrt{\lambda} a}{\sqrt{\lambda}} \cos \sqrt{\lambda} x + \sum_{i=1}^N \beta_i^2 \frac{\sin \sqrt{\lambda} a \sin \sqrt{\lambda} x}{\sqrt{\lambda} \sqrt{\lambda}} - \frac{\cot \alpha}{\lambda} \cos \sqrt{\lambda} a \cos \sqrt{\lambda} x - \frac{1}{\lambda} \cos \sqrt{\lambda} x \int_{-a}^0 \cos \sqrt{\lambda} t \cos \sqrt{\lambda} (a+t) q(t) dt + \frac{\sin \sqrt{\lambda} a}{\sqrt{\lambda}} \int_0^x \frac{\sin \sqrt{\lambda} t}{\sqrt{\lambda}} \cos \sqrt{\lambda} (x-t) q(t) dt + O\left(\frac{e^{\eta(x+a)}}{\lambda^{3/2}}\right) \right].$$

If  $\beta = \pi$  in (3.3) then

$$v(x; \lambda) = \frac{-\lambda^2}{\sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2} \left[ \frac{\sin \sqrt{\lambda} b}{\sqrt{\lambda}} \cos \sqrt{\lambda} x - \frac{1}{\lambda} \sum_{i=1}^N \beta_i^2 \cos \sqrt{\lambda} b \cos \sqrt{\lambda} x + \cos \sqrt{\lambda} x \int_0^b \frac{\sin \sqrt{\lambda} t \sin \sqrt{\lambda} (b-t)}{\sqrt{\lambda} \sqrt{\lambda}} q(t) dt + \frac{\sin \sqrt{\lambda} b}{\sqrt{\lambda}} \int_x^0 \cos \sqrt{\lambda} t \frac{\sin \sqrt{\lambda} (t-x)}{\sqrt{\lambda}} q(t) dt + O\left(\frac{e^{\eta(b-x)}}{\lambda^{3/2}}\right) \right],$$

$$v'(x; \lambda) = \frac{\lambda^3}{\sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2} \left[ \frac{\sin \sqrt{\lambda} b \sin \sqrt{\lambda} x}{\sqrt{\lambda} \sqrt{\lambda}} - \frac{1}{\lambda} \sum_{i=1}^N \beta_i^2 \cos \sqrt{\lambda} b \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} + \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} \int_0^b \frac{\sin \sqrt{\lambda} t \sin \sqrt{\lambda} (b-t)}{\sqrt{\lambda} \sqrt{\lambda}} q(t) dt + \frac{1}{\lambda} \frac{\sin \sqrt{\lambda} b}{\sqrt{\lambda}} \int_x^0 \cos \sqrt{\lambda} t \cos \sqrt{\lambda} (t-x) q(t) dt + O\left(\frac{e^{\eta(b-x)}}{\lambda^2}\right) \right]$$

if  $-a \leq x < 0$ , else if  $0 < x \leq b$ ,

$$\begin{aligned}
v(x; \lambda) &= \frac{\sin \sqrt{\lambda}(b-x)}{\sqrt{\lambda}} - \frac{q_2(x)}{\lambda} \cos \sqrt{\lambda}(b-x) \\
&\quad + \frac{1}{2\lambda} \int_x^b \cos \sqrt{\lambda}(2t-x-b)q(t)dt + O\left(\frac{e^{\eta(b-x)}}{\lambda^{3/2}}\right), \\
v'(x; \lambda) &= -\cos \sqrt{\lambda}(b-x) - q_2(x) \frac{\sin \sqrt{\lambda}(b-x)}{\sqrt{\lambda}} \\
&\quad + \int_x^b \frac{\sin \sqrt{\lambda}(2t-x-b)}{2\sqrt{\lambda}}q(t)dt + O\left(\frac{e^{\eta(b-x)}}{\lambda}\right).
\end{aligned}$$

Whereas if  $\beta \in (0, \pi)$  then

$$\begin{aligned}
v(x; \lambda) &= \frac{-\lambda^2 \sin \beta}{\sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2} \left[ \cos \sqrt{\lambda}b \cos \sqrt{\lambda}x + \left( \sum_{i=1}^N \beta_i^2 - \cot \beta \right) \frac{\sin \sqrt{\lambda}b}{\sqrt{\lambda}} \cos \sqrt{\lambda}x \right. \\
&\quad + \cos \sqrt{\lambda}x \int_0^b \frac{\sin \sqrt{\lambda}t}{\sqrt{\lambda}} \cos \sqrt{\lambda}(b-t)q(t)dt \\
&\quad \left. + \cos \sqrt{\lambda}b \int_x^0 \cos \sqrt{\lambda}t \frac{\sin \sqrt{\lambda}(t-x)}{\sqrt{\lambda}} q(t)dt + O\left(\frac{e^{\eta(b-x)}}{\lambda}\right) \right], \\
v'(x; \lambda) &= \frac{\lambda^3 \sin \beta}{\sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2} \left[ \cos \sqrt{\lambda}b \frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}} + \left( \sum_{i=1}^N \beta_i^2 - \cot \beta \right) \frac{\sin \sqrt{\lambda}b \sin \sqrt{\lambda}x}{\sqrt{\lambda}} \right. \\
&\quad + \frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}} \int_0^b \frac{\sin \sqrt{\lambda}t}{\sqrt{\lambda}} \cos \sqrt{\lambda}(b-t)q(t)dt \\
&\quad \left. + \frac{1}{\lambda} \cos \sqrt{\lambda}b \int_x^0 \cos \sqrt{\lambda}t \cos \sqrt{\lambda}(t-x)q(t)dt + O\left(\frac{e^{\eta(b-x)}}{\lambda^{3/2}}\right) \right]
\end{aligned}$$

if  $-a \leq x < 0$ , else if  $0 < x \leq b$ ,

$$\begin{aligned}
v(x; \lambda) &= \sin \beta \cos \sqrt{\lambda}(b-x) + (\sin \beta q_2(x) - \cos \beta) \frac{\sin \sqrt{\lambda}(b-x)}{\sqrt{\lambda}} \\
&\quad + \sin \beta \int_x^b \frac{\sin \sqrt{\lambda}(2t-x-b)}{2\sqrt{\lambda}}q(t)dt + O\left(\frac{e^{\eta(b-x)}}{\lambda}\right), \\
v'(x; \lambda) &= \sqrt{\lambda} \sin \beta \sin \sqrt{\lambda}(b-x) + (\cos \beta - \sin \beta q_2(x)) \cos \sqrt{\lambda}(b-x) \\
&\quad - \frac{1}{2} \sin \beta \int_x^b \cos \sqrt{\lambda}(2t-x-b)q(t)dt + O\left(\frac{e^{\eta(b-x)}}{\sqrt{\lambda}}\right).
\end{aligned}$$

*Proof.* It is easy to check that  $w_1(x; \lambda)$  and  $w_2(x; \lambda)$  satisfy the following Volterra integral equations.



If  $-a \leq x \leq 0$ ,

$$\begin{aligned} w_1(x; \lambda) &= \cos \sqrt{\lambda}x + \int_x^0 \frac{\sin \sqrt{\lambda}(t-x)}{\sqrt{\lambda}} q(t) w_1(t; \lambda) dt, \\ w_2(x; \lambda) &= \frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}} + \int_x^0 \frac{\sin \sqrt{\lambda}(t-x)}{\sqrt{\lambda}} q(t) w_2(t; \lambda) dt, \end{aligned}$$

and if  $0 \leq x \leq b$ ,

$$\begin{aligned} w_1(x; \lambda) &= \cos \sqrt{\lambda}x + \int_0^x \frac{\sin \sqrt{\lambda}(x-t)}{\sqrt{\lambda}} q(t) w_1(t; \lambda) dt, \\ w_2(x; \lambda) &= \frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}} + \int_0^x \frac{\sin \sqrt{\lambda}(x-t)}{\sqrt{\lambda}} q(t) w_2(t; \lambda) dt. \end{aligned}$$

Whence we observe that the following approximations are valid as  $|\lambda| \rightarrow \infty$

$$\begin{aligned} w_1(x; \lambda) &= \cos \sqrt{\lambda}x + O\left(\frac{e^{\eta|x|}}{\sqrt{\lambda}}\right), \\ w_1'(x; \lambda) &= -\sqrt{\lambda} \sin \sqrt{\lambda}x + O(e^{\eta|x|}), \\ w_2(x; \lambda) &= \frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}} + O\left(\frac{e^{\eta|x|}}{\lambda}\right), \\ w_2'(x; \lambda) &= -\cos \sqrt{\lambda}x + O\left(\frac{e^{\eta|x|}}{\sqrt{\lambda}}\right). \end{aligned}$$

Substituting these approximations back into the Volterra identities yields the following refined estimates. If  $-a \leq x \leq 0$  then

$$\begin{aligned} w_1(x; \lambda) &= \cos \sqrt{\lambda}x - \left[ \frac{1}{2} \int_x^0 q(t) dt \right] \frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}} + \int_x^0 \frac{\sin \sqrt{\lambda}(2t-x)}{2\sqrt{\lambda}} q(t) dt + O\left(\frac{e^{-\eta x}}{\lambda}\right), \\ w_2(x; \lambda) &= \frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}} + \frac{1}{\lambda} \left[ \frac{1}{2} \int_x^0 q(t) dt \right] \cos \sqrt{\lambda}x - \frac{1}{2\lambda} \int_x^0 \cos \sqrt{\lambda}(2t-x) q(t) dt + O\left(\frac{e^{-\eta x}}{\lambda^{3/2}}\right). \end{aligned}$$

Else if  $0 \leq x < b$  then

$$\begin{aligned} w_1(x; \lambda) &= \cos \sqrt{\lambda}x + \left[ \frac{1}{2} \int_0^x q(t) dt \right] \frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}} + \int_0^x \frac{\sin \sqrt{\lambda}(x-2t)}{2\sqrt{\lambda}} q(t) dt + O\left(\frac{e^{\eta x}}{\lambda}\right), \\ w_2(x; \lambda) &= \frac{\sin \sqrt{\lambda}x}{\sqrt{\lambda}} - \frac{1}{\lambda} \left[ \frac{1}{2} \int_0^x q(t) dt \right] \cos \sqrt{\lambda}x + \frac{1}{2\lambda} \int_0^x \cos \sqrt{\lambda}(x-2t) q(t) dt + O\left(\frac{e^{\eta x}}{\lambda^{3/2}}\right). \end{aligned}$$

Writing

$$\begin{aligned} u(x; \lambda) &= u(0^-; \lambda) w_1(x; \lambda) + u'(0^-; \lambda) w_2(x; \lambda), \quad -a \leq x < 0, \\ v(x; \lambda) &= v(0^+; \lambda) w_1(x; \lambda) + v'(0^+; \lambda) w_2(x; \lambda), \quad 0 < x \leq b, \end{aligned}$$

and inserting the definitions of  $u$  and  $v$  into the above equations we obtain

$$\begin{bmatrix} u(0^-; \lambda) \\ u'(0^-; \lambda) \end{bmatrix} = \begin{bmatrix} w_2'(-a; \lambda) & -w_2(-a; \lambda) \\ -w_1'(-a; \lambda) & w_1(-a; \lambda) \end{bmatrix} \begin{bmatrix} \sin \alpha \\ \cos \alpha \end{bmatrix},$$

$$\begin{bmatrix} v(0^+; \lambda) \\ v'(0^+; \lambda) \end{bmatrix} = \begin{bmatrix} w_2'(b; \lambda) & -w_2(b; \lambda) \\ -w_1'(b; \lambda) & w_1(b; \lambda) \end{bmatrix} \begin{bmatrix} \sin \beta \\ \cos \beta \end{bmatrix},$$

from which the approximations stated for  $u(x; \lambda)$ ,  $-a \leq x < 0$  and  $v(x; \lambda)$ ,  $0 < x \leq b$  follow. Moreover, for  $|\lambda|$  large enough we can assume that  $u_-$  and  $v_+$  are extended to solutions on  $[-a, 0) \cup (0, b]$  according to (3.24) and (3.25), from which the remaining approximations for  $u$  and  $v$  are obtained.  $\square$

# Chapter 4

## Inverse Sturm-Liouville problems with transmission conditions Herglotz dependent on the eigenparameter

### 4.1 Introduction

In this chapter we prove a uniqueness result analogous to that of Hochstadt, [39], on the determination of the potential  $q$  in the Sturm-Liouville equation

$$\ell y := -y'' + qy = \lambda y, \quad x \in [-a, 0) \cup (0, b], \quad (4.1)$$

from given spectral data. Here, as in Chapter 3, we assume that  $y|_{(-a,0)}$ ,  $y'|_{(-a,0)}$ ,  $\ell y|_{(-a,0)} \in L^2(-a, 0)$  and  $y|_{(0,b)}$ ,  $y'|_{(0,b)}$ ,  $\ell y|_{(0,b)} \in L^2(0, b)$ , where  $a, b > 0$  and  $q \in L^2(-a, b)$ . We impose separated boundary conditions

$$y(-a) \cos \alpha = y'(-a) \sin \alpha, \quad \alpha \in [0, \pi) \quad (4.2)$$

$$y(b) \cos \beta = y'(b) \sin \beta, \quad \beta \in (0, \pi] \quad (4.3)$$

and eigenparameter-dependent transmission conditions

$$y(0^+) = r(\lambda) \Delta y', \quad (4.4)$$

$$y'(0^-) = s(\lambda) \Delta y. \quad (4.5)$$

Here

$$\Delta y' = y'(0^+) - y'(0^-),$$

$$\Delta y = y(0^+) - y(0^-),$$

$$r(\lambda) = - \sum_{i=1}^N \frac{\beta_i^2}{\lambda - \gamma_i}, \quad \beta_i \in \mathbb{R}^+, \quad i = 1, \dots, N, \quad (4.6)$$

$$s(\lambda) = \sum_{j=1}^M \frac{\alpha_j^2}{\lambda - \delta_j}, \quad \alpha_j \in \mathbb{R}^+, \quad j = 1, \dots, M, \quad (4.7)$$

and

$$\gamma_1 < \gamma_2 < \dots < \gamma_N,$$

$$\delta_1 < \delta_2 < \dots < \delta_M.$$

Again, we remind the reader of the interpretation of conditions (4.4), (4.5) at zeroes and poles of  $r$  and  $s$ . With reference to the transmission condition (4.4), we note that  $r(\lambda) = 0$  reduces (4.4) at  $\lambda$  to the condition  $y(0^+) = 0$ , while if  $\lambda$  is a pole of  $r$  then (4.4) becomes  $\Delta y' = 0$ . Similarly for (4.5), if  $s(\lambda) = 0$  then (4.5) at  $\lambda$  becomes  $y'(0^-) = 0$ , while if  $\lambda$  is a pole of  $s$  then (4.5) becomes  $\Delta y = 0$ .

A notable early contribution to the inverse spectral theory of Sturm-Liouville differential operators is the work [18] by G. Borg, where the spectral data consists of two sequences of eigenvalues: the first being  $\{\lambda_n\}_{n=0}^{\infty}$ , the eigenvalues corresponding to the classical Sturm-Liouville problem (3.1) on an interval of the form  $[-a, b]$  with boundary conditions of the form (4.2)-(4.3) above, and a second sequence  $\{\tilde{\lambda}_n\}_{n=0}^{\infty}$ , obtained by changing the angle  $\beta$  in the boundary condition at  $x = b$  to  $\zeta$ , such that  $\sin(\beta - \zeta) \neq 0$ . Borg showed that these two spectra uniquely determine  $q(x)$  almost everywhere on  $[-a, b]$ . In [54], N. Levinson suggested a different method to prove Borg's results, now commonly known as the contour integral method. A related inverse problem, which is of particular interest for our purposes, was developed in the papers [38], [39] by H. Hochstadt. There, Hochstadt proves a more general uniqueness result, demonstrating the amount of freedom that  $q$  has if  $\{\lambda_n\}_{n=0}^{\infty}$  and all but finitely many of the  $\tilde{\lambda}_n$  are specified. V.A. Marčenko [56] was the first to apply the transformation operator method to the solution of the inverse Sturm-Liouville problem. This approach was also used by I.M. Gelfand and B.M. Levitan in their seminal paper [33]. A more modern class of inverse problems aim to reconstruct the potential  $q$  from so called nodal data, where, instead of a combination of eigenvalues and norming constants, the spectral data consists rather of the positions of the zeroes of the eigenfunctions (nodal positions). This was initiated in the paper [57] by J. R. Mclaughlin. For further discussions of classical results on inverse spectral problems we refer the reader to the book [27] by G. Freiling and V. Yurko.

In recent years there has been a steady increase in the literature on Sturm-Liouville operators with transmission conditions (also known as multi-point conditions, point interactions or matching conditions) at interior points. An early contribution of this type in the context of inverse spectral theory is the paper by O. Hald, [37], which generalizes the result of H. Hochstadt and B. Lieberman, [40], to show that if the potential is known on one half of the interval and one boundary condition is given then the potential on the other half and the other boundary condition is uniquely determined by the eigenvalues. Hald also shows that, under these assumptions, the position of the discontinuity and jump in the eigenfunctions are uniquely determined. This result was later extended to two interior discontinuities in the paper [90] by C. Willis. We also mention the more recent paper [74] by C. Shieh and V. Yurko, which considers an inverse nodal problem of recovering the potential and boundary conditions assuming the discontinuity conditions are known.

Of special interest are those problems where the spectral parameter appears not only in the differential equation but also in the boundary conditions and/or transmission conditions. Discontinuous inverse eigenvalue problems where the boundary conditions have either an affine or bilinear dependence on the spectral parameter and the transmission conditions are either independent of the spectral parameter or have an affine dependence have been studied in [6], [35], [36], [66], [68], [85], [86]. In [66], A.S. Ozkan et al consider a double discontinuous eigenvalue problem with eigenparameter appearing in both the boundary conditions and transmission conditions, and show that all coefficients can be obtained using either the Weyl function or two spectra. Y. P. Wang, [85] uses Weyl function techniques to recover the coefficients of the Sturm-Liouville operator with an arbitrary number of interior discontinuities and boundary conditions depending on the spectral parameter. In [86], Z. Wei and G. Wei obtain a uniqueness result using the Weyl function technique for the non self-adjoint Dirac operator with boundary conditions and jump conditions

dependent on the spectral parameter.

Inverse Sturm-Liouville problems with rational functions of the spectral parameter contained only in the boundary conditions (and not in the transmission conditions for the case of discontinuous problems) have been studied in [12], [15], [16], [22], [67]. Of the papers listed above only [67] corresponds to an eigenvalue problem with transmission conditions. Here, A. S. Ozkan extends the Hochstadt-Lieberman result, [40], to the case of a discontinuous Sturm-Liouville problem with the spectral parameter rationally contained in the boundary conditions and with affine dependence in the transmission conditions. Returning to the eigenvalue problem given by (3.1)-(4.5), this is (to the best of our knowledge) the first time an inverse result for a discontinuous Sturm-Liouville problem having rational functions contained in the transmission conditions has been presented. We prove a generalized uniqueness result analogous to that of Hochstadt, [39] (see Theorem 4.5.1). Due to notational complications we present only a sample special case here.

Let  $(\ell; \alpha, \beta; r, s)$  denote the eigenvalue problem  $\ell y = \lambda y$  with boundary conditions (4.2)-(4.3) and transmission conditions (4.4)-(4.5) as above. Let  $(\ell; \alpha, \zeta; r, s)$  denote the above eigenvalue problem, but with the boundary condition at  $x = b$  replaced by

$$y(b) \cos \zeta = y'(b) \sin \zeta,$$

where  $\sin(\beta - \zeta) \neq 0$ . Define  $(\tilde{\ell}; \alpha, \beta; r, s)$  and  $(\tilde{\ell}; \alpha, \zeta; r, s)$  in an analogous manner but with  $\ell$  replaced by  $\tilde{\ell}$ , i.e.  $q$  replaced by  $\tilde{q}$ . Finally, denote by  $\mathcal{M}_0$  the subset of  $\mathbb{N}_0$  for which  $\lambda_n$  is an eigenvalue of  $(\ell; \alpha, \beta; r, s)$  with  $r(\lambda_n) = 0$  or  $s(\lambda_n) = 0$ . Then, in particular, we obtain the following uniqueness result.

**Theorem 4.1.1.** *Suppose that the problem  $(\ell; \alpha, \beta; r, s)$  has eigenvalues  $\{\lambda_n\}_{n=0}^{\infty}$  listed in increasing order with repetition according to multiplicity. Suppose further that  $(\ell; \alpha, \beta; r, s)$  has eigenvalues  $\{\tilde{\lambda}_n\}_{n=0}^{\infty}$  listed in a like manner. Then the results given in points 1. and 2. below are independent.*

1. If  $\lambda_n = \tilde{\lambda}_n$  for all  $n \in \mathbb{N}_0$  and the eigenvalues of  $(\ell; \alpha, \zeta; r, s)$  and  $(\tilde{\ell}; \alpha, \zeta; r, s)$  coincide (up to multiplicity) then, almost everywhere

- I.  $q = \tilde{q}$  on  $(0, b]$ ,

- II.

$$q = \tilde{q} - \sum_{n \in \mathcal{M}_0} c_n [\tilde{f}_n f_n]' \tag{4.8}$$

on  $[-a, 0)$ . Here  $f_n$  and  $\tilde{f}_n$  are suitably chosen eigenfunctions of  $(\ell; \alpha, \beta; r, s)$  and  $(\tilde{\ell}; \alpha, \beta; r, s)$  corresponding to the eigenvalues  $\lambda_n$  and  $\tilde{\lambda}_n$  respectively, and  $c_n \in \mathbb{R}$ .

2. If the boundary condition at  $x = -a$  is replaced by

$$y(-a) \cos \varepsilon = y'(-a) \sin \varepsilon$$

where  $\sin(\alpha - \varepsilon) \neq 0$  we obtain the eigenvalue problems  $(\ell; \varepsilon, \beta; r, s)$  and  $(\tilde{\ell}; \varepsilon, \beta; r, s)$  respectively. If  $\lambda_n = \tilde{\lambda}_n$  for all  $n \in \mathbb{N}_0$  and, in addition, the eigenvalues of  $(\ell; \varepsilon, \beta; r, s)$  and  $(\tilde{\ell}; \varepsilon, \beta; r, s)$  coincide (with the same multiplicities) then  $q = \tilde{q}$  almost everywhere on  $[-a, 0)$ . Further, we can show that an identity similar to that of (4.8) holds on  $(0, b]$ .

With the assumptions of points 1. and 2. combined we are able to show that  $q = \tilde{q}$  almost everywhere on  $[-a, 0) \cup (0, b]$ .

The remainder of the chapter is structured as follows. In Section 4.2 we recall from Chapter 2 the definition of the Hilbert space operator eigenvalue problem whose eigenvalues are equivalent to those of (4.1)-(4.5), and define additional structures needed for the statement of the inverse problem under consideration. In Section 4.3, we define a decomposition of the eigenvalues of (4.1)-(4.5) and prove a Mittag-Leffler expansion theorem relating to the functional component of the resolvent operator of the corresponding Hilbert space operator eigenvalue problem. We define Hochstadt's transformation operator as it relates to our problem in Section 4.4. Ultimately, in Section 4.5 we prove the generalized uniqueness result alluded to above.

## 4.2 Preliminaries

Suppose that  $\beta_i, i = \overline{1, N}$  and  $\alpha_j, j = \overline{1, M}$  in (4.6) and (4.7) are positive real numbers. Let  $\mathcal{H} = L^2(-a, b) \oplus \mathbb{C}^N \oplus \mathbb{C}^M$ . Then the boundary value problem (4.1)-(4.5) can be posed in  $\mathcal{H}$  by considering the operator

$$LY = \begin{bmatrix} \ell y \\ (\gamma_i y_i^1 + \beta_i \Delta y') \\ (\delta_j y_j^2 + \alpha_j \Delta y) \end{bmatrix}, \quad \text{with } Y = \begin{bmatrix} y \\ (y_i^1) \\ (y_j^2) \end{bmatrix}, \quad (4.9)$$

and domain

$$\mathcal{D}(L) = \left\{ Y = \begin{bmatrix} y \\ (y_i^1) \\ (y_j^2) \end{bmatrix} \left| \begin{array}{l} y|_{(-a,0)}, y'|_{(-a,0)}, \ell y|_{(-a,0)} \in L^2(-a,0), \\ y|_{(0,b)}, y'|_{(0,b)}, \ell y|_{(0,b)} \in L^2(0,b), \\ y \text{ obeys (4.2) and (4.3),} \\ y'(0^-) = \sum_{j=1}^M \alpha_j y_j^2, -y(0^+) = \sum_{i=1}^N \beta_i y_i^1 \end{array} \right. \right\}.$$

We recall that  $\ell y := -y'' + qy$ .

On  $\mathcal{H}$ , we define a Hilbert space inner product,

$$\left\langle \begin{bmatrix} f \\ (f_i^1) \\ (f_j^2) \end{bmatrix}, \begin{bmatrix} g \\ (g_i^1) \\ (g_j^2) \end{bmatrix} \right\rangle = \int_{-a}^b f \bar{g} dx + \langle (f_i^1), (g_i^1) \rangle_N + \langle (f_j^2), (g_j^2) \rangle_M, \quad (4.10)$$

where  $\langle \cdot, \cdot \rangle_N$  and  $\langle \cdot, \cdot \rangle_M$  denote the Euclidean inner products in  $\mathbb{C}^N$  and  $\mathbb{C}^M$  respectively. Recall that  $\lambda$  is an eigenvalue of  $(\ell; \alpha, \beta; r, s)$  (i.e.(4.1)-(4.5)) with eigenfunction  $y$  if and only if  $\lambda$  is an eigenvalue of  $L$  with corresponding eigenvector function

$$Y = \begin{bmatrix} y \\ \left( \frac{\beta_i}{\lambda - \gamma_i} \Delta y' \right) \\ \left( \frac{\alpha_j}{\lambda - \delta_j} \Delta y \right) \end{bmatrix} \quad (4.11)$$

provided that  $\lambda \neq \gamma_i, \delta_j$  for all  $i = \overline{1, N}$  and  $j = \overline{1, M}$ . Else if  $\lambda = \gamma_p$  for some  $p$  then  $\Delta y' = 0$ , which implies that

$$y_p^1 = -\frac{y(0^+)}{\beta_p} \quad \text{and} \quad y_k^1 = 0 \quad \forall k \neq p. \quad (4.12)$$

Whereas if  $\lambda = \delta_\mu$  for some  $\mu$  then  $\Delta y = 0$ , giving

$$y_\mu^2 = \frac{y'(0^-)}{\alpha_\mu} \quad \text{and} \quad y_k^2 = 0 \quad \forall k \neq \mu. \quad (4.13)$$

By Theorem 3.3.2 in Chapter 3 the operator  $L$  is self-adjoint and densely defined on  $\mathcal{H}$ . Moreover, by Theorem 3.3.1, the eigenvalues of (3.1) - (4.5) and the Hilbert space operator eigenvalue problem  $LY = \lambda Y$ , with domain  $\mathcal{D}(L)$ , coincide. The eigenvalues are geometrically simple except at zeroes of  $r(\lambda)$  or  $s(\lambda)$ . The maximum geometric multiplicity of the eigenvalues is 2 and this occurs if and only if  $r(\lambda) = 0$  or  $s(\lambda) = 0$  and  $\lambda$  is an eigenvalue of (3.1) on  $[-a, 0)$  with boundary conditions (4.2) and  $y'(0^-) + s(\lambda)y(0^-) = 0$  and  $\lambda$  is an eigenvalue of (3.1) on  $(0, b]$  with boundary conditions  $y(0^+) - r(\lambda)y'(0^+) = 0$  and (4.3). See Lemma 3.2.1 and Theorem 3.2.2.

Except at a zero of  $r(\lambda)$  or  $s(\lambda)$ , the transmission conditions can be expressed as

$$\begin{bmatrix} y(0^+) \\ y'(0^+) \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{s(\lambda)} \\ \frac{1}{r(\lambda)} & 1 + \frac{1}{r(\lambda)s(\lambda)} \end{bmatrix} \begin{bmatrix} y(0^-) \\ y'(0^-) \end{bmatrix}. \quad (4.14)$$

We note that at poles of  $r$  or  $s$  the transfer matrix in (4.14) above has the same interpretation as in points (i)-(iii) of Lemma 3.2.1.

We denote by  $(L; \alpha, \beta; r, s)$  the Hilbert space operator eigenvalue problem defined above, and by  $(\tilde{L}; \alpha, \beta; r, s)$  the analogous problem with  $\ell$  replaced by  $\tilde{\ell}$ . The eigenvalues of  $(L; \alpha, \beta; r, s)$  (i.e. the eigenvalues of (4.1)-(4.5)) will be listed in increasing order with repetition according to multiplicity by

$$\lambda_0 \leq \lambda_1 \leq \dots < \infty, \quad (4.15)$$

and the eigenvalues of  $(\tilde{L}; \alpha, \beta; r, s)$  by

$$\tilde{\lambda}_0 \leq \tilde{\lambda}_1 \leq \dots < \infty. \quad (4.16)$$

At an eigenvalue  $\lambda_n$  of  $(L; \alpha, \beta; r, s)$  we write, for brevity,

$$F_n(x) = \begin{bmatrix} f_n(x) \\ (f_{i,n}^1) \\ (f_{j,n}^2) \end{bmatrix} := \begin{bmatrix} f(x; \lambda_n) \\ (f_i^1(\lambda_n)) \\ (f_j^2(\lambda_n)) \end{bmatrix} = F(x; \lambda_n) \quad (4.17)$$

if  $F(x; \lambda_n)$  is an eigenfunction of  $(L; \alpha, \beta; r, s)$  corresponding to  $\lambda_n$ . By assumption of the Hochstadt inverse problem the eigenvalues of  $(L; \alpha, \beta; r, s)$  and of  $(\tilde{L}; \alpha, \beta; r, s)$  agree up to multiplicity except on a finite set  $\Lambda_0$  (see Definition 4.3.1 in the next section). As a result we employ the following notation. If  $\lambda_n$  is an eigenvalue of  $(L; \alpha, \beta; r, s)$  such that  $\lambda_n = \tilde{\lambda}_m$  for some eigenvalue  $\lambda_m$  of  $(\tilde{L}; \alpha, \beta; r, s)$ , where  $\lambda_n$  and  $\tilde{\lambda}_m$  have *equal multiplicities*, then we write

$$\tilde{F}_n(x) = \begin{bmatrix} \tilde{f}_n(x) \\ (\tilde{f}_{i,n}^1) \\ (\tilde{f}_{j,n}^2) \end{bmatrix} := \begin{bmatrix} \tilde{f}(x; \lambda_n) \\ (\tilde{f}_i^1(\lambda_n)) \\ (\tilde{f}_j^2(\lambda_n)) \end{bmatrix} = \tilde{F}(x; \lambda_n) \quad (4.18)$$

for any eigenfunction  $\tilde{F}$  corresponding to  $\tilde{\lambda}_m$ . In other words,  $\tilde{F}(x; \lambda)$  is a solution of  $\tilde{L}\tilde{F} = \lambda\tilde{F}$  at  $\lambda = \lambda_n$ .

We will employ certain “base solutions” to construct the eigenfunctions of  $(L; \alpha, \beta; r, s)$  and  $(\tilde{L}; \alpha, \beta; r, s)$ . Firstly, we define fundamental solutions  $w_1(x; \lambda)$  and  $w_2(x; \lambda)$  of (4.1) on  $[-a, b]$  such that

$$w_1(0; \lambda) = 1, \quad w_2(0; \lambda) = 0 \quad (4.19)$$

$$w'_1(0; \lambda) = 0, \quad w'_2(0; \lambda) = 1. \quad (4.20)$$

Next, let  $u_-(x; \lambda)$  denote the solution of (4.1) on  $[-a, 0)$  satisfying

$$u_-(-a; \lambda) = \sin \alpha, \quad u'_-(-a; \lambda) = \cos \alpha,$$

and let  $v_+(x; \lambda)$  denote the solution of (3.1) on  $(0, b]$  satisfying

$$v_+(b; \lambda) = \sin \beta, \quad v'_+(b; \lambda) = \cos \beta,$$

as defined in Section 3.4. At values of the eigenparameter not coinciding with zeroes of  $r(\lambda)$  and  $s(\lambda)$  we extend  $u_-(x; \lambda)$  and  $v_+(x; \lambda)$  by functions  $u_+(x; \lambda)$  and  $v_-(x; \lambda)$  satisfying (4.1) on  $(0, b]$  and  $[-a, 0)$  respectively, by imposing (4.14). That is, we define

$$\begin{bmatrix} u_+(0^+; \lambda) \\ u'_+(0^+; \lambda) \end{bmatrix} := \begin{bmatrix} 1 & \frac{1}{s(\lambda)} \\ \frac{1}{r(\lambda)} & 1 + \frac{1}{r(\lambda)s(\lambda)} \end{bmatrix} \begin{bmatrix} u_-(0^-; \lambda) \\ u'_-(0^-; \lambda) \end{bmatrix} \quad (4.21)$$

and

$$\begin{bmatrix} v_-(0^-; \lambda) \\ v'_-(0^-; \lambda) \end{bmatrix} := \begin{bmatrix} 1 & \frac{1}{s(\lambda)} \\ \frac{1}{r(\lambda)} & 1 + \frac{1}{r(\lambda)s(\lambda)} \end{bmatrix}^{-1} \begin{bmatrix} v_+(0^+; \lambda) \\ v'_+(0^+; \lambda) \end{bmatrix}, \quad (4.22)$$

giving

$$\begin{aligned} u_+(x; \lambda) &= u_+(0^+; \lambda)w_1(x; \lambda) + u'_+(0^+; \lambda)w_2(x; \lambda), \quad \text{for } -a \leq x < 0, \\ v_-(x; \lambda) &= v_-(0^-; \lambda)w_1(x; \lambda) + v'_-(0^-; \lambda)w_2(x; \lambda), \quad \text{for } 0 < x \leq b, \end{aligned}$$

provided that  $r(\lambda) \neq 0$  and  $s(\lambda) \neq 0$ . At zeroes of  $r(\lambda)$  or  $s(\lambda)$  we extend  $u_+$  and  $v_-$  by continuity, defining

$$\begin{aligned} u_+(x; \lambda) &:= \lim_{\mu \rightarrow \lambda} [u_+(0^+; \mu)w_1(x; \mu) + u'_+(0^+; \mu)w_2(x; \mu)], \quad \text{for } -a \leq x < 0, \\ v_-(x; \lambda) &:= \lim_{\mu \rightarrow \lambda} [v_-(0^-; \mu)w_1(x; \mu) + v'_-(0^-; \mu)w_2(x; \mu)], \quad \text{for } 0 < x \leq b, \end{aligned}$$

if the limits exist. Note that extending solutions in this way will not necessarily straight away yield eigenfunctions of (4.1)-(4.5) with eigenvalue  $\lambda$  if  $r(\lambda)$  or  $s(\lambda) = 0$ . Such cases need to be treated with care. The procedure for constructing eigenfunctions of (4.1)-(4.5) from  $u$  and  $v$  is discussed in Note 4.2.1 at the end of this section. We define  $u$  and  $v$  in this way in order to preserve analyticity, which is crucial for our analysis.

Let

$$u(x; \lambda) = \begin{cases} u_-(x; \lambda), & \text{if } -a \leq x < 0, \\ u_+(x; \lambda), & \text{if } 0 < x \leq b, \end{cases} \quad (4.23)$$

and

$$v(x; \lambda) = \begin{cases} v_-(x; \lambda), & \text{if } -a \leq x < 0, \\ v_+(x; \lambda), & \text{if } 0 < x \leq b, \end{cases} \quad (4.24)$$



and define  $\tilde{u}(x; \lambda)$  and  $\tilde{v}(x; \lambda)$  in an analogous manner by replacing  $\ell$  with  $\tilde{\ell}$ .

Define

$$\psi(\lambda) = u(b; \lambda) \cos \beta - u'(b; \lambda) \sin \beta. \quad (4.25)$$

Then  $\psi(\lambda) = W[u, v](x)$  for  $x \in [-a, 0) \cup (0, b]$ , where  $W[\cdot, \cdot]$  denotes the Wronskian.

We can formally write

$$\omega(\lambda) = \left[ \sum_{j=1}^M \alpha_j^2 \prod_{k=1, k \neq j}^M (\lambda - \delta_k) \right] \left[ \sum_{i=1}^N \beta_i^2 \prod_{k=1, k \neq i}^N (\lambda - \gamma_k) \right] \psi(\lambda), \quad (4.26)$$

where

$$\begin{aligned} \omega(\lambda) = & \left[ \sum_{j=1}^M \alpha_j^2 \prod_{k=1, k \neq j}^M (\lambda - \delta_k) \right] \left[ \sum_{i=1}^N \beta_i^2 \prod_{k=1, k \neq i}^N (\lambda - \gamma_k) \right] [u(0^-)v'(0^+) - u'(0^-)v(0^+)] \\ & + \prod_{i=1}^N (\lambda - \gamma_i) \left[ \sum_{j=1}^M \alpha_j^2 \prod_{k=1, k \neq j}^M (\lambda - \delta_k) \right] u(0^-)v(0^+) \\ & + \prod_{j=1}^M (\lambda - \delta_j) \left[ \sum_{i=1}^N \beta_i^2 \prod_{k=1, k \neq i}^N (\lambda - \gamma_k) \right] u'(0^-)v'(0^+) \\ & + \prod_{i=1}^N (\lambda - \gamma_i) \prod_{j=1}^M (\lambda - \delta_j) u'(0^-)v(0^+). \end{aligned} \quad (4.27)$$

is the characteristic determinant corresponding to (4.1)-(4.5). That is,  $\omega(\lambda)$  has zeroes occurring at the eigenvalues of (4.1)-(4.5) (correspondingly  $(L; \alpha, \beta; r, s)$ ). Moreover, the geometric multiplicity of the eigenvalues coincide with the algebraic multiplicity as zeroes of  $\omega(\lambda)$ . See Theorem 3.4.1. Define  $\tilde{\psi}(\lambda)$  and  $\tilde{\omega}(\lambda)$  in an analogous manner by replacing  $\ell$  with  $\tilde{\ell}$ .

Let  $(L; \alpha, \zeta; r, s)$  denote the operator  $L$  with boundary condition (4.3) replaced by

$$y(b; \lambda) \cos \zeta - y'(b; \lambda) \sin \zeta = 0, \quad (4.28)$$

where we assume  $\sin(\beta - \zeta) \neq 0$ . Define

$$\nu(\lambda) = u(b; \lambda) \cos \zeta - u'(b; \lambda) \sin \zeta. \quad (4.29)$$

Similarly, define  $(\tilde{L}; \alpha, \zeta; r, s)$  and  $\tilde{\nu}(\lambda)$  as above by replacing  $\ell$  with  $\tilde{\ell}$ .

**Note 4.2.1.** Any solution of (4.1) obeying both boundary conditions (4.2) and (4.3) is necessarily of the form

$$y(x; \lambda) = \begin{cases} C(\lambda)u(x; \lambda) & \text{if } -a \leq x < 0 \\ D(\lambda)v(x; \lambda) & \text{if } 0 < x \leq b \end{cases} \quad (4.30)$$

for some  $C(\lambda)$ ,  $D(\lambda)$ . Clearly,  $\lambda$  will be an eigenvalue of  $(\ell; \alpha, \beta; r, s)$  with corresponding eigenfunction  $y(x; \lambda)$  of the form (4.30) if it also satisfies the transmission conditions (4.4) and (4.5) for appropriate  $C(\lambda)$  and  $D(\lambda)$ . At zeroes of  $r(\lambda)$  or  $s(\lambda)$  the question of multiplicity arises. In particular,

- I. if  $r(\lambda) = s(\lambda) = 0$  then the transmission conditions reduce to  $y(0^+; \lambda) = 0$ ,  $y'(0^-; \lambda) = 0$ . Thus  $\lambda$  is an eigenvalue of geometric multiplicity 2 if and only if  $u'(0^-; \lambda) = 0$  and  $v(0^+; \lambda) = 0$ . In that case, we observe from (4.26) that  $\psi(\lambda) \in \mathbb{R} \setminus \{0\}$ . If  $u'(0^-; \lambda) = 0$  and  $v(0^+; \lambda) \neq 0$  or if  $u'(0^-; \lambda) \neq 0$  and  $v(0^+; \lambda) = 0$  then  $\lambda$  has geometric multiplicity 1 and  $\psi$  has a pole at  $\lambda$ ;
- II. if precisely one of  $r(\lambda)$  and  $s(\lambda)$  is zero then  $\lambda$  has geometric multiplicity 2 if and only if  $\psi(\lambda) = 0$ . If  $\psi(\lambda) \in \mathbb{R} \setminus \{0\}$  then  $\lambda$  will have geometric multiplicity 1.

### 4.3 Expansion theorems

Let  $x, t \in [-a, 0) \cup (0, b]$ . Recall that the Green's function of  $\ell$  can be written in the form

$$G(x, t; \lambda) = \begin{cases} \frac{u(x; \lambda)v(t; \lambda)}{\psi(\lambda)}, & \text{if } x < t, \\ \frac{u(t; \lambda)v(x; \lambda)}{\psi(\lambda)}, & \text{if } t < x. \end{cases} \quad (4.31)$$

Note that the Green's operator for  $\ell$  (that is, the functional component of the resolvent operator for  $L$ ) involves both the integral operator  $g(x; \lambda) = \int_{-a}^b G(x, t; \lambda) dt$ , with kernel  $G(x, t; \lambda)$ , along with a finite summation, see Section 3.5 in Chapter 3. Define

$$\tilde{G}(x, t; \lambda) = \begin{cases} \frac{\tilde{u}(x; \lambda)v(t; \lambda)}{\psi(\lambda)}, & \text{if } x < t, \\ \frac{u(t; \lambda)\tilde{v}(x; \lambda)}{\psi(\lambda)}, & \text{if } t < x. \end{cases} \quad (4.32)$$

In the remainder of this work we refer to the following decomposition of the eigenvalues of  $(L; \alpha, \beta; r, s)$ .

**Definition 4.3.1.** We denote by

1.  $\Lambda$  the set of eigenvalues  $\lambda_n$  of  $(L; \alpha, \beta; r, s)$  such that either  $r(\lambda_n) \neq 0$  and  $s(\lambda_n) \neq 0$  or  $\lambda_n$  has geometric multiplicity 2 if  $r(\lambda_n) = 0$  or if  $s(\lambda_n) = 0$ ;
2.  $\Lambda^*$ , where  $\Lambda^* \subset \Lambda$ , the set of eigenvalues  $\lambda_n$  of  $(L; \alpha, \beta; r, s)$  of geometric multiplicity 2 such that  $r(\lambda_n) = 0$  and  $s(\lambda_n) = 0$ ;
3.  $\Lambda_1$  the set of eigenvalues  $\lambda_n$  of  $(L; \alpha, \beta; r, s)$  of geometric multiplicity 1 with  $r(\lambda_n) = 0$  or  $s(\lambda_n) = 0$ ;
4.  $\Lambda_0$  the set of eigenvalues  $\lambda_n$  of  $(L; \alpha, \beta; r, s)$  such that either  $\lambda_n \in \sigma(L; \alpha, \beta; r, s) \setminus \sigma(\tilde{L}; \alpha, \beta; r, s)$ , or  $\lambda_n = \tilde{\lambda}_m \in \sigma(\tilde{L}; \alpha, \beta; r, s)$  for some  $m$  and  $\lambda_n$  and  $\tilde{\lambda}_m$  have different geometric multiplicities. Here,  $\sigma(L; \alpha, \beta; r, s)$  (respectively  $\sigma(\tilde{L}; \alpha, \beta; r, s)$ ) denotes the spectrum of  $(L; \alpha, \beta; r, s)$  (respectively  $(\tilde{L}; \alpha, \beta; r, s)$ ). By assumption of the inverse problem,  $\Lambda_0$  is a finite set.

To clarify the notation used in Definition 4.3.1, we remark that an element  $\lambda_n$  of  $\Lambda$ ,  $\Lambda^*$ ,  $\Lambda_1$  or  $\Lambda_0$  is labelled with the subscript  $n$  corresponding to its position in the list (4.15). In particular, if  $\lambda_n = \lambda_{n+1}$ , say, then either both  $\lambda_n$  and  $\lambda_{n+1}$  or neither  $\lambda_n$  nor  $\lambda_{n+1}$  are elements of  $\Lambda$ , and similarly for the sets  $\Lambda^*$  and  $\Lambda_0$ .

A similar decomposition of the eigenvalues of  $(\tilde{L}; \alpha, \beta; r, s)$  can be found corresponding to points 1 to 3 in Definition 4.3.1 above. That is, we define  $\tilde{\Lambda}$ ,  $\tilde{\Lambda}^*$  and  $\tilde{\Lambda}_1$  in an analogous manner.

**Lemma 4.3.2.** *Let  $\lambda_n$  be an eigenvalue of  $(L; \alpha, \beta; r, s)$ .*

1. *If  $r(\lambda_n) \neq 0$  and  $s(\lambda_n) \neq 0$  then  $\lambda_n$  has geometric multiplicity 1 and both  $U_n(x)$  and  $V_n(x)$  are eigenfunctions of  $(L; \alpha, \beta; r, s)$  corresponding to  $\lambda_n$ . Consequently there exists  $k_n \in \mathbb{R} \setminus \{0\}$  such that  $V_n = k_n U_n$ .*
2. *If  $\lambda_n \in \Lambda$  with  $r(\lambda_n) = 0$  or  $s(\lambda_n) = 0$  then we can find a pair of linearly independent eigenfunctions  $Z_n^{(1)}, Z_n^{(2)}$  of  $(L; \alpha, \beta; r, s)$  corresponding to  $\lambda_n$ .*

*Furthermore, if  $\lambda_n \in \Lambda \setminus \Lambda^*$  then both  $U_n(x) = U(x; \lambda_n)$  and  $V_n(x) = V(x; \lambda_n)$  are eigenfunctions corresponding to  $\lambda_n$ . Writing*

$$U_n(x) = U_n^{(1)}(x) + U_n^{(2)}(x), \quad V_n(x) = V_n^{(1)}(x) + V_n^{(2)}(x)$$

*(where  $U_n^{(i)}(x)$  and  $V_n^{(i)}(x)$  are multiples of  $Z_n^{(i)}(x)$  for  $i = 1, 2$ ) we can find constants  $k_n^{(1)}$  and  $k_n^{(2)}$  such that*

$$V_n^{(i)}(x) = k_n^{(i)} U_n^{(i)}(x), \quad i = 1, 2. \quad (4.33)$$

*If  $\lambda_n \in \Lambda^*$  then neither  $u(x; \lambda_n)$  nor  $v(x; \lambda_n)$  are eigenfunctions of  $(\ell; \alpha, \beta; r, s)$  corresponding to  $\lambda_n$ .*

3. *If  $\lambda_n \in \Lambda_1$  then precisely one of  $\chi_{[-a,0)}u(x; \lambda_n)$  or  $\chi_{(0,b]}v(x; \lambda_n)$  is an eigenfunction of  $(\ell; \alpha, \beta; r, s)$  corresponding to  $\lambda_n$ . The corresponding eigenfunction of  $(L; \alpha, \beta; r, s)$  can be constructed using the results of Section 4.2 and will be denoted by  $Z_n$ .*

*Proof.* 1. See Lemma 3.2.1.

2. From Definition 4.3.1  $\lambda_n$  is a geometrically double eigenvalue. The existence of  $Z_n^{(i)}$ ,  $i = 1, 2$  follows from Theorem 3.2.2. The general form of  $Z_n^{(i)}$ ,  $i = 1, 2$  is determined by equations (4.11)-(4.13) (see Theorem 3.3.1 for the derivation). In particular we can make the following choices.

(i) If  $r(\lambda_n) = s(\lambda_n) = 0$  let

$$Z_n^{(1)}(x) = u(0^-; \lambda_n) \begin{bmatrix} \chi_{[-a,0)} w_1(x; \lambda_n) \\ (0) \\ \left( -\frac{\alpha_j}{\lambda_n - \delta_j} \right) \end{bmatrix}, \quad (4.34)$$

$$Z_n^{(2)}(x) = v'(0^+; \lambda_n) \begin{bmatrix} \chi_{(0,b]} w_2(x; \lambda_n) \\ \left( \frac{\beta_i}{\lambda_n - \gamma_i} \right) \\ (0) \end{bmatrix}. \quad (4.35)$$

Note that, in this case,  $Z_n^{(1)}$  and  $Z_n^{(2)}$  as defined above are not only linearly independent but also *orthogonal* with respect to the Hilbert space inner product (4.10).

(ii) If  $r(\lambda_n) = 0$  and  $\lambda_n$  is a pole of  $s(\lambda)$ , say  $\lambda_n = \delta_\mu$  for some  $\mu \in \{1, \dots, M\}$ , let

$$Z_n^{(1)}(x) = \begin{bmatrix} \chi_{[-a,0)} w_2(x; \lambda_n) \\ \left(-\frac{\beta_i}{\lambda_n - \gamma_i}\right) \\ \left(\frac{1}{\alpha_\mu} \delta_{j,\mu}\right) \end{bmatrix}, \quad Z_n^{(2)}(x) = \begin{bmatrix} \chi_{(0,b]} w_2(x; \lambda_n) \\ \left(\frac{\beta_i}{\lambda_n - \gamma_i}\right) \\ (0) \end{bmatrix}. \quad (4.36)$$

Then

$$\begin{aligned} U_n(x) &= u'_n(0^-) Z_n^{(1)}(x) + C_n Z_n^{(2)}(x), \\ V_n(x) &= D_n Z_n^{(1)}(x) + v'_n(0^+) Z_n^{(2)}(x). \end{aligned}$$

Here, we can apply Green's formula (or integrate  $W[u_-(x; \lambda_n), u_-(x; \lambda)]'$  and take the limit as  $\lambda \rightarrow \lambda_n$ ) to obtain

$$C_n = \frac{u'_n(0^-)}{\dot{r}(\lambda_n)} \left[ \int_{-a}^0 w_2^2(\tau; \lambda_n) d\tau + \dot{r}(\lambda_n) + \frac{1}{\alpha_\mu^2} \right], \quad (4.37)$$

and similarly

$$D_n = \frac{v'_n(0^+)}{\dot{r}(\lambda_n)} \left[ \int_0^b w_2^2(\tau; \lambda_n) d\tau + \dot{r}(\lambda_n) \right]. \quad (4.38)$$

Due to the choice of the functional component of  $Z_n^{(1)}$  being identically zero on  $(0, b]$  and that of  $Z_n^{(2)}$  being identically zero on  $[-a, 0)$  an expression of the form (4.33) holds. A straightforward evaluation of the functional components at  $0^\pm$  gives

$$k_n^{(1)} = \frac{D_n}{u'_n(0^-)}, \quad k_n^{(2)} = \frac{v'_n(0^+)}{C_n}. \quad (4.39)$$

(iii) If  $r(\lambda_n) = 0$  and  $s(\lambda_n) \in \mathbb{R} \setminus \{0\}$  let

$$Z_n^{(1)}(x) = \begin{bmatrix} \chi_{[-a,0)} \left[ w_2(x; \lambda_n) - \frac{w_1(x; \lambda_n)}{s(\lambda_n)} \right] \\ \left(-\frac{\beta_i}{\lambda_n - \gamma_i}\right) \\ \left(\frac{1}{s(\lambda_n)} \frac{\alpha_j}{\lambda_n - \delta_j}\right) \end{bmatrix}, \quad (4.40)$$

$$Z_n^{(2)}(x) = \begin{bmatrix} \chi_{(0,b]} w_2(x; \lambda_n) \\ \left(\frac{\beta_i}{\lambda_n - \gamma_i}\right) \\ (0) \end{bmatrix}. \quad (4.41)$$

Then

$$\begin{aligned} U_n(x) &= u'_n(0^-) Z_n^{(1)}(x) + C_n Z_n^{(2)}(x), \\ V_n(x) &= D_n Z_n^{(1)}(x) + v'_n(0^+) Z_n^{(2)}(x) \end{aligned}$$

with

$$C_n = \frac{u'_n(0^-)}{\dot{r}(\lambda_n)} \left[ \int_{-a}^0 \left[ w_2(\tau; \lambda_n) - \frac{w_1(\tau; \lambda_n)}{s(\lambda_n)} \right]^2 d\tau - \frac{\dot{s}(\lambda_n)}{s^2(\lambda_n)} + \dot{r}(\lambda_n) \right], \quad (4.42)$$

$$D_n = \frac{v'_n(0^+)}{\dot{r}(\lambda_n)} \left[ \int_0^b w_2^2(\tau; \lambda_n) d\tau + \dot{r}(\lambda_n) \right], \quad (4.43)$$

$$k_n^{(1)} = \frac{D_n}{u'_n(0^-)}, \quad k_n^{(2)} = \frac{v'_n(0^+)}{C_n}. \quad (4.44)$$

(iv) If  $s(\lambda_n) = 0$  and  $\lambda_n$  is a pole of  $r(\lambda)$ , say  $\lambda_n = \gamma_p$  for some  $p \in \{1, \dots, N\}$ , let

$$Z_n^{(1)}(x) = \begin{bmatrix} \chi_{[-a,0)} w_1(x; \lambda_n) \\ (0) \\ \left(-\frac{\alpha_j}{\lambda_n - \delta_j}\right) \end{bmatrix}, \quad Z_n^{(2)}(x) = \begin{bmatrix} \chi_{(0,b]} w_1(x; \lambda_n) \\ \left(-\frac{1}{\beta_p} \delta_{i,p}\right) \\ \left(\frac{\alpha_j}{\lambda_n - \delta_j}\right) \end{bmatrix}. \quad (4.45)$$

Then

$$\begin{aligned} U_n(x) &= u_n(0^-) Z_n^{(1)}(x) + C_n Z_n^{(2)}(x), \\ V_n(x) &= D_n Z_n^{(1)}(x) + v_n(0^+) Z_n^{(2)}(x) \end{aligned}$$

with

$$C_n = -\frac{u_n(0^-)}{\dot{s}(\lambda_n)} \left[ \int_{-a}^0 w_1^2(\tau; \lambda_n) d\tau - \dot{s}(\lambda_n) \right], \quad (4.46)$$

$$D_n = -\frac{v_n(0^+)}{\dot{s}(\lambda_n)} \left[ \int_0^b w_1^2(\tau; \lambda_n) d\tau - \dot{s}(\lambda_n) + \frac{1}{\beta_p^2} \right], \quad (4.47)$$

$$k_n^{(1)} = \frac{D_n}{u_n(0^-)}, \quad k_n^{(2)} = \frac{v_n(0^+)}{C_n}. \quad (4.48)$$

(v) If  $s(\lambda_n) = 0$  and  $r(\lambda_n) \in \mathbb{R} \setminus \{0\}$  let

$$Z_n^{(1)}(x) = \begin{bmatrix} \chi_{[-a,0)} w_1(x; \lambda_n) \\ (0) \\ \left(-\frac{\alpha_j}{\lambda_n - \delta_j}\right) \end{bmatrix}, \quad (4.49)$$

$$Z_n^{(2)}(x) = \begin{bmatrix} \chi_{(0,b]} \left[ w_1(x; \lambda_n) + \frac{w_2(x; \lambda_n)}{r(\lambda_n)} \right] \\ \left( \frac{1}{r(\lambda_n)} \frac{\beta_i}{\lambda_n - \gamma_i} \right) \\ \left( \frac{\alpha_j}{\lambda_n - \delta_j} \right) \end{bmatrix}. \quad (4.50)$$

Then

$$\begin{aligned} U_n(x) &= u_n(0^-) Z_n^{(1)}(x) + C_n Z_n^{(2)}(x), \\ V_n(x) &= D_n Z_n^{(1)}(x) + v_n(0^+) Z_n^{(2)}(x) \end{aligned}$$

with

$$C_n = -\frac{u_n(0^-)}{\dot{s}(\lambda_n)} \left[ \int_{-a}^0 w_1^2(\tau; \lambda_n) d\tau - \dot{s}(\lambda_n) \right], \quad (4.51)$$

$$D_n = -\frac{v_n(0^+)}{\dot{s}(\lambda_n)} \left[ \int_0^b \left[ w_1(\tau; \lambda_n) + \frac{w_2(\tau; \lambda_n)}{r(\lambda_n)} \right]^2 d\tau + \frac{\dot{r}(\lambda_n)}{r^2(\lambda_n)} - \dot{s}(\lambda_n) \right], \quad (4.52)$$

$$k_n^{(1)} = \frac{D_n}{u_n(0^-)}, \quad k_n^{(2)} = \frac{v_n(0^+)}{C_n}. \quad (4.53)$$

3. By Definition 4.3.1  $\lambda_n$  is geometrically simple. Since  $\chi_{[-a,0)} u(x; \lambda_n)$  and  $\chi_{(0,b]} v(x; \lambda_n)$  are linearly independent at most one of  $\chi_{[-a,0)} u(x; \lambda_n)$  and  $\chi_{(0,b]} v(x; \lambda_n)$  can be an eigenfunction corresponding to  $\lambda_n$ . To conclude we need to examine what happens at  $0^\pm$ . Consider the case when  $r(\lambda_n) = s(\lambda_n) = 0$ . Then the transmission conditions reduce to  $y(0^+) = 0$  and  $y'(0^-) = 0$  at  $\lambda_n$ . If  $u'(0^-; \lambda_n) = 0$  then  $\chi_{[-a,0)} u(x; \lambda_n)$  satisfies (4.1)-(4.5) and is thus an eigenfunction corresponding to  $\lambda_n$ . If  $u'(0^-; \lambda_n) \neq 0$  then in order to obey both (4.2) and  $y'(0^-)$  at  $\lambda_n$  the eigenfunction must be identically zero on  $[-a, 0)$ .

This implies that  $v(0^+; \lambda_n) = 0$ , making  $\chi_{(0,b]}v(x; \lambda_n)$  an eigenfunction corresponding to  $\lambda_n$ , else only the function which is identically zero on both  $[-a, 0)$  and  $(0, b]$  can satisfy the two boundary conditions at  $x = -a$  and  $x = b$ , as well as  $y(0^+) = 0, y'(0^-) = 0$ . The remaining cases can be argued in a like manner.

□

**Note 4.3.3.** I. We denote by  $\Lambda_1^-$  the subset of  $\Lambda_1$  consisting of eigenvalues  $\lambda_n$  for which  $\chi_{[-a,0)}u(x; \lambda_n)$  is an eigenfunction of  $(\ell; \alpha, \beta; r, s)$  corresponding to  $\lambda_n$  (see point 3. of Lemma 4.3.2). Furthermore, we define  $\Lambda_1^+ = \Lambda_1 \setminus \Lambda_1^-$ . Define  $\tilde{\Lambda}_1^-, \tilde{\Lambda}_1^+$  as subsets of  $\tilde{\Lambda}_1$  in an analogous manner.

II. Results analogous to that of Lemma 4.3.2 can be stated for the eigenfunctions of  $(\tilde{L}; \alpha, \beta; r, s)$ . Although, we recall that the shorthand notation used for the eigenfunctions of  $(L; \alpha, \beta; r, s)$  is different from that used for the eigenfunctions of  $(\tilde{L}; \alpha, \beta; r, s)$  (compare equations (4.17) and (4.18)). For  $(L; \alpha, \beta; r, s)$  we index eigenfunctions in reference to the corresponding list of eigenvalues (4.15). Whereas, for  $(\tilde{L}; \alpha, \beta; r, s)$  the short hand notation is only employed for functions whose eigenvalues coincide (up to multiplicity) with eigenvalues of  $(L; \alpha, \beta; r, s)$ . In this case the eigenfunction is labelled using the shorthand notation  $\tilde{F}_n(x) = \tilde{F}(x; \lambda_n)$  linking it to the corresponding element of (4.15). Thus, in order to keep notation consistent with equations (4.17) and (4.18) we state the following partial result for the eigenfunctions of  $(\tilde{L}; \alpha, \beta; r, s)$ , which is sufficient for our purposes.

**Corollary 4.3.4.** In the notation of (4.18), the following results are a consequence of Lemma 4.3.2.

1. If  $\lambda_n \in \Lambda \setminus \Lambda_0$  (so  $\lambda_n \in \tilde{\Lambda}$ ) with  $r(\lambda_n) \neq 0$  and  $s(\lambda_n) \neq 0$  then there exists  $\tilde{k}_n \in \mathbb{R} \setminus \{0\}$  such that

$$\tilde{V}_n = \tilde{k}_n \tilde{U}_n.$$

2. If  $\lambda_n \in \Lambda \setminus \Lambda_0$  with  $r(\lambda_n) = 0$  or  $s(\lambda_n) = 0$  then we can find a pair of linearly independent eigenfunctions  $\tilde{Z}_n^{(1)}, \tilde{Z}_n^{(2)}$  of  $(\tilde{L}; \alpha, \beta; r, s)$  corresponding to  $\lambda_n$ .

Furthermore, if  $\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)$  (so  $\lambda_n \in \tilde{\Lambda} \setminus \tilde{\Lambda}^*$ ) then both  $\tilde{U}_n(x) = \tilde{U}(x; \lambda_n)$  and  $\tilde{V}_n(x) = \tilde{V}(x; \lambda_n)$  are eigenfunctions of  $(\tilde{L}; \alpha, \beta; r, s)$  with eigenvalue  $\lambda_n$ . Writing

$$\tilde{U}_n(x) = \tilde{U}_n^{(1)}(x) + \tilde{U}_n^{(2)}(x), \quad \tilde{V}_n(x) = \tilde{V}_n^{(1)}(x) + \tilde{V}_n^{(2)}(x)$$

(where  $\tilde{U}_n^{(i)}(x)$  and  $\tilde{V}_n^{(i)}(x)$  are multiples of  $\tilde{Z}_n^{(i)}(x)$  for  $i = 1, 2$ ) we can find constants  $\tilde{k}_n^{(1)}$  and  $\tilde{k}_n^{(2)}$  such that

$$\tilde{V}_n^{(i)}(x) = \tilde{k}_n^{(i)} \tilde{U}_n^{(i)}(x), \quad i = 1, 2.$$

Here, for  $i = 1, 2$ ,  $\tilde{Z}_n^{(i)}$  and  $\tilde{k}_n^{(i)}$  are chosen in an analogous manner to  $Z_n^{(i)}$  and  $k_n^{(i)}$ , respectively, as in the proof of Lemma 4.3.2.

Whereas, if  $\lambda_n \in \Lambda^* \setminus \Lambda_0$  (so  $\lambda_n \in \tilde{\Lambda}^*$ ) then neither  $\tilde{u}(x; \lambda_n)$  nor  $\tilde{v}(x; \lambda_n)$  are eigenfunctions of  $(\tilde{\ell}; \alpha, \beta; r, s)$  corresponding to  $\lambda_n$ .

3. If  $\lambda_n \in \Lambda_1 \setminus \Lambda_0$  (so  $\lambda_n \in \tilde{\Lambda}_1$ ) then precisely one of  $\chi_{[-a,0)}\tilde{u}(x; \lambda_n)$  or  $\chi_{(0,b]}\tilde{v}(x; \lambda_n)$  is an eigenfunction of  $(\tilde{\ell}; \alpha, \beta; r, s)$  corresponding to  $\lambda_n$ . The corresponding eigenfunction of  $(\tilde{L}; \alpha, \beta; r, s)$  can be constructed using the results of Section 4.2 and will be denoted by  $\tilde{Z}_n$ .

**Theorem 4.3.5.** *Suppose that  $\lambda \neq \lambda_n$ ,  $n \in \mathbb{N}_0$ . Then the following expansions are valid and converge uniformly for  $x, t \in [-a, 0) \cup (0, b]$ ,  $x \neq t$ .*

If  $x < t$ ,

$$G(x, t; \lambda) = \sum_{\lambda_m \in \Lambda \setminus \Lambda^*} \frac{u_m(x)v_m(t)}{(\lambda - \lambda_m)\dot{\psi}(\lambda_m)} + \sum_{\lambda_m \in \Lambda^*} \left[ \frac{z_m^{(1)}(x)z_m^{(1)}(t)}{(\lambda - \lambda_m) \|Z_m^{(1)}\|^2} + \frac{z_m^{(2)}(x)z_m^{(2)}(t)}{(\lambda - \lambda_m) \|Z_m^{(2)}\|^2} \right] \\ + \sum_{\lambda_m \in \Lambda_1} \frac{z_m(x)z_m(t)}{(\lambda - \lambda_m) \|Z_m\|^2}$$

and

$$\tilde{G}(x, t; \lambda) = \sum_{\lambda_m \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)} \frac{\tilde{u}_m(x)v_m(t)}{(\lambda - \lambda_m)\dot{\psi}(\lambda_m)} \\ + \sum_{\lambda_m \in \Lambda^* \setminus \Lambda_0} \left[ \frac{\tilde{z}_m^{(1)}(x)z_m^{(1)}(t)}{(\lambda - \lambda_m) \|Z_m^{(1)}\|^2} + \frac{\tilde{u}(0^+; \lambda_m) v'(0^+; \lambda_m)}{u(0^+; \lambda_m) \tilde{v}'(0^+; \lambda_m)} \frac{\tilde{z}_m^{(2)}(x)z_m^{(2)}(t)}{(\lambda - \lambda_m) \|Z_m^{(2)}\|^2} \right] \\ + \sum_{\lambda_m \in \Lambda_1^- \setminus \Lambda_0} \frac{\tilde{z}_m(x)z_m(t)}{(\lambda - \lambda_m) \|Z_m\|^2} + \sum_{\lambda_m \in \Lambda_1^+ \setminus \Lambda_0} \frac{T_m^+ \tilde{z}_m(x)z_m(t)}{(\lambda - \lambda_m) \|Z_m\|^2} \\ + R(x, t; \lambda),$$

where

$$T_m^+ = \begin{cases} \frac{\tilde{u}'(0^-; \lambda_m) v'(0^+; \lambda_m)}{\tilde{u}'(0^-; \lambda_m) \tilde{v}'(0^+; \lambda_m)}, & \text{if } r(\lambda_m) = 0, s(\lambda_m) = 0, \\ \frac{\tilde{u}(0^-; \lambda_m) v'(0^+; \lambda_m)}{\tilde{u}(0^-; \lambda_m) \tilde{v}'(0^+; \lambda_m)}, & \text{if } r(\lambda_m) = 0, \lambda_m \text{ is a pole of } s(\lambda), \\ \frac{\tilde{u}(0^-; \lambda_m) + \frac{\tilde{u}'(0^-; \lambda_m)}{s(\lambda_m)} v'(0^+; \lambda_m)}{u(0^-; \lambda_m) + \frac{u'(0^-; \lambda_m)}{s(\lambda_m)} \tilde{v}'(0^+; \lambda_m)}, & \text{if } r(\lambda_m) = 0, s(\lambda_m) \in \mathbb{R} \setminus \{0\}, \\ \frac{\tilde{u}'(0^-; \lambda_m) v(0^+; \lambda_m)}{\tilde{u}'(0^-; \lambda_m) \tilde{v}(0^+; \lambda_m)}, & \text{if } s(\lambda_m) = 0, \lambda_m \text{ is a pole of } r(\lambda), \\ \frac{\tilde{u}'(0^-; \lambda_m) v(0^+; \lambda_m)}{\tilde{u}'(0^-; \lambda_m) \tilde{v}(0^+; \lambda_m)}, & \text{if } s(\lambda_m) = 0, r(\lambda_m) \in \mathbb{R} \setminus \{0\}, \end{cases}$$

and

1. if  $-a \leq x < 0 < t \leq b$  then

$$R(x, t; \lambda) = \sum_{\lambda_m \in (\Lambda \setminus \Lambda^*) \cap \Lambda_0} \frac{\tilde{u}(x; \lambda_m)v_m(t)}{(\lambda - \lambda_m)\dot{\psi}(\lambda_m)}.$$

2. if  $-a \leq x < t < 0$  then

$$R(x, t; \lambda) = \sum_{\lambda_m \in \Lambda^* \cap \Lambda_0} \frac{\tilde{u}(x; \lambda_m)z_m^{(1)}(t)}{(\lambda - \lambda_m) \|Z_m^{(1)}\|^2} + \sum_{\lambda_m \in (\Lambda \setminus \Lambda^*) \cap \Lambda_0} \frac{\tilde{u}(x; \lambda_m)v_m(t)}{(\lambda - \lambda_m)\dot{\psi}(\lambda_m)} \\ + \sum_{\lambda_m \in \Lambda_1^- \cap \Lambda_0} \frac{\tilde{u}(x; \lambda_m)z_m(t)}{(\lambda - \lambda_m) \|Z_m\|^2}.$$

3. if  $0 < x < t \leq b$  then

$$\begin{aligned}
R(x, t; \lambda) = & \sum_{\lambda_m \in (\Lambda^* \cap \Lambda_0 \cap \bar{\Lambda}_1^-)} \frac{\tilde{u}(0^+; \lambda_m) v'(0^+; \lambda_m)}{u(0^+; \lambda_m)} \frac{\tilde{w}_2(x; \lambda_m) z_m^{(2)}(t)}{(\lambda - \lambda_m) \|Z_m^{(2)}\|^2} \\
& + \sum_{\lambda_m \in (\Lambda^* \cap \Lambda_0) \setminus \bar{\Lambda}_1^-} \left\{ \frac{\Phi_m [\tilde{y}_\lambda(x; \lambda_m) z_m^{(2)}(t) + \tilde{y}(x; \lambda_m) v_\lambda(t; \lambda_m)]}{\lambda - \lambda_m} \right. \\
& \quad \left. + \left[ \dot{\Phi}_m + \frac{\Phi_m}{\lambda - \lambda_m} \right] \frac{\tilde{y}(x; \lambda_m) z_m^{(2)}(t)}{\lambda - \lambda_m} \right\} \\
& + \sum_{\lambda_m \in (\Lambda \setminus \Lambda^*) \cap \bar{\Lambda}_1^-} \frac{\tilde{u}(x; \lambda_m) v_m(t)}{(\lambda - \lambda_m) \dot{\psi}(\lambda_m)} \\
& + \sum_{\lambda_m \in (\Lambda \cap \Lambda_0) \setminus (\Lambda^* \cup \bar{\Lambda}_1^-)} \left\{ \frac{\Phi_m [\tilde{y}_\lambda(x; \lambda_m) v_m(t) + \tilde{y}(x; \lambda_m) v_\lambda(t; \lambda_m)]}{\lambda - \lambda_m} \right. \\
& \quad \left. + \left[ \dot{\Phi}_m + \frac{\Phi_m}{\lambda - \lambda_m} \right] \frac{\tilde{y}(x; \lambda_m) v_m(t)}{\lambda - \lambda_m} \right\} \\
& + \sum_{\lambda_m \in \Lambda_1 \cap \Lambda_0} \frac{\tilde{y}(x; \lambda_m) v(t; \lambda_m)}{(\lambda - \lambda_m) \dot{\phi}(\lambda_m)}
\end{aligned}$$

with  $\tilde{y}(x; \lambda)$ ,  $\phi(\lambda)$ ,  $\Phi_m$  and  $\dot{\Phi}_m$  as given in Proposition 4.6.2.

Else if  $t < x$ ,

$$\begin{aligned}
G(x, t; \lambda) = & \sum_{\lambda_m \in \Lambda \setminus \Lambda^*} \frac{u_m(t) v_m(x)}{(\lambda - \lambda_m) \dot{\psi}(\lambda_m)} + \sum_{\lambda_m \in \Lambda^*} \left[ \frac{z_m^{(1)}(x) z_m^{(1)}(t)}{(\lambda - \lambda_m) \|Z_m^{(1)}\|^2} + \frac{z_m^{(2)}(x) z_m^{(2)}(t)}{(\lambda - \lambda_m) \|Z_m^{(2)}\|^2} \right] \\
& + \sum_{\lambda_m \in \Lambda_1} \frac{z_m(x) z_m(t)}{(\lambda - \lambda_m) \|Z_m\|^2}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{G}(x, t; \lambda) = & \sum_{\lambda_m \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)} \frac{u_m(t) \tilde{v}_m(x)}{(\lambda - \lambda_m) \dot{\psi}(\lambda_m)} \\
& + \sum_{\lambda_m \in \Lambda^* \setminus \Lambda_0} \left[ \frac{\tilde{v}'(0^-; \lambda_m) u(0^-; \lambda_m)}{v'(0^-; \lambda_m) \tilde{u}(0^-; \lambda_m)} \frac{\tilde{z}_m^{(1)}(x) z_m^{(1)}(t)}{(\lambda - \lambda_m) \|Z_m^{(1)}\|^2} + \frac{\tilde{z}_m^{(2)}(x) z_m^{(2)}(t)}{(\lambda - \lambda_m) \|Z_m^{(2)}\|^2} \right] \\
& + \sum_{\lambda_m \in \Lambda_1^+ \setminus \Lambda_0} \frac{\tilde{z}_m(x) z_m(t)}{(\lambda - \lambda_m) \|Z_m\|^2} + \sum_{\lambda_m \in \Lambda_1^- \setminus \Lambda_0} \frac{T_m^- \tilde{z}_m(x) z_m(t)}{(\lambda - \lambda_m) \|Z_m\|^2} \\
& + R(x, t; \lambda),
\end{aligned}$$



where

$$T_m^- = \begin{cases} \frac{\tilde{v}(0^+; \lambda_m) u(0^-; \lambda_m)}{v(0^+; \lambda_m) \tilde{u}(0^-; \lambda_m)}, & \text{if } r(\lambda_m) = 0, s(\lambda_m) = 0, \\ \frac{\tilde{v}(0^+; \lambda_m) u'(0^-; \lambda_m)}{v(0^+; \lambda_m) \tilde{u}'(0^-; \lambda_m)}, & \text{if } r(\lambda_m) = 0, \lambda_m \text{ is a pole of } s(\lambda), \\ \frac{\tilde{v}(0^+; \lambda_m) u'(0^-; \lambda_m)}{v(0^+; \lambda_m) \tilde{u}'(0^-; \lambda_m)}, & \text{if } r(\lambda_m) = 0, s(\lambda_m) \in \mathbb{R} \setminus \{0\}, \\ \frac{\tilde{v}'(0^+; \lambda_m) u(0^-; \lambda_m)}{v'(0^+; \lambda_m) \tilde{u}(0^-; \lambda_m)}, & \text{if } s(\lambda_m) = 0, \lambda_m \text{ is a pole of } r(\lambda), \\ \frac{\tilde{v}'(0^+; \lambda_m) - \frac{\tilde{v}(0^+; \lambda_m)}{r(\lambda_m)} u(0^-; \lambda_m)}{v'(0^+; \lambda_m) - \frac{v(0^+; \lambda_m)}{r(\lambda_m)} \tilde{u}(0^-; \lambda_m)}, & \text{if } s(\lambda_m) = 0, r(\lambda_m) \in \mathbb{R} \setminus \{0\}, \end{cases}$$

and

1. if  $-a \leq t < 0 < x \leq b$  then

$$R(x, t; \lambda) = \sum_{\lambda_m \in (\Lambda \setminus \Lambda^*) \cap \Lambda_0} \frac{u_m(t) \tilde{v}(x; \lambda_m)}{(\lambda - \lambda_m) \dot{\psi}(\lambda_m)}.$$

2. if  $-a \leq t < x < 0$  then

$$\begin{aligned} R(x, t; \lambda) = & \sum_{\lambda_m \in \Lambda^* \cap \tilde{\Lambda}_1^+} \frac{\tilde{v}'(0^-; \lambda_m) u(0^-; \lambda_m)}{v'(0^-; \lambda_m)} \frac{\tilde{w}_1(x; \lambda_m) z_m^{(1)}(t)}{(\lambda - \lambda_m) \|Z_m^{(1)}\|^2} \\ & + \sum_{\lambda_m \in (\Lambda^* \cap \Lambda_0) \setminus \tilde{\Lambda}_1^+} \left\{ \frac{\Phi_m [\tilde{y}_\lambda(x; \lambda_m) z_m^{(1)}(t) + \tilde{y}(x; \lambda_m) u_\lambda(t; \lambda_m)]}{\lambda - \lambda_m} \right. \\ & \quad \left. + \left[ \dot{\Phi}_m + \frac{\Phi_m}{\lambda - \lambda_m} \right] \frac{\tilde{y}(x; \lambda_m) z_m^{(1)}(t)}{\lambda - \lambda_m} \right\} \\ & + \sum_{\lambda_m \in (\Lambda \setminus \Lambda^*) \cap \tilde{\Lambda}_1^+} \frac{u_m(t) \tilde{v}(x; \lambda_m)}{(\lambda - \lambda_m) \dot{\psi}(\lambda_m)} \\ & + \sum_{\lambda_m \in (\Lambda \cap \Lambda_0) \setminus (\Lambda^* \cup \tilde{\Lambda}_1^+)} \left\{ \frac{\Phi_m [\tilde{y}_\lambda(x; \lambda_m) u_m(t) + \tilde{y}(x; \lambda_m) u_\lambda(t; \lambda_m)]}{\lambda - \lambda_m} \right. \\ & \quad \left. + \left[ \dot{\Phi}_m + \frac{\Phi_m}{\lambda - \lambda_m} \right] \frac{\tilde{y}(x; \lambda_m) u_m(t)}{\lambda - \lambda_m} \right\} \\ & + \sum_{\lambda_m \in \Lambda_1 \cap \Lambda_0} \frac{u(t; \lambda_m) \tilde{y}(x; \lambda_m)}{(\lambda - \lambda_m) \dot{\phi}(\lambda_m)} \end{aligned}$$

with  $\tilde{y}(x; \lambda)$ ,  $\phi(\lambda)$ ,  $\Phi_m$  and  $\dot{\Phi}_m$  as given in Proposition 4.6.2.

3. if  $0 < t < x \leq b$  then

$$\begin{aligned} R(x, t; \lambda) = & \sum_{\lambda_m \in \Lambda^* \cap \Lambda_0} \frac{z_m^{(2)}(t) \tilde{v}(x; \lambda_m)}{(\lambda - \lambda_m) \|Z_m^{(2)}\|^2} + \sum_{\lambda_m \in (\Lambda \setminus \Lambda^*) \cap \Lambda_0} \frac{u_m(t) \tilde{v}(x; \lambda_m)}{(\lambda - \lambda_m) \dot{\psi}(\lambda_m)} \\ & + \sum_{\lambda_m \in \Lambda_1^+ \cap \Lambda_0} \frac{z_m(t) \tilde{v}(x; \lambda_m)}{(\lambda - \lambda_m) \|Z_m\|^2}. \end{aligned}$$

Moreover, if  $-a \leq x < 0$  then for  $i = \overline{1, N}$ ,

$$\frac{\beta_i \Delta v' u(x; \lambda)}{\lambda - \gamma_i \psi(\lambda)} = \sum_{\lambda_m \in \Lambda \setminus \Lambda^*} \frac{v_{i,m}^1 u_m(x)}{(\lambda - \lambda_m) \dot{\psi}(\lambda_m)} + \sum_{\lambda_m \in \Lambda_1^-} \frac{z_{i,m}^1 z_m(x)}{(\lambda - \lambda_m) \|Z_m\|^2},$$

$$\frac{\beta_i \Delta v' \tilde{u}(x; \lambda)}{\lambda - \gamma_i \psi(\lambda)} = \sum_{\lambda_m \in \Lambda \setminus \Lambda^*} \frac{v_{i,m}^1 \tilde{u}(x; \lambda_m)}{(\lambda - \lambda_m) \dot{\psi}(\lambda_m)} + \sum_{\lambda_m \in \Lambda_1^-} \frac{z_{i,m}^1 \tilde{u}(x; \lambda_m)}{(\lambda - \lambda_m) \|Z_m\|^2},$$

and for  $j = \overline{1, M}$ ,

$$\begin{aligned} \frac{\alpha_j \Delta v u(x; \lambda)}{\lambda - \delta_j \psi(\lambda)} &= \sum_{\lambda_m \in \Lambda \setminus \Lambda^*} \frac{v_{j,m}^2 u_m(x)}{(\lambda - \lambda_m) \dot{\psi}(\lambda_m)} + \sum_{\lambda_m \in \Lambda^*} \frac{z_{j,m}^{(1)2} z_m^{(1)}(x)}{(\lambda - \lambda_m) \|Z_m^{(1)}\|^2} \\ &+ \sum_{\lambda_m \in \Lambda_1^-} \frac{z_{j,m}^2 z_m(x)}{(\lambda - \lambda_m) \|Z_m\|^2}, \end{aligned}$$

$$\begin{aligned} \frac{\alpha_j \Delta v \tilde{u}(x; \lambda)}{\lambda - \delta_j \psi(\lambda)} &= \sum_{\lambda_m \in \Lambda \setminus \Lambda^*} \frac{v_{j,m}^2 \tilde{u}(x; \lambda_m)}{(\lambda - \lambda_m) \dot{\psi}(\lambda_m)} + \sum_{\lambda_m \in \Lambda^*} \frac{z_{j,m}^{(1)2} \tilde{u}(x; \lambda_m)}{(\lambda - \lambda_m) \|Z_m^{(1)}\|^2} \\ &+ \sum_{\lambda_m \in \Lambda_1^-} \frac{z_{j,m}^2 \tilde{u}(x; \lambda_m)}{(\lambda - \lambda_m) \|Z_m\|^2}. \end{aligned}$$

Similarly, if  $0 < x \leq b$  then for  $i = \overline{1, N}$ ,

$$\begin{aligned} \frac{\beta_i \Delta u' v(x; \lambda)}{\lambda - \gamma_i \psi(\lambda)} &= \sum_{\lambda_m \in \Lambda \setminus \Lambda^*} \frac{u_{i,m}^1 v_m(x)}{(\lambda - \lambda_m) \dot{\psi}(\lambda_m)} + \sum_{\lambda_m \in \Lambda^*} \frac{z_{i,m}^{(2)1} z_m^{(2)}(x)}{(\lambda - \lambda_m) \|Z_m^{(2)}\|^2} \\ &+ \sum_{\lambda_m \in \Lambda_1^+} \frac{z_{i,m}^1 z_m(x)}{(\lambda - \lambda_m) \|Z_m\|^2}, \end{aligned}$$

$$\begin{aligned} \frac{\beta_i \Delta u' \tilde{v}(x; \lambda)}{\lambda - \gamma_i \psi(\lambda)} &= \sum_{\lambda_m \in \Lambda \setminus \Lambda^*} \frac{u_{i,m}^1 \tilde{v}(x; \lambda_m)}{(\lambda - \lambda_m) \dot{\psi}(\lambda_m)} + \sum_{\lambda_m \in \Lambda^*} \frac{z_{i,m}^{(2)1} \tilde{v}(x; \lambda_m)}{(\lambda - \lambda_m) \|Z_m^{(2)}\|^2} \\ &+ \sum_{\lambda_m \in \Lambda_1^+} \frac{z_{i,m}^1 \tilde{v}(x; \lambda_m)}{(\lambda - \lambda_m) \|Z_m\|^2}, \end{aligned}$$

and for  $j = \overline{1, M}$ ,

$$\frac{\alpha_j \Delta u v(x; \lambda)}{\lambda - \delta_j \psi(\lambda)} = \sum_{\lambda_m \in \Lambda \setminus \Lambda^*} \frac{u_{j,m}^2 v_m(x)}{(\lambda - \lambda_m) \dot{\psi}(\lambda_m)} + \sum_{\lambda_m \in \Lambda_1^+} \frac{z_{j,m}^2 z_m(x)}{(\lambda - \lambda_m) \|Z_m\|^2},$$

$$\frac{\alpha_j \Delta u \tilde{v}(x; \lambda)}{\lambda - \delta_j \psi(\lambda)} = \sum_{\lambda_m \in \Lambda \setminus \Lambda^*} \frac{u_{j,m}^2 \tilde{v}(x; \lambda_m)}{(\lambda - \lambda_m) \dot{\psi}(\lambda_m)} + \sum_{\lambda_m \in \Lambda_1^+} \frac{z_{j,m}^2 \tilde{v}(x; \lambda_m)}{(\lambda - \lambda_m) \|Z_m\|^2}.$$

Lastly, for  $x \in [-a, 0) \cup (0, b]$ ,

$$\begin{aligned} \frac{[\tilde{v}u - \tilde{u}v](x; \lambda)}{\psi(\lambda)} &= \sum_{\lambda_m \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)} \frac{[\tilde{v}_m u_m - \tilde{u}_m v_m](x)}{(\lambda - \lambda_m) \dot{\psi}(\lambda_m)} \\ &+ \sum_{\lambda_m \in \Lambda^* \setminus \Lambda_0} \left\{ \begin{aligned} &\left[ \frac{\tilde{v}'(0^-; \lambda_m) u(0^-; \lambda_m)}{v'(0^-; \lambda_m) \tilde{u}(0^-; \lambda_m)} - 1 \right] \frac{\tilde{z}_m^{(1)}(x) z_m^{(1)}(x)}{(\lambda - \lambda_m) \|Z_m^{(1)}\|^2} \\ &+ \left[ 1 - \frac{\tilde{u}(0^+; \lambda_m) v'(0^+; \lambda_m)}{u(0^+; \lambda_m) \tilde{v}'(0^+; \lambda_m)} \right] \frac{\tilde{z}_m^{(2)}(x) z_m^{(2)}(x)}{(\lambda - \lambda_m) \|Z_m^{(2)}\|^2} \end{aligned} \right\} \\ &+ \sum_{\lambda_m \in \Lambda_1^- \setminus \Lambda_0} [T_m^- - 1] \frac{\tilde{z}_m(x) z_m(x)}{(\lambda - \lambda_m) \|Z_m\|^2} \\ &+ \sum_{\lambda_m \in \Lambda_1^+ \setminus \Lambda_0} [1 - T_m^+] \frac{\tilde{z}_m(x) z_m(x)}{(\lambda - \lambda_m) \|Z_m\|^2} + \tilde{R}(x; \lambda), \end{aligned}$$

where  $T_m^-, T_m^+$  are defined above, and

1. if  $-a \leq x < 0$ , then

$$\begin{aligned} &\tilde{R}(x; \lambda) \\ &= \sum_{\lambda_m \in \Lambda^* \cap \tilde{\Lambda}_1^+} \left[ \frac{\tilde{v}'(0^-; \lambda_m) u(0^-; \lambda_m)}{v'(0^-; \lambda_m)} \tilde{w}_1(x; \lambda_m) - \tilde{u}(x; \lambda_m) \right] \frac{z_m^{(1)}(x)}{(\lambda - \lambda_m) \|Z_m^{(1)}\|^2} \\ &+ \sum_{\lambda_m \in (\Lambda^* \cap \Lambda_0) \setminus \tilde{\Lambda}_1^+} \left\{ \begin{aligned} &\left[ \dot{\Phi}_m \tilde{y}(x; \lambda_m) + \Phi_m \tilde{y}_\lambda(x; \lambda_m) - \frac{\tilde{u}(x; \lambda_m)}{\|Z_m^{(1)}\|^2} \right] \frac{z_m^{(1)}(x)}{\lambda - \lambda_m} \\ &+ \left[ u_\lambda(x; \lambda_m) + \frac{u(x; \lambda_m)}{\lambda - \lambda_m} \right] \frac{\Phi_m \tilde{y}(x; \lambda_m)}{\lambda - \lambda_m} \end{aligned} \right\} \\ &+ \sum_{\lambda_m \in (\Lambda \setminus \Lambda^*) \cap \tilde{\Lambda}_1^+} \frac{[\tilde{v}u - \tilde{u}v](x; \lambda_m)}{(\lambda - \lambda_m) \dot{\psi}(\lambda_m)} \\ &+ \sum_{\lambda_m \in (\Lambda^* \cap \Lambda_0) \setminus \tilde{\Lambda}_1^+} \left\{ \begin{aligned} &\left[ \dot{\Phi}_m \tilde{y}(x; \lambda_m) + \Phi_m \tilde{y}_\lambda(x; \lambda_m) - \frac{k_m \tilde{u}(x; \lambda_m)}{\dot{\psi}(\lambda_m)} \right] \frac{u_m(x)}{\lambda - \lambda_m} \\ &+ \left[ u_\lambda(x; \lambda_m) + \frac{u(x; \lambda_m)}{\lambda - \lambda_m} \right] \frac{\Phi_m \tilde{y}(x; \lambda_m)}{\lambda - \lambda_m} \end{aligned} \right\} \\ &+ \sum_{\lambda_m \in \Lambda_1^- \cap \Lambda_0} \left[ \frac{\tilde{y}(x; \lambda_m)}{\dot{\phi}(\lambda_m)} - \frac{\tilde{u}(x; \lambda_m)}{\|Z_m\|^2} \right] \frac{z_m(x)}{\lambda - \lambda_m} + \sum_{\lambda_m \in \Lambda_1^+ \cap \Lambda_0} \frac{u(x; \lambda_m) \tilde{y}(x; \lambda_m)}{(\lambda - \lambda_m) \dot{\phi}(\lambda_m)}, \end{aligned}$$

2. if  $0 < x \leq b$ , then

$$\begin{aligned}
& \tilde{R}(x; \lambda) \\
&= \sum_{\lambda_m \in (\Lambda^* \cap \Lambda_0 \cap \tilde{\Lambda}_1^-)} \left[ \tilde{v}(x; \lambda_m) - \frac{\tilde{u}(0^+; \lambda_m) v'(0^+; \lambda_m)}{u(0^+; \lambda_m)} \tilde{w}_2(x; \lambda_m) \right] \frac{z_m^{(2)}(x)}{(\lambda - \lambda_m) \|Z_m^{(2)}\|^2} \\
&+ \sum_{\lambda_m \in (\Lambda^* \cap \Lambda_0) \setminus \tilde{\Lambda}_1^-} \left\{ \left[ \frac{\tilde{v}(x; \lambda_m)}{\|Z_m^{(2)}\|^2} - \dot{\Phi}_m \tilde{y}(x; \lambda_m) - \Phi_m \tilde{y}_\lambda(x; \lambda_m) \right] \frac{z_m^{(2)}(x)}{\lambda - \lambda_m} \right. \\
&\quad \left. - \left[ v_\lambda(x; \lambda_m) + \frac{v(x; \lambda_m)}{\lambda - \lambda_m} \right] \frac{\Phi_m \tilde{y}(x; \lambda_m)}{\lambda - \lambda_m} \right\} \\
&+ \sum_{\lambda_m \in (\Lambda \setminus \Lambda^*) \cap \tilde{\Lambda}_1^-} \frac{[\tilde{v}u - \tilde{u}v](x; \lambda_m)}{(\lambda - \lambda_m) \dot{\psi}(\lambda_m)} \\
&+ \sum_{\lambda_m \in (\Lambda \cap \Lambda_0) \setminus (\Lambda^* \cup \tilde{\Lambda}_1^-)} \left\{ \left[ \frac{1}{k_m} \frac{\tilde{v}(x; \lambda_m)}{\dot{\psi}(\lambda_m)} - \dot{\Phi}_m \tilde{y}(x; \lambda_m) - \Phi_m \tilde{y}_\lambda(x; \lambda_m) \right] \frac{v_m(x)}{\lambda - \lambda_m} \right. \\
&\quad \left. - \left[ v_\lambda(x; \lambda_m) + \frac{v(x; \lambda_m)}{\lambda - \lambda_m} \right] \frac{\Phi_m \tilde{y}(x; \lambda_m)}{\lambda - \lambda_m} \right\} \\
&+ \sum_{\lambda_m \in \Lambda_1^+ \cap \Lambda_0} \left[ \frac{\tilde{v}(x; \lambda_m)}{\|Z_m\|^2} - \frac{\tilde{y}(x; \lambda_m)}{\dot{\phi}(\lambda_m)} \right] \frac{z_m(t)}{\lambda - \lambda_m} - \sum_{\lambda_m \in \Lambda_1^- \cap \Lambda_0} \frac{\tilde{y}(x; \lambda_m) v(t; \lambda_m)}{(\lambda - \lambda_m) \dot{\phi}(\lambda_m)},
\end{aligned}$$

with  $\tilde{y}(x; \lambda)$ ,  $\phi(\lambda)$ ,  $\Phi_m$  and  $\dot{\Phi}_m$  as given in Proposition 4.6.2.

*Proof.* From the eigenvalue asymptotics in Theorem 4.6.1 of the appendix we deduce that there exists a sequence  $(A_n)_{n \geq N_0}$ , for some  $N_0 \in \mathbb{N}$  sufficiently large,

$$(-\infty, A_{N_0}), [A_{N_0}, A_{N_0+1}), \dots, [A_n, A_{n+1}), \dots$$

is a partition of the real line of the  $\sqrt{\lambda}$ -plane with  $\Sigma_n \subset (A_n, A_{n+1})$ , where  $\{\sqrt{\lambda_n} : n \geq N_0\} = \bigcup_{k=N_0}^{\infty} \Sigma_k$ , and  $\Sigma_k$  is defined explicitly in Theorem 4.6.1. We refer the interested reader to the proof of Theorem 3.6.2 in Chapter 3 for details of the construction of the  $A_n$ . For our purposes it is sufficient to note that  $A_n = O(n)$  for large  $n$ . Let  $\Gamma_n = \left\{ (A_n e^{i\vartheta})^2 : \vartheta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \right\}$ ,  $n \geq N_0$ . Then  $\Gamma_n$  is a path in  $\mathbb{C}$  which encloses precisely  $n(p+q) + \kappa$  eigenvalues of  $(L; \alpha, \beta; r, s)$  (again see Theorem 4.6.1).

Let  $\lambda \in \Gamma_n$  and  $\eta = |\Im(\sqrt{\lambda})|$ . Then, it follows from the approximations of Theorem 4.6.1 that

$$|\psi(\lambda)| \geq \begin{cases} \frac{A_n^3}{\sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2} \sinh \eta a \sinh \eta b + O(n^2 e^{\eta(a+b)}), & \text{if } \alpha = 0, \beta = \pi, \\ \frac{A_n^4 \sin \alpha}{\sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2} \sinh \eta a \sinh \eta b + O(n^3 e^{\eta(a+b)}), & \text{if } \alpha \in (0, \pi), \beta = \pi, \\ \frac{A_n^4 \sin \beta}{\sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2} \sinh \eta a \sinh \eta b + O(n^3 e^{\eta(a+b)}), & \text{if } \alpha = 0, \beta \in (0, \pi), \\ \frac{A_n^5 \sin \alpha \sin \beta}{\sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2} \sinh \eta a \sinh \eta b + O(n^4 e^{\eta(a+b)}), & \text{if } \alpha, \beta \in (0, \pi), \end{cases}$$

if  $\eta \neq 0$ , else

$$\psi(\lambda) = \begin{cases} O(n^3), & \text{if } \alpha = 0, \beta = \pi, \\ O(n^4), & \text{if } \alpha \in (0, \pi), \beta = \pi, \\ O(n^4), & \text{if } \alpha = 0, \beta \in (0, \pi), \\ O(n^5), & \text{if } \alpha, \beta \in (0, \pi). \end{cases}$$

Moreover, if  $-a \leq t < x < 0$  or  $0 < t < x \leq b$  then

$$u(t; \lambda)v(x; \lambda) = \begin{cases} O(n^2 e^{\eta(a+b-|x-t|)}), & \text{if } \alpha = 0, \beta = \pi, \\ O(n^3 e^{\eta(a+b-|x-t|)}), & \text{if } \alpha \in (0, \pi), \beta = \pi, \\ O(n^3 e^{\eta(a+b-|x-t|)}), & \text{if } \alpha = 0, \beta \in (0, \pi), \\ O(n^4 e^{\eta(a+b-|x-t|)}), & \text{if } \alpha, \beta \in (0, \pi), \end{cases}$$

and similarly for  $u(x; \lambda)v(t; \lambda)$  if  $-a \leq x < t < 0$  or  $0 < x < t \leq b$ .

Whereas, if  $-a \leq t < 0 < x \leq b$  then

$$u(t; \lambda)v(x; \lambda) = \begin{cases} O\left(\frac{e^{\eta(a+b-|x-t|)}}{n^2}\right), & \text{if } \alpha = 0, \beta = \pi, \\ O\left(\frac{e^{\eta(a+b-|x-t|)}}{n}\right), & \text{if } \alpha \in (0, \pi), \beta = \pi, \\ O\left(\frac{e^{\eta(a+b-|x-t|)}}{n}\right), & \text{if } \alpha = 0, \beta \in (0, \pi), \\ O(e^{\eta(a+b-|x-t|)}), & \text{if } \alpha, \beta \in (0, \pi), \end{cases}$$

and similarly for  $u(x; \lambda)v(t; \lambda)$  if  $-a \leq x < 0 < t \leq b$ .

In addition, if  $-a \leq x < 0$  then

$$\frac{\beta_i \Delta v'}{\lambda - \gamma_i} u(x; \lambda) = \begin{cases} O\left(\frac{e^{\eta(a+b+x)}}{n}\right), & \text{if } \alpha = 0, \beta = \pi, \\ O(e^{\eta(a+b+x)}), & \text{if } \alpha \in (0, \pi), \beta = \pi, \\ O(e^{\eta(a+b+x)}), & \text{if } \alpha = 0, \beta \in (0, \pi), \\ O(ne^{\eta(a+b+x)}), & \text{if } \alpha, \beta \in (0, \pi) \end{cases} \quad i = \overline{1, N},$$

$$\frac{\alpha_j \Delta v}{\lambda - \delta_j} u(x; \lambda) = \begin{cases} O(e^{\eta(a+b+x)}), & \text{if } \alpha = 0, \beta = \pi, \\ O(ne^{\eta(a+b+x)}), & \text{if } \alpha \in (0, \pi), \beta = \pi, \\ O(ne^{\eta(a+b+x)}), & \text{if } \alpha = 0, \beta \in (0, \pi), \\ O(n^2 e^{\eta(a+b+x)}), & \text{if } \alpha, \beta \in (0, \pi), \end{cases} \quad j = \overline{1, M}.$$

Whereas, if  $0 < x \leq b$  then

$$\frac{\beta_i \Delta u'}{\lambda - \gamma_i} v(x; \lambda) = \begin{cases} O(ne^{\eta(a+b+x)}), & \text{if } \alpha = 0, \beta = \pi, \\ O(n^2 e^{\eta(a+b+x)}), & \text{if } \alpha \in (0, \pi), \beta = \pi, \\ O(n^2 e^{\eta(a+b+x)}), & \text{if } \alpha = 0, \beta \in (0, \pi), \\ O(n^3 e^{\eta(a+b+x)}), & \text{if } \alpha, \beta \in (0, \pi), \end{cases} \quad i = \overline{1, N},$$

$$\frac{\alpha_j \Delta u}{\lambda - \delta_j} v(x; \lambda) = \begin{cases} O\left(\frac{e^{\eta(a+b+x)}}{n}\right), & \text{if } \alpha = 0, \beta = \pi, \\ O(e^{\eta(a+b+x)}), & \text{if } \alpha \in (0, \pi), \beta = \pi, \\ O(e^{\eta(a+b+x)}), & \text{if } \alpha = 0, \beta \in (0, \pi), \\ O(ne^{\eta(a+b+x)}), & \text{if } \alpha, \beta \in (0, \pi), \end{cases} \quad j = \overline{1, M}.$$

Let  $\mu \in \mathbb{C}$  such that  $\mu$  avoids all  $\lambda_0, \lambda_1, \dots$ . Choose  $n$  so large that  $|\mu| \ll A_n^2$ . Then we conclude by the Residue Theorem and the above bounds that

$$G(x, t; \mu) + \sum_{m=0}^{n(p+q)+\kappa-1} \text{Res} \left( \frac{G(x, t; \lambda)}{\lambda - \mu}, \lambda_m \right) = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{G(x, t; \lambda)}{\lambda - \mu} d\lambda = O\left(\frac{1}{n}\right),$$

and similarly for  $\tilde{G}(x, t; \mu)$ . Here the residues are calculated in Theorem 4.6.3 and given explicitly in the final statement of this theorem. Likewise,

$$\begin{aligned} \frac{[\tilde{v}u - \tilde{u}v](x; \mu)}{\psi(\mu)} + \sum_{m=0}^{n(p+q)+\kappa-1} \text{Res} \left( \frac{[\tilde{v}u - \tilde{u}v](x; \lambda)}{(\lambda - \mu)\psi(\lambda)}, \lambda_m \right) &= \frac{1}{2\pi i} \int_{\Gamma_n} \frac{[\tilde{v}u - \tilde{u}v](x; \lambda)}{(\lambda - \mu)\psi(\lambda)} d\lambda \\ &= O\left(\frac{1}{n}\right). \end{aligned}$$

Moreover, for all  $i = \overline{1, N}$  and  $j = \overline{1, M}$ , if  $-a \leq x < 0$  then

$$\begin{aligned} \frac{\beta_i \Delta v'(\mu) u(x; \mu)}{\mu - \gamma_i \psi(\mu)} + \sum_{m=0}^{n(p+q)+\kappa-1} \text{Res} \left( \frac{1}{\lambda - \mu} \frac{\beta_i \Delta v'(\lambda) u(x; \lambda)}{(\lambda - \gamma_i)\psi(\lambda)}, \lambda_m \right) \\ = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{1}{\lambda - \mu} \frac{\beta_i \Delta v'(\lambda) u(x; \lambda)}{(\lambda - \gamma_i)\psi(\lambda)} d\lambda \\ = O\left(\frac{1}{n^4}\right), \end{aligned}$$

$$\begin{aligned} \frac{\alpha_j \Delta v(\mu) u(x; \mu)}{\mu - \delta_j \psi(\mu)} + \sum_{m=0}^{n(p+q)+\kappa-1} \text{Res} \left( \frac{1}{\lambda - \mu} \frac{\alpha_j \Delta v(\lambda) u(x; \lambda)}{(\lambda - \delta_j)\psi(\lambda)}, \lambda_m \right) \\ = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{1}{\lambda - \mu} \frac{\alpha_j \Delta v(\lambda) u(x; \lambda)}{(\lambda - \delta_j)\psi(\lambda)} d\lambda \\ = O\left(\frac{1}{n^3}\right), \end{aligned}$$

and similarly for  $\frac{\beta_i \Delta v'(\mu)}{\mu - \gamma_i} \frac{\tilde{u}(x; \mu)}{\psi(\mu)}$  and  $\frac{\alpha_j \Delta v(\mu)}{\mu - \delta_j} \frac{\tilde{u}(x; \mu)}{\psi(\mu)}$ . Whereas if  $0 < x \leq b$  then

$$\begin{aligned} & \frac{\beta_i \Delta u'(\mu)}{\mu - \gamma_i} \frac{v(x; \mu)}{\psi(\mu)} + \sum_{m=0}^{n(p+q)+\kappa-1} \operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{\beta_i \Delta u'(\lambda) v(x; \lambda)}{(\lambda - \gamma_i) \psi(\lambda)}, \lambda_m \right) \\ &= \frac{1}{2\pi i} \int_{\Gamma_n} \frac{1}{\lambda - \mu} \frac{\beta_i \Delta u'(\lambda) v(x; \lambda)}{(\lambda - \gamma_i) \psi(\lambda)} d\lambda \\ &= O\left(\frac{1}{n^2}\right), \end{aligned}$$

$$\begin{aligned} & \frac{\alpha_j \Delta u(\mu)}{\mu - \delta_j} \frac{v(x; \mu)}{\psi(\mu)} + \sum_{m=0}^{n(p+q)+\kappa-1} \operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{\alpha_j \Delta u(\lambda) v(x; \lambda)}{(\lambda - \delta_j) \psi(\lambda)}, \lambda_m \right) \\ &= \frac{1}{2\pi i} \int_{\Gamma_n} \frac{1}{\lambda - \mu} \frac{\alpha_j \Delta u(\lambda) v(x; \lambda)}{(\lambda - \delta_j) \psi(\lambda)} d\lambda \\ &= O\left(\frac{1}{n^4}\right), \end{aligned}$$

and similarly for  $\frac{\beta_i \Delta u'(\mu)}{\mu - \gamma_i} \frac{\tilde{v}(x; \mu)}{\psi(\mu)}$  and  $\frac{\alpha_j \Delta u(\mu)}{\mu - \delta_j} \frac{\tilde{v}(x; \mu)}{\psi(\mu)}$ .  $\square$

**Theorem 4.3.6.** *Suppose that the eigenvalues of  $(L; \alpha, \zeta; r, s)$  and  $(\tilde{L}; \alpha, \zeta; r, s)$  coincide up to multiplicity. Suppose further that each eigenvalue  $\lambda_n$  of  $(L; \alpha, \beta; r, s)$  coincides with an eigenvalue of  $(\tilde{L}; \alpha, \beta; r, s)$ , up to multiplicity, except if  $\lambda_n \in \Lambda_0$ , where  $\Lambda_0$  is a finite set.*

1. *Let  $\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)$ . If  $r(\lambda_n) \neq 0$  and  $s(\lambda_n) \neq 0$  then  $k_n = \tilde{k}_n$ , else if  $r(\lambda_n) = 0$  or  $s(\lambda_n) = 0$  then  $k_n^{(2)} = \tilde{k}_n^{(2)}$ .*

2. *If  $\lambda_n \in \Lambda^* \setminus \Lambda_0$  then*

$$\frac{\tilde{u}(0^+; \lambda_n) v'(0^+; \lambda_n)}{u(0^+; \lambda_n) \tilde{v}'(0^+; \lambda_n)} = 1.$$

3. *If  $\lambda_n \in \Lambda_1^+ \setminus \Lambda_0$  then  $T_n^+ = 1$ , where  $T_n^+$  is defined in Theorem 4.3.5.*

*Proof.* 1. For an eigenvalue  $\lambda_n \in \Lambda \setminus \Lambda^*$  we have

$$\begin{bmatrix} \cos \beta & -\sin \beta \\ \cos \zeta & -\sin \zeta \end{bmatrix} \begin{bmatrix} u_n(b) \\ u'_n(b) \end{bmatrix} = \begin{bmatrix} 0 \\ \nu(\lambda_n) \end{bmatrix}. \quad (4.54)$$

Solving the above linear system gives

$$u_n(b) = \frac{\nu(\lambda_n)}{\sin(\beta - \zeta)} \sin \beta, \quad u'_n(b) = \frac{\nu(\lambda_n)}{\sin(\beta - \zeta)} \cos \beta, \quad (4.55)$$

where  $\sin \beta = v_n(b)$  and  $\cos \beta = v'_n(b)$ .

If  $r(\lambda_n) \neq 0$  and  $s(\lambda_n) \neq 0$  then  $v_n = k_n u_n$  and since  $u_n(b)$  and  $u'_n(b)$  cannot both be zero, we conclude that

$$k_n = \frac{\sin(\beta - \zeta)}{\nu(\lambda_n)}. \quad (4.56)$$

On the other hand, if either  $r(\lambda_n) = 0$  or  $s(\lambda_n) = 0$  then according to the proof of Lemma 4.3.2

$$u_n(x) = u_n^{(2)}(x), \quad v_n(x) = v_n^{(2)}(x) = k_n^{(2)} u_n^{(2)}(x)$$

for  $0 < x \leq b$ . Since  $u_n^{(2)}(b)$  and  $u_n^{(2)'}(b)$  cannot both be zero, we conclude from (4.55) above that

$$k_n^{(2)} = \frac{\sin(\beta - \zeta)}{\nu(\lambda_n)}. \quad (4.57)$$

Similarly,

$$\tilde{k}_n = \frac{\sin(\beta - \zeta)}{\tilde{\nu}(\lambda_n)} \quad (4.58)$$

if  $r(\lambda_n) \neq 0$  and  $s(\lambda_n) \neq 0$ , and

$$\tilde{k}_n^{(2)} = \frac{\sin(\beta - \zeta)}{\tilde{\nu}(\lambda_n)} \quad (4.59)$$

if  $r(\lambda_n) = 0$  or  $s(\lambda_n) = 0$ .

Using the results of Theorem 4.6.1 in the appendix, we deduce that as  $|\lambda| \rightarrow \infty$ ,

$$\nu(\lambda) = \begin{cases} \frac{\lambda^2}{\sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2} \cos \sqrt{\lambda} a \frac{\sin \sqrt{\lambda} b}{\sqrt{\lambda}} + O(\lambda e^{\eta(a+b)}), & \text{if } \alpha = 0, \zeta = \pi, \\ \frac{-\sin \alpha \lambda^3}{\sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2} \frac{\sin \sqrt{\lambda} a}{\sqrt{\lambda}} \frac{\sin \sqrt{\lambda} b}{\sqrt{\lambda}} + O(\lambda^{3/2} e^{\eta(a+b)}), & \text{if } \alpha \in (0, \pi), \zeta = \pi, \\ \frac{\sin \zeta \lambda^2}{\sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2} \cos \sqrt{\lambda} a \cos \sqrt{\lambda} b + O(\lambda^{3/2} e^{\eta(a+b)}), & \text{if } \alpha = 0, \zeta \in (0, \pi), \\ \frac{-\sin \alpha \sin \zeta \lambda^3}{\sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2} \frac{\sin \sqrt{\lambda} a}{\sqrt{\lambda}} \cos \sqrt{\lambda} b + O(\lambda^2 e^{\eta(a+b)}), & \text{if } \alpha, \zeta \in (0, \pi), \end{cases}$$

and similarly for  $\tilde{\nu}(\lambda)$ . Here  $\nu$  and  $\tilde{\nu}$  are of order  $1/2$ . Now  $\nu(\lambda)$  and  $\tilde{\nu}(\lambda)$  are meromorphic functions with zeroes occurring at eigenvalues of  $(L; \alpha, \zeta; r, s)$  and  $(\tilde{L}; \alpha, \zeta; r, s)$  respectively, and poles occurring when  $r(\lambda)$  or  $s(\lambda)$  is zero. A result by R. Nevanlinna states that a meromorphic function of finite order can be represented as the quotient of two Weierstrass canonical products in terms of its zeroes and poles (see page 220 of [64]). This is the meromorphic analogue of Hadamard's factorization theorem for entire functions (see B. Ja. Levin [49, Chapter 1]). Since the eigenvalues of  $(L; \alpha, \zeta; r, s)$  and  $(\tilde{L}; \alpha, \zeta; r, s)$  coincide and the zeroes of  $r(\lambda)$  and  $s(\lambda)$  are fixed, we deduce from Nevanlinna's result that

$$\nu(\lambda) = C \lambda^m \tilde{\nu}(\lambda)$$

for some constant  $C$  and integer  $m$ . The asymptotics given above show that

$$\nu(\lambda)/\tilde{\nu}(\lambda) = 1 + O\left(\frac{1}{\lambda}\right),$$

giving that  $\nu \equiv \tilde{\nu}$ . Hence, for all  $\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)$ ,  $k_n = \tilde{k}_n$  if  $r(\lambda_n) \neq 0$  and  $s(\lambda_n) \neq 0$ , and  $k_n^{(2)} = \tilde{k}_n^{(2)}$  if  $r(\lambda_n) = 0$  or  $s(\lambda_n) = 0$ .



2. Let  $y(x; \lambda)$  denote the solution of (4.1) satisfying

$$y(b; \lambda) = \sin \zeta, \quad y'(b; \lambda) = \cos \zeta \quad \forall \lambda.$$

Then, from the definitions

$$\psi(\lambda) = u(b; \lambda) \cos \beta - u'(b; \lambda) \sin \beta$$

and

$$\nu(\lambda) = u(b; \lambda) \cos \zeta - u'(b; \lambda) \sin \zeta$$

we obtain

$$\begin{bmatrix} u(0^+; \lambda) \\ u'(0^+; \lambda) \end{bmatrix} = \frac{1}{\sin(\beta - \zeta)} \begin{bmatrix} -y(0^+; \lambda) & v(0^+; \lambda) \\ -y'(0^+; \lambda) & v'(0^+; \lambda) \end{bmatrix} \begin{bmatrix} \psi(\lambda) \\ \nu(\lambda) \end{bmatrix}. \quad (4.60)$$

Here,  $\sin(\beta - \zeta) = W[v, y](b) = W[v, y](0^+)$ .

Suppose that  $\lambda_n \in \Lambda^*$ . Then  $r(\lambda_n) = s(\lambda_n) = 0$  and  $\lambda_n$  has geometric multiplicity 2. Hence,  $u'(0^-; \lambda_n) = 0$  and  $v(0^+; \lambda_n) = 0$  (see Note 4.2.1). From (4.21) we deduce that

$$r(\lambda)u'(0^+; \lambda) \rightarrow u(0^+; \lambda_n) \quad \text{as } \lambda \rightarrow \lambda_n.$$

Moreover,

$$\begin{aligned} \psi(\lambda) &\rightarrow u(0^+; \lambda_n)v'(0^-; \lambda_n) \\ &= \left[ u(0^-; \lambda_n) \left[ 1 - \frac{\int_{-a}^0 w_1^2(\tau; \lambda_n) d\tau}{\dot{s}(\lambda_n)} \right] \right] \left[ v'(0^+; \lambda_n) \left[ 1 + \frac{\int_0^b w_2^2(\tau; \lambda_n) d\tau}{\dot{r}(\lambda_n)} \right] \right] \end{aligned}$$

(which is finite) as  $\lambda \rightarrow \lambda_n$ . Hence, by multiplying the second equation in (4.60) by  $r(\lambda)$  and taking the limit as  $\lambda \rightarrow \lambda_n$  we obtain

$$u(0^+; \lambda_n) = v'(0^+; \lambda_n) \frac{\lim_{\lambda \rightarrow \lambda_n} r(\lambda)\nu(\lambda)}{\sin(\beta - \zeta)}.$$

Similarly, provided that  $\tilde{\lambda}_m \in \tilde{\Lambda}^*$ , we get

$$\tilde{u}(0^+; \tilde{\lambda}_m) = \tilde{v}'(0^+; \tilde{\lambda}_m) \frac{\lim_{\lambda \rightarrow \tilde{\lambda}_m} r(\lambda)\tilde{\nu}(\lambda)}{\sin(\beta - \zeta)}.$$

Since  $\tilde{\nu} \equiv \nu$  we conclude that

$$\frac{\tilde{u}(0^+; \lambda_n)}{\tilde{v}'(0^+; \lambda_n)} = \frac{u(0^+; \lambda_n)}{v'(0^+; \lambda_n)}$$

if  $\lambda_n = \tilde{\lambda}_m$ .

3. Let  $\lambda_n \in \Lambda_1^+$ . Then  $\chi_{(0,b]}v(x; \lambda_n)$  is an eigenfunction of (4.1)-(4.5) to the eigenvalue  $\lambda_n$ . We give details only for the case of  $r(\lambda_n) = s(\lambda_n) = 0$ . Here, the transmission conditions result in  $v(0^+; \lambda_n) = 0$  and  $u'(0^-; \lambda_n) \neq 0$ . By definition,

$$r(\lambda)s(\lambda)u'(0^+; \lambda) \rightarrow u'(0^-; \lambda_n) \quad \text{as } \lambda \rightarrow \lambda_n$$

(see (4.22)). Moreover,

$$\lim_{\lambda \rightarrow \lambda_n} s(\lambda)\psi(\lambda) = u'(0^-; \lambda_n)v'(0^+; \lambda_n) \left[ 1 + \frac{\int_0^b w_2^2(\tau; \lambda_n) d\tau}{\dot{r}(\lambda_n)} \right]$$

(which is finite). Thus, by multiplying the second equation in (4.60), above, it follows that

$$u'(0^-; \lambda_n) = v'(0^+; \lambda_n) \frac{\lim_{\lambda \rightarrow \lambda_n} r(\lambda)s(\lambda)\nu(\lambda)}{\sin(\beta - \zeta)}.$$

Similarly, if  $\tilde{\lambda}_m \in \tilde{\Lambda}_1^+$  and  $r(\tilde{\lambda}_m) = s(\tilde{\lambda}_m) = 0$  then

$$\tilde{u}'(0^-; \tilde{\lambda}_m) = \tilde{v}'(0^+; \tilde{\lambda}_m) \frac{\lim_{\lambda \rightarrow \tilde{\lambda}_m} r(\lambda)s(\lambda)\tilde{\nu}(\lambda)}{\sin(\beta - \zeta)}.$$

Hence, as  $\nu \equiv \tilde{\nu}$ ,

$$\frac{\tilde{u}'(0^-; \lambda_n)}{\tilde{v}'(0^+; \lambda_n)} = \frac{u'(0^-; \lambda_n)}{v'(0^+; \lambda_n)}$$

if  $\lambda_n = \tilde{\lambda}_m$ .

□

If the boundary condition at  $x = -a$  is replaced by

$$y(-a; \lambda) \cos \varepsilon - y'(-a; \lambda) \sin \varepsilon = 0,$$

where  $\sin(\alpha - \varepsilon) \neq 0$  then we obtain the eigenvalue problems  $(L; \varepsilon, \beta; r, s)$  and  $(\tilde{L}; \varepsilon, \beta; r, s)$ .

**Theorem 4.3.7.** *Suppose that the eigenvalues of  $(L; \varepsilon, \beta; r, s)$  and  $(\tilde{L}; \varepsilon, \beta; r, s)$  coincide up to multiplicity. Suppose further that each eigenvalue  $\lambda_n$  of  $(L; \alpha, \beta; r, s)$  coincides with an eigenvalue of  $(\tilde{L}; \alpha, \beta; r, s)$ , up to multiplicity, except if  $\lambda_n \in \Lambda_0$ , where  $\Lambda_0$  is a finite set.*

1. *Let  $\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)$ . If  $r(\lambda_n) \neq 0$  and  $s(\lambda_n) \neq 0$  then  $k_n = \tilde{k}_n$ , else if  $r(\lambda_n) = 0$  or  $s(\lambda_n) = 0$  then  $k_n^{(1)} = \tilde{k}_n^{(1)}$ .*

2. *If  $\lambda_n \in \Lambda^* \setminus \Lambda_0$  then*

$$\frac{\tilde{v}'(0^-; \lambda_n)u(0^-; \lambda_n)}{v'(0^-; \lambda_n)\tilde{u}(0^-; \lambda_n)} = 1.$$

3. *If  $\lambda_n \in \Lambda_1^- \setminus \Lambda_0$  then  $T_n^- = 1$ , where  $T_n^-$  is defined in Theorem 4.3.5.*

*Proof.* Similar to the proof of Theorem 4.3.6.

□

## 4.4 A transformation operator

Throughout this section we assume that each eigenvalue  $\lambda_n$  of  $(L; \alpha, \beta; r, s)$  coincides with an eigenvalue of  $(\tilde{L}; \alpha, \beta; r, s)$ , up to multiplicity, except if  $\lambda_n \in \Lambda_0$ , where  $\Lambda_0$  is a finite set. We also assume that  $k_n = \tilde{k}_n$  for all  $\lambda_n \in \Lambda \setminus \Lambda_0$  with  $r(\lambda_n) \neq 0$  and  $s(\lambda_n) \neq 0$  (this is true by the assumptions of either Theorem 4.3.6 or Theorem 4.3.7).

Define

$$\mathcal{H}_0 = \mathcal{H} \setminus \text{span} \left\{ F_n = \begin{bmatrix} f_n \\ (f_{i,n}^1) \\ (f_{j,n}^2) \end{bmatrix} \left| \begin{array}{l} F_n = U_n^{(1)} \text{ or } F_n = U_n^{(2)} \text{ if } \lambda_n \in \Lambda_0 \cap \Lambda \setminus \Lambda^* \\ F_n = Z_n^{(1)} \text{ or } F_n = Z_n^{(2)} \text{ if } \lambda_n \in \Lambda_0 \cap \Lambda^* \\ F_n = Z_n \text{ if } \lambda_n \in \Lambda_0 \cap \Lambda_1 \end{array} \right. \right\}$$

where  $\mathcal{H}$  denotes the Hilbert space defined in Section 4.2. Define  $\tilde{\mathcal{H}}_0$  in an analogous manner. Since  $L$  is self-adjoint with compact resolvent the eigenvectors to  $L$  are complete, so we can define the operator  $H : \mathcal{H}_0 \rightarrow \tilde{\mathcal{H}}_0$  as follows:

1.  $HU_n = \tilde{U}_n$  if  $\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)$ ,
2.  $HZ_n^{(1)} = \tilde{Z}_n^{(1)}$ ,  $HZ_n^{(2)} = \tilde{Z}_n^{(2)}$  if  $\lambda_n \in \Lambda^* \setminus \Lambda_0$ ,
3.  $HZ_n = \tilde{Z}_n$  if  $\lambda_n \in \Lambda_1 \setminus \Lambda_0$ ,

and extended by linearity to the linear span of the eigenspace, which is dense in  $\mathcal{H}_0$ .

**Proposition 4.4.1.** (i) *The operator  $H : \mathcal{H}_0 \rightarrow \tilde{\mathcal{H}}_0$  is bounded.*

(ii) *On  $\mathcal{H}_0$ ,*

$$H(\lambda - L)^{-1} = (\lambda - \tilde{L})^{-1}H$$

for  $\lambda \neq \lambda_n, \tilde{\lambda}_n, n \in \mathbb{N}_0$ .

*Proof.* (i) Recall from Theorem 3.6.2 that for  $n \in \mathbb{N}$  large the eigenvalues  $\lambda_n$  of  $(L; \alpha, \beta; r, s)$  satisfy  $\{\sqrt{\lambda_n} : n \geq N_0\} = \bigcup_{k=N_0}^{\infty} \Sigma^k$ , where each  $\Sigma^k$  is the disjoint union of sets  $\sigma_1^k = \{s_n^1 : 0 \leq n - (kc + \kappa_1) \leq c - 1\}$  and  $\sigma_2^k = \{s_n^2 : 0 \leq n - (kd + \kappa_2) \leq d - 1\}$  for some constants  $\kappa_1, \kappa_2, c, d$ .

Now, for  $n$  large enough we can assume that  $r(\lambda_n) \neq 0$  and  $s(\lambda_n) \neq 0$ . So  $V_n = k_n U_n$  and  $\tilde{V}_n = \tilde{k}_n \tilde{U}_n$  by Lemma 4.3.2 and Corollary 4.3.4 respectively. Also  $k_n = \tilde{k}_n$  by assumption. Hence, either  $\lambda_n = [s_{n'}^1]^2$  with  $U_n(x) = U(x; [s_{n'}^1]^2)$  and  $\tilde{U}_n(x) = \tilde{U}(x; \tilde{\lambda}_m) = \tilde{U}(x; [s_{m'}^1]^2)$ , giving

$$\frac{\|HU_n\|^2}{\|U_n\|^2} = \frac{\|\tilde{U}_n\|^2}{\|U_n\|^2} = \frac{\|\tilde{U}(x; [s_{m'}^1]^2)\|^2}{\|U(x; [s_{n'}^1]^2)\|^2}; \quad (4.61)$$

or  $\lambda_n = [s_{n'}^2]^2$  with  $V_n(x) = V(x; [s_{n'}^2]^2)$  and  $\tilde{V}_n(x) = \tilde{V}(x; \tilde{\lambda}_m) = \tilde{V}(x; [s_{m'}^2]^2)$ , giving

$$\frac{\|HU_n\|^2}{\|U_n\|^2} = \frac{\|\frac{1}{k_n} \tilde{V}_n\|^2}{\|\frac{1}{k_n} V_n\|^2} = \frac{\|\tilde{V}(x; [s_{m'}^2]^2)\|^2}{\|V(x; [s_{n'}^2]^2)\|^2}, \quad (4.62)$$

for some  $n', m'$  depending on  $n, m$  respectively, where  $n$  and  $m$  are related by  $\lambda_n = \tilde{\lambda}_m$ . By the approximations in Lemma 4.6.1 we have

$$\left\| \begin{bmatrix} \tilde{u}(x; [\tilde{s}_{m'}^1]^2) \\ \tilde{u}_i^1([\tilde{s}_{m'}^1]^2) \\ \tilde{u}_j^2([\tilde{s}_{m'}^1]^2) \end{bmatrix} \right\|^2 = \begin{cases} O((m')^4), & \text{if } \alpha = 0, \\ O((m')^6), & \text{if } \alpha \in (0, \pi), \end{cases}$$

and

$$\left\| \begin{bmatrix} u(x; [s_{n'}^1]^2) \\ u_i^1([s_{n'}^1]^2) \\ u_j^2([s_{n'}^1]^2) \end{bmatrix} \right\|^2 \geq \begin{cases} \frac{b(n'\pi)^6 \sin^2 \left[ \frac{aq_1(0)}{n'\pi} + \frac{a}{2n'\pi} \int_{-a}^0 \cos \frac{(2n'+1)\pi t}{a} q(t) dt \right]}{2a^6 \left[ \sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2 \right]^2} + O((n')^3), & \text{if } \alpha = 0, \\ \frac{b(n'\pi)^8 \sin^2 \alpha \sin^2 \left[ a \frac{q_1(0) + \cot \alpha}{n'\pi} + \frac{a}{2n'\pi} \int_{-a}^0 \cos \frac{2n'\pi t}{a} q(t) dt \right]}{2a^8 \left[ \sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2 \right]^2} + O((n')^5), & \text{if } \alpha \in (0, \pi); \end{cases}$$

together with

$$\left\| \begin{bmatrix} \tilde{v}(x; [\tilde{s}_{m'}^2]^2) \\ \tilde{v}_i^1([\tilde{s}_{m'}^2]^2) \\ \tilde{v}_j^2([\tilde{s}_{m'}^2]^2) \end{bmatrix} \right\|^2 = \begin{cases} O((m')^4), & \text{if } \beta = \pi, \\ O((m')^2), & \text{if } \beta \in (0, \pi), \end{cases}$$

and

$$\left\| \begin{bmatrix} v(x; [s_{n'}^2]^2) \\ v_i^1([s_{n'}^2]^2) \\ v_j^2([s_{n'}^2]^2) \end{bmatrix} \right\|^2 \geq \begin{cases} \frac{a \left( \frac{n'\pi}{b} \right)^6 \sin^2 \left[ \frac{q_2(0) + \sum_{i=1}^N \beta_i^2 + \frac{1}{2} \int_0^b \cos \frac{2n'\pi t}{b} q(t) dt}{\frac{n'\pi}{b}} \right]}{2 \left[ \sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2 \right]^2} + O((n')^3), & \text{if } \beta = \pi, \\ \frac{a \left( \frac{n'\pi}{b} \right)^4 \sin^2 \beta \sin^2 \left[ \frac{q_2(0) + \sum_{i=1}^N \beta_i^2 - \cot \beta + \frac{1}{2} \int_0^b \cos \frac{(2n'+1)\pi t}{b} q(t) dt}{\frac{n'\pi}{b}} \right]}{2 \left[ \sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2 \right]^2} + O(n'), & \text{if } \beta \in (0, \pi). \end{cases}$$

Since  $\Lambda_0$  is a finite set the difference between  $n$  and  $m$ , and consequently  $n'$  and  $m'$ , is bounded. Hence, the result follows by (4.61), (4.62) and the approximations above.

(ii) Suppose that  $\lambda_n \in \sigma(L; \alpha, \beta; r, s) \setminus \Lambda_0$ . Let  $F_n(x) = F(x; \lambda_n)$  and  $\tilde{F}_n(x) = \tilde{F}(x; \lambda_n)$  denote eigenfunctions of  $(L; \alpha, \beta; r, s)$  and  $(\tilde{L}; \alpha, \beta; r, s)$  respectively, where  $\lambda_n = \tilde{\lambda}_m$  for some  $m$ . In particular,

1. if  $\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)$  then  $F_n = U_n, \tilde{F}_n = \tilde{U}_n$ ,
2. if  $\lambda_n \in \Lambda^* \setminus \Lambda_0$  then  $F_n = Z_n^{(1)}$  or  $F_n = Z_n^{(2)}$  with  $\tilde{F}_n = \tilde{Z}_n^{(1)}$  or  $\tilde{F}_n = \tilde{Z}_n^{(2)}$  respectively,

3. if  $\lambda_n \in \Lambda_1 \setminus \Lambda_0$  then  $F_n = Z_n, \tilde{F}_n = \tilde{Z}_n$ .

Then

$$\begin{aligned}
H(\lambda - L)^{-1}F_n(x) &= \frac{1}{\lambda - \lambda_n}HF_n(x), \quad \lambda \neq \lambda_n \\
&= \frac{1}{\lambda - \tilde{\lambda}_m}\tilde{F}_n(x), \quad \lambda_n = \tilde{\lambda}_m, \lambda \neq \tilde{\lambda}_m \\
&= \frac{1}{\lambda - \tilde{\lambda}_m}\tilde{F}(x; \tilde{\lambda}_m), \quad \lambda_n = \tilde{\lambda}_m, \lambda \neq \tilde{\lambda}_m \\
&= (\lambda - \tilde{L})^{-1}\tilde{F}(x; \tilde{\lambda}_m) \\
&= (\lambda - \tilde{L})^{-1}\tilde{F}_n(x), \quad \lambda_n = \tilde{\lambda}_m \\
&= (\lambda - \tilde{L})^{-1}HF_n(x).
\end{aligned}$$

Hence, the result follows as the vectors  $F_n$  form a complete set in  $\mathcal{H}_0$ .

□

**Lemma 4.4.2.** *Let*

$$F = \begin{bmatrix} f \\ (f_i^1) \\ (f_j^2) \end{bmatrix} \in \mathcal{H}_0.$$

Let  $\lambda \neq \lambda_n, n \in \mathbb{N}_0, \lambda \neq \gamma_i, i = \overline{1, N}$  and  $\lambda \neq \delta_j, j = \overline{1, M}$ . Let

$$h(x; \lambda) = \int_{-a}^b \tilde{G}(x, t; \lambda)f(t)dt + \begin{cases} A(\lambda)\frac{\tilde{u}(x; \lambda)}{\psi(\lambda)}, & \text{if } -a \leq x < 0, \\ B(\lambda)\frac{\tilde{v}(x; \lambda)}{\psi(\lambda)}, & \text{if } 0 < x \leq b, \end{cases}$$

with  $\tilde{G}$  defined by equation (4.32),

$$A(\lambda) = \sum_{i=1}^N f_i^1 \left[ \frac{\beta_i}{\lambda - \gamma_i} \Delta v' \right] + \sum_{j=1}^M f_j^2 \left[ \frac{\alpha_j}{\lambda - \delta_j} \Delta v \right],$$

$$B(\lambda) = \sum_{i=1}^N f_i^1 \left[ \frac{\beta_i}{\lambda - \gamma_i} \Delta u' \right] + \sum_{j=1}^M f_j^2 \left[ \frac{\alpha_j}{\lambda - \delta_j} \Delta u \right],$$

and  $R(x, t; \lambda)$  as defined in Theorem 4.3.5. Denote by  $[Y]_0$  denotes the  $L^2$  component of  $Y$ .

1. If  $-a \leq x < 0$  then

$$\begin{aligned}
\left[ (\lambda - \tilde{L})^{-1} H F \right]_0 &= h(x; \lambda) + \sum_{\lambda_n \in \Lambda \setminus (\Lambda_0 \cup \Lambda^*)} \frac{[K_n - \tilde{K}_n] \int_{-a}^x u_n f dt}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} \tilde{u}_n(x) \\
&+ \sum_{\lambda_n \in \Lambda^* \setminus \Lambda_0} \frac{\left[ 1 - \frac{\tilde{v}'(0^-; \lambda_n) u(0^-; \lambda_n)}{v'(0^-; \lambda_n) \tilde{u}(0^-; \lambda_n)} \right] \int_{-a}^x z_n^{(1)} f dt}{(\lambda - \lambda_n) \|Z_n^{(1)}\|^2} \tilde{z}_n^{(1)}(x) \\
&+ \sum_{\lambda_n \in \Lambda_1^- \setminus \Lambda_0} \frac{[1 - T_n^-] \int_{-a}^x z_n f dt}{(\lambda - \lambda_n) \|Z_n\|^2} \tilde{z}_n(x) \\
&- \int_{-a}^b R(x, t; \lambda) f dt - \sum_{\lambda_n \in \Lambda_0 \cap \Lambda^*} \frac{\left[ \sum_{j=1}^M z_{j,n}^{(1)2} f_j^2 \right]}{(\lambda - \lambda_n) \|Z_n^{(1)}\|^2} \tilde{u}(x; \lambda_n) \\
&- \sum_{\lambda_n \in \Lambda_0 \cap \Lambda \setminus \Lambda^*} \frac{\left[ \sum_{i=1}^N v_{i,n}^1 f_i^1 + \sum_{j=1}^M v_j^2 f_j^2 \right]}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} \tilde{u}(x; \lambda_n) \\
&- \sum_{\lambda_n \in \Lambda_0 \cap \Lambda_1^-} \frac{\left[ \sum_{i=1}^N z_{i,n}^1 f_i^1 + \sum_{j=1}^M z_{j,n}^2 f_j^2 \right]}{(\lambda - \lambda_n) \|Z_n\|^2} \tilde{u}(x; \lambda_n). \tag{4.63}
\end{aligned}$$

Here, for  $\lambda_n \in \Lambda \setminus (\Lambda_0 \cup \Lambda^*)$ ,  $K_n = k_n$  and  $\tilde{K}_n = \tilde{k}_n$  if  $r(\lambda_n) \neq 0$  and  $s(\lambda_n) \neq 0$ , else  $K_n = k_n^{(1)}$  and  $\tilde{K}_n = \tilde{k}_n^{(1)}$ .

2. If  $0 < x \leq b$ ,

$$\begin{aligned}
\left[ (\lambda - \tilde{L})^{-1} H F \right]_0 &= h(x; \lambda) + \sum_{\lambda_n \in \Lambda \setminus (\Lambda_0 \cup \Lambda^*)} \frac{[\tilde{K}_n - K_n] \int_x^b u_n f dt}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} \tilde{u}_n(x) \\
&+ \sum_{\lambda_n \in \Lambda^* \setminus \Lambda_0} \frac{\left[ 1 - \frac{\tilde{u}(0^+; \lambda_n) v'(0^+; \lambda_n)}{u(0^+; \lambda_n) \tilde{v}'(0^+; \lambda_n)} \right] \int_x^b z_n^{(2)} f dt}{(\lambda - \lambda_n) \|Z_n^{(2)}\|^2} \tilde{z}_n^{(2)}(x) \\
&+ \sum_{\lambda_n \in \Lambda_1^+ \setminus \Lambda_0} \frac{[1 - T_n^+] \int_x^b z_n f dt}{(\lambda - \lambda_n) \|Z_n\|^2} \tilde{z}_n(x) \\
&- \int_{-a}^b R(x, t; \lambda) f dt - \sum_{\lambda_n \in \Lambda_0 \cap \Lambda^*} \frac{\left[ \sum_{i=1}^N z_{i,n}^{(2)1} f_i^1 \right] \tilde{v}(x; \lambda_n)}{(\lambda - \lambda_n) \|Z_n^{(2)}\|^2} \\
&- \sum_{\lambda_n \in \Lambda_0 \cap \Lambda \setminus \Lambda^*} \frac{\left[ \sum_{i=1}^N u_{i,n}^1 f_i^1 + \sum_{j=1}^M u_j^2 f_j^2 \right] \tilde{v}(x; \lambda_n)}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} \\
&- \sum_{\lambda_n \in \Lambda_0 \cap \Lambda_1^+} \frac{\left[ \sum_{i=1}^N z_{i,n}^1 f_i^1 + \sum_{j=1}^M z_{j,n}^2 f_j^2 \right] \tilde{v}(x; \lambda_n)}{(\lambda - \lambda_n) \|Z_n\|^2}. \tag{4.64}
\end{aligned}$$

Here, for  $\lambda_n \in \Lambda \setminus (\Lambda_0 \cup \Lambda^*)$ ,  $K_n = k_n$  and  $\tilde{K}_n = \tilde{k}_n$  if  $r(\lambda_n) \neq 0$  and  $s(\lambda_n) \neq 0$ , else  $K_n = k_n^{(2)}$  and  $\tilde{K}_n = \tilde{k}_n^{(2)}$ .

*Proof.* Let  $\lambda \neq \lambda_n$  for  $n \in \mathbb{N}_0$ ,  $\lambda \neq \gamma_i$  for  $i = \overline{1, N}$ ,  $\lambda \neq \delta_j$  for  $j = \overline{1, M}$ , and

$$F = \begin{bmatrix} f \\ (f_i^1) \\ (f_j^2) \end{bmatrix} \in \mathcal{H}_0.$$

Recall from Section 3.5 in Chapter 3 that  $(\lambda - L)^{-1} F = Y$  with

$$Y = \begin{bmatrix} A(\lambda) \chi_{[-a, 0)} u(x; \lambda) + B(\lambda) \chi_{(0, b]} v(x; \lambda) + \int_{-a}^b G(x, t; \lambda) f(t) dt \\ \left( \frac{f_i^1 + \beta_i \Delta y'}{\lambda - \gamma_i} \right) \\ \left( \frac{f_j^2 + \alpha_j \Delta y}{\lambda - \delta_j} \right) \end{bmatrix}, \tag{4.65}$$

where  $A(\lambda)$ ,  $B(\lambda)$  are as in the statement of this Lemma.

1. Suppose that  $-a \leq x < 0$ . Then, using the expansions of Theorem 4.3.5 we have

$$\begin{aligned}
& H(\lambda - L)^{-1}F \\
&= H \left\{ \sum_{\lambda_n \in \Lambda \setminus \Lambda^*} \frac{\langle F, V_n \rangle}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} U_n(x) + \sum_{\lambda_n \in \Lambda^*} \frac{\langle F, Z_n^{(1)} \rangle}{(\lambda - \lambda_n) \|Z_n^{(1)}\|^2} Z_n^{(1)}(x) \right. \\
&\quad \left. + \sum_{\lambda_n \in \Lambda_1^-} \frac{\langle F, Z_n \rangle}{(\lambda - \lambda_n) \|Z_n\|^2} Z_n(x) \right\} \\
&= \sum_{\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)} \frac{\left[ \int_{-a}^b f v_n dt + \sum_{i=1}^N f_i^1 v_{i,n}^1 + \sum_{j=1}^M f_j^2 v_{j,n}^2 \right]}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} \tilde{U}_n(x) \\
&\quad + \sum_{\lambda_n \in \Lambda^* \setminus \Lambda_0} \left[ \frac{\int_{-a}^x f z_n^{(1)} dt}{(\lambda - \lambda_n) \|Z_n^{(1)}\|^2} + \frac{\int_x^b f z_n^{(1)} dt + 0 + \sum_{j=1}^M f_j^2 z_{j,n}^{(1)2}}{(\lambda - \lambda_n) \|Z_n^{(1)}\|^2} \right] \tilde{Z}_n^{(1)}(x) \\
&\quad + \sum_{\lambda_n \in \Lambda_1^- \setminus \Lambda_0} \left[ \frac{\int_{-a}^x f z_n dt}{(\lambda - \lambda_n) \|Z_n\|^2} + \frac{\int_x^b f z_n dt + \sum_{i=1}^N f_i^1 z_{i,n}^1 + \sum_{j=1}^M f_j^2 z_{j,n}^2}{(\lambda - \lambda_n) \|Z_n\|^2} \right] \tilde{Z}_n(x),
\end{aligned}$$

where we have used the fact that  $\langle F, V_n \rangle = 0$  for  $\lambda_n \in \Lambda \cap \Lambda_0 \setminus \Lambda^*$ ,  $\langle F, Z_n^{(i)} \rangle = 0$  for  $\lambda_n \in \Lambda^* \cap \Lambda_0$ , and  $\langle F, Z_n \rangle = 0$  for  $\lambda_n \in \Lambda_1 \cap \Lambda_0$ , by definition of  $F \in \mathcal{H}_0$ .



On the other hand,

$$\begin{aligned}
& h(x; \lambda) \\
&= \int_{-a}^b \tilde{G}(x, t; \lambda) f dt + \left[ \sum_{i=1}^N \frac{\beta_i \Delta v'}{\lambda - \gamma_i} f_i^1 + \sum_{j=1}^M \frac{\alpha_j \Delta v}{\lambda - \delta_j} f_j^2 \right] \frac{\tilde{u}(x; \lambda)}{\psi(\lambda)} \\
&= \sum_{\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)} \frac{\tilde{K}_n \int_{-a}^x u_n f dt + \left[ \int_x^b v_n f dt + \sum_{i=1}^N v_{i,n}^1 f_i^1 + \sum_{j=1}^M v_{j,n}^2 f_j^2 \right]}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} \tilde{u}_n(x) \\
&+ \sum_{\lambda_n \in \Lambda^* \setminus \Lambda_0} \left[ \frac{\frac{\tilde{v}'(0^-; \lambda_n) u(0^-; \lambda_n)}{v'(0^-; \lambda_n) \tilde{u}(0^-; \lambda_n)} \int_{-a}^x z_n^{(1)} f dt}{(\lambda - \lambda_n) \|Z_n^{(1)}\|^2} + \frac{\int_x^0 z_n^{(1)} f dt + 0 + \sum_{j=1}^M z_{j,n}^{(1)2} f_j^2}{(\lambda - \lambda_n) \|Z_n^{(1)}\|^2} \right] \tilde{z}_n^{(1)}(x) \\
&+ \sum_{\lambda_n \in \Lambda_1^- \setminus \Lambda_0} \left[ \frac{T_n^- \int_{-a}^x z_n f dt}{(\lambda - \lambda_n) \|Z_n\|^2} + \frac{\int_x^0 z_n f dt + \sum_{i=1}^N z_{i,n}^1 f_i^1 + \sum_{j=1}^M z_{j,n}^2 f_j^2}{(\lambda - \lambda_n) \|Z_n\|^2} \right] \tilde{z}_n(x) \\
&+ \int_{-a}^b R(x, t; \lambda) f(t) dt + \sum_{\lambda_n \in \Lambda_0 \cap \Lambda \setminus \Lambda^*} \frac{\sum_{i=1}^N v_{i,n}^1 f_i^1 + \sum_{j=1}^M v_{j,n}^2 f_j^2}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} \tilde{u}(x; \lambda_n) \\
&+ \sum_{\lambda_n \in \Lambda_0 \cap \Lambda^*} \frac{0 + \sum_{j=1}^M z_{j,n}^{(1)2} f_j^2}{(\lambda - \lambda_n) \|Z_n^{(1)}\|^2} \tilde{u}(x; \lambda_n) + \sum_{\lambda_n \in \Lambda_0 \cap \Lambda_1^-} \frac{\sum_{i=1}^N z_{i,n}^1 f_i^1 + \sum_{j=1}^M z_{j,n}^2 f_j^2}{(\lambda - \lambda_n) \|Z_n\|^2} \tilde{u}(x; \lambda_n).
\end{aligned}$$

where, for  $\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)$ ,  $\tilde{K}_n = \tilde{k}_n$  if  $r(\lambda_n) \neq 0$  and  $s(\lambda_n) \neq 0$ , else  $\tilde{K}_n = \tilde{k}_n^{(1)}$ .

2. Similarly, if  $0 < x \leq b$  then

$$\begin{aligned}
& H(\lambda - L)^{-1}F \\
&= H \left\{ \sum_{\lambda_n \in \Lambda \setminus \Lambda^*} \frac{\langle F, U_n \rangle}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} K_n U_n(x) + \sum_{\lambda_n \in \Lambda^*} \frac{\langle F, Z_n^{(2)} \rangle}{(\lambda - \lambda_n) \|Z_n^{(2)}\|^2} Z_n^{(2)}(x) \right. \\
&\quad \left. + \sum_{\lambda_n \in \Lambda_1^+} \frac{\langle F, Z_n \rangle}{(\lambda - \lambda_n) \|Z_n\|^2} Z_n(x) \right\} \\
&= \sum_{\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)} \frac{\int_{-a}^b f u_n dt + \sum_{i=1}^N f_i^1 u_{i,n}^1 + \sum_{j=1}^M f_j^2 u_{j,n}^2}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} K_n \tilde{U}_n(x) \\
&\quad + \sum_{\lambda_n \in \Lambda^* \setminus \Lambda_0} \left[ \frac{\int_{-a}^x f z_n^{(2)} dt + \sum_{i=1}^N f_i^1 z_{i,n}^{(2)1} + 0}{(\lambda - \lambda_n) \|Z_n^{(2)}\|^2} + \frac{\int_x^b f z_n^{(2)} dt}{(\lambda - \lambda_n) \|Z_n^{(2)}\|^2} \right] \tilde{Z}_n^{(2)}(x) \\
&\quad + \sum_{\lambda_n \in \Lambda_1^+ \setminus \Lambda_0} \left[ \frac{\int_{-a}^x f z_n dt + \sum_{i=1}^N f_i^1 z_{i,n}^1 + \sum_{j=1}^M f_j^2 z_{j,n}^2}{(\lambda - \lambda_n) \|Z_n\|^2} + \frac{\int_x^b f z_n dt}{(\lambda - \lambda_n) \|Z_n\|^2} \right] \tilde{Z}_n(x),
\end{aligned}$$

where, for  $\lambda_n \in \Lambda \setminus \Lambda^*$ ,  $K_n = k_n$  if  $r(\lambda_n) \neq 0$  and  $s(\lambda_n) \neq 0$ , else  $K_n = k_n^{(2)}$ . Also, we have used the fact that  $\langle F, U_n \rangle = 0$  for  $\lambda_n \in \Lambda \cap \Lambda_0 \setminus \Lambda^*$ ,  $\langle F, Z_n^{(i)} \rangle = 0$  for  $\lambda_n \in \Lambda^* \cap \Lambda_0$ , and  $\langle F, Z_n \rangle = 0$  for  $\lambda_n \in \Lambda_1 \cap \Lambda_0$ , by definition of  $F \in \mathcal{H}_0$ .

Whereas,

$$\begin{aligned}
& h(x; \lambda) \\
&= \int_{-a}^b \tilde{G}(x, t; \lambda) f dt + \left[ \sum_{i=1}^N \frac{\beta_i \Delta u'}{\lambda - \gamma_i} f_i^1 + \sum_{j=1}^M \frac{\alpha_j \Delta u}{\lambda - \delta_j} f_j^2 \right] \frac{\tilde{v}(x; \lambda)}{\psi(\lambda)} \\
&= \sum_{\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)} \frac{\tilde{K}_n \left[ \int_{-a}^x u_n f dt + \sum_{i=1}^N u_{i,n}^1 f_i^1 + \sum_{j=1}^M u_{j,n}^2 f_j^2 \right] + K_n \int_x^b u_n f dt}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} \tilde{u}_n(x) \\
&+ \sum_{\lambda_n \in \Lambda^* \setminus \Lambda_0} \left[ \frac{\int_0^x z_n^{(2)} f dt + \sum_{i=1}^N z_{i,n}^{(2)1} f_i^1 + 0}{(\lambda - \lambda_n) \|Z_n^{(2)}\|^2} + \frac{\frac{\tilde{u}(0^+; \lambda_n) v'(0^+; \lambda_n)}{u(0^+; \lambda_n) \tilde{v}'(0^+; \lambda_n)} \int_x^b z_n^{(2)} f dt}{(\lambda - \lambda_n) \|Z_n^{(2)}\|^2} \right] \tilde{z}_n^{(2)}(x) \\
&+ \sum_{\lambda_n \in \Lambda_1^+ \setminus \Lambda_0} \left[ \frac{\int_0^x z_n f dt + \sum_{i=1}^N z_{i,n}^1 f_i^1 + \sum_{j=1}^M z_{j,n}^2 f_j^2}{(\lambda - \lambda_n) \|Z_n\|^2} + \frac{T_n^+ \int_x^b z_n f dt}{(\lambda - \lambda_n) \|Z_n\|^2} \right] \tilde{z}_n(x) \\
&+ \int_{-a}^b R(x, t; \lambda) f(t) dt + \sum_{\lambda_n \in \Lambda_0 \cap \Lambda \setminus \Lambda^*} \frac{\sum_{i=1}^N u_{i,n}^1 f_i^1 + \sum_{j=1}^M u_{j,n}^2 f_j^2}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} \tilde{v}(x; \lambda_n) \\
&+ \sum_{\lambda_n \in \Lambda_0 \cap \Lambda^*} \frac{\sum_{i=1}^N z_{i,n}^{(2)1} f_i^1 + 0}{(\lambda - \lambda_n) \|Z_n^{(2)}\|^2} \tilde{v}(x; \lambda_n) + \sum_{\lambda_n \in \Lambda_0 \cap \Lambda_1^+} \frac{\sum_{i=1}^N z_{i,n}^1 f_i^1 + \sum_{j=1}^M z_{j,n}^2 f_j^2}{(\lambda - \lambda_n) \|Z_n\|^2} \tilde{v}(x; \lambda_n),
\end{aligned}$$

where, for  $\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)$ ,  $\tilde{K}_n = \tilde{k}_n$  if  $r(\lambda_n) \neq 0$  and  $s(\lambda_n) \neq 0$ , else  $\tilde{K}_n = \tilde{k}_n^{(2)}$ .

Using Proposition 4.4.1 and comparing the final expressions for  $h(x; \lambda)$  with the  $L^2$  component of  $H(\lambda - L)^{-1}F$  gives the result.  $\square$

**Remark** Full expansions of  $\left[ (\lambda - \tilde{L})^{-1} H F \right]_0$  are given in Note 4.6.4 of the appendix to this chapter. This includes an expansion of the error term  $\int_{-a}^b R(x, t; \lambda) f dt$  obtained from Theorem 4.3.5.

**Theorem 4.4.3.** *Let*

$$F = \begin{bmatrix} f \\ (f_i^1) \\ (f_j^2) \end{bmatrix} \in \mathcal{H}_0.$$

*Then, in the notation of Definitions 4.3.1 and 4.6.2, we have*

$[HF]_0$

$$\begin{aligned}
& \left. \begin{aligned}
& \sum_{\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)} \frac{[K_n - \tilde{K}_n] \tilde{u}_n(x) \int_{-a}^x u_n f dt}{\dot{\psi}(\lambda_n)} \\
& + \sum_{\lambda_n \in \Lambda^* \setminus \Lambda_0} \frac{\left[1 - \frac{\tilde{v}'(0^-; \lambda_n) u(0^-; \lambda_n)}{v'(0^-; \lambda_n) \tilde{u}(0^-; \lambda_n)}\right] \tilde{z}_n^{(1)}(x) \int_{-a}^x z_n^{(1)} f dt}{\|Z_n^{(1)}\|^2} \\
& + \sum_{\lambda_n \in \Lambda_1^- \setminus \Lambda_0} \frac{[1 - T_n^-] \tilde{z}_n(x) \int_{-a}^x z_n f dt}{\|Z_n\|^2} \\
& - \sum_{\lambda_n \in \Lambda^* \cap \tilde{\Lambda}_1^+} \frac{\left[\frac{\tilde{v}'(0^-; \lambda_n) u(0^-; \lambda_n)}{v'(0^-; \lambda_n)} \tilde{w}_1 - \tilde{u}\right](x; \lambda_n) \int_{-a}^x z_n^{(1)} f dt}{\|Z_n^{(1)}\|^2} \\
& - \sum_{\lambda_n \in (\Lambda^* \cap \Lambda_0) \setminus \tilde{\Lambda}_1^+} \left\{ \begin{aligned}
& \left[ \frac{\dot{\Phi}_n \tilde{y}(x; \lambda_n) + \Phi_n \tilde{y}_\lambda(x; \lambda_n)}{\|Z_n^{(1)}\|^2} - \frac{\tilde{u}(x; \lambda_n)}{\|Z_n^{(1)}\|^2} \right] \int_{-a}^x z_n^{(1)} f dt \\
& + \Phi_n \tilde{y}(x; \lambda_n) \int_{-a}^x u_\lambda(t; \lambda_n) f dt
\end{aligned} \right\} \\
& - \sum_{\lambda_n \in (\Lambda \setminus \Lambda^*) \cap \tilde{\Lambda}_1^+} \frac{[\tilde{v} - k_n \tilde{u}](x; \lambda_n) \int_{-a}^x u_n f dt}{\dot{\psi}(\lambda_n)} \\
& - \sum_{\lambda_n \in (\Lambda \cap \Lambda_0) \setminus (\Lambda^* \cup \tilde{\Lambda}_1^+)} \left\{ \begin{aligned}
& \left[ \frac{\dot{\Phi}_n \tilde{y}(x; \lambda_n) + \Phi_n \tilde{y}_\lambda(x; \lambda_n)}{\dot{\psi}(\lambda_n)} - \frac{k_n \tilde{u}(x; \lambda_n)}{\dot{\psi}(\lambda_n)} \right] \int_{-a}^x u_n f dt \\
& + \Phi_n \tilde{y}(x; \lambda_n) \int_{-a}^x u_\lambda(t; \lambda_n) f dt
\end{aligned} \right\} \\
& - \sum_{\lambda_n \in \Lambda_1^- \cap \Lambda_0} \left[ \frac{\tilde{y}(x; \lambda_n)}{\dot{\phi}(\lambda_n)} - \frac{\tilde{u}(x; \lambda_n)}{\|Z_n\|^2} \right] \int_{-a}^x z_n f dt \\
& - \sum_{\lambda_n \in \Lambda_1^+ \cap \Lambda_0} \frac{\tilde{y}(x; \lambda_n) \int_{-a}^x u(t; \lambda_n) f dt}{\dot{\phi}(\lambda_n)}
\end{aligned} \right\} \\
& = f + \left. \begin{aligned}
& \sum_{\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)} \frac{[K_n - \tilde{K}_n] \tilde{u}_n(x) \int_{-a}^x u_n f dt}{\dot{\psi}(\lambda_n)} \\
& + \sum_{\lambda_n \in \Lambda^* \setminus \Lambda_0} \frac{\left[1 - \frac{\tilde{v}'(0^-; \lambda_n) u(0^-; \lambda_n)}{v'(0^-; \lambda_n) \tilde{u}(0^-; \lambda_n)}\right] \tilde{z}_n^{(1)}(x) \int_{-a}^x z_n^{(1)} f dt}{\|Z_n^{(1)}\|^2} \\
& + \sum_{\lambda_n \in \Lambda_1^- \setminus \Lambda_0} \frac{[1 - T_n^-] \tilde{z}_n(x) \int_{-a}^x z_n f dt}{\|Z_n\|^2} \\
& - \sum_{\lambda_n \in \Lambda^* \cap \tilde{\Lambda}_1^+} \frac{\left[\frac{\tilde{v}'(0^-; \lambda_n) u(0^-; \lambda_n)}{v'(0^-; \lambda_n)} \tilde{w}_1 - \tilde{u}\right](x; \lambda_n) \int_{-a}^x z_n^{(1)} f dt}{\|Z_n^{(1)}\|^2} \\
& - \sum_{\lambda_n \in (\Lambda^* \cap \Lambda_0) \setminus \tilde{\Lambda}_1^+} \left\{ \begin{aligned}
& \left[ \frac{\dot{\Phi}_n \tilde{y}(x; \lambda_n) + \Phi_n \tilde{y}_\lambda(x; \lambda_n)}{\|Z_n^{(1)}\|^2} - \frac{\tilde{u}(x; \lambda_n)}{\|Z_n^{(1)}\|^2} \right] \int_{-a}^x z_n^{(1)} f dt \\
& + \Phi_n \tilde{y}(x; \lambda_n) \int_{-a}^x u_\lambda(t; \lambda_n) f dt
\end{aligned} \right\} \\
& - \sum_{\lambda_n \in (\Lambda \setminus \Lambda^*) \cap \tilde{\Lambda}_1^+} \frac{[\tilde{v} - k_n \tilde{u}](x; \lambda_n) \int_{-a}^x u_n f dt}{\dot{\psi}(\lambda_n)} \\
& - \sum_{\lambda_n \in (\Lambda \cap \Lambda_0) \setminus (\Lambda^* \cup \tilde{\Lambda}_1^+)} \left\{ \begin{aligned}
& \left[ \frac{\dot{\Phi}_n \tilde{y}(x; \lambda_n) + \Phi_n \tilde{y}_\lambda(x; \lambda_n)}{\dot{\psi}(\lambda_n)} - \frac{k_n \tilde{u}(x; \lambda_n)}{\dot{\psi}(\lambda_n)} \right] \int_{-a}^x u_n f dt \\
& + \Phi_n \tilde{y}(x; \lambda_n) \int_{-a}^x u_\lambda(t; \lambda_n) f dt
\end{aligned} \right\} \\
& - \sum_{\lambda_n \in \Lambda_1^- \cap \Lambda_0} \left[ \frac{\tilde{y}(x; \lambda_n)}{\dot{\phi}(\lambda_n)} - \frac{\tilde{u}(x; \lambda_n)}{\|Z_n\|^2} \right] \int_{-a}^x z_n f dt \\
& - \sum_{\lambda_n \in \Lambda_1^+ \cap \Lambda_0} \frac{\tilde{y}(x; \lambda_n) \int_{-a}^x u(t; \lambda_n) f dt}{\dot{\phi}(\lambda_n)}
\end{aligned} \right\}
\end{aligned}$$

if  $-a \leq x < 0$ , whereas if  $0 < x \leq b$  then

$$\begin{aligned}
& [HF]_0 \\
& = f + \left\{ \begin{aligned}
& \sum_{\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)} \frac{[\tilde{K}_n - K_n] \tilde{u}_n(x) \int_x^b u_n f dt}{\dot{\psi}(\lambda_n)} \\
& + \sum_{\lambda_n \in \Lambda^* \setminus \Lambda_0} \frac{\left[1 - \frac{\tilde{u}(0^+; \lambda_n) v'(0^+; \lambda_n)}{u(0^+; \lambda_n) \tilde{v}'(0^+; \lambda_n)}\right] \tilde{z}_n^{(2)}(x) \int_x^b z_n^{(2)} f dt}{\|Z_n^{(2)}\|^2} \\
& + \sum_{\lambda_n \in \Lambda_1^+ \setminus \Lambda_0} \frac{[1 - T_n^+] \tilde{z}_n(x) \int_x^b z_n f dt}{\|Z_n\|^2} \\
& - \sum_{\lambda_n \in (\Lambda^* \cap \Lambda_0) \cap \tilde{\Lambda}_1^-} \frac{\left[\frac{\tilde{u}(0^+; \lambda_n) v'(0^+; \lambda_n)}{u(0^+; \lambda_n)} \tilde{w}_2 - \tilde{v}\right](x; \lambda_n) \int_x^b z_n^{(2)} f dt}{\|Z_n^{(2)}\|^2} \\
& - \sum_{\lambda_n \in (\Lambda^* \cap \Lambda_0) \setminus \tilde{\Lambda}_1^-} \left\{ \begin{aligned}
& \left[ \frac{\dot{\Phi}_n \tilde{y}(x; \lambda_n) + \Phi_n \tilde{y}_\lambda(x; \lambda_n)}{\|Z_n^{(2)}\|^2} - \frac{\tilde{v}(x; \lambda_n)}{\|Z_n^{(2)}\|^2} \right] \int_x^b z_n^{(2)} f dt \\
& + \Phi_n \tilde{y}(x; \lambda_n) \int_x^b v_\lambda(t; \lambda_n) f dt
\end{aligned} \right\} \\
& + \sum_{\lambda_n \in (\Lambda \setminus \Lambda^*) \cap \tilde{\Lambda}_1^-} \frac{[\tilde{v} - k_n \tilde{u}](x; \lambda_n) \int_x^b u_n f dt}{\dot{\psi}(\lambda_n)} \\
& - \sum_{\lambda_n \in (\Lambda \cap \Lambda_0) \setminus (\Lambda^* \cup \tilde{\Lambda}_1^-)} \left\{ \begin{aligned}
& \left[ \frac{\dot{\Phi}_n \tilde{y}(x; \lambda_n) + \Phi_n \tilde{y}_\lambda(x; \lambda_n)}{k_n \dot{\psi}(\lambda_n)} - \frac{\tilde{v}(x; \lambda_n)}{k_n \dot{\psi}(\lambda_n)} \right] \int_x^b v_n f dt \\
& + \Phi_n \tilde{y}(x; \lambda_n) \int_x^b v_\lambda(t; \lambda_n) f dt
\end{aligned} \right\} \\
& - \sum_{\lambda_n \in \Lambda_1^+ \cap \Lambda_0} \left[ \frac{\tilde{y}(x; \lambda_n)}{\dot{\phi}(\lambda_n)} - \frac{\tilde{v}(x; \lambda_n)}{\|Z_n\|^2} \right] \int_x^b z_n f dt \\
& - \sum_{\lambda_n \in \Lambda_1 \cap \Lambda_0} \frac{\tilde{y}(x; \lambda_n) \int_x^b v(t; \lambda_n) f dt}{\dot{\phi}(\lambda_n)}
\end{aligned} \right\}.
\end{aligned}$$

*Proof.* Using the results of Lemma 4.4.2 we obtain

$$\begin{aligned}
& \left[ (\lambda - \tilde{L})^{-1} H F \right]'_0 \\
&= \frac{\tilde{v}'(x; \lambda) \int_{-a}^x u f dt + \tilde{u}'(x; \lambda) \left[ \int_x^b v f dt + \sum_{i=1}^N f_i^1 \left[ \frac{\beta_i}{\lambda - \gamma_i} \Delta v' \right] + \sum_{j=1}^M f_j^2 \left[ \frac{\alpha_j}{\lambda - \delta_j} \Delta v \right] \right]}{\psi(\lambda)} \\
&+ \frac{[\tilde{v}u - \tilde{u}v](x; \lambda)}{\psi(\lambda)} f(x) - \frac{\partial}{\partial x} \left[ \int_{-a}^x R(x, t; \lambda) f(t) dt + \int_x^b R(x, t; \lambda) f(t) dt \right] \\
&+ \sum_{\lambda_n \in \Lambda \setminus (\Lambda_0 \cup \Lambda^*)} \frac{[K_n - \tilde{K}_n] \left[ \tilde{u}_n(x) u_n(x) f(x) + \tilde{u}'_n(x) \int_{-a}^x u_n f dt \right]}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} \\
&+ \sum_{\lambda_n \in \Lambda^* \setminus \Lambda_0} \frac{1 - \frac{\tilde{v}'(0^-; \lambda_n) u(0^-; \lambda_n)}{v'(0^-; \lambda_n) \tilde{u}(0^-; \lambda_n)}}{(\lambda - \lambda_n) \|Z_n^{(1)}\|^2} \left[ \tilde{z}_n^{(1)}(x) z_n^{(1)}(x) f(x) + \tilde{z}_n^{(1)'}(x) \int_{-a}^x z_n^{(1)} f dt \right] \\
&+ \sum_{\lambda_n \in \Lambda_1^- \setminus \Lambda_0} \frac{1 - T_n^-}{(\lambda - \lambda_n) \|Z_n\|^2} \left[ \tilde{z}_n(x) z_n(x) f(x) + \tilde{z}'_n(x) \int_{-a}^x z_n f dt \right] \\
&- \sum_{\lambda_n \in \Lambda_0 \cap \Lambda \setminus \Lambda^*} \frac{\left[ \sum_{i=1}^N v_{i,n}^1 f_i^1 + \sum_{j=1}^M v_j^2 f_j^2 \right] \tilde{u}'(x; \lambda_n)}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} - \sum_{\lambda_n \in \Lambda_0 \cap \Lambda^*} \frac{\left[ \sum_{j=1}^M z_{j,n}^{(1)2} f_j^2 \right] \tilde{u}'(x; \lambda_n)}{(\lambda - \lambda_n) \|Z_n^{(1)}\|^2} \\
&- \sum_{\lambda_n \in \Lambda_0 \cap \Lambda_1^-} \frac{\left[ \sum_{i=1}^N z_{i,n}^1 f_i^1 + \sum_{j=1}^M z_{j,n}^2 f_j^2 \right] \tilde{u}'(x; \lambda_n)}{(\lambda - \lambda_n) \|Z_n\|^2} \tag{4.66}
\end{aligned}$$

if  $-a \leq x < 0$ , where, for  $\lambda_n \in \Lambda \setminus (\Lambda_0 \cup \Lambda^*)$ ,  $K_n = k_n$  and  $\tilde{K}_n = \tilde{k}_n$  if  $r(\lambda_n) \neq 0$  and  $s(\lambda_n) \neq 0$  else  $K_n = k_n^{(1)}$  and  $\tilde{K}_n = \tilde{k}_n^{(1)}$ .

Similarly, if  $0 < x \leq b$  we have

$$\begin{aligned}
& \left[ (\lambda - \tilde{L})^{-1} H F \right]'_0 \\
&= \frac{\tilde{v}'(x; \lambda) \left[ \int_{-a}^x u f dt + \sum_{i=1}^N f_i^1 \left[ \frac{\beta_i}{\lambda - \gamma_i} \Delta u' \right] + \sum_{j=1}^M f_j^2 \left[ \frac{\alpha_j}{\lambda - \delta_j} \Delta u \right] \right] + \tilde{u}'(x; \lambda) \int_x^b v f dt +}{\psi(\lambda)} \\
&+ \frac{[\tilde{v}u - \tilde{u}v](x; \lambda)}{\psi(\lambda)} f(x) - \frac{\partial}{\partial x} \left[ \int_{-a}^x R(x, t; \lambda) f(t) dt + \int_x^b R(x, t; \lambda) f(t) dt \right] \\
&+ \sum_{\lambda_n \in \Lambda \setminus (\Lambda_0 \cup \Lambda^*)} \frac{[\tilde{K}_n - K_n] \left[ -\tilde{u}_n(x) u_n(x) f(x) dt + \tilde{u}'_n(x) \int_x^b u_n f dt \right]}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} \\
&+ \sum_{\lambda_n \in \Lambda^* \setminus \Lambda_0} \frac{1 - \frac{\tilde{u}(0^+; \lambda_n) v'(0^+; \lambda_n)}{u(0^+; \lambda_n) \tilde{v}'(0^+; \lambda_n)}}{(\lambda - \lambda_n) \|Z_n^{(2)}\|^2} \left[ -\tilde{z}_n^{(2)}(x) z_n^{(2)}(x) f(x) + \tilde{z}_n^{(2)'}(x) \int_x^b z_n^{(2)} f dt \right] \\
&+ \sum_{\lambda_n \in \Lambda_1^+ \setminus \Lambda_0} \frac{1 - T_n^+}{(\lambda - \lambda_n) \|Z_n\|^2} \left[ -\tilde{z}_n(x) z_n(x) f(x) + \tilde{z}_n'(x) \int_x^b z_n f dt \right] \\
&- \sum_{\lambda_n \in \Lambda_0 \cap \Lambda^*} \frac{\left[ \sum_{i=1}^N u_{i,n}^1 f_i^1 + \sum_{j=1}^M u_{j,n}^2 f_j^2 \right] \tilde{v}'(x; \lambda_n)}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} - \sum_{\lambda_n \in \Lambda_0 \cap \Lambda^*} \frac{\left[ \sum_{i=1}^N z_{i,n}^{(2)1} f_i^1 \right] \tilde{v}'(x; \lambda_n)}{(\lambda - \lambda_n) \|Z_n^{(2)}\|^2} \\
&- \sum_{\lambda_n \in \Lambda_0 \cap \Lambda_1^+} \frac{\left[ \sum_{i=1}^N z_{i,n}^1 f_i^1 + \sum_{j=1}^M z_{j,n}^2 f_j^2 \right] \tilde{v}'(x; \lambda_n)}{(\lambda - \lambda_n) \|Z_n\|^2}, \tag{4.67}
\end{aligned}$$

where, for  $\lambda_n \in \Lambda \setminus (\Lambda_0 \cup \Lambda^*)$ ,  $K_n = k_n$  and  $\tilde{K}_n = \tilde{k}_n$  if  $r(\lambda_n) \neq 0$  and  $s(\lambda_n) \neq 0$  else  $K_n = k_n^{(2)}$  and  $\tilde{K}_n = \tilde{k}_n^{(2)}$ .

Now, from the expansions of Theorem 4.3.5 we obtain

$$\begin{aligned}
& \frac{[\tilde{v}u - \tilde{u}v](x; \lambda)}{\psi(\lambda)} f(x) - \frac{\partial}{\partial x} \left[ \int_{-a}^x R(x, t; \lambda) f(t) dt + \int_x^b R(x, t; \lambda) f(t) dt \right] \\
&= \sum_{\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)} \frac{[\tilde{K}_n - K_n][\tilde{u}_n u_n f](x)}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} + \sum_{\lambda_n \in \Lambda^* \setminus \Lambda_0} \frac{\left[ \frac{\tilde{v}'(0^-; \lambda_n) u(0^-; \lambda_n)}{v'(0^-; \lambda_n) \tilde{u}(0^-; \lambda_n)} - 1 \right] [\tilde{z}_n^{(1)} z_n^{(1)} f](x)}{(\lambda - \lambda_n) \|Z_n^{(1)}\|^2} \\
&+ \sum_{\lambda_n \in \Lambda_1^- \setminus \Lambda_0} \frac{[T_n^- - 1][\tilde{z}_n z_n f](x)}{(\lambda - \lambda_n) \|Z_n\|^2} \\
&- \sum_{\lambda_n \in \Lambda^* \cap \tilde{\Lambda}_1^+} \frac{\frac{\tilde{v}'(0^-; \lambda_n) u(0^-; \lambda_n)}{v'(0^-; \lambda_n)} \tilde{u}'_1(x; \lambda_n) \int_{-a}^x z_n^{(1)} f dt + \tilde{u}'(x; \lambda_n) \int_x^0 z_n^{(1)} f dt}{(\lambda - \lambda_n) \|Z_n^{(1)}\|^2} \\
&- \sum_{\lambda_n \in (\Lambda^* \cap \Lambda_0) \setminus \tilde{\Lambda}_1^+} \left\{ \begin{aligned} & \left[ \dot{\Phi}_n \tilde{y}'(x; \lambda_n) + \Phi_n \tilde{y}'_\lambda(x; \lambda_n) \right] \frac{\int_{-a}^x z_n^{(1)} f dt}{\lambda - \lambda_n} \\ & + \frac{\tilde{u}'(x; \lambda_n) \int_x^0 z_n^{(1)} f dt}{(\lambda - \lambda_n) \|Z_n^{(1)}\|^2} \\ & + \frac{\Phi_n \tilde{y}'(x; \lambda_n)}{\lambda - \lambda_n} \int_{-a}^x \left[ u_\lambda(t; \lambda_n) + \frac{u(t; \lambda_n)}{\lambda - \lambda_n} \right] f dt \end{aligned} \right\} \\
&- \sum_{\lambda_n \in (\Lambda \setminus \Lambda^*) \cap \tilde{\Lambda}_1^+} \frac{\tilde{v}'(x; \lambda_n) \int_{-a}^x u_n f dt + \tilde{u}'(x; \lambda_n) k_n \int_x^b u_n f dt}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} \\
&- \sum_{\lambda_n \in (\Lambda \cap \Lambda_0) \setminus (\Lambda^* \cup \tilde{\Lambda}_1^+)} \left\{ \begin{aligned} & \left[ \dot{\Phi}_n \tilde{y}'(x; \lambda_n) + \Phi_n \tilde{y}'_\lambda(x; \lambda_n) \right] \frac{\int_{-a}^x u_n f dt}{\lambda - \lambda_n} \\ & + \frac{\tilde{u}'(x; \lambda_n) k_n \int_x^b u_n f dt}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} \\ & + \frac{\Phi_n \tilde{y}'(x; \lambda_n)}{\lambda - \lambda_n} \int_{-a}^x \left[ u_\lambda(t; \lambda_n) + \frac{u_n(t)}{\lambda - \lambda_n} \right] f dt \end{aligned} \right\} \\
&- \sum_{\lambda_n \in \Lambda_1^- \cap \Lambda_0} \left[ \frac{\tilde{y}'(x; \lambda_n) \int_{-a}^x z_n f dt}{(\lambda - \lambda_n) \dot{\phi}(\lambda_n)} + \frac{\tilde{u}'(x; \lambda_n) \int_x^0 z_n f dt}{(\lambda - \lambda_n) \|Z_n\|^2} \right] \\
&- \sum_{\lambda_n \in \Lambda_1^+ \cap \Lambda_0} \frac{\tilde{y}'(x; \lambda_n) \int_{-a}^x u(t; \lambda_n) f dt}{(\lambda - \lambda_n) \dot{\phi}(\lambda_n)} \tag{4.68}
\end{aligned}$$

if  $-a \leq x < 0$ , else if  $0 < x \leq b$  then



$$\begin{aligned}
& \frac{[\tilde{v}u - \tilde{u}v](x; \lambda)}{\psi(\lambda)} f(x) - \frac{\partial}{\partial x} \left[ \int_{-a}^x R(x, t; \lambda) f(t) dt + \int_x^b R(x, t; \lambda) f(t) dt \right] \\
= & \sum_{\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)} \frac{[\tilde{K}_n - K_n][\tilde{u}_n u_n f](x)}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} + \sum_{\lambda_n \in \Lambda^* \setminus \Lambda_0} \frac{\left[ 1 - \frac{\tilde{u}(0^+; \lambda_n) v'(0^+; \lambda_n)}{u(0^+; \lambda_n) \tilde{v}'(0^+; \lambda_n)} \right] [\tilde{z}_n^{(2)} z_n^{(2)} f](x)}{(\lambda - \lambda_n) \|Z_n^{(2)}\|^2} \\
& + \sum_{\lambda_n \in \Lambda_1^+ \setminus \Lambda_0} \frac{[1 - T_n^+][\tilde{z}_n(x) z_n f](x)}{(\lambda - \lambda_n) \|Z_n\|^2} \\
& - \sum_{\lambda_n \in \Lambda^* \cap \bar{\Lambda}_1^-} \frac{\tilde{v}'(x; \lambda_n) \int_0^x z_n^{(2)} f dt + \frac{\tilde{u}(0^+; \lambda_n) v'(0^+; \lambda_n)}{u(0^+; \lambda_n)} \tilde{w}_2'(x; \lambda_n) \int_x^b z_n^{(2)} f dt}{(\lambda - \lambda_n) \|Z_n^{(2)}\|^2} \\
& - \sum_{\lambda_n \in (\Lambda^* \cap \Lambda_0) \setminus \bar{\Lambda}_1^-} \left\{ \begin{aligned} & \left[ \dot{\Phi}_n \tilde{y}'(x; \lambda_n) + \Phi_n \tilde{y}'_\lambda(x; \lambda_n) \right] \frac{\int_x^b z_n^{(2)} f dt}{\lambda - \lambda_n} \\ & + \frac{\tilde{v}'(x; \lambda_n) \int_0^x z_n^{(2)} f dt}{(\lambda - \lambda_n) \|Z_n^{(2)}\|^2} \\ & + \frac{\Phi_n \tilde{y}'(x; \lambda_n)}{\lambda - \lambda_n} \int_x^b \left[ v_\lambda(t; \lambda_n) + \frac{v(t; \lambda_n)}{\lambda - \lambda_n} \right] f dt \end{aligned} \right\} \\
& - \sum_{\lambda_n \in (\Lambda \setminus \Lambda^*) \cap \bar{\Lambda}_1^-} \frac{\tilde{v}'(x; \lambda_n) \int_{-a}^x u_n f dt + \tilde{u}'(x; \lambda_n) k_n \int_x^b u_n f dt}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} \\
& - \sum_{\lambda_n \in (\Lambda \cap \Lambda_0) \setminus (\Lambda^* \cup \bar{\Lambda}_1^-)} \left\{ \begin{aligned} & \left[ \dot{\Phi}_n \tilde{y}'(x; \lambda_n) + \Phi_n \tilde{y}'_\lambda(x; \lambda_n) \right] \frac{\int_x^b v_n f dt}{\lambda - \lambda_n} \\ & + \frac{\tilde{v}'(x; \lambda_n) \int_{-a}^x u_n f dt}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} \\ & + \frac{\Phi_n \tilde{y}'(x; \lambda_n) f(x)}{\lambda - \lambda_n} \int_x^b \left[ v_\lambda(t; \lambda_n) + \frac{v(t; \lambda_n)}{\lambda - \lambda_n} \right] f dt \end{aligned} \right\} \\
& - \sum_{\lambda_n \in \Lambda_1^+ \cap \Lambda_0} \left[ \frac{\tilde{y}'(x; \lambda_n) \int_x^b v_n f dt}{(\lambda - \lambda_n) \dot{\phi}(\lambda_n)} + \frac{\tilde{v}'(x; \lambda_n) \int_0^x z_n f dt}{(\lambda - \lambda_n) \|Z_n\|^2} \right] \\
& - \sum_{\lambda_n \in \Lambda_1^- \cap \Lambda_0} \frac{\tilde{y}'(x; \lambda_n) \int_x^b v(t; \lambda_n) f dt}{(\lambda - \lambda_n) \dot{\phi}(\lambda_n)}. \tag{4.69}
\end{aligned}$$

Here we have used the fact that for  $\lambda_n \in \Lambda^*$ ,  $z_n^{(2)}(x) = 0$  if  $-a \leq x < 0$ , and  $z_n^{(1)}(x) = 0$  if  $0 < x \leq b$ . Also, if  $\lambda_n \in \Lambda_1^-$  then  $z_n(x) = 0$  for  $0 < x \leq b$ , whereas if  $\lambda_n \in \Lambda_1^+$  then  $z_n(x) = 0$  for  $-a \leq x < 0$ .

Substituting the above expressions, (4.68) and (4.69), into (4.66) and (4.67), respectively, gives

$$\begin{aligned}
& [(\lambda - \tilde{L})^{-1}HF]_0' \\
&= \frac{\tilde{v}'(x; \lambda) \int_{-a}^x u f dt + \tilde{u}'(x; \lambda) \left[ \int_x^b v f dt + \sum_{i=1}^N f_i^1 \left[ \frac{\beta_i}{\lambda - \gamma_i} \Delta v' \right] + \sum_{j=1}^M f_j^2 \left[ \frac{\alpha_j}{\lambda - \delta_j} \Delta v \right] \right]}{\psi(\lambda)} \\
&+ \sum_{\lambda_n \in \Lambda \setminus (\Lambda_0 \cup \Lambda^*)} \frac{[K_n - \tilde{K}_n] \tilde{u}'_n(x) \int_{-a}^x u_n f dt}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} + \sum_{\lambda_n \in \Lambda_1^- \setminus \Lambda_0} \frac{[1 - T_n^-] \tilde{z}'_n(x) \int_{-a}^x z_n f dt}{(\lambda - \lambda_n) \|Z_n\|^2} \\
&+ \sum_{\lambda_n \in \Lambda^* \setminus \Lambda_0} \frac{\left[ 1 - \frac{\tilde{v}'(0^-; \lambda_n) u(0^-; \lambda_n)}{v'(0^-; \lambda_n) \tilde{u}(0^-; \lambda_n)} \right] \tilde{z}_n^{(1)'}(x) \int_{-a}^x z_n^{(1)} f dt}{(\lambda - \lambda_n) \|Z_n^{(1)}\|^2} \\
&- \sum_{\lambda_n \in \Lambda^* \cap \tilde{\Lambda}_1^+} \frac{\frac{\tilde{v}'(0^-; \lambda_n) u(0^-; \lambda_n)}{v'(0^-; \lambda_n)} \tilde{w}'_1(x; \lambda_n) \int_{-a}^x z_n^{(1)} f dt - \tilde{u}'(x; \lambda_n) \int_{-a}^x z_n^{(1)} f dt}{(\lambda - \lambda_n) \|Z_n^{(1)}\|^2} \\
&- \sum_{\lambda_n \in (\Lambda^* \cap \Lambda_0) \setminus \tilde{\Lambda}_1^+} \left\{ \begin{aligned} & \frac{\left[ \dot{\Phi}_n \tilde{y}'(x; \lambda_n) + \Phi_n \tilde{y}'_\lambda(x; \lambda_n) \right] \int_{-a}^x z_n^{(1)} f dt}{\lambda - \lambda_n} \\ & - \frac{\tilde{u}'(x; \lambda_n) \int_{-a}^x z_n^{(1)} f dt}{(\lambda - \lambda_n) \|Z_n^{(1)}\|^2} \\ & + \frac{\Phi_n \tilde{y}'(x; \lambda_n) \int_{-a}^x \left[ u_\lambda(t; \lambda_n) + \frac{u(t; \lambda_n)}{\lambda - \lambda_n} \right] f dt}{\lambda - \lambda_n} \end{aligned} \right\} \\
&- \sum_{\lambda_n \in (\Lambda \setminus \Lambda^*) \cap \tilde{\Lambda}_1^+} \frac{\tilde{v}'(x; \lambda_n) \int_{-a}^x u_n f dt - \tilde{u}'(x; \lambda_n) \int_{-a}^x v_n f dt}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} \\
&- \sum_{\lambda_n \in (\Lambda \cap \Lambda_0) \setminus (\Lambda^* \cup \tilde{\Lambda}_1^+)} \left\{ \begin{aligned} & \frac{\left[ \dot{\Phi}_n \tilde{y}'(x; \lambda_n) + \Phi_n \tilde{y}'_\lambda(x; \lambda_n) \right] \int_{-a}^x u_n f dt}{\lambda - \lambda_n} \\ & - \frac{\tilde{u}'(x; \lambda_n) \int_{-a}^x v_n f dt}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} \\ & + \frac{\Phi_n \tilde{y}'(x; \lambda_n) \int_{-a}^x \left[ u_\lambda(t; \lambda_n) + \frac{u_n(t)}{\lambda - \lambda_n} \right] f dt}{\lambda - \lambda_n} \end{aligned} \right\} \\
&- \sum_{\lambda_n \in \Lambda_1^- \cap \Lambda_0} \left[ \frac{\tilde{y}'(x; \lambda_n) \int_{-a}^x z_n f dt}{(\lambda - \lambda_n) \dot{\phi}(\lambda_n)} - \frac{\tilde{u}'(x; \lambda_n) \int_{-a}^x z_n f dt}{(\lambda - \lambda_n) \|Z_n\|^2} \right] \\
&- \sum_{\lambda_n \in \Lambda_1^+ \cap \Lambda_0} \frac{\tilde{y}'(x; \lambda_n) \int_{-a}^x u(t; \lambda_n) f dt}{(\lambda - \lambda_n) \dot{\phi}(\lambda_n)}
\end{aligned}$$

if  $-a \leq x < 0$ , else

$$\begin{aligned}
& \left[ (\lambda - \tilde{L})^{-1} HF \right]'_0 \\
& \tilde{v}'(x; \lambda) \left[ \int_{-a}^x u f dt + \sum_{i=1}^N f_i^1 \left[ \frac{\beta_i}{\lambda - \gamma_i} \Delta u' \right] + \sum_{j=1}^M f_j^2 \left[ \frac{\alpha_j}{\lambda - \delta_j} \Delta u \right] \right] + \tilde{u}'(x; \lambda) \int_x^b v f dt \\
= & \frac{\psi(\lambda)}{\psi(\lambda)} \\
& + \sum_{\lambda_n \in \Lambda \setminus (\Lambda_0 \cup \Lambda^*)} \frac{[\tilde{K}_n - K_n] \tilde{u}'_n(x) \int_x^b u_n f dt}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} + \sum_{\lambda_n \in \Lambda_1^+ \setminus \Lambda_0} \frac{[1 - T_n^+] \tilde{z}'_n(x) \int_x^b z_n f dt}{(\lambda - \lambda_n) \|Z_n\|^2} \\
& + \sum_{\lambda_n \in \Lambda^* \setminus \Lambda_0} \frac{\left[ 1 - \frac{\tilde{u}(0^+; \lambda_n) v'(0^+; \lambda_n)}{u(0^+; \lambda_n) \tilde{v}'(0^+; \lambda_n)} \right] \tilde{z}_n^{(2)'}(x) \int_x^b z_n^{(2)} f dt}{(\lambda - \lambda_n) \|Z_n^{(2)}\|^2} \\
& + \sum_{\lambda_n \in \Lambda^* \cap \tilde{\Lambda}_1^-} \frac{\tilde{v}'(x; \lambda_n) \int_x^b z_n^{(2)} f dt - \frac{\tilde{u}(0^+; \lambda_n) v'(0^+; \lambda_n)}{u(0^+; \lambda_n)} \tilde{w}'_2(x; \lambda_n) \int_x^b z_n^{(2)} f dt}{(\lambda - \lambda_n) \|Z_n^{(2)}\|^2} \\
& - \sum_{\lambda_n \in (\Lambda^* \cap \Lambda_0) \setminus \tilde{\Lambda}_1^-} \left\{ \begin{aligned} & \frac{\left[ \dot{\Phi}_n \tilde{y}'(x; \lambda_n) + \Phi_n \tilde{y}'_\lambda(x; \lambda_n) \right] \int_x^b z_n^{(2)} f dt}{\lambda - \lambda_n} \\ & - \frac{\tilde{v}'(x; \lambda_n) \int_x^b z_n^{(2)} f dt}{(\lambda - \lambda_n) \|Z_n^{(2)}\|^2} \\ & + \frac{\Phi_n \tilde{y}'(x; \lambda_n) \int_x^b \left[ v_\lambda(t; \lambda_n) + \frac{v(t; \lambda_n)}{\lambda - \lambda_n} \right] f dt}{\lambda - \lambda_n} \end{aligned} \right\} \\
& + \sum_{\lambda_n \in (\Lambda \setminus \Lambda^*) \cap \tilde{\Lambda}_1^-} \frac{\tilde{v}'(x; \lambda_n) \int_x^b u_n f dt - \tilde{u}'(x; \lambda_n) k_n \int_x^b u_n f dt}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} \\
& - \sum_{\lambda_n \in (\Lambda \cap \Lambda_0) \setminus (\Lambda^* \cup \tilde{\Lambda}_1^-)} \left\{ \begin{aligned} & \frac{\left[ \dot{\Phi}_n \tilde{y}'(x; \lambda_n) + \Phi_n \tilde{y}'_\lambda(x; \lambda_n) \right] \int_x^b v_n f dt}{\lambda - \lambda_n} \\ & - \frac{\tilde{v}'(x; \lambda_n) \int_x^b u_n f dt}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} \\ & + \frac{\Phi_n \tilde{y}'(x; \lambda_n) f(x) \int_x^b \left[ v_\lambda(t; \lambda_n) + \frac{v(t; \lambda_n)}{\lambda - \lambda_n} \right] f dt}{\lambda - \lambda_n} \end{aligned} \right\} \\
& - \sum_{\lambda_n \in \Lambda_1^+ \cap \Lambda_0} \left[ \frac{\tilde{y}'(x; \lambda_n) \int_x^b v_n f dt}{(\lambda - \lambda_n) \dot{\phi}(\lambda_n)} - \frac{\tilde{v}'(x; \lambda_n) \int_x^b z_n f dt}{(\lambda - \lambda_n) \|Z_n\|^2} \right] \\
& - \sum_{\lambda_n \in \Lambda_1^- \cap \Lambda_0} \frac{\tilde{y}'(x; \lambda_n) \int_x^b v(t; \lambda_n) f dt}{(\lambda - \lambda_n) \dot{\phi}(\lambda_n)}
\end{aligned}$$

if  $0 < x \leq b$ .

Since

$$[HF]_0 = (\lambda - \tilde{\ell}) \left[ (\lambda - \tilde{L})^{-1} HF \right]_0,$$

differentiating a second time with respect to  $x$  and comparing with Lemma 4.4.2 we see that, for

$$-a \leq x < 0,$$

[HF]<sub>0</sub>

$$\begin{aligned}
& \left( \sum_{\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)} \frac{[K_n - \tilde{K}_n] \int_{-a}^x u_n f dt}{\dot{\psi}(\lambda_n)} \tilde{u}_n(x) + \sum_{\lambda_n \in \Lambda_1^- \setminus \Lambda_0} \frac{[1 - T_n^-] \int_{-a}^x z_n f dt}{\|Z_n\|^2} \tilde{z}_n(x) \right. \\
& + \sum_{\lambda_n \in \Lambda^* \setminus \Lambda_0} \frac{\left[ 1 - \frac{\tilde{v}'(0^-; \lambda_n) u(0^-; \lambda_n)}{v'(0^-; \lambda_n) \tilde{u}(0^-; \lambda_n)} \right] \int_{-a}^x z_n^{(1)} f dt}{\|Z_n^{(1)}\|^2} \tilde{z}_n^{(1)}(x) \\
& - \sum_{\lambda_n \in \Lambda^* \cap \tilde{\Lambda}_1^+} \frac{\left[ \frac{\tilde{v}'(0^-; \lambda_n) u(0^-; \lambda_n)}{v'(0^-; \lambda_n)} \tilde{w}_1 - \tilde{u} \right] (x; \lambda_n) \int_{-a}^x z_n^{(1)} f dt}{\|Z_n^{(1)}\|^2} \\
& - \sum_{\lambda_n \in (\Lambda^* \cap \Lambda_0) \setminus \tilde{\Lambda}_1^+} \left\{ \left[ \dot{\Phi}_n \tilde{y}(x; \lambda_n) + \Phi_n \tilde{y}_\lambda(x; \lambda_n) - \frac{\tilde{u}(x; \lambda_n)}{\|Z_n^{(1)}\|^2} \right] \int_{-a}^x z_n^{(1)} f dt \right. \\
& \quad \left. + \Phi_n \tilde{y}(x; \lambda_n) \int_{-a}^x u_\lambda(t; \lambda_n) f dt \right\} \\
& - \sum_{\lambda_n \in (\Lambda \setminus \Lambda^*) \cap \tilde{\Lambda}_1^+} \frac{[\tilde{v} - k_n \tilde{u}](x; \lambda_n) \int_{-a}^x u_n f dt}{\dot{\psi}(\lambda_n)} \\
& - \sum_{\lambda_n \in (\Lambda \cap \Lambda_0) \setminus (\Lambda^* \cup \tilde{\Lambda}_1^+)} \left\{ \left[ \dot{\Phi}_n \tilde{y}(x; \lambda_n) + \Phi_n \tilde{y}_\lambda(x; \lambda_n) - \frac{k_n \tilde{u}(x; \lambda_n)}{\dot{\psi}(\lambda_n)} \right] \int_{-a}^x u_n f dt \right. \\
& \quad \left. + \Phi_n \tilde{y}(x; \lambda_n) \int_{-a}^x u_\lambda(t; \lambda_n) f dt \right\} \\
& - \sum_{\lambda_n \in \Lambda_1^- \cap \Lambda_0} \left[ \frac{\tilde{y}(x; \lambda_n)}{\dot{\phi}(\lambda_n)} - \frac{\tilde{u}(x; \lambda_n)}{\|Z_n\|^2} \right] \int_{-a}^x z_n f dt - \sum_{\lambda_n \in \Lambda_1^+ \cap \Lambda_0} \frac{\tilde{y}(x; \lambda_n) \int_{-a}^x u(t; \lambda_n) f dt}{\dot{\phi}(\lambda_n)} \Bigg) \\
& \tag{4.70}
\end{aligned}$$

is equal to

$$\begin{aligned}
& \frac{[\tilde{v}'u - \tilde{u}'v](x; \lambda)}{\psi(\lambda)} f(x) \\
& + \left\{ \begin{aligned}
& \sum_{\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)} \frac{[K_n - \tilde{K}_n][\tilde{u}'_n u_n](x)}{(\lambda - \lambda_n)\dot{\psi}(\lambda_n)} f(x) + \sum_{\lambda_n \in \Lambda_1^- \setminus \Lambda_0} \frac{[1 - T_n^-][\tilde{z}'_n z_n](x)}{(\lambda - \lambda_n)\|Z_n\|^2} f(x) \\
& + \sum_{\lambda_n \in \Lambda^* \setminus \Lambda_0} \frac{\left[1 - \frac{\tilde{v}'(0^-; \lambda_n)u(0^-; \lambda_n)}{v'(0^-; \lambda_n)\tilde{u}(0^-; \lambda_n)}\right] [\tilde{z}_n^{(1)'} z_n^{(1)}](x)}{(\lambda - \lambda_n)\|Z_n^{(1)}\|^2} f(x) \\
& - \sum_{\lambda_n \in \Lambda^* \cap \tilde{\Lambda}_1^+} \frac{\left[\frac{\tilde{v}'(0^-; \lambda_n)u(0^-; \lambda_n)}{v'(0^-; \lambda_n)} \tilde{w}'_1 - \tilde{u}'\right](x; \lambda_n) z_n^{(1)}(x)}{(\lambda - \lambda_n)\|Z_n^{(1)}\|^2} f(x) \\
& - \sum_{\lambda_n \in (\Lambda^* \cap \Lambda_0) \setminus \tilde{\Lambda}_1^+} \left\{ \begin{aligned}
& \left[ \dot{\Phi}_n \tilde{y}'(x; \lambda_n) + \Phi_n \tilde{y}'_\lambda(x; \lambda_n) - \frac{\tilde{u}'(x; \lambda_n)}{\|Z_n^{(1)}\|^2} \right] \frac{z_n^{(1)}(x)}{\lambda - \lambda_n} f(x) \\
& + \Phi_n \frac{\tilde{y}'(x; \lambda_n) \left[ u_\lambda(x; \lambda_n) + \frac{u(x; \lambda_n)}{\lambda - \lambda_n} \right]}{\lambda - \lambda_n} f(x) \\
& - \sum_{\lambda_n \in (\Lambda \setminus \Lambda^*) \cap \tilde{\Lambda}_1^+} \frac{[\tilde{v} - k_n \tilde{u}]'(x; \lambda_n) u_n(x)}{(\lambda - \lambda_n)\dot{\psi}(\lambda_n)} f(x) \\
& \left[ \dot{\Phi}_n \tilde{y}'(x; \lambda_n) + \Phi_n \tilde{y}'_\lambda(x; \lambda_n) - \frac{k_n \tilde{u}'(x; \lambda_n)}{\dot{\psi}(\lambda_n)} \right] \frac{u_n(x)}{\lambda - \lambda_n} f(x) \\
& + \Phi_n \frac{\tilde{y}'(x; \lambda_n) \left[ u_\lambda(x; \lambda_n) + \frac{u_n(x)}{\lambda - \lambda_n} \right]}{\lambda - \lambda_n} f(x)
\end{aligned} \right\} \\
& - \sum_{\lambda_n \in \Lambda_1^+ \cap \Lambda_0} \left[ \frac{\tilde{y}'(x; \lambda_n)}{\dot{\phi}(\lambda_n)} - \frac{\tilde{u}'(x; \lambda_n)}{\|Z_n\|^2} \right] \frac{z_n(x)}{\lambda - \lambda_n} f(x) - \sum_{\lambda_n \in \Lambda_1^+ \cap \Lambda_0} \frac{\tilde{y}'(x; \lambda_n) u(x; \lambda_n)}{(\lambda - \lambda_n)\dot{\phi}(\lambda_n)} f(x)
\end{aligned} \right\}. \tag{4.71}
\end{aligned}$$

Notice that the first expression is independent of  $\lambda$ . From Theorem 4.6.1 in the appendix to this chapter we observe that

$$[\tilde{v}'u - \tilde{u}'v](x; \lambda) = \psi(\lambda) + O(\lambda), \quad \lambda \in \mathbb{R}.$$

Furthermore, in (4.71) all of the summations are finite as  $K_n = \tilde{K}_n$  for all but finitely many  $n$  (corresponding to  $r(\lambda_n) = 0$  or  $s(\lambda_n) = 0$ ) for  $\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)$ , and because  $\Lambda^*$ ,  $\Lambda_1^\pm$  and  $\Lambda_0$  are finite sets. Hence, the expression in (4.71) is of the form

$$[1 + O(\lambda/\psi(\lambda)) + O(1/\lambda)] f(x), \quad \lambda \in \mathbb{R}$$

But comparing with (4.70), the expression in (4.71) must be independent of  $\lambda$ . Setting  $\lambda = A_m$ , where  $A_m$ ,  $m \in \mathbb{N}$  is defined in Theorem 3.6.2, and taking the limits as  $m \rightarrow \infty$  we see that  $\lambda/\psi(\lambda) = O(1/\sqrt{\lambda}) \rightarrow 0$ .

If  $0 < x \leq b$  then we have

$[HF]_0$

$$\left[ \begin{aligned}
& \sum_{\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)} \frac{[\tilde{K}_n - K_n] \int_x^b u_n f dt}{\dot{\psi}(\lambda_n)} \tilde{u}_n(x) + \sum_{\lambda_n \in \Lambda_1^+ \setminus \Lambda_0} \frac{[1 - T_n^+] \int_x^b z_n f dt}{\|Z_n\|^2} \tilde{z}_n(x) \\
& + \sum_{\lambda_n \in \Lambda^* \setminus \Lambda_0} \frac{\left[1 - \frac{\tilde{u}(0^+; \lambda_n) v'(0^+; \lambda_n)}{u(0^+; \lambda_n) \tilde{v}'(0^+; \lambda_n)}\right] \int_x^b z_n^{(2)} f dt}{\|Z_n^{(2)}\|^2} \tilde{z}_n^{(2)}(x) \\
& - \sum_{\lambda_n \in (\Lambda^* \cap \Lambda_0) \cap \tilde{\Lambda}_1^-} \frac{\left[\frac{\tilde{u}(0^+; \lambda_n) v'(0^+; \lambda_n)}{u(0^+; \lambda_n)} \tilde{w}_2 - \tilde{v}\right](x; \lambda_n)}{\|Z_n^{(2)}\|^2} \int_x^b z_n^{(2)} f dt \\
& - \sum_{\lambda_n \in (\Lambda^* \cap \Lambda_0) \setminus \tilde{\Lambda}_1^-} \left\{ \left[ \dot{\Phi}_n \tilde{y}(x; \lambda_n) + \Phi_n \tilde{y}_\lambda(x; \lambda_n) - \frac{\tilde{v}(x; \lambda_n)}{\|Z_n^{(2)}\|^2} \right] \int_x^b z_n^{(2)} f dt \right. \\
& \quad \left. + \Phi_n \tilde{y}(x; \lambda_n) \int_x^b \left[ v_\lambda(t; \lambda_n) + \frac{z_n^{(2)}(t)}{\lambda - \lambda_n} \right] f dt \right\} \\
& + \sum_{\lambda_n \in (\Lambda \setminus \Lambda^*) \cap \tilde{\Lambda}_1^-} \frac{[\tilde{v} - k_n \tilde{u}](x; \lambda_n) \int_x^b u_n f dt}{\dot{\psi}(\lambda_n)} \\
& - \sum_{\lambda_n \in (\Lambda \cap \Lambda_0) \setminus (\Lambda^* \cup \tilde{\Lambda}_1^-)} \left\{ \left[ \dot{\Phi}_n \tilde{y}(x; \lambda_n) + \Phi_n \tilde{y}_\lambda(x; \lambda_n) - \frac{\tilde{v}(x; \lambda_n)}{k_n \dot{\psi}(\lambda_n)} \right] \int_x^b v_n f dt \right. \\
& \quad \left. + \Phi_n \tilde{y}(x; \lambda_n) \int_x^b \left[ v_\lambda(t; \lambda_n) + \frac{v_n(t)}{\lambda - \lambda_n} \right] f dt \right\} \\
& - \sum_{\lambda_n \in \Lambda_1^+ \cap \Lambda_0} \left[ \frac{\tilde{y}(x; \lambda_n)}{\dot{\phi}(\lambda_n)} - \frac{\tilde{v}(x; \lambda_n)}{\|Z_n\|^2} \right] \int_x^b z_n f dt - \sum_{\lambda_n \in \Lambda_1 \cap \Lambda_0} \frac{\tilde{y}(x; \lambda_n) \int_x^b v(t; \lambda_n) f dt}{\dot{\phi}(\lambda_n)}
\end{aligned} \right]$$

equal to

$$\begin{aligned}
& \frac{[\tilde{v}'u - \tilde{u}'v](x; \lambda)}{\psi(\lambda)} f(x) \\
& - \left( \sum_{\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_n)} \frac{[\tilde{K}_n - K_n][\tilde{u}'_n u_n](x)}{(\lambda - \lambda_n)\dot{\psi}(\lambda_n)} f(x) + \sum_{\lambda_n \in \Lambda_1^+ \setminus \Lambda_0} \frac{[1 - T_n^+][\tilde{z}'_n z_n](x)}{(\lambda - \lambda_n) \|Z_n\|^2} f(x) \right. \\
& \quad + \sum_{\lambda_n \in \Lambda^* \setminus \Lambda_0} \frac{\left[1 - \frac{\tilde{u}(0^+; \lambda_n)v'(0^+; \lambda_n)}{u(0^+; \lambda_n)\tilde{v}'(0^+; \lambda_n)}\right] [\tilde{z}_n^{(2)'} z_n^{(2)}](x)}{(\lambda - \lambda_n) \|Z_n^{(2)}\|^2} f(x) \\
& \quad - \sum_{\lambda_n \in \Lambda^* \cap \tilde{\Lambda}_1^-} \frac{\left[\frac{\tilde{u}(0^+; \lambda_n)v'(0^+; \lambda_n)}{u(0^+; \lambda_n)} \tilde{w}'_2 - \tilde{v}'\right](x; \lambda_n) z_n^{(2)}(x)}{(\lambda - \lambda_n) \|Z_n^{(2)}\|^2} f(x) \\
& \quad - \sum_{\lambda_n \in (\Lambda^* \cap \Lambda_0) \setminus \tilde{\Lambda}_1^-} \left\{ \left[ \dot{\Phi}_n \tilde{y}'(x; \lambda_n) + \Phi_n \tilde{y}'_\lambda(x; \lambda_n) - \frac{\tilde{v}(x; \lambda_n)}{\|Z_n^{(2)}\|^2} \right] \frac{z_n^{(2)}(x)}{\lambda - \lambda_n} f(x) \right. \\
& \quad \quad \left. + \Phi_n \frac{\tilde{y}'(x; \lambda_n) \left[ v_\lambda(x; \lambda_n) + \frac{v(x; \lambda_n)}{\lambda - \lambda_n} \right]}{\lambda - \lambda_n} f(x) \right\} \\
& \quad + \sum_{\lambda_n \in (\Lambda \setminus \Lambda^*) \cap \tilde{\Lambda}_1^-} \frac{[\tilde{v}' - k_n \tilde{u}'](x; \lambda_n) u_n(x)}{(\lambda - \lambda_n)\dot{\psi}(\lambda_n)} f(x) \\
& \quad - \sum_{\lambda_n \in (\Lambda \cap \Lambda_0) \setminus (\Lambda^* \cup \tilde{\Lambda}_1^-)} \left\{ \left[ \dot{\Phi}_n \tilde{y}'(x; \lambda_n) + \Phi_n \tilde{y}'_\lambda(x; \lambda_n) - \frac{\tilde{v}'(x; \lambda_n)}{k_n \dot{\psi}(\lambda_n)} \right] \frac{v_n(x)}{\lambda - \lambda_n} f(x) \right. \\
& \quad \quad \left. + \Phi_n \frac{\tilde{y}'(x; \lambda_n) \left[ v_\lambda(x; \lambda_n) + \frac{v(x; \lambda_n)}{\lambda - \lambda_n} \right]}{\lambda - \lambda_n} f(x) \right\} \\
& \quad - \sum_{\lambda_n \in \Lambda_1^+ \cap \Lambda_0} \left[ \frac{\tilde{y}'(x; \lambda_n)}{\dot{\phi}(\lambda_n)} - \frac{\tilde{v}'(x; \lambda_n)}{\|Z_n\|^2} \right] \frac{z_n(x)}{\lambda - \lambda_n} f(x) - \sum_{\lambda_n \in \Lambda_1^- \cap \Lambda_0} \frac{\tilde{y}'(x; \lambda_n) v(x; \lambda_n)}{(\lambda - \lambda_n)\dot{\phi}(\lambda_n)} f(x) \Big)
\end{aligned}$$

and the result follows as per the previous case.  $\square$

We are finally in a position to prove the main results.

## 4.5 Main theorems

Let  $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \dots$  denote the eigenvalues of the Hilbert space operator eigenvalue problem  $(L; \alpha, \beta; r, s)$  (i.e. the eigenvalues of  $\ell y = \lambda y$  with boundary conditions (4.2)-(4.3) and transmission conditions (4.4)-(4.5)). Similarly, denote by  $\tilde{\lambda}_0 \leq \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \dots$  the eigenvalues of  $(\tilde{L}; \alpha, \beta; r, s)$  with  $L$  replaced by  $\tilde{L}$  (i.e.  $\ell$  replaced by  $\tilde{\ell}$ ). Let  $(L; \alpha, \zeta; r, s)$  and  $(\tilde{L}; \alpha, \zeta; r, s)$  denote the corresponding boundary value problems with the boundary condition at  $x = b$  replaced by

$$y(b) \cos \zeta = y'(b) \sin \zeta, \quad \zeta \in (0, \pi], \quad (4.72)$$

where  $\sin(\beta - \zeta) \neq 0$ . Let  $(L; \varepsilon, \beta; r, s)$  and  $(\tilde{L}; \varepsilon, \beta; r, s)$  denote the boundary value problems with the boundary condition at  $x = -a$  replaced by

$$y(-a) \cos \varepsilon = y'(-a) \sin \varepsilon, \quad \varepsilon \in [0, \pi), \quad (4.73)$$

where  $\sin(\alpha - \varepsilon) \neq 0$ .

**Theorem 4.5.1.** *Assume that each eigenvalue  $\lambda_n$  of  $(L; \alpha, \beta; r, s)$  corresponds with an eigenvalue of  $(\tilde{L}; \alpha, \beta; r, s)$ , up to multiplicity, except if  $\lambda_n \in \Lambda_0$ , where  $\Lambda_0$  is a finite set.*

I. *If the eigenvalues of  $(L; \alpha, \zeta; r, s)$  and  $(\tilde{L}; \alpha, \zeta; r, s)$  coincide up to multiplicity then, almost everywhere*

1. *on  $[-a, 0)$ ,*

$$[q - \tilde{q}](x) = \left\{ \begin{aligned} & - \sum_{\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)} \frac{2[K_n - \tilde{K}_n][u_n \tilde{u}_n]'(x)}{\dot{\psi}(\lambda_n)} \\ & - \sum_{\lambda_n \in \Lambda_1^- \setminus \Lambda_0} \frac{2[1 - T_n^-][z_n \tilde{z}_n]'(x)}{\|Z_n\|^2} \\ & - \sum_{\lambda_n \in \Lambda^* \setminus \Lambda_0} \frac{2 \left[ 1 - \frac{\tilde{v}'(0^-; \lambda_n) u(0^-; \lambda_n)}{v'(0^-; \lambda_n) \tilde{u}(0^-; \lambda_n)} \right] [z_n^{(1)} \tilde{z}_n^{(1)}]'(x)}{\|Z_n^{(1)}\|^2} \\ & + \sum_{\lambda_n \in \Lambda^* \cap \tilde{\Lambda}_1^+} \frac{2 \left[ \left[ \frac{\tilde{v}'(0^-; \lambda_n) u(0^-; \lambda_n)}{v'(0^-; \lambda_n)} \tilde{w}_1 - \tilde{u} \right] (x; \lambda_n) z_n^{(1)}(x) \right]'}{\|Z_n^{(1)}\|^2} \\ & + \sum_{\lambda_n \in (\Lambda^* \cap \Lambda_0) \setminus \tilde{\Lambda}_1^+} \left\{ 2 \left[ \left[ \dot{\Phi}_n \tilde{y} + \Phi_n \tilde{y}_\lambda - \frac{\tilde{u}}{\|Z_n^{(1)}\|^2} \right] (x; \lambda_n) z_n^{(1)}(x) \right]' \right. \\ & \quad \left. + 2\Phi_n [\tilde{y} u_\lambda]'(x; \lambda_n) \right\} \\ & + \sum_{\lambda_n \in (\Lambda \setminus \Lambda^*) \cap \tilde{\Lambda}_1^+} \frac{2 \left[ [\tilde{v} - k_n \tilde{u}] (x; \lambda_n) u_n(x) \right]'}{\dot{\psi}(\lambda_n)} \\ & + \sum_{\lambda_n \in (\Lambda \cap \Lambda_0) \setminus (\Lambda^* \cup \tilde{\Lambda}_1^+)} \left\{ 2 \left[ \left[ \dot{\Phi}_n \tilde{y} + \Phi_n \tilde{y}_\lambda - \frac{k_n \tilde{u}}{\dot{\psi}(\lambda_n)} \right] (x; \lambda_n) u_n(x) \right]' \right. \\ & \quad \left. + 2\Phi_n [\tilde{y} u_\lambda]'(x; \lambda_n) \right\} \\ & + \sum_{\lambda_n \in \Lambda_1^- \cap \Lambda_0} 2 \left[ \left[ \frac{\tilde{y}(x; \lambda_n)}{\dot{\phi}(\lambda_n)} - \frac{\tilde{u}(x; \lambda_n)}{\|Z_n\|^2} \right] z_n(x) \right]' + \sum_{\lambda_n \in \Lambda_1^+ \cap \Lambda_0} \frac{2 [\tilde{y} u]'(x; \lambda_n)}{\dot{\phi}(\lambda_n)} \end{aligned} \right\},$$

where for  $\lambda_n \in \Lambda \setminus (\Lambda_0 \cup \Lambda^*)$ , if  $r(\lambda_n) = 0$  or  $s(\lambda_n) = 0$  then  $K_n - \tilde{K}_n = k_n^{(1)} - \tilde{k}_n^{(1)}$ , else  $K_n - \tilde{K}_n = 0$ ,



2. on  $(0, b]$ ,

$$\begin{aligned}
& [q - \tilde{q}](x) \\
& = \left\{ \begin{aligned}
& - \sum_{\lambda_n \in \Lambda^* \cap \tilde{\Lambda}_1^-} \frac{2 \left[ \left[ \frac{\tilde{u}(0^+; \lambda_n) v'(0^+; \lambda_n)}{u(0^+; \lambda_n)} \tilde{w}_2 - \tilde{v} \right] (x; \lambda_n) z_n^{(2)}(x) \right]'}{\|Z_n^{(2)}\|^2} \\
& - \sum_{\lambda_n \in (\Lambda^* \cap \Lambda_0) \setminus \tilde{\Lambda}_1^-} \left\{ 2 \left[ \left[ \dot{\Phi}_n \tilde{y} + \Phi_n \tilde{y}_\lambda - \frac{\tilde{v}}{\|Z_n^{(2)}\|^2} \right] (x; \lambda_n) z_n^{(2)}(x) \right]'} \right. \\
& \quad \left. + 2\Phi_n [\tilde{y} v_\lambda]'(x; \lambda_n) \right\} \\
& + \sum_{\lambda_n \in (\Lambda \setminus \Lambda^*) \cap \tilde{\Lambda}_1^-} \frac{2 \left[ [\tilde{v} - k_n \tilde{u}] (x; \lambda_n) v_n(x) \right]'}{\dot{\psi}(\lambda_n)} \\
& - \sum_{\lambda_n \in (\Lambda \cap \Lambda_0) \setminus (\Lambda^* \cup \tilde{\Lambda}_1^-)} \left\{ 2 \left[ \left[ \dot{\Phi}_n \tilde{y} + \Phi_n \tilde{y}_\lambda - \frac{\tilde{v}}{k_n \dot{\psi}(\lambda_n)} \right] (x; \lambda_n) v_n(x) \right]'} \right\} \\
& - \sum_{\lambda_n \in \Lambda_1^+ \cap \Lambda_0} 2 \left[ \left[ \frac{\tilde{y}(x; \lambda_n)}{\dot{\phi}(\lambda_n)} - \frac{\tilde{v}(x; \lambda_n)}{\|Z_n\|^2} \right] z_n(x) \right]' - \sum_{\lambda_n \in \Lambda_1^- \cap \Lambda_0} 2 \frac{[\tilde{y} v]'(x; \lambda_n)}{\dot{\phi}(\lambda_n)}
\end{aligned} \right\},
\end{aligned}$$

in the notation of Definition 4.3.1, Lemma 4.3.2 and Proposition 4.6.2.

II. If the eigenvalues of  $(L; \varepsilon, \beta; r, s)$  and  $(\tilde{L}; \varepsilon, \beta; r, s)$  coincide up to multiplicity then, almost everywhere

1. on  $[-a, 0)$ ,

$$\begin{aligned}
& [q - \tilde{q}](x) \\
& = \left\{ \begin{aligned}
& \sum_{\lambda_n \in \Lambda^* \cap \tilde{\Lambda}_1^+} \frac{2 \left[ \left[ \frac{\tilde{v}'(0^-; \lambda_n) u(0^-; \lambda_n)}{v'(0^-; \lambda_n)} \tilde{w}_1 - \tilde{u} \right] (x; \lambda_n) z_n^{(1)}(x) \right]'}{\|Z_n^{(1)}\|^2} \\
& + \sum_{\lambda_n \in (\Lambda^* \cap \Lambda_0) \setminus \tilde{\Lambda}_1^+} \left\{ 2 \left[ \left[ \dot{\Phi}_n \tilde{y} + \Phi_n \tilde{y}_\lambda - \frac{\tilde{u}}{\|Z_n^{(1)}\|^2} \right] (x; \lambda_n) z_n^{(1)}(x) \right]'} \right. \\
& \quad \left. + 2\Phi_n [\tilde{y} u_\lambda]'(x; \lambda_n) \right\} \\
& + \sum_{\lambda_n \in (\Lambda \setminus \Lambda^*) \cap \tilde{\Lambda}_1^+} \frac{2 \left[ [\tilde{v} - k_n \tilde{u}] (x; \lambda_n) u_n(x) \right]'}{\dot{\psi}(\lambda_n)} \\
& + \sum_{\lambda_n \in (\Lambda \cap \Lambda_0) \setminus (\Lambda^* \cup \tilde{\Lambda}_1^+)} \left\{ 2 \left[ \left[ \dot{\Phi}_n \tilde{y} + \Phi_n \tilde{y}_\lambda - \frac{k_n \tilde{u}}{\dot{\psi}(\lambda_n)} \right] (x; \lambda_n) u_n(x) \right]'} \right\} \\
& + \sum_{\lambda_n \in \Lambda_1^- \cap \Lambda_0} 2 \left[ \left[ \frac{\tilde{y}(x; \lambda_n)}{\dot{\phi}(\lambda_n)} - \frac{\tilde{u}(x; \lambda_n)}{\|Z_n\|^2} \right] z_n(x) \right]' + \sum_{\lambda_n \in \Lambda_1^+ \cap \Lambda_0} 2 \frac{[\tilde{y} u]'(x; \lambda_n)}{\dot{\phi}(\lambda_n)}
\end{aligned} \right\},
\end{aligned}$$

2. on  $(0, b]$ ,

$$\begin{aligned}
& [q - \tilde{q}](x) \\
& = \left\{ \begin{aligned}
& \sum_{\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)} \frac{2[\tilde{K}_n - K_n][\tilde{u}_n u_n]'(x)}{\dot{\psi}(\lambda_n)} \\
& + \sum_{\lambda_n \in \Lambda_1^+ \setminus \Lambda_0} \frac{2[1 - T_n^+][\tilde{z}_n z_n]'(x)}{\|Z_n\|^2} \\
& + \sum_{\lambda_n \in \Lambda^* \setminus \Lambda_0} \frac{2 \left[ 1 - \frac{\tilde{u}(0^+; \lambda_n) v'(0^+; \lambda_n)}{u(0^+; \lambda_n) \tilde{v}'(0^+; \lambda_n)} \right] [\tilde{z}_n^{(2)} z_n^{(2)}]'(x)}{\|Z_n^{(2)}\|^2} \\
& - \sum_{\lambda_n \in \Lambda^* \cap \tilde{\Lambda}_1^-} \frac{2 \left[ \left[ \frac{\tilde{u}(0^+; \lambda_n) v'(0^+; \lambda_n)}{u(0^+; \lambda_n)} \tilde{w}_2 - \tilde{v} \right] (x; \lambda_n) z_n^{(2)}(x) \right]'}{\|Z_n^{(2)}\|^2} \\
& - \sum_{\lambda_n \in (\Lambda^* \cap \Lambda_0) \setminus \tilde{\Lambda}_1^-} \left\{ 2 \left[ \left[ \dot{\Phi}_n \tilde{y} + \Phi_n \tilde{y}_\lambda - \frac{\tilde{v}}{\|Z_n^{(2)}\|^2} \right] (x; \lambda_n) z_n^{(2)}(x) \right]'} \right. \\
& \quad \left. + 2\Phi_n [\tilde{y} v_\lambda]'(x; \lambda_n) \right\} \\
& + \sum_{\lambda_n \in (\Lambda \setminus \Lambda^*) \cap \tilde{\Lambda}_1^-} \frac{2 [[\tilde{v} - k_n \tilde{u}](x; \lambda_n) v_n(x)]'}{\dot{\psi}(\lambda_n)} \\
& - \sum_{\lambda_n \in (\Lambda \cap \Lambda_0) \setminus (\Lambda^* \cup \tilde{\Lambda}_1^-)} \left\{ 2 \left[ \left[ \dot{\Phi}_n \tilde{y} + \Phi_n \tilde{y}_\lambda - \frac{\tilde{v}}{k_n \dot{\psi}(\lambda_n)} \right] (x; \lambda_n) v_n(x) \right]'} \right. \\
& \quad \left. + 2\Phi_n [\tilde{y} v_\lambda]'(x; \lambda_n) \right\} \\
& - \sum_{\lambda_n \in \Lambda_1^+ \cap \Lambda_0} 2 \left[ \left[ \frac{\tilde{y}(x; \lambda_n)}{\dot{\phi}(\lambda_n)} - \frac{\tilde{v}(x; \lambda_n)}{\|Z_n\|^2} \right] z_n(x) \right]' - \sum_{\lambda_n \in \Lambda_1^- \cap \Lambda_0} 2 \frac{[\tilde{y} v]'(x; \lambda_n)}{\dot{\phi}(\lambda_n)}
\end{aligned} \right\},
\end{aligned}$$

where for  $\lambda_n \in \Lambda \setminus (\Lambda_0 \cup \Lambda^*)$ , if  $r(\lambda_n) = 0$  or  $s(\lambda_n) = 0$  then  $\tilde{K}_n - K_n = \tilde{k}_n^{(2)} - k_n^{(2)}$ , else  $\tilde{K}_n - K_n = 0$ ,

in the notation of Definition 4.3.1, Lemma 4.3.2 and Proposition 4.6.2.

*Proof.* We start off assuming only that  $k_n = \tilde{k}_n$  for  $\lambda_n \in \Lambda \setminus \Lambda_0$  with  $r(\lambda_n) \neq 0$  and  $s(\lambda_n) \neq 0$ . This is a result of both Lemma 4.3.6 and Lemma 4.3.7. The final identities in *I* and *II* will follow from the arguments below after applying the remaining conclusions of Lemmas 4.3.6 and 4.3.7 respectively. Let  $F \in \mathcal{D}(L) \cap \mathcal{H}_0$ .

Suppose that  $-a \leq x < 0$ . Replacing  $F$  by  $LF$  in the result of Theorem 4.4.3 we get

$$\begin{aligned}
& [HLF]_0 \\
& = -f'' + qf \\
& + \left\{ \begin{aligned}
& \sum_{\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)} \frac{[K_n - \tilde{K}_n] \left[ fu'_n - f'u_n + \lambda_n \int_{-a}^x u_n f dt \right]}{\dot{\psi}(\lambda_n)} \tilde{u}_n \\
& + \sum_{\lambda_n \in \Lambda_1^- \setminus \Lambda_0} \frac{[1 - T_n^-] \left[ fz'_n - f'z_n + \lambda_n \int_{-a}^x z_n f dt \right]}{\|Z_n\|^2} \tilde{z}_n \\
& + \sum_{\lambda_n \in \Lambda^* \setminus \Lambda_0} \frac{\left[ 1 - \frac{\tilde{v}'(0^-; \lambda_n) u(0^-; \lambda_n)}{v'(0^-; \lambda_n) \tilde{u}(0^-; \lambda_n)} \right] \left[ fz_n^{(1)'} - f'z_n^{(1)} + \lambda_n \int_{-a}^x z_n^{(1)} f dt \right]}{\|Z_n^{(1)}\|^2} \tilde{z}_n^{(1)} \\
& - \sum_{\lambda_n \in \Lambda^* \cap \tilde{\Lambda}_1^+} \frac{\left[ \frac{\tilde{v}'(0^-; \lambda_n) u(0^-; \lambda_n)}{v'(0^-; \lambda_n)} \tilde{w}_1 - \tilde{u} \right] (x; \lambda_n) \left[ fz_n^{(1)'} - f'z_n^{(1)} + \lambda_n \int_{-a}^x z_n^{(1)} f dt \right]}{\|Z_n^{(1)}\|^2} \\
& - \sum_{\lambda_n \in (\Lambda^* \cap \Lambda_0) \setminus \tilde{\Lambda}_1^+} \left\{ \begin{aligned}
& \left[ \dot{\Phi}_n \tilde{y} + \Phi_n \tilde{y}_\lambda - \frac{\tilde{u}}{\|Z_n^{(1)}\|^2} \right] (x; \lambda_n) \left[ \begin{aligned}
& fz_n^{(1)'} - f'z_n^{(1)} \\
& + \lambda_n \int_{-a}^x z_n^{(1)} f dt
\end{aligned} \right] \\
& + \Phi_n \tilde{y}(x; \lambda_n) \left[ \begin{aligned}
& [fu'_\lambda - f'u_\lambda] (x; \lambda_n) \\
& + \int_{-a}^x [z_n^{(1)} + \lambda_n u_\lambda(t; \lambda_n)] f dt
\end{aligned} \right]
\end{aligned} \right\} \\
& - \sum_{\lambda_n \in (\Lambda \setminus \Lambda^*) \cap \tilde{\Lambda}_1^+} \frac{[\tilde{v} - k_n \tilde{u}] (x; \lambda_n) \left[ fu'_n - f'u_n + \lambda_n \int_{-a}^x u_n f dt \right]}{\dot{\psi}(\lambda_n)} \\
& - \sum_{\lambda_n \in (\Lambda \cap \Lambda_0) \setminus (\Lambda^* \cup \tilde{\Lambda}_1^+)} \left\{ \begin{aligned}
& \left[ \dot{\Phi}_n \tilde{y} + \Phi_n \tilde{y}_\lambda - \frac{k_n \tilde{u}}{\dot{\psi}(\lambda_n)} \right] (x; \lambda_n) \left[ \begin{aligned}
& fu'_n - f'u_n \\
& + \lambda_n \int_{-a}^x u_n f dt
\end{aligned} \right] \\
& + \Phi_n \tilde{y}(x; \lambda_n) \left[ \begin{aligned}
& [fu'_\lambda - f'u_\lambda] (x; \lambda_n) \\
& + \int_{-a}^x [u_n + \lambda_n u_\lambda(t; \lambda_n)] f dt
\end{aligned} \right]
\end{aligned} \right\} \\
& - \sum_{\lambda_n \in \Lambda_1^- \cap \Lambda_0} \left[ \frac{\tilde{y}(x; \lambda_n)}{\dot{\phi}(\lambda_n)} - \frac{\tilde{u}(x; \lambda_n)}{\|Z_n\|^2} \right] \left[ fz'_n - f'z_n + \lambda_n \int_{-a}^x z_n f dt \right] \\
& - \sum_{\lambda_n \in \Lambda_1^+ \cap \Lambda_0} \frac{\tilde{y}(x; \lambda_n) \left[ [fu' - f'u] (x; \lambda_n) + \lambda_n \int_{-a}^x u(t; \lambda_n) f dt \right]}{\dot{\phi}(\lambda_n)}
\end{aligned} \right\}
\end{aligned}$$

where we have used the fact that

$$\int_{-a}^x u(t; \lambda_n) [-f'' + qf](t) dt = f(x)u'(x; \lambda_n) - f'(x)u(x; \lambda_n) + \lambda_n \int_{-a}^x u(t; \lambda_n) f dt$$

(recall that if  $-a \leq x < 0$ ,  $z_n^{(1)}(x) = u(x; \lambda_n)$  for  $\lambda_n \in \Lambda^*$ , and  $z_n(x) = u(x; \lambda_n)$  for  $\lambda_n \in \Lambda_1^-$ ),

and also that

$$\begin{aligned} \int_{-a}^x u_\lambda(t; \lambda_n) [-f'' + qf] dt &= f(x) u'_\lambda(x; \lambda_n) - f'(x) u_\lambda(x; \lambda_n) \\ &\quad + \int_{-a}^x [u(t; \lambda_n) + \lambda_n u_\lambda(t; \lambda_n)] f dt \end{aligned}$$

since  $u(-a; \lambda) = \sin \alpha$  and  $u'(-a; \lambda) = \cos \alpha$  for all  $\lambda$  by definition of  $u(x; \lambda)$ , and since  $f(-a) \cos \alpha - f'(-a) \sin \alpha = 0$  as  $f$  must obey (4.2) by the domain condition.

Moreover, operating with  $-\frac{d^2}{dx^2} + \tilde{q}(x)$  on  $[HF]_0$  we obtain

$$\begin{aligned} & \left[ \tilde{L}HF \right]_0 \\ &= -f'' + \tilde{q}f \\ &+ \left\{ \begin{aligned} & \sum_{\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)} \frac{[K_n - \tilde{K}_n] \left[ [-[u_n f]' + \lambda_n \int_{-a}^x u_n f dt] \tilde{u}_n - 2u_n f \tilde{u}'_n \right]}{\dot{\psi}(\lambda_n)} \\ & + \sum_{\lambda_n \in \Lambda_1^- \setminus \Lambda_0} \frac{[1 - T_n^-] \left[ [-[z_n f]' + \lambda_n \int_{-a}^x z_n f dt] \tilde{z}_n - 2z_n f \tilde{z}'_n \right]}{\|Z_n\|^2} \\ & + \sum_{\lambda_n \in \Lambda^* \setminus \Lambda_0} \frac{\left[ 1 - \frac{\tilde{v}'(0^-; \lambda_n) u(0^-; \lambda_n)}{v'(0^-; \lambda_n) \tilde{u}(0^-; \lambda_n)} \right] \left[ \begin{aligned} & [-[z_n^{(1)} f]' + \lambda_n \int_{-a}^x z_n^{(1)} f dt] \tilde{z}_n^{(1)} \\ & - 2z_n^{(1)} f \tilde{z}_n^{(1)'} \end{aligned} \right]}{\|Z_n^{(1)}\|^2} \\ & - \sum_{\lambda_n \in \Lambda^* \cap \tilde{\Lambda}_1^+} \left\{ \frac{\frac{\tilde{v}'(0^-; \lambda_n) u(0^-; \lambda_n)}{v'(0^-; \lambda_n)} \left[ \begin{aligned} & [-[z_n^{(1)} f]' + \lambda_n \int_{-a}^x z_n^{(1)} f dt] \tilde{w}_1 \\ & - 2z_n^{(1)} f \tilde{w}'_1 \end{aligned} \right] (x; \lambda_n)}{\|Z_n^{(1)}\|^2} \right. \\ & \quad \left. - \frac{\left[ [-[z_n^{(1)} f]' + \lambda_n \int_{-a}^x z_n^{(1)} f dt] \tilde{u} - 2z_n^{(1)} f \tilde{u}' \right] (x; \lambda_n)}{\|Z_n^{(1)}\|^2} \right\} \\ & - \sum_{\lambda_n \in (\Lambda^* \cap \Lambda_0) \setminus \tilde{\Lambda}_1^+} \left\{ \begin{aligned} & \lambda_n \int_{-a}^x z_n^{(1)} f dt \left[ \begin{aligned} & \left[ \dot{\Phi}_n + \frac{\Phi_n}{\lambda_n} \right] \tilde{y} + \Phi_n \tilde{y}_\lambda - \frac{\tilde{u}}{\|Z_n^{(1)}\|^2} \end{aligned} \right] (x; \lambda_n) \\ & - 2z_n^{(1)} f \left[ \begin{aligned} & \dot{\Phi}_n \tilde{y}' + \Phi_n \tilde{y}'_\lambda - \frac{\tilde{u}'}{\|Z_n^{(1)}\|^2} \end{aligned} \right] (x; \lambda_n) \\ & - [z_n^{(1)} f]' \left[ \begin{aligned} & \dot{\Phi}_n \tilde{y} + \Phi_n \tilde{y}_\lambda - \frac{\tilde{u}}{\|Z_n^{(1)}\|^2} \end{aligned} \right] (x; \lambda_n) \\ & + \Phi_n \tilde{y}(x; \lambda_n) \left[ \begin{aligned} & \lambda_n \int_{-a}^x u_\lambda(t; \lambda_n) f dt - [u_\lambda(x; \lambda_n) f]' \\ & - 2\Phi_n \tilde{y}'(x; \lambda_n) u_\lambda(x; \lambda_n) f \end{aligned} \right] \end{aligned} \right\} \end{aligned} \right\} \end{aligned}$$

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$$\begin{aligned}
& \left. \begin{aligned}
& - \sum_{\lambda_n \in (\Lambda \setminus \Lambda^*) \cap \bar{\Lambda}_1^+} \left\{ \frac{\left[ -[u_n f]' + \lambda_n \int_{-a}^x u_n f dt \right] [\tilde{v} - k_n \tilde{u}] - 2u_n f [\tilde{v} - k_n \tilde{u}]'}{\dot{\psi}(\lambda_n)} (x; \lambda_n) \right\} \\
& - \sum_{\lambda_n \in (\Lambda \cap \Lambda_0) \setminus (\Lambda^* \cup \bar{\Lambda}_1^+)} \left\{ \begin{aligned}
& \lambda_n \int_{-a}^x u_n f dt \left[ \left[ \dot{\Phi}_n + \frac{\Phi_n}{\lambda_n} \right] \tilde{y} + \Phi_n \tilde{y}_\lambda - \frac{k_n \tilde{u}}{\dot{\psi}(\lambda_n)} \right] (x; \lambda_n) \\
& - 2u_n f \left[ \dot{\Phi}_n \tilde{y}' + \Phi_n \tilde{y}'_\lambda - \frac{k_n \tilde{u}'}{\dot{\psi}(\lambda_n)} \right] (x; \lambda_n) \\
& - [u_n f]' \left[ \dot{\Phi}_n \tilde{y} + \Phi_n \tilde{y}_\lambda - \frac{k_n \tilde{u}}{\dot{\psi}(\lambda_n)} \right] (x; \lambda_n) \\
& + \Phi_n \tilde{y}(x; \lambda_n) \left[ \lambda_n \int_{-a}^x u_\lambda(t; \lambda_n) f dt - [u_\lambda(x; \lambda_n) f]' \right] \\
& - 2\Phi_n \tilde{y}'(x; \lambda_n) u_\lambda(x; \lambda_n) f
\end{aligned} \right\} \\
& - \sum_{\lambda_n \in \Lambda_1^- \cap \Lambda_0} \left\{ \begin{aligned}
& \frac{\left[ -[z_n f]' + \lambda_n \int_{-a}^x z_n f dt \right] \tilde{y}(x; \lambda_n) - 2[z_n f] \tilde{y}'(x; \lambda_n)}{\dot{\phi}(\lambda_n)} \\
& - \frac{\left[ -[z_n f]' + \lambda_n \int_{-a}^x z_n f dt \right] \tilde{u}(x; \lambda_n) - 2[z_n f] \tilde{u}'(x; \lambda_n)}{\|Z_n\|^2}
\end{aligned} \right\} \\
& - \sum_{\lambda_n \in \Lambda_1^+ \cap \Lambda_0} \left\{ \frac{\left[ -[u(x; \lambda_n) f]' + \lambda_n \int_{-a}^x u(t; \lambda_n) f dt \right] \tilde{y}(x; \lambda_n) - 2u(x; \lambda_n) f \tilde{y}'(x; \lambda_n)}{\dot{\phi}(\lambda_n)} \right\}
\end{aligned} \right\}.
\end{aligned}$$

Similarly, if  $0 < x \leq b$ , we have

$$\begin{aligned}
& [HLF]_0 \\
& = -f'' + qf \\
& + \left\{ \begin{aligned}
& \sum_{\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)} \frac{[\tilde{K}_n - K_n] \left[ -[u'_n f - u_n f'] + \lambda_n \int_x^b u_n f dt \right]}{\dot{\psi}(\lambda_n)} \tilde{u}_n \\
& + \sum_{\lambda_n \in \Lambda_1^+ \setminus \Lambda_0} \frac{[1 - T_n^+] \left[ -[f z'_n - f' z_n] + \lambda_n \int_x^b z_n f dt \right]}{\|Z_n\|^2} \tilde{z}_n \\
& + \sum_{\lambda_n \in \Lambda^* \setminus \Lambda_0} \frac{\left[ 1 - \frac{\tilde{u}(0^+; \lambda_n) v'(0^+; \lambda_n)}{u(0^+; \lambda_n) \tilde{v}'(0^+; \lambda_n)} \right] \left[ -[f z_n^{(2)'} - f' z_n^{(2)}] + \lambda_n \int_x^b z_n^{(2)} f dt \right]}{\|Z_n^{(2)}\|^2} \tilde{z}_n^{(2)} \\
& - \sum_{\lambda_n \in (\Lambda^* \cap \Lambda_0) \cap \bar{\Lambda}_1^-} \frac{\left[ \frac{\tilde{u}(0^+; \lambda_n) v'(0^+; \lambda_n)}{u(0^+; \lambda_n)} \tilde{w}_2 - \tilde{v} \right] (x; \lambda_n) \left[ -[f z_n^{(2)'} - f' z_n^{(2)}] + \lambda_n \int_x^b z_n^{(2)} f dt \right]}{\|Z_n^{(2)}\|^2} \\
& + \sum_{\lambda_n \in (\Lambda^* \cap \Lambda_0) \setminus \bar{\Lambda}_1^-} \left\{ \begin{aligned}
& \left[ \dot{\Phi}_n \tilde{y} + \Phi_n \tilde{y}_\lambda - \frac{\tilde{v}}{\|Z_n^{(2)}\|^2} \right] (x; \lambda_n) \left[ -[f z_n^{(2)'} - f' z_n^{(2)}] + \lambda_n \int_x^b z_n^{(2)} f dt \right] \\
& + \Phi_n \tilde{y}(x; \lambda_n) \left[ -[f v'_\lambda - f' v_\lambda] (x; \lambda_n) + \int_x^b [z_n^{(2)} + \lambda_n v_\lambda(t; \lambda_n)] f dt \right]
\end{aligned} \right\} \\
& + \sum_{\lambda_n \in (\Lambda \setminus \Lambda^*) \cap \bar{\Lambda}_1^-} \frac{[\tilde{v} - k_n \tilde{u}] (x; \lambda_n) \left[ -[f v'_n - f' v_n] + \lambda_n \int_x^b v_n f dt \right]}{\dot{\psi}(\lambda_n)} \\
& - \sum_{\lambda_n \in (\Lambda \cap \Lambda_0) \setminus (\Lambda^* \cup \bar{\Lambda}_1^-)} \left\{ \begin{aligned}
& \left[ \dot{\Phi}_n \tilde{y} + \Phi_n \tilde{y}_\lambda - \frac{\tilde{v}}{k_n \dot{\psi}(\lambda_n)} \right] (x; \lambda_n) \left[ -[f v'_n - f' v_n] + \lambda_n \int_x^b v_n f dt \right] \\
& + \Phi_n \tilde{y}(x; \lambda_n) \left[ -[f v'_\lambda - f' v_\lambda] (x; \lambda_n) + \int_x^b [v_n + \lambda_n v_\lambda(t; \lambda_n)] f dt \right]
\end{aligned} \right\} \\
& - \sum_{\lambda_n \in \Lambda_1^+ \cap \Lambda_0} \left[ \frac{\tilde{y}(x; \lambda_n)}{\dot{\phi}(\lambda_n)} - \frac{\tilde{v}(x; \lambda_n)}{\|Z_n\|^2} \right] \left[ -[f z'_n - f' z_n] + \lambda_n \int_x^b z_n f dt \right] \\
& - \sum_{\lambda_n \in \Lambda_1^- \cap \Lambda_0} \frac{\tilde{y}(x; \lambda_n) \left[ -[f v' - f' v](x; \lambda_n) + \lambda_n \int_x^b v(t; \lambda_n) f dt \right]}{\dot{\phi}(\lambda_n)}
\end{aligned} \right\}
\end{aligned}$$

where we have used the fact that

$$\int_x^b v(t; \lambda_n) [-f'' + qf](t) dt = -[f(x)v'(x; \lambda_n) - f'(x)v(x; \lambda_n)] + \lambda_n \int_x^b v(t; \lambda_n) f dt$$

and

$$\begin{aligned} & \int_x^b v_\lambda(t; \lambda_n)[-f'' + qf](t)dt \\ &= -[f(x)v'_\lambda(x; \lambda_n) - f'(x)v_\lambda(x; \lambda_n)] + \int_x^b [v(t; \lambda_n) + \lambda_n v_\lambda(t; \lambda_n)] f dt \end{aligned}$$

since  $v(b; \lambda) = \sin \beta$  and  $v'(b; \lambda) = \cos \beta$  for all  $\lambda$ , and  $f(b) \cos \beta - f'(b) \sin \beta = 0$  as  $f$  must obey (4.3). Moreover, as  $u_n(x)$  as an eigenfunction must also obey the boundary condition at  $x = b$ , giving

$$\int_x^b u_n(t)[-f'' + qf](t)dt = [u_n f' - u'_n f](x) + \lambda_n \int_x^b v(t; \lambda_n) f dt.$$

Again, operating with  $-\frac{d^2}{dx^2} + \tilde{q}(x)$  on  $[HF]_0$  we obtain

$$\begin{aligned} & \left[ \tilde{L}HF \right]_0 \\ &= -f'' + \tilde{q}f \\ &+ \left\{ \begin{aligned} & \sum_{\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)} \frac{[\tilde{K}_n - K_n] \left[ [u_n f]' + \lambda_n \int_x^b u_n f dt \right] \tilde{u}_n + 2\tilde{u}_n u_n f}{\dot{\psi}(\lambda_n)} \\ & + \sum_{\lambda_n \in \Lambda_1^+ \setminus \Lambda_0} \frac{[1 - T_n^+] \left[ [z_n f]' + \lambda_n \int_x^b z_n f dt \right] \tilde{z}_n + 2\tilde{z}'_n z_n f}{\|Z_n\|^2} \\ & + \sum_{\lambda_n \in \Lambda^* \setminus \Lambda_0} \frac{\left[ 1 - \frac{\tilde{u}(0^+; \lambda_n) v'(0^+; \lambda_n)}{u(0^+; \lambda_n) \tilde{v}'(0^+; \lambda_n)} \right] \left[ [z_n^{(2)} f]' + \lambda_n \int_x^b z_n^{(2)} f dt \right] \tilde{z}_n^{(2)} + 2\tilde{z}_n^{(2)'} z_n^{(2)} f}{\|Z_n^{(2)}\|^2} \end{aligned} \right\} \\ &- \left\{ \begin{aligned} & \sum_{\lambda_n \in (\Lambda^* \cap \Lambda_0) \cap \tilde{\Lambda}_1^-} \frac{\frac{\tilde{u}(0^+; \lambda_n) v'(0^+; \lambda_n)}{u(0^+; \lambda_n)} \left[ [z_n^{(2)} f]' + \lambda_n \int_x^b z_n^{(2)} f dt \right] \tilde{w}_2(x; \lambda_n) + 2z_n^{(2)} f \tilde{w}'_2(x; \lambda_n)}{\|Z_n^{(2)}\|^2} \\ & - \frac{\tilde{v}(x; \lambda_n) \left[ [z_n^{(2)} f]' + \lambda_n \int_x^b z_n^{(2)} f dt \right] + 2\tilde{v}'(x; \lambda_n) z_n^{(2)} f}{\|Z_n^{(2)}\|^2} \end{aligned} \right\} \end{aligned}$$

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$$\begin{aligned}
& - \sum_{\lambda_n \in (\Lambda^* \cap \Lambda_0) \setminus \tilde{\Lambda}_1^-} \left\{ \begin{aligned} & \lambda_n \int_x^b z_n^{(2)} f dt \left[ \dot{\Phi}_n + \frac{\Phi_n}{\lambda_n} \right] \tilde{y} + \Phi_n \tilde{y}_\lambda - \frac{\tilde{v}}{\|Z_n^{(2)}\|^2} (x; \lambda_n) \\ & + [z_n^{(2)} f]' \left[ \dot{\Phi}_n \tilde{y} + \Phi_n \tilde{y}_\lambda - \frac{\tilde{v}}{\|Z_n^{(2)}\|^2} (x; \lambda_n) \right] \\ & + 2z_n^{(2)} f \left[ \dot{\Phi}_n \tilde{y}' + \Phi_n \tilde{y}'_\lambda - \frac{\tilde{v}'}{\|Z_n^{(2)}\|^2} (x; \lambda_n) \right] \\ & + \Phi_n \tilde{y}(x; \lambda_n) \left[ v_\lambda(x; \lambda_n) f' + \lambda_n \int_x^b v_\lambda(t; \lambda_n) f dt \right] \\ & + 2\Phi_n \tilde{y}'(x; \lambda_n) v_\lambda(x; \lambda_n) f \end{aligned} \right\} \\
& + \sum_{\lambda_n \in (\Lambda \setminus \Lambda^*) \cap \tilde{\Lambda}_1^-} \left\{ \frac{[[u_n f]' + \lambda_n \int_x^b u_n f dt] [\tilde{v} - k_n \tilde{u}] + 2u_n f [\tilde{v} - k_n \tilde{u}']}{\dot{\psi}(\lambda_n)} (x; \lambda_n) \right\} \\
& + \left\{ \begin{aligned} & - \sum_{\lambda_n \in (\Lambda \cap \Lambda_0) \setminus (\Lambda^* \cup \tilde{\Lambda}_1^-)} \left\{ \begin{aligned} & \lambda_n \int_x^b v_n f dt \left[ \dot{\Phi}_n + \frac{\Phi_n}{\lambda_n} \right] \tilde{y} + \Phi_n \tilde{y}_\lambda - \frac{\tilde{v}}{k_n \dot{\psi}(\lambda_n)} (x; \lambda_n) \\ & + [v_n f]' \left[ \dot{\Phi}_n \tilde{y} + \Phi_n \tilde{y}_\lambda - \frac{\tilde{v}}{k_n \dot{\psi}(\lambda_n)} (x; \lambda_n) \right] \\ & + 2v_n f \left[ \dot{\Phi}_n \tilde{y}' + \Phi_n \tilde{y}'_\lambda - \frac{\tilde{v}'}{k_n \dot{\psi}(\lambda_n)} (x; \lambda_n) \right] \\ & + \Phi(\lambda_n) \tilde{y}(x; \lambda_n) \left[ v_\lambda(x; \lambda_n) f' + \lambda_n \int_x^b v_\lambda(t; \lambda_n) f dt \right] \\ & + 2\Phi_n \tilde{y}'(x; \lambda_n) v_\lambda(x; \lambda_n) f \end{aligned} \right\} \\ & - \sum_{\lambda_n \in \Lambda_1^+ \cap \Lambda_0} \left\{ \begin{aligned} & \frac{\tilde{y}(x; \lambda_n) [z_n f]' + \lambda_n \int_x^b z_n f dt}{\dot{\phi}(\lambda_n)} + 2\tilde{y}'(x; \lambda_n) z_n f \\ & - \frac{\tilde{v}(x; \lambda_n) [z_n f]' + \lambda_n \int_x^b z_n f dt}{\|Z_n\|^2} + 2\tilde{v}'(x; \lambda_n) z_n f \end{aligned} \right\} \\ & - \sum_{\lambda_n \in \Lambda_1^- \cap \Lambda_0} \frac{\tilde{y}(x; \lambda_n) [v(x; \lambda_n) f]' + \lambda_n \int_x^b v(t; \lambda_n) f dt}{\dot{\phi}(\lambda_n)} + 2\tilde{y}'(x; \lambda_n) v(x; \lambda_n) f \end{aligned} \right\}.
\end{aligned}$$

Applying part (ii) of Proposition 4.4.1 to the elements of  $\mathcal{D}(L) \cap \mathcal{H}_0$  we obtain  $HL = \tilde{L}H$ . Comparing  $[HLF]_0$  with  $[\tilde{L}HF]_0$  above, and observing that  $F$  can be chosen so that  $[F]_0 = f$  is non-zero a.e. in  $[-a, 0)$ , the results for  $-a \leq x < 0$  in *I* and *II* follow after applying the conclusions of Lemma 4.3.6 and Lemma 4.3.7, respectively. In particular, we get simplified expressions in case *I* for  $0 < x \leq b$ . Assuming that the eigenvalues of  $(L; \alpha, \zeta; r, s)$  and  $(\tilde{L}; \alpha, \zeta; r, s)$  coincide fully, this is achieved by applying the following results of Lemma 4.3.6: if  $\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)$  and  $r(\lambda_n) = 0$  or  $s(\lambda_n) = 0$  then  $k_n^{(2)} = \tilde{k}_n^{(2)}$ ; if  $\lambda_n \in \Lambda^* \setminus \Lambda_0$  then  $\frac{\tilde{u}(0^+; \lambda_n) v'(0^+; \lambda_n)}{u(0^+; \lambda_n) \tilde{v}'(0^+; \lambda_n)} = 1$ ; if  $\lambda_n \in \Lambda_1^+ \setminus \Lambda_0$  then  $T_n^+ = 1$ . Similarly, a simplified expression is obtained in case *II* for



$-a \leq x < 0$ . Assuming that the eigenvalues of  $(L; \varepsilon, \beta; r, s)$  and  $(\tilde{L}; \varepsilon, \beta; r, s)$  coincide fully, we can apply the following results of Lemma 4.3.7: if  $\lambda_n \in \Lambda \setminus (\Lambda^* \cup \Lambda_0)$  and  $r(\lambda_n) = 0$  or  $s(\lambda_n) = 0$  then  $k_n^{(1)} = \tilde{k}_n^{(1)}$ ; if  $\lambda_n \in \Lambda^* \setminus \Lambda_0$  then  $\frac{\tilde{v}'(0^-; \lambda_n)u(0^-; \lambda_n)}{v'(0^-; \lambda_n)\tilde{u}(0^-; \lambda_n)} = 1$ ; if  $\lambda_n \in \Lambda_1^- \setminus \Lambda_0$  then  $T_n^- = 1$ .  $\square$

**Remark** Note that,  $\tilde{w}_1(x; \lambda_n)$ ,  $\tilde{w}_2(x; \lambda_n)$ ,  $\tilde{u}(x; \lambda_n)$ ,  $\tilde{v}(x; \lambda_n)$  together with  $\tilde{y}(x; \lambda_n)$  as in Theorem 4.5.1 above are merely solutions of  $\tilde{\ell}\tilde{y} = \lambda_n\tilde{y}$ , not eigenfunctions.

**Corollary 4.5.2.** *Suppose that the eigenvalues of  $(L; \alpha, \zeta; r, s)$  and  $(\tilde{L}; \alpha, \zeta; r, s)$  coincide up to multiplicity. If  $\lambda_n = \tilde{\lambda}_n$  for all  $n \in \mathbb{N}_0$  then, almost everywhere*

1. on  $(0, b]$ ,  $q = \tilde{q}$  and
2. on  $[-a, 0)$ ,

$$q(x) = \tilde{q}(x) - \left\{ \begin{aligned} & \sum_{\lambda_n \in \Lambda \setminus \Lambda^*} \frac{2[K_n - \tilde{K}_n][u_n \tilde{u}_n]'}{\dot{\psi}(\lambda_n)} \\ & + \sum_{\lambda_n \in \Lambda^*} \frac{2 \left[ 1 - \frac{\tilde{v}'(0^-; \lambda_n)u(0^-; \lambda_n)}{v'(0^-; \lambda_n)\tilde{u}(0^-; \lambda_n)} \right] [z_n^{(1)} \tilde{z}_n^{(1)}]'(x)}{\|Z_n^{(1)}\|^2} \\ & + \sum_{\lambda_n \in \Lambda_1^-} \frac{2[1 - T_n^-][z_n \tilde{z}_n]'(x)}{\|Z_n\|^2} \end{aligned} \right\}$$

in the notation of Definition 4.3.1 and Lemma 4.3.2, where for  $\lambda_n \in \Lambda \setminus \Lambda^*$ , if  $r(\lambda_n) = 0$  or  $s(\lambda_n) = 0$  then  $K_n - \tilde{K}_n = k_n^{(1)} - \tilde{k}_n^{(1)}$ , else  $K_n - \tilde{K}_n = 0$ .

**Corollary 4.5.3.** *Suppose that the eigenvalues of  $(L; \varepsilon, \beta; r, s)$  and  $(\tilde{L}; \varepsilon, \beta; r, s)$  coincide up to multiplicity. If  $\lambda_n = \tilde{\lambda}_n$  for all  $n \in \mathbb{N}_0$  then, almost everywhere*

1. on  $[-a, 0)$ ,  $q = \tilde{q}$  and
2. on  $[(0, b]$ ,

$$q(x) = \tilde{q}(x) + \left\{ \begin{aligned} & \sum_{\lambda_n \in \Lambda \setminus \Lambda^*} \frac{2[\tilde{K}_n - K_n][\tilde{u}_n u_n]'}{\dot{\psi}(\lambda_n)} \\ & + \sum_{\lambda_n \in \Lambda^*} \frac{2 \left[ 1 - \frac{\tilde{u}(0^+; \lambda_n)v'(0^+; \lambda_n)}{u(0^+; \lambda_n)\tilde{v}'(0^+; \lambda_n)} \right] [\tilde{z}_n^{(2)} z_n^{(2)}]'(x)}{\|Z_n^{(2)}\|^2} \\ & + \sum_{\lambda_n \in \Lambda_1^+} \frac{2[1 - T_n^+][\tilde{z}_n z_n]'(x)}{\|Z_n\|^2} \end{aligned} \right\}$$

in the notation of Definition 4.3.1 and Lemma 4.3.2, where for  $\lambda_n \in \Lambda \setminus \Lambda^*$ , if  $r(\lambda_n) = 0$  or  $s(\lambda_n) = 0$  then  $\tilde{K}_n - K_n = \tilde{k}_n^{(2)} - k_n^{(2)}$ , else  $\tilde{K}_n - K_n = 0$ .

## 4.6 Appendix

The following lemma summarises the asymptotic approximations required for Sections 4.3 and 4.4. For details, we refer the reader to Sections 3.7 and 3.6 of Chapter 3.

**Lemma 4.6.1.** *Let  $\eta = |\Im(\sqrt{\lambda})|$  then as  $|\lambda| \rightarrow \infty$  the following approximations are valid.*

If  $-a \leq x < 0$

$$u(x; \lambda) = \begin{cases} \frac{\sin \sqrt{\lambda}(x+a)}{\sqrt{\lambda}} + O\left(\frac{e^{\eta(x+a)}}{\lambda}\right) & \text{if } \alpha = 0 \\ \sin \alpha \cos \sqrt{\lambda}(x+a) + O\left(\frac{e^{\eta(x+a)}}{\sqrt{\lambda}}\right) & \text{if } \alpha \in (0, \pi) \end{cases}$$

$$u'(x; \lambda) = \begin{cases} \cos \sqrt{\lambda}(x+a) + O\left(\frac{e^{\eta(x+a)}}{\sqrt{\lambda}}\right) & \text{if } \alpha = 0 \\ -\lambda \sin \alpha \frac{\sin \sqrt{\lambda}(x+a)}{\sqrt{\lambda}} + O\left(e^{\eta(x+a)}\right) & \text{if } \alpha \in (0, \pi) \end{cases}$$

$$v(x; \lambda) = \begin{cases} \frac{-\lambda^2}{\sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2} \frac{\sin \sqrt{\lambda} b}{\sqrt{\lambda}} \cos \sqrt{\lambda} x + O\left(\lambda e^{\eta(b-x)}\right) & \text{if } \beta = \pi \\ \frac{-\lambda^2 \sin \beta}{\sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2} \cos \sqrt{\lambda} b \cos \sqrt{\lambda} x + O\left(\lambda^{3/2} e^{\eta(b-x)}\right) & \text{if } \beta \in (0, \pi) \end{cases}$$

$$v'(x; \lambda) = \begin{cases} \frac{\lambda^3}{\sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2} \frac{\sin \sqrt{\lambda} b}{\sqrt{\lambda}} \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} + O\left(\lambda^{3/2} e^{\eta(b-x)}\right) & \text{if } \beta = \pi \\ \frac{\lambda^3 \sin \beta}{\sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2} \cos \sqrt{\lambda} b \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} + O\left(\lambda^2 e^{\eta(b-x)}\right) & \text{if } \beta \in (0, \pi) \end{cases}.$$

If  $0 < x \leq b$

$$u(x; \lambda) = \begin{cases} \frac{-\lambda^2}{\sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2} \cos \sqrt{\lambda} a \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} + O\left(\lambda e^{\eta(x+a)}\right) & \text{if } \alpha = 0 \\ \frac{\lambda^3 \sin \alpha}{\sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2} \frac{\sin \sqrt{\lambda} a}{\sqrt{\lambda}} \frac{\sin \sqrt{\lambda} x}{\sqrt{\lambda}} + O\left(\lambda^{3/2} e^{\eta(x+a)}\right) & \text{if } \alpha \in (0, \pi) \end{cases}$$

$$u'(x; \lambda) = \begin{cases} \frac{-\lambda^2}{\sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2} \cos \sqrt{\lambda} a \cos \sqrt{\lambda} x + O\left(\lambda^{3/2} e^{\eta(x+a)}\right) & \text{if } \alpha = 0 \\ \frac{\lambda^3 \sin \alpha}{\sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2} \frac{\sin \sqrt{\lambda} a}{\sqrt{\lambda}} \cos \sqrt{\lambda} x + O\left(\lambda^2 e^{\eta(x+a)}\right) & \text{if } \alpha \in (0, \pi) \end{cases}$$

$$v(x; \lambda) = \begin{cases} \frac{\sin \sqrt{\lambda}(b-x)}{\sqrt{\lambda}} + O\left(\frac{e^{\eta(b-x)}}{\lambda}\right) & \text{if } \beta = \pi \\ \sin \beta \cos \sqrt{\lambda}(b-x) + O\left(\frac{e^{\eta(b-x)}}{\sqrt{\lambda}}\right) & \text{if } \beta \in (0, \pi) \end{cases}$$

$$v'(x; \lambda) = \begin{cases} -\cos \sqrt{\lambda}(b-x) + O\left(\frac{e^{\eta(b-x)}}{\sqrt{\lambda}}\right) & \text{if } \beta = \pi \\ \lambda \sin \beta \frac{\sin \sqrt{\lambda}(b-x)}{\sqrt{\lambda}} + O\left(\frac{e^{\eta(b-x)}}{\lambda}\right) & \text{if } \beta \in (0, \pi) \end{cases}.$$

$$\psi(\lambda) = \begin{cases} \frac{\lambda^2}{\sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2} \cos \sqrt{\lambda} a \frac{\sin \sqrt{\lambda} b}{\sqrt{\lambda}} + O(\lambda e^{\eta(a+b)}) & \text{if } \alpha = 0, \beta = \pi, \\ \frac{-\sin \alpha \lambda^3}{\sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2} \frac{\sin \sqrt{\lambda} a}{\sqrt{\lambda}} \frac{\sin \sqrt{\lambda} b}{\sqrt{\lambda}} + O(\lambda^{3/2} e^{\eta(a+b)}) & \text{if } \alpha \in (0, \pi), \beta = \pi, \\ \frac{\sin \beta \lambda^2}{\sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2} \cos \sqrt{\lambda} a \cos \sqrt{\lambda} b + O(\lambda^{3/2} e^{\eta(a+b)}) & \text{if } \alpha = 0, \beta \in (0, \pi), \\ \frac{-\sin \alpha \sin \beta \lambda^3}{\sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2} \frac{\sin \sqrt{\lambda} a}{\sqrt{\lambda}} \cos \sqrt{\lambda} b + O(\lambda^2 e^{\eta(a+b)}) & \text{if } \alpha, \beta \in (0, \pi). \end{cases}$$

Furthermore, the eigenvalues  $\lambda_n$  of  $(L; \alpha, \beta; r, s)$  satisfy  $\{\sqrt{\lambda_n} : n \geq N_0\} = \bigcup_{k=N_0}^{\infty} \Sigma^k$  for some  $N_0 \in \mathbb{N}$  sufficiently large, where

$$\Sigma^k = \left\{ \sqrt{\lambda_n} : 0 \leq n - (k(c+d) + \kappa) \leq c+d-1 \right\}$$

and each  $\Sigma^k$  is the disjoint union  $\sigma_1^k \dot{\cup} \sigma_2^k$  with

$$\begin{aligned} \sigma_1^k &= \{s_n^1 : 0 \leq n - (kc + \kappa_1) \leq c-1\} \\ \sigma_2^k &= \{s_n^2 : 0 \leq n - (kd + \kappa_2) \leq d-1\} \end{aligned}$$

for some constants  $\kappa, \kappa_1, \kappa_2 \in \mathbb{N}$  and, moreover,

$$\begin{aligned} s_n^1 &= \begin{cases} \frac{(n+1/2)\pi}{a} + \frac{q_1(0)}{n\pi} + \frac{1}{2n\pi} \int_{-a}^0 \cos \frac{(2n+1)\pi t}{a} q(t) dt + O\left(\frac{1}{n^2}\right) & \text{if } \alpha = 0 \\ \frac{n\pi}{a} + \frac{q_1(0) + \cot \alpha}{n\pi} + \frac{1}{2n\pi} \int_{-a}^0 \cos \frac{2n\pi t}{a} q(t) dt + O\left(\frac{1}{n^2}\right) & \text{if } \alpha \in (0, \pi) \end{cases} \\ s_n^2 &= \begin{cases} \frac{n\pi}{b} + \frac{q_2(0) + \sum_{i=1}^N \beta_i^2}{n\pi} + \frac{1}{2n\pi} \int_0^b \cos \frac{2n\pi t}{b} q(t) dt + O\left(\frac{1}{n^2}\right) & \text{if } \beta = \pi \\ \frac{(n+1/2)\pi}{b} + \frac{q_2(0) + \sum_{i=1}^N \beta_i^2 - \cot \beta}{n\pi} + \frac{1}{2n\pi} \int_0^b \cos \frac{(2n+1)\pi t}{b} q(t) dt + O\left(\frac{1}{n^2}\right) & \text{if } \beta \in (0, \pi) \end{cases} \end{aligned}$$

where

$$q_1(x) = \frac{1}{2} \int_{-a}^x q(t) dt, \quad q_2(x) = \frac{1}{2} \int_x^b q(t) dt.$$

Then

$$\int_{-a}^b u^2(\tau; [s_n^1]^2) d\tau = \begin{cases} \frac{b(n\pi)^6 \sin^2 \left[ \frac{aq_1(0)}{n\pi} + \frac{a}{2n\pi} \int_{-a}^0 \cos \frac{(2n+1)\pi t}{a} q(t) dt \right]}{2a^6 \left[ \sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2 \right]^2} + O(n^3) & \text{if } \alpha = 0 \\ \frac{b(n\pi)^8 \sin^2 \alpha \sin^2 \left[ \frac{aq_1(0) + \cot \alpha}{n\pi} + \frac{a}{2n\pi} \int_{-a}^0 \cos \frac{2n\pi t}{a} q(t) dt \right]}{2a^8 \left[ \sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2 \right]^2} + O(n^5) & \text{if } \alpha \in (0, \pi) \end{cases}$$

$$\frac{\beta_i \Delta u'([s_n^1]^2)}{[s_n^1]^2 - \gamma_i} = \begin{cases} O(n) & \text{if } \alpha = 0 \\ O(n^2) & \text{if } \alpha \in (0, \pi) \end{cases}$$

$$\frac{\alpha_j \Delta u([s_n^1]^2)}{[s_n^1]^2 - \delta_j} = \begin{cases} O\left(\frac{1}{n}\right) & \text{if } \alpha = 0 \\ O(n) & \text{if } \alpha \in (0, \pi) \end{cases}$$

and

$$\int_{-a}^b v^2(\tau; [s_n^2]^2) d\tau = \begin{cases} \frac{a \left(\frac{n\pi}{b}\right)^6 \sin^2 \left[ \frac{q_2(0) + \sum_{i=1}^N \beta_i^2 + \frac{1}{2} \int_0^b \cos \frac{2n\pi t}{b} q(t) dt}{\frac{n\pi}{b}} \right]}{2 \left[ \sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2 \right]^2} + O(n^3) & \text{if } \beta = \pi \\ \frac{a \left(\frac{n\pi}{b}\right)^4 \sin^2 \beta \sin^2 \left[ \frac{q_2(0) + \sum_{i=1}^N \beta_i^2 - \cot \beta + \frac{1}{2} \int_0^b \cos \frac{(2n+1)\pi t}{b} q(t) dt}{\frac{n\pi}{b}} \right]}{2 \left[ \sum_{j=1}^M \alpha_j^2 \sum_{i=1}^N \beta_i^2 \right]^2} + O(n) & \text{if } \beta \in (0, \pi) \end{cases}$$

$$\frac{\beta_i \Delta v'([s_n^2]^2)}{[s_n^2]^2 - \gamma_i} = \begin{cases} O(1) & \text{if } \beta = \pi \\ O(n) & \text{if } \beta \in (0, \pi) \end{cases}$$

$$\frac{\alpha_j \Delta v([s_n^2]^2)}{[s_n^2]^2 - \delta_j} = \begin{cases} O(1) & \text{if } \beta = \pi \\ O(n) & \text{if } \beta \in (0, \pi) \end{cases}$$

**Proposition 4.6.2.** *With reference to items 1-5 below, we can find  $\epsilon > 0$  such that the functions  $y_1(x; \lambda), \dots, y_5(x; \lambda)$ , and  $\phi_1(\lambda), \dots, \phi_5(\lambda)$  are continuous with respect to  $\lambda$  on the sets  $J_1, \dots, J_5$ , respectively. Define,  $\tilde{y}(x; \lambda)$  and  $\phi(\lambda)$  such that  $\tilde{y}(x; \lambda) = \tilde{y}_j(x; \lambda)$  and  $\phi(\lambda) = \phi_j(\lambda)$  for  $\lambda \in \overline{J_j}$ ,  $j = 1, \dots, 5$ .*

1. For each  $n \in \mathbb{N}_0$  with  $r(\lambda_n) = s(\lambda_n) = 0$ , let  $J_1^n = (\lambda_n - \epsilon, \lambda_n + \epsilon) \setminus \{\lambda_n\}$ . For  $\lambda \in J_1 := \cup J_1^n$  define

$$\tilde{y}_1(x; \lambda) = \begin{cases} r(\lambda)s(\lambda)\tilde{v}(x; \lambda), & \text{if } -a \leq x < 0, \\ r(\lambda)s(\lambda)\tilde{u}(x; \lambda), & \text{if } 0 < x \leq b, \end{cases}$$

$$\phi_1(\lambda) = r(\lambda)s(\lambda)\psi(\lambda) = r(\lambda)s(\lambda)W[u, v],$$

with

$$\tilde{y}_1(x; \lambda_n) = \lim_{\lambda \rightarrow \lambda_n} \tilde{y}_1(x; \lambda),$$

$$\phi_1(\lambda_n) = \lim_{\lambda \rightarrow \lambda_n} \phi_1(\lambda).$$

2. For each  $n \in \mathbb{N}_0$  with  $r(\lambda_n) = 0$  and  $\frac{1}{s(\lambda_n)} = 0$ , let  $J_2^n = (\lambda_n - \epsilon, \lambda_n + \epsilon) \setminus \{\lambda_n\}$ . For  $\lambda \in J_2 := \cup J_2^n$  define

$$\tilde{y}_2(x; \lambda) = \begin{cases} r(\lambda)\tilde{v}(x; \lambda), & \text{if } -a \leq x < 0, \\ r(\lambda)\tilde{u}(x; \lambda), & \text{if } 0 < x \leq b, \end{cases}$$

$$\phi_2(\lambda) = r(\lambda)\psi(\lambda) = r(\lambda)W[u, v],$$

with

$$\tilde{y}_2(x; \lambda_n) = \lim_{\lambda \rightarrow \lambda_n} \tilde{y}_2(x; \lambda),$$

$$\phi_2(\lambda_n) = \lim_{\lambda \rightarrow \lambda_n} \phi_2(\lambda).$$

3. For each  $n \in \mathbb{N}_0$  with  $r(\lambda_n) = 0$  and  $s(\lambda_n) \in \mathbb{R} \setminus \{0\}$ , let  $J_3^n = (\lambda_n - \epsilon, \lambda_n + \epsilon) \setminus \{\lambda_n\}$ . For  $\lambda \in J_3 := \cup J_3^n$  define

$$\tilde{y}_3(x; \lambda) = \begin{cases} r(\lambda)\tilde{v}(x; \lambda), & \text{if } -a \leq x < 0, \\ r(\lambda)\tilde{u}(x; \lambda), & \text{if } 0 < x \leq b, \end{cases},$$

$$\phi_3(\lambda) = r(\lambda)\psi(\lambda) = r(\lambda)W[u, v],$$

with

$$\tilde{y}_3(x; \lambda_n) = \lim_{\lambda \rightarrow \lambda_n} \tilde{y}_3(x; \lambda),$$

$$\phi_3(\lambda_n) = \lim_{\lambda \rightarrow \lambda_n} \phi_3(\lambda).$$

4. For each  $n \in \mathbb{N}_0$  with  $s(\lambda_n) = 0$  and  $\frac{1}{r(\lambda_n)} = 0$ , let  $J_4^n = (\lambda_n - \epsilon, \lambda_n + \epsilon) \setminus \{\lambda_n\}$ . For  $\lambda \in J_4 := \cup J_4^n$  define

$$\tilde{y}_4(x; \lambda) = \begin{cases} s(\lambda)\tilde{v}(x; \lambda), & \text{if } -a \leq x < 0, \\ s(\lambda)\tilde{u}(x; \lambda), & \text{if } 0 < x \leq b, \end{cases},$$

$$\phi_4(\lambda) = s(\lambda)\psi(\lambda) = s(\lambda)W[u, v],$$

with

$$\tilde{y}_4(x; \lambda_n) = \lim_{\lambda \rightarrow \lambda_n} \tilde{y}_4(x; \lambda),$$

$$\phi_4(\lambda_n) = \lim_{\lambda \rightarrow \lambda_n} \phi_4(\lambda).$$

5. For each  $n \in \mathbb{N}_0$  with  $s(\lambda_n) = 0$  and  $r(\lambda_n) \in \mathbb{R} \setminus \{0\}$ , let  $J_5^n = (\lambda_n - \epsilon, \lambda_n + \epsilon) \setminus \{\lambda_n\}$ . For  $\lambda \in J_5 := \cup J_5^n$  define

$$\tilde{y}_5(x; \lambda) = \begin{cases} s(\lambda)\tilde{v}(x; \lambda), & \text{if } -a \leq x < 0, \\ s(\lambda)\tilde{u}(x; \lambda), & \text{if } 0 < x \leq b, \end{cases},$$

$$\phi_5(\lambda) = s(\lambda)\psi(\lambda) = s(\lambda)W[u, v],$$

with

$$\tilde{y}_5(x; \lambda_n) = \lim_{\lambda \rightarrow \lambda_n} \tilde{y}_5(x; \lambda),$$

$$\phi_5(\lambda_n) = \lim_{\lambda \rightarrow \lambda_n} \phi_5(\lambda).$$

Furthermore, the following limits exist.

I. If  $\lambda_n \in \Lambda \cap \Lambda_0$ ,

$$\Phi_n := \lim_{\lambda \rightarrow \lambda_n} \frac{(\lambda - \lambda_n)^2}{\phi(\lambda)},$$

$$\dot{\Phi}_n := \lim_{\lambda \rightarrow \lambda_n} \frac{d}{d\lambda} \frac{(\lambda - \lambda_n)^2}{\phi(\lambda)},$$

$$\tilde{y}_\lambda(x; \lambda_n) = \lim_{\lambda \rightarrow \lambda_n} \frac{\tilde{y}(x; \lambda) - \tilde{y}(x; \lambda_n)}{\lambda - \lambda_n}.$$

II. If  $\lambda_n \in \Lambda_1 \cap \Lambda_0$ ,

$$\dot{\phi}(\lambda_n) = \lim_{\lambda \rightarrow \lambda_n} \frac{\phi(\lambda) - \phi(\lambda_n)}{\lambda - \lambda_n} \neq 0.$$

*Proof.* To prove the first assertion, consider case 1. Let  $\lambda_n$  be an eigenvalue of  $(L; \alpha, \beta; r, s)$  with  $r(\lambda_n) = s(\lambda_n) = 0$ . For  $\lambda$  close to  $\lambda_n$  we have

$$\begin{aligned} \tilde{u}(0^+; \lambda) &= \tilde{u}(0^-; \lambda) + \frac{1}{s(\lambda)} \tilde{u}'(0^-; \lambda) \\ \tilde{u}'(0^+; \lambda) &= \tilde{u}'(0^-; \lambda) + \frac{\tilde{u}(0^-; \lambda) + \frac{1}{s(\lambda)} \tilde{u}'(0^-; \lambda)}{r(\lambda)} \end{aligned}$$

and

$$\begin{aligned} \tilde{v}'(0^-; \lambda) &= \tilde{v}'(0^+; \lambda) - \frac{1}{r(\lambda)} \tilde{v}(0^+; \lambda) \\ \tilde{v}(0^-; \lambda) &= \tilde{v}(0^+; \lambda) - \frac{\tilde{v}'(0^+; \lambda) - \frac{1}{r(\lambda)} \tilde{v}(0^+; \lambda)}{s(\lambda)}. \end{aligned}$$

Hence, for  $-a \leq x < 0$

$$\begin{aligned} r(\lambda)s(\lambda)\tilde{v}(x; \lambda) &= r(\lambda)s(\lambda)\tilde{v}(0^-; \lambda)\tilde{w}_1(x; \lambda) + r(\lambda)s(\lambda)\tilde{v}'(0^-; \lambda)\tilde{w}_2(x; \lambda) \\ &\rightarrow \tilde{v}(0^+; \lambda_n)\tilde{w}_1(x; \lambda_n) \end{aligned}$$

as  $\lambda \rightarrow \lambda_n$ . Similarly, for  $0 < x \leq b$ ,

$$\begin{aligned} r(\lambda)s(\lambda)\tilde{u}(x; \lambda) &= r(\lambda)s(\lambda)\tilde{u}(0^+; \lambda)\tilde{w}_1(x; \lambda) + r(\lambda)s(\lambda)\tilde{u}'(0^+; \lambda)\tilde{w}_2(x; \lambda) \\ &\rightarrow \tilde{u}'(0^-; \lambda_n)\tilde{w}_2(x; \lambda_n) \end{aligned}$$

as  $\lambda \rightarrow \lambda_n$ . Moreover,

$$\begin{aligned} r(\lambda)s(\lambda)\psi(\lambda) &= r(\lambda)s(\lambda) [u(0^+; \lambda)v'(0^+; \lambda) - u'(0^+; \lambda)v(0^+; \lambda)] \\ &= r(\lambda) [s(\lambda)u(0^-; \lambda) + u'(0^-; \lambda)] v'(0^+; \lambda) \\ &\quad - [r(\lambda)s(\lambda)u'(0^-; \lambda) + s(\lambda)u(0^-; \lambda) + u'(0^-; \lambda)] v(0^+; \lambda) \\ &\rightarrow -u'(0^-; \lambda_n)v(0^+; \lambda_n) \end{aligned} \tag{4.74}$$

as  $\lambda \rightarrow \lambda_n$ . Here we note that  $u'(0^-; \lambda)$ ,  $\tilde{u}'(0^-; \lambda)$ ,  $v(0^+; \lambda)$ ,  $\tilde{v}(0^+; \lambda)$  are entire functions, hence the above limits exist. Choosing  $\epsilon_1 > 0$  small enough so that  $r(\lambda)$  and  $s(\lambda)$  are bounded on  $[\lambda_n - \epsilon_1, \lambda_n + \epsilon_1]$  for all eigenvalues  $\lambda_n$  with  $r(\lambda_n) = s(\lambda_n)$ , we conclude that  $\tilde{y}_1(x; \lambda)$  and  $\phi_1(\lambda)$  are continuous for all  $\lambda \in (\lambda_n - \epsilon_1, \lambda_n + \epsilon_1)$  for all  $\lambda_n$  with  $r(\lambda_n) = s(\lambda_n)$ . Similar arguments can be made for the remaining cases, yielding values  $\epsilon_2, \dots, \epsilon_5 > 0$ . Choosing  $\epsilon = \min_{j=1, \dots, 5} \epsilon_j$  the result follows.

Lastly, to prove the final claim we again consider only case 1, as remaining cases are similar. Recall that the transmission conditions reduce to  $y(0^+) = y'(0^-) = 0$  at  $\lambda_n$  if  $r(\lambda_n) = s(\lambda_n) = 0$ .

I. If  $\lambda_n \in \Lambda \cap \Lambda_0$  then  $\lambda_n$  has geometric multiplicity 2. Thus  $u'(0^-; \lambda_n) = 0$  and  $v(0^+; \lambda_n) = 0$ , resulting in a double zero of  $\phi(\lambda)$  at  $\lambda = \lambda_n$  (see (4.74) above). Since each meromorphic

function admits a factorization in terms of its zeroes and poles (see [64], page 220) we deduce that  $\Phi_n$  exists.  $\dot{\Phi}_n$  is the residue of  $\phi^{-1}$  at  $\lambda_n$ . Moreover, if  $-a \leq x < 0$  then

$$\begin{aligned} & \lim_{\lambda \rightarrow \lambda_n} \frac{\tilde{y}_1(x; \lambda) - \tilde{y}_1(x; \lambda_n)}{\lambda - \lambda_n} \\ &= \lim_{\lambda \rightarrow \lambda_n} \left\{ \frac{s(\lambda) [r(\lambda)\tilde{v}'(0^+; \lambda) - \tilde{v}(0^+; \lambda)] \tilde{w}_2(x; \lambda)}{\lambda - \lambda_n} \right. \\ & \quad \left. + \frac{r(\lambda) [s(\lambda)\tilde{v}(0^+; \lambda) - \tilde{v}'(0^+; \lambda)] \tilde{w}_1(x; \lambda)}{\lambda - \lambda_n} \right. \\ & \quad \left. + \frac{\tilde{v}(0^+; \lambda)\tilde{w}_1(x; \lambda) - \tilde{v}(0^+; \lambda_n)\tilde{w}_1(x; \lambda_n)}{\lambda - \lambda_n} \right\} \\ &= [-\dot{r}(\lambda_n)\tilde{v}'(0^+; \lambda_n) + \tilde{v}_\lambda(0^+; \lambda_n)] \tilde{w}_1(x; \lambda_n) + \tilde{v}(0^+; \lambda_n)\tilde{w}_{1\lambda}(x; \lambda_n), \end{aligned}$$

whereas if  $0 < x \leq b$  then

$$\begin{aligned} & \lim_{\lambda \rightarrow \lambda_n} \frac{\tilde{y}_1(x; \lambda) - \tilde{y}_1(x; \lambda_n)}{\lambda - \lambda_n} \\ &= \lim_{\lambda \rightarrow \lambda_n} \left\{ \frac{r(\lambda) [s(\lambda)\tilde{u}(0^-; \lambda) + \tilde{u}'(0^-; \lambda)] \tilde{w}_1(x; \lambda)}{\lambda - \lambda_n} \right. \\ & \quad \left. + \frac{s(\lambda) [r(\lambda)\tilde{u}'(0^-; \lambda) + \tilde{u}(0^-; \lambda)] \tilde{w}_2(x; \lambda)}{\lambda - \lambda_n} \right. \\ & \quad \left. + \frac{\tilde{u}'(0^-; \lambda)\tilde{w}_2(x; \lambda) - \tilde{u}'(0^-; \lambda_n)\tilde{w}_2(x; \lambda_n)}{\lambda - \lambda_n} \right\} \\ &= [\dot{s}(\lambda_n)\tilde{u}(0^-; \lambda_n) + \tilde{u}'_\lambda(0^-; \lambda_n)] \tilde{w}_2(x; \lambda_n) + \tilde{u}'(0^-; \lambda_n)\tilde{w}_{2\lambda}(x; \lambda_n). \end{aligned}$$

II. If  $\lambda_n \in \Lambda_1 \cap \Lambda_0$  then, in particular,  $\lambda_n$  has geometric multiplicity 1. From equation (4.26) we deduce that  $\psi(\lambda)$  has a simple pole at  $\lambda = \lambda_n$ . Hence,  $\phi(\lambda) = r(\lambda)s(\lambda)\psi(\lambda)$  has a zero of order 1 at  $\lambda = \lambda_n$ , which implies that  $\dot{\phi}(\lambda_n) \neq 0$ . In particular,

(i) if  $\lambda_n \in \Lambda_1^-$ , then  $u'(0^-; \lambda_n) = 0$  and  $v(0^+; \lambda_n) \neq 0$ . Observe that

$$r(\lambda)\psi(\lambda) \rightarrow - \left[ u(0^-; \lambda_n) + \frac{u'_\lambda(0^-; \lambda_n)}{\dot{s}(\lambda_n)} \right] v(0^+; \lambda_n)$$

as  $\lambda \rightarrow \lambda_n$ . Thus,

$$\begin{aligned} \lim_{\lambda \rightarrow \lambda_n} \frac{\phi(\lambda) - \phi(\lambda_n)}{\lambda - \lambda_n} &= \lim_{\lambda \rightarrow \lambda_n} \frac{s(\lambda)}{\lambda - \lambda_n} [r(\lambda)\psi(\lambda)] \\ &= -\dot{s}(\lambda_n) \left[ u(0^-; \lambda_n) + \frac{u'_\lambda(0^-; \lambda_n)}{\dot{s}(\lambda_n)} \right] v(0^+; \lambda_n). \end{aligned}$$

(ii) if  $\lambda_n \in \Lambda_1^-$ , then  $u'(0^-; \lambda_n) \neq 0$  and  $v(0^+; \lambda_n) = 0$ . In this case,

$$s(\lambda)\psi(\lambda) \rightarrow u'(0^-; \lambda_n) \left[ v'(0^+; \lambda_n) - \frac{v_\lambda(0^+; \lambda_n)}{\dot{r}(\lambda_n)} \right]$$

as  $\lambda \rightarrow \lambda_n$ . Hence,

$$\begin{aligned} \lim_{\lambda \rightarrow \lambda_n} \frac{\phi(\lambda) - \phi(\lambda_n)}{\lambda - \lambda_n} &= \lim_{\lambda \rightarrow \lambda_n} \frac{r(\lambda)}{\lambda - \lambda_n} [s(\lambda)\psi(\lambda)] \\ &= \dot{r}(\lambda_n)u'(0^-; \lambda_n) \left[ v'(0^+; \lambda_n) - \frac{v_\lambda(0^+; \lambda_n)}{\dot{r}(\lambda_n)} \right]. \end{aligned}$$

□

**Theorem 4.6.3.** *Suppose that  $\lambda_n$  is an eigenvalue of  $(L; \alpha, \beta; r, s)$  coinciding with a zero of  $r(\lambda)$  or  $s(\lambda)$ . Let  $\mu \in \mathbb{C}$  with  $\mu \neq \lambda_m$  for all  $m \in \mathbb{N}_0$ .*

1. *If  $\lambda_n \in \Lambda^*$  then*

$$Z_n^{(1)}(\tau) = u(0^-; \lambda_n) \begin{bmatrix} \chi_{[-a,0)} w_1(\tau; \lambda_n) \\ (0) \\ \left(-\frac{\alpha_j}{\lambda_n - \delta_j}\right) \end{bmatrix}, \quad Z_n^{(2)}(\tau) = v'(0^+; \lambda_n) \begin{bmatrix} \chi_{(0,b]} w_2(\tau; \lambda_n) \\ \left(\frac{\beta_i}{\lambda_n - \gamma_i}\right) \\ (0) \end{bmatrix}$$

*are a pair of linearly independent eigenfunctions of  $(L; \alpha, \beta; r, s)$  corresponding to  $\lambda_n$ .*

*Moreover*

$$\text{Res} \left( \frac{u(x; \lambda)v(t; \lambda)}{(\lambda - \mu)\psi(\lambda)}, \lambda = \lambda_n \right) = \begin{cases} 0, & \text{if } -a \leq x < 0 < t \leq b, \\ \frac{1}{\lambda_n - \mu} \frac{z_n^{(1)}(x)z_n^{(1)}(t)}{\|Z_n^{(1)}\|^2}, & \text{if } -a \leq x < t < 0, \\ \frac{1}{\lambda_n - \mu} \frac{z_n^{(2)}(x)z_n^{(2)}(t)}{\|Z_n^{(2)}\|^2}, & \text{if } 0 < x < t \leq b, \end{cases}$$

$$\text{Res} \left( \frac{\tilde{u}(x; \lambda)v(t; \lambda)}{(\lambda - \mu)\psi(\lambda)}, \lambda = \lambda_n \right) = \begin{cases} 0, & \text{if } -a \leq x < 0 < t \leq b, \\ \frac{1}{\lambda_n - \mu} \frac{\tilde{u}(x; \lambda_n)z_n^{(1)}(t)}{\|Z_n^{(1)}\|^2}, & \text{if } -a \leq x < t < 0, \\ \frac{1}{\lambda_n - \mu} \frac{\tilde{u}(0^+; \lambda_n)}{u(0^+; \lambda_n)} \frac{\tilde{w}_2(x; \lambda_n)w_2(t; \lambda_n)}{\int_0^b w_2^2(\tau; \lambda_n) d\tau + \hat{r}(\lambda_n)}, & \text{if } 0 < x < t \leq b, \lambda_n \in \tilde{\Lambda}^* \cup \tilde{\Lambda}_1^-, \\ \frac{\Phi_n [\tilde{y}_\lambda(x; \lambda_n)z_n^{(2)}(t) + \tilde{y}(x; \lambda_n)v_\lambda(t; \lambda_n)]}{\lambda_n - \mu} \\ + \left[ \dot{\Phi}_n - \frac{\Phi_n}{\lambda_n - \mu} \right] \frac{\tilde{y}(x; \lambda_n)z_n^{(2)}(t)}{\lambda_n - \mu}, & \text{if } 0 < x < t \leq b, \lambda_n \notin \tilde{\Lambda}^* \cup \tilde{\Lambda}_1^-, \end{cases}$$

$$\text{Res} \left( \frac{u(t; \lambda)\tilde{v}(x; \lambda)}{(\lambda - \mu)\psi(\lambda)}, \lambda = \lambda_n \right) = \begin{cases} 0, & \text{if } -a \leq t < 0 < x \leq b, \\ \frac{1}{\lambda_n - \mu} \frac{\tilde{v}'(0^-; \lambda_n)}{v'(0^-; \lambda_n)} \frac{\tilde{w}_1(x; \lambda_n)w_1(t; \lambda_n)}{\int_{-a}^0 w_1^2(\tau; \lambda_n) d\tau - \hat{s}(\lambda_n)}, & \text{if } -a \leq t < x < 0, \tilde{v}(0^+; \lambda_n) = 0, \\ \frac{\Phi_n}{\lambda_n - \mu} [u_\lambda(t; \lambda_n)\tilde{y}(x; \lambda_n) + z_n^{(1)}(t)\tilde{y}_\lambda(x; \lambda_n)] \\ + \left[ \dot{\Phi}_n - \frac{\Phi_n}{\lambda_n - \mu} \right] \frac{z_n^{(1)}(t)\tilde{y}(x; \lambda_n)}{\lambda_n - \mu}, & \text{if } -a \leq t < x < 0, \tilde{v}(0^+; \lambda_n) \neq 0, \\ \frac{1}{\lambda_n - \mu} \frac{z_n^{(2)}(t)\tilde{v}(x; \lambda_n)}{\|Z_n^{(2)}\|^2}, & \text{if } 0 < t < x \leq b. \end{cases}$$

*If  $-a \leq x < 0$  then*

$$\text{Res} \left( \frac{1}{\lambda - \mu} \frac{\beta_i \Delta v' u(x; \lambda)}{\lambda - \gamma_i \psi(\lambda)}, \lambda = \lambda_n \right) = 0, \quad \text{Res} \left( \frac{1}{\lambda - \mu} \frac{\beta_i \Delta v' \tilde{u}(x; \lambda)}{\lambda - \gamma_i \psi(\lambda)}, \lambda = \lambda_n \right) = 0;$$



$$\text{Res} \left( \frac{1}{\lambda - \mu} \frac{\alpha_j \Delta v}{\lambda - \gamma_j} \frac{u(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) = \frac{1}{\lambda_n - \mu} \frac{z_n^{(1)2} z_n^{(1)}(x)}{\|Z_n^{(1)}\|^2},$$

$$\text{Res} \left( \frac{1}{\lambda - \mu} \frac{\alpha_j \Delta v}{\lambda - \gamma_j} \frac{\tilde{u}(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) = \frac{1}{\lambda_n - \mu} \frac{z_n^{(1)2} \tilde{u}(x; \lambda_n)}{\|Z_n^{(1)}\|^2}.$$

If  $0 < x \leq b$  then

$$\text{Res} \left( \frac{1}{\lambda - \mu} \frac{\beta_i \Delta u'}{\lambda - \gamma_i} \frac{v(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) = \frac{1}{\lambda_n - \mu} \frac{z_n^{(2)1} z_n^{(2)}(x)}{\|Z_n^{(2)}\|^2},$$

$$\text{Res} \left( \frac{1}{\lambda - \mu} \frac{\beta_i \Delta u'}{\lambda - \gamma_i} \frac{\tilde{v}(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) = \frac{1}{\lambda_n - \mu} \frac{z_n^{(2)1} \tilde{v}(x; \lambda_n)}{\|Z_n^{(2)}\|^2};$$

$$\text{Res} \left( \frac{1}{\lambda - \mu} \frac{\alpha_j \Delta u}{\lambda - \delta_j} \frac{v(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) = 0, \quad \text{Res} \left( \frac{1}{\lambda - \mu} \frac{\alpha_j \Delta u}{\lambda - \delta_j} \frac{\tilde{v}(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) = 0.$$

2. If  $\lambda_n \in \Lambda \setminus \Lambda^*$  then both

$$U_n(\tau) = \begin{bmatrix} u_n(\tau) \\ (u_{i,n}^1) \\ (u_{j,n}^2) \end{bmatrix} \quad \text{and} \quad V_n(\tau) = \begin{bmatrix} v_n(\tau) \\ (v_{i,n}^1) \\ (v_{j,n}^2) \end{bmatrix}$$

are eigenfunctions of  $(L; \alpha, \beta; r, s)$  corresponding to  $\lambda_n$ .

Moreover,

$$\text{Res} \left( \frac{1}{\lambda - \mu} \frac{u(x; \lambda)v(t; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) = \frac{1}{\lambda_n - \mu} \frac{u_n(x)v_n(t)}{\psi(\lambda_n)}$$

for all  $x, t \in [-a, 0) \cup (0, b]$ ,  $x < t$ ;

$$\text{Res} \left( \frac{1}{\lambda - \mu} \frac{\tilde{u}(x; \lambda)v(t; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) = \begin{cases} \frac{1}{\lambda_n - \mu} \frac{\tilde{u}(x; \lambda_n)v_n(t)}{\psi(\lambda_n)}, & \text{if } -a \leq x < 0 < t \leq b \\ \frac{1}{\lambda_n - \mu} \frac{\tilde{u}(x; \lambda_n)v_n(t)}{\psi(\lambda_n)}, & \text{or } -a \leq x < t < 0, \\ \frac{\Phi_n [\tilde{y}_\lambda(x; \lambda_n)v_n(t) + \tilde{y}(x; \lambda_n)v_\lambda(t; \lambda_n)]}{\lambda_n - \mu} & \text{if } 0 < x < t \leq b, \lambda_n \in (\tilde{\Lambda} \cup \tilde{\Lambda}_1^-), \\ + \left[ \dot{\Phi}_n - \frac{\Phi_n}{\lambda_n - \mu} \right] \frac{\tilde{y}(x; \lambda_n)v_n(t)}{\lambda_n - \mu}, & \text{if } 0 < x < t \leq b, \lambda_n \notin (\tilde{\Lambda} \cup \tilde{\Lambda}_1^-); \end{cases}$$

$$\text{Res} \left( \frac{1}{\lambda - \mu} \frac{u(t; \lambda)\tilde{v}(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) = \begin{cases} \frac{1}{\lambda_n - \mu} \frac{u_n(t)\tilde{v}(x; \lambda_n)}{\psi(\lambda_n)}, & \text{if } -a \leq t < 0 < x \leq b \\ \frac{1}{\lambda_n - \mu} \frac{u_n(t)\tilde{v}(x; \lambda_n)}{\psi(\lambda_n)}, & \text{or } 0 < t < x \leq b, \\ \frac{\Phi_n [u_\lambda(t; \lambda_n)\tilde{y}(x; \lambda_n) + u_n(t)\tilde{y}_\lambda(x; \lambda_n)]}{\lambda_n - \mu} & \text{if } -a \leq t < x < 0, \lambda_n \in (\tilde{\Lambda} \cup \tilde{\Lambda}_1^+), \\ + \left[ \dot{\Phi}_n - \frac{\Phi_n}{\lambda_n - \mu} \right] \frac{u_n(t)\tilde{y}(x; \lambda_n)}{\lambda_n - \mu}, & \text{if } -a \leq t < x < 0, \lambda_n \notin (\tilde{\Lambda} \cup \tilde{\Lambda}_1^+). \end{cases}$$

If  $-a \leq x < 0$  then

$$\begin{aligned} \operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{\beta_i \Delta v'}{\lambda - \gamma_i} \frac{u(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) &= \frac{1}{\lambda_n - \mu} \frac{v_{i,n}^1 u_n(x)}{\dot{\psi}(\lambda_n)}, \\ \operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{\beta_i \Delta v'}{\lambda - \gamma_i} \frac{\tilde{u}(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) &= \frac{1}{\lambda_n - \mu} \frac{v_{i,n}^1 \tilde{u}(x; \lambda_n)}{\dot{\psi}(\lambda_n)}; \\ \operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{\alpha_j \Delta v}{\lambda - \gamma_j} \frac{u(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) &= \frac{1}{\lambda_n - \mu} \frac{v_{j,n}^2 u_n(x)}{\dot{\psi}(\lambda_n)}, \\ \operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{\alpha_j \Delta v}{\lambda - \gamma_j} \frac{\tilde{u}(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) &= \frac{1}{\lambda_n - \mu} \frac{v_{j,n}^2 \tilde{u}(x; \lambda_n)}{\dot{\psi}(\lambda_n)}. \end{aligned}$$

If  $0 < x \leq b$  then

$$\begin{aligned} \operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{\beta_i \Delta u'}{\lambda - \gamma_i} \frac{v(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) &= \frac{1}{\lambda_n - \mu} \frac{u_{i,n}^1 v_n(x)}{\dot{\psi}(\lambda_n)}, \\ \operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{\beta_i \Delta u'}{\lambda - \gamma_i} \frac{\tilde{v}(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) &= \frac{1}{\lambda_n - \mu} \frac{u_{i,n}^1 \tilde{v}(x; \lambda_n)}{\dot{\psi}(\lambda_n)}; \\ \operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{\alpha_j \Delta u}{\lambda - \delta_j} \frac{v(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) &= \frac{1}{\lambda_n - \mu} \frac{u_{j,n}^2 v_n(x)}{\dot{\psi}(\lambda_n)}, \\ \operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{\alpha_j \Delta u}{\lambda - \delta_j} \frac{\tilde{v}(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) &= \frac{1}{\lambda_n - \mu} \frac{u_{j,n}^2 \tilde{v}(x; \lambda_n)}{\dot{\psi}(\lambda_n)}. \end{aligned}$$

3. If  $\lambda_n \in \Lambda_1^-$  then  $\chi_{[-a,0)} u(x; \lambda_n)$  is an eigenfunction of  $(\ell; \alpha, \beta; r, s)$  and the corresponding eigenfunction of  $(L; \alpha, \beta; r, s)$  is found by extending  $\chi_{[-a,0)} u(x; \lambda_n)$  to  $L^2(-a, b) \oplus \mathbb{C}^N \oplus \mathbb{C}^M$  using the rules of Section 4.2, we denote this eigenfunction by  $Z_n(x)$ .

Moreover,

$$\operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{u(x; \lambda) v(t; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) = \frac{1}{\lambda_n - \mu} \frac{z_n(x) z_n(t)}{\|Z_n\|^2}$$

for all  $x, t \in [-a, 0) \cup (0, b]$ ,  $x < t$ ;

$$\operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{\tilde{u}(x; \lambda) v(t; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) = \begin{cases} 0, & \text{if } -a \leq x < 0 < t \leq b, \\ \frac{1}{\lambda_n - \mu} \frac{\tilde{u}(x; \lambda_n) z_n(t)}{\|Z_n\|^2}, & \text{if } -a \leq x < t < 0, \\ \frac{1}{\lambda_n - \mu} \frac{\tilde{y}(x; \lambda_n) v(t; \lambda_n)}{\phi(\lambda_n)}, & \text{if } 0 < x < t \leq b, \end{cases}$$

$$\operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{u(t; \lambda) \tilde{v}(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) = \begin{cases} 0, & \text{if } -a \leq t < 0 < x \leq b, \\ \frac{1}{\lambda_n - \mu} \frac{u(t; \lambda_n) \tilde{y}(x; \lambda_n)}{\phi(\lambda_n)}, & \text{if } -a \leq t < x < 0, \\ 0, & \text{if } 0 < t < x \leq b. \end{cases}$$

If  $-a \leq x < 0$  then

$$\begin{aligned} \operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{\beta_i \Delta v'}{\lambda - \gamma_i} \frac{u(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) &= \frac{1}{\lambda_n - \mu} \frac{z_n^1 z_n(x)}{\|Z_n\|^2}, \\ \operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{\beta_i \Delta v'}{\lambda - \gamma_i} \frac{\tilde{u}(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) &= \frac{1}{\lambda_n - \mu} \frac{z_n^1 \tilde{u}(x; \lambda_n)}{\|Z_n\|^2}; \end{aligned}$$

$$\begin{aligned} \operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{\alpha_j \Delta v}{\lambda - \gamma_j} \frac{u(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) &= \frac{1}{\lambda_n - \mu} \frac{z_{n_j}^2 z_n(x)}{\|Z_n\|^2}, \\ \operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{\alpha_j \Delta v}{\lambda - \gamma_j} \frac{\tilde{u}(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) &= \frac{1}{\lambda_n - \mu} \frac{z_{n_j}^2 \tilde{u}(x; \lambda_n)}{\|Z_n\|^2}. \end{aligned}$$

If  $0 < x \leq b$  then

$$\begin{aligned} \operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{\beta_i \Delta u'}{\lambda - \gamma_i} \frac{v(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) &= 0, \quad \operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{\beta_i \Delta u'}{\lambda - \gamma_i} \frac{\tilde{v}(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) = 0; \\ \operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{\alpha_j \Delta u}{\lambda - \delta_j} \frac{v(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) &= 0, \quad \operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{\alpha_j \Delta u}{\lambda - \delta_j} \frac{\tilde{v}(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) = 0. \end{aligned}$$

4. If  $\lambda_n \in \Lambda_1^+$  then  $\chi_{(0,b]} v(x; \lambda_n)$  is an eigenfunction of  $(\ell; \alpha, \beta; r, s)$  and the corresponding eigenfunction of  $(L; \alpha, \beta; r, s)$  is found by extending  $\chi_{(0,b]} v(x; \lambda_n)$  to  $L^2 \oplus \mathbb{C}^N \oplus \mathbb{C}^M$  using the rules of Section 4.2, we denote this eigenfunction by  $Z_n(x)$ .

Moreover,

$$\operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{u(x; \lambda) v(t; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) = \frac{1}{\lambda_n - \mu} \frac{z_n(x) z_n(t)}{\|Z_n\|^2}$$

for all  $x, t \in [-a, 0) \cup (0, b]$ ,  $x < t$ ;

$$\operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{\tilde{u}(x; \lambda) v(t; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) = \begin{cases} 0, & \text{if } -a \leq x < 0 < t \leq b, \\ 0, & \text{if } -a \leq x < t < 0, \\ \frac{1}{\lambda_n - \mu} \frac{\tilde{y}(x; \lambda_n) v(t; \lambda_n)}{\phi(\lambda_n)}, & \text{if } 0 < x < t \leq b, \end{cases}$$

$$\operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{u(t; \lambda) \tilde{v}(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) = \begin{cases} 0, & \text{if } -a \leq t < 0 < x \leq b, \\ \frac{1}{\lambda_n - \mu} \frac{u(t; \lambda_n) \tilde{y}(x; \lambda_n)}{\phi(\lambda_n)}, & \text{if } -a \leq t < x < 0, \\ \frac{1}{\lambda_n - \mu} \frac{z_n(t) \tilde{v}(x; \lambda_n)}{\|Z_n\|^2}, & \text{if } 0 < t < x \leq b. \end{cases}$$

If  $-a \leq x < 0$  then

$$\begin{aligned} \operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{\beta_i \Delta v'}{\lambda - \gamma_i} \frac{u(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) &= 0, \quad \operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{\beta_i \Delta v'}{\lambda - \gamma_i} \frac{\tilde{u}(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) = 0; \\ \operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{\alpha_j \Delta v}{\lambda - \delta_j} \frac{u(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) &= 0, \quad \operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{\alpha_j \Delta v}{\lambda - \delta_j} \frac{\tilde{u}(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) = 0. \end{aligned}$$

If  $0 < x \leq b$  then

$$\begin{aligned} \operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{\beta_i \Delta u'}{\lambda - \gamma_i} \frac{v(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) &= \frac{1}{\lambda_n - \mu} \frac{z_{n_i}^1 z_n(x)}{\|Z_n\|^2}, \\ \operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{\beta_i \Delta u'}{\lambda - \gamma_i} \frac{\tilde{v}(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) &= \frac{1}{\lambda_n - \mu} \frac{z_{n_i}^1 \tilde{v}(x; \lambda_n)}{\|Z_n\|^2}; \\ \operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{\alpha_j \Delta u}{\lambda - \gamma_j} \frac{v(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) &= \frac{1}{\lambda_n - \mu} \frac{z_{n_j}^2 z_n(x)}{\|Z_n\|^2}, \\ \operatorname{Res} \left( \frac{1}{\lambda - \mu} \frac{\alpha_j \Delta u}{\lambda - \gamma_j} \frac{\tilde{v}(x; \lambda)}{\psi(\lambda)}, \lambda = \lambda_n \right) &= \frac{1}{\lambda_n - \mu} \frac{z_{n_j}^2 \tilde{v}(x; \lambda_n)}{\|Z_n\|^2}. \end{aligned}$$

*Proof.* The only new results contained in this theorem are the residues, and the only interesting cases occur when an eigenvalue coincides with a zero of  $r$  or  $s$  as these cases can yield double poles. We consider only one case as the calculations are similar in the remaining cases.

We begin by recalling the following fundamental results from Sturm-Liouville theory. Let  $u(x; \lambda)$  and  $v(x; \lambda)$  be solutions of (3.1) as defined at the beginning of Section 3.2. Let  $\lambda_n$  denote an eigenvalue of  $(L; \alpha, \beta; r, s)$  or correspondingly  $(\ell; \alpha, \beta; r, s)$ . Then for  $-a \leq \tau < 0$ ,

$$\frac{d}{d\tau} [u(\tau; \lambda)u'(\tau; \lambda_n) - u'(\tau; \lambda)u(\tau; \lambda_n)] = (\lambda - \lambda_n)u(\tau; \lambda)u(\tau; \lambda_n),$$

which if integrated from  $-a$  to  $0^-$  yields

$$\int_{-a}^0 u(\tau; \lambda)u(\tau; \lambda_n)d\tau = \frac{u(0^-; \lambda)u'(0^-; \lambda_n) - u'(0^-; \lambda)u(0^-; \lambda_n)}{\lambda - \lambda_n}, \quad \lambda \neq \lambda_n. \quad (4.75)$$

Similarly,

$$\int_0^b v(\tau; \lambda)v(\tau; \lambda_n)d\tau = -\frac{v(0^+; \lambda)v'(0^+; \lambda_n) - v'(0^+; \lambda)v(0^+; \lambda_n)}{\lambda - \lambda_n}, \quad \lambda \neq \lambda_n. \quad (4.76)$$

Now suppose that  $\lambda_n$  denotes an eigenvalue  $(L; \alpha, \beta; r, s)$  coinciding with a zero of  $r(\lambda)$  or  $s(\lambda)$ . Then  $\lambda_n$  has geometric multiplicity equal to either 1 or 2 corresponding to the algebraic multiplicity of  $\omega(\lambda)$  at  $\lambda = \lambda_n$  (see Theorem 3.4.1 in Chapter 3). We give details only for the case when  $r(\lambda_n) = s(\lambda_n) = 0$ , as the calculations for the remaining cases are similar. Recall that at such an eigenvalue the transmission conditions give  $y(0^+) = 0$ ,  $y'(0^-) = 0$ . So either  $u'(0^-; \lambda_n) = 0$  or  $v(0^+; \lambda_n) = 0$ .

For  $\lambda$  close to  $\lambda_n$ :

$$\begin{aligned} u(0^+; \lambda) &= u(0^-; \lambda) + \frac{1}{s(\lambda)}u'(0^-; \lambda) \\ u'(0^+; \lambda) &= u'(0^-; \lambda) + \frac{u(0^-; \lambda) + \frac{1}{s(\lambda)}u'(0^-; \lambda)}{r(\lambda)} \end{aligned}$$

and

$$\begin{aligned} v'(0^-; \lambda) &= v'(0^+; \lambda) - \frac{1}{r(\lambda)}v(0^+; \lambda) \\ v(0^-; \lambda) &= v(0^+; \lambda) - \frac{v'(0^+; \lambda) - \frac{1}{r(\lambda)}v(0^+; \lambda)}{s(\lambda)}. \end{aligned}$$

If  $u'(0^-; \lambda_n) = 0$  then, as  $\lambda \rightarrow \lambda_n$ ,

$$u(0^+; \lambda) = u(0^-; \lambda) + \frac{1}{s(\lambda)}u'(0^-; \lambda) \rightarrow u(0^-; \lambda_n) + \frac{u'_\lambda(0^-; \lambda_n)}{\dot{s}(\lambda_n)},$$

where  $\dot{s}(\lambda_n) = -\sum_{j=1}^M \frac{\alpha_j^2}{(\lambda_n - \delta_j)^2} < 0$  and  $u'_\lambda(0^-; \lambda_n) = -u(0^-; \lambda_n) \int_{-a}^0 w_1(\tau; \lambda_n)^2 d\tau$  by equation (4.75). Thus, as  $\lambda \rightarrow \lambda_n$ ,

$$u(0^+; \lambda) \rightarrow u(0^-; \lambda_n) \left[ 1 - \frac{\int_{-a}^0 w_1(\tau; \lambda_n)^2 d\tau}{\dot{s}(\lambda_n)} \right] \neq 0, \quad u'(0^+; \lambda) = u'(0^-; \lambda) + \frac{u(0^+; \lambda)}{r(\lambda)} \rightarrow \pm\infty.$$

If  $u'(0^-; \lambda_n) \neq 0$  then as  $\lambda \rightarrow \lambda_n$ ,

$$u(0^+; \lambda), u'(0^+; \lambda) \rightarrow \pm\infty.$$

If  $v(0^+; \lambda_n) = 0$ , then as  $\lambda \rightarrow \lambda_n$ ,

$$v'(0^-; \lambda) = v'(0^+; \lambda) - \frac{1}{r(\lambda)}v(0^+; \lambda) \rightarrow v'(0^+; \lambda_n) - \frac{v_\lambda(0^+; \lambda_n)}{\dot{r}(\lambda_n)}$$

where  $\dot{r}(\lambda_n) = \sum_{i=1}^N \frac{\beta_i^2}{(\lambda_n - \gamma_i)^2} > 0$  and  $v_\lambda(0^+; \lambda_n) = -v'(0^+; \lambda_n) \int_0^b w_2^2(\tau; \lambda_n) d\tau$  by equation (4.76). So

$$v'(0^-; \lambda) \rightarrow v'(0^+; \lambda_n) \left[ 1 + \frac{\int_0^b w_2^2(\tau; \lambda_n) d\tau}{\dot{r}(\lambda_n)} \right] \neq 0, \quad v(0^-; \lambda) = v(0^+; \lambda) - \frac{v'(0^-; \lambda)}{s(\lambda)} \rightarrow \pm\infty$$

as  $\lambda \rightarrow \lambda_n$ .

If  $v(0^+; \lambda_n) \neq 0$ , then

$$v'(0^-; \lambda), v(0^-; \lambda) \rightarrow \pm\infty \text{ as } \lambda \rightarrow \lambda_n.$$

Now, for  $\lambda$  close to  $\lambda_n$ ,

$$\begin{aligned} W[u, v](0^-) &= u(0^-; \lambda)v'(0^-; \lambda) - u'(0^-; \lambda) \left[ v(0^+; \lambda) - \frac{v'(0^-; \lambda)}{s(\lambda)} \right] \\ &= \left[ u(0^-; \lambda) + \frac{u'(0^-; \lambda)}{s(\lambda)} \right] v'(0^-; \lambda) - u'(0^-; \lambda)v(0^+; \lambda) \\ &= u(0^+; \lambda)v'(0^-; \lambda) - u'(0^-; \lambda)v(0^+; \lambda), \end{aligned}$$

likewise

$$\begin{aligned} W[u, v](0^+) &= u(0^+; \lambda) \left[ v'(0^-; \lambda) + \frac{v(0^+; \lambda)}{r(\lambda)} \right] - u'(0^+; \lambda)v(0^+; \lambda) \\ &= u(0^+; \lambda)v'(0^-; \lambda) - \left[ -\frac{u(0^+; \lambda)}{r(\lambda)} + u'(0^+; \lambda) \right] v(0^+; \lambda) \\ &= u(0^+; \lambda)v'(0^-; \lambda) - u'(0^-; \lambda)v(0^+; \lambda). \end{aligned}$$

Hence

$$\psi(\lambda) = W[u, v](b) = W[u, v](0^+) = W[u, v](0^-).$$

If  $u'(0^-; \lambda_n) = v(0^+; \lambda_n) = 0$  (i.e.  $\lambda_n$  has geometric multiplicity 2) then as  $\lambda \rightarrow \lambda_n$ ,

$$\begin{aligned} \psi(\lambda) &= u(0^+; \lambda)v'(0^-; \lambda) - u'(0^-; \lambda)v(0^+; \lambda) \\ &\rightarrow u(0^-; \lambda_n)v'(0^+; \lambda_n) \left[ 1 - \frac{\int_{-a}^0 w_1^2(\tau; \lambda_n) d\tau}{\dot{s}(\lambda_n)} \right] \left[ 1 + \frac{\int_0^b w_2^2(\tau; \lambda_n) d\tau}{\dot{r}(\lambda_n)} \right] \neq 0. \end{aligned}$$

Whereas, if  $u'(0^-; \lambda_n) = 0$  and  $v(0^+; \lambda_n) \neq 0$  or if  $u'(0^-; \lambda_n) \neq 0$  and  $v(0^+; \lambda_n) = 0$  (i.e.  $\lambda_n$  has geometric multiplicity 1) then <sup>1</sup>

$$\psi(\lambda) \rightarrow \pm\infty \quad \text{as } \lambda \rightarrow \lambda_n.$$

<sup>1</sup>Note that this applies only to this particular case of  $r(\lambda_n) = s(\lambda_n) = 0$ . If precisely one of  $r(\lambda_n)$  and  $s(\lambda_n)$  is zero then  $\psi(\lambda)$  will have a zero at  $\lambda_n$  in the case of a geometrically double eigenvalue, whereas for geometrically simple eigenvalues  $\psi(\lambda)$  will have a finite non-zero limit as  $\lambda \rightarrow \lambda_n$ . This can be deduced from equation (4.26).

Since  $u(x; \lambda)$  and  $v(t; \lambda)$  (respectively  $\tilde{u}(x; \lambda)$  and  $\tilde{v}(t; \lambda)$ ) are entire for  $x < 0$  and  $t > 0$  respectively,  $G(x, t; \lambda)$  (respectively  $\tilde{G}(x, t; \lambda)$ ) will have no poles for  $-a \leq x < 0 < t \leq b$  and similarly for  $-a \leq t < 0 < x \leq b$ .

Let  $\mu \neq \lambda_n$ . Here we give the calculations of the residues of  $\frac{G(x, t; \lambda)}{\lambda - \mu}$  and  $\frac{\tilde{G}(x, t; \lambda)}{\lambda - \mu}$  at  $\lambda_n$  for the case of  $r(\lambda_n) = s(\lambda_n) = 0$  under consideration. Remaining calculations are similar and are omitted.

I. Suppose that  $u'(0^-; \lambda_n) = v(0^+; \lambda_n) = 0$ .

- If  $-a \leq x < t < 0$  then

$$\begin{aligned} & \lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) \frac{u(x; \lambda)v(t; \lambda)}{(\lambda - \mu)\psi(\lambda)} \\ &= \frac{1}{\lambda_n - \mu} \lim_{\lambda \rightarrow \lambda_n} \frac{\lambda - \lambda_n}{s(\lambda)} \lim_{\lambda \rightarrow \lambda_n} \frac{u(x; \lambda)}{\psi(\lambda)} \lim_{\lambda \rightarrow \lambda_n} [s(\lambda)v(t; \lambda)] \\ &= -\frac{1}{\lambda_n - \mu} \frac{1}{\dot{s}(\lambda_n)} \frac{w_1(x; \lambda_n)w_1(t; \lambda_n)}{1 - \frac{\int_{-a}^0 w_1^2(\tau; \lambda_n) d\tau}{\dot{s}(\lambda_n)}} \\ &= \frac{1}{\lambda_n - \mu} \frac{w_1(x; \lambda_n)w_1(t; \lambda_n)}{\int_{-a}^0 w_1^2(\tau; \lambda_n) d\tau - \dot{s}(\lambda_n)}, \end{aligned}$$

and similarly

$$\lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) \frac{\tilde{u}(x; \lambda)v(t; \lambda)}{(\lambda - \mu)\psi(\lambda)} = \frac{1}{\lambda_n - \mu} \frac{\tilde{u}(x; \lambda_n)}{u(0^-; \lambda_n)} \frac{w_1(t; \lambda_n)}{\int_{-a}^0 w_1^2(\tau; \lambda_n) d\tau - \dot{s}(\lambda_n)}.$$

- If  $-a \leq t < x < 0$  then

$$\lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) \frac{u(t; \lambda)v(x; \lambda)}{(\lambda - \mu)\psi(\lambda)} = \frac{1}{\lambda_n - \mu} \frac{w_1(t; \lambda_n)w_1(x; \lambda_n)}{\int_{-a}^0 w_1^2(\tau; \lambda_n) d\tau - \dot{s}(\lambda_n)}$$

as above. Moreover, if  $\tilde{v}(0^+; \lambda_n) = 0$  (i.e.  $\lambda_n \in \tilde{\Lambda}^* \cup \tilde{\Lambda}_1^+$ ) then  $\tilde{G}(x, t; \lambda)$  will have a simple pole at  $\lambda_n$ , giving

$$\begin{aligned} & \lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) \frac{u(t; \lambda)\tilde{v}(x; \lambda)}{(\lambda - \mu)\psi(\lambda)} \\ &= \frac{1}{\lambda_n - \mu} \lim_{\lambda \rightarrow \lambda_n} \frac{\lambda - \lambda_n}{s(\lambda)} \lim_{\lambda \rightarrow \lambda_n} \frac{u(t; \lambda)}{\psi(\lambda)} \lim_{\lambda \rightarrow \lambda_n} [s(\lambda)\tilde{v}(x; \lambda)] \\ &= \frac{1}{\lambda_n - \mu} \frac{\tilde{v}'(0^+; \lambda_n) \left[ \int_0^b \tilde{w}_2^2(\tau; \lambda_n) d\tau + \dot{r}(\lambda_n) \right]}{v'(0^+; \lambda_n) \left[ \int_0^b w_2^2(\tau; \lambda_n) d\tau + \dot{r}(\lambda_n) \right]} \frac{\tilde{w}_1(x; \lambda_n)w_1(t; \lambda_n)}{\int_{-a}^0 w_1^2(\tau; \lambda_n) d\tau - \dot{s}(\lambda_n)}. \end{aligned}$$

If  $\tilde{v}(0^+; \lambda_n) \neq 0$  (so  $\lambda_n \in \Lambda_0$ ) then  $\tilde{G}(x, t; \lambda)$  will have a double pole at  $\lambda_n$ . Let  $\tilde{y}(x; \lambda)$ ,  $\phi(\lambda)$  be as defined in Proposition 4.6.2. Then

$$\begin{aligned} \lim_{\lambda \rightarrow \lambda_n} \frac{\partial}{\partial \lambda} \left[ \frac{(\lambda - \lambda_n)^2 \tilde{G}(x, t; \lambda)}{\lambda - \mu} \right] &= \lim_{\lambda \rightarrow \lambda_n} \frac{\partial}{\partial \lambda} \left[ \frac{(\lambda - \lambda_n)^2 u(t; \lambda) \tilde{y}(x; \lambda)}{\phi(\lambda) (\lambda - \mu)} \right] \\ &= \left[ \dot{\Phi}_n - \frac{\Phi_n}{\lambda_n - \mu} \right] \frac{u(t; \lambda_n) \tilde{y}(x; \lambda_n)}{\lambda_n - \mu} \\ &\quad + \Phi_n \frac{u_\lambda(t; \lambda_n) \tilde{y}(x; \lambda_n) + u(t; \lambda_n) \tilde{y}_\lambda(x; \lambda_n)}{\lambda_n - \mu}, \end{aligned}$$

where

$$\begin{aligned}\Phi_n &= \lim_{\lambda \rightarrow \lambda_n} \frac{(\lambda - \lambda_n)^2}{r(\lambda)s(\lambda)\psi(\lambda)} \\ &= \frac{-1}{u(0^-; \lambda_n)v'(0^+; \lambda_n) \left[ \int_{-a}^0 w_1^2(\tau; \lambda_n) d\tau - \dot{s}(\lambda_n) \right] \left[ \int_0^b w_2^2(\tau; \lambda_n) d\tau + \dot{r}(\lambda_n) \right]}\end{aligned}$$

and

$$\begin{aligned}\dot{\Phi}_n &= \lim_{\lambda \rightarrow \lambda_n} \frac{d}{d\lambda} \frac{(\lambda - \lambda_n)^2}{r(\lambda)s(\lambda)\psi(\lambda)} \\ &= \frac{\lim_{\lambda \rightarrow \lambda_n} \frac{d}{d\lambda} \frac{s(\lambda)}{\lambda - \lambda_n} u(0^-; \lambda_n) + \dot{s}(\lambda_n) u_\lambda(0^-; \lambda_n) + \lim_{\lambda \rightarrow \lambda_n} \frac{d}{d\lambda} \frac{u'(0^-; \lambda)}{\lambda - \lambda_n}}{u^2(0^-; \lambda_n)v'(0^+; \lambda_n) \left[ \int_{-a}^0 w_1^2(\tau; \lambda_n) d\tau - \dot{s}(\lambda_n) \right]^2 \left[ \int_0^b w_2^2(\tau; \lambda_n) d\tau + \dot{r}(\lambda_n) \right]} \\ &\quad - \frac{v'_\lambda(0^+; \lambda_n)\dot{r}(\lambda_n) + v'(0^+; \lambda_n) \lim_{\lambda \rightarrow \lambda_n} \frac{d}{d\lambda} \frac{r(\lambda)}{\lambda - \lambda_n} - \lim_{\lambda \rightarrow \lambda_n} \frac{d}{d\lambda} \frac{v(0^+; \lambda)}{\lambda - \lambda_n}}{u(0^-; \lambda_n)v'^2(0^+; \lambda_n) \left[ \int_{-a}^0 w_1^2(\tau; \lambda_n) d\tau - \dot{s}(\lambda_n) \right] \left[ \int_0^b w_2^2(\tau; \lambda_n) d\tau + \dot{r}(\lambda_n) \right]^2}.\end{aligned}$$

- If  $0 < x < t \leq b$

$$\begin{aligned}\lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) \frac{u(x; \lambda)v(t; \lambda)}{(\lambda - \mu)\psi(\lambda)} &= \frac{1}{\lambda_n - \mu} \lim_{\lambda \rightarrow \lambda_n} \frac{\lambda - \lambda_n}{r(\lambda)} \lim_{\lambda \rightarrow \lambda_n} [r(\lambda)u(x; \lambda)] \lim_{\lambda \rightarrow \lambda_n} \frac{v(t; \lambda)}{\psi(\lambda)} \\ &= \frac{1}{\lambda_n - \mu} \frac{w_2(x; \lambda_n)w_2(t; \lambda_n)}{\dot{r}(\lambda_n) + \int_0^b w_2^2(\tau; \lambda_n) d\tau}.\end{aligned}$$

Moreover, if  $\tilde{u}'(0^-; \lambda_n) = 0$  (i.e.  $\lambda_n \in \tilde{\Lambda}^* \cup \tilde{\Lambda}_1^-$ ) then  $\tilde{G}(x, t; \lambda)$  will have a simple pole at  $\lambda = \lambda_n$ , giving

$$\begin{aligned}\lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) \frac{\tilde{u}(x; \lambda)v(t; \lambda)}{(\lambda - \mu)\psi(\lambda)} &= \frac{1}{\lambda_n - \mu} \lim_{\lambda \rightarrow \lambda_n} \frac{\lambda - \lambda_n}{r(\lambda)} \lim_{\lambda \rightarrow \lambda_n} [r(\lambda)\tilde{u}(x; \lambda)] \lim_{\lambda \rightarrow \lambda_n} \frac{v(t; \lambda)}{\psi(\lambda)} \\ &= \frac{1}{\lambda_n - \mu} \frac{\tilde{u}(0^-; \lambda_n) \left[ \int_{-a}^0 \tilde{w}_1^2(\tau; \lambda_n) d\tau - \dot{s}(\lambda_n) \right]}{u(0^-; \lambda_n) \left[ \int_{-a}^0 w_1^2(\tau; \lambda_n) d\tau - \dot{s}(\lambda_n) \right]} \frac{\tilde{w}_2(x; \lambda_n)w_2(t; \lambda_n)}{\dot{r}(\lambda_n) + \int_0^b w_2^2(\tau; \lambda_n) d\tau},\end{aligned}$$

whereas if  $\tilde{u}'(0^-; \lambda_n) \neq 0$  (so  $\lambda_n \in \Lambda_0$ ) then  $\tilde{G}(x, t; \lambda)$  has a double pole at  $\lambda = \lambda_n$ . Let  $\tilde{y}(x; \lambda)$ ,  $\phi(\lambda)$ ,  $\Phi_n$ ,  $\dot{\Phi}_n$  be as defined in Proposition 4.6.2. Then

$$\begin{aligned}\lim_{\lambda \rightarrow \lambda_n} \frac{\partial}{\partial \lambda} \left[ \frac{(\lambda - \lambda_n)^2 \tilde{G}(x, t; \lambda)}{\lambda - \mu} \right] &= \lim_{\lambda \rightarrow \lambda_n} \frac{\partial}{\partial \lambda} \left[ \frac{(\lambda - \lambda_n)^2 \tilde{y}(x; \lambda)v(t; \lambda)}{\phi(\lambda) \lambda - \mu} \right] \\ &= \left[ \dot{\Phi}_n - \frac{\Phi_n}{\lambda_n - \mu} \right] \frac{\tilde{y}(x; \lambda_n)v(t; \lambda_n)}{\lambda_n - \mu} \\ &\quad + \Phi_n \frac{\tilde{y}_\lambda(x; \lambda_n)v(t; \lambda_n) + \tilde{y}(x; \lambda_n)v_\lambda(t; \lambda_n)}{\lambda_n - \mu}.\end{aligned}$$

- If  $0 < t < x \leq b$  then

$$\lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) \frac{u(t; \lambda)v(x; \lambda)}{(\lambda - \mu)\psi(\lambda)} = \frac{1}{\lambda_n - \mu} \frac{w_2(t; \lambda_n)w_2(x; \lambda_n)}{\dot{r}(\lambda_n) + \int_0^b w_2^2(\tau; \lambda_n)d\tau}$$

and

$$\begin{aligned} & \lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) \frac{u(t; \lambda)\tilde{v}(x; \lambda)}{(\lambda - \mu)\psi(\lambda)} \\ &= \frac{1}{\lambda_n - \mu} \lim_{\lambda \rightarrow \lambda_n} \frac{\lambda - \lambda_n}{r(\lambda)} \lim_{\lambda \rightarrow \lambda_n} [r(\lambda)u(t; \lambda)] \lim_{\lambda \rightarrow \lambda_n} \frac{\tilde{v}(x; \lambda)}{\psi(\lambda)} \\ &= \frac{1}{\lambda_n - \mu} \frac{w_2(t; \lambda_n)}{\int_0^b w_2^2(\tau; \lambda_n)d\tau + \dot{r}(\lambda_n)} \frac{\tilde{v}(x; \lambda_n)}{v'(0^+; \lambda_n)}. \end{aligned}$$

II. (i) Suppose  $v(0^+; \lambda_n) \neq 0$  and  $u'(0^-; \lambda_n) = 0$ .

- If  $-a \leq x < t < 0$  then

$$\begin{aligned} & \lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) \frac{u(x; \lambda)v(t; \lambda)}{(\lambda - \mu)\psi(\lambda)} \\ &= \frac{1}{\lambda_n - \mu} \lim_{\lambda \rightarrow \lambda_n} \frac{\lambda - \lambda_n}{s(\lambda)} \lim_{\lambda \rightarrow \lambda_n} \frac{u(x; \lambda)[s(\lambda)r(\lambda)v(t; \lambda)]}{r(\lambda)\psi(\lambda)} \\ &= \frac{w_1(x; \lambda_n)w_1(t; \lambda_n)}{\int_{-a}^0 w_1^2(\tau; \lambda_n)d\tau - \dot{s}(\lambda_n)}, \end{aligned}$$

and similarly

$$\lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) \frac{\tilde{u}(x; \lambda)v(t; \lambda)}{(\lambda - \mu)\psi(\lambda)} = \frac{1}{\lambda_n - \mu} \frac{\tilde{u}(x; \lambda_n)}{u(0^-; \lambda_n)} \frac{w_1(t; \lambda_n)}{\int_{-a}^0 w_1^2(\tau; \lambda_n)d\tau - \dot{s}(\lambda_n)}.$$

- If  $-a \leq t < x < 0$  then

$$\lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) \frac{u(t; \lambda)v(x; \lambda)}{(\lambda - \mu)\psi(\lambda)} = \frac{1}{\lambda_n - \mu} \frac{w_1(t; \lambda_n)w_1(x; \lambda_n)}{\int_{-a}^0 w_1^2(\tau; \lambda_n)d\tau - \dot{s}(\lambda_n)},$$

(as above) and, moreover,

$$\begin{aligned} \lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) \frac{u(t; \lambda)\tilde{v}(x; \lambda)}{(\lambda - \mu)\psi(\lambda)} &= \frac{1}{\lambda_n - \mu} \lim_{\lambda \rightarrow \lambda_n} \frac{\lambda - \lambda_n}{s(\lambda)} \lim_{\lambda \rightarrow \lambda_n} \frac{u(x; \lambda)[s(\lambda)r(\lambda)\tilde{v}(t; \lambda)]}{r(\lambda)W[u, v]} \\ &= \frac{1}{\lambda_n - \mu} \frac{w(t; \lambda_n)\tilde{w}_1(x; \lambda_n)}{\int_{-a}^0 w_1^2(\tau; \lambda_n)d\tau - \dot{s}(\lambda_n)} \frac{\tilde{v}(0^+; \lambda_n)}{v(0^+; \lambda_n)}. \end{aligned}$$

- If  $0 < x < t \leq b$  then

$$\lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) \frac{u(x; \lambda)v(t; \lambda)}{(\lambda - \mu)\psi(\lambda)} = \frac{1}{\lambda_n - \mu} \lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) \frac{[r(\lambda)u(x; \lambda)]v(t; \lambda)}{r(\lambda)\psi(\lambda)} = 0.$$

Similarly, if  $\tilde{u}'(0^-; \lambda_n) = 0$  (i.e.  $\lambda_n \in \tilde{\Lambda}_1^- \cup \tilde{\Lambda}^*$ ) then

$$\lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) \frac{\tilde{u}(x; \lambda)v(t; \lambda)}{(\lambda - \mu)\psi(\lambda)} = 0,$$



Whereas, if  $\tilde{u}'(0^-; \lambda_n) \neq 0$  (so  $\lambda_n \in \Lambda_0$ ) then

$$\begin{aligned} & \lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) \frac{\tilde{u}(x; \lambda)v(t; \lambda)}{(\lambda - \mu)\psi(\lambda)} \\ &= \frac{1}{\lambda_n - \mu} \frac{\tilde{y}(x; \lambda_n)v(t; \lambda_n)}{\dot{\phi}(\lambda_n)} \quad (\text{see Proposition 4.6.2}) \\ &= \frac{1}{\lambda_n - \mu} \frac{\tilde{u}'(0^-; \lambda_n)v(t; \lambda_n)}{u(0^-; \lambda_n)v(0^+; \lambda_n) \left[ \int_{-a}^0 w_1^2(\tau; \lambda_n) d\tau - \dot{s}(\lambda_n) \right]}. \end{aligned}$$

- If  $0 < t < x \leq b$  then

$$\lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) \frac{u(t; \lambda)v(x; \lambda)}{(\lambda - \mu)\psi(\lambda)} = 0$$

(as above). Similarly,

$$\lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) \frac{u(t; \lambda)\tilde{v}(x; \lambda)}{(\lambda - \mu)\psi(\lambda)} = 0.$$

II. (ii) Suppose that  $u'(0^-; \lambda_n) \neq 0$  and  $v(0^+; \lambda_n) = 0$ .

- If  $-a \leq x < t < 0$  then

$$\lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) \frac{u(x; \lambda)v(t; \lambda)}{(\lambda - \mu)\psi(\lambda)} = \frac{1}{\lambda_n - \mu} \lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) \frac{u(x; \lambda)[s(\lambda)v(t; \lambda)]}{s(\lambda)W[u, v]} = 0,$$

and similarly,

$$\lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) \frac{\tilde{u}(x; \lambda)v(t; \lambda)}{(\lambda - \mu)\psi(\lambda)} = 0.$$

- If  $-a \leq t < x < 0$  then

$$\lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) \frac{u(t; \lambda)v(x; \lambda)}{(\lambda - \mu)\psi(\lambda)} = 0$$

as above, and similarly, if  $\tilde{v}(0^+; \lambda_n) = 0$  (i.e.  $\lambda_n \in \tilde{\Lambda}_1^+ \cup \tilde{\Lambda}^*$ ) then

$$\lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) \frac{u(t; \lambda)\tilde{v}(x; \lambda)}{(\lambda - \mu)\psi(\lambda)} = 0.$$

If  $\tilde{v}(0^+; \lambda_n) \neq 0$  (so  $\lambda_n \in \Lambda_0$ ) then

$$\begin{aligned} & \lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) \frac{u(t; \lambda)\tilde{v}(x; \lambda)}{(\lambda - \mu)\psi(\lambda)} \\ &= \frac{1}{\lambda_n - \mu} \frac{u(t; \lambda_n)\tilde{y}(x; \lambda_n)}{\dot{\phi}(\lambda_n)} \quad (\text{see Proposition 4.6.2}) \\ &= \frac{1}{\lambda_n - \mu} \frac{u(t; \lambda_n)\tilde{v}(0^+; \lambda_n)\tilde{w}_1(x; \lambda_n)}{u'(0^-; \lambda_n)v'(0^+; \lambda_n) \left[ \int_0^b w_2^2(\tau; \lambda_n) d\tau + \dot{r}(\lambda_n) \right]}. \end{aligned}$$

- If  $0 < x < t \leq b$  then

$$\begin{aligned} \lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) \frac{u(x; \lambda)v(t; \lambda)}{(\lambda - \mu)\psi(\lambda)} &= \frac{1}{\lambda_n - \mu} \lim_{\lambda \rightarrow \lambda_n} \frac{\lambda - \lambda_n}{r(\lambda)} \lim_{\lambda \rightarrow \lambda_n} \frac{[s(\lambda)r(\lambda)u(x; \lambda)]v(t; \lambda)}{s(\lambda)W[u, v]} \\ &= \frac{1}{\lambda_n - \mu} \frac{w_2(x; \lambda_n)w_2(t; \lambda_n)}{\int_0^b w_2^2(\tau; \lambda_n) d\tau + \dot{r}(\lambda_n)} \end{aligned}$$

and similarly,

$$\begin{aligned} \lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) \frac{\tilde{u}(x; \lambda)v(t; \lambda)}{(\lambda - \mu)\psi(\lambda)} &= \frac{1}{\lambda_n - \mu} \lim_{\lambda \rightarrow \lambda_n} \frac{\lambda - \lambda_n}{r(\lambda)} \lim_{\lambda \rightarrow \lambda_n} \frac{[s(\lambda)r(\lambda)\tilde{u}(x; \lambda)]v(t; \lambda)}{s(\lambda)W[u, v]} \\ &= \frac{1}{\lambda_n - \mu} \frac{\tilde{u}'(0^-; \lambda_n)}{u'(0^-; \lambda_n)} \frac{\tilde{w}_2(x; \lambda_n)w_2(t; \lambda_n)}{\int_0^b w_2^2(\tau; \lambda_n)d\tau + \dot{r}(\lambda_n)}. \end{aligned}$$

- If  $0 < t < x \leq b$  then

$$\lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) \frac{u(t; \lambda)v(x; \lambda)}{(\lambda - \mu)\psi(\lambda)} = \frac{1}{\lambda_n - \mu} \frac{w_2(t; \lambda_n)w_2(x; \lambda_n)}{\dot{r}(\lambda_n) + \int_0^b w_2^2(\tau; \lambda_n)d\tau}$$

as above, and similarly,

$$\lim_{\lambda \rightarrow \lambda_n} (\lambda - \lambda_n) \frac{u(t; \lambda)\tilde{v}(x; \lambda)}{(\lambda - \mu)\psi(\lambda)} = \frac{1}{\lambda_n - \mu} \frac{w_2(t; \lambda_n)}{\int_0^b w_2^2(\tau; \lambda_n)d\tau + \dot{r}(\lambda_n)} \frac{\tilde{v}(x; \lambda_n)}{v'(0^+; \lambda_n)}.$$

□

**Note 4.6.4.** Let  $F, h$  be defined as in Lemma 4.4.2. Let  $\lambda \neq \lambda_n, n \in \mathbb{N}_0, \lambda \neq \gamma_i, i = \overline{1, N}$  and  $\lambda \neq \delta_j, j = \overline{1, M}$ . Then, in particular,

1. if  $-a \leq x < 0$

$$\begin{aligned}
& \left[ (\lambda - \tilde{L})^{-1} H F \right]_0 \\
&= h(x; \lambda) + \sum_{\lambda_n \in \Lambda \setminus (\Lambda_0 \cup \Lambda^*)} \frac{[K_n - \tilde{K}_n] \int_{-a}^x u_n f dt}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} \tilde{u}_n(x) \\
&+ \sum_{\lambda_n \in \Lambda^* \setminus \Lambda_0} \frac{\left[ 1 - \frac{\tilde{v}'(0^-; \lambda_n) u(0^-; \lambda_n)}{v'(0^-; \lambda_n) \tilde{u}(0^-; \lambda_n)} \right] \int_{-a}^x z_n^{(1)} f dt}{(\lambda - \lambda_n) \|Z_n^{(1)}\|^2} \tilde{z}_n^{(1)}(x) \\
&+ \sum_{\lambda_n \in \Lambda_1^- \setminus \Lambda_0} \frac{[1 - T_n^-] \int_{-a}^x z_n f dt}{(\lambda - \lambda_n) \|Z_n\|^2} \tilde{z}_n(x) \\
&- \sum_{\lambda_n \in \Lambda^* \cap \tilde{\Lambda}_1^+} \frac{\left[ \frac{\tilde{v}'(0^-; \lambda_n) u(0^-; \lambda_n)}{v'(0^-; \lambda_n)} \right] \tilde{w}_1(x; \lambda_n) - \tilde{u}(x; \lambda_n)}{(\lambda - \lambda_n) \|Z_n^{(1)}\|^2} \int_{-a}^x z_n^{(1)} f dt \\
&- \sum_{\lambda_n \in (\Lambda^* \cap \Lambda_0) \setminus \tilde{\Lambda}_1^+} \left\{ \begin{aligned} & \left[ \dot{\Phi}_n \tilde{y}(x; \lambda_n) + \Phi_n \tilde{y}_\lambda(x; \lambda_n) - \frac{\tilde{u}(x; \lambda_n)}{\|Z_n^{(1)}\|^2} \right] \frac{\int_{-a}^x z_n^{(1)} f dt}{\lambda - \lambda_n} \\ & + \frac{\Phi_n \tilde{y}(x; \lambda_n)}{\lambda - \lambda_n} \int_{-a}^x \left[ u_\lambda(t; \lambda_n) + \frac{z_n^{(1)}(t)}{\lambda - \lambda_n} \right] f dt \end{aligned} \right\} \\
&- \sum_{\lambda_n \in (\Lambda \setminus \Lambda^*) \cap \tilde{\Lambda}_1^+} \frac{\tilde{v}(x; \lambda_n) \int_{-a}^x u_n f dt - \tilde{u}(x; \lambda_n) \int_{-a}^x v_n f dt}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} \\
&- \sum_{\lambda_n \in (\Lambda \cap \Lambda_0) \setminus (\Lambda^* \cup \tilde{\Lambda}_1^+)} \left\{ \begin{aligned} & \left[ \dot{\Phi}_n \tilde{y}(x; \lambda_n) + \Phi_n \tilde{y}_\lambda(x; \lambda_n) - \frac{k_n \tilde{u}(x; \lambda_n)}{\dot{\psi}(\lambda_n)} \right] \frac{\int_{-a}^x u_n f dt}{\lambda - \lambda_n} \\ & + \frac{\Phi_n \tilde{y}(x; \lambda_n)}{\lambda - \lambda_n} \int_{-a}^x \left[ u_\lambda(t; \lambda_n) + \frac{u_n(t)}{\lambda - \lambda_n} \right] f dt \end{aligned} \right\} \\
&- \sum_{\lambda_n \in \Lambda_1^- \cap \Lambda_0} \left[ \frac{\tilde{y}(x; \lambda_n)}{\dot{\phi}(\lambda_n)} - \frac{\tilde{u}(x; \lambda_n)}{\|Z_n\|^2} \right] \frac{\int_{-a}^x z_n f dt}{\lambda - \lambda_n} \\
&- \sum_{\lambda_n \in \Lambda_1^+ \cap \Lambda_0} \frac{\tilde{y}(x; \lambda_n) \int_{-a}^x u(t; \lambda_n) f dt}{(\lambda - \lambda_n) \dot{\phi}(\lambda_n)},
\end{aligned}$$

where for  $\lambda_n \in \Lambda \setminus (\Lambda_0 \cup \Lambda^*)$ ,  $K_n = k_n$  and  $\tilde{K}_n = \tilde{k}_n$  if  $r(\lambda_n) \neq 0$  and  $s(\lambda_n) \neq 0$  else  $K_n = k_n^{(1)}$  and  $\tilde{K}_n = \tilde{k}_n^{(1)}$ .

2. if  $0 < x \leq b$

$$\begin{aligned}
& \left[ (\lambda - \tilde{L})^{-1} H F \right]_0 \\
&= h(x; \lambda) + \sum_{\lambda_n \in \Lambda \setminus (\Lambda_0 \cup \Lambda^*)} \frac{[\tilde{K}_n - K_n] \int_x^b u_n f dt}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} \tilde{u}_n(x) \\
&+ \sum_{\lambda_n \in \Lambda^* \setminus \Lambda_0} \frac{\left[ 1 - \frac{\tilde{u}(0^+; \lambda_n) v'(0^+; \lambda_n)}{u(0^+; \lambda_n) \tilde{v}'(0^+; \lambda_n)} \right] \int_x^b z_n^{(2)} f dt}{(\lambda - \lambda_n) \|Z_n^{(2)}\|^2} \tilde{z}_n^{(2)}(x) \\
&+ \sum_{\lambda_n \in \Lambda_1^+ \setminus \Lambda_0} \frac{[1 - T_n^+] \int_x^b z_n f dt}{(\lambda - \lambda_n) \|Z_n\|^2} \tilde{z}_n(x) \\
&- \sum_{\lambda_n \in (\Lambda^* \cap \Lambda_0 \cap \tilde{\Lambda}_1^-)} \frac{\left[ \frac{\tilde{u}(0^+; \lambda_n) v'(0^+; \lambda_n)}{u(0^+; \lambda_n)} \right] \tilde{w}_2(x; \lambda_n) - \tilde{v}(x; \lambda_n)}{(\lambda - \lambda_n) \|Z_n^{(2)}\|^2} \int_x^b z_n^{(2)} f dt \\
&- \sum_{\lambda_n \in (\Lambda^* \cap \Lambda_0) \setminus \tilde{\Lambda}_1^-} \left\{ \begin{aligned} & \left[ \dot{\Phi}_n \tilde{y}(x; \lambda_n) + \Phi_n \tilde{y}_\lambda(x; \lambda_n) - \frac{\tilde{v}(x; \lambda_n)}{\|Z_n^{(2)}\|^2} \frac{\int_x^b z_n^{(2)} f dt}{\lambda - \lambda_n} \right] \\ & + \frac{\Phi_n \tilde{y}(x; \lambda_n)}{\lambda - \lambda_n} \int_x^b \left[ v_\lambda(t; \lambda_n) + \frac{z_n^{(2)}(t)}{\lambda - \lambda_n} \right] f dt \end{aligned} \right\} \\
&+ \sum_{\lambda_n \in (\Lambda \setminus \Lambda^*) \cap \tilde{\Lambda}_1^-} \frac{\tilde{v}(x; \lambda_n) \int_x^b u_n f dt - \tilde{u}(x; \lambda_n) \int_x^b v_n f dt}{(\lambda - \lambda_n) \dot{\psi}(\lambda_n)} \\
&- \sum_{\lambda_n \in (\Lambda \cap \Lambda_0) \setminus (\Lambda^* \cup \tilde{\Lambda}_1^-)} \left\{ \begin{aligned} & \left[ \dot{\Phi}_n \tilde{y}(x; \lambda_n) + \Phi_n \tilde{y}_\lambda(x; \lambda_n) - \frac{\tilde{v}(x; \lambda_n)}{k_n \dot{\psi}(\lambda_n)} \frac{\int_x^b v_n f dt}{\lambda - \lambda_n} \right] \\ & + \frac{\Phi_n \tilde{y}(x; \lambda_n)}{\lambda - \lambda_n} \int_x^b \left[ v_\lambda(t; \lambda_n) + \frac{v_n(t)}{\lambda - \lambda_n} \right] f dt \end{aligned} \right\} \\
&+ \sum_{\lambda_n \in \Lambda_1^+ \cap \Lambda_0} \left[ \frac{\tilde{v}(x; \lambda_n)}{\|Z_n\|^2} - \frac{\tilde{y}(x; \lambda_n)}{\dot{\phi}(\lambda_n)} \right] \frac{\int_x^b z_n f dt}{\lambda - \lambda_n} \\
&- \sum_{\lambda_m \in \Lambda_1 \cap \Lambda_0} \frac{\tilde{y}(x; \lambda_m) \int_x^b v(t; \lambda_m) f dt}{(\lambda - \lambda_m) \dot{\phi}(\lambda_m)},
\end{aligned}$$

where for  $\lambda_n \in \Lambda \setminus (\Lambda_0 \cup \Lambda^*)$ ,  $K_n = k_n$  and  $\tilde{K}_n = \tilde{k}_n$  if  $r(\lambda_n) \neq 0$  and  $s(\lambda_n) \neq 0$  else  $K_n = k_n^{(2)}$  and  $\tilde{K}_n = \tilde{k}_n^{(2)}$ .

# Chapter 5

## Oscillation theory for Sturm-Liouville operators with point transfer conditions

### 5.1 Introduction

In this chapter we consider the problem of extending Sturm's oscillation theorem, regarding the number of zeroes of eigenfunctions, to the case of discontinuous Sturm-Liouville problems with constant coefficient transmission conditions. In particular, we study the generalized Sturm-Liouville equation

$$-(py')' + q(x)y = \lambda ry, \quad (5.1)$$

with  $x \in [-a, 0) \cup (0, b]$ , subject to separated boundary conditions

$$y(-a) \cos \alpha = (py')(-a) \sin \alpha, \quad \alpha \in [0, \pi), \quad (5.2)$$

$$y(b) \cos \beta = (py')(b) \sin \beta, \quad \beta \in (0, \pi], \quad (5.3)$$

and a point transfer condition

$$\begin{bmatrix} y(0^+) \\ (py')(0^+) \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} y(0^-) \\ (py')(0^-) \end{bmatrix}. \quad (5.4)$$

Here,  $T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$  has  $t_{ij} \in \mathbb{R}$  and  $\det T > 0$ . We assume that  $p, q, r \in L^2(-a, b)$ ,  $q$  is real function, and  $p(x), r(x) > 0$  for all  $x \in [-a, b]$ .

In the case where  $T$  is the identity matrix, the boundary value problem (5.1)-(5.4) reduces to the classical setting, without discontinuity. Classical Sturmian oscillation theory is well known. Topics studied include the number of zeroes of eigenfunctions, positions of such zeroes, and perturbation of the positions of zeroes when the coefficients  $p, q, r$  and/or the parameter  $\lambda$  are changed. See E. A. Coddington and N. Levinson [23] for an introductory account. These problems are usually treated under smoothness assumptions on the coefficients. For example,  $p, p', q, r$  continuous with  $p, r > 0$  on  $[-a, b]$ . The results can be generalized when less stringent conditions on the coefficients are enforced. In [26], W. N. Everitt, M K. Kwong and A. Zettl consider (5.1) under minimal restrictions on the coefficients, and assume that the weight function  $r$  is allowed to vanish on a subset of  $[-a, b]$  of positive measure (in the Lebesgue sense). This generalization has a significant effect on the oscillatory properties of the eigenfunctions. In particular, they show that the eigenfunction corresponding to the smallest eigenvalue can have one or more zeroes in  $(-a, b)$ , contrasting with the classical case when  $r > 0$ . This result was later generalized by P. Binding and H. Volkmer in [17], where they deduced the minimal oscillation number associated with the eigenfunctions. For other semi-definite generalizations see F. V. Atkinson [8].

A certain discontinuous problem was studied by P. A. Binding, P. J. Browne and B. A. Watson in [14], [13]. In [14] the authors considered asymptotics for the case where  $r$  is allowed to change sign at some interior point  $c \in (-a, b)$ , such that  $r|_{(-a,c)}, -r|_{(c,b)} > 0$  with finite discontinuity at  $c$ . There, the main aim was to investigate how  $r(-a^+), r(c^-), r(c^+)$  and  $r(b^-)$  determine the values of  $C$  and  $\kappa$  in

$$\varsigma \sqrt{\lambda_n} = n\pi + C + O(n^{-\kappa}), \quad \kappa > 0 \quad (5.5)$$

for positive eigenvalues  $\lambda_n$ , where  $\varsigma = \int_{-a}^c \sqrt{r} dx$ . A similar relation holds for the negative eigenvalues, this time depending on negative values of  $r$ . These results were generalized to multiple turning points in [13]. In comparison, the following approximation holds for classical eigenvalues (when  $r$  is smooth and of one sign):

$$D\lambda_n = (n + E)^2 \pi^2 + F + o(1). \quad (5.6)$$

Here  $D, E, F$  can be determined explicitly from the coefficients in the differential equation and the boundary conditions (see for example [39]).

We will determine asymptotics for the eigenvalues of (5.1)-(5.4) in terms of generalized oscillation counts of the corresponding eigenfunctions. In particular, we show that

$$\xi \sqrt{\lambda_{N,M}} = (N + M)\pi + C + O\left(\frac{1}{N + M}\right), \quad (5.7)$$

where  $\xi = \int_{-a}^b (r/p)^{1/2} dx$  and  $\lambda_{N,M}$  denotes the eigenvalue of (5.1)-(5.4) whose corresponding eigenfunction has  $N$  zeroes in  $(-a, 0]$  and  $M$  zeroes in  $[0, b)$ , with special treatment at the point of discontinuity,  $x = 0$  (see Theorem 5.5.2). We will show that the constant  $C$  is determined solely by the angles  $\alpha, \beta$  in the boundary conditions, and that the form of the transmission matrix  $T$  effects the possible counts  $N, M$ .

We note that oscillation theory for discontinuous problems of the type (5.1)-(5.4) considered here is not entirely new. Some partial results have been obtained for Schrödinger operators on graph domains (see [9], [70], [71]). There, the transmission conditions are replaced by matching conditions at the interior vertices of the graph. For vertices of degree 2 these matching conditions are equivalent to transmission conditions of the form (5.4). However, it is important to note that in order to obtain nodal counts (that is, oscillation counts) such graph problems have required the continuity of the eigenfunction,  $y$ , at the vertices, only allowing for discontinuities in  $y'$ . By contrast, we will obtain results for general non-singular  $2 \times 2$  transfer matrices.

To study the oscillatory properties of the boundary value problem (5.1)-(5.4) we make use of a novel parametrization of the transfer matrix  $T$ . In particular, we use the Iwasawa decomposition of  $SL(2, \mathbb{R})$ , which gives each  $g \in SL(2, \mathbb{R})$  a unique representation in the form

$$g = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} \gamma & 0 \\ 0 & 1/\gamma \end{bmatrix} \begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix}. \quad (5.8)$$

Here  $\gamma \in \mathbb{R}^+, \delta \in \mathbb{R}$  and we restrict  $\phi \in [-\pi, \pi)$ . In particular, writing  $T = g\sqrt{\det T}$  with  $g = (1/\sqrt{\det T})T \in SL(2, \mathbb{R})$  it can be shown that  $\phi, \gamma$  and  $\delta$  are determined uniquely by the following formulae

$$\cos \phi = \frac{t_{11}}{\sqrt{t_{11}^2 + t_{21}^2}}, \quad \sin \phi = \frac{t_{21}}{\sqrt{t_{11}^2 + t_{21}^2}}, \quad (5.9)$$

$$\gamma = \sqrt{\frac{t_{11}^2 + t_{21}^2}{\det T}}, \quad \delta = \frac{t_{11}t_{12} + t_{21}t_{22}}{t_{11}^2 + t_{21}^2}. \quad (5.10)$$

Our plan is as follows: We make use of two base solutions of (5.1), namely  $u(x; \lambda)$  (defined for  $x \in [-a, 0)$ ) and  $v(x; \lambda)$  (defined for  $x \in (0, b]$ ) which satisfy the boundary conditions at  $x = -a$  and  $x = b$  respectively. We use standard Prüfer transformations to convert  $u$  and  $v$  into angles  $\theta(x; \lambda)$  and  $\varphi(x; \lambda)$ , defined for  $x \in [-a, 0)$  and  $x \in (0, b]$  respectively. The eigencondition takes the form of a matching condition for the angles  $\theta(0; \lambda)$  and  $\varphi(0; \lambda)$ , which is deduced from the transmission condition (5.4). The eigencondition in question is given by

$$\tan(\varphi(0; \lambda) + \phi) = \gamma^2 [\tan \theta(0; \lambda) + \delta]. \quad (5.11)$$

Here,  $\phi$ ,  $\gamma$  and  $\delta$  are given by equations (5.9)-(5.10) above. This is derived in Section 5.2. In this preliminary section, we also introduce certain modifying functions  $\Theta$  and  $\Delta$  (borrowed from [14] and [13]) which we use to simplify lengthy calculations later on.

In Section 5.3 we introduce modifications of the Prüfer angles of  $\theta(x; \lambda)$  and  $\varphi(x; \lambda)$ , which will play a central role in our analysis. Using the functions  $\Theta$  and  $\Delta$  introduced in Section 5.2 we separately analyse the effect of each matrix in the Iwasawa decomposition on the modified Prüfer angles corresponding to  $u$  and  $v$ . Here, the strategy is apply the inverse of the rotation matrix,  $\begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}^{-1}$ , to  $\begin{bmatrix} v(0^+) \\ pv'(0^+) \end{bmatrix}$ , and the shear matrix,  $\begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix}$  to  $\begin{bmatrix} u(0^-) \\ pu'(0^-) \end{bmatrix}$ , and thereby obtain translated modified Prüfer angles. These results are derived in Section 5.4. The inner factor,  $\begin{bmatrix} \gamma & 0 \\ 0 & 1/\gamma \end{bmatrix}$ , in (5.8) produces only a scaling effect and thus will not change oscillation counts. It will be combined into the final matching condition for the translated modified Prüfer angles (see Theorem 5.5.1).

Finally, in Section 5.5 we are able prove the main oscillation theorems. These include formulae for asymptotics of eigenvalues in terms of generalized oscillation counts of eigenfunctions (Theorem 5.5.2). We also consider the problem of indexing eigenvalues in terms of oscillation counts (Theorems 5.5.4 and 5.5.5).

## 5.2 Preliminary considerations

For  $-a \leq x < 0$ , let  $u(x; \lambda)$  denote the solution to (5.1) satisfying

$$u(-a; \lambda) = \sin \alpha, \quad p(-a)u'(-a; \lambda) = \cos \alpha, \quad \forall \lambda \in \mathbb{R}, \quad (5.12)$$

and let  $\theta(x; \lambda)$  denote the Prüfer angle associated with  $u$  (i.e.  $\cot \theta = pu'/u$ , see Coddington and Levinson [23, Chapter 8]). Then  $\theta$  satisfies the initial condition  $\theta(-a; \lambda) = \alpha$  for all  $\lambda \in \mathbb{R}$ .

For  $0 < x \leq b$ , let  $v(x; \lambda)$  denote the solution to (5.1) satisfying

$$v(b; \lambda) = \sin \beta, \quad p(b)v'(b; \lambda) = \cos \beta, \quad \forall \lambda \in \mathbb{R}, \quad (5.13)$$

and let  $\varphi(x; \lambda)$  denote the Prüfer angle associated with  $v$  (i.e.  $\cot \varphi = pv'/v$ ). Then  $\varphi$  satisfies the terminal condition  $\varphi(b; \lambda) = \beta$  for all  $\lambda \in \mathbb{R}$ .

We note that  $y(x; \lambda)$  is an eigenfunction to the eigenvalue  $\lambda$  if and only if  $y$  is of the form

$$y(x, \lambda) = \begin{cases} A(\lambda)u(x; \lambda), & \text{if } -a \leq x < 0, \\ B(\lambda)v(x; \lambda), & \text{if } 0 < x \leq b, \end{cases} \quad (5.14)$$

for some  $A(\lambda), B(\lambda) \in \mathbb{R} \setminus \{0\}$ , and (5.4) is satisfied. The corresponding eigen-condition can be written in terms of  $\theta(0, \lambda)$  and  $\varphi(0, \lambda)$  as follows,

$$\tan(\varphi(0; \lambda) + \phi) = \gamma^2 [\tan \theta(0; \lambda) + \delta]. \quad (5.15)$$

We will make use of a modified Prüfer angle (see Definition 5.3.1 below). In order to describe the effects of the transfer condition on this angle in an efficient manner we make use of the following functions. These functions were introduced in the papers [14] and [13] by P. A. Binding, P. J. Browne and B. A. Watson.

**Definition 5.2.1.** Let  $\Theta(\omega; k)$ ,  $k > 0$  denote the angle depending continuously on  $\omega$  such that  $\Theta(0; k) = 0$  and  $\tan \Theta(\omega; k) = k \tan \omega$ .

**Definition 5.2.2.** Let  $\Delta(\omega; k)$ ,  $k \in (-\frac{\pi}{4}, \frac{\pi}{4})$  denote the angle depending continuously on  $\omega$  such that

$$\tan \Delta(\omega; k) = \frac{\sin(\omega - k)}{\cos(\omega + k)} \quad \text{and} \quad \Delta(0; k) = -k.$$

We will make use of the following results from [14], [13].

**Lemma 5.2.3.** (i)  $\Theta(\omega + m\pi; k) = \Theta(\omega; k) + m\pi$  for all  $m \in \mathbb{Z}$ .

(ii)  $\Theta(\frac{m\pi}{2}; k) = \frac{m\pi}{2}$  for all  $m \in \mathbb{Z}$ .

(iii)  $\omega \in [\frac{m\pi}{2}, \frac{(m+1)\pi}{2}]$  if and only if  $\Theta(\omega; k) \in [\frac{m\pi}{2}, \frac{(m+1)\pi}{2}]$ .

(iv)  $\Theta(\Theta(\omega; k); l) = \Theta(\omega; kl)$  and  $\Theta(\omega; 1) = \omega$ .

(v)  $\Theta(\omega; k)$  is  $C^\infty$  with respect to both  $\omega$  and  $k$ .

(vi)  $\Delta(\omega + m\pi; k) = \Delta(\omega; k) + m\pi$  for all  $m \in \mathbb{Z}$ .

(vii)  $\Delta(\omega; 0) = \omega$ .

Lastly, with reference to (5.1), we define positive quantities

$$\begin{aligned} \epsilon_- &= (rp)^{1/4}(-a), & \epsilon_+ &= (rp)^{1/4}(b), \\ \sigma_- &= (rp)^{1/4}(0^-), & \sigma_+ &= (rp)^{1/4}(0^+), \\ \xi^- &= \int_{-a}^0 (r(x)/p(x))^{1/2} dx & \text{and} & \quad \xi^+ = \int_0^b (r(x)/p(x))^{1/2} dx. \end{aligned}$$

### 5.3 A modified Prüfer angle

We define a modified Prüfer angle similar to the one used in [14] and [13] as follows.

**Definition 5.3.1.** Let  $\Omega(x, s; \chi, \omega)$  be the angle depending continuously on  $x$  such that

$$\tan \Omega(x, s; \chi, \omega) = \frac{sy(x)}{(py')(x)} \quad \text{and} \quad \Omega(\chi, s; \chi, \omega) = \omega$$

for  $x, \chi \in [-a, 0)$  or  $x, \chi \in (0, b]$ , where  $s = \sqrt{\lambda}$  and  $y$  is the solution to (5.1) (on either  $[-a, 0)$  or  $(0, b]$ ) satisfying  $y(\chi) = \sin \omega$ ,  $(py')(\chi) = s \cos \omega$ .



In particular, for  $x \in [-a, 0)$  we make use of the modified Prüfer angle corresponding to  $u(x; \lambda)$ , namely  $\Omega(x, s; -a, \Theta(\alpha; s)) = \Theta(\theta(x; \lambda); s)$ . For  $x \in (0, b]$  we make use of the modified Prüfer angle corresponding to  $v(x; \lambda)$ , namely  $\Omega(x, s; b, \Theta(\beta; s)) = \Theta(\varphi(x; \lambda); s)$ .

In general, the modified Prüfer angle has the following properties in common with the usual Prüfer angle.

**Lemma 5.3.2.** 1.  $\Omega(x, s; \chi, \omega + m\pi) = \Omega(x, s; \chi, \omega) + m\pi$  for all  $m \in \mathbb{Z}$ .

2.  $\Omega(x, s; \chi, \omega)$  is monotonically increasing in  $\omega$ .

**Proposition 5.3.3.** Let  $y$  be any non-zero solution of (5.1). Let  $s = \sqrt{\lambda} \in \mathbb{R}$ . Then as  $s \rightarrow \infty$ ,

$$\begin{bmatrix} y(0^-) \\ \frac{(py')(0^-)}{s} \end{bmatrix} = \left[ \begin{bmatrix} \frac{\epsilon_-}{\sigma_-} \cos \xi^- s & \frac{1}{\epsilon_- \sigma_-} \sin \xi^- s \\ -\epsilon_- \sigma_- \sin \xi^- s & \frac{\sigma_-}{\epsilon_-} \cos \xi^- s \end{bmatrix} + O\left(\frac{1}{s}\right) \right] \begin{bmatrix} y(-a) \\ \frac{(py')(-a)}{s} \end{bmatrix} \quad (5.16)$$

and

$$\begin{bmatrix} y(0^+) \\ \frac{(py')(0^+)}{s} \end{bmatrix} = \left[ \begin{bmatrix} \frac{\epsilon_+}{\sigma_+} \cos \xi^+ s & -\frac{1}{\epsilon_+ \sigma_+} \sin \xi^+ s \\ \epsilon_+ \sigma_+ \sin \xi^+ s & \frac{\sigma_+}{\epsilon_+} \cos \xi^+ s \end{bmatrix} + O\left(\frac{1}{s}\right) \right] \begin{bmatrix} y(b) \\ \frac{(py')(b)}{s} \end{bmatrix}. \quad (5.17)$$

*Proof.* We refer the reader to [39]. The formulae above can be derived using a similar approach to that used by Hochstadt in the appendix of [39].

**Lemma 5.3.4.** (a) If  $-a \leq x < 0$ , then as  $s \rightarrow \infty$

$$\Omega(0^-, s; -a, \Omega_{-a}) = \Theta\left(\xi^- s + \Theta(\Omega_{-a}, \epsilon_-^2), \frac{1}{\sigma_-^2}\right) + O\left(\frac{1}{s}\right). \quad (5.18)$$

(b) If  $0 < x \leq b$ , then as  $s \rightarrow \infty$

$$\Omega(0^+, s; b, \Omega_b) = \Theta\left(\Theta(\Omega_b, \epsilon_+^2) - \xi^+ s, \frac{1}{\sigma_+^2}\right) + O\left(\frac{1}{s}\right). \quad (5.19)$$

*Proof.* Let  $y$  be as in Definition 5.3.1. Substituting  $y(-a) = \sin \Omega_{-a}$ ,  $(py')(-a) = s \cos \Omega_{-a}$  into equation (5.16) gives

$$\begin{aligned} \begin{bmatrix} y(0^-) \\ \frac{(py')(0^-)}{s} \end{bmatrix} &= \begin{bmatrix} \frac{\epsilon_-}{\sigma_-} \cos \xi^- s \sin \Omega_{-a} + \frac{1}{\epsilon_- \sigma_-} \sin \xi^- s \cos \Omega_{-a} \\ -\epsilon_- \sigma_- \sin \xi^- s \sin \Omega_{-a} + \frac{\sigma_-}{\epsilon_-} \cos \xi^- s \cos \Omega_{-a} \end{bmatrix} + O\left(\frac{1}{s}\right) \\ &= \frac{\sigma_-}{\epsilon_-} \begin{bmatrix} \frac{1}{\sigma_-^2} (\sin \xi^- s \cos \Omega_{-a} + \cos \xi^- s (\epsilon_-^2 \sin \Omega_{-a})) \\ \cos \xi^- s \cos \Omega_{-a} - \sin \xi^- s (\epsilon_-^2 \sin \Omega_{-a}) \end{bmatrix} + O\left(\frac{1}{s}\right) \\ &= \frac{\sigma_-}{\epsilon_-} \begin{bmatrix} \frac{1}{\sigma_-^2} \sin(\xi^- s + \Theta(\Omega_{-a}, \epsilon_-^2)) \\ \cos(\xi^- s + \Theta(\Omega_{-a}, \epsilon_-^2)) \end{bmatrix} + O\left(\frac{1}{s}\right), \end{aligned}$$

from which the first equation follows. Substituting  $y(b) = \sin \Omega_b$  and  $(py')(b) = s \cos \Omega_b$  into equation (5.17), yields the second equation after similar manipulation.  $\square$

## 5.4 Effect of the transfer condition on modified Prüfer angles

From this point onwards, let

$$\Omega_x^- := \Omega(x, s; -a, \Theta(\alpha; s)) = \Theta(\theta(x, \lambda); s), \quad -a \leq x \leq 0 \quad (5.20)$$

denote the modified Prüfer angle corresponding to  $u(x; \lambda)$  and let

$$\Omega_x^+ := \Omega(x, s; b, \Theta(\beta; s)) = \Theta(\varphi(x, \lambda); s), \quad 0 \leq x \leq b \quad (5.21)$$

denote the modified Prüfer angle corresponding to  $v(x; \lambda)$ .

Using the decomposition of the transfer matrix given by equations (5.9) and (5.10) in Section 3.2, we restructure the transfer condition in the following way,

$$\left[ \begin{array}{cc} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{array} \right] \left[ \begin{array}{c} y(0^+) \\ (py')(0^+) \end{array} \right] = \sqrt{\det T} \left[ \begin{array}{cc} \gamma & 0 \\ 0 & 1/\gamma \end{array} \right] \left[ \begin{array}{cc} 1 & \delta \\ 0 & 1 \end{array} \right] \left[ \begin{array}{c} y(0^-) \\ (py')(0^-) \end{array} \right]. \quad (5.22)$$

Before enforcing this modified transfer condition, we examine the effect of the shear matrix,  $\begin{bmatrix} 1 & \delta \\ 0 & 1 \end{bmatrix}$ , on  $\Omega_0^-$  and the effect of the rotation matrix,  $\begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix}$ , on  $\Omega_0^+$  individually. These results are summarised in Lemmas 5.4.1 and 5.4.2 below. We assume throughout that  $\tan^{-1}$  yields values in  $[-\pi/2, \pi/2)$  and that  $\cot^{-1}$  yields values in  $(0, \pi]$ .

**Lemma 5.4.1.** *Let  $\Omega_0^{*-}$  denote the angle obtained after applying the shear matrix to  $[u(0^-), (pu')(0^-)]^T$ , then*

$$\Omega_0^{*-} = \Theta \left( \Delta \left( \Omega_0^- + \frac{1}{2} \tan^{-1} \delta s, -\frac{1}{2} \tan^{-1} \delta s \right), \sec \tan^{-1} \delta s \right). \quad (5.23)$$

Moreover, if  $\Omega_0^- \in [(n - \frac{1}{2})\pi, (n + \frac{1}{2})\pi)$ ,  $n \in \mathbb{N}$  then

$$(i) \quad \Omega_0^{*-} \in [n\pi, (n + \frac{1}{2})\pi) \text{ if } \delta + \frac{u(0^-)}{(pu')(0^-)} \geq 0, \text{ i.e. } \Omega_0^- \in [n\pi - \tan^{-1} \delta s, (n + \frac{1}{2})\pi).$$

$$(ii) \quad \Omega_0^{*-} \in [(n - \frac{1}{2})\pi, n\pi) \text{ if } \delta + \frac{u(0^-)}{(pu')(0^-)} < 0, \text{ i.e. } \Omega_0^- \in [(n - \frac{1}{2})\pi, n\pi - \tan^{-1} \delta s).$$

*Proof.* First of all, writing  $u(0^-) = \rho_0^- \sin \Omega_0^-$ ,  $(pu')(0^-) = \rho_0^- s \cos \Omega_0^-$  and applying the shear matrix gives

$$\begin{aligned} \rho_0^{*-} \begin{bmatrix} \sin \Omega_0^{*-} \\ \cos \Omega_0^{*-} \end{bmatrix} &= \rho_0^- \begin{bmatrix} 1 & \delta s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sin \Omega_0^- \\ \cos \Omega_0^- \end{bmatrix} \\ &= \rho_0^- \begin{bmatrix} \sec(\tan^{-1} \delta s) \sin(\Omega_0^- + \tan^{-1} \delta s) \\ \cos \Omega_0^- \end{bmatrix} \\ &= \rho_0^- \begin{bmatrix} \sec(\tan^{-1} \delta s) \sin \left( (\Omega_0^- + \frac{1}{2} \tan^{-1} \delta s) + \frac{1}{2} \tan^{-1} \delta s \right) \\ \cos \left( (\Omega_0^- + \frac{1}{2} \tan^{-1} \delta s) - \frac{1}{2} \tan^{-1} \delta s \right) \end{bmatrix}, \end{aligned}$$

from which it follows that

$$\Omega_0^{*-} = \Theta \left( \Delta \left( \Omega_0^- + \frac{1}{2} \tan^{-1} \delta s, -\frac{1}{2} \tan^{-1} \delta s \right), \sec \tan^{-1} \delta s \right) + 2k\pi$$

for some  $k \in \mathbb{Z}$ . By Lemma 5.2.3 parts (i) and (vi) we can write the above line as

$$\Omega_0^{*-} = \Theta \left( \Delta \left( (\Omega_0^- + 2k\pi) + \frac{1}{2} \tan^{-1} \delta s, -\frac{1}{2} \tan^{-1} \delta s \right), \sec \tan^{-1} \delta s \right),$$

which amounts to first shifting  $\Omega_0^-$  by  $2k\pi$  and then applying the shear matrix. The shift by  $2k\pi$  is therefore independent of this transformation. Thus it follows by definition of  $\Omega_0^{*-}$  that  $k = 0$ .

Next, let  $\Omega_0^- \in ((n - \frac{1}{2})\pi, (n + \frac{1}{2})\pi)$  for some  $n \in \mathbb{N}$ . Observe that

$$\begin{aligned} \tan \Delta \left( \Omega_0^- + \frac{1}{2} \tan^{-1} \delta s, -\frac{1}{2} \tan^{-1} \delta s \right) &= \frac{\sin(\Omega_0^- + \tan^{-1} \delta s)}{\cos \Omega_0^-} \\ &= \cos \tan^{-1} \delta s [\tan \Omega_0^- + \delta s], \end{aligned}$$

which for brevity will be denoted  $\tan \Delta$ . Since  $\cos \tan^{-1} \delta s > 0$ ,  $\tan \Delta$  is, for fixed  $\delta$ , a continuous function of  $s$  with exactly one intercept which occurs when  $\tan \Omega_0^- + \delta s = 0$  - i.e. when  $\frac{u(0^-)}{(pv')(0^-)} + \delta = 0$ . Hence, the range of values of  $\Delta$  must occupy only one period of the  $\tan$  graph. Since for  $\Omega_0^- = n\pi$ ,  $\Omega_0^- + \tan^{-1} \delta s \in (n\pi - \pi/2, n\pi + \pi/2)$ ,  $\Delta$  must occupy a range of values within  $((n - \frac{1}{2})\pi, (n + \frac{1}{2})\pi)$ . So if  $\delta + \frac{u(0^-)}{(pv')(0^-)} \geq 0$  then  $\Delta \in [n\pi, (n + \frac{1}{2})\pi)$  and if  $\delta + \frac{u(0^-)}{(pv')(0^-)} < 0$  then  $\Delta \in ((n - \frac{1}{2})\pi, n\pi)$ . Moreover,  $\tan \Delta \rightarrow -\infty$  as  $\Omega_0^- \rightarrow (n - \frac{1}{2})\pi$ . Thus by Lemma 5.2.3 (4), we conclude that, for  $\Omega_0^- \in [(n - \frac{1}{2})\pi, (n + \frac{1}{2})\pi)$ ,  $\delta + \frac{u(0^-)}{(pv')(0^-)} \geq 0$  implies that  $\Omega_0^{*-} \in [n\pi, (n + \frac{1}{2})\pi)$  and  $\delta + \frac{u(0^-)}{(pv')(0^-)} < 0$  implies that  $\Omega_0^{*-} \in [(n - \frac{1}{2})\pi, n\pi)$ .  $\square$

**Lemma 5.4.2.** *Let  $\Omega_0^{*+}$  denote the angle obtained after applying the rotation matrix to  $(v(0^+), (pv')(0^+))^T$ , then*

$$\Omega_0^{*+} = \Theta \left( \phi + \Theta \left( \Omega_0^+, \frac{1}{s} \right), s \right). \quad (5.24)$$

Moreover, if  $m \in \mathbb{Z}$  and if

(i)  $\phi \in [-\pi, -\frac{\pi}{2})$  then

$$\begin{aligned} \Omega_0^+ &\in \left[ (m-1)\pi + \Theta \left( \frac{\pi}{2} - \phi, s \right), \left( m + \frac{1}{2} \right) \pi \right) \cup \left[ \left( m + \frac{1}{2} \right) \pi, m\pi + \Theta(-\phi, s) \right) \\ \Rightarrow \Omega_0^{*+} &\in \left[ \left( m - \frac{1}{2} \right) \pi, m\pi \right), \text{ with } \cot \phi \leq \frac{v(0^+)}{(pv')(0^+)} \text{ or } \frac{v(0^+)}{(pv')(0^+)} < -\tan \phi; \end{aligned}$$

$$\begin{aligned} \Omega_0^+ &\in [m\pi + \Theta(-\phi, s), (m+1)\pi) \cup \left[ (m+1)\pi, m\pi + \Theta \left( \frac{\pi}{2} - \phi, s \right) \right) \\ \Rightarrow \Omega_0^{*+} &\in \left[ m\pi, \left( m + \frac{1}{2} \right) \pi \right), \text{ with } -\tan \phi \leq \frac{v(0^+)}{(pv')(0^+)} < \cot \phi; \end{aligned}$$

(ii)  $\phi \in [-\frac{\pi}{2}, 0)$  then

$$\begin{aligned} \Omega_0^+ &\in \left[ (m-1)\pi + \Theta \left( \frac{\pi}{2} - \phi, s \right), m\pi \right) \cup [m\pi, m\pi + \Theta(-\phi, s)) \\ \Rightarrow \Omega_0^{*+} &\in \left[ \left( m - \frac{1}{2} \right) \pi, m\pi \right), \text{ with } \cot \phi \leq \frac{v(0^+)}{(pv')(0^+)} < -\tan \phi; \end{aligned}$$

$$\begin{aligned}\Omega_0^+ &\in \left[ m\pi + \Theta(-\phi, s), \left(m + \frac{1}{2}\right)\pi \right) \cup \left[ \left(m + \frac{1}{2}\right)\pi, m\pi + \Theta\left(\frac{\pi}{2} - \phi, s\right) \right) \\ \Rightarrow \Omega_0^{*+} &\in \left[ m\pi, \left(m + \frac{1}{2}\right)\pi \right), \text{ with } -\tan\phi \leq \frac{v(0^+)}{(pv')(0^+)} \text{ or } \frac{v(0^+)}{(pv')(0^+)} < \cot\phi;\end{aligned}$$

(iii)  $\phi \in [0, \frac{\pi}{2})$  then

$$\begin{aligned}\Omega_0^+ &\in \left[ (m-1)\pi + \Theta\left(\frac{\pi}{2} - \phi, s\right), \left(m - \frac{1}{2}\right)\pi \right) \cup \left[ \left(m - \frac{1}{2}\right)\pi, m\pi + \Theta(-\phi, s) \right) \\ \Rightarrow \Omega_0^{*+} &\in \left[ \left(m - \frac{1}{2}\right)\pi, m\pi \right), \text{ with } \cot\phi \leq \frac{v(0^+)}{(pv')(0^+)} \text{ or } \frac{v(0^+)}{(pv')(0^+)} < -\tan\phi;\end{aligned}$$

$$\begin{aligned}\Omega_0^+ &\in [m\pi + \Theta(-\phi, s), m\pi) \cup [m\pi, m\pi + \Theta\left(\frac{\pi}{2} - \phi, s\right)) \\ \Rightarrow \Omega_0^{*+} &\in \left[ m\pi, \left(m + \frac{1}{2}\right)\pi \right), \text{ with } -\tan\phi \leq \frac{v(0^+)}{(pv')(0^+)} < \cot\phi;\end{aligned}$$

(iv)  $\phi \in [\frac{\pi}{2}, \pi)$  then

$$\begin{aligned}\Omega_0^+ &\in \left[ (m-1)\pi + \Theta\left(\frac{\pi}{2} - \phi, s\right), (m-1)\pi \right) \cup [(m-1)\pi, m\pi + \Theta(-\phi, s)) \\ \Rightarrow \Omega_0^{*+} &\in \left[ \left(m - \frac{1}{2}\right)\pi, m\pi \right), \text{ with } \cot\phi \leq \frac{v(0^+)}{(pv')(0^+)} < -\tan\phi;\end{aligned}$$

$$\begin{aligned}\Omega_0^+ &\in \left[ m\pi + \Theta(-\phi, s), \left(m - \frac{1}{2}\right)\pi \right) \cup \left[ \left(m - \frac{1}{2}\right)\pi, m\pi + \Theta\left(\frac{\pi}{2} - \phi, s\right) \right) \\ \Rightarrow \Omega_0^{*+} &\in \left[ m\pi, \left(m + \frac{1}{2}\right)\pi \right), \text{ with } -\tan\phi \leq \frac{v(0^+)}{(pv')(0^+)} \text{ or } \frac{v(0^+)}{(pv')(0^+)} < \cot\phi.\end{aligned}$$

*Proof.* Writing  $v(0^+) = \rho_0^+ \sin \Omega_0^+$ ,  $(pv')(0^+) = \rho_0^+ s \cos \Omega_0^+$  and applying the rotation matrix gives

$$\begin{aligned}\rho_0^{*+} \begin{bmatrix} \sin \Omega_0^{*+} \\ \cos \Omega_0^{*+} \end{bmatrix} &= \rho_0^+ \begin{bmatrix} \cos \phi & s \sin \phi \\ -\frac{1}{s} \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} \sin \Omega_0^+ \\ \cos \Omega_0^+ \end{bmatrix} \\ &= \rho_0^+ \begin{bmatrix} s (\sin \phi \cos \Omega_0^+ + \cos \phi \frac{1}{s} \sin \Omega_0^+) \\ \cos \phi \cos \Omega_0^+ - \sin \phi \frac{1}{s} \sin \Omega_0^+ \end{bmatrix} \\ &= \frac{\rho_0^+}{\sqrt{s^2 \sin^2 \phi + \cos^2 \phi}} \begin{bmatrix} s \sin(\phi + \varphi(0, \lambda)) \\ \cos(\phi + \varphi(0, \lambda)) \end{bmatrix} \\ &= \frac{\rho_0^+ \sqrt{s^2 \sin^2(\phi + \varphi) + \cos^2(\phi + \varphi)}}{\sqrt{s^2 \sin^2 \phi + \cos^2 \phi}} \begin{bmatrix} \sin \Theta(\phi + \varphi(0, \lambda), s) \\ \cos \Theta(\phi + \varphi(0, \lambda), s) \end{bmatrix},\end{aligned}$$

where

$$\sin \Omega_0^+ = \sin \Theta(\varphi(0, \lambda), s) = \frac{s \sin \varphi(0, \lambda)}{\sqrt{s^2 \sin^2 \varphi + \cos^2 \varphi}},$$

and

$$\cos \Omega_0^+ = \cos \Theta(\varphi(0, \lambda), s) = \frac{\cos \varphi(0, \lambda)}{\sqrt{s^2 \sin^2 \varphi + \cos^2 \varphi}},$$

by definition of  $\Theta$ . Since  $\Theta(\Omega_0^+, \frac{1}{s}) = \varphi(0, \lambda)$  by Lemma 5.2.3, 5, it follows from the above that

$$\begin{aligned}\Omega_0^{*+} &= \Theta(\phi + \varphi(0, \lambda), s) + 2k\pi \\ &= \Theta\left(\phi + \Theta\left(\Omega_0^+, \frac{1}{s}\right), s\right) + 2k\pi,\end{aligned}$$

for some  $k \in \mathbb{Z}$ . But  $k = 0$  by definition of  $\Omega_0^{*+}$ .

For the angle decompositions, we only give the proof for  $\phi \in [-\pi, -\frac{\pi}{2})$ , as the other cases are similar. Consider

$$\tan \Omega_0^{*+} = \frac{s \left( \tan \phi + \frac{v(0^+)}{(pv')(0^+)} \right)}{1 - \tan \phi \frac{v(0^+)}{(pv')(0^+)}}.$$

Zeroes of  $\tan \Omega_0^{*+}$  occur when  $\varphi(0; \lambda) = m\pi - \phi$ , for some  $m \in \mathbb{Z}$  (i.e. when  $\tan \varphi(0; \lambda) = -\tan \phi$ ). Poles occur when  $\varphi(0; \lambda) = (m - 1/2)\pi - \phi$ , for some  $m \in \mathbb{Z}$  (i.e. when  $\tan \varphi(0; \lambda) = \cot \phi$ ). We consider four separate cases determined by the relative geometry of the graphs of  $y = \tan \varphi(0; \lambda)$ ,  $y = -\tan \phi$ ,  $y = 0$  and  $y = \cot \phi$ . If  $\phi = \pi$  then case II below dissolves.

- I. If  $\frac{v(0^+)}{(pv')(0^+)} < -\tan \phi$  then  $\varphi(0, \lambda) \in [(m + 1/2)\pi, m\pi - \phi)$  for some  $m \in \mathbb{Z}$ ,  
 $\Rightarrow \phi + \varphi(0, \lambda) \in [(m + 1/2)\pi + \phi, m\pi) \subset [(m - 1/2)\pi, m\pi)$ .
- II. If  $-\tan \phi \leq \frac{v(0^+)}{(pv')(0^+)} < 0$  then  $\varphi(0, \lambda) \in [m\pi - \phi, (m + 1)\pi)$  for some  $m \in \mathbb{Z}$ ,  
 $\Rightarrow \phi + \varphi(0, \lambda) \in [m\pi, (m + 1)\pi + \phi) \subset [m\pi, (m + 1/2)\pi)$ .
- III. If  $0 \leq \frac{v(0^+)}{(pv')(0^+)} < \cot \phi$  then  $\varphi(0, \lambda) \in [(m + 1)\pi, (m + 1/2)\pi - \phi)$  for some  $m \in \mathbb{Z}$ ,  
 $\Rightarrow \phi + \varphi(0, \lambda) \in [(m + 1)\pi + \phi, (m + 1/2)\pi) \subset [m\pi, (m + 1/2)\pi)$ .
- IV. If  $\cot \phi \leq \frac{v(0^+)}{(pv')(0^+)}$  then  $\varphi(0, \lambda) \in [(m - 1/2)\pi - \phi, (m + 1/2)\pi]$  for some  $m \in \mathbb{Z}$ ,  
 $\Rightarrow \phi + \varphi(0, \lambda) \in [(m - 1/2)\pi, (m + 1/2)\pi + \phi) \subset [(m - 1/2)\pi, m\pi)$ .

□

## 5.5 Generalized oscillation counts and asymptotics of eigenvalues

This section contains our main results. Here, we determine generalized oscillation counts of eigenfunctions, taking into account the effect of the transmission condition at  $x = 0$ . By “generalized oscillation count” of an eigenfunction we mean the sum of the number of zeroes of the eigenfunction in  $(-a, 0) \cup (0, b)$  together with so called “half zeroes” at  $x = 0$ . These half zeroes occur when either  $u(0^-; \lambda) = 0$  or  $v(0^+; \lambda) = 0$  at an eigenvalue  $\lambda$ , and each contribute 1/2 to the total count. Furthermore, we consider the problem of indexing eigenvalues in terms of generalized oscillation counts. We give asymptotics for eigenvalues up to order  $1/(N + M)$ , where  $N + M$  is the value of the generalized oscillation count of the corresponding eigenvalue.

**Theorem 5.5.1.** Let  $\Omega_0^{*-}$  and  $\Omega_0^{*+}$  be defined as in Lemmas 5.4.1 and 5.4.2. Then the eigencondition (5.15) written in terms of the modified Prüfer angles is

$$\Theta\left(\Omega_0^{*+}; \frac{1}{s}\right) = \Theta\left(\Omega_0^{*-}; \frac{\gamma^2}{s}\right) - k\pi. \quad (5.25)$$

Moreover, there is precisely one solution to (5.25) for each  $k \in \mathbb{N}$ , large enough.

*Proof.* The first claim follows directly from equations (5.23) and (5.24) after straightforward manipulation.

To prove the second assertion, we need the following results concerning the usual Prüfer angles  $\theta$  and  $\varphi$ . Derivations of equations (5.26) and (5.27) can be found in [8], Theorem 8.4.2, while (5.28) and (5.29) can be proved in a similar manner.

If  $p(0^-)u'(0^-; \lambda) \neq 0$ ,

$$\frac{\partial}{\partial \lambda} \tan \theta(0; \lambda) = [p(0^-)u'(0^-; \lambda)]^{-2} \int_{-a}^0 r(t) [u(t; \lambda)]^2 dt, \quad (5.26)$$

while if  $u(0^-; \lambda) \neq 0$ ,

$$\frac{\partial}{\partial \lambda} \cot \theta(0; \lambda) = -[u(0^-; \lambda)]^{-2} \int_{-a}^0 r(t) [u(t; \lambda)]^2 dt. \quad (5.27)$$

If  $p(0^+)v'(0^+; \lambda) \neq 0$ ,

$$\frac{\partial}{\partial \lambda} \tan \varphi(0; \lambda) = -[p(0^+)v'(0^+; \lambda)]^{-2} \int_0^b r(t) [v(t; \lambda)]^2 dt, \quad (5.28)$$

while if  $v(0^+; \lambda) \neq 0$ ,

$$\frac{\partial}{\partial \lambda} \cot \varphi(0; \lambda) = [v(0^+; \lambda)]^{-2} \int_0^b r(t) [v(t; \lambda)]^2 dt. \quad (5.29)$$

From  $\tan \Theta\left(\Omega_0^{*-}; \frac{\gamma^2}{s}\right) = \gamma^2 [\tan \theta(0; \lambda) + \delta]$ , we obtain

$$\frac{\partial}{\partial s} \Theta\left(\Omega_0^{*-}; \frac{\gamma^2}{s}\right) = \cos^2 \Theta\left(\Omega_0^{*-}; \frac{\gamma^2}{s}\right) 2s\gamma^2 \frac{\partial}{\partial \lambda} \tan \theta(0; \lambda) > 0, \quad (5.30)$$

if  $p(0^-)u'(0^-; \lambda) \neq 0$ . Note that  $\cos \Theta\left(\Omega_0^{*-}; \frac{\gamma^2}{s}\right) = 0$  if and only if  $\cos \theta(0; \lambda) = 0$ , so the above expression is indeed non-zero. Whereas, if  $u(0^-; \lambda) \neq 0$  then

$$\frac{\partial}{\partial s} \Theta\left(\Omega_0^{*-}; \frac{\gamma^2}{s}\right) = \frac{-\sin^2 \Theta\left(\Omega_0^{*-}; \frac{\gamma^2}{s}\right)}{\gamma^2 [1 + \delta \cot \theta(0; \lambda)]^2} 2s \frac{\partial}{\partial \lambda} \cot \theta(0; \lambda). \quad (5.31)$$

From (5.31), we are only interested in the case where  $p(0^-)u'(0^-; \lambda) = 0$ . Setting  $\cos \theta(0; \lambda) = 0$  in (5.31) and observing that  $\sin \Theta\left(\Omega_0^{*-}; \frac{\gamma^2}{s}\right) \neq 0$  we obtain the desired result. Hence,  $\frac{\partial}{\partial s} \Theta\left(\Omega_0^{*-}; \frac{\gamma^2}{s}\right)$  is positive for all  $s$ .

On the other hand, differentiating  $\Theta\left(\Omega_0^{*+}; \frac{1}{s}\right) = (\phi + \varphi(0; \lambda))$  gives

$$\frac{\partial}{\partial s} \Theta\left(\Omega_0^{*+}; \frac{1}{s}\right) = 2s \cos^2 \varphi(0; \lambda) \frac{\partial}{\partial \lambda} \tan \varphi(0; \lambda) < 0, \quad (5.32)$$

if  $p(0^+)v'(0^+; \lambda) \neq 0$ , else

$$\frac{\partial}{\partial s} \Theta\left(\Omega_0^{*+}; \frac{1}{s}\right) = -2s \sin^2 \varphi(0; \lambda) \frac{\partial}{\partial \lambda} \cot \varphi(0; \lambda) < 0, \quad (5.33)$$

if  $v(0^+; \lambda) \neq 0$ .

From standard properties of the Prüfer angle we know that  $\theta(0; \lambda) \rightarrow 0$  as  $\lambda \rightarrow -\infty$ , and that  $\theta(0; \lambda)$  is monotonically increasing in  $\lambda$ , with  $\theta(0; \lambda) \rightarrow \infty$  as  $\lambda \rightarrow \infty$ . See for example [8, Section 8.4]. From these results we deduce that for each  $n \in \mathbb{N}$ , large there is  $\lambda_{\sigma(n)}^{N-} > 0$  such  $\lambda \in [\lambda_{\sigma(n)}^{N-}, \lambda_{\sigma(n+1)}^{N-})$  implies that  $\theta(0; \lambda) \in [(n-1/2)\pi, (n+1/2)\pi)$ . From part (iii) of Lemma 5.2.3 we have  $\Omega_0^- = \Theta(\theta(0; \lambda); s) \in [(n-1/2)\pi, (n+1/2)\pi)$  for  $s = \sqrt{\lambda} \in \left[\sqrt{\lambda_{\sigma(n)}^{N-}}, \sqrt{\lambda_{\sigma(n+1)}^{N-}}\right)$ . Finally, Lemma 5.4.1 combined with part (iii) of Lemma 5.2.3 shows that  $\Theta\left(\Omega_0^{*-}; \frac{\gamma^2}{s}\right) \geq (n-1/2)\pi$  for  $s \geq \sqrt{\lambda_{\sigma(n)}^{N-}}$ . So  $\Theta\left(\Omega_0^{*-}; \frac{\gamma^2}{s}\right) \rightarrow \infty$  as  $s \rightarrow \infty$ .

Since  $\varphi(x; \lambda)$  is initialized by  $\varphi(b; \lambda) = \beta$ , with  $\beta \in (0, \pi]$ , at the right-endpoint of the interval  $(0, b)$ , standard Prüfer theory can be adapted to show that  $\varphi(0; \lambda) \rightarrow \pi$  as  $\lambda \rightarrow -\infty$ , and that  $\varphi(0; \lambda)$  is monotonically decreasing in  $\lambda$ , with  $\varphi(0; \lambda) \rightarrow -\infty$  as  $\lambda \rightarrow \infty$ . To show that  $\Theta\left(\Omega_0^{*+}; \frac{1}{s}\right) \rightarrow -\infty$  as  $s \rightarrow \infty$  we consider only the case when  $\phi \in [-\pi, \pi/2)$  as the remaining cases are similar. For each  $m \in \mathbb{Z}$ , with  $-m$  large, there are  $\lambda_{\tau(m)}^{D+}, \lambda_{\tau(m+1)}^{N+} > 0$  such that  $\lambda \in \left(\lambda_{\tau(m)}^{D+}, \lambda_{\tau(m+1)}^{N+}\right]$  implies that  $\varphi(0; \lambda) \in [(m+1/2)\pi, (m+1)\pi)$ . By part (iii) of Lemma 5.2.3 we have  $\Omega_0^+ \in [(m+1/2)\pi, (m+1)\pi)$  for  $s = \sqrt{\lambda} \in \left(\sqrt{\lambda_{\tau(m)}^{D+}}, \sqrt{\lambda_{\tau(m+1)}^{N+}}\right]$ . By applying Lemma 5.4.2 together with part (iii) of Lemma 5.2.3 we see that  $\Theta\left(\Omega_0^{*+}; \frac{1}{s}\right) \in [(m-1/2)\pi, (m+1/2)\pi)$  for  $s \in \left(\sqrt{\lambda_{\tau(m)}^{D+}}, \sqrt{\lambda_{\tau(m+1)}^{N+}}\right]$ . Note, if  $\Omega_0^+ = (m+1)\pi$  then  $\Omega_0^{*+} = \Theta\left(\phi + (m+1)\pi; \sqrt{\lambda_{\tau(m)}^{D+}}\right) \in [m\pi, (m+1/2)\pi)$ . Thus, in particular, we have shown that  $s > \sqrt{\lambda_{\tau(m)}^{D+}}$  implies that  $\Theta\left(\Omega_0^{*+}; \frac{1}{s}\right) < (m+1/2)\pi$ .

Since  $\Theta\left(\Omega_0^{*-}; \frac{\gamma^2}{s}\right)$  is monotonically increasing with  $\Theta\left(\Omega_0^{*-}; \frac{\gamma^2}{s}\right) \rightarrow \infty$  as  $\lambda \rightarrow \infty$ , whilst  $\Theta\left(\Omega_0^{*+}; \frac{1}{s}\right)$  is monotonically decreasing with  $\Theta\left(\Omega_0^{*+}; \frac{1}{s}\right) \rightarrow -\infty$  as  $\lambda \rightarrow \infty$ , the result follows.  $\square$

The next theorem gives asymptotics for the eigenvalues of (5.1)-(5.4) in terms of the number of zeroes of the eigenfunctions in  $(-a, b)$ , with special treatment at  $x = 0$  as follows: if  $u(0^-; \lambda) = 0$  for the eigenvalue  $\lambda$  then this counts as half a zero, likewise  $v(0^+; \lambda) = 0$  counts as half a zero. If  $\tan \phi = \gamma^2 \delta$ , an eigenfunction will have either two half zeroes at  $x = 0$  or no half zeroes (since  $u(0^-; \lambda) = 0$  if and only if  $v(0^+; \lambda) = 0$  in this case). Otherwise, if  $\tan \phi \neq \gamma^2 \delta$ , an eigenfunction will have at most one half zero at  $x = 0$ .

**Theorem 5.5.2.** *Let  $\xi = \xi^- + \xi^+$ , where  $\xi^-$  and  $\xi^+$  are defined at the end of section 5.2. Let  $s_{N,M}^2 = \lambda_{N,M}$  denote an eigenvalue of (5.1)-(5.4) with oscillation count  $N$  in  $(-a, 0]$  (including a possible half zero when  $u(0^-) = 0$ ) and  $M$  in  $[0, b)$  (including a possible half zero when*

$v(0^+) = 0$ ). Here  $2N, 2M \in \mathbb{N}_0$ . Then

$$s_{N,M} = \frac{(N+M)\pi + C}{\xi} + O\left(\frac{1}{N+M}\right), \quad (5.34)$$

where the constant  $C$  is given in the table below.

$C$	$\beta \in (0, \pi)$	$\beta = \pi$
$\alpha = 0$	$\pi/2$	$\pi$
$\alpha \in (0, \pi)$	$0$	$\pi/2$

*Proof.* Let  $k \in \mathbb{N}$ , large. From Theorem 5.5.1 we have that

$$\Theta\left(\Omega_0^{*+}; \frac{1}{s}\right) = \Theta\left(\Omega_0^{*-}; \frac{\gamma^2}{s}\right) - k\pi, \quad (5.35)$$

has a unique solution for some  $s = \sqrt{\lambda}$ , where  $\lambda$  is an eigenvalue of (5.1)-(5.4). At the point of intersection assume that  $\Omega_0^{*-} \in [(n-1/2)\pi, (n+1/2)\pi)$ , for some  $n \in \mathbb{N}$ , and that  $\Omega_0^{*+} \in [(m-1/2)\pi, (m+1/2)\pi)$ , for some  $m \in \mathbb{Z}$ . Then

$$0 \leq |k - (n - m)|\pi = \left| \Theta\left(\Omega_0^{*-} - n\pi, \frac{\gamma^2}{s}\right) - \Theta\left(\Omega_0^{*+} - m\pi; \frac{1}{s}\right) \right| \leq \frac{\pi}{2},$$

since  $\Theta\left(\Omega_0^{*-} - n\pi, \frac{\gamma^2}{s}\right), \Theta\left(\Omega_0^{*+} - m\pi; \frac{1}{s}\right) \in [-\frac{\pi}{2}, \frac{\pi}{2})$  and have the same sign, implying that  $n - m = k$ .

Using Lemmas 5.4.1 and 5.4.2 and considering the relative geometry of the graphs of  $y = \tan(\varphi(0; \lambda) + \phi)$  and  $y = \gamma^2(\tan\theta(0; \lambda) + \delta)$  at the point of intersection  $\lambda$ , we show how the counts  $N$  and  $M$  are related to  $n$  and  $m$  above. Knowing this, asymptotics for the eigenvalue is easily determined from Lemma 5.3.4. This is shown only for the case of  $\phi \in [-\pi, -\frac{\pi}{2})$ , as all other cases are similar.

Since  $\Omega_0^{*-} \in [(n-1/2)\pi, (n+1/2)\pi)$  by the above assumption, it follows from Lemma 5.4.1 that  $\Omega_0^- \in [(n-\frac{1}{2})\pi, (n+\frac{1}{2})\pi)$ .

Case I:  $\tan\phi < \gamma^2\delta$

(a) Suppose that  $\Omega_0^{*-} \in [(n-\frac{1}{2})\pi, n\pi)$  and  $\Omega_0^{*+} \in [(m-\frac{1}{2})\pi, m\pi)$ . Then  $\gamma^2\left(\frac{u(0^-)}{(pu')(0^-)} + \delta\right) < 0$  by Lemma 5.4.1. There are two sub-cases:

- (i)  $-\cot\phi \leq \gamma^2\left(\frac{u(0^-)}{(pu')(0^-)} + \delta\right) < 0$ . For an intersection we need  $\frac{v(0^+)}{(pv')(0^+)} < -\tan\phi$  by Lemma 5.4.2, in which case  $\Omega_0^+ \in [(m+\frac{1}{2})\pi, m\pi + \Theta(-\phi, s)] \subset [(m+\frac{1}{2})\pi, (m+1)\pi)$ . Moreover,  $\Omega_0^- \in [(n-1/2)\pi, n\pi - \tan^{-1}\delta s] \subset [(n-1/2)\pi, n\pi)$  by Lemma 5.4.1. Notice that the total oscillation count (i.e. the oscillation count in  $(-a, b) = (-a, 0] \cup [0, b)$ ) is  $(n-1) - m = k-1$ .



Thus

$$\begin{aligned}
\xi^- s &= \Theta(\Omega_0^-, \sigma_-^2) - \Theta(\alpha, \epsilon_-^2 s) + O\left(\frac{1}{s}\right) \\
&= n\pi + \arctan\left(s \frac{u(0^-)}{(pu')(0^-)} \sigma_-^2\right) - \Theta(\alpha, \epsilon_-^2 s) + O\left(\frac{1}{s}\right) \\
&= \begin{cases} n\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \\ n\pi - \frac{\pi}{2} - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha \in (0, \pi), \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\xi^+ s &= \Theta(\beta, \epsilon_+^2 s) - \Theta(\Omega_0^+, \sigma_+^2) + O\left(\frac{1}{s}\right) \\
&= \Theta(\beta, \epsilon_+^2 s) - \left\{ (m+1)\pi + \arctan\left(s \frac{v(0^+)}{(pv')(0^+)} \sigma_+^2\right) \right\} + O\left(\frac{1}{s}\right) \\
&= \begin{cases} \frac{\pi}{2} - m\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \beta \in (0, \pi), \\ \pi - m\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \beta = \pi. \end{cases}
\end{aligned}$$

Hence,

$$\xi s_{n-1, -m} = \begin{cases} (n-1-m)\pi + O\left(\frac{1}{n-1-m}\right), & \text{if } \alpha, \beta \in (0, \pi), \\ (n-1-m)\pi + \pi + O\left(\frac{1}{n-1-m}\right), & \text{if } \alpha = 0, \beta = \pi, \\ (n-1-m)\pi + \frac{\pi}{2} + O\left(\frac{1}{n-1-m}\right), & \text{otherwise.} \end{cases} \quad (5.36)$$

- (ii)  $\gamma^2 \left( \frac{u(0^-)}{(pu')(0^-)} + \delta \right) < -\cot \phi$ , in which case  $\cot \phi \leq \frac{v(0^+)}{(pv')(0^+)}$  and  $\Omega_0^+ \in [(m-1)\pi + \Theta(\frac{\pi}{2} - \phi, s), (m + \frac{1}{2})\pi) \subset (m\pi, (m + \frac{1}{2})\pi)$  by Lemma 5.4.2 and, again,  $\Omega_0^- \in [(n - \frac{1}{2})\pi, n\pi)$ . Here the total oscillation count is again  $(n-1) - m = k-1$ .

Then,

$$\begin{aligned}
\xi^- s &= \Theta(\Omega_0^-, \sigma_-^2) - \Theta(\alpha, \epsilon_-^2 s) + O\left(\frac{1}{s}\right) \\
&= n\pi + \arctan\left(s \frac{u(0^-)}{(pu')(0^-)} \sigma_-^2\right) - \Theta(\alpha, \epsilon_-^2 s) + O\left(\frac{1}{s}\right) \\
&= \begin{cases} n\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \\ n\pi - \frac{\pi}{2} - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha \in (0, \pi), \end{cases}
\end{aligned}$$

(as before) and

$$\begin{aligned}
\xi^+ s &= \Theta(\beta, \epsilon_+^2 s) - \Theta(\Omega_0^+, \sigma_+^2) + O\left(\frac{1}{s}\right) \\
&= \Theta(\beta, \epsilon_+^2 s) - \left\{ m\pi + \arctan\left(s \frac{v(0^+)}{(pv')(0^+)} \sigma_+^2\right) \right\} + O\left(\frac{1}{s}\right) \\
&= \begin{cases} \frac{\pi}{2} - m\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \beta \in (0, \pi), \\ \pi - m\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \beta = \pi. \end{cases}
\end{aligned}$$

In this case we get the same asymptotics as (5.36), above.

(b) On the other hand, if  $\Omega_0^{*-} \in [n\pi, (n + \frac{1}{2})\pi)$  and  $\Omega_0^{*+} \in [m\pi, (m + \frac{1}{2})\pi)$  for some  $m \in \mathbb{Z}$  with  $k = n - m$  then  $\gamma^2 \left( \frac{u(0^-)}{(pu')(0^-)} + \delta \right) \geq 0$  by Lemma 5.4.1. In this instance there are five possible subcases:

- (i)  $0 \leq \gamma^2 \left( \frac{u(0^-)}{(pu')(0^-)} + \delta \right) < \tan \phi$ , in which case  $-\tan \phi \leq \frac{v(0^+)}{(pv')(0^+)} < 0$  and  $\Omega_0^+ \in [m\pi + \Theta(-\phi, s), (m + 1)\pi) \subset ((m + \frac{1}{2})\pi, (m + 1)\pi)$  by Lemma 5.4.2. From this we get that

$$\gamma^2 \frac{u(0^-)}{(pu')(0^-)} < \tan \phi - \gamma^2 \delta < 0,$$

giving  $\Omega_0^- \in [n\pi - \tan^{-1} \delta s, n\pi) \subset ((n - 1/2)\pi, n\pi)$ . Thus, we deduce that the oscillation count is  $n - 1 - m = k - 1$ .

Then

$$\begin{aligned} \xi^- s &= \Theta(\Omega_0^-, \sigma_-^2) - \Theta(\alpha, \epsilon_-^2 s) + O\left(\frac{1}{s}\right) \\ &= n\pi + \arctan\left(s \frac{u(0^-)}{(pu')(0^-)} \sigma_-^2\right) - \Theta(\alpha, \epsilon_-^2 s) + O\left(\frac{1}{s}\right) \\ &= \begin{cases} n\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \\ n\pi - \frac{\pi}{2} - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha \in (0, \pi), \end{cases} \end{aligned}$$

and

$$\begin{aligned} \xi^+ s &= \Theta(\beta, \epsilon_+^2 s) - \Theta(\Omega_0^+, \sigma_+^2) + O\left(\frac{1}{s}\right) \\ &= \Theta(\beta, \epsilon_+^2 s) - \left\{ (m + 1)\pi + \arctan\left(s \frac{v(0^+)}{(pv')(0^+)} \sigma_+^2\right) \right\} + O\left(\frac{1}{s}\right) \\ &= \begin{cases} \frac{\pi}{2} - m\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \beta \in (0, \pi), \\ \pi - m\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \beta = \pi. \end{cases} \end{aligned}$$

As before we get,

$$\xi s_{n-1, -m} = \begin{cases} (n - 1 - m)\pi + O\left(\frac{1}{n-1-m}\right), & \text{if } \alpha, \beta \in (0, \pi), \\ (n - 1 - m)\pi + \pi + O\left(\frac{1}{n-1-m}\right), & \text{if } \alpha = 0, \beta = \pi, \\ (n - 1 - m)\pi + \frac{\pi}{2} + O\left(\frac{1}{n-1-m}\right), & \text{otherwise.} \end{cases}$$

- (ii)  $\gamma^2 \left( \frac{u(0^-)}{(pu')(0^-)} + \delta \right) = \tan \phi$ . In this case  $v(0^+) = 0$  i.e.  $\Omega_0^+ = (m + 1)\pi$  by Lemma 5.4.2 and  $\Omega_0^- \in [n\pi - \tan^{-1} \delta s, n\pi) \subset ((n - \frac{1}{2})\pi, n\pi)$  as in (b)(i) above. Here, the total count is  $(n - 1) - (m + 1) + 1/2 = (k - 1) - 1/2$ , corresponding to usual oscillation counts  $(n - 1)$  and  $-(m + 1)$  in  $(-a, 0)$  and  $(0, b)$  respectively, together with an extra half-zero at  $x = 0$ .

Then

$$\xi^- s = \begin{cases} n\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \\ n\pi - \frac{\pi}{2} - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha \in (0, \pi), \end{cases}$$

and

$$\begin{aligned}
\xi^+ s &= \Theta(\beta, \epsilon_+^2 s) - \Theta(\Omega_0^+, \sigma_+^2) + O\left(\frac{1}{s}\right) \\
&= \Theta(\beta, \epsilon_+^2 s) - (m+1)\pi + O\left(\frac{1}{s}\right) \\
&= \begin{cases} \frac{\pi}{2} - (m+1)\pi + O\left(\frac{1}{s}\right), & \text{if } \beta \in (0, \pi), \\ \pi - (m+1)\pi + O\left(\frac{1}{s}\right), & \text{if } \beta = \pi, \end{cases}
\end{aligned}$$

giving

$$\begin{aligned}
&\xi s_{n-1, -(m+1)+1/2} \\
&= \begin{cases} (n-1 - (m+1) + 1/2)\pi + O\left(\frac{1}{n-1-m}\right), & \text{if } \alpha, \beta \in (0, \pi), \\ (n-1 - (m+1) + 1/2)\pi + \pi + O\left(\frac{1}{n-1-m}\right), & \text{if } \alpha = 0, \beta = \pi, \\ (n-1 - (m+1) + 1/2)\pi + \frac{\pi}{2} + O\left(\frac{1}{n-1-m}\right), & \text{otherwise.} \end{cases} \quad (5.37)
\end{aligned}$$

- (iii)  $\tan \phi < \gamma^2 \left( \frac{u(0^-)}{(pu')(0^-)} + \delta \right) < \gamma^2 \delta$ , in which case  $0 < \frac{v(0^+)}{(pv')(0^+)} < \cot \phi$  and  $\Omega_0^+ \in ((m+1)\pi, m\pi + \Theta(\frac{\pi}{2} - \phi, s)) \subset ((m+1)\pi, (m + \frac{3}{2})\pi)$  by Lemma 5.4.2. Moreover,  $\Omega_0^- \in (n\pi - \tan^{-1} \delta s, n\pi) \subset ((n-1/2), n\pi)$ . This case is different from previous cases without half-zeroes in that the total oscillation count is  $n - m - 2 = k - 2$ .

Moreover,

$$\xi^- s = \begin{cases} n\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \\ n\pi - \frac{\pi}{2} - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha \in (0, \pi), \end{cases}$$

and

$$\begin{aligned}
\xi^+ s &= \Theta(\beta, \epsilon_+^2 s) - \Theta(\Omega_0^+, \sigma_+^2) + O\left(\frac{1}{s}\right) \\
&= \Theta(\beta, \epsilon_+^2 s) - \left\{ (m+1)\pi + \arctan \left( s \frac{v(0^+)}{(pv')(0^+)} \sigma_+^2 \right) \right\} + O\left(\frac{1}{s}\right) \\
&= \begin{cases} \frac{\pi}{2} - m\pi - \frac{3\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \beta \in (0, \pi), \\ \pi - m\pi - \frac{3\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \beta = \pi. \end{cases}
\end{aligned}$$

Hence,

$$\xi s_{n-1, -m-1} = \begin{cases} (n-1 - m - 1)\pi + O\left(\frac{1}{n-m-2}\right), & \text{if } \alpha, \beta \in (0, \pi), \\ (n-1 - m - 1)\pi + \pi + O\left(\frac{1}{n-m-2}\right), & \text{if } \alpha = 0, \beta = \pi, \\ (n-1 - m - 1)\pi + \frac{\pi}{2} + O\left(\frac{1}{n-m-2}\right), & \text{otherwise.} \end{cases}$$

- (iv)  $\gamma^2 \left( \frac{u(0^-)}{(pu')(0^-)} + \delta \right) = \gamma^2 \delta \implies u(0^-) = 0$  and  $\Omega_0^- = n\pi$  by Lemma 5.4.1. In this case, we still have  $0 < \frac{v(+0)}{(pv')(+0)} < \cot \phi$  and  $\Omega_0^+ \in ((m+1)\pi, m\pi + \Theta(\frac{\pi}{2} - \phi, s))$  as in case (b), (iii) above. Here, the total count is  $(n-1) + 1/2 - (m+1) = (k-1) - 1/2$ , corresponding to usual oscillation counts  $(n-1)$  and  $-(m+1)$  in  $(-a, 0)$  and  $(0, b)$  respectively together with an extra half-zero at  $x = 0$ .

Then

$$\begin{aligned}\xi^-s &= n\pi - \Theta(\alpha, \epsilon_-^2s) + O\left(\frac{1}{s}\right) \\ &= \begin{cases} n\pi + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \\ n\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha \in (0, \pi), \end{cases}\end{aligned}$$

and (as above)

$$\xi^+s = \begin{cases} \frac{\pi}{2} - m\pi - \frac{3\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \beta \in (0, \pi), \\ \pi - m\pi - \frac{3\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \beta = \pi. \end{cases}$$

So

$$\begin{aligned}\xi^{s(n-1)+1/2, -(m+1)} &= \begin{cases} [(n-1) + 1/2 - (m+1)]\pi + O\left(\frac{1}{n-m-2}\right), & \text{if } \alpha, \beta \in (0, \pi), \\ [(n-1) + 1/2 - (m+1)]\pi + \pi + O\left(\frac{1}{n-m-2}\right), & \text{if } \alpha = 0, \beta = \pi, \\ [(n-1) + 1/2 - (m+1)]\pi + \frac{\pi}{2} + O\left(\frac{1}{n-m-2}\right), & \text{otherwise.} \end{cases} \quad (5.38)\end{aligned}$$

- (v)  $\gamma^2 \left( \frac{u(0^-)}{(pu')(0^-)} + \delta \right) > \gamma^2\delta$ , so  $\gamma^2 \frac{u(0^-)}{(pu')(0^-)} > 0$  i.e.  $\Omega_0^- \in (n\pi, (n + \frac{1}{2})\pi)$ . Again,  $0 < \frac{v(0^+)}{(pv')(0^+)} < \cot \phi$  and  $\Omega_0^+ \in ((m+1)\pi, m\pi + \Theta(\frac{\pi}{2} - \phi, s))$  as above. Then the total oscillation count is  $n - (m+1) = k - 1$ .

Thus

$$\begin{aligned}\xi^-s &= \Theta(\Omega_0^-, \sigma_-^2) - \Theta(\alpha, \epsilon_-^2s) + O\left(\frac{1}{s}\right) \\ &= n\pi + \arctan\left(s \frac{u(0^-)}{(pu')(0^-)} \sigma_-^2\right) - \Theta(\alpha, \epsilon_-^2s) + O\left(\frac{1}{s}\right) \\ &= \begin{cases} n\pi + \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \\ n\pi + \frac{\pi}{2} - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha \in (0, \pi), \end{cases} \\ \xi^+s &= \begin{cases} \frac{\pi}{2} - m\pi - \frac{3\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \beta \in (0, \pi), \\ \pi - m\pi - \frac{3\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \beta = \pi, \end{cases}\end{aligned}$$

giving

$$\xi^{s_{n, -(m+1)}} = \begin{cases} (n - (m+1))\pi + O\left(\frac{1}{n-m-1}\right), & \text{if } \alpha, \beta \in (0, \pi), \\ (n - (m+1))\pi + \pi + O\left(\frac{1}{n-m-1}\right), & \text{if } \alpha = 0, \beta = \pi, \\ (n - (m+1))\pi + \frac{\pi}{2} + O\left(\frac{1}{n-m-1}\right), & \text{otherwise.} \end{cases}$$

Case II:  $\tan \phi = \gamma^2\delta$

- (a) Suppose that  $\Omega_0^{*-} \in [(n - \frac{1}{2})\pi, n\pi)$  and  $\Omega_0^{*+} \in [(m - \frac{1}{2})\pi, m\pi)$ . Then by Lemma 5.4.1,  $\gamma^2 \frac{u(0^-)}{(pu')(0^-)} < -\gamma^2\delta = -\tan \phi \leq 0$ . Lemma 5.4.2 give us two possible subcases:

- (i)  $\frac{v(0^+)}{(pv')(0^+)} \geq \cot \phi > 0$ , in which case  $\Omega_0^+ \in [(m-1)\pi + \Theta(\frac{\pi}{2} - \phi, s), (m + \frac{1}{2})\pi)$  and  $\Omega_0^- \in [(n - \frac{1}{2})\pi, n\pi)$  by above. Then the total oscillation count is  $(n-1) - m = k - 1$ .

Moreover,

$$\xi^- s = \begin{cases} n\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \\ n\pi - \frac{\pi}{2} - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha \in (0, \pi), \end{cases}$$

and

$$\begin{aligned} \xi^+ s &= \Theta(\beta, \epsilon_+^2 s) - \Theta(\Omega_0^+, \sigma_+^2) + O\left(\frac{1}{s}\right) \\ &= \Theta(\beta, \epsilon_+^2 s) - \left\{ m\pi + \arctan\left(s \frac{v(0^+)}{(pv')(0^+)} \sigma_+^2\right) \right\} + O\left(\frac{1}{s}\right) \\ &= \begin{cases} \frac{\pi}{2} - m\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \beta \in (0, \pi), \\ \pi - m\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \beta = \pi. \end{cases} \end{aligned}$$

This gives

$$\xi s_{n-1, -m} = \begin{cases} (n-1-m)\pi + O\left(\frac{1}{n-1-m}\right), & \text{if } \alpha, \beta \in (0, \pi), \\ (n-1-m)\pi + \pi + O\left(\frac{1}{n-1-m}\right), & \text{if } \alpha = 0, \beta = \pi, \\ (n-1-m)\pi + \frac{\pi}{2} + O\left(\frac{1}{n-1-m}\right), & \text{otherwise.} \end{cases}$$

- (ii)  $\frac{v(0^+)}{(pv')(0^+)} < -\tan \phi \leq 0$ , in which case  $\Omega_0^+ \in [(m+1/2)\pi, m\pi + \Theta(-\phi, s)]$ . Again,  $\Omega_0^- \in [(n-\frac{1}{2})\pi, n\pi)$  by above. In this case the oscillation counts are the same as in (a), (i) above.

Here,

$$\xi^- s = \begin{cases} n\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \\ n\pi - \frac{\pi}{2} - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha \in (0, \pi), \end{cases}$$

and

$$\begin{aligned} \xi^+ s &= \Theta(\beta, \epsilon_+^2 s) - \Theta(\Omega_0^+, \sigma_+^2) + O\left(\frac{1}{s}\right) \\ &= \Theta(\beta, \epsilon_+^2 s) - \left\{ (m+1)\pi + \arctan\left(s \frac{v(0^+)}{(pv')(0^+)} \sigma_+^2\right) \right\} + O\left(\frac{1}{s}\right) \\ &= \begin{cases} \frac{\pi}{2} - m\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \beta \in (0, \pi), \\ \pi - m\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \beta = \pi, \end{cases} \end{aligned}$$

giving

$$\xi s_{n-1, -m} = \begin{cases} (n-1-m)\pi + O\left(\frac{1}{n-1-m}\right), & \text{if } \alpha, \beta \in (0, \pi), \\ (n-1-m)\pi + \pi + O\left(\frac{1}{n-1-m}\right), & \text{if } \alpha = 0, \beta = \pi, \\ (n-1-m)\pi + \frac{\pi}{2} + O\left(\frac{1}{n-1-m}\right), & \text{otherwise.} \end{cases}$$

(b) On the other hand, if  $\Omega_0^{*-} \in [n\pi, (n+\frac{1}{2})\pi)$  and  $\Omega_0^{*+} \in [m\pi, (m+\frac{1}{2})\pi)$  then there are three possible sub-cases:

- (i)  $-\tan \phi \leq \frac{v(0^+)}{(pv')(0^+)} < 0$ , in which case  $\Omega_0^+ \in [m\pi + \Theta(-\phi, s), (m+1)\pi)$  by Lemma 5.4.2 and

$$\gamma^2 \frac{u(0^-)}{(pu')(0^-)} = \frac{\frac{v(0^+)}{(pv')(0^+)} (1 + \tan^2 \phi)}{1 - \tan \phi \frac{v(0^+)}{(pv')(0^+)}} < 0$$

so  $\Omega_0^- \in [n\pi - \tan^{-1} \delta s, n\pi) \subset ((n-1/2)\pi, n\pi)$  by Lemma 5.4.1. Thus the total oscillation count is  $n-1-m=k-1$ .

Moreover,

$$\xi^-_s = \begin{cases} n\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \\ n\pi - \frac{\pi}{2} - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha \in (0, \pi), \end{cases}$$

and

$$\begin{aligned} \xi^+_s &= \Theta(\beta, \epsilon_+^2 s) - \Theta(\Omega_0^+, \sigma_+^2) + O\left(\frac{1}{s}\right) \\ &= \Theta(\beta, \epsilon_+^2 s) - \left\{ (m+1)\pi + \arctan\left(s \frac{v(0^+)}{(pv')(0^+)} \sigma_+^2\right) \right\} + O\left(\frac{1}{s}\right) \\ &= \begin{cases} \frac{\pi}{2} - m\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \beta \in (0, \pi), \\ \pi - m\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \beta = \pi, \end{cases} \end{aligned}$$

giving

$$\xi_{s_{n-1, -m}} = \begin{cases} (n-1-m)\pi + O\left(\frac{1}{n-1-m}\right), & \text{if } \alpha, \beta \in (0, \pi), \\ (n-1-m)\pi + \pi + O\left(\frac{1}{n-1-m}\right), & \text{if } \alpha = 0, \beta = \pi, \\ (n-1-m)\pi + \frac{\pi}{2} + O\left(\frac{1}{n-1-m}\right), & \text{otherwise.} \end{cases}$$

- (ii)  $v(0^+) = 0$  so  $\Omega_0^+ = (m+1)\pi$  by Lemma 5.4.1. This forces  $u(0^-) = 0 \implies \Omega_0^- = n\pi$  by Lemma 5.4.1, giving a total oscillation count of  $(n-1) + 1/2 - (m+1) + 1/2 = k-1$ , corresponding to usual oscillation counts  $(n-1)$  and  $-(m+1)$  in  $(-a, 0)$  and  $(0, b)$  respectively and two half zeroes at  $x = 0$ .

Hence,

$$\begin{aligned} \xi^-_s &= \begin{cases} n\pi + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \\ n\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha \in (0, \pi), \end{cases} \\ \xi^+_s &= \begin{cases} \frac{\pi}{2} - (m+1)\pi + O\left(\frac{1}{s}\right), & \text{if } \beta \in (0, \pi), \\ \pi - (m+1)\pi + O\left(\frac{1}{s}\right), & \text{if } \beta = \pi, \end{cases} \end{aligned}$$

and

$$\begin{aligned} &\xi^{s_{(n-1)+1/2, -(m+1)+1/2}} \\ &= \begin{cases} [(n-1) + 1/2 - (m+1) + 1/2]\pi + O\left(\frac{1}{n-m-1}\right), & \text{if } \alpha, \beta \in (0, \pi), \\ [(n-1) + 1/2 - (m+1) + 1/2]\pi + \pi + O\left(\frac{1}{n-m-1}\right), & \text{if } \alpha = 0, \beta = \pi, \\ [(n-1) + 1/2 - (m+1) + 1/2]\pi + \frac{\pi}{2} + O\left(\frac{1}{n-m-1}\right), & \text{otherwise.} \end{cases} \end{aligned} \tag{5.39}$$

- (iii)  $0 < \frac{v(0^+)}{(pv')(0^+)} < \cot \phi$ , in which case  $\Omega_0^+ \in ((m+1)\pi, m\pi + \Theta(\frac{\pi}{2} - \phi, s))$  by Lemma 5.4.2, giving

$$\gamma^2 \frac{u(0^-)}{(pu')(0^-)} = \frac{\frac{v(0^+)}{(pv')(0^+)}(1 + \tan^2 \phi)}{1 - \tan \phi \frac{v(0^+)}{(pv')(0^+)}} > 0$$

so  $\Omega_0^- \in (n\pi, (n + \frac{1}{2})\pi)$ . Then the total oscillation count is  $n - m - 1 = k - 1$ .

Moreover,

$$\begin{aligned}\xi^- s &= \Theta(\Omega_0^-, \sigma_-^2) - \Theta(\alpha, \epsilon_-^2 s) + O\left(\frac{1}{s}\right) \\ &= n\pi + \arctan\left(s \frac{u(0^-)}{(pu')(0^-)} \sigma_-^2\right) - \Theta(\alpha, \epsilon_-^2 s) + O\left(\frac{1}{s}\right) \\ &= \begin{cases} n\pi + \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \\ n\pi + \frac{\pi}{2} - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha \in (0, \pi), \end{cases}\end{aligned}$$

and

$$\begin{aligned}\xi^+ s &= \Theta(\beta, \epsilon_+^2 s) - \Theta(\Omega_0^+, \sigma_+^2) + O\left(\frac{1}{s}\right) \\ &= \Theta(\beta, \epsilon_+^2 s) - \left\{ (m+1)\pi + \arctan\left(s \frac{v(0^+)}{(pv')(0^+)} \sigma_+^2\right) \right\} + O\left(\frac{1}{s}\right) \\ &= \begin{cases} \frac{\pi}{2} - m\pi - \frac{3\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \beta \in (0, \pi), \\ \pi - m\pi - \frac{3\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \beta = \pi, \end{cases}\end{aligned}$$

giving

$$\xi s_{n, -(m+1)} = \begin{cases} (n-m-1)\pi + O\left(\frac{1}{n-m-1}\right), & \text{if } \alpha, \beta \in (0, \pi), \\ (n-m-1)\pi + \pi + O\left(\frac{1}{n-m-1}\right), & \text{if } \alpha = 0, \beta = \pi, \\ (n-m-1)\pi + \frac{\pi}{2} + O\left(\frac{1}{n-m-1}\right), & \text{otherwise.} \end{cases}$$

**Case III:  $\tan \phi > \gamma^2 \delta$**

Again we consider only  $\phi \in [-\pi, -\pi/2)$ . Throughout we have to consider three possibilities: 1)  $0 \leq \gamma^2 \delta < \tan \phi$ , 2)  $-\cot \phi \leq \gamma^2 \delta < 0$  and 3)  $\gamma^2 \delta < -\cot \phi$ .

(a) Suppose that  $\Omega_0^{*-} \in [(n - \frac{1}{2})\pi, n\pi)$  and  $\Omega_0^{*+} \in [(m - \frac{1}{2})\pi, m\pi)$  with  $k = n - m$ . Then  $\Omega_0^- \in [(n - \frac{1}{2})\pi, n\pi - \tan^{-1} \delta s)$  by Lemma 5.4.1 and Lemma 5.4.2 gives us two possible sub-cases:

(i)  $\cot \phi \leq \frac{v(0^+)}{(pv')(0^+)}$  i.e.  $\Omega_0^+ \in [(m-1)\pi + \Theta(\frac{\pi}{2} - \phi, s), (m + \frac{1}{2})\pi)$ , which implies that

$$\gamma^2 \delta + \gamma^2 \frac{u(0^-)}{(pu')(0^-)} < -\cot \phi.$$

So if (1),  $\delta \geq 0$  or (2),  $-\cot \phi \leq \gamma^2 \delta < 0$  then  $\Omega_0^- \in [(n - \frac{1}{2})\pi, n\pi)$  at the point of intersection. Then the total oscillation count is  $n - 1 - m = k - 1$ .

Moreover,

$$\begin{aligned}\xi^- s &= n\pi + \arctan\left(s \frac{u(0^-)}{(pu')(0^-)} \sigma_-^2\right) - \Theta(\alpha, \epsilon_-^2 s) + O\left(\frac{1}{s}\right) \\ &= \begin{cases} n\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \\ n\pi - \frac{\pi}{2} - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha \in (0, \pi), \end{cases}\end{aligned}$$

and

$$\begin{aligned}\xi^+ s &= \Theta(\beta, \epsilon_+^2) - \left\{ m\pi + \arctan \left( s \frac{v(0^+)}{(pv')(0^+)} \sigma_+^2 \right) \right\} + O\left(\frac{1}{s}\right) \\ &= \begin{cases} \frac{\pi}{2} - m\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \beta \in (0, \pi), \\ \pi - m\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \beta = \pi, \end{cases}\end{aligned}$$

from above, giving

$$\xi s_{n-1, -m} = \begin{cases} (n-1-m)\pi + O\left(\frac{1}{n-1-m}\right), & \text{if } \alpha, \beta \in (0, \pi), \\ (n-1-m)\pi + \pi + O\left(\frac{1}{n-1-m}\right), & \text{if } \alpha = 0, \beta = \pi, \\ (n-1-m)\pi + \frac{\pi}{2} + O\left(\frac{1}{n-1-m}\right), & \text{otherwise.} \end{cases}$$

Whereas if (3),  $\gamma^2 \delta < -\cot \phi$  then there are three possibilities. Either

$$\gamma^2 \delta + \gamma^2 \frac{u(0^-)}{(pu')(0^-)} < \gamma^2 \delta,$$

in which case the total oscillation count is again  $n-1-m = k-1$ , and

$$\xi^- s = \begin{cases} n\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \\ n\pi - \frac{\pi}{2} - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha \in (0, \pi), \end{cases}$$

giving

$$\xi s_{n-1, -m} = \begin{cases} (n-1-m)\pi + O\left(\frac{1}{n-1-m}\right), & \text{if } \alpha, \beta \in (0, \pi), \\ (n-1-m)\pi + \pi + O\left(\frac{1}{n-1-m}\right), & \text{if } \alpha = 0, \beta = \pi, \\ (n-1-m)\pi + \frac{\pi}{2} + O\left(\frac{1}{n-1-m}\right), & \text{otherwise,} \end{cases}$$

as above. Or

$$\gamma^2 \delta + \gamma^2 \frac{u(0^-)}{(pu')(0^-)} = \gamma^2 \delta,$$

so  $\Omega_0^- = n\pi$ , giving a total oscillation count of  $(n-1) + 1/2 - m = k - 1/2$ . Moreover,

$$\begin{aligned}\xi^- s &= n\pi - \Theta(\alpha, \epsilon_-^2) + O\left(\frac{1}{s}\right) \\ &= \begin{cases} n\pi + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \\ n\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha \in (0, \pi), \end{cases}\end{aligned}$$

which yields

$$\xi s_{(n-1)+\frac{1}{2}, -m} = \begin{cases} ((n-1) + \frac{1}{2} - m)\pi + O\left(\frac{1}{n-1-m}\right), & \text{if } \alpha, \beta \in (0, \pi), \\ ((n-1) + \frac{1}{2} - m)\pi + \pi + O\left(\frac{1}{n-1-m}\right), & \text{if } \alpha = 0, \beta = \pi, \\ ((n-1) + \frac{1}{2} - m)\pi + \frac{\pi}{2} + O\left(\frac{1}{n-1-m}\right), & \text{otherwise.} \end{cases}$$

Or

$$\gamma^2 \delta < \gamma^2 \delta + \gamma^2 \frac{u(0^-)}{(pu')(0^-)} < -\cot \phi,$$



in which case the total oscillation count is  $n - m = k$  and

$$\begin{aligned}\xi^- s &= n\pi + \frac{\pi}{2} - \Theta(\alpha, \epsilon_-^2 s) + O\left(\frac{1}{s}\right) \\ &= \begin{cases} n\pi + \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \\ n\pi + \frac{\pi}{2} - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha \in (0, \pi), \end{cases}\end{aligned}$$

giving

$$\xi s_{n,-m} = \begin{cases} (n-m)\pi + O\left(\frac{1}{n-m}\right), & \text{if } \alpha, \beta \in (0, \pi), \\ (n-m)\pi + \pi + O\left(\frac{1}{n-m}\right), & \text{if } \alpha = 0, \beta = \pi, \\ (n-m)\pi + \frac{\pi}{2} + O\left(\frac{1}{n-m}\right), & \text{otherwise.} \end{cases}$$

(ii) if  $\frac{v(0^+)}{(pv')(0^+)} < -\tan \phi$ , i.e.  $\Omega_0^+ \in \left[(m + \frac{1}{2})\pi, m\pi + \Theta(-\phi, s)\right)$  then at the point of intersection

$$-\cot \phi \leq \gamma^2 \delta + \gamma^2 \frac{u(0^-)}{(pu')(0^-)} < 0.$$

So if (1),  $\delta \geq 0$  then  $\Omega_0^- \in \left((n - \frac{1}{2})\pi, n\pi - \tan^{-1} \delta s\right)$  and the total oscillation count is  $n - 1 - m$ . Moreover,

$$\xi^- s = \begin{cases} n\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \\ n\pi - \frac{\pi}{2} - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha \in (0, \pi), \end{cases}$$

and

$$\begin{aligned}\xi^+ s &= \Theta(\beta, \epsilon_+^2 s) - \left\{ (m+1)\pi + \arctan\left(s \frac{v(0^+)}{(pv')(0^+)} \sigma_+^2\right) \right\} + O\left(\frac{1}{s}\right) \\ &= \begin{cases} \frac{\pi}{2} - (m+1)\pi + \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \beta \in (0, \pi), \\ \pi - (m+1)\pi + \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \beta = \pi, \end{cases}\end{aligned}$$

giving

$$\xi s_{n-1,-m} = \begin{cases} (n-1-m)\pi + O\left(\frac{1}{n-1-m}\right), & \text{if } \alpha, \beta \in (0, \pi), \\ (n-1-m)\pi + \pi + O\left(\frac{1}{n-1-m}\right), & \text{if } \alpha = 0, \beta = \pi, \\ (n-1-m)\pi + \frac{\pi}{2} + O\left(\frac{1}{n-1-m}\right), & \text{otherwise.} \end{cases}$$

Else if (2),  $-\cot \phi \leq \gamma^2 \delta < 0$  then there are three possibilities. Either

$$-\cot \phi \leq \gamma^2 \delta + \gamma^2 \frac{u(0^-)}{(pu')(0^-)} < \gamma^2 \delta,$$

in which case the total oscillation count is  $n - 1 - m = k - 1$ , and

$$\xi^- s = \begin{cases} n\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \\ n\pi - \frac{\pi}{2} - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha \in (0, \pi), \end{cases}$$

giving

$$\xi s_{n-1,-m} = \begin{cases} (n-1-m)\pi + O\left(\frac{1}{n-1-m}\right), & \text{if } \alpha, \beta \in (0, \pi), \\ (n-1-m)\pi + \pi + O\left(\frac{1}{n-1-m}\right), & \text{if } \alpha = 0, \beta = \pi, \\ (n-1-m)\pi + \frac{\pi}{2} + O\left(\frac{1}{n-1-m}\right), & \text{otherwise.} \end{cases}$$

Or

$$\gamma^2\delta + \gamma^2 \frac{u(0^-)}{(pu')(0^-)} = \gamma^2\delta,$$

which gives at total oscillation count of  $(n-1) + 1/2 - m = k - 1/2$ . Moreover,

$$\xi^-_s = \begin{cases} n\pi + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \\ n\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha \in (0, \pi), \end{cases}$$

and

$$\xi^{s_{(n-1)+\frac{1}{2}, -m}} = \begin{cases} \left( (n-1) + \frac{1}{2} - m \right) \pi + O\left(\frac{1}{n-1-m}\right), & \text{if } \alpha, \beta \in (0, \pi), \\ \left( (n-1) + \frac{1}{2} - m \right) \pi + \pi + O\left(\frac{1}{n-1-m}\right), & \text{if } \alpha = 0, \beta = \pi, \\ \left( (n-1) + \frac{1}{2} - m \right) \pi + \frac{\pi}{2} + O\left(\frac{1}{n-1-m}\right), & \text{otherwise.} \end{cases}$$

Else

$$\gamma^2\delta < \gamma^2\delta + \gamma^2 \frac{u(0^-)}{(pu')(0^-)} < 0,$$

in which case the total oscillation count is  $n - m = k$ , and

$$\xi^-_s = \begin{cases} n\pi + \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \\ n\pi + \frac{\pi}{2} - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha \in (0, \pi), \end{cases}$$

giving

$$\xi^{s_{n, -m}} = \begin{cases} (n-m)\pi + O\left(\frac{1}{n-m}\right), & \text{if } \alpha, \beta \in (0, \pi), \\ (n-m)\pi + \pi + O\left(\frac{1}{n-m}\right), & \text{if } \alpha = 0, \beta = \pi, \\ (n-m)\pi + \frac{\pi}{2} + O\left(\frac{1}{n-m}\right), & \text{otherwise.} \end{cases}$$

Lastly if (3),  $\gamma^2\delta < -\cot\phi$  then  $\Omega_0^- \in (n\pi, n\pi - \tan^{-1}\delta s)$ , giving a total oscillation count of  $n - m = k$ . Moreover,

$$\xi^-_s = \begin{cases} n\pi + \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \\ n\pi + \frac{\pi}{2} - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha \in (0, \pi), \end{cases}$$

giving

$$\xi^{s_{n, -m}} = \begin{cases} (n-m)\pi + O\left(\frac{1}{n-m}\right), & \text{if } \alpha, \beta \in (0, \pi), \\ (n-m)\pi + \pi + O\left(\frac{1}{n-m}\right), & \text{if } \alpha = 0, \beta = \pi, \\ (n-m)\pi + \frac{\pi}{2} + O\left(\frac{1}{n-m}\right), & \text{otherwise.} \end{cases}$$

(b) On the other hand, if  $\Omega_0^{*-} \in [n\pi, (n + \frac{1}{2})\pi)$  and  $\Omega_0^{*+} \in [m\pi, (m + \frac{1}{2})\pi)$  then we must consider three possibilities: (1)  $0 \leq \gamma^2\delta < \tan\phi$ , (2)  $-\cot\phi \leq \gamma^2\delta < 0$  and (3)  $\gamma^2\delta < -\cot\phi$ . Furthermore, as above, Lemma 5.4.2 gives us two possible subcases:

(i)  $\Omega_0^+ \in [m\pi + \Theta(-\phi, s), (m+1)\pi)$ , i.e.  $-\tan\phi \leq \frac{v(0^+)}{(pv')(0^+)} < 0$ . This implies that

$$-\gamma^2\delta \leq \gamma^2 \frac{u(0^-)}{(pu')(0^-)} < \tan\phi - \gamma^2\delta.$$

If (1),  $0 \leq \gamma^2 \delta < \tan \phi$  then there are three possibilities. Either

$$-\gamma^2 \delta \leq \gamma^2 \frac{u(0^-)}{(pu')(0^-)} < 0,$$

in which case the total oscillation count is  $n - 1 - m = k - 1$ . Moreover,

$$\begin{aligned} \xi^- s &= n\pi + \arctan \left( s \frac{u(0^-)}{(pu')(0^-)} \sigma_-^2 \right) - \Theta(\alpha, \epsilon_-^2 s) + O\left(\frac{1}{s}\right) \\ &= \begin{cases} n\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \\ n\pi - \frac{\pi}{2} - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha \in (0, \pi), \end{cases} \end{aligned}$$

and

$$\begin{aligned} \xi^+ s &= \Theta(\beta, \epsilon_+^2 s) - \left( (m+1)\pi + \arctan \left( s \frac{v(0^+)}{(pv')(0^+)} \sigma_+^2 \right) \right) \\ &= \begin{cases} \frac{\pi}{2} - (m+1)\pi + \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \beta \in (0, \pi) \\ \pi - (m+1)\pi + \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \beta = \pi, \end{cases} \end{aligned}$$

giving

$$\xi s_{n-1, -m} = \begin{cases} (n-m-1)\pi + O\left(\frac{1}{s}\right), & \text{if } \alpha, \beta \in (0, \pi), \\ (n-m-1)\pi + \pi + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \beta = \pi, \\ (n-m-1)\pi + \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{otherwise.} \end{cases}$$

Else  $\frac{u(0^-)}{(pu')(0^-)} = 0$ . Then  $\Omega_0^- = n\pi$  and the total oscillation count is  $(n-1) + 1/2 - m = k - 1/2$ . Also,

$$\xi^- s = \begin{cases} n\pi + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \\ n\pi \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha \in (0, \pi), \end{cases}$$

giving

$$\xi s_{n-1+\frac{1}{2}, -m} = \begin{cases} (n-1+\frac{1}{2}-m)\pi + O\left(\frac{1}{s}\right), & \text{if } \alpha, \beta \in (0, \pi), \\ (n-1+\frac{1}{2}-m)\pi + \pi + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \beta = \pi, \\ (n-1+\frac{1}{2}-m)\pi \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{otherwise.} \end{cases}$$

Otherwise,  $0 < \frac{u(0^-)}{(pu')(0^-)} < \tan \phi - \gamma^2 \delta$  and the total oscillation count is  $n - m = k$ . Then

$$\xi^- s = \begin{cases} n\pi + \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \\ n\pi + \frac{\pi}{2} - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha \in (0, \pi), \end{cases}$$

giving

$$\xi s_{n, -m} = \begin{cases} (n-m)\pi + O\left(\frac{1}{s}\right), & \text{if } \alpha, \beta \in (0, \pi), \\ (n-m)\pi + \pi + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \beta = \pi, \\ (n-m)\pi + \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{otherwise.} \end{cases}$$

(ii)  $\Omega_0^+ \in [(m+1)\pi, m\pi + \Theta(\frac{\pi}{2} - \phi, s)]$  i.e.  $0 \leq \frac{v(0^+)}{(pv')(0^+)} < \cot \phi$ . Then

$$\gamma^2 \frac{u(0^-)}{(pu')(0^-)} \geq \tan \phi - \gamma^2 \delta > 0,$$

and

$$\xi^- s = \begin{cases} n\pi + \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \\ n\pi + \frac{\pi}{2} - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \alpha \in (0, \pi). \end{cases}$$

If  $v(0^+) = 0$  then the total oscillation count is  $n - (m + 1) + 1/2 = k - 1/2$ , and

$$\xi^+ s = \begin{cases} \frac{\pi}{2} - (m + 1)\pi + O\left(\frac{1}{s}\right), & \text{if } \beta \in (0, \pi), \\ \pi - (m + 1)\pi + O\left(\frac{1}{s}\right), & \text{if } \beta = \pi, \end{cases}$$

so that

$$\xi s_{n, -m-1+1/2} = \begin{cases} (n - m - 1 + 1/2)\pi + O\left(\frac{1}{s}\right), & \text{if } \alpha, \beta \in (0, \pi), \\ (n - m - 1 + 1/2)\pi + \pi + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \beta = \pi, \\ (n - m - 1 + 1/2)\pi + \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{otherwise.} \end{cases}$$

Else if  $0 < \frac{v(0^+)}{(pv')(0^+)} < \cot \phi$  then the total oscillation count is  $n - m - 1 = k - 1$ , and

$$\begin{aligned} \xi^+ s &= \Theta(\beta, \epsilon_+^2 s) - \left( (m + 1)\pi + \arctan \left( s \frac{v(0^+)}{(pv')(0^+)} \sigma_+^2 \right) \right) + O\left(\frac{1}{s}\right) \\ &= \begin{cases} \frac{\pi}{2} - (m + 1)\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \beta \in (0, \pi), \\ \pi - (m + 1)\pi - \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{if } \beta = \pi, \end{cases} \end{aligned}$$

giving

$$\xi s_{n, -m-1} = \begin{cases} (n - m - 1)\pi + O\left(\frac{1}{s}\right), & \text{if } \alpha, \beta \in (0, \pi), \\ (n - m - 1)\pi + \pi + O\left(\frac{1}{s}\right), & \text{if } \alpha = 0, \beta = \pi, \\ (n - m - 1)\pi + \frac{\pi}{2} + O\left(\frac{1}{s}\right), & \text{otherwise.} \end{cases}$$

□

**Note 5.5.3.** In Theorem 5.5.1 we showed that for each  $k \in \mathbb{N}$ , large enough, we obtain precisely one eigenvalue of (5.1)-(5.4). In Theorem 5.5.2, we determined that the oscillation count of each eigenvalue is dependent on the value of  $k$  and possibly the values of the incident angles  $\Omega_0^-$  and  $\Omega_0^+$  prior to the applications of the sheer and rotation matrices respectively. In particular, for the case of  $\phi \in [-\pi, -\frac{\pi}{2}]$  (presented in the proof of Theorem 5.5.2), we observe that the oscillation count of the eigenvalue corresponding to  $k$  is always  $k - 1$  if  $\tan \phi = \gamma^2 \delta$ ; is equal to  $k - 1$ ,  $k - 3/2$  or  $k - 2$  if  $\tan \phi < \gamma^2 \delta$ ; and equals  $k - 1$ ,  $k - 1/2$  or  $k$  if  $\tan \phi > \gamma^2 \delta$ . Thus, in particular for the case of  $\tan \phi = \gamma^2 \delta$  we have precisely one eigenvalue corresponding to each integer oscillation count, and the oscillation count is always integer valued since  $u(0^-; \lambda) = 0$  if and only if  $v(0^+; \lambda) = 0$ . However, in the two cases where  $\tan \phi \neq \gamma^2 \delta$  it is possible to have at two adjacent eigenvalues with the same oscillation counts.

The final results link the oscillation count of an eigenvalue with its position in the list of eigenvalues, labelled according to increasing magnitude. We consider separately the cases of  $\tan \phi = \gamma^2 \delta$  and  $\tan \phi \neq \gamma^2 \delta$ .

Let  $\eta = |\Im(\sqrt{\lambda})|$ ,  $\lambda \in \mathbb{C}$ . It can be shown that as  $\sqrt{\lambda} \rightarrow \infty$ ,

$$u(0^-; \lambda) = \frac{\epsilon_- \sin \alpha}{\sigma_-} \cos \xi^- \sqrt{\lambda} + \frac{\cos \alpha \sin \xi^- \sqrt{\lambda}}{\epsilon_- \sigma_- \sqrt{\lambda}} + O\left(\frac{e^{\eta \xi^-}}{\lambda}\right), \quad (5.40)$$

$$p(0^-)u'(0^-; \lambda) = -\lambda \epsilon_- \sigma^- \sin \alpha \frac{\sin \xi^- \sqrt{\lambda}}{\sqrt{\lambda}} + \frac{\sigma_- \cos \alpha}{\epsilon_-} \cos \xi^- \sqrt{\lambda} + O\left(\frac{e^{\eta \xi^-}}{\sqrt{\lambda}}\right), \quad (5.41)$$

$$v(0^+; \lambda) = \frac{\epsilon_+ \sin \beta}{\sigma_+} \cos \xi^+ \sqrt{\lambda} - \frac{\cos \beta \sin \xi^+ \sqrt{\lambda}}{\epsilon_+ \sigma_+ \sqrt{\lambda}} + O\left(\frac{e^{\eta \xi^+}}{\lambda}\right), \quad (5.42)$$

$$p(0^+)v'(0^+; \lambda) = \lambda \epsilon_+ \sigma^+ \sin \beta \frac{\sin \xi^+ \sqrt{\lambda}}{\sqrt{\lambda}} + \frac{\sigma_+ \cos \beta}{\epsilon_+} \cos \xi^+ \sqrt{\lambda} + O\left(\frac{e^{\eta \xi^+}}{\sqrt{\lambda}}\right). \quad (5.43)$$

The above approximations reduce to those stated in Proposition 5.3.3 if  $\lambda \in \mathbb{R}$ . For the proof we again refer the reader to Hochstadt, [39].

Let

$$\omega(\lambda) = \det \left[ \begin{bmatrix} \gamma & \gamma \delta \\ 0 & \frac{1}{\gamma} \end{bmatrix} \begin{bmatrix} u(0^-) \\ p u'(0^-) \end{bmatrix} : \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} v(0^+) \\ p v'(0^+) \end{bmatrix} \right]. \quad (5.44)$$

Then the zeroes of  $\omega(\lambda)$  coincide with the eigenvalues of (5.1)-(5.4), making  $\omega(\lambda)$  a characteristic determinant for (5.1)-(5.4). Let

$$\lambda_0 < \lambda_1 < \lambda_2 < \dots < \infty \quad (5.45)$$

denote the list of eigenvalues of (5.1)-(5.4).

Consider

$$-(py')' + qy = \lambda ry, \quad x \in [-a, 0] \quad (5.46)$$

with boundary condition at  $x = -a$  given by (5.2). We denote by  $\lambda_n^{D-}$ ,  $n = 0, 1, 2, \dots$  the eigenvalues of the boundary value problem consisting of (5.46), (5.2) with Dirichlet boundary condition  $y(0^-) = 0$ . Setting  $\lambda_{-1}^{D-} = -\infty$ , we note that if  $\lambda \in (\lambda_{n-1}^{D-}, \lambda_n^{D-}]$  then  $y(x; \lambda)$  satisfying (5.46), (5.2) has  $n$  zeroes in  $(-a, 0)$ . Denote by  $\lambda_n^{N-}$ ,  $n = 0, 1, 2, \dots$  the eigenvalues of the boundary value problem consisting of (5.46), (5.2) with Neumann boundary condition  $y'(0^-) = 0$ . Here  $\lambda_n^{N-} \in (\lambda_{n-1}^{D-}, \lambda_n^{D-})$ .

Similarly, considering

$$-(py')' + qy = \lambda ry, \quad x \in (0, b] \quad (5.47)$$

with boundary condition at  $x = b$  given by (5.3), we denote by  $\lambda_n^{D+}$ ,  $n = 0, 1, 2, \dots$  the eigenvalues of the boundary value problem consisting of (5.47), (5.3) with Dirichlet boundary condition  $y(0^+) = 0$ . Set  $\lambda_{-1}^{D+} = -\infty$ . If  $\lambda \in (\lambda_{n-1}^{D+}, \lambda_n^{D+}]$  then  $y(x; \lambda)$  satisfying (5.47), (5.3) has  $n$  zeroes in  $(0, b)$ . Let the eigenvalues of the boundary value problem (5.47), (5.3) with Neumann boundary condition  $y'(0^+) = 0$  be denoted by  $\lambda_n^{N+}$ , where  $\lambda_n^{N+} \in (\lambda_{n-1}^{D+}, \lambda_n^{D+})$ .

**Theorem 5.5.4.** *Suppose that  $\tan \phi = \gamma^2 \delta$ . Let  $\lambda_{N,M} = (s_{N,M})^2$  be the eigenvalue of (5.1)-(5.4) whose eigenfunction has oscillation count  $N$  in  $(-a, 0]$  and  $M$  in  $[0, b)$ , with zeroes at  $0^-$  and  $0^+$*

counting  $1/2$  each. Then either  $N, M \in \mathbb{N}_0$  or both  $N$  and  $M$  are odd-integer multiples of  $1/2$ , and, moreover,

$$\lambda_{N,M} = \lambda_{N+M}. \quad (5.48)$$

That is,  $\lambda_{N,M}$  is the  $(N + M + 1)$ th eigenvalue in the list (5.45).

*Proof.* Expanding the right hand side of equation (5.44), we observe that

$$\frac{\omega(\lambda)}{\cos \phi \sqrt{\gamma^4 \delta^2 + 1}} = \frac{u(0^-)pv'(0^+)}{\sqrt{\gamma^2 \delta^2 + \frac{1}{\gamma^2}}} - \sqrt{\gamma^2 \delta^2 + \frac{1}{\gamma^2}} pu'(0^-)v(0^+) - \frac{\tan \phi}{\sqrt{\gamma^2 \delta^2 + \frac{1}{\gamma^2}}} u(0^-)v(0^+), \quad (5.49)$$

where we have used the fact that  $\gamma \delta \cos \phi - \frac{1}{\gamma} \sin \phi = 0$ , since  $\tan \phi = \gamma^2 \delta$ . Let

$$\Delta_1(\lambda) = \frac{u(0^-)pv'(0^+)}{\sqrt{\gamma^2 \delta^2 + \frac{1}{\gamma^2}}} - \sqrt{\gamma^2 \delta^2 + \frac{1}{\gamma^2}} pu'(0^-)v(0^+).$$

Notice that

$$\Delta_1(\lambda) = \det \left[ \begin{bmatrix} \frac{1}{\sqrt{\gamma^2 \delta^2 + \frac{1}{\gamma^2}}} & 0 \\ 0 & \sqrt{\gamma^2 \delta^2 + \frac{1}{\gamma^2}} \end{bmatrix} \begin{bmatrix} u(0^-) \\ pu'(0^-) \end{bmatrix} : \begin{bmatrix} v(0^+) \\ pv'(0^+) \end{bmatrix} \right]. \quad (5.50)$$

Comparing (5.50) and (5.44) we observe that the zeroes of  $\Delta_1(\lambda)$  correspond to the eigenvalues of (5.1)-(5.3) with transfer condition

$$\begin{bmatrix} y(0^+) \\ (py')(0^+) \end{bmatrix} = \tilde{T} \begin{bmatrix} y(0^-) \\ (py')(0^-) \end{bmatrix}. \quad (5.51)$$

Here  $\tilde{T} = \tilde{g} \sqrt{\det \tilde{T}}$ , with

$$\tilde{g} = \begin{bmatrix} \cos \tilde{\phi} & -\sin \tilde{\phi} \\ \sin \tilde{\phi} & \cos \tilde{\phi} \end{bmatrix} \begin{bmatrix} \tilde{\gamma} & 0 \\ 0 & 1/\tilde{\gamma} \end{bmatrix} \begin{bmatrix} 1 & \tilde{\delta} \\ 0 & 1 \end{bmatrix},$$

and  $\tilde{\gamma} = \frac{1}{\sqrt{\gamma^2 \delta^2 + \frac{1}{\gamma^2}}}$ ,  $\tilde{\delta} = 0$  and  $\tilde{\phi} = 0$  (see equations (5.8)-(5.10)). Thus the zeroes of  $\Delta_1(\lambda)$  can be found by solving

$$\tan \varphi(0; \lambda) = \frac{1}{\gamma^2 \delta^2 + \frac{1}{\gamma^2}} \tan \theta(0; \lambda)$$

(see equation (5.15) with  $\gamma$ ,  $\delta$  and  $\phi$  replaced with  $\tilde{\gamma}$ ,  $\tilde{\delta}$  and  $\tilde{\phi}$  as above). If the graphs of  $y = \tan \varphi(0; \lambda)$  and  $y = \tan \theta(0; \lambda)$  have a common vertical asymptote then such a value of  $\lambda$  corresponds to a zero of  $\Delta_1(\lambda)$  with  $u'(0^-; \lambda) = 0$  and  $v'(0^+; \lambda) = 0$ .

On the other hand, intersections of the graphs of  $y = \tan \varphi(0; \lambda)$  and  $y = \tan \theta(0; \lambda)$  correspond to eigenvalues of the classical Sturm Liouville problem, consisting of

$$-(py')' + qy = \lambda ry, \quad [-a, b] \quad (5.52)$$

with boundary conditions (5.2), (5.3). Note that this eigenvalue problem can be recast as an eigenvalue problem of the form (5.1) - (5.4) with transfer matrix  $M$  given by the identity. In this way, zeroes at  $x = 0$  are counted as two half zeroes corresponding to  $y(0^-) = 0$  and  $y(0^+) = 0$ .

Let  $\lambda_k^c$ ,  $k = 0, 1, 2, \dots$  denote the eigenvalues of (5.52), (5.2), (5.3). Applying Theorem 5.5.2, with  $k = N + M$  being the total count in  $(-a, b)$  (including zeroes at  $x = 0$ ), we obtain

$$\xi \sqrt{\lambda_k^c} = \begin{cases} k\pi + \frac{\pi}{2} + O\left(\frac{1}{k}\right) & \text{if } \alpha = 0, \beta \neq \pi, \\ k\pi + \pi + O\left(\frac{1}{k}\right) & \text{if } \alpha = 0, \beta = \pi, \\ k\pi + O\left(\frac{1}{k}\right) & \text{if } \alpha \neq 0, \beta \neq \pi, \\ k\pi + \frac{\pi}{2} + O\left(\frac{1}{k}\right) & \text{if } \alpha \neq 0, \beta = \pi. \end{cases} \quad (5.53)$$

Note that the results of Theorem 5.5.2 apply to any transfer matrix, not specifically the original  $M$ . In (5.53), the subscript  $k$  in  $\lambda_k^c$  labels the oscillation count, but from classical Sturm-Liouville theory we know that the eigenfunction of the  $(k + 1)$ th eigenvalue of (5.52), (5.2), (5.3) has  $k$  zeroes in  $(-a, b)$ .

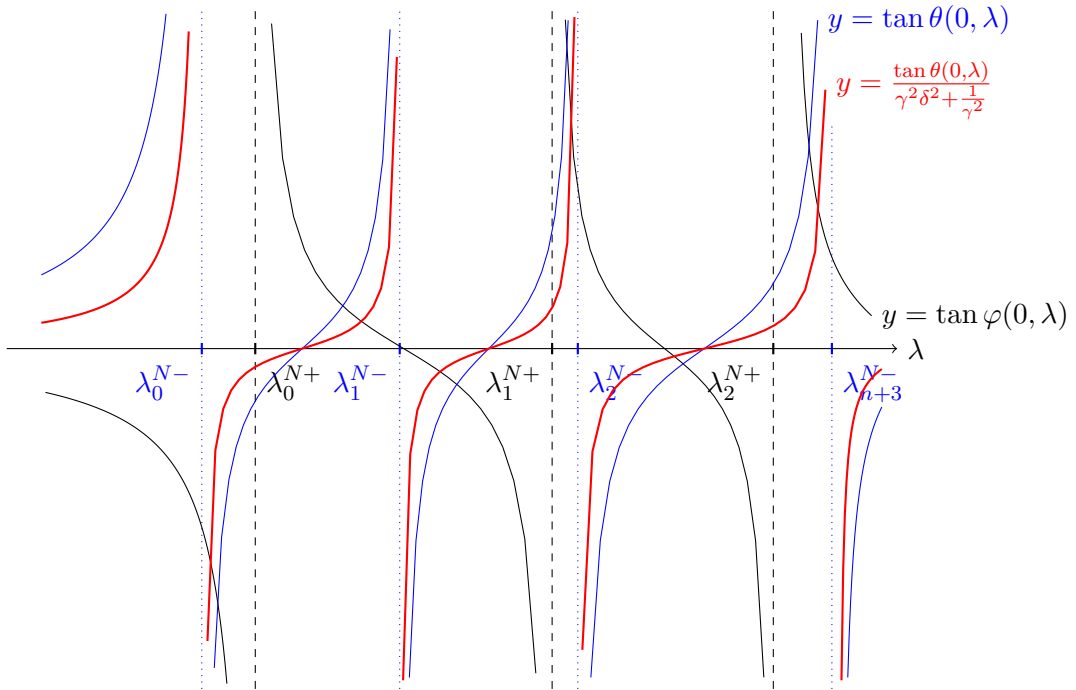


Figure 1: Example graphs of  $y = \tan \varphi(0, \lambda)$ ,  $y = \tan \theta(0, \lambda)$  and  $y = \frac{\tan \theta(0, \lambda)}{\gamma^2 \delta^2 + \frac{1}{\gamma^2}}$ .

Now, since the zeroes and poles of  $\tan \theta(0, \lambda)$  and  $\frac{\tan \theta(0, \lambda)}{\gamma^2 \delta^2 + \frac{1}{\gamma^2}}$  coincide, we observe that there is a one to one correspondence between eigenvalues of (5.1)-(5.3), (5.51) and the eigenvalues of (5.52), (5.2), (5.3) according to oscillation count. That is, if the graphs of  $y = \tan \varphi(0, \lambda)$  and  $y = \frac{\tan \theta(0, \lambda)}{\gamma^2 \delta^2 + \frac{1}{\gamma^2}}$  intersect for some  $\lambda \in (\lambda_{n-1}^{D-}, \lambda_n^{N-}]$ , say, then the graphs of  $y = \tan \varphi(0, \lambda)$  and  $y = \tan \theta(0, \lambda)$  will also intersect in the interval  $(\lambda_{n-1}^{D-}, \lambda_n^{N-}]$ , giving respective eigenvalues whose corresponding eigenfunctions have  $n$  zeroes in  $(-a, 0)$ . Such intersections must correspond to intersections in an interval of the form  $[\lambda_m^{N+}, \lambda_m^{D+})$  for some  $m$ . Hence, the respective eigenfunctions will have  $m$  zeroes in  $(0, b)$ , giving total oscillation counts of  $n + m$  in  $(-a, b)$ . A similar argument can be made for intersections below the line  $y = 0$ . The case of intersections at Dirichlet eigenvalues,  $\lambda_{n-1}^{D-}$  equal to  $\lambda_m^{D+}$  say, is trivial. Since the oscillation count of an eigenvalue determines its asymptotic form according to Theorem 5.5.2, not only do the corresponding

eigenvalues of (5.1)-(5.3), (5.51) and of (5.52), (5.2), (5.3) have the same oscillation counts, but also the same asymptotic form. Using this information we can conclude that the  $(k + 1)$ th eigenvalue of (5.1)-(5.3), (5.51) must have a total oscillation count of  $k$  in  $(-a, b)$ . Simply put, we can count the zeroes of  $\Delta_1(\lambda)$  by counting the intersections of the graphs of  $y = \tan \varphi(0, \lambda)$  and  $y = \tan \theta(0, \lambda)$  which is done by identifying oscillation counts.

We now consider the eigenvalues of (5.1)-(5.4).

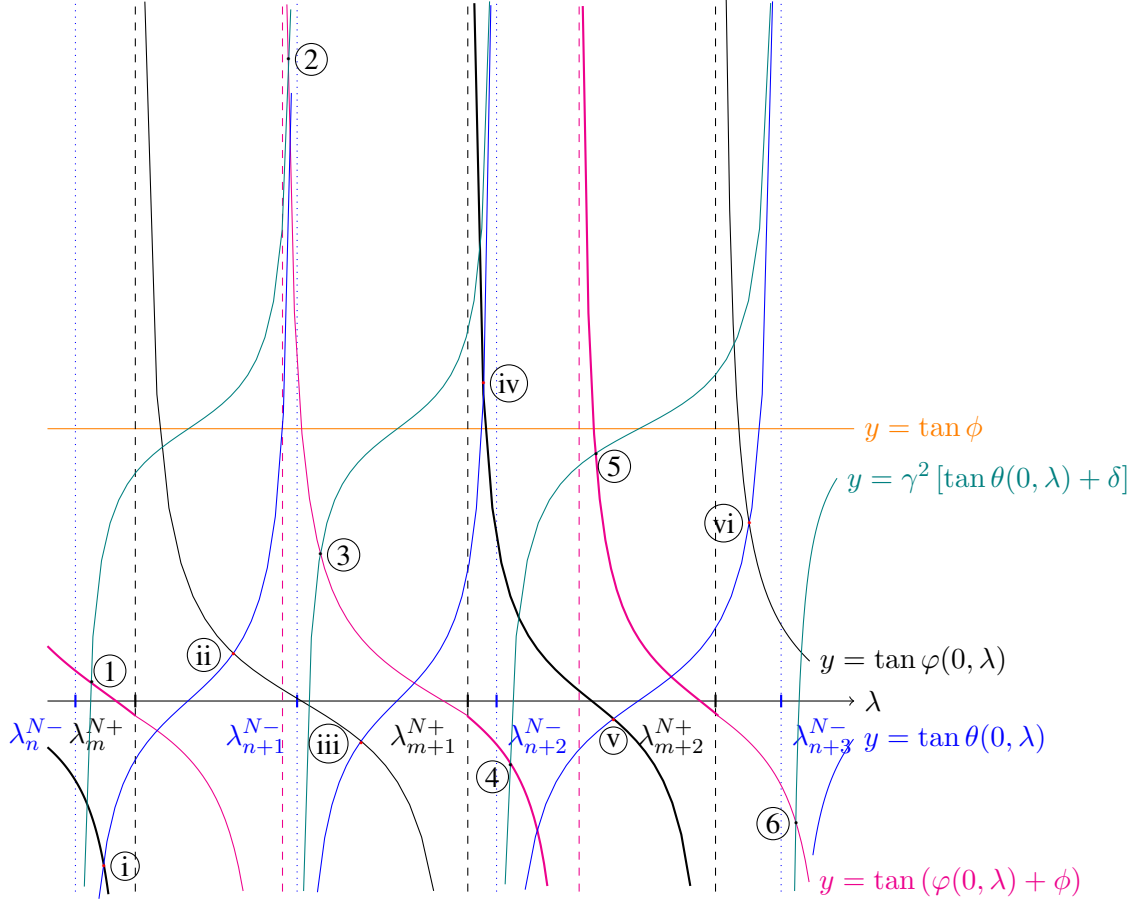


Figure 2: Comparison of intersections of  $y = \tan(\varphi(0, \lambda) + \phi)$  and  $y = \gamma^2 [\tan \theta(0, \lambda) + \delta]$  with intersections of  $y = \tan \varphi(0, \lambda)$  and  $y = \tan \theta(0, \lambda)$ .

In Figure 2 we have labelled intersections of the graphs of  $y = \tan(\varphi(0, \lambda) + \phi)$  and  $y = \gamma^2 [\tan \theta(0, \lambda) + \delta]$  with arabic numbers  $1, 2, \dots, 6$ , and intersections of the graphs of  $y = \tan \varphi(0, \lambda)$  and  $y = \tan \theta(0, \lambda)$  by roman numerals  $i, ii, \dots, vi$ . Comparing to the vertical asymptotes and  $\lambda$ -intercepts of  $y = \tan \varphi(0, \lambda)$  and  $y = \tan \theta(0, \lambda)$  respectively, we deduce that the intersections as labelled above are (1)  $\lambda = \lambda_{n,m}$ , (2)  $\lambda = \lambda_{n+1,m}$ , (3)  $\lambda = \lambda_{n+1,m+1}$ , (4)  $\lambda = \lambda_{n+2,m+1}$ , (5)  $\lambda = \lambda_{n+2,m+2}$ , (6)  $\lambda = \lambda_{n+3,m+2}$  (in the notation of Theorem 5.5.2) and (i)  $\lambda = \lambda_{n+m}^c$ , (ii)  $\lambda = \lambda_{(n+1)+m}^c$ , (iii)  $\lambda = \lambda_{(n+1)+(m+1)}^c$ , (iv)  $\lambda = \lambda_{(n+2)+(m+1)}^c$ , (v)  $\lambda = \lambda_{(n+2)+(m+2)}^c$ , (vi)  $\lambda = \lambda_{(n+3)+(m+2)}^c$  (in the notation of (5.53)).



Let  $\Gamma_k = \left\{ \lambda = [A_k e^{i\vartheta}]^2 : \vartheta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right) \right\}$  where  $A_k$  is chosen to satisfy

$$\xi A_k = \begin{cases} k\pi & \text{if } \alpha = 0, \beta \neq \pi, \\ (k + 1/2)\pi & \text{if } \alpha = 0, \beta = \pi, \\ (k - 1/2)\pi & \text{if } \alpha \neq 0, \beta \neq \pi, \\ k\pi & \text{if } \alpha \neq 0, \beta = \pi. \end{cases}$$

Then for large  $k$ ,  $\Gamma_k$  encloses  $\lambda_0^c, \dots, \lambda_{k-1}^c$  (see equation (5.53)) and hence  $k$  zeroes of  $\Delta_1(\lambda)$  by the discussion above. Since

$$|\Delta_1(\lambda)| > \left| \frac{\tan \phi}{\sqrt{\gamma^2 \delta^2 + \frac{1}{\gamma^2}}} u(0^-) v(0^+) \right|$$

in equation (5.49) for  $\lambda \in \Gamma_k$ , we conclude by Rouché's Theorem that  $\omega(\lambda)$  also has  $k$  zeroes inside  $\Gamma_k$ . Now from (5.53) we have that

$$\xi \sqrt{\lambda_{k-1}^c} = \begin{cases} k\pi - \frac{\pi}{2} + O\left(\frac{1}{k-1}\right) & \text{if } \alpha = 0, \beta \neq \pi, \\ k\pi + O\left(\frac{1}{k-1}\right) & \text{if } \alpha = 0, \beta = \pi, \\ (k-1)\pi + O\left(\frac{1}{k-1}\right) & \text{if } \alpha \neq 0, \beta \neq \pi, \\ k\pi - \frac{\pi}{2} + O\left(\frac{1}{k-1}\right) & \text{if } \alpha \neq 0, \beta = \pi. \end{cases} \quad (5.54)$$

On the other hand,

$$\xi s_{N,M} = \begin{cases} (N+M)\pi + \frac{\pi}{2} + O\left(\frac{1}{N+M}\right) & \text{if } \alpha = 0, \beta \neq \pi, \\ (N+M+1)\pi + O\left(\frac{1}{N+M}\right) & \text{if } \alpha = 0, \beta = \pi, \\ (N+M)\pi + O\left(\frac{1}{N+M}\right) & \text{if } \alpha \neq 0, \beta \neq \pi, \\ (N+M)\pi + \frac{\pi}{2} + O\left(\frac{1}{N+M}\right) & \text{if } \alpha \neq 0, \beta = \pi, \end{cases} \quad (5.55)$$

by Theorem 5.5.2. Comparing the above approximations, (5.54) and (5.55), we conclude that the  $k$ th zero of  $\omega(\lambda)$  (i.e. the  $k$ th largest eigenvalue of (5.1)-(5.4)) must have total oscillation count of  $(N+M)$  equal to  $k-1$ . Thus we can conclude that  $\lambda_{N,M} = [s_{N,M}]^2$ , which has oscillation count  $N+M$ , is the  $(N+M+1)$ th largest eigenvalue of (5.1)-(5.4). That is

$$\lambda_{N,M} = \lambda_{N+M}.$$

□

**Theorem 5.5.5.** *Suppose that  $\tan \phi \neq \gamma^2 \delta$ . Let  $\lambda_{N,M} = (s_{N,M})^2$  be an eigenvalue of (5.1)-(5.4) whose eigenfunction has oscillation count  $N$  in  $(-a, 0]$  and  $M$  in  $[0, b)$ , with a zero at  $x = 0^-$  or  $x = 0^+$  contributing  $1/2$ . Then, in the notation of (5.45),*

- (i) *either  $N+M$  is an odd-integer multiple of  $1/2$  and  $\lambda_{N,M} = \lambda_{\lceil N+M \rceil}$ ,*
- (ii) *or  $N, M \in \mathbb{N}_0$  and either  $\lambda_{N,M} = \lambda_{N+M}$  or  $\lambda_{N,M} = \lambda_{N+M+1}$ .*

Here,  $\lceil t \rceil$  denotes the smallest integer greater than or equal to  $t$ .

*Proof.* We observe that the characteristic determinant (5.44) can be written in the form

$$\omega(\lambda) = \Delta_2(\lambda) + f(\lambda), \quad (5.56)$$

with

$$\Delta_2(\lambda) = \left[ \gamma \delta \cos \phi - \frac{1}{\gamma} \sin \phi \right] pu'(0^-)pv'(0^+) \quad (5.57)$$

and

$$\begin{aligned} f(\lambda) = & \gamma \cos \phi u(0^-)pv'(0^+) - \left[ \gamma \delta \sin \phi + \frac{1}{\gamma} \cos \phi \right] pu'(0^-)v(0^+) \\ & - \gamma \sin \phi u(0^-)v(0^+). \end{aligned} \quad (5.58)$$

Here, the zeroes of  $\Delta_2(\lambda)$  are the Neumann eigenvalues  $\lambda_n^{N^-}$  (corresponding to (5.46), (5.2) with  $y'(0^-) = 0$ ) and  $\lambda_n^{N^+}$  (corresponding to (5.47), (5.3) with  $y'(0^+) = 0$ ). By methods similar to those used in the proof of Theorem 5.5.2, we can show that

$$\xi^- \sqrt{\lambda_n^{N^-}} = \begin{cases} n\pi + \frac{\pi}{2} + O\left(\frac{1}{n}\right), & \text{if } \alpha = 0, \\ n\pi + O\left(\frac{1}{n}\right), & \text{if } \alpha \in (0, \pi), \end{cases} \quad (5.59)$$

$$\xi^+ \sqrt{\lambda_m^{N^+}} = \begin{cases} m\pi + O\left(\frac{1}{m}\right), & \text{if } \beta \in (0, \pi), \\ m\pi + \frac{\pi}{2} + O\left(\frac{1}{m}\right), & \text{if } \beta = \pi. \end{cases} \quad (5.60)$$

Here, the subscripts  $n$  and  $m$  in  $\lambda_n^{N^-}$  and  $\lambda_m^{N^+}$  respectively, denote Neumann eigenvalues of oscillation count  $n$  in  $(-a, 0)$  and  $m$  in  $(0, b)$  respectively. To compare, the Dirichlet eigenvalues  $\lambda_n^{D^-}$  (corresponding to (5.46), (5.2) with  $y(0^-) = 0$ ) and  $\lambda_n^{D^+}$  (corresponding to (5.47), (5.3) with  $y(0^+) = 0$ ) have the following approximations:

$$\xi^- \sqrt{\lambda_n^{D^-}} = \begin{cases} n\pi + \pi + O\left(\frac{1}{n}\right), & \text{if } \alpha = 0, \\ n\pi + \frac{\pi}{2} + O\left(\frac{1}{n}\right), & \text{if } \alpha \in (0, \pi), \end{cases} \quad (5.61)$$

$$\xi^+ \sqrt{\lambda_m^{D^+}} = \begin{cases} m\pi + \frac{\pi}{2} + O\left(\frac{1}{m}\right), & \text{if } \beta \in (0, \pi), \\ m\pi + \pi + O\left(\frac{1}{m}\right), & \text{if } \beta = \pi. \end{cases} \quad (5.62)$$

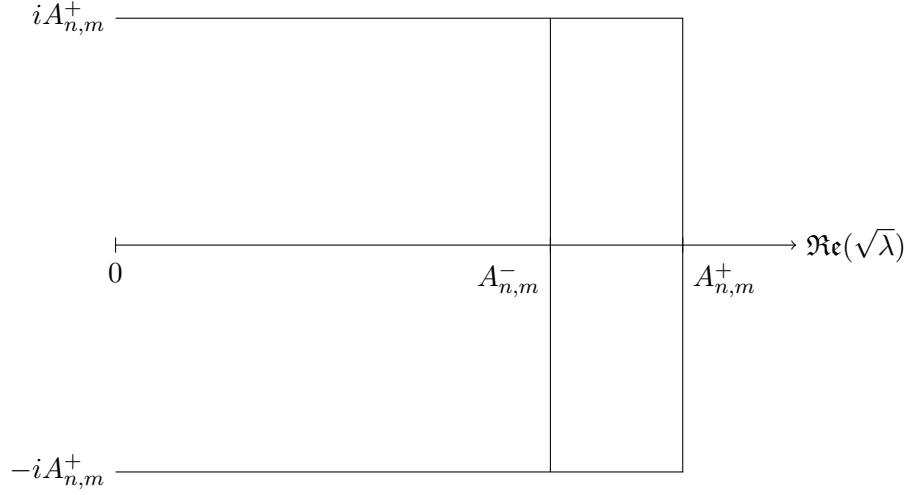
Now, let  $\lambda_{N,M}$  be an eigenvalue of (5.1)-(5.4) with corresponding oscillation count  $N$  in  $(-a, 0]$  and  $M$  in  $[0, b)$ , with  $N, M$  large. Then the oscillation count includes at most one half-zero at  $x = 0$ .

- (i) Suppose that the oscillation count includes one half-zero at  $x = 0$ . We present only one case, say,  $N = n$  and  $M = m + 1/2$  where  $n, m \in \mathbb{N}$ . Then  $\lambda_{N,M} = \lambda_m^{D^+}$  and we know from Theorem 5.5.2 that  $\xi^- \left[ \sqrt{\lambda_{N,M}} - \sqrt{\lambda_n^{N^-}} \right] = O\left(\frac{1}{N+M}\right)$ . Let

$$A_{n,m}^- = \begin{cases} \max \left\{ \frac{(n+1/4)\pi}{\xi^-}, \frac{(m+1/4)\pi}{\xi^+} \right\}, & \text{if } \alpha = 0, \beta \neq \pi, \\ \max \left\{ \frac{(n+1/4)\pi}{\xi^-}, \frac{(m+3/4)\pi}{\xi^+} \right\}, & \text{if } \alpha = 0, \beta = \pi, \\ \max \left\{ \frac{(n-1/4)\pi}{\xi^-}, \frac{(m+1/4)\pi}{\xi^+} \right\}, & \text{if } \alpha \neq 0, \beta \neq \pi, \\ \max \left\{ \frac{(n-1/4)\pi}{\xi^-}, \frac{(m+3/4)\pi}{\xi^+} \right\}, & \text{if } \alpha \neq 0, \beta = \pi, \end{cases} \quad (5.63)$$

and

$$A_{n,m}^+ = \begin{cases} \min \left\{ \frac{(n+3/4)\pi}{\xi^-}, \frac{(m+3/4)\pi}{\xi^+} \right\}, & \text{if } \alpha = 0, \beta \neq \pi, \\ \min \left\{ \frac{(n+3/4)\pi}{\xi^-}, \frac{(m+5/4)\pi}{\xi^+} \right\}, & \text{if } \alpha = 0, \beta = \pi, \\ \min \left\{ \frac{(n+1/4)\pi}{\xi^-}, \frac{(m+3/4)\pi}{\xi^+} \right\}, & \text{if } \alpha \neq 0, \beta \neq \pi, \\ \min \left\{ \frac{(n+1/4)\pi}{\xi^-}, \frac{(m+5/4)\pi}{\xi^+} \right\}, & \text{if } \alpha \neq 0, \beta = \pi. \end{cases} \quad (5.64)$$



Let  $\Gamma_{n,m} = \bigcup_{j=1}^5 \Gamma_{n,m}^j$  and  $\Gamma_{n,m}^* = \Gamma_{n,m}^2 \cup \Gamma_{n,m}^3 \cup \Gamma_{n,m}^4 \cup \Gamma_{n,m}^6$ , where

$$\begin{aligned} \Gamma_{n,m}^1 &= \left\{ (\zeta - iA_{n,m}^+)^2 : \zeta \in [0, A_{n,m}^-] \right\}, \\ \Gamma_{n,m}^2 &= \left\{ (\zeta - iA_{n,m}^+)^2 : \zeta \in [A_{n,m}^-, A_{n,m}^+] \right\}, \\ \Gamma_{n,m}^3 &= \left\{ (A_{n,m}^+ + i\zeta)^2 : \zeta \in [-A_{n,m}^+, A_{n,m}^+] \right\}, \\ \Gamma_{n,m}^4 &= \left\{ (\zeta + iA_{n,m}^+)^2 : \zeta \in [A_{n,m}^-, A_{n,m}^+] \right\}, \\ \Gamma_{n,m}^5 &= \left\{ (\zeta + iA_{n,m}^+)^2 : \zeta \in [0, A_{n,m}^-] \right\}, \\ \Gamma_{n,m}^6 &= \left\{ (A_{n,m}^- + i\zeta)^2 : \zeta \in [-A_{n,m}^+, A_{n,m}^+] \right\}. \end{aligned}$$

Then  $|\Delta_2(\lambda)| > |f(\lambda)|$  if  $\lambda \in \Gamma_{n,m}^*$  or if  $\lambda \in \Gamma_{n,m}$ . By Rouché's Theorem we conclude that  $\lambda_{n,m+1/2} = \lambda_{n+m+1}$  ( $\Gamma_{n,m}$  contains precisely  $n + m + 2$  zeroes of  $\Delta_2(\lambda)$ ).

- (ii) Suppose that  $N = n$ ,  $M = m$  with  $n, m \in \mathbb{N}$ . We know from Note 5.5.3 that there could possibly be a second eigenvalue with the same oscillation count, and hence the same asymptotic form by Theorem 5.5.2. Further, we know from Theorem 5.5.2 that  $\xi^- \left[ \sqrt{\lambda_{n,m}} - \sqrt{\lambda_n^{N^-}} \right] = O\left(\frac{1}{n+m}\right)$  and, likewise,  $\xi^+ \left[ \sqrt{\lambda_{n,m}} - \sqrt{\lambda_m^{N^+}} \right] = O\left(\frac{1}{n+m}\right)$ . Define

$$A_{n,m}^- = \begin{cases} \max \left\{ \frac{(n-1/4)\pi}{\xi^-}, \frac{(m-3/4)\pi}{\xi^+} \right\}, & \text{if } \alpha = 0, \beta \neq \pi, \\ \max \left\{ \frac{(n-1/4)\pi}{\xi^-}, \frac{(m-1/4)\pi}{\xi^+} \right\}, & \text{if } \alpha = 0, \beta = \pi, \\ \max \left\{ \frac{(n-3/4)\pi}{\xi^-}, \frac{(m-3/4)\pi}{\xi^+} \right\}, & \text{if } \alpha \neq 0, \beta \neq \pi, \\ \max \left\{ \frac{(n-3/4)\pi}{\xi^-}, \frac{(m-1/4)\pi}{\xi^+} \right\}, & \text{if } \alpha \neq 0, \beta = \pi, \end{cases} \quad (5.65)$$

and

$$A_{n,m}^+ = \begin{cases} \min \left\{ \frac{(n+5/4)\pi}{\xi^-}, \frac{(m+3/4)\pi}{\xi^+} \right\}, & \text{if } \alpha = 0, \beta \neq \pi, \\ \min \left\{ \frac{(n+5/4)\pi}{\xi^-}, \frac{(m+5/4)\pi}{\xi^+} \right\}, & \text{if } \alpha = 0, \beta = \pi, \\ \min \left\{ \frac{(n+3/4)\pi}{\xi^-}, \frac{(m+3/4)\pi}{\xi^+} \right\}, & \text{if } \alpha \neq 0, \beta \neq \pi, \\ \min \left\{ \frac{(n+3/4)\pi}{\xi^-}, \frac{(m+5/4)\pi}{\xi^+} \right\}, & \text{if } \alpha \neq 0, \beta = \pi. \end{cases} \quad (5.66)$$

Let  $\Gamma_{n,m}$  and  $\Gamma_{n,m}^*$  be defined as in case (i) above but with  $A_{n,m}^\pm$  as defined here. Considering  $\lambda \in \Gamma_{n,m}^*$ , we have  $|\Delta_2(\lambda)| > |f(\lambda)|$  with precisely 2 zeroes of  $\Delta_2$  contained in  $\Gamma_{n,m}^*$ . By Rouché's Theorem we conclude that  $\omega(\lambda)$  must have 2 zeroes inside  $\Gamma_{n,m}^*$ . One is  $\lambda_{n,m}$  the second eigenvalue can have maximum generalized oscillation count  $n + m + 2$  and minimum generalized oscillation count  $n + m - 2$  (in  $(-a, 0] \cup [0, b)$ ). Now considering  $\lambda \in \Gamma_{n,m}$ , we again have  $|\Delta_2(\lambda)| > |f(\lambda)|$ . Moreover, there are a total of  $n + m + 2$  zeroes of  $\Delta_2$  inside  $\Gamma_{n,m}$ , thus we conclude from Rouché's Theorem that  $\lambda_{n,m}$  is either the  $(n + m + 2)$ th or the  $(n + m + 1)$ th largest zero of  $\omega(\lambda)$ . That is, either  $\lambda_{n,m} = \lambda_{n+m+1}$  or  $\lambda_{n,m} = \lambda_{n+m}$ .

□

# Chapter 6

## Further work

In this thesis we have presented new work on discontinuous Sturm-Liouville problems involving two particular classes of transmission conditions. We extended Sturm's oscillation theorem to the case of discontinuous problems with constant coefficient transmission conditions. Our methods enabled us to analyse general real non-singular  $2 \times 2$  transfer matrices. Up to now, nodal counts have been studied only for very specific types of transfer matrices. Furthermore, the work presented in Chapters 3 and 4 is the first to deal with transmission conditions having a rational dependence on the spectral parameter. Here, we showed that the double geometric multiplicity of certain eigenvalues combined with the discontinuity in the eigenfunctions introduced some interesting challenges with regards to the analysis. However, there is still much more to consider.

We have begun investigations into an interesting inverse problem pertaining to transmission conditions of the form discussed in Chapter 5. A well-studied inverse problem for continuous Sturm-Liouville operators deals with the question of determining the coefficients of the Sturm-Liouville equation from two spectra. The first spectrum consists of eigenvalues associated with the boundary value problem under consideration, and the second spectrum is obtained by changing one of the two end conditions. We propose the following alteration. Consider the Sturm-Liouville problem:

$$-y'' + qy = \lambda y, \quad x \in (-a, 0) \cup (0, b), \quad (6.1)$$

with boundary conditions

$$y(-a) \cos \alpha = (py')(-a) \sin \alpha, \quad \alpha \in [0, \pi), \quad (6.2)$$

$$y(b) \cos \beta = (py')(b) \sin \beta, \quad \beta \in (0, \pi], \quad (6.3)$$

and transmission conditions

$$\begin{bmatrix} y(0^+) \\ y'(0^+) \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} y(0^-) \\ y'(0^-) \end{bmatrix} \quad (6.4)$$

where  $T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$  has  $t_{ij} \in \mathbb{R}$  and  $\det T > 0$ . We ask, is it possible to determine  $q$  from two spectra, where the second spectrum is obtained not by changing one of the end boundary conditions, but rather by changing the coefficients of the transfer matrix  $T$ ? The question arises whether the second spectrum gives us enough information to solve the inverse problem or if additional data is needed.

We have also made initial investigations into developing oscillation theory for Sturm-Liouville equations with rationally dependent transmission conditions of the type considered in Chapters 3 and 4. The possibility also exists to extend the results of Chapter 5 to the case of finitely many transmission conditions, and to the case of  $\det T < 0$ .

# References

- [1] S. AGMON, *Lectures on elliptic boundary value problems*, AMS Chelsea Publishing, (2010).
- [2] Z. AKDOGAN, M. DEMIRCI, O. SH. MUKHTAROV, Discontinuous Sturm-Liouville problems with eigenparameter dependent boundary and transmissions conditions, *Acta Applicandae Mathematicae*, **86** (2005), 329-344.
- [3] S. ALBEVERIO, F. GESZTESY, R. HOEGH-KROHN, H. HOLDEN, *Solvable models in quantum mechanics*, Springer Verlag, Berlin, (1988); 2nd ed. with an Appendix by P. Exner, Chelsea, AMS, Providence, 2005.
- [4] S. ALBEVERIO, P. KURASOV, *Singular perturbations of singular operators: Solvable Schrödinger type operators*, Cambridge University Press, (1999).
- [5] S. ALBERIO, J. BRASCHE, M. RÖCKNER, Dirichlet forms and generalized Schrödinger operators, *Lecture Notes in Physics*, **345** (2005), 1–42.
- [6] R. KH. AMIROV , A. S. OZKAN, B. KESKIN, Inverse problems for impulsive Sturm–Liouville operator with spectral parameter linearly contained in boundary conditions, *Integral Transforms And Special Functions*, **20**, iss. 8 (2009), 607-618.
- [7] R. S. ANDERSSON, The effect of discontinuities in density and shear velocity on the asymptotic overtone structure of torsional eigenfrequencies of the Earth, *Geophys. J. R. astr. Soc.* **50** (1977), 303–309.
- [8] F. V. ATKINSON, *Discrete and continuous boundary problems*, Academic Press, New York (1964).
- [9] R. BAND, The nodal count  $\{0, 1, 2, 3, \dots\}$  implies the graph is a tree, *Phil. Trans. R. Soc. A*, **372** 20120504, (2014).
- [10] P. BINDING, P. BROWNE, B. A. WATSON, Inverse spectral problems for Sturm–Liouville equations with eigenparameter dependent boundary conditions, *J. London Math. Soc.*, **62** (2000), 161–182.
- [11] P. BINDING, P. BROWNE, B. A. WATSON, Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter, I, *Proc. Edinb. Math. Soc.*, **45**, iss. 3 (2002), 631–645.
- [12] P. A. BINDING, P. J. BROWNE, B. A. WATSON, Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter, II, *Journal of Computational and Applied Mathematics*, **148** (2002), 147-168.
- [13] P. A. BINDING, P. J. BROWNE, B. A. WATSON, Spectral asymptotics for Sturm–Liouville equations with indefinite weight, *Transactions of the American Mathematical Society*, **354** (2002), 4043–4065.

- [14] P.A. BINDING, P.J. BROWNE, B.A. WATSON, Decomposition of spectral asymptotics Sturm–Liouville equations with a turning point, *Advances in Differential Equations*, **8** (2003), 491–511.
- [15] P. A. BINDING, P. J. BROWNE, B. A. WATSON, Equivalence of inverse Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter, *Journal of Mathematical Analysis and Applications*, **291**, iss. 1(2004), 246–261.
- [16] P. A. BINDING, P. J. BROWNE, B. A. WATSON, Recovery of the  $m$ -function from spectral data for generalized Sturm–Liouville problems, *Journal of Computational and Applied Mathematics*, **171** (2004), 73–91.
- [17] P. BINDING, H. VOLKMER, Prüfer angle asymptotics for Atkinson’s semi-definite Sturm-Liouville eigenvalue problem, *Math. Nachr.*, **278**, iss. 12 - 13 (2005), 1458–1475.
- [18] G. BÖRG, Eine Umkehrung der Sturm-Liouvilleschen eigenwertaufgabe, *Acta Math.* **78** (1946), 1-96.
- [19] J. F. BRASCHE, L. NIZHNIK, One-dimensional Schrödinger operators with  $\delta'$ -interactions on a set of Lebesgue measure zero, *Oper. Matrices*, **7** (2013), 887-904.
- [20] D. BUSCHMANN, G. STOLZ, J. WEIDMANN, One-dimensional Schrödinger operators with local point interactions, *J. Reine Angew. Math.* **467** (1995), 169–186.
- [21] A. CHERNOZHUKOVA, G. FREILING, A uniqueness theorem for the boundary value problems with non-linear dependence on the spectral parameter in the boundary conditions, *Inverse Problems in Science and Engineering*, **17**, iss. 6 (2009), 777–785.
- [22] M. V. CHUGUNOVA, Inverse spectral problem for the Sturm–Liouville operator with eigenvalue parameter dependent boundary conditions, *Operator Theory: Advances and Applications*, **123** (2001), 187–194.
- [23] E. A. CODDINGTON, N. LEVINSON, *Theory of Ordinary Differential Equations*, McGraw-Hill, (1955).
- [24] I. DEGHANI, A. J. AKBARFAM, Resolvent operator and self-adjointness of Sturm-Liouville operators with a finite number of transmission conditions, *Mediterr. J. Math.*, **11** (2014), 447-462.
- [25] J. ECKHARDT, A. KOSTENKO, M. MALAMUD, G. TESCHL, One-dimensional Schrödinger operators with  $\delta'$ -interactions on Cantor-type sets, *J. of Differential Equations*, **257**, iss. 2 (2014), 415-449.
- [26] W. N. EVERITT, M. K. KWONG, A. ZETTL, Oscillation of eigenfunctions of weighted regular Sturm–Liouville problems, *J. London Math. Soc.* , **27** (1983), 106–120.
- [27] G. FREILING, V. YURKO, *Inverse Sturm-Liouville Problems and their Applications*, Nova Science, New York (2001).
- [28] G. FREILING, AND V. A. YURKO, Inverse problems for Sturm-Liouville equations with boundary conditions polynomially dependent on the spectral parameter, (2010), *Inverse Problems*, **26**, 055003.
- [29] C. T. FULTON, Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions, *Proc. Roy. Soc. Edinburgh*, **77**, iss. 3-4 (1977), 293–308.

- [30] C. T. FULTON, Singular eigenvalue problems with eigenvalue parameter contained in the boundary conditions, *Proc. Roy. Soc. Edinburgh*, **87**, iss. 1-2 (1980), 1–34.
- [31] M. G. GASIMOV, The inverse problem for the Sturm–Liouville problem, *Dokl. Akad. Nauk SSSR*, **150** (1963), 68–71.
- [32] M. G. GASIMOV, B.M. LEVITAN, The inverse problem for a Dirac system , *Dokl. Akad. Nauk SSSR*, **167** (1966), 967–970.
- [33] I. M. GELFAND, B.M. LEVITAN, On the determination of a differential equation from its spectral function, *Izv. Akad. Nauk SSSR Ser. Mat.*, **15**, iss. 4 (1951), 309–360. Translated in *Amer. Math. Soc. Transl. ser. 2*, **1**, (1955), 253–304.
- [34] N. A. GORDON, D.B. PEARSON, Point transfer matrices for the Schrödinger equation: The algebraic theory, *Proceedings of the Royal Society of Edinburgh*, **129A** (1999), 717–732.
- [35] Y. GÜLDÜ, Inverse eigenvalue problems for a discontinuous Sturm–Liouville operator with two discontinuities, *Bound. Value Probl.* **2013** (2013), 209–221.
- [36] N. J. GULIYEV, Inverse eigenvalue problems for Sturm-Liouville equations with spectral parameter linearly contained in one of the boundary conditions, *Inverse Problems*, **21** (2005), 1315-1330.
- [37] O. H. HALD, Discontinuous inverse eigenvalue problems, *Comm. Pure Appl. Math.* **37**, iss. 5, (1984), 539-577.
- [38] H. HOCHSTADT, The inverse Sturm–Liouville problem, *Comm. Pure Appl. Math.* **26** (1973), 715–729.
- [39] H. HOCHSTADT, On inverse problems associated with Sturm–Liouville operators, *J. Differential Equations*, **17** (1975), 220–235.
- [40] H. HOCHSTADT, B. LIEBERMAN, An inverse Sturm–Liouville problem with mixed given data, *SIAM J. Appl. Math.* **34** (1978), 676-680.
- [41] B. KESKIN, Spectral problems for impulsive Dirac operators with spectral parameters entering via polynomials in the boundary and discontinuity conditions, *Appl. Math. Sci.* **6**, iss. 38 (2012), 1893–1899.
- [42] M. G. KREIN, Solution of the inverse Sturm–Liouville problem, *Dokl. Akad. Nauk SSSR* , **76** (1951), 21–24.
- [43] M. G. KREIN, On the transfer function of a one-dimensional boundary problem of second order, *Dokl. Akad. Nauk SSSR* , **88** (1953), 405–408.
- [44] M. G. KREIN, On some case of effective determination of the density of an inhomogenous cord from its spectral function, *Dokl. Akad. Nauk SSSR* , **93** (1953), 617–620.
- [45] L. KONG, Q. KONG, M. K. KWONG, AND J. S. W. WONG, Linear Sturm–Liouville problems with multi-point boundary conditions, *Math. Nachr.*, **286**, iss. 11-12 (2013), 1167–1179.
- [46] Q. KONG, Q. R. WANG, Using time scales to study multi-interval Sturm-Liouville problems with interface conditions, *Results Math.*, **63** (2013), 451-465.



- [47] R. J. KRUEGER, An inverse problem for an absorbing medium with multiple discontinuities, *Quarterly of Applied Mathematics*, **36**, iss. 3 (1978), 235–253.
- [48] R. J. KRUEGER, Inverse problems for nonabsorbing media with discontinuous material properties, *J. Math. Phys.* **23**, iss. 3 (1982), 396–404.
- [49] B. JA. LEVIN, *Distribution of zeroes of entire functions*, American Mathematical Society, Revised edition (1980).
- [50] B. M. LEVITAN, The application of generalized displacement operators to linear differential equations of the second order, *Usp. Mat. Nauk* , **4** (1949), 3–112. Translated by F.V. Atkinson *Amer. Math. Soc. Transl.*, **59**, 1–135.
- [51] B. M. LEVITAN, Certain questions in the spectral theory of self-adjoint differential operators, *Usp. Mat. Nauk* , **11** (1956), 117–144. Translated in *Amer. Mat. Soc. Transl.*, **18**, 49–80.
- [52] B. M. LEVITAN, I.S. SARGSIAN, *Introduction to Spectral Theory*, American Mathematical Society, (1975).
- [53] B. M. LEVITAN, *Inverse Sturm–Liouville Problems*, VNU Science Press, (1987).
- [54] N. LEVINSON, The inverse Sturm–Liouville problem, *Math. Tidsskr.* **13** (1949), 25–30.
- [55] W. S. LOUD, Self-Adjoint multi-point boundary value problems, *Pacific J. Math.*, **24** (1968), 303–317.
- [56] V. A. MARČENKO, Some problems in the theory of a second-order differential operator, *Dokl. Akad. Nauk SSSR*, **72** (1950), 457–460.
- [57] J. R. MCLAUGHLIN, Inverse spectral theory using nodal points as data A uniqueness result, *Journal of Differential Equations*, **73**, iss. 2 (1988), 354–362.
- [58] R. MENNICKEN, H. SCHMI, A. A. SHKALIKOV, On the eigenvalue accumulation of Sturm–Liouville problems depending nonlinearly on the spectral parameter, *Mathematische Nachrichten*, **189**, iss. 1 (1998), 157–170.
- [59] O. SH. MUKHTAROV, M. KANDEMIR, N. KURUOGLU, Distribution of eigenvalues for the discontinuous boundary value problem with functional–manypoint conditions, *Israel Journal of Mathematics*, **129** (2002), 143–156.
- [60] O. SH. MUKHTAROV, M. KADAKAL, F. S. MUHTAROV, Eigenvalues and normalized eigenfunctions of discontinuous Sturm–Liouville problem with transmission conditions, *Reports on Mathematical Physics*, **54** (2004), 41–56.
- [61] O. SH. MUKHTAROV, M. KADAKAL AND F. S. MUHTAROV, On discontinuous Sturm–Liouville problems with transmission conditions, *J. Math. Kyoto Univ. (JMKYAZ)*, **44**, iss. 4 (2004), 779–798.
- [62] O. SH. MUKHTAROV, C. TUNC, Eigenvalue problems for Sturm–Liouville equations with transmission conditions, *Israel Journal of Mathematics*, **144** (2004), 367–380.
- [63] M. A. NAIMARK, *Linear Differential Operators, Part II*, Frederick Ungar Publishing, (1968).

- [64] R. NEVANLINNA, *Analytic Functions*, Springer-Verlag Berlin, (1970).
- [65] A. S. OZKAN, B. KESKIN, Spectral problems for Sturm-Liouville operator with boundary and jump conditions linearly dependent on the eigenparameter, *Inverse Problems in Science and Engineering*, **20**, iss. 6 (2012), 799–808.
- [66] A. S. OZKAN, B. KESKIN, AND Y. CAKMAK, Double discontinuous inverse problems for Sturm–Liouville operator with parameter dependent conditions, *Abstract and Applied Analysis*, vol. 2013, Article ID 794262, 7 pages, (2013).
- [67] A. S. OZKAN, Half-inverse Sturm–Liouville problem with boundary and discontinuity conditions dependent on the spectral parameter, *Inverse Problems in Science and Engineering*, **22**, iss. 5 (2014), 848–859.
- [68] A. S. OZKAN, B. KESKIN, Inverse nodal problems for Sturm-Liouville equation with eigenparameter–dependent boundary and jump conditions, *Inverse Problems in Science and Engineering*, **23**, iss. 8 (2015), 1306–1312.
- [69] YU. V. POKORNYI, A. V. BOROVSKIKH, Differential equations on networks (geometric graphs), *J. of Math. Sci.*, **119**, iss. 6 (2004), 691–718.
- [70] YU. V. POKORNYI, V. L. PRYADIEV, A. AL-OBEID, Oscillation properties of the spectrum of a boundary value problem on a graph, *Mat. Zametki*, **60**, iss. 3 (1996), 468–470.
- [71] P. SCHAPOTSCHNIKOW, Eigenvalue and nodal properties on quantum graph trees, *Waves in Random and Complex Media*, **16** (2006), 167–178.
- [72] H. SCHMID, C. TRETTER, Singular Dirac systems and Sturm-Liouville problems nonlinear in the spectral parameter, *Journal of Differential Equations*, **181**, iss. 2 (2002), 511–542.
- [73] E. SEN, O. SH. MUKHTAROV, Spectral properties of discontinuous Sturm-Liouville problems with a finite number of transmission conditions, *Mediterr. J. Math.*, **13**, iss. 1 (2016), 153–170.
- [74] C. SHIEH, V.A. YURKO, Inverse nodal and inverse spectral problems for discontinuous boundary value problems, *J. Math. Anal. Appl.* **347** (2008), 266–272.
- [75] C. STURM, Mémoire sur les équations différentielles linéaires du second ordre, *J. Math. Pures Appl.* **1** (1836), 106–186.
- [76] C. STURM, Mémoire sur une classe d'équations à différences partielles, *J. Math. Pures Appl.* **1** (1836), 373–444.
- [77] C. STURM, J. LIOUVILLE, Extrait d'un Mémoire sur le développement des fonctions en séries dont les différents termes sont assujettis à satisfaire à une même équation différentielle linéaire, contenant un paramètre variable, *J. Math. Pures Appl.* **2** (1837), 220–223.
- [78] M. M. THARWAT, Discontinuous Sturm–Liouville problems and associated sampling theories, *Abstract and Applied Analysis*, vol. 2011, Article ID 610232, 30 pages, 2011. doi:10.1155/2011/610232.
- [79] A. N. TIKHONOV, A. A. SAMARSKII, *Equations of mathematical physics*, Dover, New York (1990).

- [80] I. TITEUX, Y. YAKUBOV, Completeness of root functions for thermal conduction in a strip with piecewise continuous coefficients, *Math. Models Methods Appl. Sc.*, **7**, iss. 7 (1997), 1035-1050.
- [81] A. WANG, J. SUN, A. ZETTL, Two-interval Sturm-Liouville operators in modified Hilbert spaces, *J. Math. Anal. Appl.*, **328** (2007), 390-399.
- [82] A. WANG, J. SUN, X. HAO, AND S. YAO, Completeness of eigenfunctions of Sturm-Liouville problems with transmission conditions, *Methods and Applications of Analysis*, **16** (2009), 299-312.
- [83] A. WANG, A. ZETTL, Self-adjoint Sturm-Liouville operators with discontinuous conditions, *Methods and Appl. of Analysis*, **22**, iss. 1 (2015), 37-66.
- [84] C. WANG, J. F. GETTRUST AND J. R. CLEARY, Asymptotic overtone structure in eigenfrequencies of torsional normal modes of the Earth: a model study, *Mathematics Geophysical Journal International*, **50**, iss. 2 (1876), 289-302.
- [85] Y. P. WANG, Inverse problems for Sturm-Liouville operators with interior discontinuities and boundary conditions dependent on the spectral parameter, *Math. Meth. Appl. Sci.* **36** (2013), 857-868.
- [86] Z. WEI, G. WEI, Inverse spectral problem for non selfadjoint Dirac operator with boundary and jump conditions dependent on the spectral parameter, *J. of Computational and Applied Mathematics*, **308** (2016), 199-214.
- [87] J. WEIDMANN, *Spectral theory of ordinary differential operators*, Springer-Verlag (1980).
- [88] J. WEIDMANN, *Linear operators in Hilbert spaces*, Springer-Verlag (1980).
- [89] H. WEYL, Über gewöhnliche differentialgleichungen mit singularitäten und die zugehörigen entwicklungen willkürlicher, *Funktionen. Math. Annalen*, **68** (1910), 220-269.
- [90] C. WILLIS, Inverse Sturm-Liouville problems with two discontinuities, *Inverse Problems*, **1**, iss. 3 (1985), 263-289.
- [91] C. YANG, Reconstruction of the Sturm-Liouville operators on a graph with  $\delta'_s$  couplings, *Tankang J. of Mathematics*, **42**, iss. 3 (2011), 329-342.
- [92] C. YANG, Trace and inverse problem of a discontinuous Sturm-Liouville operator with retarded argument, *J. Math. Anal. Appl.*, **395**, iss. 1 (2012), 30-41.
- [93] V. YURKO, Integral transforms connected with discontinuous boundary value problems, *J. Integral Transforms and Special Functions*, **10**, iss. 2 (2000), 141-164.
- [94] V. YURKO, Inverse spectral problems for Sturm-Liouville operators on graphs, *Inverse Problems*, **21** (2005), 1075-1086.
- [95] A. ZETTL, Adjoint and self-adjoint boundary value problems with interface conditions, *SIAM Journal on Applied Mathematics*, **16**, iss. 4 (1968), 851-859.
- [96] Y. ZHAO, J. SUN, A. ZETTL, Self-adjoint Sturm-Liouville problems with an infinite number of boundary conditions, *Math. Nachr.*, **289**, iss. 8-9 (2016), 1148-1169.

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