Inspection and Replacement Models for Reliability and Maintenance: Filling in gaps

BY

Honest Walter Chipoyera

Supervisor: Professor F. Beichelt

Submitted in fulfillment of the requirements for the Degree of Doctor of Philosophy

SCHOOL OF STATISTICS AND ACTUARIAL SCIENCE
Faculty of Science
University of Witwatersrand

February, 2017

DECLARATION

I declare that this thesis is my own, unaided work. It is being submitted for the Degree of Doctor of Philosophy at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

(Signature of candidate)_________________________________

28^{th} day of **February, 2017**.

Abstract

The work done in this thesis on finite planning horizon inspection models has demonstrated that with the advent of powerful computers these days it is possible to easily find an optimal inspection schedule when the lifetime distribution is known. For the case of system time to failure following a uniform distribution, a result for the maximum number of inspections for the finite planning models has been derived. If the time to failure follows an exponential distribution, it has been noted that periodically carrying out inspections may not result in maximization of expected profit. For the Weibull distributions family (of which the exponential distribution is a special case), evenly spreading the inspections over a given finite planning horizon may not lead to any serious prejudice in profit.

The case of inspection models where inspections are of non-negligible duration has also been explored. The conditions necessary for inspections that are evenly spread over the entire planning horizon to be near-optimal when system time to failure either follows a uniform distribution or exponential distribution have been explored.

Finite and infinite planning horizon models where inspections are imperfect have been researched on. Interesting observations on the impact of Type I and Type II errors in inspection have been made. These observations are listed on page 174.

A clear and easy to implement road map on how to get an optimal inspection permutation in problems first discussed by Zuckerman (1989) and later reviewed by Qiu (1991) for both the undiscounted and discounted cases has been given. The only challenge envisaged when a system has a large number of components is that of computer memory requirements - which nowadays is fast being overcome. In particular, it has been clearly demonstrated that the impact of repair times and per unit of time repair costs on the optimal inspection permutation cannot be ignored.

The ideas and procedures of determining optimal inspection permutations which have been developed in this thesis will no doubt lead to huge cost savings especially for systems where the cost of inspecting components is huge.

Key words: false negative, false positive, Inspection permutation, long-run average reward, mixed non-linear integer programming, net-income-rate, non-linear integer programming, optimal inspection times, optimality criterion, optimal planning horizon, stochastic deterioration.

Contents

List of Tables

List of Figures

Acknowledgments

The author wishes to express deep gratitude to

1. his supervisor, Professor Frank Beichelt who aroused his interest in the area of Inspection and Replacement Models and provided much needed guidance and motivation for him to be able to compile this PhD thesis,

2. his colleagues Byron Jacobs and Mark Dowdeswell (for their extensive assistance and guidance with Mathematica),

3. his colleagues in the School of Statistics and Actuarial Science at the University of the Witwatersrand; in particular Jacob Majakwara, Salha Mamane, Charles Chimedza, Herbert Hove (for assistance with type-setting and many other aspects related to perfecting the thesis) and both his former and present head of school, Professor Peter Fridjhon and Professor Stephen Jurisch, respectively, for structuring his work load in such a way that he would have time on his hands to work on the PhD research,

4. the anonymous referees (for two papers published so far out of material drawn from Chapter 3 and Chapter 6) for their valuable comments and suggestions,

5. the Applied Stochastic Models in Business and Industry journal for granting him copyright clearance to use the material in two of the papers that he published in the journal in November 2016[1],

6. his employer, the University of the Witwatersrand who in addition to giving him a bursary to do this doctoral work also granted him sabbatical leave in the second half of 2016 so that he could focus more on his PhD work,

7. his wife (Pellagia) and children (Primrose, Tapiwa, Charlene, Walter (jnr) and Tinei) who sacrificed family time for this work to be possible,

8. his parents (Mark and Rosemary) and siblings as well as his aunt Annie Chipoyera and uncles Bybit and Sebastian Chipoyera who played an important role in making him believe in himself.

[1] Chipoyera (2016a) and Chipoyera (2016b)

Chapter 1

Introduction

1.1 Introduction

Deciding how long it is most *profitable* to operate a system (which deteriorates over time) and when to schedule inspections is critically important in business. Since the pioneering work on inspection policy models for deteriorating systems by Barlow et al. (1963) [1], a lot of research resulting in a wealth of publications has evolved. The major contributors to system degradation are wear, corrosion, erosion, fatigue and crack generation (Clifton, 1974). Initially the system will be in a perfect working state and it is possible that over time the system can be in one of several states of degradation as the efficiency of the system progressively decreases (e.g. Ohnishi et al. (1986a),

[1] The main focus of this thesis is on inspection and replacement models; one cannot, however, omit the classical paper that laid the foundation of this direction in modern reliability theory by Barlow and Hunter (1960)

Ohnishi et al. (1986b), Milioni and Pliska (1988), Hontelez et al. (1996), Yeh (1997), Li and Pham (2005), Wang et al. (2014)). System failure occurs if the degradation level exceeds a particular level.

A reliability inspection problem inevitably exists whenever equipment such as machinery is running over a specified time horizon which may be finite or infinite. Mangalam and Feo (2006) give examples of equipment and items where public safety laws relating to inspection (in Canada) give birth to inspection models; these include amusement devices, elevating devices, boilers and pressure vessels, fuels, upholstery and stuffed articles [2]. Inspection models may also be needed for human beings such as operating engineers, aircraft pilots, drivers of heavy vehicles and operators of heavy duty equipment for purposes of re-licencing from time to time as it is a fact that things like visual impairment in people increase with age in general.

Where inspection and replacement policies are at work, the underlying idea is that there is need for scheduled inspection of facilities, equipment, etc.

[2] according to the https://www.ontario.ca/laws/statute/90u04 website, an "upholstered or stuffed article" means an article with any part which contains stuffing e.g. mattresses, beds, upholstery, pillows, plush toys, teddy bears etc.

during operation to ensure that devices continue to operate not only safely but also in an optimal way.

Durango-Cohen and Madanat (2008) point out the importance of inspection models by citing the United States's (US) recent shift in policy on infrastructure investment. They say that the US has placed more emphasis on maintenance as opposed to new construction as reflected by the rising proportion of the budget that is channeled towards maintenance and rehabilitation. In their introductory section they say "the critical issue facing public works agencies today is how to allocate the limited resources available for maintenance and rehabilitation so as to obtain the best return for their expenditure". Further evidence of the importance of maintenance at national level is given by Christer and Lee (1999); in their paper, Christer and Lee mention that some 14% of the Netherlands's Gross Domestic Product (GDP) is consumed by maintenance activities.

Thomas et al. (1991) say that *inspection* involves examining deteriorating systems to try to identify their state, in order to effect some repair, replacement and maintenance action. They go on to say that in most inspection policy models, one is able to monitor the system at no extra cost and with no interference to the system (resulting in a costless inspection process) while in other inspection policies, inspection involves stopping the normal running of the system to carry out some costly procedures.

1.2 Some Related Classes of Inspection Policies

Inspection policies vary a lot depending on the underlying assumptions. For instance, there are inspection policies

1. where time to failure denoted by T, has a known probability distribution function (e.g. Savage (1962), Barlow et al. (1963), Munford and Shahani (1972), Munford and Shahani (1973), Luss and Kander (1974) , Tadikamalla (1979), Kabir and El Tamimi (1988), Chelbi and Ait-Kadi (1999) and more recent papers such as Wang (2009), Tan et al. (2010), Ahmadi and Newby (2011), Wang (2011), Wang (2013), Caballé et al. (2015), Sheu et al. (2015), etc.) on one hand and inspection policies such as the ones discussed by Derman (1961) where

T's distribution is completely unknown on the other. Beichelt (1981) extends the work by Derman and researched on the case of the mean of the lifetime distribution being known. Roeloffs (1963), Roeloffs (1967), Kander and Raviv (1974) and Beichelt (1981) separately looked at the case where T's distribution is partially known (only one percentile of T being known).

2. where an inspection is assumed to be instantaneous (this is the case with most models) on one hand and models where the duration of checking or inspecting and the duration of repair are assumed to be

   (a) of non-negligible fixed duration (e.g. Luss and Kander (1974), Parmigiani (1993), Zuckerman (1989)),

   (b) non-negligible stochastic variables (e.g. Fang and Liu (2006)).

3. where an inspection gives a perfect diagnosis of the state of the system (e.g. Barlow et al. (1963)) on one hand and those where an inspection may give a diagnosis that may be erroneous on the other hand (e.g. Morey (1968), Christer (1988), Kaio and Osaki (1988), Milioni and Pliska (1988), Devooght et al. (1990), Parmigiani (1993), Hontelez et al. (1996), Ghasemi et al. (2008), Flage (2014)). Devooght et al. (1990)

develop models where an inspection may be imperfect as a result of a combination of any of the following: human error, instrumentation failure and incomplete information. Where inspection errors occur, the errors may arise as follows: a) an inspection may erroneously declare a normally operating system faulty (error of the first kind or Type I error) or b) an inspection may fail to detect that the system is in a failed state (error of the second kind or Type II error).

4. where the objective may be any of the following:

    - to minimize expected cost per unit of time of running the system in cases where the planning horizon is infinite (e.g. Barlow et al. (1963), Luss and Kander (1974), Anbar (1976a), Nakagawa (1976), Luss (1977), Zuckerman (1978), Nakagawa (1984), Badia et al. (2001), Wang (2009), Zhao et al. (2010), Wang (2013), Flage (2014), Wang et al. (2014), Caballé et al. (2015));
    - to maximize revenue (e.g. Ahmadi and Newby (2011)) or the expected profit per unit of time (for the case of a planning horizon that is infinite). This criterion though has received relatively little attention in the construction of inspection and replacement mod-

els. (e.g. Savage (1962), Luss (1983), Mohandas et al. (1992) and Zuckerman (1989) are some of the few authors who have looked at maximization of expected profit per unit of time as the optimization criterion in their work;

- to minimize the costs of operating the system (for a system that will be operated over a finite length of time (e.g. Usher et al. (1998) have discussed the case of a finite planning horizon with minimization of costs as their objective function);
- to minimize the cost per cycle[3] (e.g. Taghipour et al. (2010) have used this criterion to determine an ideal inter-inspection time);
- to maximize *safety* (e.g. Kabir and El Tamimi (1988) discuss inspection models based on specified *fractional dead time* - the proportion of time for which the system is in the failed state while Christer and Lee (1999) discuss the case of delay-time-based preventative maintenance (PM) inspection models which account for the downtime incurred at failures over a PM inspection period; their decision criterion is the minimization of expected downtime per unit time). Christer (1988) looks at organizations maintaining

[3] the length of a cycle is the time between two successive replacements

major civil engineering structures such as bridges and dams and explores the case of two competing objectives: 1) reduction of risks to users due to failures and 2) to control the cost of inspection and maintenance;

- minimization of expected present value of total cost (e.g. Yun and Nakagawa (2010) developed periodic replacement models using this optimization criterion);
- etc.

Remark 1.1 *The cornerstone of all research works that have used their criteria as minimization of expected value of cost per unit of time or maximization of expected value of profit per unit of time is the well-known result from Renewal Theory which states that the expected value of cost/profit per unit of time (for the case of an infinite planning horizon) is obtained by dividing the expected cost/profit per cycle by the expected cycle time (refer to page 203 of Karlin and Taylor (1975) or Vlasiou (2010) or Wang (2009)).*

Other papers use the minimization of cost per cycle[4] to determine the ideal inter-inspection times (e.g. Munford and Shahani (1972), Munford

[4]the length of a cycle is the time between two successive replacements

and Shahani (1973), Luss and Kander (1974), Tadikamalla (1979), Munford (1981), Sheu et al. (2015)); in the case of Luss (1983), the criterion is maximization of expected profit per cycle. Butler (1979) has developed inspection models where the criterion is the maximization of the expected life of the system. More recently, Yun and Nakagawa (2010) developed periodic replacement models with minimal repair and used minimization of expected present value of total cost as their optimization criterion.

A good number of policies assume that an inspection does not affect the state of the system; some policies assume that an inspection itself may induce failure (e.g. Butler (1979), looks at inspection models where an inspection is potentially harmful to the “device” under consideration; he gives the example of a cancer patient who has to be treated with X-rays and argues that exposure to X-ray radiation itself may actually cause cancer. The objective in Butler’s Hazardous Inspection Model is to determine inspection policies which maximizes the expected lifetime of the system or device.) Earlier on Wattanapanom and Shaw (1979) had developed a model for failure detection where tests hasten failures.

Replacement policies is another class of policies that have been extensively researched on by many researchers and they form the basis of this PhD works. Replacement policies specify the time when a system should be replaced regardless of whether it has failed or not; there are no scheduled inspections. A lot of publications on replacement models have been churned out by many researchers; the list includes the pioneers: Taylor (1923), Hotelling (1925), and more recent research papers by Taylor (1975), Nakagawa (1976), Zuckerman (1978), Boland and Proschan (1983), Gottlieb (1982), Aven and Gaarder (1987), Lai and Yuan (1993), Beichelt (2001a), Beichelt (2001b) and Tan et al. (2010).

Other works that have combined the element of inspection with replacement to form **Inspection and Replacement Policies** include Zuckerman (1980), Kawai (1984), Parmigiani (1993), Yeh (1997), Ghasemi et al. (2008), Scarf et al. (2009), Golmakani and Fattahipour (2011). A sub-class of inspection and replacement policies which take the age of a system as the most critical factor, called **Age-based Inspection and Replacement Policies**, have also been developed; according to Geurts (1983), the rule for this sub-class of policies is that a unit is replaced by a new one if it has survived a certain age (preventive replacement) or has failed (corrective replacement), whichever occurs first. Yet another sub-class of policies which is a rival of Age-based Inspection Policies, called **Condition Based Replacement Policies** have been developed. Geurts (1983) says "condition based replacement is gradually becoming a feasible alternative to age replacement, especially with the advent of more varied and more sophisticated condition monitoring equipement".

There are other interesting dimensions of inspection and replacement policies that have been explored. For instance, Lee and Rosenblatt (1987) develop an inspection model for a machine used in a production process by fusing the classical ideas of an *Economic Production Model* and the classical ideas of Inspection Models. In a typical production process, there are production cycles and each cycle lasts a time T'. Each cycle has an associated set-up or start-up cost and it consists of two phases: the production phase (lasting an amount of time T and the "selling phase" where depletion of stock takes place at a steady and uniform demand rate and there is no production taking place so that $T \leq T'$. At the beginning of a production cycle, the machine starts off in the "in-control" state and after a period of production (which is a random variable following an exponential distribution) the machine gets into the "out-of-control" state. During the time the machine is in-control, no defectives are produced while its being in the "out-of-control" state is accompanied by some proportion of defectives being produced. The total costs incurred by the production system has the following components: set-up costs, holding costs, costs of inspections and the cost of defective items. The objective in Lee and Rosenblatt (1987) is that of determining an optimal production run time T^*, the optimal number of inspections, n^* and the associated inspection schedule (i.e. specifying inspection times $\tau_o < \tau_1 < \cdots < \tau_n \leq T$ which minimize the total costs incurred by the production system.

1.3 Basic Theory and Notation for Inspection Models

Most inspection models assume that the system to be inspected deteriorates over time and its time to failure may be modelled by a known (or unknown) probability distribution function $F(t)$ and probability density function $f(t)$. Some models deal with systems with non-self announcing failures so that detection of a failed unit only occurs after inspection while for other models failure may be noted at the very instant when it occurs. The net utility function in most situations is the overall cost of running the equipment whose components may include cost of inspections, cost associated with undetected failure and idleness of the system, etc.

The following common notation will apply throughout this thesis:

- c_I = cost of a single inspection (assumed fixed);
- c_F = cost per unit of time when the system is not working properly or is idle;
- C_S - salvage value of the system upon disposal;
- $R_T(.)$ - system reliability function (or survival function) so that $R_T(t) = P(T > t)$.

1.4 Statement of the problem

Nowadays it makes business sense to dispose of a system when it malfunctions either because the cost of repairing it is astronomical or the system is simply not repairable. Examples of typical systems which come to mind include

- an electronic gadget/component which is disposed off once it is found to no longer function - a new and perfectly functioning gadget/component replaces it

- a battery which is sent for recycling once it starts malfunctioning - a new and perfectly functioning battery is put in its place
- a machine bearing which is sent for re-cycling once it starts malfunctioning - a new and perfectly functioning bearing is put in its place
- an electric iron or kettle once it packs up; a new and perfectly functioning iron or kettle is brought in to fill the void
- a building which is demolished once it has been condemned, etc.

Some systems may be operated for a finite length of time and may require inspections to be carried out during the course of operation and the need to plan on how long operation of such a system should be becomes imperative. Other systems may be operated in such a way that in addition to inspections being carried out at planned times, minimal repairs are done at certain recommended times. From a business perspective, if malfunctioning of a gadget is not life-threatening, then maximization of profit becomes the default objective function. The recommended length of time over which a system is planned to be operated is called the finite planning horizon.

Research on finite planning horizon models has been fairly scarce. Research on extensions of finite planning horizon models which take into account the fact that inspections are not instantaneous (i.e. they take time) are even more scarce. Also, models which deal with inspections that are imperfect are hard to come by. This research is an attempt to fill the void.

Another pool of inspection models (called Hierarchical Inspection Models) is discussed in this thesis. Like in the paper by Anbar (1976a), this pool of inspection models deals with a system that has a number of components whose times to failure are independent and identically distributed random variables. The difference with most other inspection models will be that the issue at stake is not the times at which inspections need to be scheduled but rather to specify the order in which inspections should be carried out in the event of the system being in the failed state.

1.5 Aims and objectives

The main aim in this work is to identify some gaps in literature on inspection and replacement models and develop new models that address some of the gaps. Identifying the gaps has involved reviewing methods used at arriving at optimal solutions, especially given the ever improving computer technology.

In this PhD project, the objectives are:

1. to develop new models to complement the rich pool of inspection and replacement models available;
2. to derive mathematical results related to the models derived;
3. to develop computer programs (in Mathematica) to address the problem of arriving at optimal solutions that was faced in the past because of lack of computer technology with capacity to help solve the problem; and
4. where appropriate, to compare the performance of existing models and the models that have been developed in this study.

1.6 Layout of the thesis

The rest of the thesis is arranged as follows. Chapter 2 gives a survey of all the relevant literature reviewed. Chapter 3 looks at the development of some finite planning horizon inspection and replacement models with inspections that are instantaneous while Chapter 4 discusses finite planning inspection and replacement models with inspections that take up time. Chapter 5 explores finite planning inspection and replacement models with imperfect inspections. Chapter 6 explores the problem of Hierarchical Inspection Models; as discussed earlier on, this is a class of models which has not been extensively researched on. Chapter 7 gives the conclusions and recommendations.

1.7 Milestones and novelty of this thesis

The author of this thesis has contributed new results in the subject of Reliability Theory; specifically in the area of Inspection and Replacement Models. He has published two papers in the Applied Stochastic Models in Business and Industry journal (an ISI accredited journal): Chipoyera (2016a) (based on Chapter 3) and Chipoyera (2016b) (based on Chapter 6). He envisages publishing two other papers (one based on Chapter 4 and another based on Chapter 5 material) in ISI accredited journals in 2017. The work done in this thesis is likely to be readily embraced in the engineering world because it has obvious applications in engineering.

The work done in Chapter 3 was presented at the The Eighth International Conference on Mathematical Methods in Reliability (MMR2013) held in Stellenbosch in 2013 and the work done in Chapter 6 was presented at the Twentieth International Conference of the International Federation of Operational Research Societies held in Barcelona in 2014. The work done in Chapter 5 will hopefully be presented at the 21st International Conference of the International Federation of Operational Research Societies to be held in Quebec City in July 2017.

The work in Chapter 2 and Chapter 6 was also presented separately in seminars in 2013 and 2016, respectively, in the School of Statistics and Actuarial Science, University of the Witwatersrand seminar series.

Chapter 2

Literature Review

This chapter gives a detailed review of some of the research papers which are closely related to the work in this thesis. The review covers papers which go way back to the pioneering works by Barlow et al. (1963).

2.1 Pioneering works - Barlow et al Inspection Model

For convenience purposes, the model by Barlow et al. (1963) will sometimes be referred to as $\mathbf{X}_{BP}$ policies. The ideas discussed in chapters 3, 4 and 5 are given birth to by the ideas in the paper by Barlow et al. (1963).

2.1.1 Introduction to X_{BP} policies

In their paper, Barlow et al. (1963) focus on a system whose deterioration is stochastic and whose condition may only be known upon inspection or checking because the systems are assumed to have non-announcing failures. The assumptions they make follow:

1. if the system has a problem, the problem ends at the point of inspection,

2. checking does not degrade the system, and

3. the system cannot fail while being checked

Each check is accompanied by a fixed cost c_I while the cost per unit time when the system is not working properly is c_F. An optimum inspection policy would involve specifying checking or inspection times $x_1^*, x_2^*, \cdots$, which minimize the expected total cost of inspections. The total cost of operating the system comprise of 1) cost of carrying out inspections and 2) cost incurred during the time the system is in a faulty state. The system will be in the faulty state from the time it becomes faulty up to the time when the next scheduled inspection reveals that it is indeed faulty. The total cost for a given time interval $[0, t]$ can be viewed as a loss function $\mathcal{L}$:

$$\mathcal{L} = c_I[N(t) + 1] + c_F\gamma_t \tag{2.1}$$

where $N(t)$ is a random variable denoting the number of inspections before and up to time t and γ_t is the time to the next inspection (when the faulty state of the system is detected) if the system fails at time t.

2.1.2 Key Results of X_{BP} Policies

The key results derived in the paper are given in this section.

1. If inspections prior to time t are carried out at times $x_1 < x_2 < \cdots < x_k$ and one more inspection is conducted at time x_{k+1} given that the system failed at time t such that $x_k < t \leq x_{k+1}$, the cost incurred would be $c_I(k+1) + c_F(x_{k+1} - t)$; hence, the expected loss would be:

$$E[\mathcal{L}] = \sum_{k=0}^{\infty} \int_{x_k}^{x_{k+1}} [c_I(k+1) + c_F(x_{k+1} - t)] dF(t). \qquad (2.2)$$

   Barlow et al. (1963) define a checking procedure that minimizes the expected loss (or objective) function $E[\mathcal{L}]$ as an *optimum checking procedure.*

2. If the failure distribution $F(x)$ is continuous with a finite mean μ, there exists an optimum degenerate checking procedure. A necessary condition that a sequence $\{x_k\}$ be a minumum cost checking procedure is that

$$\frac{\partial E[\mathcal{L}]}{\partial x_k} = 0; \text{ for all } k \qquad (2.3)$$

and consequent of (2.3) and (2.2)we have

$$x_{k+1} - x_k = \frac{F(x_k) - F(x_{k-1})}{f(x_k)} - \frac{c_I}{c_F}. \tag{2.4}$$

Also if the failure density f is a Polya frequency function of order 2 (abbreviated PF_2) and $f(x) > 0$ for $x > 0$, the optimum checking intervals $\delta_k^* = x_{k+1}^* - x_k^*$ are non-increasing.

3. For a system that is known to fail within a finite time interval, sometimes a single check may be adequate. Let $F(t) = 1$ for $t \geq T$. If

$$F(t) \leq \frac{1}{1 + (c_F/c_I)(T - t)} \quad for\ 0 \leq t \leq T,$$

then the optimum checking policy will consist of a single check at time T. Conversely, if

$$F(t) > \frac{1}{1 + (c_F/c_I)(T - t)} \quad for\ 0 \leq t \leq T,$$

for some $0 \leq t \leq T$, then the optimum checking policy will require, in addition to the check at time T, at least one more check before time T.

4. In their paper, Barlow et al. (1963) present the tool that may be used for estimating the sequence of checking times $\{x_k^*\}$ in the $\mathbf{X}_{BP}$ policy (to any degree of accuracy desired) as Theorem 6 on page 1091. It

reads as follows: "Let the failure density f be PF_2, $f(t) > 0$ for $t > 0$, and not of the form ae^{bt} for any interval $0 < t_1 < t < t_2$. Then for

(a) $x_1 < x_1^*$, $\Delta\delta_n = \delta_n - \delta_{n-1} > 0$ for some positive integer n and

(b) for $x_1 < x_1^*$, $\delta_n < 0$ for some positive integer n".

5. For a system whose time to failure follows an exponential distribution, the optimal inspection policy has inter-inspection times which are constant; that is, checking times are periodic. Suppose the failure density of a system is $f(t) = \theta e^{-\theta t}.I_{[0,\infty)}(t)$ and that the system is subjected to an inspection after every x units of time. If the system fails at time t such that $kx < t \leq (k+1)x$,

   the cost incurred is

$$\mathcal{L} = c_I(k+1) + c_F[(k+1)x - t]$$

   and the expected loss is

$$\begin{aligned} E[\mathcal{L}] &= \sum_{k=0}^{\infty} \int_{kx}^{(k+1)x} [(k+1)c_I + c_F\{(k+1)x - t)\}]dF(t) \\ &= \frac{c_I + c_F x}{1 - e^{-\theta x}} - \frac{c_F}{\theta}. \end{aligned} \tag{2.5}$$

   Equation (2.5) gives us an expression for the optimum periodic checking policy with inter-check times x^* as a solution of

$$e^{\theta x^*} - \theta x^* = 1 + \frac{\theta c_I}{c_F}. \tag{2.6}$$

6. For a failure distribution with a large mean (or small θ),

$$x^* \approx \sqrt{\frac{2c_I}{\theta c_F}}. \tag{2.7}$$

2.1.3 Identified Gaps in X_{BP} Policies

1. Many authors who have reviewed the paper by Barlow et al. (1963) lament that finding an optimal inspection schedule is not a tractable process (e.g. Munford and Shahani (1973), Nakagawa and Yasui (1980), Kaio and Osaki (1989)); Kaio and Osaki (1989) say, " ... the algorithm by Barlow et al. (1963) is complicated to execute, because one must apply trial and error to find the first inspection time t_1, and the assumption on $f(t)$ is restrictive". The author contends that the advent of more powerful computers these days should go a long way in making the process much simpler. To this end, it would undoubtedly be useful if computer programs (in a software like Mathematica) for X_{BP} policies for a number of commonly used distributions for modelling time to failure such as the Weibull and Gamma distributions were to be developed.

2. The assumption of inspection times being instantaneous certainly holds for some systems and of course breaks down for many other systems. An extension of X_{BP} policies in which inspection times are taken as non-negligible random variables is worth exploring.

3. The assumption of inspections giving perfect diagnosis of the state of the system whenever conducted, just like the assumption of inspections being instantaneous, may not hold true for some systems. Chapter 5 addresses the case where this assumption breaks down.

2.2 $\mathbf{X}_p$ Inspection Policies

Munford and Shahani (1972) pioneered the work on this class of inspection policies. In their paper, State 0 is defined as a state in which a system is in a working state and State 1 as the state in which a system will have failed. It is assumed that the transition from State 0 to State 1 can only be detected through inspection of the System (because system failure is assumed to be non-announcing). Also, the system may not move from State 1 back to State 0 on its own accord - it can only do so through repair.

2.2.1 $\mathbf{X}_p$ inspection policies explained

Definition 2.1 *Let a system be such that its time to failure has probability density function $f(t) = f_T(t)$ and failure distribution function $F(t) = F_T(t) = P(T \leq t)$ and let $p \in (0,1)$ be some constant. Then an $\mathbf{X}_p$ inspection policy has inspection times $x_1, x_2, \cdots$ such that*

$$\frac{F_T(x_i) - F_T(x_{i-1})}{1 - F_T(x_{i-1})} = p, \; i = 1, 2, \cdots . \tag{2.8}$$

An optimal $\mathbf{X}_p$ inspection policy is one such that the inspection times $\mathbf{X}_p = \mathbf{X}_p^* = (x_1^*, x_2^*, \cdots)$ minimize the expected cost of running the system. Just like X_{BP} policies, the costs incurred in running the system comprise of the cost of inspections and cost due to system idleness when the system is in State 1.

Remark 2.1 $\frac{F_T(x_i)-F_T(x_{i-1})}{1-F_T(x_{i-1})}$ *is the probability of transition from State 0 to State 1 in the interval (x_{i-1}, x_i) given that the system was in State 0 at time x_{i-1}. Also, $x_o = 0$ and $F_T(x_o) = 0$ so that $F(x_1) = p$. Further, from Equation (2.8), we have*

$$F(x_i) = p\,[1 - F(x_{i-1})] + F(x_{i-1}) \tag{2.9}$$

and it can easily be deduced that

$$F(x_i) = 1 - (1 - p)^i = 1 - q^i \tag{2.10}$$

where $p = 1 - q$.

2.2.2 Determination of optimal $\mathbf{X}_p$ inspection policy

If the transition from State 0 to State 1 occurs at time t such that

$x_{i-1} < t \leq x_i$, and the total cost until a failure is detected at time x_i, C:

$$C = ic_I + c_F(x_i - t). \tag{2.11}$$

Remark 2.2 *An optimal* $\mathbf{X}_p$ *policy is one which results in the minimization of the expected cost function*

$$E[C] = E[c_I i + c_F(x_i - t)] = c_I E[N] + c_F E[\textit{System idleness time}]. \tag{2.12}$$

where N *is the random variable denoting the number of inspections required.*

To proceed with the minimization of $E[C]$, one needs to know the results for $E[N]$ and the expected value of the system idleness time.

Munford and Shahani show that

1. Result for $E[N]$: For the event $N = i$ to occur, that means a transition from State 0 to State 1 during the time interval $(x_{i-1}, x_i])$, meaning State 0 is preserved in time interval $(0, x_1), (x_1, x_2), \cdots, (x_{i-2}, x_{i-1})$

with probability q^{i-1}.

$$P(N=i)=q^{i-1}p,\ i=1,2,3,\cdots \Rightarrow \quad E[N]=\sum_{i=1}^{\infty} iq^{i-1}p=\frac{1}{p}. \quad (2.13)$$

2. E[<u>System idleness time</u>]:

$$\begin{aligned} E[\text{System idleness time}] &= \sum_{i=1}^{\infty}\int_{x_{i-1}}^{x_i}(x_i-t)f(t)dt \\ &= \sum_{i=1}^{\infty} x_i\left[F(x_i)-F(x_{i-1})\right]-\int_0^{\infty} tf(t)dt \\ &= \sum_{i=1}^{\infty} x_i\left[(1-q^i)-(1-q^{i-1}\right]-\int_0^{\infty} tf(t)dt \\ &= \sum_{i=1}^{\infty} x_i q^{i-1}p-E[T] \qquad (2.14) \end{aligned}$$

Thus, combining equations (2.12), (2.13) and (2.14) we get:

$$C(p)=E[C]=\frac{c_I}{p}+c_F\left[\sum_{i=1}^{\infty} x_i q^{i-1}p-E[T]\right]. \qquad (2.15)$$

Remark 2.3 *E[C] may be denoted by $C(p)$ to emphasize that the expected cost is a function of p; consequently, the problem of minimizing $C(p)$ is essentially reduced to that of finding the value of p that minimizes $C(p)$.*

2.2.3 X_p inspection policies for some distributions

Munford and Shahani (1972) wind up their paper by drawing a comparison of $\mathbf{X}_p$ policies and $\underset{\sim}{X}_{BP}$ policies and make the following observations:

1. Just like $\mathbf{X}_{BP}$ policies, for a failure distribution with a monotonic increasing failure rate $r(t) = f(t)/(1-F(t))$, the inter-inspection intervals $\delta_i = x_i - x_{i-1}$ are monotonic decreasing.

2. for the exponential distribution with parameter θ (i.e. the distribution where $r(t)$ is a constant),

   (a) the inspection times are equally spaced and $\delta_i = -\frac{1}{\theta}\ln q$.

   (b) the optimal inter-inspection time, $\delta_i = \delta$, just like with $\mathbf{X}_{BP}$ models, is a solution of the equation

   $$e^{\theta\delta} - \theta\delta = 1 + \theta\frac{c_I}{c_F}.$$

3. for the case of a system with a normally distributed time to failure, i.e. $T \sim N(\mu, \sigma^2)$, the $\mathbf{X}_P$ policy yields

   $$E[C] = \sigma c_F\left[\frac{c_I}{\sigma p c_F} + \sum_{i=1}^{\infty}\left(\frac{x_i - \mu}{\sigma}\right)q^{i-1}p\right] = \sigma c_F\left[\frac{\gamma}{p} + \sum_{i=1}^{\infty} z_i q^{i-1}p\right] \tag{2.16}$$

   where $\gamma = \frac{1}{\sigma}c_I/c_F$ and $z_i = (x_i - \mu)/\sigma$. The authors try out different combinations of γ and p when comparing expected cost for the $\mathbf{X}_{BP}$ and $\mathbf{X}_P$ policies; in all instances explored, the ratio of the expected cost was more than 0.9 leading them to conclude that $\mathbf{X}_P$ policies are just marginally more expensive compared to $\mathbf{X}_{BP}$ policies.

In a follow up paper, Munford and Shahani (1973) focus on $\mathbf{X}_P$ policies where the time to failure, T follows a Weibull distribution so that the probability density function of T, $f_T(t)$ is given by

$$f_T(t) = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} \exp\left\{-\left(\frac{t}{\alpha}\right)^{\beta}\right\} .I_{(0,\infty)}(t), \quad \alpha > 0, \beta > 0 \qquad (2.17)$$

where α is a scaling parameter and β is the shape parameter. They deduce that the value of $p = p_{opt}$ which minimizes the function $g(p)$:

$$g(p) = \frac{c_I}{\alpha c_F p} + \{-\ln(1-p)\}^{1/\beta}\, p \sum_{i=1}^{\infty} i^{1/\beta}(1-p)^{i-1} \qquad (2.18)$$

is key to finding the optimal X_p inspection schedule. They propose the use of a nomogram (a graph) in finding p_{opt} and contend that the use of a method outlined by Lyle (1954) usually lead to quite accurate nomograms.

Tadikamalla (1979) explores X_p policies when the time to failure is a Gamma distributed random variable. He laments the problem of the non-existence of a closed form of the cdf of a gamma distribution; a key component in finding an accurate value of p_{opt}. He however uses an approximation given by Goldstein (1973) for finding the inverse of the cdf F, F^{-}.

2.2.4 Gaps and Criticism of X_p policies

The author (of this thesis) has reservations about the use of nomograms as means to determining optimal inspection schedules, especially given that they (Munford and Shahani) mention that nomograms do not always yield accurate results but rather usually give accurate results. The author contends that one can write simple computer program (in a software like Mathematica) which takes advantage of the fast convergence of $\sum_{i=1}^{\infty} i^{1/\beta}(1-p)^{i-1}$, a term in (2.18), to calculate the value of p_{opt} to any desired degree of accuracy, given values of the parameters β, α, c_I, c_F, etc.

2.3 Inspection Models for System with Components Connected in Parallel

2.3.1 Introduction

Anbar (1976b) develops an inspection and replacement model for a system with n identical items/components connected in parallel. The respective lifetimes of the n items, $T_1, T_2, \cdots, T_n$ are identically and independently distributed random variables following an exponential distribution with an unknown parameter θ; i.e. $F_{T_i}(t) = P(T_i \leq t) = F_T(t) = 1 - e^{-\theta t}$ where $t > 0$, $\theta > 0$.

The assumptions of the model are as follows:

1. Failures are non-announcing and are only detected through inspection (and we use the notation c_F for the cost incurred per unit of time owing to a component being idle as a result of failure)

2. all units are inspected when an inspection takes place and all failing units are replaced by new ones so that the system becomes "new" immediately after an inspection is complete (and we use the notation c_I for the cost of inspecting a single unit so that total cost of an inspection of the system each time is nc_I)

3. Inspection and replacement are instantaneous (and the replacement cost of a failed unit is C_o)

2.3.2 Key Results from Anbar's Model

Anbar's model policy is that the time between successive inspections is a constant, τ, i.e. inspections are periodic. The net utility function in the model is the expected cost per unit of time $c(\tau)$ given by

$$c(\tau) = \frac{1}{\tau}E\left[nc_I + c_F\sum_{i=1}^{n}(\tau - T)^+ + C_oN(\tau)\right] \tag{2.19}$$

where $N(\tau)$ is the random variable denoting the number of failures per cycle and

$$(\tau - T)^+ = \begin{cases} \tau - T, & \text{if } \tau - T > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Anbar deduces that

$$c(\tau) = \frac{n}{\tau}\left[c_I + c_F\left(\tau - \frac{F(\tau)}{\theta}\right) + C_o F(\tau)\right] \tag{2.20}$$

and defining $\eta(x) = (x+1)e^{-x},\ x \geq 0$, Anbar notes the following:

1. A necessary condition for τ^* to minimize $c(\tau)$ is that it is a solution of the equation

$$(1 + \theta\tau^*)e^{-\theta\tau^*} = 1 - \frac{c_I}{c_F/\theta - C_o} \tag{2.21}$$

2. A finite value of τ^* exists if and only if

$$0 < \theta < c_F/(c_I + C_o); \tag{2.22}$$

   if $\theta > c_F/(c_I + C_o)$, the optimal inspection schedule is a *no inspection schedule.*

3. Assuming Equation (2.22) holds, the optimal inter-inspection interval, τ_o:

$$\tau^* = \frac{1}{\theta}\eta^{-1}\left(1 - \frac{c_I}{c_F/\theta - C_o}\right) \tag{2.23}$$

Remark 2.4 *The major pitfall with Anbar's model is that the value of θ is assumed unknown and hence a direct calculation of τ_o is not possible. Anbar suggests an iterative method based on a stochastic approximation procedure developed by Albert and Gardner (1967) that circumverts this pitfall. Essentially, the method hinges on the fact that if $\{\hat{\theta}_n\}_{n=1}^{\infty}$ is a sequence of random variables that converge to θ with probability 1, then the sequence $\{\tau_n^*\}_{n=1}^{\infty}$ such that*

$$\tau_n^* = \frac{1}{\hat{\theta}_n}\eta^{-1}\left(1 - \frac{c_I}{c_F/\hat{\theta}_n - C_o}\right) \tag{2.24}$$

simultaneously converges to τ^* provided $\hat{\theta}_n > c_F/(c_I + C_o)$ with probability 1 for all n.

Anbar then utilizes Theorem (2.1) and Lemma (2.1) to develop an algorithm that can help one iteratively arrive at the optimal inspection schedule for the Anbar Inspection Models pool.

Theorem 2.1 *Let $a_n, N \geq 1$ be a nonnegative $\mathcal{F}_n$ measurable random variable such that*

$$a)\ \sum a_n = \infty \ \ and \ \ \sum a_n^2 < \infty \tag{2.25}$$

with probability 1.

Let θ_1 be an $\mathcal{F}_1$ measurable random variable. For $j \geq 1$ define

$$\theta_{j+1} = [\theta_j - a_j(\theta_1, \cdots, \theta_j)[F_j(\theta_j) - N_j/n]]_a^b. \tag{2.26}$$

where $F_k(x) = \max(0, 1 - e^{-x\tau k})$, $N_1, N_2, \cdots$ is some sequence of random variables and

$$[x]_a^b = \begin{cases} a, & \text{if } x \leq a \\ x, & \text{if } a < x < b \\ b, & \text{if } x \geq b \end{cases}$$

Then $\theta_n \to \theta$ with probability 1.

Lemma 2.1 *Suppose that there are two constants a and b such that $a < b$ and $0 \leq a \leq \theta \leq b \leq \frac{c_F}{C_o + c_I}$. If A and B are defined as follows:*

$$A = \frac{1}{b\theta}\left(1 - \frac{c_I}{c_F/a - C_o}\right) \quad \text{and} \quad B = \frac{1}{a\theta}\left(1 - \frac{c_I}{c_F/b - C_o}\right)$$

then $0 < A < \tau < B < \infty$.

The algorithm

The algorithm works as follows:

1. Calculate an initial estimate of θ, $\hat{\theta}_1$.

2. Using $\hat{\theta}_1$, calculate an initial estimate of τ, τ_1:

$$\tau_1 = \frac{1}{\hat{\theta}_1}\eta^{-1}\left(1 - \frac{c_I}{c_F/\hat{\theta}_1 - C_o}\right).$$

3. Sequentially generate estimates of θ and τ as follows. For $k \geq 1$,

$$\hat{\theta}_{k+1} = [\hat{\theta}_k - a_k(\hat{\theta}_1, \cdots, \hat{\theta}_k)[F_k(\hat{\theta}_k) - N(\tau_k)/n]]_a^b \tag{2.27}$$

and

$$\tau_{k+1} = \frac{1}{\hat{\theta}_{k+1}}\eta^{-1}\left(1 - \frac{c_I}{c_F/\hat{\theta}_{k+1} - C_o}\right) \tag{2.28}$$

bearing in mind that the sequences $\{\hat{\theta}_k\}$ and $\{\tau_k\}$ converge to θ and τ, respectively, with probability 1.

Challenges in implementing the Anbar Algorithm

Anbar (1976b) concedes that the major challenge when it comes to the implementation of his algorithm relates to the choice of the sequence $\{a_k\}$ and he asserts that he does not have an adequate answer for this. He only mentions that the form $a_k = Dk^{-1}$ yields what he terms some interesting numerical results. To this end, Anbar says that when one restricts themselves to this choice of $\{a_k\}$, then perhaps an appropriate choice of D, D_{opt} (where D_{opt} may depend on the ratio $c_F/(c_I + C_o)$ would have to be settled on; cost minimization would be of paramount importance in doing so.

2.3.3 Matters Arising from Anbar's Model

1. Models with a system having n identical components connected in parallel (like in Anbar's case) whose times to failure are iid random vari-

ables following some other distribution other than the exponential distribution beg to be developed. Typical distributions that come to mind include the more versatile Weibull and Gamma distributions.

2. A similar inspection and replacement policy for a complex system with n components that are not identical is mooted here. The difference with Anbar's approach is that for the system under consideration, the times to failure $T_i, \cdots, T_n$ need not be independent and identically distributed random variables. An example that comes to mind is that of a circuit board with a number of different components.

2.4 Hierarchical Inspection Model for a System With Components Connected in Series

Zuckerman (1989) explores the case of a system/machine with n components/units (presumably connected in series) whose times to failure are independent exponentially distributed random variables. By virtue of them being connected in series, the machine breaks down the moment any one of the N components fails and machine failure is attributed to just that component which will have failed.

2.4.1 Introduction

The basic assumptions in Zuckerman's model are:

1. When in operation, the system/machine generates income at a rate of I rand[1] per unit of time.

2. The machine has N units such that the lifetime of the i^{th} unit, $S_i \sim Expo(\theta_i), i = 1, \cdots, N$ and the random variables $S_1, \cdots, S_N$ are stochastically independent.

3. The system's status is observed continuously at zero cost (by a controller) and a failure is due to exactly one component having failed. In the event of a breakdown, a series of inspections (in a hierarchical manner and one unit at a time) is performed in order to identify the failed unit. Once the failed unit has been identified[2], it is repaired and immediately thereafter the machine starts working again. The cost of inspecting the n^{th} unit is C_n rand per unit of time and the inspection time for the n^{th} unit is T_n while the expected repair time for the n^{th} unit is Z_n and the expected repair cost for the unit is denoted by R_n.

[1] or some other appropriate monetary unit

[2] the inherent assumption is that two or more units may not fail simultaneously

Remark 2.5 *The objective in Zuckerman's model is the formulation of an optimal inspection permutation or strategy (i.e. the order in which the N units are inspected) in order to maximize either the long-run average net income or total discounted net income. An inspection permutation* $\sigma = (\sigma(1), \cdots, \sigma(N))$ *spells out the order in which the units are inspected so that* $\sigma(j)$ *is the* j^{th} *unit of the machine to be inspected.*

2.4.2 Main results in Zuckerman (1989)

Zuckerman (1989) uses the notation $E_\sigma[.]$ and $P_\sigma(.)$ to refer to the expectation and probability, respectively, when an inspection strategy σ is used.

Letting $\theta = \sum_{i=1}^{N} \theta_i$, the main results from Zuckerman (1989) are:

1. If a machine has broken down, the probability that the breakdown is due to the n^{th} unit, P_n is given as:

$$P_n = P\left(S_n = \min_{1 \le i \le N} \{S_i\}\right) = \frac{\theta_n}{\theta}. \tag{2.29}$$

2.

$$E[\min_{1 \le i \le N} \{S_i\}] = \frac{1}{\theta}. \tag{2.30}$$

Letting C be the accumulated inspection cost over a cycle and T be the time to identify the failed unit (total inspection time per cycle), we have

$$E_\sigma[C] = \sum_{i=1}^{N} P_{\sigma(i)} \left[\sum_{j \le i} C_{\sigma(j)} T_{\sigma(j)} \right] \tag{2.31}$$

and

$$E_\sigma[T] = \sum_{i=1}^{N} P_{\sigma(i)} \left[\sum_{j \le i} T_{\sigma(j)} \right] \tag{2.32}$$

resulting in the long run average net income[3] for inspection strategy σ being

$$\psi(\sigma) = \frac{\frac{I}{\lambda} - E_\sigma[C] - \sum_{n=1}^{N} P_n R_n}{\frac{1}{\theta} + E_\sigma[T] + \sum_{n=1}^{N} P_n Z_n}. \tag{2.33}$$

3. Zuckerman goes on to give a result (listed as Theorem 1 in his paper) which is deemed critical for determining the optimal inspection permutation **for the undiscounted case**; it says that in the undiscounted case, the units are inspected in an increasing order of the indices

$$e_j = \frac{T_j C_j + \psi^* T_j}{P_j}, \quad j = 1, 2, \cdots, N, \tag{2.34}$$

where $\psi^* = \max_\sigma \psi(\sigma)$ is the optimal net-income-rate.

Remark 2.6 *Zuckerman laments that since ψ^* is unkonwn, his procedure is not tractable as the indices $e_1, \cdots, e_N$ cannot be computed explicitly. He proposes a graphical computational procedure for the optimal inspection permutation which is quite involved!*

[3] a result arrived at by invoking the Renewal Reward Theorem

Zuckerman's works are then reviewed by Qiu (1991); Qiu suggests that some of the results (results for the discounted case) arrived at by Zuckerman are not correct.

2.4.3 Main results in Qiu (1991)

Qiu looks at the simplified case where the repair times and repair costs are assumed to be negligible. He denotes the inspection cost rate at time t by $C(t)$ and denotes the obtaining continuous discount factor by α. Both Zuckerman and Qiu give the total discounted net income per cycle when an inspection strategy σ is adopted as

$$\eta(\sigma) = \frac{\frac{I}{\lambda+\alpha} - \frac{\lambda}{\lambda+\alpha} E_\sigma \left[\int_0^T C(t) e^{-\alpha t} dt \right]}{1 - \frac{\lambda}{\lambda+\alpha} E_\sigma [e^{-\alpha T}]} \tag{2.35}$$

Letting $\eta^* = \max \eta(\sigma)$ and $Q_i = 1 - P_i$, Zuckerman states that an optimal inspection strategy would inspect the units in an increasing order of the indices g_i:

$$g_i = \frac{(\eta^* + C_i/\alpha)(1 - \exp(-\alpha T_i))}{1 - Q_i \exp(-\alpha T_i)}. \tag{2.36}$$

Qiu disputes Result (2.36) and uses a counterexample to demonstrate that result is not correct. He ends his paper by giving necessary conditions for an inspection strategy to be optimal.

2.4.4 Identified Gaps in Zuckerman and Qiu's works

1. It must be stressed that while the results obtained by Zuckerman (1989) and Qiu (1991) are appealing in that they deal with commonly encountered practical important problems, implementation is unfortunately not easy. In particular, the fact that one has to resort to linear graphs in order to arrive at the optimal hierarchical inspection permutation. The models are also a departure from the classical inspection models in the sense that the objective here is not to recommend times as to when inspections should take place but rather to set out an order or hierarchy in which the components of a machine may be inspected in the event of a failure.

2. In this PhD thesis, a Mathematica program which makes use of (2.33) (for the undiscounted case) and (6.12) (for the discounted case) makes it easy to obtain an optimal inspection strategy for the Zuckerman-Qiu policies are developed (see Appendix A.2). The procedure involves simply computing income per cycle values for all possible inspection permutations.

3. For the same system, it appears it would be very useful to develop

an inspection schedule which outlines times at which the system needs to be inspected in line with classical inspection policies where system failure is non-announcing and can only be detected the next time an inspection of the system takes place.

2.5 Replacement Models

2.5.1 Introduction

Taylor (1923) is credited with being the pioneer on Replacement Models. Taylor's approach takes into account the cost of a new system/machine, C, the average operating expenses per year of the system/machine, O the average repair costs per year, R and the average number of units output per year, I as well as the obtaining interest rate, i (for purpose of factoring depreciation of the machine over time) as well as salvage value of the machine after N years to derive the average cost of production over N years, C_N. The recommendation then becomes that the machine should be replaced after N_o years where N_o is the value of N that minimizes C_N. Hotelling (1925) published a follow up paper to Taylor's works where the objective function is based on profit considerations (maximization of the present value of the machine's out minus its operating costs) and uses the notion of continuously compounded interest and some continuous functions in his derivations.

Remark 2.7 *One apparent weakness in Taylor and Hotelling's works is that they assume that the machine's life is indefinite. This is obviously not realistic and recent works have tried to address this anomaly.*

Ever since the pioneering works of Taylor and Hotelling, many research papers have been published. More recent works have tended to focus on replacement models for a machine that is subjected to shocks over time. The justification for the increased attention is provided by Nakagawa (1976) who says "it is of great importance to avoid a failure of an item when its failure during operation is costly and/or dangerous"; he lists tyres and railway lines as items that are ideal candidates for this pool of models. According to Nakagawa, degradation of a system may occur in the form of any one of the following: wear, fatigue, corrosion or erosion. Summaries of the papers reviewed are given in subsequent sections.

Remark 2.8 *The term* **control limit policy** *applies to any replacement policy where the system is replaced with a new one upon reaching a specified age T or failure, which ever comes first. This normally applies for situations where replacement after failure is far much more expensive compared to replacing before failure.*

2.5.2 Replacement Models by Taylor

Taylor (1975) develops a cumulative damage model for system failure where shocks occur to the system or machine in accordance with a Poisson process having rate λ; each shock causes a random amount of damage or wear and the damage a system incurs accumulates additively. The amounts of damage $Y_1, Y_2, \cdots$ are assumed to be independent and identically distributed random variables. Replacement is recommended either when failure occurs or when the cumulative damage first exceeds a critical control level, ϵ^*. In his paper, Taylor explores

1. **Optimal Planned Replacement Model** - the cumulative damage process up to failure time ζ is a Markov process $\{X(t); 0 \leq t < \zeta\}$ and a controller is allowed to institute planned replacement at Markov time $T < \zeta$. This model assumes that replacement is instanteneous. The long run average cost per unit of time if the replacement time is T, ψ_T:

   $$\psi_T = \frac{c + C_o Pr(T = \zeta)}{E[T]} \tag{2.37}$$

   The objective then becomes that of finding T^* such that

   $\psi^* = \psi_{T^*} = \inf \psi_T$.

2. **The Threshold Model** - (Taylor says that this is the cumulative damage model that has received the most attention.) For this model replacement is again assumed to be instantaneous and system failure occurs when cumulative damage z first exceeds a fixed threshold of size L, so that the *survivor-ship function*

$$r(z) = \begin{cases} 1, & \text{for } 0 \leq z \leq L \\ 0, & \text{for } z \geq L \end{cases} \tag{2.38}$$

Given $F(z) = P(Y_k \leq z)$, the renewal function $M(z) = \sum_{n=0}^{\infty} P(Y_1 + Y_2 + \cdots + Y_n \leq z) = \sum_{n=0}^{\infty} F^{(n)}(z)$ (where $F^{(n)}$ denotes the n-fold convolution of F) is used to derive results for the optimal strategy.

3. **More General Cost and Income Model** - The model deals with the case of a system that requires a non-negligible amount of time to carry out the replacement job and during the replacement process, a cost is incurred as a result of lost income. Letting I denote the mean rate of income accrued per unit of time when the system is operating, τ_f be the downtime associated with a failure replacement and τ_p be the downtime associated with a planned replacement, the long-run net

income per unit of time Δ_T:

$$\Delta_T = \frac{IE[T] - c - C_o Pr(T = \zeta)}{E[T] + \tau_p Pr(T < \zeta) + \tau_f Pr(T = \zeta)}. \qquad (2.39)$$

All efforts are then directed at finding the value of ϵ, ϵ^* which results in $\Delta^* = \sup_T \Delta_T$.

4. **A More General Failure Model** - The assumptions of this model are:

    - shocks occur to a machine or production system in accordance with a Poisson process, $\{N(t); 0 \leq t < \infty\}$, having a known rate λ.
    - the system survives k or more shocks with a known probability $P_k, k = 0, 1, 2, \cdots$ where $\{P_k\}$ is a decreasing sequence of probability values such that $\sum_{k=0}^{\infty} P_k < \infty$.
    - the shocks, when they occur, are observable by the controller, and a decision for replacement is only made number of shocks have reached a control limit N_o that will have occurred.

The system survival probability function, i.e. the probability that the system failure time $\zeta > t$, $P(\zeta > t)$:

$$Pr(\zeta > t) = \sum_{k=0}^{\infty} \frac{\lambda t)^k e^{-\lambda t}}{k!} P_k \tag{2.40}$$

and the mean time to failure

$$E[\zeta] = \int_0^{\infty} Pr(\zeta > t)dt = \frac{1}{\lambda} \sum_0^{\infty} P_k < \infty;$$

the Markov process $\{N(t); 0 \leq t < \zeta\}$ is a terminating pure birth process. If replacement is to take place at any Markov time $T < \zeta$, the long run average cost per unit time, ψ_T:

$$\psi_T = \frac{c + C_R Pr(T = \zeta)}{E[T]} = (c + C_R(1 - P_{N_o})) \,/ \left(\frac{1}{\lambda} \sum_{k=0}^{A-1} P_k \right) \tag{2.41}$$

The only outstanding issue involves a search for the optimal value of N_o, N_o^* the value that results in $\inf \psi_T$.

2.5.3 Nakagawa's Replacement Models

The work done by Nakagawa (1976) is very much similar to what Taylor (1975) did when he developed his *Optimal Planned Replacement Model* discussed above.

2.5.4 Zuckerman's Works

The approach taken by Zuckerman (1978) in his paper is not very different from Taylor's approach only that in Zuckerman's case, the damage process is an increasing one with stationary independent increments - it is a one-sided Levy Process. The system fails when the accumulated damage first exceeds V, a random variable which has a known absolutely continuous distribution B (called the killing distribution). Denoting the accumulated damage in time $[0, t]$ by $Z(t)$ and the time to failure by ζ we have:

$$\zeta = \inf\{t \geq 0, Z(t) \geq V\}.$$

The long-run average associated with a Markov (replacement) time T, $\psi(t)$:

$$\psi_T = \frac{c + c_R P(T = \zeta)}{E[T]} \tag{2.42}$$

In another paper on inspection and replacement models, Zuckerman (1980) explores inspection and replacement policies where the status of a system or device can only be determined by a physical inspection; upon detection of failure, the system is replaced by a new identical one and a failure cost is incurred. The time between two successive inspections τ is a constant; an optimal inspection and replacement policy is achieved by finding a value of τ and replacement time T^* which result in the minimization of the total long-run average cost per unit of time.

2.5.5 Gottlieb's Replacement Model

Gottlieb (1982) studies the problem of a device which is subjected to a series of shocks which arrive as a non-negative and non-decreasing semi-Markov process $\{Z_t, t \geq 0\}$; unlike in the case of Taylor (1975) and Nakagawa (1976), no assumption is made about the monotonicity of the failure rate or possible times of replacement; the only assumption in his model is that failure only occurs at times of jumps of Z_t. The following are assumed: $Z_o = X_o$, $T_o = 0$, $Y_o = 0$ and $S_n = Z_{T_n}$. (Figure 2.1 is meant to make it easy to appreciate the notation used here.) If time to failure is denoted by ζ, and its assumed

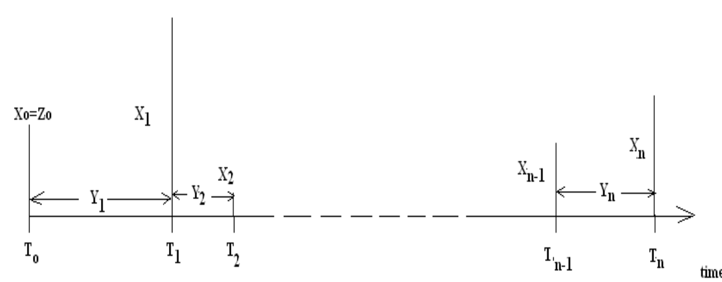

Figure 2.1: Illustration of Semi-Markov Shock Process Notation

that failure occurs at the n^{th} shock, then the objective is to find τ^* which minimizes the long-run average cost of a replacement policy (if replacement is recommended after time τ from the time a new cycle begins or upon failure), ψ_τ:

$$\psi_\tau = \frac{c + c_R P(\tau \geq \zeta)}{E[\min(\tau, \zeta)} \tag{2.43}$$

where the calculation of $P(\tau \geq \zeta)$ can be achieved using

$$P(\zeta > T_n | X_o, X_1, \cdots, X_n, Y_o, Y_1, \cdots, Y_n) = \prod_{i=0}^{n} r(S_i) \tag{2.44}$$

and $r(x)$ is the probability that a functioning device will survive a shock which increases the cumulative damage to x. Gottlieb asserts that the problem of finding an optimal τ^* can be viewed as a Markov decision process. He winds the paper by outlining an algorithm for computing an optimal replacement policy.

2.5.6 Aven and Gaarder's Replacement Model

Aven and Gaarder (1987) model is very similar to the planned Replacement Model discussed by Taylor (1975); the only difference is that in Taylor's work, shocks occur at any time wheareas in Aven and Gaarder's model shocks occur at discrete times $n = 1, 2, \cdots$. The failure time is denoted by ζ. In the model, replacement is again assumed to be instantaneous and replacement is done at the integer-valued stopping time $N \leq \zeta$; if replacement is done before failure, a cost c is incurred whilst if replacement is a post-failure exercise, the cost will be $c + c_R$. Essentially, the optimal replacement policy is arrived at by finding the stopping time N^* which minimizes the expected cost per replacement cycle, ψ_N:

$$\psi_N = \frac{c + c_R Pr(N = \zeta)}{E[N]}, \quad N \leq \zeta. \tag{2.45}$$

$$r(z) = \begin{cases} 1, & \text{for } 0 \leq z \leq L \\ 0, & \text{for } z \geq L \end{cases} \tag{2.46}$$

2.5.7 Beichelt's Replacement Model

In his paper, Beichelt (2001a) focuses on determining cost-optimum replacement times for complex technical systems. One interesting policy developed in his paper is the repair cost limit replacement policy where the policy says "a system is replaced after failure by a new one if the corresponding repair cost reaches or exceeds a certain level". He says that a common replacement policy for technical systems involves replacing a system by a new one after its *economic lifetime* is reached.

2.5.8 Some Recent Works on Condition Based Maintenance

A good treatise of research done in the area of Condition Based Maintenance as applicable to inspection and replacement models is given by Ghasemi et al. (2008).

2.5.9 Gaps identified in Current Replacement Models

Nakagawa's observation that *it is of great importance to avoid a failure of an item when its failure during operation is costly and/or dangerous* needs to be taken very seriously especially when one ponders the ramifications of failure of a critical component for an aeroplane flying in mid-air or a space aircraft in space; one can argue that the same applies for a country's missile defence system - the list is essentially endless. One way of reducing the incidence of failure is to have parallel connection of two or more of the critical components. An inspection and replacement model where this parallel connection is in place will prove useful.

Chapter 3

Scheduling of Inspection Times over a Finite Planning Horizon

3.1 Introduction

The focus on inspection and replacement models has mainly been confined to the infinite planning horizon because of the fact that the majority of papers dealing with inspection and replacement models, according to Berrade et al. (2013), assume that systems are required indefinitely and the major concern is that of cost of running the system. Another reason why many papers have focused on maintenance of systems for an infinite span as opposed to a finite span, according to Nakagawa and Mizutani (2009), is that the latter are theoretically more difficult to study.

Nakagawa and Mizutani (2009), however, point out that finite planning horizon models are particularly useful for such things as power plants which are becoming obsolete in Japan. Other systems cited in their paper which are good candidates for the application of the latter models include public infrastructure which encompass bridges, railroads, water supply and drainage in advanced nations. Wang and Christer (1997) extend earlier works which assume an infinite planning horizon and makes use of asymptotic results from Renewal and Renewal Reward processes to arrive at pertinent results for the finite planning horizon case. They assert that in practice, the time horizon over which a component or system may be used is finite and they give the need to move to upgraded systems from time to time to support their assertion. Other research work that has focused on a finite planning horizon include

1. Morey (1968) who researched on finite planning horizon models using minimization of cost of operating the system as his criterion. He has derived results for determining when it is prudent to carry out at least one inspection over a finite planning horizon for the case of imperfect inspections.

2. Usher et al. (1998) who discussed the case of a finite planning horizon with minimization of costs as their objective function.

3. Nakagawa et al. (2004) who discussed the application of basic inspection policies over a finite time span to five models: back-up for a hard disk, checkpoint for double modular redundancy, job partition, garbage collection and network partition.

4. Nakagawa and Mizutani (2009) who developed three replacement policies for a one-unit system; for the replacement policies, n identical units are sequentially replaced over a finite period $[0, L]$ in accordance with some set rules.

5. Taghipour et al. (2010) who proposed a model to find the optimal periodic inspection interval on a finite time horizon for a complex repairable system. The system has components which can experience "hard failures" (which are detected as and when they occur) and "soft failures" which are only detected when an inspection is carried out.

6. Ahmadi and Newby (2011) who use a new approach (which they defined as the intensity control model) at determining an optimal inspection schedule over a production run of finite length L with the sole objective of minimizing overall costs.

7. Berrade et al. (2013) who researched on periodic inspections being conducted on a system over a finite planning horizon of length L. The inspections in their paper are imperfect and the criterion they use is minimization of total cost over the planning horizon.

In this chapter, the optimization criterion used is maximization of profit (similar to the paper by Antelman and Savage (1965)) and the goal of models discussed in this thesis is that of determining procedures to answer the questions:

1. what is the ideal planning horizon (denoted by L in this thesis) for operating the system? How many inspections should be carried out over this ideal planning horizon and at which points in time should the inspections be scheduled?

2. when is it prudent to evenly spread inspections over the planning horizon?

As has been discussed in Chapter 2, Barlow et al. (1963) showed that inspection times that are equally interspaced (over a planning horizon that is not finite) generally do not result in minimization of per unit of time maintenance costs; in their work they state that for the class of lifetime distributions which are Polya frequency functions of order two (PF_2), it only happens for a system whose lifetime distribution is an exponential distribution. One would ask the question: is it also the case for models where the planning horizon is finite, i.e. would periodic inspections result in maximization of profit for a system that is operated over a finite planning horizon when the system lifetime distribution is an exponential distribution? The works of Nakagawa (1984) and Taghipour et al. (2010) are somewhat similar to what is done in this thesis when dealing with inspections that are evenly spread across the entire planning horizon.

3.2 A simple finite planning horizon inspection model

3.2.1 The model

This model applies to a situation where one plans to operate a system (whose purchase price is C_o) over a finite period of time, call it a finite planning horizon of length L. During this period, there are no planned inspections or $n \in \mathbb{N}$ planned inspections at times $x_1, x_2, \cdots, x_N$ such that

$0 < x_1 < x_2 < \cdots < x_n < L$.

Assumptions

It is assumed that at the start of operation, the system is in a working state (i.e. functioning perfectly) and producing products that are of acceptable quality and will continue to do so at a steady rate (thus enabling the owner or company which owns the system to generate income at a steady rate) until it gets into a failed state. The system's time to failure T is a continuous random variable with probability density function and cumulative distribution function $f_T(t)$ and $F_T(t)$, respectively. Like in most papers listed in Section 1.2, in this thesis we assume that T's distribution is completely known. The assumptions of the model are:

1. when working, the system generates or brings in revenue at a constant rate of c_R per unit of time,

2. the time it takes to complete each and every inspection is negligible and each inspection costs an amount of c_I - this is a common assumption in most papers on inspection replacement models that have been published. As will be seen in Chapter 4, Luss and Kander (1974), Luss (1977), Stadje and Zuckerman (1990), Badia et al. (2001) and Wang (2009) are some of the few papers which work on the premise that

inspections are of non-negligible duration.

3. no inspections or a finite number of inspections $n \in \mathbb{N}$ are scheduled to be done at times $x_1, x_2, \cdots, x_n$ such that $0 < x_1 < x_2 < \cdots x_n < L$; the notation $x_{jn}, j = 1, \cdots, n$ or $x_{j,n}, j = 1, \cdots, n$ is used to emphasize that x_{jn} is the time at which the j^{th} inspection out of a total of n inspections is to be conducted. Any scheduled inspections after the system has gone into the failed state are not done and the owner of the system only pays for those inspections which will have been done. When n inspections have been scheduled at times $x_1, x_2, \cdots, x_n$ as described above, the actual number of inspections done in a cycle N_C is therefore a random variable which depends on T and

$$P(N_C = k) = P(x_{k-1} \leq T \leq x_k) = \begin{cases} F_T(x_k) - F_T(x_{k-1}), & k = 1, 2, \cdots, n-1 \\ R_T(x_{k-1}), & k = n \\ 0, & \text{otherwise} \end{cases} \quad (3.1)$$

where $x_o = 0$.

4. if the system fails during operation, this is only detected at the next scheduled inspection at which point in time the project will be decommissioned and the system is sold (as scrap) to a recycling plant at

a salvage value of C_S[1]. Other examples of such systems which come to mind may include a) a system whose main and critically important component is a battery, b) a system whose main and critically important component is a bearing, c) a refrigerator where the main and critically important component is the compressor, d) an electric circuit which has one or more components which burn out when it gets into a failed state and are therefore not repairable, etc.

5. inspections will accurately report on the state of the system; i.e. there is no error on the part of inspections whereby a system that is functioning perfectly may be reported as being in a failed state (i.e. *false positive*) or a failed system being reported as functional (*false negative*). As will be seen in Chapter 5, Morey (1968), Badia et al. (2001), Wang (2009), Berrade et al. (2013), Wang et al. (2014) and Flage (2014) are some of the researchers who have developed models which deal with imperfect inspections. Badia et al. (2001) says that an inspection that erroneously reports that a non-faulty system is faulty commits a Type I error while an inspection that erroneously reports that a faulty system

[1] In this chapter C_S does not affect the optimal inspection schedule; it would, however, be an important factor in the case of a financial environment where the discount factor is non-zero

is not faulty commits a Type II error.

6. the system cannot fail while being checked and in addition, checking does not degrade the system. This assumption is in contrast to the works of Wattanapanom and Shaw (1979) and Butler (1979) who have separately worked on the case of systems where inspections may destroy or damage the system. Flage (2014) too has developed models with imperfect inspections that are in addition, failure-inducing.

7. if the system operates until the planned horizon L, it is de-commissioned and sold (as scrap) to a recycling plant at a salvage value of C_S,

8. the company or owner of the system incurs a cost of c_F for each unit of time the system is idle so that the cost associated with the system being idle for a time period of γ_t is $c_F\gamma_t$. Munford (1981) has developed a novel model which requires that the entire produce in a cycle where failure occurred needs to be reworked so that the cost associated with the downtime when the system was in a failed state discovered at the k^{th} inspection is $c_F(x_k - x_{k-1})$.

Remark 3.1 *The rules for decommissioning the system (given in Assumptions 4. and 7.) are very similar to the replacement rules in the papers* *by Zuckerman (1980) and Sheu et al. (2015) as well as the conditions for retiring the system in Berrade et al. (2013). Rangan and Grace (1989)'s "Replacement Policy T" rules for replacing a system are somewhat similar to the replacement rules used in this chapter.*

Remark 3.2 *If there exists at least one finite value t_o such that $F_T(t_o) = 1$, then models discussed in chapters 3, 4 and 5 assume that the planning horizon $L \leq t_o$.*

Notation

The following is a comprehensive list of the notation used in this thesis:

c_F - per unit of time cost of system idleness

c_I - cost of carrying out an inspection

C_o - cost of buying and installing the system

c_R - rate at which revenue accrues per unit of time

C_S - salvage value of the system upon disposal

T time to failure of the system

$f_T(.)$ - probability density of system time to failure T

$F_T(.)$ - cdf of system time to failure T

$R_T(.)$ - system reliability function (or survival function) and

$R_T(t) = P(T > t)$

$H_T(.)$ - hazard function (or failure rate function) for the system.

L - length of a finite planning horizon[2] ($L = x_{n+1}$)

x_{in} - i^{th} ($i = 1, \cdots, n$) inspection time when n inspections have been scheduled

τ - time between two successive inspections when inspections are equidistant and evenly spread across the planning horizon [3]

Theoretical results

If no inspection is planned and the system is to be operated up to a planning horizon of L, the expected value of the profit $G_{E.0}$[4]:

$$G_{E.0} = (c_R + c_F)\left[\int_0^L t f_T(t)dt - LF_T(L)\right] + c_R L - (C_o - C_S) \qquad (3.2)$$

[2]no inspection is scheduled at time $L = x_{n+1}$ as such

[3]when n inspections are evenly spread over the planning horizon of length L, then $L = (n+1)\tau$.

[4]in this chapter, $C_o - C_S$ does not impact on the optimal solution; it would, however, affect the optimal solution if there was a non-zero discount rate

Defining $x_o = 0$, the cost of operating the system up to the time it fails, T (if it fails before time L) or for the maximum time period of L (if it does not fail by time L, i.e. if $T > L$), $C(\mathbf{x}, T, L)$:

$$C(\mathbf{x}, T, L) = \begin{cases} ic_I + c_F(x_i - T) + (C_o - C_S), & x_{i-1} \leq T < x_i, i = 1, 2, \cdots, n \\ nc_I + c_F(L - T) + (C_o - C_S), & x_n \leq T < L \\ nc_I + (C_o - C_S), & T \geq L \end{cases} \tag{3.3}$$

The revenue generated, $R(T, L)$:

$$R(T, L) = \begin{cases} c_R T, & T < L \\ c_R L, & T \geq L \end{cases} \tag{3.4}$$

The net profit if the system fails after a time T, thus, $\mathcal{G}(\mathbf{x}, T, L)$:

$$\mathcal{G}(\mathbf{x}, T, L) = \begin{cases} c_R T - ic_I - c_F(x_i - T) - (C_o - C_S), & x_{i-1} \leq T < x_i, i = 1, 2, \cdots, n \\ c_R T - nc_I - c_F(L - T) - (C_o - C_S), & x_n \leq T < L \\ c_R L - nc_I - (C_o - C_S), & T > L \end{cases} \tag{3.5}$$

The expected profit for the simple finite planning horizon inspection model,

$\mathcal{G}_{E.n} = E[\mathcal{G}(\mathbf{x}, T, L)]$, thus, is such that:

$$\mathcal{G}_{E.1} = (c_R + c_F)\left[\int_0^L t f_T(t)dt - LF_T(L)\right] + c_F(L - x_1)F_T(x_1) + c_R L - c_I - (C_o - C_S) \tag{3.6}$$

(i.e. for the case where only one inspection is scheduled) and

$$\begin{aligned}\mathcal{G}_{E.n} &= (c_R + c_F)\left[\int_0^L t f_T(t)dt - LF_T(L)\right] + c_F(L - x_n)F_T(x_n) \\ &\quad + \sum_{i=1}^{n-1}\left[c_I + c_F(x_{i+1} - x_i)\right]F_T(x_i) + c_R L - nc_I - (C_o - C_S) \quad (3.7)\end{aligned}$$

for the case where at least two inspections are scheduled.

3.2.2 Properties of $\mathcal{G}_{E.n}$

The following four properties of $\mathcal{G}_{E.n}$, given as Lemmas (3.1) to (3.4), hold for a given value of n and any finite inspection times $x_1, x_2, \cdots, x_n$ such that $0 < x_1 < \cdots < x_n < L$.

Lemma 3.1 $\lim_{L \to 0^+} \mathcal{G}_{E.n} = C_S - C_o - nc_I$.

Proof:

As $L \to 0^+$, $F_T(L) \to 0$, *and* $F_T(x_i) \to 0$, $i = 1, \cdots, n$. *Thus,*

$\lim_{L \to 0^+} \mathcal{G}_{E.n} = -(C_0 - C_S + nc_I)$.

Lemma 3.2 *If the time to failure has a finite mean* μ_T *and* n *inspections are scheduled at specific finite times* $x_1, \cdots, x_n$, *then* $\lim_{L \to +\infty} \mathcal{G}_{E.n} = -\infty$.

Proof:

$$\mathcal{G}_{E.n} = (c_R + c_F)\int_0^L t f_T(t)dt - c_F L\left[F_T(L) - F_T(x_n)\right] + c_R L\left[1 - F_T(L)\right] + \kappa(x_1, \cdots, x_n) \tag{3.8}$$

where

$\kappa(x_1, \cdots, x_n) = -c_F F_T(x_n) + \sum_{i=1}^{n-1}[c_I + c_F(x_{i+1} - x_i)]F_T(x_i) - nc_I - (C_o - C_S)$

is finite. Now, since

$\lim_{L\to\infty}[F_T(L) - F_T(x_n)] = 1 - F_T(x_n) > 0$ *we have*

$\lim_{L\to\infty} c_F L[F_T(L) - FT(x_n)] = +\infty$ *while* $\lim_{L\to\infty} c_R L\left[1 - F_T(L)\right] = 0.$

Thus, $\lim_{L\to\infty} \mathcal{G}_{E.n} = -\infty.$

Lemma 3.3

$$\frac{\partial \mathcal{G}_{E.n}}{\partial L} = -c_F\left[F_T(L) - F_T(x_n)\right] + c_R\left[1 - F_T(L)\right] \tag{3.9}$$

and since $\lim_{L\to 0} F_T(L) = \lim_{L\to 0} F_T(x_n) = 0$, *we have*

$$\lim_{L\to 0} \frac{\partial \mathcal{G}_{E.n}}{\partial L} = c_R. \tag{3.10}$$

Remark 3.3 *For any given number of inspections n taking place at specific fixed times*

$0 < x_1 < x_2 \cdots, < x_n < L$, *initially the expected profit increases as the planned horizon L for operating the system increases.*

Lemma 3.4 *For any given number of inspections n taking place at specified fixed times $0 < x_1 < x_2 \cdots, < x_n < L$, there exists a unique planning horizon*[5] L^* *which maximizes the profit function $\mathcal{G}_{E.n}$ which is given by*

$$L^* = F_T^{-1}\left[\frac{c_R + c_F F(x_n)}{c_R + c_F}\right] \tag{3.11}$$

Proof: From Equation (3.9),

$$\begin{aligned}
\frac{\partial \mathcal{G}_{E.n}}{\partial L} &= -c_F\left[F_T(L) - F_T(x_n)\right] + c_R\left[1 - F_T(L)\right] = 0 \text{ when } L = L^* \\
\Rightarrow F(L^*) &= \frac{c_R + c_F F(x_n)}{c_R + c_F} \\
\Rightarrow L^* &= F^{-1}\left[\frac{c_R + c_F F(x_n)}{c_R + c_F}\right]
\end{aligned}$$

$$\text{Further}, \frac{\partial^2 \mathcal{G}_{E.n}}{\partial L^2} = -(c_F + c_R) f_T(L) < 0 \text{ when } L = L^*.$$

Remark 3.4 *If n inspections are planned at specific fixed times $x_{1n}, \cdots, x_{nn}$, the length of the optimal planning horizon L_n^* depends only on the time at which the very last inspection takes place and the time to failure distribution.*

Bearing in mind the properties of $\mathcal{G}_{E.n}$ discussed above, it is imperative to formulate the following as objectives in the search for a scenario that will maximize profit:

[5] provided $F_T(t)$ is a strictly monotonic increasing function of t

1. For each $n \in \mathbb{N}$, to find the n optimal inspection times $x^*_{1n}, x^*_{2n}, \cdots, x^*_{nn}$ such that $0 < x^*_{1n}, x^*_{2n}, \cdots, x^*_{nn}$ and optimal planning horizon L^*_n which maximize $\mathcal{G}_{E.n}$ (see Section 3.2.4.

2. For a pre-set value of $L > 0$, to find the optimal number of inspection times, $n = n^*$ and the inspection times $0 < x^*_{1n^*} < x^*_{2n^*}, \cdots < x^*_{n^*n^*}$ which maximize $\mathcal{G}_{E.n}$; the algorithm for finding n^* and the accompanying inspection times involves starting off with 0 inspections and computing $\mathcal{G}_{E.0}$; next consider 1 inspection and calculate $\mathcal{G}_{E,1}$ and compare with $\mathcal{G}_{E.0}$; of the two, opt for one that results in a higher profit. If 1 inspection has been found to be better, calculate $\mathcal{G}_{E.2}$ and compare with $\mathcal{G}_{E,1}$ and so on. The process is repeated iteratively until n is such that $\mathcal{G}_{E.n} > \mathcal{G}_{E.n+1}$ or for some $\epsilon > 0$, $|\mathcal{G}_{E.n+1} - \mathcal{G}_{E.n}| < \epsilon$ in the case of $G_{E.n}$ being a monotonic increasing function of n.

3. To find the values $n = n^{**}$, $L = L^{**}$, $x_1 = x^{**}_1, x_1 = x^{**}_1, \cdots, x_n = x^{**}_{n^*}$, which jointly result in the global maxima of $\mathcal{G}_{E.n}$.

4. To assess the impact of evenly spreading inspections across the planning horizon on profit. When inspections are evenly spread across the planning horizon, the i^{th} inspection takes place at time $x_i = i\tau,\ i = 1, \cdots, n$

where $L = (n+1)\tau$.

Remark 3.5 *From Equation (3.7), it can be deduced that for equally inter-spaced inspections such that $x_i - x_{i-1} = \tau$ $(i = 1, \cdots, n)$ and $L = (n+1)\tau$*

$$\begin{aligned}\mathcal{G}_{E.n}(L) &= (c_R + c_F)\left[\int_0^L t f_T(t)dt - LF_T(L)\right] + \frac{c_F L}{n+1} F_T\left(\frac{nL}{n+1}\right) \\ &\quad + \left(c_I + \frac{c_F L}{n+1}\right)\sum_{i=1}^{n-1} F_T\left(\frac{iL}{n+1}\right) + c_R L - nc_I - (C_o - C_S)\end{aligned} \tag{3.12}$$

3.2.3 Optimal inspection times and optimal planning horizon

Using differential calculus and Equation (3.2) we deduce that the optimal planning horizon when no inspection is scheduled is of length L_0^*:

$$L_0^* = F_T^{-1}\left(\frac{c_R}{c_R + c_F}\right). \tag{3.13}$$

From Equation (3.6), we see that when a single inspection is to be scheduled, the optimal inspection time x_{11}^* and optimal planning horizon L_1^* are solutions of system of equations (3.14):

$$\left.\begin{array}{ll} L_1^* = x_{11}^* + \frac{F_T(x_{11}^*)}{f_T(x_{11}^*)} & [1] \\ F_T(x_{11}^*) = \frac{(c_R + c_F)F_T(L_1^*) - c_R}{c_F} & [2] \end{array}\right\} \tag{3.14}$$

Lemma 3.5 *The optimal planning horizon when no inspection is scheduled is shorter than the optimal planning horizon when a single inspection is scheduled, i.e. $L_0^* < L_1^*$.*

Proof:

$$x_{11}^* > 0 \;\Rightarrow\; F_T(x_{11}^*) > 0 \Rightarrow \frac{(c_R + c_F)F_T(L_1^*) - c_R}{c_F} > 0$$

$$\Rightarrow F_T(L_1^*) > \frac{c_R}{c_R + c_F} = F_T(L_o^*) \Rightarrow L_1^* > L_0^*.$$

In the general case of $n \in \mathbb{N}$ ($n \geq 2$) inspections being scheduled, the optimal inspection times $x_{1n}^*, \cdots, x_{nn}^*$ and optimal planning horizon L_n^* are solutions of the system of equations (3.15) (the equations arise from the $n+1$ partial derivatives of $\mathcal{G}_{E.n}$ in Equation (3.7) with respect to $x_1, \cdots, x_n$ and L being set to 0 each):

$$\left.\begin{array}{lll} x_{2n}^* & = x_{1n}^* + \frac{F_T(x_{1n}^*)}{f_T(x_{1n}^*)} - \frac{c_I}{c_F}, & [1] \\ x_{k+1,n}^* & = x_{kn}^* + \frac{F_T(x_{kn}^*) - F_T(x_{k-1,n}^*)}{f_T(x_{kn}^*)} - \frac{c_I}{c_F},\ k = 2, \cdots, n-1 & [2] - [n-1] \\ L_n^* & = x_{nn}^* + \frac{F_T(x_{nn}^*) - F_T(x_{n-1,n}^*)}{f_T(x_{nn}^*)} & [n] \\ F_T(x_{nn}^*) & = \frac{(c_R + c_F)F_T(L_n^*) - c_R}{c_F} & [n+1] \end{array}\right\} \tag{3.15}$$

The results arrived at in this thesis are somewhat similar to those arrived at by Barlow et al. (1963) (see Equation (7) in their paper) and Munford (1981) in the section entitled "Optimal Inspection Policy, Model 2". The latter two papers look at a continuous production process where inspections take place at certain designated times over an infinite planning horizon.

Remark 3.6 *Just like in the papers by Barlow et al. (1963), Luss (1977) and*

Luss (1983), Equations (3.14) and (3.15) do not present tractable solutions for determining optimal inspection times and the corresponding optimal planning horizon for a specified number of inspections n. Two methods (which rely on computer programs) for determining optimal inspection times and the optimal planning horizon are suggested in this thesis (see Section 3.2.4 and Section 3.2.5).

3.2.4 Method 1 - Iterative procedure for calculating optimal inspection times

The iterative procedure suggested in this chapter (which makes use of Equation (3.15) for determining estimates of the optimal inspection times and optimal planning horizon for a given number of inspections) is as follows:

Step 1 : Start with a guesstimate of x^*_{1n}, call it $\chi^{(1)}_{1n}$ and use it in Result [1] of Equation (3.15) to calculate an estimate $\chi^{(1)}_{2n}$ of x^*_{2n} which in turn is used to calculate estimates $\chi^{(1)}_{3n}, \cdots, \chi^{(1)}_{nn}$ and $\mathcal{L}_n$ of $x^*_{3n}, \cdots, x^*_{nn}$ and L^*_n, respectively, by making use of results [2] to [n] of Equation (3.15).

Step 2 : Using the value of $\mathcal{L}_n$ found in Iteration 1, determine a new estimate $\chi^{(2)}_{nn}$ of x^*_{nn} using Equation (4.23):

$$F_T(\chi^{(2)}_{nn}) = \frac{(c_R + c_F)F_T(\mathcal{L}_n) - c_R}{c_F} \tag{3.16}$$

and then calculate the difference $\Delta = \chi_{nn}^{(2)} - \chi_{nn}^{(1)}$. If $\Delta \approx 0$, then the guesstimate of x_{1n}^*, $\chi_{1n}^{(1)}$ as well as estimates of $x_{3n}^*, \cdots, x_{nn}^*$ and L_n^* are acceptably good. If $\Delta < 0$, go back to Step 1 and use a larger value of the guesstimate $\chi_{1n}^{(1)}$. On the other hand, If $\Delta > 0$, go back to Step 1 and use a smaller value of the guesstimate $\chi_{1n}^{(1)}$.

The algorithm is summarized diagramatically using a flowchart in Figure 3.1.

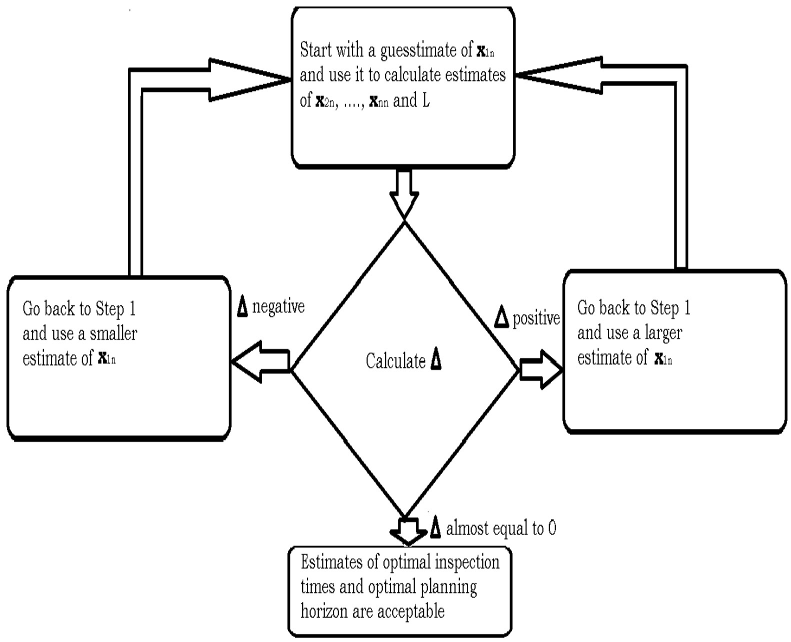

Figure 3.1: An algorithm for calculating optimal inspection times and the optimal planning horizon

Remark 3.7 *If one starts with a guesstimate* $\chi_{1n}^{(1)}$ *which is too small in comparison to* x_{1n}^**, i.e.* $\chi_{1n}^{(1)} - x_{1n}^* << 0$*, Equation (4.23) will be infeasible because the value of* $\frac{(c_R+c_F)F_T(\mathcal{L}_n)-c_R}{c_F}$ *will be negative. On the other hand, a guesstimate* $\chi_{1n}^{(1)}$ *which is too large in comparison to* x_{1n}^**, i.e.* $\chi_{1n}^{(1)} - x_{1n}^* >> 0$ *results in some values of* $\chi_{jn}^{(1)}, j > 1$ *approaching infinity and computation of* $\chi_{nn}^{(2)}$ *being rendered impossible.*

3.2.5 Method 2 - Nonlinear optimization procedure for calculating optimal inspection times

The second method hinges on the formulation of the problem as a non-linear optimization problem with the usual non-negativity constraints as in Equation (3.17):

$$\begin{aligned}
\text{Maximize } \mathcal{G}_{E.n} \quad &= \quad (c_R + c_F)\left[\int_0^{x_{n+1}} t f_T(t)dt - x_{n+1}F_T(x_{n+1})\right] + c_F(x_{n+1} - x_n)F_T(x_n) \\
&\quad + \sum_{i=1}^{n-1}\left[c_I + c_F(x_{i+1} - x_i)\right]F_T(x_i) + c_R x_{n+1} - nc_I - (C_o - C_S) \\
\text{subject to:} \quad &\quad x_i - x_{i+1} \leq 0;\ i = 1, \cdots, n \qquad (3.17)
\end{aligned}$$

3.3 Applications and Examples

In this section, the theoretical results and examples applicable to time to failure following a continuous uniform distribution or being a member of the Weibull family of probability distributions are explored.

Remark 3.8 *Barlow et al. (1963) have stated the following (Theorem 5 of their paper) with proof (for the case of an infinite planning horizon where the cost per unit of time is the optimization criterion):* **If the failure density** f **is a Polya frequency function of order 2 (**PF_2**), and** $f_T(t) > 0$ **for** $t > 0$**, then the optimal checking intervals are non-decreasing.**

3.3.1 Time to failure following a continuous uniform distribution

For the case $T \sim U[0, L_o]$ (i.e. T being uniformly distributed over the interval $[0, L_o]$), if inspections are evenly spread across a preset planning horizon $L < L_o$, from Equation (3.12) we have:

$$G_{E.n}(L) = \frac{nc_F L^2}{(n+1)^2 L_o} + \frac{n(n-1)\left(c_I + \frac{c_F L}{n+1}\right) L}{2(n+1)L_o} - nc_I + c_R L - \frac{(c_R + c_F)L^2}{2L_o} - (C_o - C_S) \tag{3.18}$$

Remark 3.9 *It does not make business sense to have a planning horizon L which is longer than the maximum possible length for which the system may operate and hence if $T \sim U[0, L_o]$, we consider the scenario $L < L_o$ only.*

Lemma 3.6 *The optimal number of inspections for this sub-class of inspection models is the least integer n such that*

$$n \geq \sqrt{\frac{c_F L^2}{c_I(2L_o - L)} - \frac{4L_o}{2L_o - L} + \frac{9}{4}} - \frac{3}{2} \tag{3.19}$$

Proof:

$$\begin{aligned}
G_{E.n}(L) - G_{E.n+1}(L) &= \frac{[2(n^2+3n+2)L_o - (n+3)nL]\, c_I - c_F L^2}{2(n+1)(n+2)L_o} \\
\textit{and}\; G_{E.n}(L) - G_{E.n+1}(L) &\geq 0 \\
\Leftrightarrow 2(n^2+3n+2)L_o - (n+3)nL &\geq \frac{c_F}{c_I} L^2 \\
\Leftrightarrow n &\geq \sqrt{\frac{c_F L^2}{c_I(2L_o - L)} - \frac{4L_o}{2L_o - L} + \frac{9}{4}} - \frac{3}{2}.
\end{aligned}$$

For the case where the planning horizon is not preset, the optimal planning horizon when n inspections are to be evenly spread, $L_{opt}(n)$:

$$L_{opt}(n) = \begin{cases} \frac{2(n+1)c_R L_o + n(n-1)c_I}{2(c_F + (n+1)c_R)}, & \text{if } n \leq \sqrt{2\frac{c_F}{c_I}L_o + \frac{1}{4}} + \frac{1}{2} \\ L_o, & \text{if } n > \sqrt{2\frac{c_F}{c_I}L_o + \frac{1}{4}} + \frac{1}{2} \end{cases} \qquad (3.20)$$

Remark 3.10 *It does not make any business sense to schedule more than* $\sqrt{2\frac{c_F}{c_I}L_o + \frac{1}{4}} + \frac{1}{2}$ *inspections if the time to failure* $T \sim U[0, L_o]$.

Example 3.1 *Suppose a system is such that its time to failure follows a continuous uniform distribution over the interval* $[0, 100]$. *Other attributes of the system are:* $C_o = \$10000$, $C_S = \$2500$, $c_R = \$1000$, $c_F = \$200$ *and* $c_I = \$400$. *The optimum inspection schedules and optimal planning horizon (found with the aid of a Mathematica program similar to the one in Appendix A.1) are given in Table 3.1.*

What is interesting is that for a given number of inspections n, the optimal planninng horizon and the optimal planning horizon for the case of uniformly spread inspections are approximately equal. The global optimal inspections schedule requires that 7 inspections be done at optimal times $19.07, 36.15, \cdots, 91.51$. For the subclass of inspection models where inspections are evenly spread over a finite planning horizon, the optimal inspection model requires that 6 inspections be carried out at unformly interspaced times $14.01, 28.02, \cdots, 84.05$. For any specified number of inspections, the frequency of inspection rises with the ageing of the system (consistent with Remark 3.8 made by Barlow et al. (1963)) .

Barlow et al. (1963) say that in the case of time to failure being a uniformly distributed random variable over the interval $[0, \tau_o]$ and the criterion being minimization of costs, the optimal number of inspections is the largest integer n^* such that $n^*(n^* - 1) < 2c_F\tau_o/c_I$; thus, n^* would have been 10 - which is different when the optimization criterion is maximization of expected profit. However, the inspection schedules for the two models when the number of inspections is $n = 10$ are almost the same.

Table 3.1: Optimal inspection times and the optimal planning horizon when time to failure is uniformly distributed

n	Inspection times which are optimally scheduled/ Inspection times which are evenly spread	Optimal Planning horizon	Expected Profit for Optimal scheduling/Expected profit for Uniformly spread inspections
0	-	83.33	34166.70
	-	83.33	34166.70
1	45.46	90.91	37554.55
	45.46	90.91	37554.55
2	32.62, 63.25	93.88	38702.75
	31.29, 62.58	93.88	38700.10
3	26.38, 50.76, 73.14	95.52	39216.19
	23.88, 47.76, 71.64	95.52	39205.20
4	22.92, 43.85, 62.77, 79.69	96.62	39466.77
	19.32, 38.65, 57.97, 77.29	96.62	39439.57
5	20.90, 39.81, 56.71, 71.61, 84.52	97.42	39587.74
	16.24, 32.47, 48.71, 64.95, 81.18	97.42	39534.40
6	19.72, 37.44, 53.17, 66.89, 78.61, 88.33	98.06	39639.40
	14.01, 28.02, 42.02, 56.03, 70.04, 84.05	98.06	39548.00
7	19.07, 36.15, 51.22, 64.29, 75.37, 84.44, 91.51	98.59	39653.75
	12.32, 24.65, 36.97, 49.29, 61.62, 73.94, 86.26	98.59	39510.30
8	18.78, 35.57, 50.35, 63.13, 73.91, 82.70, 89.48, 94.26	99.04	39649.57
	11.00, 22.01, 33.01, 44.02, 55.02, 66.03, 77.03, 88.04	99.04	39438.00
9	18.75, 35.49, 50.24, 62.98, 73.73, 82.47, 89.22, 93.96, 96.71	99.45	39639.10
	09.95, 19.89, 29.84, 39.78, 49.73, 59.67, 69.62, 79.56, 89.51	99.45	39341.50
10	18.89, 35.79, 50.68, 63.57, 74.46, 83.36, 90.25, 95.14, 98.04, 98.93	99.82	39631.07
	09.07, 18.15, 27.22, 36.30, 45.37, 54.45,63.52, 72.60, 81.67, 90.75	99.82	39227.40

3.3.2 Time to failure an exponentially or Weibull distributed random variable

Sun et al. (1993) and Smith and Naylor (1987) say that the Weibull distribution [6] (WD) is perhaps the distribution that has the widest acclaim in Reliability Theory owing to its flexibility which makes it able to fit a wide range of life-time data.

Time to failure exponentially distributed

In the case of T following an exponential distribution with parameter θ, $f_T(t) = \theta e^{-\theta t}.I_{(0,\infty)}(t)$ and $F_T(t) = (1 - e^{-\theta t}).I_{(0,\infty)}(t)$.

Example 3.2 *Suppose the time to failure of a system follows an exponential distribution with $\theta = 0.05$ year^{-1}. The machine has $C_o = \$10000$, $C_S = \$2500$, $c_R = \$1000$, $c_F = \$200$ and $c_I = \$400$. Using a Mathematica program, the various optimal inspection times and optimal planning horizons for different values of n are given in Table 3.2 below.*

The observations made are

- From Table 3.2, one observes that as the number of inspections increases, the expected profit obtained by evenly scheduling inspections

[6] the exponential distribution is a special case of the Weibull distribution

over the planning horizon and the expected profit obtained from optimally scheduled inspections converge.

- Figure 3.5 gives an illustration of the typical variation of $\mathcal{G}_{E.n}$ with L for a fixed number of inspections ($n = 3$) which are evenly spread over the planning horizon for a system whose time to failure follows an exponential distribution. The graph of $G_{E.n}$ versus L (for an exponential distribution) follows the same pattern in the case of optimal inspection policies.

- Figure 3.2 demonstrates the typical dependence of $G_{E.n}$ on the inspection time and planning horizon if a single inspection were to be scheduled; the values of x_1 and L which jointly maximize the profit function are unique and the origin can be used as an initial feasible solution when trying to find the values of the variables which jointly maximize the profit using non-linear programming.

- Figure 3.3 illustrates the fact that a larger planning horizon will certainly result in a larger value of $\mathcal{G}_{E.n}$ provided an optimum and higher number of inspections at optimally set inspection times are done. What is observed is that the net gain of increasing the planning horizon by

1 unit of time, however, progressively diminishes with larger values of L and this translates to $\mathcal{G}_{E.n}$ converging to its supremum when values of L and n are appropriately increased. From this observation, it may be prudent to recommend that, for some desirable $\epsilon > 0$, an optimal value of n is the least value of n such that $\mathcal{G}_{E.n+1} - \mathcal{G}_{E.n} < \epsilon$

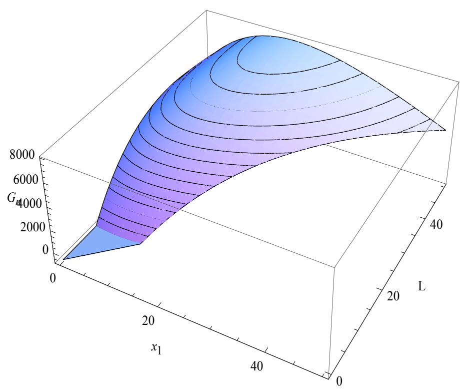

Figure 3.2: Profit when a single inspection is scheduled

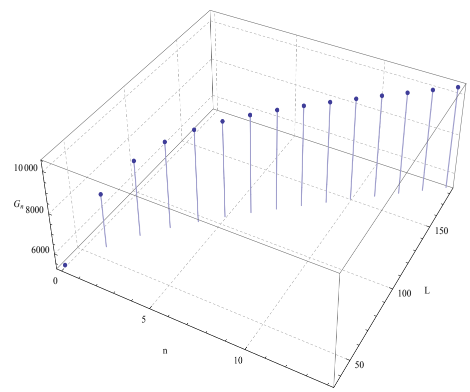

Figure 3.3: Maximum of G_E versus n and L^*(lifetime exponentially distributed)

- Also from Figure 3.3 the impression one gets is that shorter planning horizons have a lower number of inspections per unit of time (i.e. n^*/L^*) compared to longer planning horizons; initially L^* versus n has a higher slope which appears to stabilize as L^* (or n) increases.

- higher values of c_I favour fewer and relatively more evenly spread inspections.

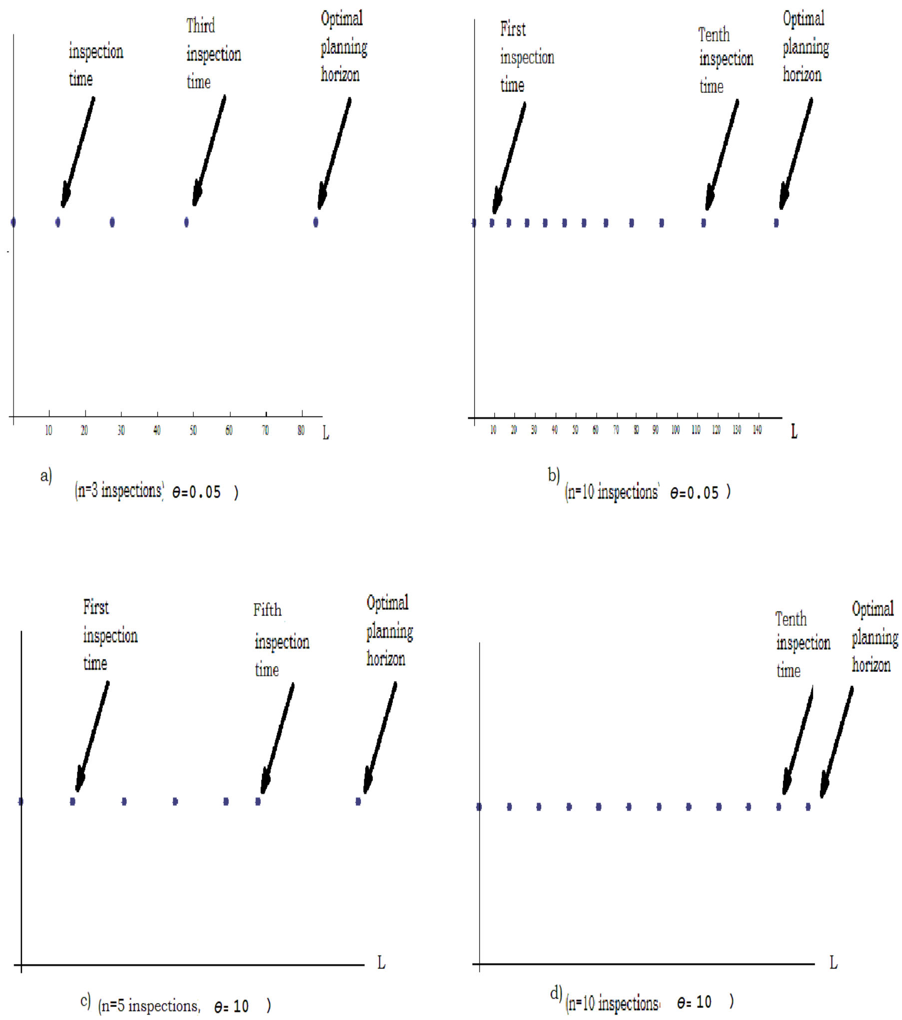

Figure 3.3: The dependence of optimal inspection times on θ

- Inference drawn from Figure 3.3 is that the time between successive inspections increases progressively over time for low values of θ (i.e. for systems which generally have a longer survival time). A reverse trend (whereby the inter-inspection time progressively decreases) occurs for systems with much larger values of θ (i.e. systems whose time to failure are generally shorter).

- For systems with shorter time to failure (i.e. higher values of θ), evenly spreading the inspections over the planning horizon, if the number of inspections is large, will not result in much lower profits compared to the optimal inspection schedule [7].

[7] a comparison of the first few rows and last row of Table 3.2 attest to this observation

Table 3.2: Comparison of evenly spread and optimally set inspection times for failure time following a Weibull distribution ($k = 1, \theta = 0.05, c_R = 1000; c_o = 10000; c_S = 2500; c_I = 400; c_F = 200$)

n	L(evenly spread inspection times)	Optimal profit value $G_{E.n}$(for evenly spread inspections)	Optimal inspection times (not evenly spread)	Optima planning horizon L_{opt}	Optimal profit, $G_{E.n}$
0	35.8352	5332.96	-	35.8352	5332.96
1	51.3335	7701.17	$x_1 = 20.5334$	56.3686	7993.31
2	61.8151	8749.17	$x_1 = 15.0912, x_2 = 35.6246$	71.4598	9081.77
3	69.9111	9318.25	$x_1 = 12.3529, x_2 = 27.4441, x_3 = 47.9775,$	83.8127	9629.41
4	76.7377	9665.01	$,\{x_1 = 10.8191, x_2 = 23.1720, x_3 = 38.2632, x_4 = 58.7966$	94.6318	9936.18
5	82.8427	9891.52	$9.9056, x_2 = 20.7247, x_3 = 33.0776, x_4 = 48.1688, x_5 = 68.7022$	104.5374	10118.88
6	88.5342	10046.17	$x_1 = 9.3410, x_2 = 19.2465, x_3 = 30.0656, x_4 = 42.4186, x_5 = 57.5097, x_6 = 78.0432$	113.8784	10231.81
7	94.0051	10154.84	$x_1 = 8.9839, x_2 = 18.3248, x_3 = 28.2304, x_4 = 39.0495, x_5 = 51.4024, x_6 = 66.4936, x_7 = 87.0271$	122.8622	**10303.22**
8	99.3868	10232.64	$x_1 = 8.7547, x_2 = 17.7386, x_3 = 27.0795, x_4 = 36.9851, x_5 = 47.8042, x_6 = 60.1571, x_7 = 75.248302922565727613, x_8 = 95.7817$	131.6169	**10349.06**
9	104.7755	10288.96	$x_1 = 8.6062, x_2 = 17.3609, x_3 = 26.3448, x_4 = 35.6857, x_5 = 45.5913, x_6 = 56.4104, x_7 = 68.7633, x_8 = 83.8545, x_9 = 104.3879$	140.2231	10378.76
10	110.2450	**10329.96**[1]	$\{x_1 = 8.5094, x_2 = 17.1156, x_3 = 25.8703, x_4 = 34.8542, x_5 = \cdots, x_8 = 77.2728, x_9 = 92.3639, x_{10} = 112.8974\}$	148.7325	10398.12
11				157.1786	10410.78
12				165.5831	10419.10
13				173.9603	10424.57
14				182.3194	10428.17
15				190.6668	10430.53
20	176.7356	10432.48	$x_1 = 8.3272, x_2 = 16.6560, x_3 = 24.9870, x_4 = 33.3214, \cdots, x_{15} = 127.7903, x_{16} = 137.6958, x_{17} = 148.5150, x_{18} = 160.8679, x_{19} = 175.9590, x_{20} = 196.4925$	232.3276	10434.54
30	258.1276	10435.0727[2]	$x_1 = 8.3245, x_2 = 16.6490, x_3 = 24.9735, x_4 = 33.2981, x_5 = 41.6227, , \cdots, x_{24} = 201.6988, x_{25} = 211.0396, x_{26} = 220.9450, x_{27} = 231.7638, x_{28} = 244.1161, x_{29} = 259.2062, x_{30} = 279.7372$	315.5638	10435.11

[1] This profit value is achieved with 10 evenly spread inspections over a planning horizon of about 110 time units; about the same profit level is achieved by having between 7 and 8 inspections which are optimally set over a planning horizon of about 125 time units.

[2] This profit value is achieved with 30 evenly spread inspections over a planning horizon of about 258 time units; about the same profit level is achieved by about 20inspections which are optimally set over a planning horizon of about 232 time units. For this example, evenly spread inspections are ideal for shorter planning horizons where inspections comparatively more frequent.

Example 3.3 *Suppose a system is such that its time to failure follows an exponential distribution with* $\theta = 0.05$ *year*$^{-1}$. *Other attributes of the system are:* $C_o = \$10000$, $C_S = \$2500$, $c_R = \$1000$, $c_F = \$200$ *and* $c_I = \$400$. *The optimum inspection schedules when it is planned that the system will be operated a) for 100 and b) for 200 time units are illustrated in Figures 3.4.*

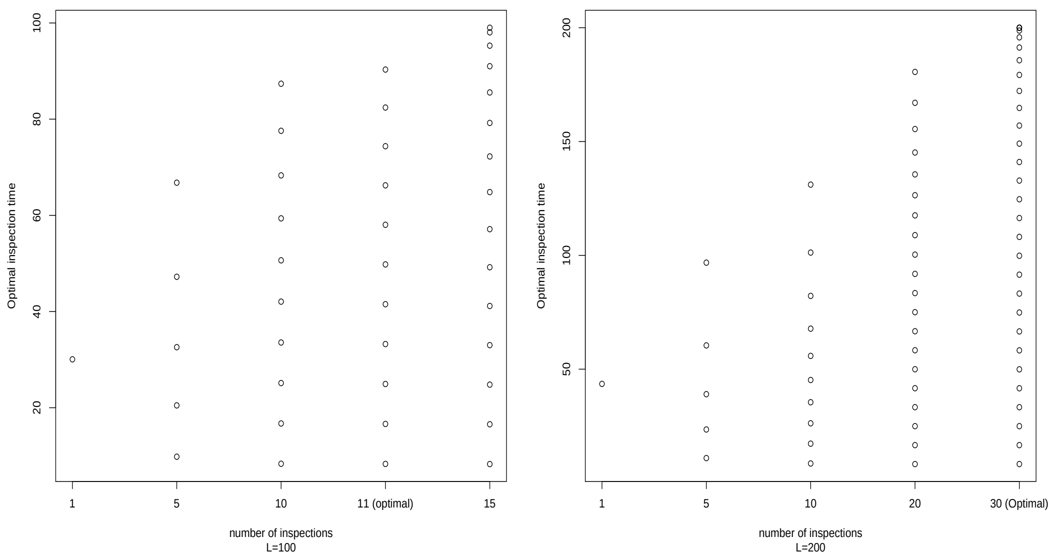

Figure 3.4: Optimal inspection schedules when $L = 100$ and 200 time units

What is observed (see Figure 3.4) is that for a given fixed planning horizon,

- if the number of inspections is small, the time between successive inspections starts off small and progressively increases with time and most of the inspections are crammed in the beginning. On the other hand, if the number of inspections is large, the time between two successive inspections starts off large and progressively gets diminished over time such that a lot of inspections are crammed in the time interval leading up to the planning horizon.

- if the optimal number of inspections is used, the inspections appear to be somewhat evenly spread over the planning horizon.

Time to failure following a Weibull distribution (WD)

If the lifetime of a system follows a Weibull distribution with parameters θ and k [8], i.e. $T \sim WD(k, \theta)$, its pdf and cdf, respectively are

$$\begin{aligned} f_T(t; \theta, k) &= k\theta(\theta t)^{k-1} e^{-(\theta t)^k} . I_{(0,\infty)}(t) \text{ and} \\ F_T(t; k, \theta) &= \left(1 - e^{-(\theta t)^k}\right) . I_{(0,\infty)}(t) \end{aligned} \tag{3.21}$$

where the indicator function $I_{(0,\infty)}(t) = \begin{cases} 1, & t > 0 \\ 0, & t \leq 0 \end{cases}$

The dependence of $\mathcal{G}_{E.n}$ on the planning horizon for $k = 1$ and $k = 7$ (when inspections are evenly spread) are separately illustrated in Figure 3.5(a) and Figure 3.5(b). In particular, it is observed that for low values of k, e.g. $k = 1$

[8] a random variable $T \sim WD(1, \theta)$ is essentially a t-distributed random variable with parameter θ

(i.e. when the time to failure is an exponential distribution), there is one distinct peak for the graph of $\mathcal{G}_{E.n}$ versus L while for higher values of k, it is observed that $\mathcal{G}_{E.n}$ initially increases with increasing values of L to a certain local maxima; from this point on the pattern is rugged and a couple of local maximas are observed after which the expected profit then starts the downward trend as expected profits decline with increasing values of L.

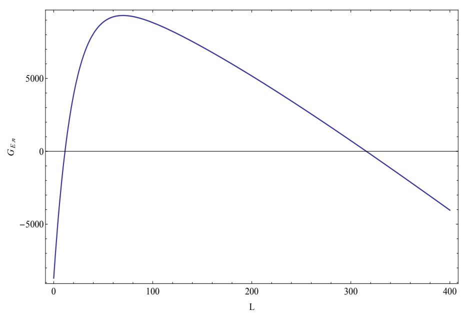

Figure 3.5: (a)$G_{E.n}$ vs L for evenly spread inspections (when $n = 3$) and $T \sim WD(1, 0.05)$)

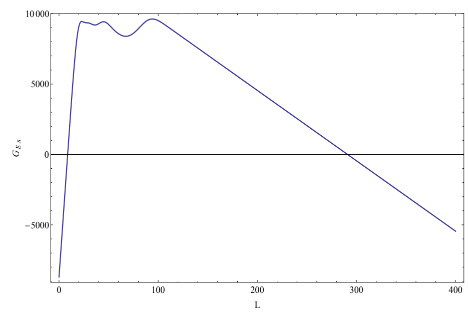

Figure 3.5: (b)$G_{E.n}$ vs L for evenly spread inspections (when $n = 3$) and $T \sim WD(7, 0.05)$

Example 3.4 *Suppose the time to failure of a system follows a Weibull distribution with* $\theta = 0.05$ *year*$^{-1}$ *and* $k = 5$*. The machine has* $C_o = \$10000$*,* $C_S = \$2500$*,* $c_R = \$1000$*,* $c_F = \$200$ *and* $c_I = \$400$*.*

The observations made are

- from Figure 3.8 (when a single inspection is to be done) profit initially increases with both increasing L and inspection time x_1. The origin is not a suitable initial feasible solution as the graph of $G_{E.1}$ versus x_1 and L has more than one local maxima. Figure 3.8 clearly demostrates that there exist optimal values of L and x_1 which together jointly maximize the profit.

- The shape of the plot in Figure 3.9 is similar to the one in Figure 3.2. Just like in the case of the exponential distribution, Figure 3.9 illustrates the fact that a larger planning horizon will certainly result in a larger value of $\mathcal{G}_{n.L}$ provided an optimum and higher number of inspections at optimally set inspection times are done. It is again observed that the net gain of increasing the planning horizon by 1 unit of time progressively gets diminished with larger values of L and this translates to $\mathcal{G}_{E.n}$ converging to its supremum when values of L and n are appropriately increased. From this observation, it is again appropriate to recommend that for some desirable $\epsilon > 0$, an optimal value of n is the least value of n such that $\mathcal{G}_{E.n+1} - \mathcal{G}_{E.n} < \epsilon$

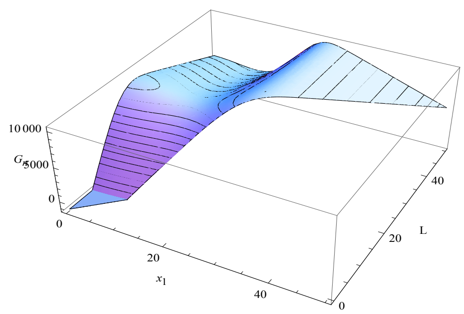

Figure 3.8: $G_{E.n}$ vs L and inspection time (x_1) for $T \sim WD(5, 0.05)$)

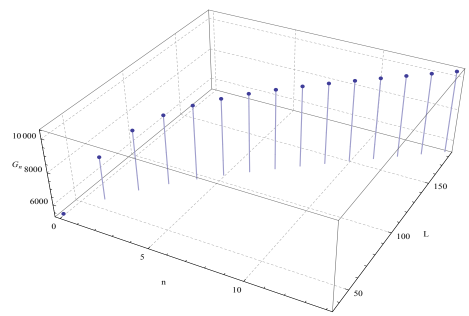

Figure 3.9: $G_{E.n}$ vs L and n for optimally placed inspection times when $T \sim WD(5, 0.05)$

Optimal inspection schedules for different forms of Weibull distributions

An investigation of the dependence of optimal inspection schedule on the form of the Weibull distribution (specifically for different coefficient of skewness, α_3 values) was carried out with the aid of Example (3.5) below. According to Lindsay et al. (1996), the coefficient of skewness of the Weibull distribution depends on k and

$$\alpha_3 = \frac{\Gamma\left(1+\frac{3}{k}\right) - 3\Gamma\left(1+\frac{1}{k}\right)\Gamma\left(1+\frac{2}{k}\right) + 2\Gamma^3\left(1+\frac{1}{k}\right)}{\left[\Gamma\left(1+\frac{2}{k}\right) - \Gamma^2\left(1+\frac{1}{k}\right)\right]^{\frac{3}{2}}} \tag{3.22}$$

When $k \approx 3.6$, $\alpha_3 \approx 0$ and the pdf of T is near symmetrical; the distribution is positively skewed when k has lower values while larger values of k entail negative skewness.

Example 3.5 *Suppose the time to failure of a system follows a Weibull distribution with $\theta = 0.05$ year^{-1} and $k > 1$. The machine has $C_o = \$10000$, $C_S = \$2500$, $c_R = \$1000$, $c_F = \$200$ and $c_I = \$400$. The optimal inspection schedules for $n = 1$ and $n = 3$ are given in Table 3.3 below.*

Table 3.3: Optimal inspection schedules for different forms of the Weibull Distribution

k	μ_T	σ_T^2	α_3	x_1	L	Optimal value of $G_{E.1}$	x_1	x_2	x_3	L	Optimal value of $G_{E.3}$
0.5	40.00	8000.00	6.619	37.86	200.68	\$15879.20	16.95	70.59	189.32	474.03	\$23798.10
1	20.00	400.00	2.000	20.53	56.37	\$7993.31	12.35	27.44	47.98	83.81	\$9629.41
2	17.72	85.84	0.631	18.20	32.37	\$8021.60	14.59	22.23	29.40	39.76	\$8442.28
3	17.86	42.13	0.168	18.23	27.32	\$8755.01	16.48	21.84	26.01	31.72	\$8880.45
3.6	18.02	30.91	0.00056	18.35	25.87	\$9111.24	17.30	21.86	25.10	29.51	\$9149.23
10	19.03	5.24	-0.6376	19.16	21.87	\$10745.60	20.22	21.94	22.62	23.59	\$10592.70
20	19.47	1.46	-0.868	19.54	20.904	\$11375.40	20.59	21.44	21.66	22.00	\$11242.10
30	19.64	0.67	-0.953	19.69	20.60	\$11605.70	5.21	6.75	19.69	20.60	\$10805.70

Distribution is near symmetrical when $k = 3.6$

With the exception of the first row, the mean for all other rows is robust - lying between 17.72 and 20.00. The variance and skewness progressively decrease as k increases. When the distribution is near symmetrical, the inspection times appear to be somewhat evenly spread over the optimal planning horizon.

Other time to failure distributions

Other distributions touted as good for modelling time to failure include the lognormal distribution (see http://www.weibull.nl/weibullstatistics.htm website), generalized gamma distribution (see Khodabin and Ahmadabadi (2010)), log-logistic distribution (see Kus and Kaya (2006) and Rao et al.

(2009)), inverse Gaussian distribution (see Folks and Chhikara (1978)) and Log-EIG distribution (see Saw et al. (2002)).

Chapter 4

Finite planning horizon models with inspection times that are of non-negligible duration

4.1 Introduction

As has been noted in the last three chapters, inspection models are developed with the sole goal of deciding when it is most ideal to schedule inspections of a system that is known to deteriorate over time and ultimately fail at some point in time, T. The time to failure is a random variable and if the system were to be operated continuously until it fails at time T, then T is assumed to be having a probability density function (pdf) $f_T(t)$ and cumulative distribution function (cdf) $F_T(t)$ (which may be known or unknown).

Most of the research papers that have been cited in the first three chapters make the assumption that inspections are instantaneous. The assumption may indeed hold for many systems and may not hold for many others. An inspection may actually turn out to be a process which takes up a non-negligible fixed amount of time to complete as noted by Luss and Kander (1974), Zuckerman (1989), Thomas et al. (1991), Parmigiani (1993), Christer and Lee (1999), (and a few others) or may infact be of a non-negligible duration which is a stochastic variable as discussed by Fang and Liu (2006). Some inspections may indeed be carried out while the system is operational whilst for others, an inspection may require that the system be switched off during the period that an inspection is carried out. In this chapter, like in the previous chapter, we discuss the case of a system which generates revenue at the rate of c_R per unit of time during the time that it is non-faulty; from the time it fails until the next scheduled inspection when it is detected that the system is faulty, the system incurs a cost at the rate of c_F per every unit of time it is in the faulty state. The optimization criterion used is maximization of profit (similar to the papers by Antelman and Savage (1965) and Fang and Liu (2006). An attempt to give a solution to the following questions is made this chapter:

1. how many inspections and when should the inspections be scheduled

for a given planning horizon (denoted by L in this thesis) in such a way that profit is maximized?

2. what are the conditions necessary for evenly spread inspections to be near-optimal?

4.2 Assumptions and notation for finite planning horizon inspection models with inspection times that are of equal and fixed duration

The models apply to situations where one plans to operate a system (whose purchase price is C_o) over a finite period of time, call it a finite planning horizon of length L; each inspection takes a fixed amount of Δ_i units of time to complete. During this period, there are n planned inspections at times $x_1, x_2, \cdots, x_N$ such that $0 < x_1$, $x_1 + \Delta_i < x_2$, $x_2 + \Delta_i < x_3, \cdots, x_{n-1} + \Delta_i < x_n$, $x_n + \Delta_i < L$.

The notation and assumptions of the model are essentially the same as the assumptions made in Section 3.2.1 with only a few more assumptions added and some slight modification of one or two of the assumptions:

1. when working, the system generates or brings in revenue at the rate of

c_R per unit of time,

2. The time it takes to complete each and every inspection is Δ_i and each inspection costs an amount of c_I,

3. An inspection does not affect the level of degradation; i.e. it is neither failure hastening nor otherwise,

4. if the system fails during operation, this is only detected at the end of the next scheduled inspection at which point in time the project will be de-commissioned and the system is sold to a recycling plant at a salvage value of C_S,

5. if the system's lifetime exceeds L then it is operated until the end of the planning horizon L whereupon it is de-commissioned and sold to a recycling plant at a salvage value of C_S,

6. the owners of the system only pay for the actual inspections done as opposed to the inspections scheduled, and

7. the company incurs a cost of c_F for each unit of time the system is idle so that the cost associated with the system being idle for a time period of γ_t is $c_F\gamma_t$.

4.3 Theoretical results

If no inspection is scheduled over the planning horizon and the system fails at time T, the net profit will be

$$\mathcal{G}_o = \begin{cases} c_R T - (C_o - C_S), & T < L \\ c_R L - (C_o - C_S), & T \geq L \end{cases} \tag{4.1}$$

Thus, expected profit for the finite planning horizon inspection model with no inspections, $\mathcal{G}_{E.o}$ is given in Equation (4.2).

$$\mathcal{G}_{E.o} = (c_R + c_F)\left[\int_0^L t f_T(t)dt - LF_T(L)\right] + c_R L - (C_o - C_s) \tag{4.2}$$

4.3.1 Modeling inspections which take place when the system is running

If the plan is to have only one inspection taking place during the time interval $(x_1, x_1 + \Delta_i)$, then the net profit if failure occurs at time T is

$$\mathcal{G}_1 = \begin{cases} c_R T - c_I - c_F(x_1 + \Delta_i - T) - (C_o - C_s), & T < x_1 + \Delta_i \\ c_R T - c_I - c_F(L - T) - (C_o - C_s), & x_1 + \Delta_i \leq T < L \\ c_R L - c_I - (C_o - C_s), & T \geq L \end{cases} \tag{4.3}$$

and the expected value of the net profit $\mathcal{G}_{E.1}$:

$$\begin{aligned} \mathcal{G}_{E.1} \;=\; & (c_R + c_F)\left[\int_0^L t f_T(t)dx - LF_T(L)\right] + c_F(L - \Delta_i - x_1)F_T(x_1 + \Delta_i) + c_R L \\ & - c_I - (C_o - C_s) \end{aligned} \tag{4.4}$$

The optimal commencement time for a single inspection when the planning horizon is preset at L, thus, is such that

$$\begin{aligned} \frac{\partial \mathcal{G}}{\partial x_1} &= c_F(L-\Delta_i-x_1)\frac{\partial F(x_1+\Delta_i)}{\partial x_1} - c_F F_T(x_1+\Delta_i) = 0 \\ &\Leftrightarrow (L-\Delta_i-x_1)f(x_1+\Delta_i) - F_T(x_1+\Delta_i) = 0. \end{aligned} \tag{4.5}$$

Remark 4.1 *The optimal inspection time for the single inspection when the planning horizon is preset at L depends on the length of the planning horizon and the probability distribution of T only.*

Lemma 4.1 *When the planning horizon is a variable, the joint optimal inspection time and optimal planning horizon are a joint solution to the pair of equations (4.6):*

$$\left.\begin{array}{lll} L_1^* & = x_{11}^* + \Delta_i + \frac{F_T(x_{11}^*+\Delta_i)}{f_T(x_{11}^*+\Delta_i)} & [1] \\ F_T(x_{11}^*+\Delta_i) & = \frac{(c_R+c_F)F_T(L_1^*)-c_R}{c_F} & [2] \end{array}\right\} \tag{4.6}$$

If the system has been set to operate over a finite planning horizon L with n ($n \in \mathbb{N}$ such that $n \geq 2$) inspections scheduled in time intervals $(x_1, x_1+\Delta_i), (x_2, x_2+\Delta_i), \cdots, (x_n, x_n+\Delta_i)$, then the net profit if the system fails after a time T, $\mathcal{G}(\mathbf{x}, T, L)$:

$$\mathcal{G}(\mathbf{x}, X, L) = \begin{cases} c_R T - c_F(x_1 + \Delta_i - T) - c_I - (C_o - C_S), & 0 \leq T < x_1 + \Delta_i \\ c_R T - c_F(x_i + \Delta_i - T) - ic_I - (C_o - C_S), & x_{i-1} + \Delta_i \leq T < x_i + \Delta_i; \\ & i = 2, 3, \cdots, n \\ c_R T - c_F(L - T) - nc_I - (C_o - C_S), & x_n + \Delta_i \leq T < L \\ c_R L - nc_I - (C_o - C_S), & T \geq L \end{cases} \tag{4.7}$$

Thus, if n ($n \in \mathbb{N}$ and $n \geq 2$) inspections are scheduled such that each inspection lasts a time of Δ_i time units and can be conducted while the system is running, the expected profit $\mathcal{G}_{E.n} = E[\mathcal{G}(\mathbf{x}, T, L)]$:

$$\begin{aligned} \mathcal{G}_{E.n} &= (c_R + c_F)\left[\int_0^L t f_T(t)dt - LF_T(L)\right] + c_F(L - x_n - \Delta_i)F_T(x_n + \Delta_i) \\ &\quad + \sum_{i=1}^{n-1}\left[c_I + c_F(x_{i+1} - x_i)\right]F_T(x_i + \Delta_i) + c_R L - nc_I - (C_o - C_s) \end{aligned} \tag{4.8}$$

In the general case of $n \in \mathbb{N}$ ($n \geq 2$) inspections being scheduled, the optimal times at which the inspections should commence $x^*_{1n}, \cdots, x^*_{nn}$ and optimal planning horizon L^*_n are solutions of the system of equations (4.9) (the equations arise from the $n + 1$ partial derivatives of $\mathcal{G}_{E.n}$ in Equation (4.8) with respect to $x_1, \cdots, x_n$ and L being set to 0 each):

$$\left.\begin{array}{llr} x^*_{2n} & = x^*_{1n} + \frac{F_T(x^*_{1n}+\Delta_i)}{f_T(x^*_{1n}+\Delta_i)} - \frac{c_I}{c_F}, & [1] \\ x^*_{k+1,n} & = x^*_{kn} + \frac{F_T(x^*_{kn}+\Delta_i)-F_T(x^*_{k-1,n}+\Delta_i)}{f_T(x^*_{kn}+\Delta_i)} - \frac{c_I}{c_F},\ k=2,\cdots,n-1 & [2]-[n-1] \\ L^*_n & = x^*_{nn} + \Delta_i + \frac{F_T(x^*_{nn}+\Delta_i)-F_T(x^*_{n-1,n}+\Delta_i)}{f_T(x^*_{nn}+\Delta_i)} & [n] \\ F_T(x^*_{nn}+\Delta_i) & = \frac{(c_R+c_F)F_T(L^*_n)-c_R}{c_F} & [n+1] \end{array}\right\} \quad (4.9)$$

Remark 4.2 *If one defines $y_i,\ i = 1, \cdots, n$ as the time at which the i^{th} inspection ends, the results in 4.6 and 4.9 trivially become the same as the results in 3.14 and 3.15, respectively and therefore the process of obtaining an optimal inspection strategy is the same as that in the case of Simple Finite Planning Horizon Inspection and Replacement Model discussed in Chapter 3.*

4.3.2 System is shutdown when inspections take place

The assumptions in Section 4.2 apply; the only two additional assumptions for this model are:

1. whenever there is need for carrying out an inspection, the system must be completely shut down during the inspection period and

2. an inspection does not affect the level of degradation of the system.

Remark 4.3 *The implication of the shutdowns is that if the pdf and cdf of the time to failure is $f_T(t)$ and $F_T(t)$ (i.e. had the system operated uninterrupted until it fails at time T), respectively, then the cdf and pdf of the actual time to failure with inspections scheduled at times $\{x_i\}$ (denoted by $H_X(.)$ and $h_X(.)$, respectively):*

$$H_X(x) = \begin{cases} F_T(x), & 0 \leq x < x_1 \\ F_T(x_i), & x_i \leq x < x_i + \Delta_i; i = 1, \cdots, n \\ F_T\left(x - (i-1)\Delta_i\right), & x_{i-1} + \Delta_i \leq x < x_i; i = 2, \cdots, n \\ F_T(x - n\Delta_i), & x_n + \Delta_i \leq x \\ 0, & \textit{otherwise} \end{cases} \tag{4.10}$$

$$h_X(x) = \begin{cases} f_T(x), & 0 \leq x < x_1 \\ f_T\left(x - (i-1)\Delta_i\right), & x_{i-1} + \Delta_i \leq x < x_i; i = 2, \cdots, n \\ f_T(x - n\Delta_i), & x_n + \Delta_i \leq x \\ 0, & \textit{otherwise} \end{cases} \tag{4.11}$$

If the plan is to have only one inspection taking place during the time interval $(x_1, x_1 + \Delta_i)$, then the net profit if failure occurs at time X is

$$\mathcal{G}_1 = \begin{cases} c_R X - c_F(x_1 - X) - c_I - (C_o - C_s), & X < x_1 \\ c_R(X - \Delta_i) - c_F(L - X) - c_I - (C_o - C_s), & x_1 + \Delta_i \leq X < L \\ c_R(L - \Delta_i) - c_I - (C_o - C_s), & X \geq L \end{cases} \tag{4.12}$$

and the expected value of the net profit $\mathcal{G}_{E.1}$:

$$\begin{aligned} \mathcal{G}_{E.1} &= (c_R + c_F)\left[\int_0^{L-\Delta_i} x f_T(x)dx - (L - \Delta_i)F_T(L - \Delta_i)\right] \\ &\quad + c_F\left(L - x_1 - \Delta_i\right)F_T(x_1) + c_R(L - \Delta_i) - c_I - (C_o - C_s) \end{aligned} \tag{4.13}$$

The optimal commencement time for a single inspection when the planning horizon is preset at L, thus, is such that

$$\begin{aligned}\frac{\partial \mathcal{G}}{\partial x_1} &= -c_F F_T(x_1) + c_F(L - x_1 - \Delta_i) f_T(x_1) = 0 \\ &\Leftrightarrow (L - \Delta_i - x_1) f(x_1) - F_T(x_1) = 0. \end{aligned} \tag{4.14}$$

Lemma 4.2 *When the planning horizon is a variable, the joint optimal inspection time and optimal planning horizon are a joint solution to the pair of equations (4.15):*

$$\left.\begin{array}{lll} L_1^* & = x_{11}^* + \Delta_i + \frac{F_T(x_{11}^*)}{f_T(x_{11}^*)} & [1] \\ F_T(x_{11}^*) & = \frac{(c_R + c_F) F_T(L_1^* - \Delta_i) - c_R}{c_F} & [2] \end{array}\right\} \tag{4.15}$$

If the system has been set to operate over a finite planning horizon L with n ($n \in \mathbb{N}$ such that $n \geq 2$) inspections scheduled in time intervals $(x_1, x_1 + \Delta_i), (x_2, x_2 + \Delta_i), \cdots, (x_n, x_n + \Delta_i)$, then the net profit if the system fails at time X, $\mathcal{G}(\mathbf{x}, X, L)$:

$$\mathcal{G}(\mathbf{x}, X, L) = \begin{cases} c_R X - c_I - c_F(x_1 - X) - (C_o - C_S), & 0 \leq X < x_1 \\ c_R(X - (i-1)\Delta_i) - i c_I - c_F(x_i - X) - (C_o - C_S), & x_{i-1} + \Delta_i \leq X < x_i; \\ & i = 2, 3, \cdots, n \\ c_R(X - n\Delta_i) - n c_I - c_F(L - X) - (C_o - C_S), & x_n + \Delta_i \leq X < L \\ c_R(L - n\Delta_i) - n c_I - (C_o - C_S), & X \geq L \end{cases} \tag{4.16}$$

If n ($n \in \mathbb{N}$ and $n \geq 2$) inspections are scheduled such that each inspection lasts a time of Δ_i time units, the expected profit $\mathcal{G}_{E.n} = E[\mathcal{G}(\mathbf{x}, T, L)]$:

$$
\begin{aligned}
\mathcal{G}_{E.n} \quad = \quad & (c_R + c_F)\left[\int_0^{L-n\Delta_i} x f_T(x)dx - (L-n\Delta_i)F_T(L-n\Delta_i)\right] \\
& + c_F(L - x_n - \Delta_i)F_T(x_n - (n-1)\Delta_i) \\
& + \sum_{i=1}^{n-1}\left[c_I + c_F(x_{i+1} - x_i - \Delta_i)\right] F_T(x_i - (i-1)\Delta_i) \\
& + c_R(L - n\Delta_i) - nc_I - (C_o - C_s) \qquad (4.17)
\end{aligned}
$$

In the general case of $n \in \mathbb{N}$ $(n \geq 2)$ inspections being scheduled, the optimal times at which the inspections should commence $x^*_{1n}, \cdots, x^*_{nn}$ when the planning horizon is preset at L are solutions of the system of equations (4.18) (the equations arise from the n partial derivatives of $\mathcal{G}_{E.n}$ in Equation (4.17) with respect to $x_1, \cdots, x_n$ being set to 0 each):

$$
\left.
\begin{array}{lll}
x^*_{2n} & = x^*_{1n} + \Delta_i + \frac{F_T(x^*_{1n})}{f_T(x^*_{1n})} - \frac{c_I}{c_F}, & [1] \\
x^*_{k+1,n} & = x^*_{kn} + \Delta_i + \frac{F_T(x^*_{kn}-(k-1)\Delta_i) - F_T(x^*_{k-1,n}-(k-2)\Delta_i)}{f_T(x^*_{kn}-(k-1)\Delta_i)} - \frac{c_I}{c_F}, & k = 2, \cdots, n-1 \\
 & & [2] - [n-1] \\
L & = x^*_{nn} + \Delta_i + \frac{F_T(x^*_{nn}-(n-1)\Delta_i) - F_T(x^*_{n-1,n}-(n-2)\Delta_i)}{f_T(x^*_{nn}-(n-1)\Delta_i)} & [n]
\end{array}
\right\} \qquad (4.18)
$$

In the general case of $n \in \mathbb{N}$ $(n \geq 2)$ inspections being scheduled, the optimal times at which the inspections should commence $x^*_{1n}, \cdots, x^*_{nn}$ and optimal planning horizon L^*_n are solutions of the system of equations (4.19) (the equations arise from the $n+1$ partial derivatives of $\mathcal{G}_{E.n}$ in Equation (4.17) with respect to $x_1, \cdots, x_n$ and L being set to 0 each):

$$\left.\begin{array}{lll}
x_{2n}^* & = x_{1n}^* + \Delta_i + \frac{F_T(x_{1n}^*)}{f_T(x_{1n}^*)} - \frac{c_I}{c_F}, & [1] \\
x_{k+1,n}^* & = x_{kn}^* + \Delta_i + \frac{F_T(x_{kn}^* - (k-1)\Delta_i) - F_T(x_{k-1,n}^* - (k-2)\Delta_i)}{f_T(x_{kn}^* - (k-1)\Delta_i)} - \frac{c_I}{c_F}, & k = 2, \cdots, n-1 \\
 & & [2] - [n-1] \\
L_n^* & = x_{nn}^* + \Delta_i + \frac{F_T(x_{nn}^* - (n-1)\Delta_i) - F_T(x_{n-1,n}^* - (n-2)\Delta_i)}{f_T(x_{nn}^* - (n-1)\Delta_i)} & [n] \\
F_T\left(x_{nn}^* - (n-1)\Delta_i\right) & = \frac{(c_R + c_F) F_T(L_n^* - n\Delta_i) - c_R}{c_F} & [n+1]
\end{array}\right\} \tag{4.19}$$

For $n \geq 2$ inspections that are evenly spread over the planning horizon such that the first inspection commences at time $\tau - \frac{1}{2}$ and any two successive inspections have their midpoints being τ units of time apart and $L = (n+1)\tau$, the profit is a function of one variable, τ:

$$\begin{aligned}
\mathcal{G}_{E.n}(\tau) \quad = \quad & (c_R + c_F) \left[\int_0^{(n+1)\tau - n\Delta_i} x f_T(x) dx - \left\{ (n+1)\tau - n\Delta_i \right\} F_T((n+1)\tau - n\Delta_i) \right] \\
& + c_F(\tau - \frac{1}{2}\Delta_i) F_T \left(n(\tau - \Delta_i) + \frac{1}{2}\Delta_i \right) \\
& + \sum_{i=1}^{n-1} \left[c_I + c_F(\tau - \Delta_i) \right] F_T \left(i(\tau - \Delta_i) + \frac{1}{2}\Delta_i \right) \\
& + c_R \left[(n+1)\tau - n\Delta_i) \right] - n c_I - (C_o - C_s).
\end{aligned} \tag{4.20}$$

For a single inspection scheduled before the end of the planning horizon, the optimal inspection time $\tau = \tau^*$ is a solution of Equation (4.21)

$$\frac{d\mathcal{G}_{E.1}(\tau)}{d\tau} = -2(c_R + c_F) F_T(2\tau - \Delta_i) + c_F F_T(\tau) + c_F(\tau - \Delta_i) f_T(\tau) + 2c_R = 0 \tag{4.21}$$

while for $n \in \mathbb{N}$ such that $n > 1$, the optimal inter-inspection time $\tau = \tau^*$ is a solution of Equation (4.22)

$$\begin{aligned}
\frac{d\mathcal{G}_{E.n}(\tau)}{d\tau} &= -(n+1)(c_R + c_F)F_T((n+1)\tau - \Delta_i) + c_F F_T((n+1)\tau - \Delta_i) \\
&\quad + nc_F(\tau - \Delta_i) f_T(n\tau - (n-1)\Delta_i)) + c_F \tau \sum_{i=1}^{n-1} F_T(i(\tau - \Delta_i) + \Delta_i) \\
&\quad + [c_I + c_F(\tau - \Delta_i)] \sum_{i=1}^{n-1} i f_T(i(\tau - \Delta_i) + \Delta_i) + (n+1)c_R = 0
\end{aligned} \tag{4.22}$$

Remark 4.4 *The solutions to Equations (4.21) and (4.22) provide a good starting point in the search for a global optimal inspection schedule and optimal finite planning horizon. Approximate solutions to the equations are easily obtainable through the use a software such as Mathematica.*

4.3.3 Proposed methods for calculating optimal inspection times when shutdowns are necessary for inspections

Just like in Chapter 3, two methods explained below are explored.

Method 1 - Iterative procedure for calculating optimal inspection times

The iterative procedure suggested in this chapter (which makes use of Equation (4.9) and has already been summarized diagramatically by means of the flowchart in Figure 3.1) for determining estimates of the optimal inspection times and optimal planning horizon for a given number of inspections) is essentially the same the procedure discussed in Section 3.2.4:

Step 1 : Start with a guesstimate of x^*_{1n}, call it $\chi^{(1)}_{1n}$ and use it in Result [1] of Equation (4.9) to calculate an estimate $\chi^{(1)}_{2n}$ of x^*_{2n} which in turn is used to calculate estimates $\chi^{(1)}_{3n}, \cdots, \chi^{(1)}_{nn}$ and $\mathcal{L}_n$ of $x^*_{3n}, \cdots, x^*_{nn}$ and L^*_n, respectively, by making use of results [2] to [n] of Equation (4.19).

Step 2 : Using the value of $\mathcal{L}_n$ found in Iteration 1, determine a new estimate $\chi^{(2)}_{nn}$ of x^*_{nn} using Equation (4.23):

$$F_T(\chi^{(2)}_{nn}) = \frac{(c_R + c_F)F_T(\mathcal{L}_n) - c_R}{c_F} \tag{4.23}$$

and then calculate the difference $\Delta = \chi^{(2)}_{nn} - \chi^{(1)}_{nn}$. If $\Delta \approx 0$, then the guesstimate of x^*_{1n}, $\chi^{(1)}_{1n}$ as well as estimates of $x^*_{3n}, \cdots, x^*_{nn}$ and L^*_n are acceptably good. If $\Delta < 0$, go back to Step 1 and use a larger value of the guesstimate $\chi^{(1)}_{1n}$. On the other hand, if $\Delta > 0$, go back to Step 1 and use a smaller value of the guesstimate $\chi^{(1)}_{1n}$.

Remark 4.5 *If one starts with a guesstimate $\chi^{(1)}_{1n}$ which is too small in comparison to x^*_{1n}, i.e. $\chi^{(1)}_{1n} - x^*_{1n} << 0$, Equation (4.23) will be infeasible because* *the value of* $\frac{(c_R+c_F)F_T(\mathcal{L}_n)-c_R}{c_F}$ *will be negative. On the other hand, a guesstimate* $\chi_{1n}^{(1)}$ *which is too large in comparison to* x_{1n}^**, i.e.* $\chi_{1n}^{(1)} - x_{1n}^* >> 0$ *results in some values of* $\chi_{jn}^{(1)}, j > 1$ *approaching infinity and computation of* $\chi_{nn}^{(2)}$ *being rendered impossible.*

Method 2 - Nonlinear optimization procedure for calculating optimal inspection times

The second method hinges on the formulation of the problem as a nonlinear optimization problem with the usual non-negativity constraints as in Equation (4.24):

$$\begin{aligned}
\text{Maximize } \mathcal{G}_{E.n} \quad &= \quad (c_R + c_F)\left[\int_0^{x_{n+1}-n\Delta_i} t f_T(t)dt - (x_{n+1} - n\Delta_i)F_T(x_{n+1} - n\Delta_i)\right] \\
&\quad + c_F(x_{n+1} - x_n - \Delta_i)F_T\left(x_n - (n-1)\Delta_i\right) \\
&\quad + \sum_{i=1}^{n-1}\left[c_I + c_F(x_{i+1} - x_i - \Delta_i)\right]F_T\left(x_i - (i-1)\Delta_i\right) \\
&\quad + c_R(x_{n+1} - n\Delta_i) - nc_I - (C_o - C_S) \\
\text{subject to:} \quad &\quad -x_i + x_{i+1} \geq \Delta_i;\ i = 1, \cdots, n \qquad (4.24)
\end{aligned}$$

4.4 Applications

The applications of the results derived in Section 4.3 for the case of time to failure T following a continuous uniform distribution (i.e. $T \sim U[0, L_o]$) or an exponential distribution are discussed in this section.

4.4.1 Time to failure following a uniform distribution

Lemma 4.3 *If the time to failure $T \sim U[0, L_o]$ and the planning horizon L is fixed such that $L < L_o$, then the optimal time at which the inspection should commence if a single inspection is planned is*

$$x_1^* = \frac{1}{2}(L - \Delta_i). \tag{4.25}$$

Proof: From Equation (4.14), since $f_T(x_1 + \Delta_i) = \frac{1}{L_o}$ and
$F_T(x_1 + \Delta_i) = \frac{x_1+\Delta_i}{L_o}$,

$$\begin{aligned} \frac{(L - \Delta_i - x_1)}{L_o} - \frac{x_1}{L_o} &= 0 \\ \Rightarrow x_1 &= \frac{1}{2}(L - \Delta_i). \end{aligned}$$

Remark 4.6 *If a single inspection is to be done, the midpoint of the scheduled inspection coincides with the halfway mark of the planning horizon.*

Lemma 4.4 *If $T \sim U[0, L_o]$ and $\Delta_i \approx \frac{c_I}{c_F}$,*

1. *then $x_{kn}^* \approx kx_{1n}^*$, $k = 2, \cdots, n$, i.e. the time between any two inspections is a constant*

2. *and in addition, if $\Delta_i \approx 0$ the optimal inspections are almost evenly spread over the planning horizon.*

Proof:

1. From 4.19, if If $T \sim U[0, L_o]$ and $\Delta_i \approx \frac{c_I}{c_F}$,

$$\begin{aligned} x^*_{2n} &\approx x^*_{1n} + \frac{x^*_{1n}/L_o}{1/L_o} = 2x^*_{1n} \\ x^*_{k+1,n} &\approx x^*_{kn} + \frac{(x^*_{kn} - x^*_{k-1,n})/L_o}{1/L_o} \Rightarrow x^*_{k+1,n} - x^*_{kn} = x^*_{kn} - x^*_{k-1,n}, k = 2, \cdots, n-1 \text{ and} \end{aligned}$$

2. $L^*_n \approx x^*_{nn} + \frac{(x^*_{nn} - x^*_{n-1,n})/L_o}{1/L_o} \Rightarrow L^*_n - x^*_{nn} \approx x^*_{nn} - x^*_{n-1,n}$ and

   $L^*_n \approx \frac{c_F x^*_{nn} + c_R L_o}{c_F + c_R}$.

4.4.2 Time to failure following an exponential distribution

The Maclaurin series expansion of $g(x) = e^{-\theta x}$,

$$g(x) = 1 - \theta x + \sum_{k=2}^{\infty} \frac{(-\theta x)^k}{k!}$$

is used to prove Lemma 4.5 below.

Lemma 4.5 *The condition necessary for the inter-inspection times to be approximately constant are $\Delta_i \approx 0$ and $\frac{c_I}{c_F} \approx 0$; if both the latter conditions hold, then as $\theta \to 0$, the optimal inter-inspection times approach a constant and*

$x^*_{in} \approx i x^*_{1n}$.

Proof:

From Equation 4.19, if $\Delta_i \approx \frac{c_I}{c_F}$,

$$x^*_{2n} \approx x^*_{1n} + \frac{F_T(x^*_{1n})}{f_T(x^*_{1n})} = x^*_{1n} + \frac{1 - e^{-\theta x^*_{1n}}}{\theta e^{-\theta x^*_{1n}}} = x^*_{1n} + \frac{1 - \left(1 - \theta x^*_{1n} + \sum_{k=2}^{\infty} \frac{(-\theta x^*_{1n})^k}{k!}\right)}{\theta} \to 2x^*_{1n}$$

as $\theta \to 0$.

$$\begin{aligned} x^*_{i+1,n} &= x^*_{in} + \frac{e^{-\theta x^*_{i-1,n}} - e^{-\theta x^*_{in}}}{\theta e^{-\theta x^*_{in}}} \\ &= x^*_{in} + \frac{e^{\theta(x^*_{kn} - x_{i-1,n})} - 1}{\theta} \\ &= x^*_{in} + \frac{\left(1 + \theta(x^*_{in} - x^*_{i-1,n}) + \sum_{k=2}^{\infty} \frac{\left(\theta(x^*_{in} - x^*_{i-1,n})\right)^k}{k!}\right) - 1}{\theta}, \ i = 2, \cdots, n-1 \end{aligned}$$

Clearly, $x^*_{i+1,n} \to 2x^*_{in} - x^*_{i-1,n}$ as $\theta \to 0$ or $x^*_{i+1,n} - x^*_{in} \to x^*_{in} - x^*_{i-1,n}$ as $\theta \to 0$.

$$\text{Similarly, } L^*_n = x^*_{nn} + \frac{\left(1 + \theta(x^*_{nn} - x^*_{n-1,n}) + \sum_{k=2}^{\infty} \frac{\left(\theta(x^*_{nn} - x^*_{n-1,n})\right)^k}{k!}\right) - 1}{\theta}$$

and $L^*_n - x^*_{nn} \to x^*_{nn} - x^*_{n-1,n}$ as $\theta \to 0$.

4.5 Conclusions and Recommendations

In this chapter, a solution to the problem of scheduling inspections which are of fixed non-negligible duration has been proffered. The conditions necessary for inspections that are evenly spread over the entire planning horizon to be near-optimal have been found for a system whose time to failure either follows a uniform distribution or an exponential distribution. Admittedly, this only a preliminary treatise of the problem.

Any discussion of the problem which falls short of looking at the case of time to failure following other distributions like the Weibull distribution and others mentioned in Section 3.3.2 can be considered incomplete. The discussion (in this chapter) also falls shy of looking at infinite planning horizon models with inspection times that are of non-negligible duration.

Chapter 5

Finite and Infinite Planning Horizon Models with Imperfect Inspections

5.1 Introduction

According to Devooght et al. (1990), an inspection may be imperfect as a result of a combination of any of the following: human error, instrumentation failure and incomplete information. Where inspection errors occur, the errors may arise as follows: a) an inspection may erroneously declare a normally operating system faulty (error of the first kind or Type I error) or b) an inspection may fail to detect that the system is in a failed state (error of the second kind or Type II error). Some of the few papers published on the subject of inspection and replacement models with imperfect inspections include the following.

1. Morey (1968) has researched on finite planning horizon models using minimization of cost of operating the system as his criterion. His work focuses on models with a finite planning horizon when inspections are imperfect and the time to failure of the system has a known probability distribution. The models have a provision for one type of error - an inspection wrongfully saying a failed system is not faulty (commonly referred to as the Type II error or *false negative* error). His work is limited in scope in that the major result given in his thesis only gives a condition when it is meritorious to conduct at least a single inspection as opposed to no inspection.

2. Kaio and Osaki (1986) deal with a system whose time to failure follows an exponential distribution. They start off on the premise that two errors may occur when an inspection takes place. The errors may arise as follows: a) An inspection may erroneously declare a normally operating system faulty (error of the first kind or Type I error) or b) an inspection may fail to detect that the system is in a failed state (error of the second kind or Type II error). In their paper, the optimal inspection policy is one which minimizes the total expected cost up to the detection of system failure. Their work deals with the case of an

infinite planning horizon and inspections are carried out periodically.

3. Yun and Bai (1988) develop a replacement policy with minimal repair cost limit. The problem studied by Yun and Bai (1988) is radically different from the problem studied in this thesis in the sense that the inspections are done only when the system goes into the failed state and the purpose of each inspection is to determine an estimate of the repair cost to have the system up and running again. If the estimated cost does not exceed a cost limit L, the system is made to undergo minimal repair, otherwise it is replaced. The assumption in their paper is that the inspections are imperfect and therefore repair cost of a failed system cannot be accurately determined.

4. Badia et al. (2001) deal with models where inspections (which are imperfect) are periodically conducted. In their work, whenever an inspection reveals that the system has gone into the failed state, corrective maintenance is done to restore the system into the good as new state. The inspections are assumed to be non-negligible in duration. They deal with the case of an infinite planning horizon and their objective function is the cost per unit of time; they make use of the Renewal

Reward Theorem in Stochastic Processes theory to appropriately determine the objective function. Their work focuses on systems whose time to failure follow an exponential distribution or Pareto distribution.

5. Wang (2009) has researched on the problem of a production process which has two types of inspections: minor inspections (which are carried out periodically) and major inspections (which are carried out periodically and less frequently in comparison to minor inspections). The major inspections are not perfect and have a finite probability for correctly identifying a defect. The paper deals with the case of an infinite planning horizon. The objective in Wang's paper is to determine the optimum time intervals between two successive minor inspections as well as major inspections.

6. Berrade et al. (2012) who have researched on the problem of imperfect inspections with particular application in a beverage manufacturing process. There are two phases of inspections with inspections in each phase being carried out periodically. They discuss two models: a) a model in which an alarm is further investigated to check if it is a false positive, and if it is detected that the alarm is a false positive, the

system is put back into operation and b) a model in which an alarm leads to the renewal of the system without any interrogation to establish whether the alarm was a false positive or not.

7. Berrade et al. (2013) who have researched on periodic and aperiodic inspections being conducted on a system over a finite planning horizon. The inspections in their paper are imperfect and the criterion they use is minimization of total cost over the planning horizon. In the paper, two types of inspections are considered: a) one where a positive inspection (i.e. an inspection saying that the system is faulty) results in the system being summarily retired and b) where a positive inspection is followed by a check to verify the authenticity of the inspection result (at an additional cost) and if the system is certified to be non-faulty it is put back in operation, otherwise it is immediately retired. For the finite planning horizon case, in their paper, they have an additional penalty being the cost incurred if the system is retired rather prematurely (i.e. before the planning horizon) and this cost is proportional to the downtime period up to the planning horizon. This additional cost, they argue, may be for instance the cost of leasing a system to fill in the void left by the system that would have been retired. An example

given in the paper is that of a company running a train service; if a train is retired, there would obviously be a need to lease a train for the rest of the time up to the end of the planning horizon. A similar situation would apply to a power generating utility in the event of a power generating unit being retired prematurely.

8. Flage (2014) discusses the case of periodic inspections which are imperfect and in addition, are also potentially failure-inducing where an inspection may either introduce new failure modes or affect the time to system failure. The system is assumed to start off in the *perfectly functioning* state and then progressively move into the *defective* state and then failed state with time. The duration of the perfectly functioning state (denoted by X) and the duration of the defective state (denoted by Y) are independent random variables. Y is defined as the delay time. The failed state is assumed announcing while the other two states are not, i.e. if the system gets into the failed state, this is immediately detected. The system is correctively replaced upon failure or preventatively replaced at the N^{th} inspection time or when an inspection reveals that it is defective. Thus the errors which may occur as a result of an inspection are: Type I error which occurs when the

system which is perfectly functioning is identified as defective and Type II error which occurs when a system which has gone in the defective state is identified as perfectly functioning. The planning horizon for the models discussed in the paper is infinite. The problem in the models discussed by Flage is that of finding the optimal periodic inspection interval and optimal preventive age replacement limit.

Remark 5.1 *The work done in this chapter is very close to work done in the paper by Berrade et al. (2013); in particular, to the first case where a positive inspection means an automatic and immediate retirement of the system.*

This chapter discusses inspection models with imperfect inspections for both the finite and infinite planning horizons.

5.2 A finite planning horizon inspection model with imperfect inspections

5.2.1 The model

This model applies to a situation where one plans to operate a system (whose purchase price is C_o) over a finite period of time, call it a finite planning horizon of length L. In the general case, there are n planned inspections at times $x_1, x_2, \cdots, x_n$ such that $0 < x_1 < x_2 < \cdots < x_n < L$. In this thesis,

$x_o = 0$.

5.2.2 Assumptions

It is assumed that at the start of operation, the system is in a working state (i.e. functioning perfectly) and producing products that are of acceptable quality and will continue to do so at a steady rate until it gets into a failed state. The system's time to failure T is a continuous random variable with probability density function and cumulative distribution function $f_T(t)$ and $F_T(t)$, respectively, and the reliability function is denoted by $R_T(t)$. As mentioned earlier on, we assume that T's distribution is completely known. The notation and assumptions of the model are:

1. When working, the system generates or brings in revenue at a constant rate of c_R per unit of time,

2. The time it takes to complete each and every inspection is negligible and each inspection costs an amount of c_I,

3. The results of the inspections are independent events,

4. This work follows in the footsteps of Morey (1968)[1], Badia et al. (2001),

[1] Morey (1968) only deals with one type of error - an inspection wrongfully saying a failed system is ok

Berrade et al. (2013) and Flage (2014) in that the inspections are not perfect. Inspection errors may arise as follows: 1) a system that is functioning perfectly may be reported as being in a failed state (i.e. *false positive*) or 2) a failed system being reported as functional (*false negative*). If an inspection says that a properly functioning system is faulty then it is said to have committed a Type I error; an inspection which reports that a malfunctioning system is ok, on the other hand, is said to have committed a Type II error. The probability that an inspection will confirm that a system is in good working condition is $1 - \alpha$ and hence, the probability of a "false positive" is α. The probability of an inspection detecting that a faulty system is faulty is $1 - \beta$ and hence, the probability of a "false negative" is denoted by β,

5. The system cannot fail while being checked,

6. For the finite planning horizon models, the system is decommissioned immediately after the first inspection that says it is faulty (regardless of whether it is indeed faulty or not) or at the latest at the end of the planning horizon L (on condition all the n inspections have each independently reported that the system is ok). When it is decommissioned

it is sold (as scrap) to a recycling plant at a salvage value of C_S. This is similar to an *age replacement policy* described by Jiang (2009) and Rangan and Grace (1989); the difference between the work done in the latter papers and what is done in this chapter is that in the latter two papers, there are no inspections carried out whilst in this chapter there is a provision for inspections being carried out,

7. In the case of the infinite planning horizon models, the system is replaced by a new one immediately after the first inspection that says it is faulty (regardless of whether it is indeed faulty or not) or at the latest at the end of the cycle of length L (on condition all the n inspections have each independently reported that the system is ok). Upon replacement, it is sold (as scrap) to a recycling plant at a salvage value of C_S. The work done by Flage is similar when the planning horizon is infinite; in Flage's paper, the system is correctively replaced upon failure or preventatively replaced at the N^{th} inspection time or when an inspection reveals that it is defective,

8. The company or owner of the system incurs a cost of c_F for each unit of time the system is idle so that the cost associated with the sys-

tem being idle for a time period of γ_t is $c_F\gamma_t$. This is unlike in the paper by Berrade et al. (2013) where the cost per unit of time due to system unavailability pre-retirement and the cost of unavailability post-retirement are different.

5.2.3 Notation

I_i - event of a false positive error at the i^{th} $(i = 1, \cdots, n)$ inspection; thus, I'_i is the event of the i^{th} inspection correctly detecting that the system is not in the failed state

II_i - event of a false negative error at the i^{th} inspection; thus, II'_i is the event of the i^{th} inspection correctly detecting that the system is in a failed state

$\mathcal{A}_j$ - event of system failure occurring in interval $[x_{j-1}, x_j), j = 1, \cdots, n+1$

ψ_j - probability of event $\mathcal{A}_j$ occurring $(\psi_j = P(\mathcal{A}_j) = F_T(x_j) - F_T(x_{j-1}))$

$\mathcal{A}_{n+2}$ - event of system lifetime being greater than $L = x_{n+1}$ so that $\psi_{n+2} = P(\mathcal{A}_{n+2}) = R_T(L)$; $R_T(.)$ is the system reliability function

$\underline{\psi}$ - an $(n+2)$-dimensional vector such that $\underline{\psi} = (\psi_1, \cdots, \psi_{n+2})^T$

$\mathcal{B}_i$ - event of the system being decommissioned[2]/replaced[3] at the i^{th} inspection time $x_i, i = 1, \cdots, n+1$

p_i - the probability of event $\mathcal{B}_i$ occurring

α - the probability of a false positive inspection occurring - α is the conditional probability that an inspection says that the system is not ok when the system is ok; i.e $\alpha = P(B_k|A_i), k < i$

β - the probability of a false negative inspection occurring $-\beta$ is the conditional probability that an inspection says that the system is ok when the system is not ok; i.e $\beta = P(B_k|A_i), k > i$

c_F - per unit of time cost of system idleness

c_I - cost of carrying out an inspection

C_o - cost of buying and installing the system

c_R - rate at which revenue accrues per unit of time

C_S - salvage value of the system upon disposal

T - time to failure of the system

[2]for finite planning horizon models
[3]for infinite planning horizon models

$f_T(.)$ - probability density of system time to failure T

$F_T(.)$ - cdf of system time to failure T

$R_T(.)$ - system reliability function (or survival function) and

$R_T(t) = P(T > t)$

$H_T(.)$ - hazard function (or failure rate function) for the system.

L - length of a finite planning horizon[4] or length of cycle if no inspection declares the system faulty in the case of infinite planning horizon models

5.2.4 Theoretical results

If no inspection is planned and the system is to be operated up to a planning horizon of L, the expected value of the profit $G_{E.0}$

$$G_{E.0} = (c_R + c_F)\left[\int_0^L t f_T(t)dt - LF_T(L)\right] + c_R L - (C_o - C_S) \qquad (5.1)$$

From the assumptions of the model, the system can be decommissioned at any one of the times $x_1, x_2, \cdots, x_n, x_{n+1}$ (in the case of a finite planning horizon) or replaced (in the case of an infinite planning horizon). Denoting the event that the system is decommissioned at time $x_i, i = 1, \cdots, n+1$ by B_i

[4]no inspection is scheduled at time $L = x_{n+1}$ as such

and the event that $T \in [x_{j-1}, x_j), j = 1, \cdots, n+1$ by A_j and $P(A_j) = \psi_j$ and the event that $T > x_{n+1}$ by A_{n+2} so that $P(A_{n+2}) = \psi_{n+2}$, the Law of Total Probability can applied to get an expression for $P(B_i)$.

If a single inspection is planned at time x_1 which is before the end of the planning horizon, decommissioning can take place at either x_1 (Event B_1) or $L = x_2$ (Event B_2) with probabilities $\alpha+(1-\alpha-\beta)\psi_1$ and $1-\alpha-(1-\alpha-\beta)\psi_1$, respectively. The net profit will depend on the time the system fails as well as the result of an inspection at x_1. The expressions for conditional expected profit are given in Table 5.1.

Table 5.1: Conditional expected profit when $n = 1$ inspection is scheduled

Time to failure	Conditional expected profit $E[G_1 \mid T]$	Conditional probability
$0 \leq T < x_1$	$c_R T - c_I - c_F(x_1 - T) - (C_o - C_S)$	$1-\beta$
	$c_R T - c_I - c_F(L - T) - (C_o - C_S)$	β
$x_1 \leq T < L$	$c_R T - c_I - c_F(L - T) - (C_o - C_S)$	$1-\alpha$
	$c_R x_1 - c_I - (C_o - C_S)$	α
$L \leq T$	$c_R x_1 - c_I - (C_o - C_S)$	α
	$c_R L - c_I - (C_o - C_S)$	$1-\alpha$

The expected profit for the finite planning horizon inspection model with a single imperfect inspection scheduled at time x_1, $\mathcal{G}_{E.1}$:

$$
\begin{aligned}
\mathcal{G}_{E.1} &= E[E[G_1|T]] \\
&= (c_R + c_F)\left(\int_0^{x_1} t f_T(t)dt + (1-\alpha)\int_{x_1}^{L} t f_T(t)dt\right) - (1-\beta)c_F x_1 \psi_1 - \beta c_F x_2 \psi_1 \\
&\quad + \alpha c_R x_1 \psi_2 - (1-\alpha)c_F x_2 \psi_2 + \alpha c_R x_1 \psi_3 + (1-\alpha)c_R x_2 \psi_3 - c_I - (C_o - C_S) \\
&= (c_R + c_F)\left(\int_0^{L} t f_T(t)dt - \alpha\int_{x_1}^{L} t f_T(t)dt\right) - \underline{\psi}_1^T \mathbf{\Phi}_1 \mathbf{x}_1 - c_I - (C_o - C_S)
\end{aligned}
\tag{5.2}
$$

where $\underline{\psi}_1 = (\psi_1, \psi_2, \psi_3)^T$, $\mathbf{x}_1 = (x_1, L)^T$ and ϕ_1 is a 3×2 matrix:

$$
\mathbf{\Phi}_1 = \begin{pmatrix} c_F(1-\beta) & \beta c_F \\ -\alpha c_R, & (1-\alpha)c_F \\ -\alpha c_R & -(1-\alpha)c_R \end{pmatrix} \tag{5.3}
$$

Remark 5.2 *Substitution of $\alpha = \beta = 0$ in Equation (5.2) gives an expression of $G_{E.1}$ that is the same as the one for an inspection model with a finite planning horizon when inspections are perfect (see Equation (3.6)!).*

Remark 5.3 *The partial derivatives of $\mathcal{G}_{E.1}$ with respect to α and β given in Equation (5.4) and (5.5), respectively, present a very interesting scenario.*

$$
\frac{\partial \mathcal{G}_{E.1}}{\partial \alpha} = -(c_R + c_F)\int_{x_1}^{L} t f_T(t)dt + c_R x_1 \psi_2 + c_F x_2 \psi_2 + c_R x_1 \psi_3 - c_R x_2 \psi_3 \tag{5.4}
$$

$$
\frac{\partial \mathcal{G}_{E.1}}{\partial \beta} = c_F(x_1\psi_1 - x_2\psi_2) = c_F x_1 \psi_1 \left(1 - \frac{x_2\psi_2}{x_1\psi_1}\right) \tag{5.5}
$$

When a single inspection is scheduled at a fixed time x_1 for a system operated over a fixed planning horizon $L = x_2$, one observes that

1. *for a fixed value of α, if $\frac{x_2\psi_2}{x_1\psi_1} > 1$ then $\frac{\partial \mathcal{G}_{E.1}}{\partial \beta} < 0$ and $\mathcal{G}_{E.1}$ is a monotonic decreasing function of β, i.e. an increase in β is accompanied by a lower value of $\mathcal{G}_{E.1}$*

2. *for a fixed value of β, an increase in α is not necessarily accompanied by a lower value of $\mathcal{G}_{E.1}$.*

If n inspections ($n \geq 2$) are scheduled within a planning horizon of length L at times $x_1, \cdots, x_n$, the expected profit conditional on the event A_1 having occurred:

$$\begin{aligned}
G_n|A_1 &= c_R T - c_F \left[\sum_{k=1}^{n}(1-\beta)\beta^{k-1}(x_k - T) + \beta^n(x_{n+1} - T)\right] \\
&\quad - c_I \left[\left(\sum_{k=1}^{n} k(1-\beta)\beta^{k-1} + n\beta^n\right)\right] - (C_o - C_S) \\
&= (c_R + c_F)T - c_F \left[\sum_{k=1}^{n}(1-\beta)\beta^{k-1}x_k + \beta^n L\right] \\
&\quad - c_I \left[\sum_{k=1}^{n} k(1-\beta)\beta^{k-1} + n\beta^n\right] - (C_o - C_S) \qquad (5.6)
\end{aligned}$$

while the expected profit conditional on the event $A_i, i = 2, \cdots, n$ having occurred:

$$\begin{aligned}
G_n|A_i &= \sum_{k=1}^{i-1} \alpha(1-\alpha)^{k-1}\left[c_R x_k - kc_I - (C_o - C_S)\right] \\
&\quad + \sum_{k=i}^{n} (1-\alpha)^{i-1}\beta^{k-i}(1-\beta)\left[c_R T - c_F(x_k - T) - kc_I - (C_o - C_S)\right] \\
&\quad + (1-\alpha)^{i-1}\beta^{n-i+1}\left[c_R T - c_F(L-T) - nc_I - (C_o - C_S)\right] \\
&= (c_R + c_F)(1-\alpha)^{i-1} T \\
&\quad + \left[\sum_{k=1}^{i-1} c_R\alpha(1-\alpha)^{k-1}x_k - \sum_{k=i}^{n} c_F(1-\alpha)^{i-1}\beta^{k-i}(1-\beta)x_k - c_F(1-\alpha)^{i-1}\beta^{n-i+1}L\right] \\
&\quad - c_I\left[\sum_{k=1}^{i-1} k(1-\alpha)^{k-1}\alpha + (1-\alpha)^{i-1}\left(\sum_{k=i}^{n} k(1-\beta)\beta^{k-i} + n\beta^{n-i+1}\right)\right] - (C_o - C_S)
\end{aligned} \tag{5.7}$$

and the conditional expected profit given events A_{n+1} and A_{n+2} are:

$$\begin{aligned}
G_n|A_{n+1} &= \sum_{k=1}^{n} \alpha(1-\alpha)^{k-1}\left[c_R x_k - kc_I - (C_o - C_S)\right] \\
&\quad + (1-\alpha)^{n}\left[c_R T - c_F(L-T) - nc_I - (C_o - C_S)\right] \\
&= (c_R + c_F)(1-\alpha)^{n} T + \left[\sum_{k=1}^{n} c_R\alpha(1-\alpha)^{k-1}x_k - c_F(1-\alpha)^{n}L\right] \\
&\quad - c_I\left[\sum_{k=1}^{n} k(1-\alpha)^{k-1}\alpha + n(1-\alpha)^{n}\right] - (C_o - C_S)
\end{aligned} \tag{5.8}$$

$$
\begin{aligned}
\text{and } G_n|A_{n+2} &= \sum_{k=1}^{n} \alpha(1-\alpha)^{k-1}\left[c_R x_k - kc_I - (C_o - C_S)\right] + (1-\alpha)^n \left[c_R L - nc_I - (C_o - C_S)\right] \\
&= \left[\sum_{k=1}^{n} \alpha(1-\alpha)^{k-1} x_k + c_R(1-\alpha)^n L\right] \\
&\quad -c_I \left[\sum_{k=1}^{n} k(1-\alpha)^{k-1}\alpha + n(1-\alpha)^n\right] - (C_o - C_S),
\end{aligned} \tag{5.9}
$$

respectively.

Thus, if n inspections ($n \geq 2$) are scheduled within a planning horizon of length L at times $x_1, \cdots, x_n$, the expected value of the profit $\mathcal{G}_{E.n}$:

$$
\begin{aligned}
\mathcal{G}_{E.n} &= \sum_{i=1}^{n+1} \int_{x_{i-1}}^{x_i} (G_n|A_i)\, f_T(t)dt + \int_{x_{n+1}}^{\infty} (G_n|A_{n+2})\, f_T(t)dt \\
&= (c_R + c_F) \int_0^{x_1} t f_T(t)dt - c_F \left[\sum_{k=1}^{n} (1-\beta)\beta^{k-1} x_k + \beta^n L \right] \psi_1 \\
&\quad - c_I \left[\sum_{k=1}^{n} k(1-\beta)\beta^{k-1} + n\beta^n \right] \psi_1 - (C_o - C_S)\psi_1 \\
&\quad + \sum_{i=2}^{n} (c_R + c_F)(1-\alpha)^{i-1} \int_{x_{i-1}}^{x_i} t f_T(t)dt \\
&\quad + \sum_{i=2}^{n} \left[\sum_{k=1}^{i-1} c_R \alpha (1-\alpha)^{k-1} x_k - \sum_{k=i}^{n} c_F (1-\alpha)^{i-1} \beta^{k-i} (1-\beta) x_k - c_F (1-\alpha)^{i-1} \beta^{n-i+1} L \right] \psi_i \\
&\quad - \sum_{i=2}^{n} c_I \left[\sum_{k=1}^{i-1} k(1-\alpha)^{k-1}\alpha + (1-\alpha)^{i-1} \left(\sum_{k=i}^{n} k(1-\beta)\beta^{k-i} + n\beta^{n-i+1} \right) \right] \psi_i \\
&\quad - \sum_{i=2}^{n} (C_o - C_S)\psi_i \\
&\quad + (c_R + c_F) \left[(1-\alpha)^n \right] \int_{x_n}^{L} t f_T(t)dt \\
&\quad + \left[\sum_{k=1}^{n} c_R \alpha (1-\alpha)^{k-1} x_k - c_F (1-\alpha)^n L \right] \psi_{n+1} \\
&\quad - c_I \left[\sum_{k=1}^{n} k(1-\alpha)^{k-1}\alpha + n(1-\alpha)^n \right] \psi_{n+1} - (C_o - C_S)\psi_{n+1} \\
&\quad + \left[\sum_{k=1}^{n} c_R \alpha (1-\alpha)^{k-1} x_k + c_R (1-\alpha)^n L \right] \psi_{n+2} \\
&\quad - c_I \left[\sum_{k=1}^{n} k(1-\alpha)^{k-1}\alpha + n(1-\alpha)^n \right] \psi_{n+2} - (C_o - C_S)\psi_{n+2} \\
&= (c_R + c_F) \sum_{i=1}^{n+1} \left[(1-\alpha)^{i-1} \int_{x_{i-1}}^{x_i} t f_T(t)dt \right] - \underline{\psi}_n^T \underline{\lambda}_n - \underline{\psi}_n^T \mathbf{\Phi}_n \mathbf{x}_n - (C_o - C_S) \qquad (5.10)
\end{aligned}
$$

$$\text{where } \lambda_i \quad = \quad \begin{cases} c_I \left[\alpha \sum_{k=0}^{i-1} k(1-\alpha)^{k-1} + (1-\beta)(1-\alpha)^{i-1} \sum_{k=i}^{n} k\beta^{k-i}\right] & \\ +nc_I(1-\alpha)^{i-1}\beta^{n-i+1}, & i = 1, \cdots, n \\ & \\ c_I \left[\alpha \sum_{k=0}^{n} k(1-\alpha)^{k-1} + n(1-\alpha)^n\right], & i = n+1, n+2 \end{cases} \tag{5.11}$$

and $\mathbf{\Phi}_n = \{\phi_{ij}\}$ is an $(n+2) \times (n+1)$ matrix such that

$$\text{and } \phi_{ij} \quad = \quad (\mathbf{\Phi}_n) = \begin{cases} -\alpha(1-\alpha)^{j-1}c_R, & 1 \leq j < i \leq n \\ c_F(1-\alpha)^{i-1}(1-\beta)\beta^{j-i}, & 1 \leq i \leq j \leq n \\ c_F(1-\alpha)^{i-1}\beta^{n-i}, & j = n+1, 1 \leq i \leq n \\ -\alpha(1-\alpha)^{j-1}c_R, & i = n+1, n+2, 1 \leq j \leq n \\ c_F(1-\alpha)^n, & i = n+1, j = n+1 \\ -(1-\alpha)^n c_R, & i = n+2, j = n+1 \end{cases} \tag{5.12}$$

Remark 5.4 *The expression for $\mathcal{G}_{E.n}$ in Equation (5.10) when $\alpha = \beta = 0$ is the same as that for $\mathcal{G}_{E.n}$ in Equation(3.7) when inspections are perfect.*

5.3 Optimal inspection times and optimal planning horizon

Definition 5.1 *For a given fixed number of inspections n, an inspection schedule which results in the maximization of the expected profit $\mathcal{G}_{E.n}$ (for the finite planning horizon case) or the maximization of the ratio of the expected profit per cycle and expected cycle length (as espoused by the Renewal Reward Theorem) is called the* **optimal inspection schedule** *under n.*

5.3.1 Optimal inspection schedule for finite planning horizon

The optimal planning horizon when no inspections are scheduled L_0^* is given in Equation (5.13):

$$L_0^* = F_T^{-1}\left(\frac{c_R}{c_R + c_F}\right). \tag{5.13}$$

From Equation (5.2), we see that when a single inspection is to be scheduled, the optimal inspection time x_{11}^* and optimal planning horizon L_1^* are solutions of the system of equations (5.14) (on condition $\alpha + \beta < 1$):

$$\left.\begin{array}{ll} L_1^* = x_{11}^* + \left(\frac{(1-\beta)c_F+\alpha c_R}{c_F(1-\alpha-\beta)}\right)\frac{F_T(x_{11}^*)}{f_T(x_{11}^*)} - \frac{\alpha c_R}{c_F(1-\alpha-\beta)f_T(x_{11}^*)} & [1] \\ F_T(x_{11}^*) = \frac{(c_R+c_F)F_T(L_1^*)-(1-\alpha)c_R}{c_F(1-\alpha-\beta)} & [2] \end{array}\right\} \tag{5.14}$$

In general, when $n \geq 2$ inspections are scheduled at times $x_1, x_2, \cdots, x_n$ over a finite planning horizon of length L, the optimal inspection schedule $x_{1n}^*, \cdots, x_{nn}^*$ and optimal finite planning horizon L_n^* for given error sizes α and β are solutions of the non-linear optimization problem:

$$\begin{aligned} \text{Maximize } \mathcal{G}_{E.n} &= (c_R + c_F)\sum_{i=1}^{n+1}\left[(1-\alpha)^{i-1}\int_{x_{i-1}}^{x_i} t f_T(t)dt\right] - \underline{\psi}_n^T\underline{\lambda}_n - \underline{\psi}_n^T\mathbf{\Phi}_n\mathbf{x}_n - (C_o - C_S) \\ \text{subject to } x_{i-1} &\leq x_i; i = 1, \cdots, n+1. \end{aligned} \tag{5.15}$$

If n inspections ($n \geq 2$) are scheduled within a finite planning horizon of length L at times $x_1, \cdots, x_n$, the probability that the system is decommissioned at time x_i $(i = 2, \cdots, n+1)$, p_i:

$$p_i = \begin{cases} (1-\beta)\sum_{k=1}^{i}(1-\alpha)^{k-1}\beta^{i-k}\psi_k & \\ +\alpha(1-\alpha)^{i-1}\sum_{k=i+1}^{n+2}\psi_k, & i = 1, \cdots, n \\ & \\ (1-\beta)\sum_{k=1}^{n+1}(1-\alpha)^{k-1}\beta^{n+1-k}\psi_k & \\ +(1-\alpha)^n\psi_{n+2}, & i = n+1 \end{cases} \tag{5.16}$$

5.3.2 Optimal inspection schedule for infinite planning horizon case

If no inspection is planned and a system is replaced by a new one after a fixed length of time L so that the planning horizon is infinite, the optimal value of L is found as follows. Let

$$\mathcal{G}'_{E.0} = \frac{\mathcal{G}_{E.0}}{L}, \tag{5.17}$$

the optimal value of L, L_0^* is a solution of the equation

$$\frac{\partial \mathcal{G}'_{E.0}}{\partial L} = \frac{-(c_R + c_F)\int_0^L t f_T(t)dt - c_R L + (C_o - C_S)}{L^2} = 0 \tag{5.18}$$

so that L_0^* is such that

$$(c_R + c_F)\int_0^{L_0^*} t f_T(t)dt + c_R L_0^* - (C_o - C_S) = 0 \tag{5.19}$$

For the case where $n \geq 1$ inspections being scheduled, if a system is replaced by a new one each time a failure occurs (and failure is detected through inspection), then the planning horizon is infinite. If the length of a cycle when n inspections have been scheduled is denoted by W_n, the expected length of a cycle when a single inspection is scheduled (i.e.e when $n = 1$) is

$$E[W_1] = (\psi_1, \psi_2, \psi_3) \begin{pmatrix} 1-\beta & \beta \\ \alpha & 1-\alpha \\ \alpha & 1-\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ L \end{pmatrix}. \tag{5.20}$$

while when $n \geq 2$,

$$E[W_n] = \sum_{i=1}^{n+1} x_i p_i = \underline{\psi}_n^T \mathbf{A}_n \mathbf{x} \tag{5.21}$$

where $\mathbf{A}_n$ is an $n+2 \times n+1$ matrix such that

$$\text{and } a_{ij} \quad = \quad (\mathbf{A}_n) = \begin{cases} \alpha(1-\alpha)^{j-1}, & 1 \leq j < i \leq n \\ (1-\alpha)^{i-1}(1-\beta)\beta^{j-i}, & 1 \leq i \leq j \leq n \\ (1-\alpha)^{i-1}\beta^{n-i}, & j = n+1, 1 \leq i \leq n \\ \alpha(1-\alpha)^{j-1}, & i = n+1, n+2, 1 \leq j \leq n \\ (1-\alpha)^n, & i = n+1, j = n+1 \\ (1-\alpha)^n, & i = n+2, j = n+1 \end{cases} \tag{5.22}$$

Remark 5.5 *If all inspections are perfect, then*

$$E[W_1] = \psi_1 x_1 + (1-\psi_1)L \tag{5.23}$$

and

$$E[W_n] = \sum_{j=1}^{n} \psi_j x_j + \left(1 - \sum_{j=1}^{n} \psi_j\right) L \tag{5.24}$$

Remark 5.6 *Using the Renewal Reward Theorem, if n inspection are scheduled, the optimal inspection times and maximum cycle length $L = x_{n+1}$ under* *n are solutions of the non-linear programming problem*

$$\text{Maximize } \mathcal{G}'_{E.n} = \frac{\mathcal{G}_{E.n}}{E[W_n]} = \frac{\mathcal{G}_{E.n}}{\underline{\psi}_n^T \mathbf{A}_n \mathbf{x}} \tag{5.25}$$

$$\text{subject to } x_i < x_{i+1}; i = 1, \cdots, n \tag{5.26}$$

5.4 Impact of sizes of errors

In this section, a preliminary examination of the impact of error sizes α and β on the expected profit and distribution of inspections and optimal planning horizon is carried out. Only the cases of time to failure following a continuous uniform distribution or an exponential probability distribution are explored.

5.4.1 Impact of sizes of errors on the optimal expected profit when a fixed number of inspections is planned

For the case $T \sim U[0, L_o]$ (i.e. T being uniformly distributed over the interval $[0, L_o]$), it has already been argued that it is not prudent to have a planning horizon $L > L_o$. The impact of values of probabilities of error in inspections, α and β are explored with the aid of Example (5.1) and Example (5.2) below. The impact of changing values of α and β on 1) the maximum expected profit (for the finite planning horizon case) and maximum expected profit per unit of time (for the infinite planning horizon case), 2) the optimal inspection times and the optimal planning horizon (for the finite planning horizon case) when the number of inspections is $(n = 4)$[5] is seen in Table 5.2[6] and Table 5.3 as well as Figure 5.1 and Figure 5.2.

Example 5.1 *(*T **following a uniform distribution***) Suppose a system is such that its time to failure follows a continuous uniform distribution over the interval* $[0, 100]$*. Other attributes of the system are:* $C_o = \$10000$*,* $C_S = \$2500$*,* $c_R = \$1000$*,* $c_F = \$200$ *and inspections are imperfect with* $c_I = \$400$*.*

Example 5.2 *(*T **following an exponential distribution***) Suppose a system is such that its time to failure follows an exponential distribution with parameter* $\theta = \frac{1}{50}$*. Other attributes of the system are:* $C_o = \$10000$*,* $C_S = \$2500$*,* $c_R = \$1000$*,* $c_F = \$200$ *and inspections are imperfect with* $c_I = \$400$*.*

[5] This behavior is also the same for any other number of inspections

[6] expected profit of operating the system until the optimal planning horizon for the finite planning horizon case is at the top while the profit per unit of time for the infinite planning horizon case is at the bottom

Table 5.2: Optimal profit values and per unit of time profit for different values α and β for uniformly distributed system time to failure

α	β										
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.99
0.0	39467	39128	38748	38321	37837	37281	36637	35883	34986	33902	32567
	712.40	699.78	686.56	672.59	657.72	641.74	624.39	605.35	58416	560.22	535.62
0.1	36958	36584	36195	35784	35343	34865	34342	33778	33205	32791	32624
	689.16	675.41	661.01	645.86	629.78	612.59	594.12	574.29	553.72	538.47	536.46
0.2	35957	35586	35215	34840	34455	34058	33653	33264	32986	32824	32657
	667.81	653.05	637.82	622.00	605.52	588.38	570.85	554.13	543.56	541.61	539.61
0.3	35354	35000	34657	34322	33991	33668	33639	33154	33014	32852	32686
	648.70	633.20	617.54	601.68	585.73	570.19	556.11	547.99	546.31	544.36	542.36
0.4	34937	34603	34290	33996	33719	33472	33296	33177	33038	32876	32710
	631.90	615.90	600.23	585.01	570.70	558.47	551.79	550.37	548.69	546.74	544.74
0.5	34625	34312	34032	33780	33565	33417	33316	33198	33058	32897	32731
	617.17	600.93	585.75	572.03	560.84	555.03	553.82	552.40	550.72	548.77	546.77
0.6	34381	34091	33845	33647	33517	33433	33333	33215	33076	32914	32729
	604.21	588.02	574.01	563.10	557.74	556.74	555.54	554.11	552.43	550.48	548.48
0.7	34184	33918	33719	33600	33531	33447	33347	33229	33090	32929	32763
	592.71	576.98	565.36	560.00	559.17	558.16	556.96	555.53	553.86	551.91	549.90
0.8	34021	33788	33668	33611	33542	33459	33359	33240	33101	32940	32755
	582.40	568.01	561.84	561.17	560.34	559.34	558.13	556.70	555.03	553.08	551.07
0.9	33884	33722	33677	33620	33551	33468	33368	33250	33111	32950	32765
	573.07	563.35	562.80	562.12	561.30	560.29	559.09	557.66	555.98	554.04	552.03
0.99	33767	33730	33684	33628	33559	33476	33376	33257	33119	32958	32773
	565.40	564.05	563.51	562.83	562.00	561.00	559.79	558.37	556.69	554.74	552.74

Table 5.3: Optimal profit values and per unit of time profit for different values of α and β for exponentially distributed system time to failure

	β										
α	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.99
0.0	37346	36479	35523	34466	33295	31997	30555	28953	27171	25188	23214
	650.14	631.35	611.97	591.79	570.57	548.08	524.00	497.98	469.56	438.17	406.76
0.1	33698	32768	31775	30703	29541	28281	26925	25499	24107	23207	23040
	621.59	600.93	579.69	557.65	534.60	510.30	484.58	457.44	429.77	409.63	406.72
0.2	31494	30472	29436	28367	27258	26120	24992	23790	23402	23239	23072
	593.68	571.32	548.61	525.37	501.50	477.05	452.49	432.46	415.43	412.58	409.66
0.3	29910	28818	27770	26743	25741	24798	24005	23569	23430	23267	23101
	567.27	543.51	519.90	496.39	473.15	450.87	431.42	420.48	418.02	415.17	412.24
0.4	28670	27534	26508	25569	24733	24064	23712	23593	23454	23291	23125
	542.94	518.14	494.35	471.75	451.00	433.86	424.83	422.73	420.27	417.41	414.47
0.5	27648	26494	25533	24735	24134	23832	23732	23613	23474	23312	23145
	520.90	495.50	472.44	452.25	436.41	428.54	426.76	424.66	422.20	419.33	416.38
0.6	26777	25636	24793	24206	23933	23849	23749	23630	23491	23329	23162
	501.09	475.62	454.56	438.95	431.65	430.17	428.40	426.29	423.82	420.95	417.97
0.7	26017	24934	24286	24016	23947	23863	23763	23644	23505	23343	23176
	483.31	458.59	441.59	434.24	433.02	431.53	429.75	427.65	425.17	422.30	419.34
0.8	25342	24397	24083	24027	23958	23874	23774	23665	23516	23354	23187
	467.29	444.95	436.37	435.37	434.14	432.66	430.87	428.76	426.28	423.41	420.44
0.9	24735	24138	24092	24036	23967	23884	23783	23665	23525	23363	23197
	452.77	438.10	437.29	436.29	435.06	433.57	431.79	429.67	427.19	424.31	421.35
0.99	24235	24145	24099	24043	23974	23890	23790	23671	23532	23370	23204
	440.78	438.78	437.97	436.97	435.77	434.25	432.46	430.35	427.87	424.98	422.01

The following observations are made:

- For a fixed value of α, the maximum achievable expected value of profit $\mathcal{G}_{E.4}$ decreases monotonically with increasing β while the same cannot be said about varying α when β is fixed. When β is fixed, as α is increased from zero to $1 - \beta$, $\mathcal{G}_{E.4}$ monotonically decreases to some minima and then starts to slowly increase.

- a fixed increase in α is more lethal to profit compared to the same increase in β

5.4.2 Optimal inspection times for different sizes of errors when the number of inspections is fixed

The data in Table 5.4 below relate to Example 5.1 while the data in Table 5.5 relate to Example 5.2. In both examples, a fixed number of inspections ($n = 4$) need to be scheduled. In Example 5.1 the system time to failure follows a uniform distribution (i.e. $T \sim U(0, 100)$) while in the case of Example 5.2, the time to failure follows an exponential distribution.

It is observed that for both the infinite planning and finite planning horizon cases,

- When α is small and fixed, increasing the value of β results in a delay

Table 5.4: Optimal inspection times for different values of α and β when system time to failure follows a uniform distribution ($n = 4$ inspections)

Planning horizon	α	β	Optimal Inspection times				Optimal planning horizon	Profit
Finite	0.0	0.0	22.92	43.85	62.77	79.69	96.62	39466.80
		0.4	27.53	42.63	56.86	70.24	93.49	37836.50
		0.8	36.67	42.89	48.89	54.67	88.00	34986.20
	0.4	0.0	71.22	92.01	98.02	99.43	99.91	34936.50
		0.4	77.80	83.60	87.86	90.94	94.98	33719.30
		0.8	83.07	83.07	83.07	83.07	83.07	33038.00
	0.8	0.0	80.71	96.34	99.37	99.88	99.99	34936.50
		0.4	83.21	83.21	83.21	83.21	83.21	33719.30
		0.8	82.77	82.77	82.77	82.77	82.77	33101.40
Infinite	0.0	0.0	13.23	26.02	38.37	50.29	62.20	712.40
		0.4	14.52	22.90	31.07	39.05	52.57	657.72
		0.8	17.48	20.69	23.84	26.94	43.57	584.16
	0.4	0.0	24.90	43.76	58.12	69.23	78.67	631.90
		0.4	29.96	36.97	43.29	48.99	57.86	570.70
		9.8	37.35	37.35	37.35	37.35	37.35	548.69
	0.8	0.0	33.10	55.27	70.12	80.03	86.98	582.40
		0.4	36.51	36.51	36.51	36.51	36.51	560.34
		0.8	36.51	36.51	36.51	36.51	36.51	555.03

in starting inspections

- For a fixed value of α, increasing the value of β results in a shorter optimal planning horizon with the net effect that inspections are scheduled over a smaller and smaller range

- When β is small and fixed, increasing the value of α results in a delay in starting inspections and a longer planning horizon

- When β is large and fixed, increasing α is accompanied by a shorter optimal planning horizon

- Larger values for both α and β result in optimal inspections which are chronologically very close and comparatively shorter planning horizons.

Table 5.5: Optimal inspection times for different values α and β for system time to failure that follows an exponential distribution ($n = 4$ inspections)

Planning horizon	α	β	Optimal Inspection times				Optimal planning horizon	Profit Profit per unit of time[7]
Finite	0.0	0.0	23.77	52.20	88.50	139.84	229.42	37345.80
		0.4	28.59	47.20	68.50	93.42	161.31	33295.10
		0.8	34.65	40.96	47.49	54.26	111.38	27170.80
	0.4	0.0	67.15	134.37	201.99	272.23	361.82	28669.80
		0.4	75.46	90.50	105.54	120.60	149.72	24732.50
		0.8	89.33	89.33	89.33	89.33	89.33	23453.50
	0.8	0.0	83.79	167.58	251.38	335.31	424.90	25341.80
		0.4	89.46	89.46	89.46	89.46	89.46	23957.90
		0.8	89.02	89.02	89.02	89.02	89.02	23515.90
Infinite	0.0	0.0	11.28	23.47	36.80	51.61	68.85	650.14
		0.4	12.12	19.57	27.37	35.59	51.69	570.57
		0.8	14.43	17.09	19.76	22.46	39.54	469.56
	0.4	0.0	20.13	40.38	60.96	82.35	106.32	542.94
		0.4	24.95	31.96	38.98	45.99	58.64	451.00
		9.8	32.74	32.74	32.74	32.74	32.74	420.27
	0.8	0.0	28.08	56.15	84.24	112.39	141.73	467.29
		0.4	31.76	31.76	31.76	31.76	31.76	434.14
		0.8	31.94	31.94	31.94	31.94	31.94	426.28

Again, it is observed that when system time to failure follows an exponential distribution, just like when it follows a uniform distribution, for both the infinite planning and finite planning horizon cases,

- When α is small and fixed, increasing the value of β results in a delay in starting inspections;

- For a fixed value of α, increasing the value of β results in a shorter optimal planning horizon with the net effect that inspections are scheduled over a smaller and smaller range;

- When β is small and fixed, increasing the value of α results in a delay in starting inspections and a longer planning horizon; and in addition

- When β is large and fixed, increasing α is accompanied by a shorter optimal planning horizon

- Larger values for both α and β result in optimal inspections which are chronologically very close and a comparatively shorter planning horizon

5.4.3 Impact of sizes of errors on the optimal number of inspections and global optimal inspection times

An investigation of the impact of sizes of errors on the optimal inspection times was conducted. Tables B.1 to C.4 in Appendices B and C were compiled from this exercise.

The following observations were made:

- the times at which the first inspection and a few subsequent inspections are to be done are not very sensitive to changes in the the number of planned inspections; in fact, beyond a certain threshold number of inspections, the time at which the i^{th} inspection has to be done hardly changes as the number of inspections is increased.

- When the time to failure of a system follows a uniform distribution, i.e. for $T \sim U(0, L_o)$, large values of n will result in more and more inspections being crammed towards the time L_o.

- for a given number of inspections per cycle (for the case of an infinite planning horizon) or planning horizon (for the case of a finite planning horizon), the cycle length is substantially shorter than the finite planning horizon.

- For the case where T follows a uniform distribution, the inter-inspection times get shorter and shorter over time

- The optimal number of inspections decreases with an increase in α and/or β.

- For the time to failure following an exponential distribution, the inter-inspection times for the infinite planning horizon case are almost constant while for the finite planning horizon case they very gradually get close to being a constant when a larger number of inspections is planned.

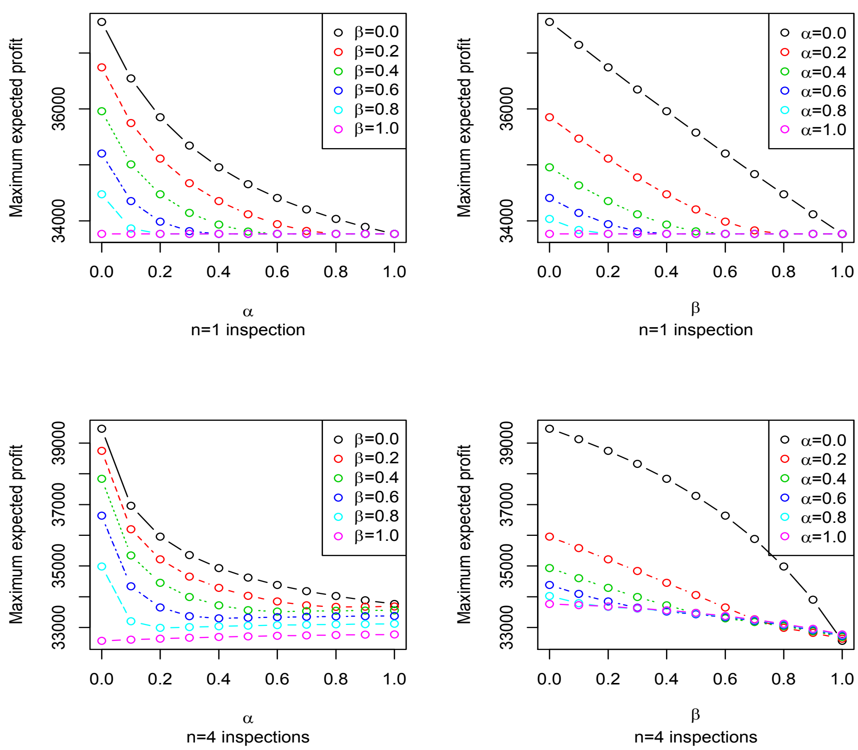

Figure 5.1: Impact of α and β on maximum expected profit for a fixed number of inspections

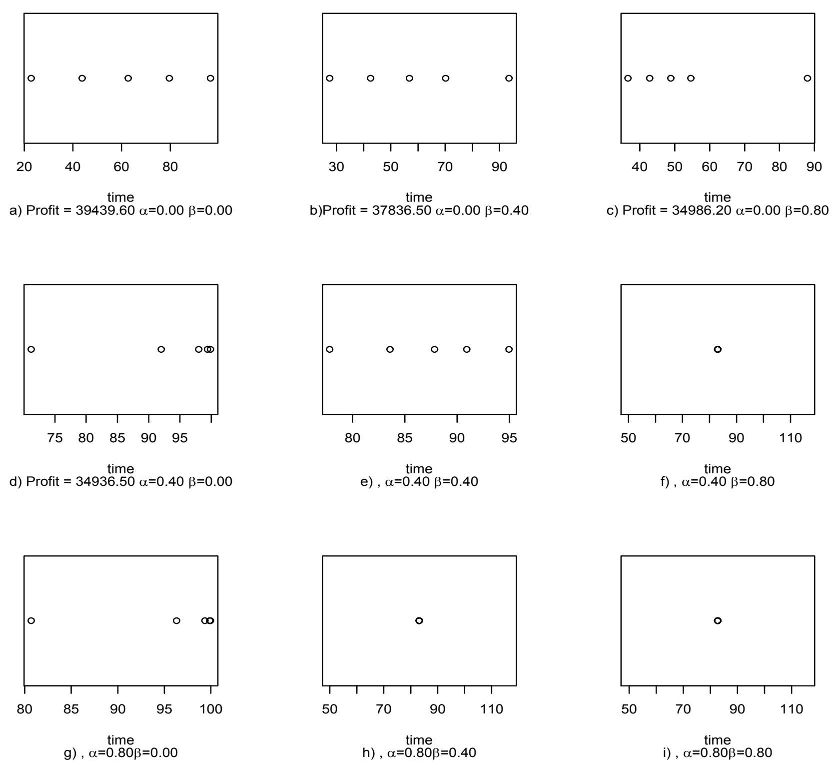

Figure 5.2: Impact of α and β on optimal inspection times ($n = 4$ inspections)

Chapter 6

Hierarchical Inspection Models

6.1 Introduction

The problem that most classical inspection and replacement models seek to address is that of finding ideal times when inspections of a stochastically deteriorating system should be scheduled [1]. The problem[2] discussed in this paper are a departure from the latter school of models.

The problem discussed in this thesis was first explored by Zuckerman (1989) who looked at the case of a system with N components/units (presumably connected in series) whose times to failure are independent exponentially distributed random variables. By virtue of them being connected in series,

[1]see also a detailed literature survey by Beichelt and Tittmann (2012)

[2]this problem has apparently been somewhat ignored as only a handful of papers cite the two pioneering papers. Only Levner (1994) and Qiu and Cox Jr (1994) have researched on problems bearing some similarities to the problem discussed in the pioneering paper by Zuckerman (1989) which was reviewed by Qiu (1991)

the system fails the moment any one of the N components fails and system failure is attributed to just that component which will have failed. The model discussed by Zuckerman assumes that the system's status is observed continuously at zero cost (by a controller) and a failure is due to exactly one component having failed. In the event of a breakdown, a series of inspections (in a hierarchical manner with one unit being inspected at a time) is performed in order to identify the failed unit. Once the failed unit has been identified, it is repaired and immediately thereafter the system starts working again. Both the processes 1) of inspecting the units and 2) of repairing the failed unit will result in costs being incurred as explained in Section 6.1.1.

The order in which the N units are to be inspected is called an inspection permutation or strategy; there are a total of N *factorial* ($N!$) distinct inspection permutations and the one that results in the maximum long run average net income per unit of time or total discounted net income per unit of time is called the optimal inspection permutation/strategy.

Remark 6.1 *In this thesis, from this point on, the term inspection permutation is consistently used to refer to an inspection permutation or strategy. Also, the term net-income-rate is used in place of long-run average net* *income-rate.*

A detailed literature search apparently suggests that Levner (1994) and Qiu and Cox Jr (1994) are the only two research works which have looked at problems which are similar to the one discussed in this thesis. Levner (1994) researched on a system with N independent stochastically failing modules. When system failure occurs, the decision-maker has to perform a series of sequential inspections. For the problem that Levner looks at, the decision-maker is given a chance to inspect a module infinitely many times because inspections are not perfect. The inspection process ends when the failed module is identified and repaired.

Qiu and Cox Jr (1994) researched on a coherent multi-component system with units that have constant failure rates and operate independently of each other. Their works are very similar to Zuckerman (1989) in that when the system is working, it produces revenue at a constant rate and the system ceases to work if and only if all components in one of its *cut sets* fail. The difference between the problem of Zuckerman (1989) and Qiu (1991) and the problem of Qiu and Cox Jr (1994) is that in the former two papers only one failed unit may trigger failure of the system while in the latter paper (by Qiu and Cox Jr (1994)) more than one failed unit may trigger failure of the system. Their paper (Qiu and Cox) presents a heuristic approach for determining an optimal inspection permutation of such general coherent systems.

Remark 6.2 *The inspection models discussed in the latter three papers are clearly a departure from the classical inspection models such as the ones in the works of Barlow et al. (1963), Munford and Shahani (1972), Luss and Kander (1974), Anbar (1976b), Butler (1979), Wattanapanom and Shaw (1979), Nakagawa and Yasui (1980), Zuckerman (1980), Beichelt (1981), Kawai (1984), Milioni and Pliska (1988), Christer (1988), Teramoto et al. (1990) Devooght et al. (1990), Chelbi and Ait-Kadi (1999), Ghasemi et al. (2008), Scarf et al. (2009), Wang (2009), Ahmadi and Newby (2011), Golmakani and Fattahipour (2011), Golmakani and Moakedi (2012), Wang (2013), Wang et al. (2014), Flage (2014) and many others in the sense that the objective here is not to recommend times as to when inspections should take place but rather to set out an order or hierarchy in which the components of a system may be inspected in the event of a system failure.*

6.1.1 Assumptions

The basic assumptions in Zuckerman's model are:

1. When in operation, the system generates income at a rate of I dollars[3] per unit of time.

2. The system has N components whose lifetimes are stochastically independent random variables which follow exponential distributions so that the lifetime of the j^{th} unit, $S_j \sim Expo(\theta_j), j = 1, \cdots, N$ and the cumulative distribution function of the lifetime of the j^{th} component

$$F_{S_j}(t) = \begin{cases} 1 - e^{-\theta_j t}, & t > 0 \\ 0, & \text{otherwise} \end{cases}$$

3. The cost of inspecting the j^{th} unit is $\$C_j$ per unit of time and the inspection time for the j^{th} unit is T_j while the repair time for the j^{th} unit is Z_j and the expected repair cost for the unit is denoted by $\$R_j$.

Remark 6.3 *The objective in Zuckerman's model is the formulation of an optimal inspection permutation (i.e. the order in which the N units are inspected - or rather an optimal inspection permutation for the units) in order to maximize either the net-income-rate or total discounted net income. An*

[3]or any other monetary unit as applicable

inspection permutation $\sigma = (\sigma(1), \cdots, \sigma(N))$ specifies the order in which the units are inspected so that $\sigma(j)$ is the j^{th} system to be inspected.

6.1.2 Main results in Zuckerman (1989)

The notation $E_\sigma[.]$ and $P_\sigma(.)$ refer to the expectation and probability, respectively, when an inspection permutation σ is used.

Letting $\theta = \sum_{j=1}^{N} \theta_j$, the main results from Zuckerman (1989) are:

1. If a system has gone into the failed state, the probability that the breakdown is due to the j^{th} unit, P_j:

$$P_j = P\left(S_j = \min_{1\le n\le N}\{S_n\}\right) = \frac{\theta_j}{\theta}. \tag{6.1}$$

2. The time the system operates, $S = \min_{1\le n\le N}\{S_n\}$ is an exponentially distributed random variable with parameter θ;

$$i.e.\ \ S \sim Expo(\theta) \tag{6.2}$$

and $E[S] = \frac{1}{\theta}$.

Letting C be the accumulated inspection cost over a cycle and T be the time to identify the failed unit (total inspection time per cycle), we have (for the undiscounted case)

$$E_\sigma[C] = \sum_{n=1}^{N} P_{\sigma(j)}\left[\sum_{n\le j} C_{\sigma(n)} T_{\sigma(n)}\right] \tag{6.3}$$

and

$$E_\sigma[T] = \sum_{j=1}^{N} P_{\sigma(j)} \left[\sum_{n \le j} T_{\sigma(n)} \right] \tag{6.4}$$

resulting in the net-income-rate for inspection permutation σ being

$$\psi(\sigma) = \frac{\frac{I}{\theta} - E_\sigma[C] - \sum_{j=1}^{N} P_j R_j}{\frac{1}{\theta} + E_\sigma[T] + \sum_{j=1}^{N} P_j Z_j}. \tag{6.5}$$

3. Zuckerman goes on to give a result (listed as Theorem 1 in his paper) which is deemed critical for determining the optimal inspection permutation **for the undiscounted case**; it says that in the undiscounted case, the units are inspected in an increasing order of the indices

$$e_j = \frac{T_j C_j + \psi^* T_j}{P_j}, \quad j = 1, 2, \cdots, N, \tag{6.6}$$

where $\psi^* = \max_\sigma \psi(\sigma)$ is the optimal net-income-rate.

Remark 6.4 *Zuckerman laments that since ψ^* is unknown, his procedure is not tractable as the indices $e_1, \cdots, e_N$ cannot be computed explicitly. He proposes a graphical computational procedure for the optimal inspection permutation which is quite involved!*

Remark 6.5 *Zuckerman's assertion that the optimal inspection permutation does not depend on the repair times and repair costs is not correct; this is shown with the aid of a counter-example (see Example (6.2)). This then casts* *doubt on the role of Result (6.6) in being the cornerstone of determining an optimal inspection permutation.*

Zuckerman's works are then reviewed by Qiu (1991); Qiu suggests that some of the results (results for the discounted case) arrived at by Zuckerman are indeed not correct.

6.1.3 Main results in Qiu (1991)

Qiu looks at the simplified case where the repair times and repair costs are assumed to be negligible. He denotes the inspection cost rate at time t by $C(t)$ and the obtaining continuous discount factor by α. Both Zuckerman and Qiu give the total discounted net income per cycle when an inspection permutation σ is adopted as

$$\eta(\sigma) = \frac{\frac{I}{\theta+\alpha} - \frac{\theta}{\theta+\alpha}E_\sigma\left[\int_0^T C(t)e^{-\alpha t}dt\right]}{1 - \frac{\theta}{\theta+\alpha}E_\sigma[e^{-\alpha T}]} \tag{6.7}$$

Letting $\eta^* = \max \eta(\sigma)$ and $Q_n = 1-P_n$, Zuckerman states that an optimal inspection permutation would inspect the units in an increasing order of the indices g_n:

$$g_n = \frac{(\eta^* + C_n/\alpha)(1 - \exp(-\alpha T_n))}{1 - Q_n exp(-\alpha T_n)},\ n = 1, \cdots, N. \tag{6.8}$$

Qiu disputes Result (6.8) and uses a counter-example to support his argument that the result is not correct. He completes his paper by giving necessary conditions for an inspection permutation to be optimal.

6.2 Limitations of Zuckerman and Qiu's works

Just like in Zuckerman's case, Qiu's paper falls short of giving a clear roadmap which outlines how to obtain an optimal inspection permutation with ease. Further, one needs to be wary of Zuckerman and Qiu's model assumptions that the repair times and and repair costs are not important in finding an optimal inspection permutation. It must be stressed that while the results obtained by Zuckerman (1989) and Qiu (1991) are appealing in that they deal with commonly encountered practical problems, implementation is unfortunately not easy. In particular, the fact that one has to resort to linear graphs in order to arrive at the optimal hierarchical inspection schedule makes their works less appealing.

In this thesis Mathematica programs which make use of Result (6.5) (for the undiscounted case) and Result (6.12) (for the discounted case) are used to showcase that it is easy to obtain an optimal inspection permutation for the Zuckerman-Qiu policies. The procedure employed in both cases involves simply computing net-income-rate values for all possible inspection permutations and obtaining the optimal inspection permutation by inspection.

6.3 New results for the discounted case

For an inspection permutation $\sigma = (\sigma_1, \cdots, \sigma_N)$, of an N-unit system with inspection cost rates C_{σ_i} inspection times T_{σ_i} as well as repair times Z_i and repair costs of $R_i, i = 1, \cdots, N$ per unit of time, respectively, the total discounted net income per cycle attributable to the j^{th} unit inspected under inspection permutation σ (i.e. when failure of the j^{th} unit is what triggered system failure), $G_j(S, \sigma)$:

$$\begin{aligned}
G_j(S,\sigma) &= \int_0^S I e^{-\alpha t} dt \\
&\quad - e^{-\alpha S}\left[\int_0^{T_{\sigma_1}} C_{\sigma_1} e^{-\alpha t} dt + e^{-\alpha T_{\sigma_1}} \int_0^{T_{\sigma_2}} C_{\sigma_2} e^{-\alpha t} dt + \cdots + e^{-\alpha \sum_{k=1}^{j-1} T_{\sigma_k}} \int_0^{T_{\sigma_j}} C_{\sigma_j} e^{-\alpha t} dt\right] \\
&\quad - e^{-\alpha\left(S + \sum_{n=1}^{j} T_{\sigma_n}\right)} \left[\int_0^{Z_{\sigma_j}} R_{\sigma_j} e^{-\alpha t} dt\right] \\
&= \frac{1}{\alpha}\left[I\left(1 - e^{-\alpha S}\right) - e^{-\alpha S} \sum_{n=1}^{j} C_{\sigma_n} e^{-\alpha \zeta_{n-1}} \left(1 - e^{-\alpha T_{\sigma_n}}\right) - R_{\sigma_j} e^{-\alpha(S+\zeta_j)} \left(1 - e^{-\alpha Z_{\sigma_j}}\right)\right] \\
&= \frac{1}{\alpha}\left[I\left(1 - e^{-\alpha S}\right) - e^{-\alpha S} \mathcal{H}_j(\sigma)\right]
\end{aligned} \tag{6.9}$$

where $\zeta_{\sigma_n} = \sum_{k=0}^{n} T_{\sigma_k}$ and $\mathcal{H}_j(\sigma) = \sum_{n=1}^{j} C_{\sigma_n} e^{-\alpha\zeta_{\sigma_{n-1}}} \left(1 - e^{-\alpha T_{\sigma_n}}\right) + R_{\sigma_j} e^{-\alpha\zeta_{\sigma_j}} \left(1 - e^{-\alpha Z_{\sigma_j}}\right)$ and $T_{\sigma_0} = 0$.

A cycle involves an operating time S, an inspection time and a repair time for the failed unit so that if the duration of a cycle is denoted by T_C, the expected duration of a cycle $E[T_C]$:

$$E[T_C] = \frac{1}{\theta} + \sum_{j=1}^{N} P_{\sigma_j} \left(\sum_{n=1}^{j} T_{\sigma_n} + Z_{\sigma_j} \right) = \frac{1}{\theta} \left[1 + \sum_{j=1}^{N} \theta_j \left(\sum_{n=1}^{j} T_{\sigma_n} + Z_{\sigma_j} \right) \right]. \tag{6.10}$$

$$E[G(S,\sigma)|S] = \sum_{n=1}^{N} G_n(S,\sigma) P_{\sigma_n} = \frac{1}{\alpha} \left[I\left(1 - e^{-\alpha S}\right) - \frac{1}{\theta} e^{-\alpha S} \sum_{j=1}^{N} \theta_j \mathcal{H}_j(\sigma) \right]$$

and hence, the expected net discounted income per cycle, $G(\sigma)$:

$$\begin{aligned} G(\sigma) &= \frac{1}{\alpha} \int_0^\infty \left[I\left(1 - e^{-\alpha s}\right) - \frac{1}{\theta} e^{-\alpha s} \sum_{j=1}^{N} \theta_j \mathcal{H}_j(\sigma) \right] \theta e^{-\theta s} ds \\ &= \frac{1}{\alpha(\alpha+\theta)} \left[\alpha I - \sum_{j=1}^{N} \theta_j \mathcal{H}_j(\sigma) \right] \end{aligned} \tag{6.11}$$

An optimal inspection permutation maximizes η:

$$\eta(\sigma) = \frac{G(\sigma)}{E[T_C]} = \frac{\frac{1}{\alpha(\alpha+\theta)} \left[\alpha I - \sum_{j=1}^{N} \theta_j \mathcal{H}_j(\sigma) \right]}{\frac{1}{\theta} \left[1 + \sum_{j=1}^{N} \theta_j \left(\sum_{n=1}^{j} T_{\sigma_n} + Z_{\sigma_j} \right) \right]} = \frac{\left(\frac{I}{\alpha+\theta} - \frac{1}{\alpha(\alpha+\theta)} \sum_{j=1}^{N} \theta_j \mathcal{H}_j(\sigma) \right)}{\left[\frac{1}{\theta} + \sum_{j=1}^{N} \frac{\theta_j}{\theta} \left(\sum_{n=1}^{j} T_{\sigma_n} + Z_{\sigma_j} \right) \right]} \tag{6.12}$$

Theorem 6.1

$$\lim_{\alpha \to 0^+} \eta(\sigma) = \frac{\frac{I}{\theta} - E_\sigma[C] - \sum_{n=1}^{N} P_n R_n}{\frac{1}{\theta} + E_\sigma[T] + \sum_{n=1}^{N} P_n Z_n} = \psi(\sigma).$$

Proof*:*

$$\begin{aligned}\frac{d\mathcal{H}_j(\sigma)}{d\alpha} &= \mathcal{H}'_j(\sigma) = \sum_{n=1}^{j} C_{\sigma_n}\left((\zeta_{n-1}+\zeta_n)e^{-\alpha(\zeta_{n-1}+\zeta_n)} - \zeta_{n-1}e^{-\alpha\zeta_{n-1}}\right) \\ &\quad + R_{\sigma_j}\left((\zeta_j + Z_{\sigma_j})e^{-\alpha(\zeta_j+Z_{\sigma_j})} - \zeta_j e^{-\alpha\zeta_j}\right)\end{aligned}$$

$$\textit{and} \lim_{\alpha\to 0+} \mathcal{H}'_j(\sigma) = \sum_{n=1}^{j} C_{\sigma_n}\zeta_n + R_{\sigma_j} Z_{\sigma_j}$$

Applying the L' Hospital's rule:

$$\begin{aligned}\lim_{\alpha\to 0^+} G(\sigma) &= \frac{\frac{I}{\theta} - \frac{\sum_{j=1}^{N}\theta_j \lim_{\alpha\to 0+}\mathcal{H}'_j(\sigma)}{\lim_{\alpha\to 0+}(2\alpha+\theta)}}{\left[\frac{1}{\theta} + \sum_{j=1}^{N}\frac{\theta_j}{\theta}\left(\sum_{n=1}^{j} T_{\sigma_n} + Z_{\sigma_j}\right)\right]} \\ &= \frac{\frac{I}{\theta} - \sum_{j=1}^{N}\frac{\theta_j}{\theta}\left(\sum_{n=1}^{j} C_{\sigma_n}\zeta_n + R_{\sigma_j}Z_{\sigma_j}\right)}{\left[\frac{1}{\theta} + \sum_{j=1}^{N}\frac{\theta_j}{\theta}\left(\sum_{n=1}^{j} T_{\sigma_n} + Z_{\sigma_j}\right)\right]} \\ &= \psi(\sigma)\end{aligned}$$

6.4 The ideal method for obtaining an optimal inspection permutation

6.4.1 (The undiscounted case)

The numerical example given by Zuckerman is used to demostrate how the proposed procedure works.

Example 6.1 *Consider a system which is composed of $N = 6$ independent units (call them $a_1, \cdots, a_6$) and generates income at the rate of $I = \$20$. Assume the parameters given in Table 6.1. We discuss the case of $R_i = 0$*

and $Z_i = 0$, for $i = 1, \cdots, 6$, i.e. the case of a system with units that take a negligible amount of time to repair in the event of failure.

Table 6.1: Costs and other constants associated with a system

Unit	T_i	C_i	θ_i	$P_i = \frac{\theta_i}{\sum_{i=1}^{6} \theta_i}$
a_1	3	4	$\frac{1}{100}$	$\frac{6}{35}$
a_2	2	3	$\frac{1}{150}$	$\frac{4}{35}$
a_3	4	5	$\frac{1}{200}$	$\frac{3}{35}$
a_4	6	3	$\frac{1}{150}$	$\frac{4}{35}$
a_5	5	7	$\frac{1}{50}$	$\frac{12}{35}$
a_6	4	4	$\frac{1}{100}$	$\frac{6}{35}$

To determine the optimal inspection permutation, the following steps are taken:

Step 0 Assign a value to the rate at which income is generated, I as well as define the vectors of repair costs and repair times **R** and **Z** using the commands

$I = 20$;

R=Table[0,{i,1,6}];

Z=Table[0,{i,1,6}];

Step 1 Create an $N! \times N = 720 \times 6$ matrix consisting of all the different 720 permutations (each row is a different permutation) using the command

$\text{AA} = \text{Permutations}\left[\{a_1, a_2, a_3, a_4, a_5, a_6\}\right];.$

Step 2 Create a vector of inspection times denoted by $T_1, \cdots, T_6$ using the command

$T = \text{Table}\left[T_k, \{k, 1, 6\}\right] /. \ \{T_1 \to 3, T_2 \to 2, T_3 \to 4, T_4 \to 6, T_5 \to 5, T_6 \to 4\};.$

Also create a corresponding 720×6 matrix of permutations, TT of $T1, \cdots, T6$ using the command:

$\text{TT} = \text{Permutations}[\{\text{T1}, \text{T2}, \text{T3}, \text{T4}, \text{T5}, \text{T6}\}];$

and then use the replacement rule which sets $T1 = 3, T2 = 2, \cdots, T6 = 4$ using the command

$\text{TT} = \text{TT}/. \{\text{T1} \to 3, \text{T2} \to 2, \text{T3} \to 4, \text{T4} \to 6, \text{T5} \to 5, \text{T6} \to 4\};$

Step 3 Create a vector of inspection cost rates denoted by $C_1, \cdots, C_6$ using the command

$C = \text{Table}\left[C_k, \{k, 1, 6\}\right] /. \ \{C_1 \to 4, C_2 \to 3, C_3 \to 5, C_4 \to 3, C_5 \to 7, C_6 \to 4\}$

Also create a corresponding 720×6 matrix of permutations, CC of $C1, \cdots, C6$ using the command:

$\text{CC} = \text{Permutations}[\{\text{C1}, \text{C2}, \text{C3}, \text{C4}, \text{C5}, \text{C6}\}]$ and then use the replacement rule which sets $C1 = 4, C2 = 3, \cdots, C6 = 4$ using the command

$\text{CC} = \text{CC}/. \{\text{C1} \to 4, \text{C2} \to 3, \text{C3} \to 5, \text{C4} \to 3, \text{C5} \to 7, \text{C6} \to 4\};$

Step 4 Calculate the probability of the i^{th} unit ($i = 1, \cdots, 6$) being the cause of system failure

Step 5 Next calculate a matrix (of cumulative cost of inspections) $V_{720\times 6}$ such that

$v_{ij} = \sum_{k\leq j} C_{\sigma_i(k)} T_{\sigma_i(k)}$ using the command

$V = \text{Table}[0.0, \{720\}, \{6\}];$

$$For\left[i = 1, i \leq 720, i{+}{+}, \text{For}\left[j = 1, j \leq 6, j{+}{+}, V[[i,j]] = N\left[\sum_{k=1}^{j} \text{CC}[[i,k]]\text{TT}[[i,k]]\right]\right]\right];$$

$V//\text{MatrixForm};$

and a vector (or list) of size 720 (call it $ECost$), whose elements are the expected costs of the inspection permutations $\sigma_1, \cdots, \sigma_{720}$, $E_{\sigma_1}, \cdots, E_{\sigma_{720}}$, respectively, using the commands

$\text{ECost} = \text{Table}[0.0, \{720\}];$

$$\text{For}\left[i = 1, i \leq 720, i{+}{+}, \text{ECost}[[i]] = N\left[\sum_{j=1}^{6} V[[i,j]]\text{PP}[[i,j]]\right]\right];$$

$\text{ECost}//\text{MatrixForm};$

Step 6 Calculate an $N! \times n$ (720×6) matrix of cumulative inspection times $Ecum$; (under the i^{th} inspection permutation σ_i, the $j^{th}, j = 1, \cdots, 6$ cumulative inspection time is $\sum_{k=1}^{j} TT_{ik}$). The following commands are used to compute Tcum:

$\text{Tcum} = \text{Table}[0.0, \{720\}, \{6\}];$

$$\text{For}\left[i = 1, i \leq 720, i{+}{+}, \text{For}\left[j = 1, j \leq 6, j{+}{+}, \text{Tcum}[[i,j]] = N\left[\sum_{k=1}^{j} \text{TT}[[i,k]]\right]\right]\right];$$

and then calculate the vector or list of expected values of inspection times for all $N! = 720$ inspection permutations (call it ETime) using the commands

$\text{ETime} = \text{Table}[0.0, \{720\}];$

$$\text{For}\left[i = 1, i \leq 720, i{+}{+}, \text{ETime}[[i]] = N\left[\sum_{j=1}^{6} \text{Tcum}[[i,j]]\ \text{PP}[[i,j]]\right]\right];$$

ETime//MatrixForm;

Step 7 Calculate the vector or list of net-income-rate for each inspection permutation $\psi(\sigma_i), i = 1 \cdots, 720$ using the commands

$\psi = \text{Table}[0.0, \{720\}, \{1\}];$

$$\text{For}\left[i = 1, i \leq 720, i{+}{+}, \psi[[i]] = N\left[\frac{\frac{I}{\theta} - \text{ECost}[[i]] - \sum_{k=1}^{6} P[[k]]R[[k]]}{\frac{1}{\theta} + \text{ETime}[[i]] + \sum_{k=1}^{6} P[[k]]Z[[k]]}\right]\right];$$

ETime//MatrixForm;

ψ//MatrixForm

Step 8 Merge the matrix of permutations AA with the list ψ so that each permutation appears in the same row as its net-income-rate and then sort/reorder the resultant matrix so that the inspection permutations are sorted by the magnitude of ψ. This is done by using the commands

$\text{Optsol} = \text{Transpose}[\text{Join}[\text{Transpose}[\text{AA}], \{\psi\}]];$

$\text{Optsol//MatrixForm};$

$\text{Optsorted} = \text{SortBy}[\text{Optsol}, \text{Last}];$

$\text{Optsorted//MatrixForm}$

The results obtained for the sorted matrix are as follows:

$$Optsorted = \begin{pmatrix} a_3 & a_4 & a_6 & a_5 & a_1 & a_2 & 7.85403 \\ a_4 & a_3 & a_6 & a_5 & a_1 & a_2 & 7.86265 \\ a_3 & a_4 & a_6 & a_5 & a_2 & a_1 & 7.86409 \\ a_3 & a_4 & a_6 & a_1 & a_5 & a_2 & 7.86978 \\ a_4 & a_3 & a_6 & a_5 & a_2 & a_1 & 7.87270 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_5 & a_1 & a_2 & a_6 & a_4 & a_3 & 10.1657 \\ a_2 & a_5 & a_1 & a_6 & a_3 & a_4 & 10.1998 \\ a_2 & a_1 & a_5 & a_6 & a_3 & a_4 & 10.2046 \\ a_2 & a_5 & a_1 & a_6 & a_4 & a_3 & 10.2055 \\ a_2 & a_1 & a_5 & a_6 & a_4 & a_3 & 10.2103 \end{pmatrix}$$

From the matrix *Optsorted* above, we deduce that the optimal inspection permutation is to inspect the units of the system in the order $(a_2 \to a_1 \to a_5 \to a_6 \to a_4 \to a_3)$ and the associated net-income-rate, $\psi^* = 10.2103$ - the same result obtained by Zuckerman. The worst inspection permutation is $(a_3 \to a_4 \to a_6 \to a_5 \to a_1 \to a_2)$ and its associated net-income-rate of

$\psi^* = 7.85403$.

Example 6.2 *Consider Example (6.1) discussed above with all repair times and repair costs set at 0 except R_2 and Z_2.*

Table 6.2: Some optimal inspection permutations for different inspection costs

R_2	Z_2	Best strategy/ Worst strategy	Net-income-rate per unit of time
9	2279	$(a_2 \to a_1 \to a_5 \to a_6 \to a_4 \to a_3)$/ $(a_3 \to a_4 \to a_5 \to a_6 \to a_1 \to a_2)$	1.00030 0.904568
10	2279	$(a_2 \to a_1 \to a_6 \to a_5 \to a_4 \to a_3)$/ $(a_3 \to a_4 \to a_5 \to a_6 \to a_1 \to a_2)$	0.999901 0.904179
6242	2279	$(a_2 \to a_1 \to a_6 \to a_5 \to a_4 \to a_3)$/ $(a_3 \to a_5 \to a_4 \to a_6 \to a_1 \to a_2)$	-1.46139 -1.51866
6243	2279	$(a_2 \to a_1 \to a_6 \to a_4 \to a_5 \to a_3)$/ $(a_3 \to a_5 \to a_4 \to a_6 \to a_1 \to a_2)$	-1.46179 -1.51905

*If Z_2 is fixed at 2279 then the optimal inspection permutation, as R_2 is increased from 0 upto a threshold value that is just below 10, remains $(a_2 \to a_1 \to \mathbf{a_5} \to \mathbf{a_6} \to a_4 \to a_3)$. However, values of R_2 beyond this threshold upto a value between 6242 and 6243 dictate a different optimal strategy - $(a_2 \to a_1 \to \mathbf{a_6} \to \mathbf{a_5} \to a_4 \to a_3)$; from a value just below 6243 on, the optimal strategy changes to $(a_2 \to a_1 \to a_6 \to a_4 \to a_5 \to a_3)$. Am inspection of Table ** also reveals that changes in the worst strategy do not occur simulataneously with changes in the optimal inspection permutation as the value of the repair cost R_2 increases. An interesting observation made* *from the first row and the third row of Table ** is that, unlike in rows 2 and 4, the worst strategy is not necessarily a result of reading of the optimal strategy in reverse order.*

Remark 6.6 *What clearly stands out from Example (6.2) is that Zuckerman's assertion that the optimal inspection permutation does not depend on repair costs and repair times is erroneous - inspection times and inspection costs infact play a role in the determination of the optimal inspection permutation!*

6.4.2 Discounted case

Example 6.3 (The discounted case) *Consider the system which is composed of six independent units (call them $a_1, \cdots, a_6$) and generates income at the rate of $I = \$20$. Assume the parameters given in Table 6.1. We again look at the case of repair times and repair costs being negligible so that $R_1 = \cdots = R_6 = 0$ and the respective repair costs being $Z_1 = \cdots = Z_6 = 0$.*

Table 6.3 gives the results obtained using a Mathematica computer program.

Figure 6.1 contains plots of the discounted maximum and minimum net income-rate. For low discount rates, the difference between the income associated with the optimal inspection permutation and the income associated with the worst inspection permutation is large and progressively gets smaller and smaller with increasing discount rate. The impression created is that for low discount rates, one particularly has to insist on using an optimal inspection permutation; however, for high discount rates one may as well be indifferent on what inspection permutation to use as the penalty for using a non-ptimal inspection permutation becomes negligible.

Remark 6.7 *Example 6.4 highlights the fact that optimal solutions of hierarchical inspection problems are very sensitive to changes in per unit of time repair costs and repair times of the units. In Zuckerman (1989) and Qiu (1991), the impact of these per unit of time repair costs and inspection times is downplayed.*

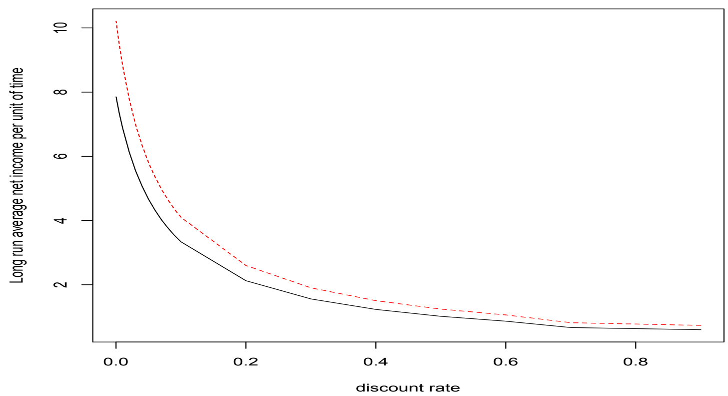

Figure 6.1: Plot of discounted maximum and minimum net-income-rate

Example 6.4 *Consider the system which is composed of six independent units (call them $a_1, \cdots, a_6$) and generates income at the rate of $I = \$20$. Assume the parameters given in Table 6.1 and a discount rate of $\alpha = 0.1$. We now examine the case of some repair times and repair costs being non-negligible; Table 6.4 gives the results obtained using a Mathematica computer program; all repair costs and repair times not mentioned in the table should be assumed negligible.*

The changes in the optimal solutions when changes in the repair times *and per unit of time repair costs take place are in line with the intuitive reasoning that those units whose total repair costs are large should roughly be inspected last. In the first row of Table 6.4, we see that Unit a_2, according to the optimal solution should be inspected first. In Row 2 of the table, Unit 2 is the only one with a non-zero total repair cost (a whooping total repair cost of* 100×100*=\$10000) and in the optimal solution given in Row 2, it is relegated to the last position. A similar change in the repair time and per unit of time repair cost of Unit 1 (see Row 2) which is in first position also results in it being relegated to the last position in the line up. Again similar changes in the repair time and per unit of time repair cost of Unit 5 sees it being similarly relegated to the last position in the line up.*

Table 6.3: Some optimal inspection permutations for different inspection costs and varying discount rate (when repair costs and repair times are negligible)

α	**Worst inspection permutation/ strategy**	**Net income per unit of time of worst strategy**	**Optimal inspection permutation**	**Net income per unit of time of optimal strategy**
0.90		\$0.59796		\$0.73235
0.70		\$0.66634		\$0.81616
0.60		\$0.86398		\$1.05794
0.50	$a_3, a_4, a_6, a_1, a_2, a_5$	\$1.01446	$a_2, a_5, a_1, a_6, a_3, a_4$	\$1.24149
0.40		\$1.22846		\$1.50177
0.30		\$1.55682		\$1.90343
0.20		\$2.12376		\$2.59713
0.10		\$3.33335	$a_2, a_1, a_5, a_6, a_3, a_4$	\$4.09587
0.09		\$3.53618		\$4.34917
0.08		\$3.76130		\$4.63688
0.07		\$4.01966		\$4.96677
0.06		\$4.31670		\$5.34918
0.05	$a_3, a_4, a_6, a_1, a_5, a_2$	\$4.66130		\$5.79823
0.04	$a_3, a_4, a_6, a_5, a_1, a_2$	\$5.06534		\$6.33370
0.03		\$5.54747		\$6.98422
0.02		\$6.13806		\$7.79297
0.01		\$6.88193	$a_2, a_1, a_5, a_6, a_4, a_3$	\$8.83034
0.005		\$7.33294		\$9.46748
0.001		\$7.74338		\$10.05200
0.0001		\$7.84281		\$10.19420
10^{-13}		\$7.85398		\$10.2103
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
0.0000		\$7.85403		\$10.2103

Table 6.4: Some optimal inspection permutations when some repair costs and repair times are non-negligible

R_1	Z_1	R_2	Z_2	R_5	Z_5	**Optimal strategy/ Worst strategy**	**Net income per unit of time**
0	0	0	0	0	0	$a_2 \to a_1 \to a_5 \to a_6 \to a_3 \to a_4$	4.09587
						$a_3 \to a_4 \to a_6 \to a_1 \to a_2 \to a_5$	3.33335
0	0	100	100	0	0	$a_1 \to a_5 \to a_6 \to a_3 \to a_4 \to a_2$	2.72885
						$a_2 \to a_3 \to a_4 \to a_6 \to a_1 \to a_5$	1.81072
100	100	100	100	0	0	$a_6 \to a_5 \to a_4 \to a_3 \to a_1 \to a_2$	1.74119
						$a_2 \to a_1 \to a_3 \to a_4 \to a_6 \to a_5$	0.72044
100	100	100	100	100	100	$a_6 \to a_4 \to a_3 \to a_1 \to a_2 \to a_5$	0.89803
						$a_5 \to a_2 \to a_1 \to a_3 \to a_6 \to a_4$	-0.08414

Chapter 7

Conclusions and Recommendations

7.1 Finite planning horizon models

The advent of powerful computers these days has made it possible to demonstrate that finding a finite optimal inspection model when the lifetime distribution is known is a process easily achievable by employing either the iterative procedure given in Section 3.2.4 or the nonlinear optimization procedure in Section 3.2.5.

If the time to failure of a system follows a uniform distribution over the interval $[0, L_o]$, according to Remark 3.9, it does not make business sense to have a planning horizon which is longer than the maximum possible time for which the system may operate, i.e. having $L > L_o$ is ruled out. It also may not make business sense to schedule n inspections such that $n > \sqrt{2\frac{c_F}{c_I}L_o + \frac{1}{4}} + \frac{1}{2}$ (see Remark 3.10).

If the time to failure follows an exponential distribution, it has been noted that periodically carrying out inspections may not result in maximization of expected profit. The recommended pattern of scheduling inspections for a given number of inspections depends on the parameter θ; a high value of θ (which in turn means the system at hand generally has a shorter time to failure) will favour inspections that are"back loaded" while systems with a lower value of θ will require inspections being"front loaded" for maximization of expected profit to be achieved. When the number of inspections is large, uniformly spreading inspections over a recommended planning horizon may not be seriously prejudicial to the expected profit. For the Weibull distributions family (of which the exponential distribution is a special case), evenly spreading the inspections over a given finite planning horizon may not lead to any serious prejudice in profit.

For a system whose time to failure follows a Weibull distribution, Example 3.2 and Example 3.4 demonstrated that an increase in the length of the planning horizon will result in an increase in the expected profit provided that an optimal number of inspections which are at optimally set times are carried out. There is, however, a saturation point to which the profit may asymptotically converge if the planning horizon keeps increasing accompanied with an optimal number of inspections which are also optimally scheduled. In view of this, the recommendation given in Chapter 3 is that the number of inspections and corresponding optimal planning horizon may be deemed optimal if for some desirable $\epsilon > 0$, n is the least value such that $|\mathcal{G}_{E.n+1} - \mathcal{G}_{E.n}| < \epsilon$.

7.2 Inspection models with Inspection times that are of non-negligible duration

Chapter 4 explores the case of inspection models where each inspection takes a fixed amount of time. It is observed that

1. for a system whose time to failure follows a continuous uniform distribution, the conditions necessary for inspections that are evenly spread over the entire planning horizon to be near-optimal are 1) that the cost of carrying out an inspection, c_I is negligible in comparison to the per

unit cost when the system is idle, c_F and 2) $\Delta_i \to 0$

2. for a system whose time to failure follows a an exponential distribution, the conditions necessary for inspections that are evenly spread over the entire planning horizon to be near-optimal are 1) that the cost of carrying out an inspection, c_I is negligible in comparison to the per unit cost when the system is idle, c_F and 2) $\Delta_i \to 0$ and 3) the mean of the time to failure $E[T] \to \infty$.

The discussion in Chapter 4 is limited in that it does not look at the case of time to failure following other distributions like the Weibull distribution and others mentioned in Section 3.3.2. The discussion also falls shy of looking at infinite planning horizon models with inspection times that are of non-negligible duration.

7.3 Finite and infinite planning horizon models where inspections are imperfect

Chapter 5 deals with finite and infinite planning horizon models where inspections are imperfect. The following observations were made for this pool of models:

1. increasing the value of a Type I error (i.e. increasing α) when the value

of a Type II error (β) is held constant will result in the expected profits initially plummeting. An interestingly anomalous observation is that the expected profit curve reaches its minima when $\alpha + \beta = 1$ and for values of α greater than $1 - \beta$, increasing α results in the expected profit slowly increasing

2. increasing the value of β when α is held constant will result in the expected profits consistently plummeting

3. When α is small and fixed, increasing the value of β results in a delay in starting inspections

4. For a fixed value of α, increasing the value of β results in a shorter optimal planning horizon with the net effect that inspections are scheduled over a smaller and smaller range

5. When β is small and fixed, increasing the value of α results in a delay in starting inspections and a longer planning horizon

6. When β is large and fixed, increasing α is accompanied by a shorter optimal planning horizon

7. Larger values for both α and β result in optimal inspections which are

chronologically very close and comparatively shorter planning horizons.

7.4 Hierarchical inspection models

Chapter 6 gives a clear and easy to implement road map on how to get an optimal inspection permutation in problems first discussed by Zuckerman (1989) and later reviewed by Qiu (1991) for both the undiscounted and discounted cases; the only challenge envisaged when a system has a large number of components is that of computer memory requirements - which nowadays is fast being overcome. In particular, it has been clearly demonstrated that the impact of repair times and per unit of time repair costs on the optimal inspection permutation cannot be ignored.

The ideas and procedures of determining optimal inspection permutations which have been developed in this thesis will no doubt lead to huge cost savings especially for systems where the cost of inspecting components is huge.

7.5 Scope for further research

In the case of finite planning horizon models discussed in Chapter 3, the author of this thesis notes that

1. An extension of the theoretical results given in this thesis which takes on board actuarial considerations such as making C_S a time-dependent variable will no doubt lead to results with wider applications in industry. The dimension taken by Usher et al. (1998) whereby they determined an optimal preventive maintenance schedule by considering the time value of money in all future costs (and incomes when profit maximization is the objective function) is worth exploring.

2. A possible avenue of research would involve developing finite planning horizon models for a system whose time to failure follows a Bernstein distribution. According to Ahmad and Sheikh (1984) the distribution is useful in modeling life characteristics of machine components which deteriorate according to a scheme of non-stationary linear wear processes (such as cutting tools, slideways and rotating parts of machine tools used in precision machining).

3. Other envisaged research works involve the development of finite plan-

ning horizon inspection models applicable to systems that are subjected to shocks of random magnitudes at random times as explained by Finkelstein and Marais (2010).

4. Another interesting dimension is to develop a finite planning horizon model based on three-stage failure process as done by Wang (2011).

The work done in Chapter 4 centered on finite planning horizon inspection models where inspections are of non-negligible fixed duration using a similar approach to the one taken by Luss and Kander (1974). In real-life, inspections may turn out to be random variables and the development of models where inspection times are random variables may prove very useful.

Another limitation of the work done in Chapter 4 is that only the cases of system time to failure following a uniform distribution or an exponential distribution are explored fully. Cases of time to failure following other distributions like the Weibull distribution and others mentioned in Section 3.3.2 need to be explored. In particular, the generalized gamma distribution which is a family of four distributions of which some have been discussed in this thesis will most likely yield interesting results. The discussion also falls shy of looking at infinite planning horizon models with inspection times that are of non-negligible duration. This is one area which the author hopes to explore in future.

Chapter 5 deals with finite and infinite planning horizon models where inspections are imperfect. Extensions of the ideas discussed in the chapter to encompass inspections that are failure inducing, just like in the paper by Flage (2014) also need to be explored.

The Hierarchical inspection models discussed in Chapter 6 assume that the system's status is observed continuously at zero cost (by a controller) and a failure is detected as and when it occurs. For most classical inspection and replacement models, the system's status (i.e. whether it is still functioning properly or it is in a failed state) is only known after an inspection has been carried out. For classical inspection and replacement models scheduling times at which the system needs to be inspected is therefore imperative. It would be helpful if models that encompass both the specification of inspection cycles (to determine a system's status) as well as the optimal inspection permutation (for the units that make up the system) were developed for those systems where continuous monitoring is not possible.

Models of a similar nature where the time to failure of the units follow some other distributions such as the Weibull distribution need to be developed. Further research could also consider cases when the life time distribution is either partially unknown or entirely unknown is also possible.

Other research work to be done would be on inspection and replacement models for a system that is subjected to shocks over time. The literature in Section 2.5.2 provide good ideas on how to proceed in this vain.

References

Ahmad, M. and Sheikh, A. (1984). Bernstein reliability model: derivation and estimation of parameters. *Reliability Engineering*, 8(3):131–148.

Ahmadi, R. and Newby, M. (2011). Maintenance scheduling of a manufacturing system subject to deterioration. *Reliability Engineering & System Safety*, 96(10):1411–1420.

Albert, A. E. and Gardner, L. A. (1967). *Stochastic approximation and nonlinear regression*. MIT-Pr.

Anbar, D. (1976a). An asymptotically optimal inspection policy. *Naval Research Logistics Quarterly*, 23(2):211–218.

Anbar, D. (1976b). An asymptotically optimal inspection policy. *Nav Res Logist Q*, pages 211–218.

Antelman, G. R. and Savage, I. R. (1965). Surveillance problems: Wiener processes. *Naval Research Logistics Quarterly*, 12(1):35–55.

Aven, T. and Gaarder, S. (1987). Optimal replacement in a shock model: discrete time. *Journal of Applied Probability*, pages 281–287.

Badia, F., Berrade, M., Campos, C. A., et al. (2001). Optimization of inspection intervals based on cost. *Journal of Applied Probability*, 38(4):872–881.

Barlow, R. and Hunter, L. (1960). Optimum preventive maintenance policies. *Operations research*, 8(1):90–100.

Barlow, R. E., Hunter, L. C., and Proschan, F. (1963). Optimum checking procedures. *Journal of the Society for Industrial & Applied Mathematics*, 11(4):1078–1095.

Beichelt, F. (1981). Minimax inspection strategies for single unit systems. *Naval Research Logistics Quarterly*, 28(3):375–381.

Beichelt, F. (2001a). A first-passage time problem in Reliability Theory. *Economic Quality Control*, 16(1):65–73.

Beichelt, F. (2001b). A replacement policy based on limiting the cumulative maintenance cost. *International Journal of Quality & Reliability Management*, 18(1):76–83.

Beichelt, F. and Tittmann, P. (2012). *Reliability and maintenance: networks and systems*. CRC Press.

Berrade, M. D., Cavalcante, C. A., and Scarf, P. A. (2012). Maintenance scheduling of a protection system subject to imperfect inspection and replacement. *European Journal of Operational Research*, 218(3):716–725.

Berrade, M. D., Cavalcante, C. A., and Scarf, P. A. (2013). Modelling imperfect inspection over a finite horizon. *Reliability Engineering & System Safety*, 111:18–29.

Boland, P. J. and Proschan, F. (1983). Optimum replacement of a system subject to shocks. *Operations Research*, 31(4):697–704.

Butler, D. A. (1979). A hazardous-inspection model. *Management Science*, 25(1):79–89.

Caballé, N., Castro, I., Pérez, C., and Lanza-Gutiérrez, J. M. (2015). A condition-based maintenance of a dependent degradation-threshold-shock model in a system with multiple degradation processes. *Reliability Engineering & System Safety*, 134:98–109.

Chelbi, A. and Ait-Kadi, D. (1999). An optimal inspection strategy for randomly failing equipment. *Reliability Engineering & System Safety*, 63(2):127–131.

Chipoyera, H. (2016a). Optimal scheduling of inspection times in a production process with a finite planning horizon. *Applied Stochastic Models in Business and Industry*, 32(6):775–791.

Chipoyera, H. W. (2016b). An ideal way of obtaining an optimal inspection permutation for a system with components connected in series. *Applied Stochastic Models in Business and Industry*, 32(6):825–835.

Christer, A. (1988). Condition-based inspection models of major civil-engineering structures. *Journal of the Operational Research Society*, pages 71–82.

Christer, A. and Lee, C. (1999). Refining the delay-time-based pm inspection model with non-negligible system downtime estimates of the expected number of failures. *International Journal of Production Economics*, 67(1):77–85.

Clifton, R. H. (1974). *Principles of planned maintenance*. Arnold.

Derman, C. (1961). On minimax surveillance schedules. *Naval Research Logistics Quarterly*, 8(4):415–419.

Devooght, J., Dubus, A., and Smidts, C. (1990). Suboptimal inspection policies for imperfectly observed realistic systems. *European Journal of Operational Research*, 45(2):203–218.

Durango-Cohen, P. L. and Madanat, S. M. (2008). Optimization of inspection and maintenance decisions for infrastructure facilities under performance model uncertainty: a quasi-Bayes approach. *Transportation Research Part A: Policy and Practice*, 42(8):1074–1085.

Fang, Y.-t. and Liu, B.-y. (2006). Preventive repair policy and replacement policy of repairable system taking non-zero preventive repair time. *Journal of Zhejiang University SCIENCE A*, 7(2):207–212.

Finkelstein, M. and Marais, F. (2010). On terminating poisson processes in some shock models. *Reliability Engineering & System Safety*, 95(8):874–879.

Flage, R. (2014). A delay time model with imperfect and failure-inducing inspections. *Reliability Engineering & System Safety*, 124:1–12.

Folks, J. and Chhikara, R. (1978). The inverse gaussian distribution and its statistical application–a review. *Journal of the Royal Statistical Society. Series B (Methodological)*, pages 263–289.

Geurts, J. H. (1983). Optimal Age Replacement versus Condition Based Replacement: Some Theoretical and Practical Considerations. *Journal of Quality Technology*, 15(4):171–179.

Ghasemi, A., Yacout, S., and Ouali, M.-S. (2008). Optimal inspection period and replacement policy for cbm with imperfect information using phm. In *World Congress on Engineering and Computer Science*, volume 1007, pages 247–266. AIP Publishing.

Goldstein, R. B. (1973). Algorithm 451: chi-square quantiles [g1]. *Communications of the ACM*, 16(8):483–485.

Golmakani, H. R. and Fattahipour, F. (2011). Optimal replacement policy and inspection interval for condition-based maintenance. *International Journal of Production Research*, 49(17):5153–5167.

Golmakani, H. R. and Moakedi, H. (2012). Optimal non-periodic inspection scheme for a multi-component repairable system using a search algorithm. *Computers & Industrial Engineering*, 63(4):1038–1047.

Gottlieb, G. (1982). Optimal replacement for shock models with general failure rate. *operations Research*, 30(1):82–92.

Hontelez, J. A., Burger, H. H., and Wijnmalen, D. J. (1996). Optimum condition-based maintenance policies for deteriorating systems with partial information. *Reliability Engineering & System Safety*, 51(3):267–274.

Hotelling, H. (1925). A general mathematical theory of depreciation. *Journal of the American Statistical Association*, 20(151):340–353.

Jiang, R. (2009). An accurate approximate solution of optimal sequential age replacement policy for a finite-time horizon. *Reliability Engineering & System Safety*, 94(8):1245–1250.

Kabir, A. Z. and El Tamimi, A. A. (1988). Inspection policy based on specified fractional dead time. *Reliability Engineering & System Safety*, 21(3):231–238.

Kaio, N. and Osaki, S. (1986). Optimal inspection policy with two types of imperfect inspection probabilities. *Microelectronics Reliability*, 26(5):935–942.

Kaio, N. and Osaki, S. (1988). Inspection policies: Comparisons and modifications. *Revue française d'automatique, d'informatique et de recherche opérationnelle. Recherche opérationnelle*, 22(4):387–400.

Kaio, N. and Osaki, S. (1989). Comparison of inspection policies. *Journal of the Operational Research Society*, pages 499–503.

Kander, Z. and Raviv, A. (1974). Maintenance policies when failure distribution of equipment is only partially known. *Naval Research Logistics Quarterly*, 21(3):419–429.

Karlin, S. and Taylor, H. M. (1975). A first course in stochastic processes. *Academic, San Diego*.

Kawai, H. (1984). An optimal inspection and replacement policy of a markovian deterioration system. In *Stochastic Models in Reliability Theory*, pages 177–186. Springer.

Khodabin, M. and Ahmadabadi, A. (2010). Some properties of generalized gamma distribution. *Mathematical Sciences*, 4(1):9–28.

Kus, C. and Kaya, M. F. (2006). Estimation of parameters of the loglogistic distribution based on progressive censoring using the em algorithm. *Hacettepe Journal of Mathematics and Statistics*, 35(2).

Lai, M.-T. and Yuan, J. (1993). Cost-optimal periodical replacement policy for a system subjected to shock damage. *Microelectronics Reliability*, 33(8):1159–1168.

Lee, H. L. and Rosenblatt, M. J. (1987). Simultaneous determination of production cycle and inspection schedules in a production system. *Management Science*, 33(9):1125–1136.

Levner, E. (1994). Infinite-horizon scheduling algorithms for optimal search for hidden objects. *International Transactions in Operational Research*, 1(2):241–250.

Li, W. and Pham, H. (2005). Reliability modeling of multi-state degraded systems with multi-competing failures and random shocks. *Reliability, IEEE Transactions on*, 54(2):297–303.

Lindsay, S., Wood, G., and Woollons, R. (1996). Stand table modelling through the weibull distribution and usage of skewness information. *Forest Ecology and Management*, 81(1):19–23.

Luss, H. (1977). Inspection policies for a system which is inoperative during inspection periods. *AIIE Transactions*, 9(2):189–194.

Luss, H. (1983). An inspection policy model for production facilities. *Management Science*, 29(9):1102–1109.

Luss, H. and Kander, Z. (1974). Inspection policies when duration of checkings is non-negligible. *Operational Research Quarterly*, pages 299–309.

Lyle, P. (1954). The construction of nomograms for use in statistics: Part i. true and empirical nomograms. *Applied Statistics*, 3(2):116–125.

Mangalam, S. and Feo, R. (2006). Risk informed decision making by a public safety regulatory authority in Canada: a case study involving risk based scheduling of periodic inspections. In *Proceedings of the Waste Management Symposium: Global Accomplishments in Environmental and Radioactive Waste Management.¡ http://www. wmsym. org/archives/2006/prof6364. html*.

Milioni, A. Z. and Pliska, S. R. (1988). Optimal inspection under semimarkovian deterioration: Basic results. *Naval Research Logistics (NRL)*, 35(5):373–392.

Mohandas, K., Chaudhuri, D., and Rao, B. (1992). Optimal periodic replacement for a deteriorating production system with inspection and minimal repair. *Reliability Engineering & System Safety*, 37(1):73–77.

Morey, R. C. (1968). Letter to the editor—a criterion for the economic application of imperfect inspections. *Operations Research*, 16(3):695–698.

Munford, A. (1981). Comparison among certain inspection policies. *Management Science*, 27(3):260–267.

Munford, A. and Shahani, A. (1972). A nearly optimal inspection policy. *Operational Research Quarterly*, 23(3):373–379.

Munford, A. and Shahani, A. (1973). An inspection policy for the weibull case. *Operational Research Quarterly*, 24(3):453–458.

Nakagawa, T. (1976). On a replacement problem of cumulative damage model. *Operat. Resear. Quart.*, 27(4):895–900.

Nakagawa, T. (1984). Periodic inspection policy with preventive maintenance. *Naval Research Logistics Quarterly*, 31(1):33–40.

Nakagawa, T. and Mizutani, S. (2009). A summary of maintenance policies for a finite interval. *Reliability Engineering & System Safety*, 94(1):89–96.

Nakagawa, T. and Yasui, K. (1980). Approximate calculation of optimal inspection times. *Journal of the Operational Research Society*, pages 851–853.

Nakagawa, T., Yasui, K., and Sandoh, H. (2004). Note on optimal partition problems in reliability models. *Journal of Quality in Maintenance Engineering*, 10(4):282–287.

Ohnishi, M., Kawai, H., and Mine, H. (1986a). An optimal inspection and replacement policy for a deteriorating system. *Journal of applied probability*, pages 973–988.

Ohnishi, M., Kawai, H., and Mine, H. (1986b). An optimal inspection and replacement policy under incomplete state information. *European Journal of Operational Research*, 27(1):117–128.

Parmigiani, G. (1993). Optimal inspection and replacement policies with age-dependent failures and fallible tests. *Journal of the Operational Research Society*, pages 1105–1114.

Qiu, Y. (1991). A note on optimal inspection policy for stochastically deteriorating series systems. *Journal of applied probability*, pages 934–939.

Qiu, Y. and Cox Jr, L. A. (1994). Optimal search for failed components in renewable coherent systems. In *Systems, Man, and Cybernetics, 1994. Humans, Information and Technology., 1994 IEEE International Conference on*, volume 2, pages 1240–1245. IEEE.

Rangan, A. and Grace, R. E. (1989). Optimal replacement policies for a deteriorating system with imperfect maintenance. *Advances in Applied Probability*, pages 949–951.

Rao, G. S., Kantam, R., and Rosaiah, K. (2009). Reliability estimation in log-logistic distribution from censored samples. In *ProbStat Forum*, volume 2, pages 52–67.

Roeloffs, R. (1963). Minimax surveillance schedules with partial information. *Naval Research Logistics Quarterly*, 10(1):307–322.

Roeloffs, R. (1967). Minimax surveillance schedules for replaceable units. *Naval Research Logistics Quarterly*, 14(4):461–471.

Savage, I. R. (1962). Surveillance problems. *Naval Research Logistics Quarterly*, 9(3-4):187–209.

Saw, S. L., Balasooriya, U., and Chong Tan, K. (2002). The log-eig distribution: a new probability model for lifetime data. *Communications in Statistics*, 31(11):1913–1926.

Scarf, P., Cavalcante, C. A., Dwight, R., Gordon, P., et al. (2009). An age-based inspection and replacement policy for heterogeneous components. *Reliability, IEEE Transactions on*, 58(4):641–648.

Sheu, S.-H., Tsai, H.-N., Wang, F.-K., and Zhang, Z. G. (2015). An extended optimal replacement model for a deteriorating system with inspections. *Reliability Engineering & System Safety*, 139:33–49.

Smith, R. L. and Naylor, J. (1987). A comparison of maximum likelihood and bayesian estimators for the three-parameter weibull distribution. *Applied Statistics*, pages 358–369.

Stadje, W. and Zuckerman, D. (1990). Optimal strategies for some repair replacement models. *Advances in Applied Probability*, pages 641–656.

Sun, Y., Xie, M., Goh, T., and Ong, H. (1993). Development and applications of a three-parameter weibull distribution with load-dependent location and scale parameters. *Reliability Engineering & System Safety*, 40(2):133–137.

Tadikamalla, P. R. (1979). An inspection policy for the gamma failure distributions. *Journal of the Operational Research Society*, pages 77–80.

Taghipour, S., Banjevic, D., and Jardine, A. K. (2010). Periodic inspection optimization model for a complex repairable system. *Reliability Engineering & System Safety*, 95(9):944–952.

Tan, L., Cheng, Z., Guo, B., and Gong, S. (2010). Condition-based maintenance policy for gamma deteriorating systems. *Systems Engineering and Electronics, Journal of*, 21(1):57–61.

Taylor, H. M. (1975). Optimal replacement under additive damage and other failure models. *Naval Research Logistics Quarterly*, 22(1):1–18.

Taylor, J. S. (1923). A statistical theory of depreciation: Based on unit cost. *Journal of the American Statistical Association*, 18(144):1010–1023.

Teramoto, K., Nakagawa, T., and Motoori, M. (1990). Optimal inspection policy for a parallel redundant system. *Microelectronics Reliability*, 30(1):151–155.

Thomas, L. C., Gaver, D., and Jacobs, P. (1991). Inspection models and their application. *IMA Journal of Management Mathematics*, 3(4):283–303.

Usher, J. S., Kamal, A. H., and Syed, W. H. (1998). Cost optimal preventive maintenance and replacement scheduling. *IIE transactions*, 30(12):1121–1128.

Vlasiou, M. (2010). Renewal processes with costs and rewards. *Wiley Encyclopedia of Operations Research and Management Science*.

Wang, W. (2009). An inspection model for a process with two types of inspections and repairs. *Reliability Engineering & System Safety*, 94(2):526–533.

Wang, W. (2011). An inspection model based on a three-stage failure process. *Reliability Engineering & System Safety*, 96(7):838–848.

Wang, W. (2013). Models of inspection, routine service, and replacement for a serviceable one-component system. *Reliability Engineering & System Safety*, 116:57–63.

Wang, W. and Christer, A. (1997). A modelling procedure to optimize component safety inspection over a finite time horizon. *Quality and reliability engineering international*, 13(4):217–224.

Wang, W., Zhao, F., and Peng, R. (2014). A preventive maintenance model with a two-level inspection policy based on a three-stage failure process. *Reliability Engineering & System Safety*, 121:207–220.

Wattanapanom, N. and Shaw, L. (1979). Optimal inspection schedules for failure detection in a model where tests hasten failures. *Operations Research*, 27(2):303–317.

Yeh, R. H. (1997). Optimal inspection and replacement policies for multistate deteriorating systems. *European Journal of Operational Research*, 96(2):248–259.

Yun, W. and Bai, D. (1988). Repair cost limit replacement policy under imperfect inspection. *Reliability Engineering & System Safety*, 23(1):59–64.

Yun, W. Y. and Nakagawa, T. (2010). Replacement and inspection policies for products with random life cycle. *Reliability Engineering & System Safety*, 95(3):161–165.

Zhao, X., Fouladirad, M., Bérenguer, C., and Bordes, L. (2010). Condition-based inspection/replacement policies for non-monotone deteriorating systems with environmental covariates. *Reliability Engineering & System Safety*, 95(8):921–934.

Zuckerman, D. (1978). Optimal replacement policy for the case where the damage process is a one-sided lévy process. *Stochastic Processes and their Applications*, 7(2):141–151.

Zuckerman, D. (1980). Inspection and replacement policies. *Journal of Applied Probability*, pages 168–177.

Zuckerman, D. (1989). Optimal inspection policy for a multi-unit machine. *Journal of Applied Probability*, pages 543–551.

Appendices

Appendix A

Mathematica programs

These Mathematica programs can be provided by the author upon request sent to any one of the emails:

honest.chipoyera@wits.ac.za or *hwchipoyera@gmail.com*

A.1 Mathematica program for imperfect inspections

```
Clear["Global`*"]
Clear[n]
Clear[L]
```

$\mathrm{Clear}[H]$

$n = 20;$

$\alpha = 0.2;$

$\beta = 0.2;$

$\mathrm{cR} = 1000.;$

$\mathrm{Co} = 10000.;$

$\mathrm{Cs} = 2500.;$

$\mathrm{Ci} = 2000.;$

$\mathrm{cF} = 200.;$

$H = 100.;$

$\mathrm{udist} = \mathrm{UniformDistribution}[\{0, H\}];$

$\mathrm{updfdist} = \mathrm{PDF}[\mathrm{udist}, t];$

$\mathrm{ucdfdist} = \mathrm{CDF}[\mathrm{udist}, t];$

$$\mathrm{Go}[\mathrm{L\_}] = \mathrm{Assuming}\left[\{L > 0, L \leq H\}, (\mathrm{cR} + \mathrm{cF})\left(\int_0^L t\mathrm{PDF}[\mathrm{udist}, t]dt - L\ \mathrm{CDF}[\mathrm{udist}, L]\right) + \mathrm{cR}L - (\mathrm{Co} - \mathrm{Cs})\right];$$

$\{\text{maxvalnoinspectfin}, \text{reprperf0fin}\} = \text{NMaximize}[\{\text{Go}[L], L > 0\}, \{L\}];$

$\{\text{maxvalnoinspectinf}, \text{reprperf0inf}\} = \text{NMaximize}[\{\text{Go}[L]/L, L > 0\}, \{L\}];$

$\text{G1}[\text{x1\_}, \text{x2\_}] = \text{Assuming}\left[\{\text{x2} > \text{x1}, \text{x1} > 0, \text{x2} < H\}, (\text{cR} + \text{cF})\left(\frac{\text{x1}^2}{2H} + (1-\alpha)\left(\frac{\text{x2}^2 - \text{x1}^2}{2H}\right)\right) - (1-\beta)\text{cFx1CDF}[\text{udist}, \text{x1}] - \beta\text{cFx2CDF}[\text{udist}, \text{x1}] + \right.$

$(\alpha\text{cRx1} - (1-\alpha)\text{cFx2})(\text{CDF}[\text{udist}, \text{x2}] - \text{CDF}[\text{udist}, \text{x1}]) + (\alpha\text{cRx1} - (1-\alpha)\text{cRx2})(1 - \text{CDF}[\text{udist}, \text{x2}]) - \text{Ci} - (\text{Co} - \text{Cs})];$

$\{\text{maxvalsingleinspect}, \text{reprsingleimperf1}\} = \text{NMaximize}[\{\text{G1}[\text{x1}, \text{x2}], \text{x1} > 0, \text{x1} < \text{x2}, \text{x2} < H\}, \{\text{x1}, \text{x2}\}];$

(

$\text{xvec} = \text{Table}\left[x_i, \{i, 1, n+1\}\right];$

$\text{xvec}//\text{MatrixForm};$

$\text{Clear}[\psi\text{vec}];$

$\psi\text{vec} = \text{Table}[0, \{i, 1, n+2\}];$

$\psi\text{vec}[[1]] = \text{CDF}\left[\text{udist}, x_1\right];$

$\text{For}[i = 2, i \leq n+1, i\text{++}, \psi\text{vec}[[i]] = \text{Assuming}[\{x_1 > 0, x_i > x_{i-1}, x_i < H\}, \text{CDF}\left[\text{udist}, x_i\right] - \text{CDF}\left[\text{udist}, x_{i-1}\right]]];$

$\psi\text{vec}[[n+2]] = 1 - \text{CDF}\left[\text{udist}, x_{n+1}\right];$

$\psi\text{vecc} = \text{Transpose}[\psi\text{vec}, \{1\}];)$

(

$\lambda\text{vec} = \text{Table}[0, \{i, 1, n+2\}];$

$\lambda\text{vec}[[1]] = \text{Ci}(1-\beta)\ \sum_{k=1}^{n} k\beta^{k-1} + \text{Ci}n\beta^{n};$

$\text{For}\left[i = 2, i \leq n, i{+}{+}, \lambda\text{vec}[[i]] = \text{Ci}\left(\alpha \sum_{k=1}^{i-1} k(1-\alpha)^{k-1} + (1-\beta)(1-\alpha)^{i-1} \sum_{k=i}^{n} k\beta^{k-i} + n(1-\alpha)^{i-1}\beta^{n-i+1}\right)\right];$

$\lambda\text{vec}[[n+1]] = \text{Ci}\alpha \sum_{k=1}^{n} k(1-\alpha)^{k-1} + n\text{Ci}(1-\alpha)^{n};$

$\lambda\text{vec}[[n+2]] = \text{Ci}\alpha \sum_{k=1}^{n} k(1-\alpha)^{k-1} + n\text{Ci}(1-\alpha)^{n};$

$\text{FullSimplify}[\lambda\text{vec}]//\text{MatrixForm};)$

$(n = 20;$

$v = \text{Table}[x_i, \{i, 1, n+1\}];$

$\text{constraints1} = \text{Table}[x_i > 0, \{i, n+1\}];$

$\text{constraints2} = \text{Table}[x_{i+1} - x_i > 0, \{i, n\}];$

$\text{constraints3} = \text{Table}[x_i \in \text{Reals}, \{i, 1, n+1\}];$

$\text{uvec} = \text{Table}[0, \{i, 1, n+1\}];$

$\text{uvec}[[1]] = \text{Assuming}[\{\text{constraints1}, x_1 < H, x_{n+1} < H\}, (\text{cR} + \text{cF})(\frac{(x_1)^2}{2H})];$

$\text{For}\left[i=2,i\leq n+1,i\text{++},\text{uvec}[[i]]=\text{Simplify}\left[\text{Assuming}\left[\{x_{i-1}<x_i,x_{i-1}>0,x_i>0,x_{n+1}<H\},(\text{cR}+\text{cF})\left((1-\alpha)^{i-1}\right)\left(\frac{(x_i)^2-(x_{i-1})^2}{2H}\right)\right]\right]\right];$

$\text{uvec}//\text{MatrixForm};$

$u=\sum_{i=1}^{n+1}\text{uvec}[[i]];$

$\text{FullSimplify}[u];)$

$(n=20;$

$\Phi=\text{Table}[0.0,\{n+2\},\{n+1\}];$

$\text{For}\left[j=1,j\leq n,j\text{++},\Phi[[1,j]]=\text{cF}(1-\beta)\beta^{j-1}\right];$

$\Phi[[2,1]]=-\text{cR}\alpha;$

$\text{For}\left[i=2,i\leq n,i\text{++}\left[\text{For}\left[j=1,j\leq i-1,j\text{++},\Phi[[i,j]]=-\text{cR}\alpha(1-\alpha)^{j-1}\right]\right]\right];$

$\text{For}\left[i=1,i\leq n,i\text{++}\left[\text{For}\left[j=i,j\leq n,j\text{++},\Phi[[i,j]]=\text{cF}(1-\alpha)^{i-1}(1-\beta)\beta^{j-i}\right]\right]\right];$

$\text{For}\left[i=1,i\leq n,i\text{++},\Phi[[i,n+1]]=\text{cF}(1-\alpha)^{i-1}\beta^{n+1-i}\right];$

$\text{For}\left[j=1,j\leq n,j\text{++},\Phi[[n+1,j]]=-\text{cR}\alpha(1-\alpha)^{j-1}\right];$

$\text{For}\left[j=1,j\leq n,j\text{++},\Phi[[n+2,j]]=-\text{cR}\alpha(1-\alpha)^{j-1}\right];$

$\Phi[[n+1,n+1]]=\text{cF}(1-\alpha)^n;$

$\Phi[[n+2,n+1]]=-\text{cR}(1-\alpha)^n;$

$\Phi//\text{MatrixForm};)$

$(n = 20;$

$Q = \text{Table}[0.0, \{n+2\}, \{n+1\}];$

$\text{For}\left[j = 1, j \leq n, j{+}{+}, Q[[1, j]] = (1-\beta)\beta^{j-1}\right];$

$Q[[2, 1]] = \alpha;$

$\text{For}\left[i = 2, i \leq n, i{+}{+}\left[\text{For}\left[j = 1, j \leq i-1, j{+}{+}, Q[[i, j]] = \alpha(1-\alpha)^{j-1}\right]\right]\right];$

$\text{For}\left[i = 1, i \leq n, i{+}{+}\left[\text{For}\left[j = i, j \leq n, j{+}{+}, Q[[i, j]] = (1-\alpha)^{i-1}(1-\beta)\beta^{j-i}\right]\right]\right];$

$\text{For}\left[i = 1, i \leq n, i{+}{+}, Q[[i, n+1]] = (1-\alpha)^{i-1}\beta^{n+1-i}\right];$

$\text{For}\left[j = 1, j \leq n, j{+}{+}, Q[[n+1, j]] = \alpha(1-\alpha)^{j-1}\right];$

$\text{For}\left[j = 1, j \leq n, j{+}{+}, Q[[n+2, j]] = \alpha(1-\alpha)^{j-1}\right];$

$Q[[n+1, n+1]] = (1-\alpha)^{n};$

$Q[[n+2, n+1]] = (1-\alpha)^{n};$

$Q//\text{MatrixForm};)$

(*OPTIMAL INSPECTION TIMES OF PERFECT INSPECTIONS WHICH ARE EVENLY SPREAD*)

$(\text{udist} = \text{UniformDistribution}[\{0, H\}];$

$\mathrm{updfdist} = \mathrm{PDF}[\mathrm{udist}, t];$

$\mathrm{ucdfdist} = \mathrm{CDF}[\mathrm{udist}, t];$

$\mathrm{Clear}[\mathrm{Uunif}];$

$\mathrm{Clear}[\mathrm{LL}];$

$v = \mathrm{Table}\left[x_i, \{i, 1, n+1\}\right];$

$\mathrm{constraints1} = \mathrm{Table}\left[x_i > 0, \{i, n+1\}\right];$

$\mathrm{constraints2} = \mathrm{Table}\left[x_{i+1} - x_i > 0, \{i, n\}\right];$

$\mathrm{constraints3} = \mathrm{Table}\left[x_i \in \mathrm{Reals}, \{i, 1, n+1\}\right];$

$$\mathrm{Uunif}[\mathrm{L\_}] = \mathrm{Assuming}\left[\{L > 0, L \leq H\}, (\mathrm{cR} + \mathrm{cF})\left(\int_0^L t\mathrm{PDF}[\mathrm{udist}, t]dt - L\ \mathrm{CDF}[\mathrm{udist}, L]\right) + \mathrm{cF}\left(L - \frac{nL}{n+1}\right)\mathrm{CDF}\left[\mathrm{udist}, \frac{nL}{n+1}\right] + \right.$$
$$\left.\sum_{i=1}^{n-1}\left(\mathrm{Ci} + \mathrm{cF}\left(\frac{(i+1)L}{n+1} - \frac{iL}{n+1}\right)\right)\mathrm{CDF}\left[\mathrm{udist}, \frac{iL}{n+1}\right] + \mathrm{cR}L - n\mathrm{Ci} - (\mathrm{Co} - \mathrm{Cs})\right];$$

$\{\mathrm{maxvalunif}, \mathrm{repr1}\} = \mathrm{NMaximize}[\{\mathrm{Uunif}[L], L > 0, L \leq H\}, \{L\}]$

$\mathrm{Clear}[\mathrm{LL}];$

$\text{LL} = L/.\text{Last}[\text{NMaximize}[\{\text{Uunif}[L], L > 0, L \leq H\}, \{L\}]];$

Clear[startsol];

$\text{startsol} = \text{Table}[0.0, \{1\}, \{n+1\}];$

$\text{startsol} = \text{Table}\left[N\left[\frac{j\text{LL}}{n+1}\right], \{j, 1, n+1\}\right];$

)

(*Globaloptimalplanninghorizonwhen$n$PERFECTinspectionsarescheduled*)

$(n = 20;$

Clear[gwperfect];

Print[“Profit function for n inspections at x1, x2, ..., xn”];

$\text{gwperfect}[\text{n\_}] = \text{Assuming}\left[\text{Flatten@}\{\text{constraints1}, \text{constraints2}, x_{n+1} \leq H, x_1 \geq 0\},\right.$

$(\text{cR} + \text{cF})\left(\int_0^{x_{n+1}} t\text{PDF}[\text{udist}, t]dt - x_{n+1}\ \text{CDF}[\text{udist}, x_{n+1}]\right) + \text{cF}(x_{n+1} - x_n)\,\text{CDF}[\text{udist}, x_n] + \sum_{i=1}^{n-1}(\text{Ci} + \text{cF}(x_{i+1} - x_i))\,\text{CDF}[\text{udist}, x_i] +$

$\mathrm{cR}x_{n+1} - n\mathrm{Ci} - (\mathrm{Co} - \mathrm{Cs})]\,;$

$\{\mathrm{maxvalperf}, \mathrm{reprperf}\} = \mathrm{NMaximize}\,[\{\mathrm{gwperfect}[n], \mathrm{Flatten}\,[\{\mathrm{constraints1}, \mathrm{constraints2}, x_{n+1} \leq H\}]\}\,, v, \mathrm{MaxIterations} \rightarrow 10000,$

$\mathrm{AccuracyGoal} \rightarrow 9, \mathrm{PrecisionGoal} \rightarrow 8, \mathrm{Method} \rightarrow \{\text{“NelderMead”}, \text{“InitialPoints”} \rightarrow \{\mathrm{startsol}\}\}];$

(*Global optimal planning horizon when n PERFECT inspections are scheduled*)

$\mathrm{Clear}[\mathrm{gwimperfect}];$

$v = \mathrm{Table}\,[x_i, \{i, 1, n+1\}]\,;$

$\mathrm{constraints1} = \mathrm{Table}\,[x_i < H, \{i, 1, n+1\}]\,;$

$\mathrm{constraints2} = \mathrm{Table}\,[x_{i+1} - x_i > 0, \{i, 1, n\}]\,;$

$\mathrm{constraints3} = \mathrm{Table}\,[x_i > 0, \{i, 1, n+1\}]\,;$

$\mathrm{gwimperfect}[\mathrm{n\_}] = \mathrm{Assuming}\,[\mathrm{Flatten@}\,\{\mathrm{constraints1}, \mathrm{constraints2}, \mathrm{constraints3}, x_{n+1} < H\}\,, u - \psi\mathrm{vecc}.\lambda\mathrm{vec} - \psi\mathrm{vecc}.\Phi.\mathrm{xvec} - (\mathrm{Co} - \mathrm{Cs})]\,;$

$\{\text{maxvalimperf}, \text{reprimperf}\} = \text{NMaximize}[\{\text{gwimperfect}[n], \text{Flatten}[\{\text{constraints1}, \text{constraints2}, \text{constraints3}\}]\}, v, \text{MaxIterations} \to 10000,$

$\text{AccuracyGoal} \to 9, \text{PrecisionGoal} \to 8, \text{Method} \to \{\text{“NelderMead”}, \text{“InitialPoints”} \to \{\text{startsol}\}\}];)$

Profit function for n inspections at x1, x2, ..., xn

$(n = 20;$

$W = \psi\text{vecc}.Q.\text{xvec};$

$\text{Clear}[\text{gwimperfectinf}];$

$v = \text{Table}\left[x_i, \{i, 1, n+1\}\right];$

$\text{constraints1} = \text{Table}\left[x_i < H, \{i, 1, n+1\}\right];$

$\text{constraints2} = \text{Table}\left[x_{i+1} - x_i > 0, \{i, 1, n\}\right];$

$\text{constraints3} = \text{Table}\left[x_i > 0, \{i, 1, n+1\}\right];$

$\text{gwimperfectinf}[\text{n\_}] = \text{Assuming}\left[\text{Flatten@}\{\text{constraints1}, \text{constraints2}, \text{constraints3}, x_{n+1} < H\}, (u - \psi\text{vecc}.\lambda\text{vec} - \psi\text{vecc}.\Phi.\text{xvec} - (\text{Co} - \text{Cs}))/W\right];$

$\{\text{maxvalimperfinf}, \text{reprimperfinf}\} = \text{NMaximize}[\{\text{gwimperfectinf}[n], \text{Flatten}[\{\text{constraints1}, \text{constraints2}, \text{constraints3}\}]\}, v,$

$\text{MaxIterations} \to 10000, \text{AccuracyGoal} \to 9, \text{PrecisionGoal} \to 8, \text{Method} \to \{\text{“NelderMead”}, \text{“InitialPoints”} \to \{\text{startsol}\}\}];)$

“Profit function for n inspections at x1, x2, ..., xn”

```
Print["(value of n)"n]

Print["(value of α)"α]

Print["(value of β)"β]

Print["Profit when no inspection is scheduled - finite planning horizon"]

maxvalnoinspectfin

reprperf0fin

Print["Profit when no inspection is scheduled - infinite planning horizon"]

maxvalnoinspectinf

reprperf0inf


Print["Profit when a single inspection is scheduled - finite planning horizon"]

maxvalsingleinspect

reprsingleimperf1


Print["Uniformly spread inspection times of PERFECT INSPECTIONS"]

maxvalunif

startsol
```

Print[“Global optimal inspection times of PERFECT INSPECTIONS”];

maxvalperf

reprperf

Print[“Global optimal inspection times of IMPERFECT INSPECTIONS over a finite planning horizon”];

maxvalimperf

reprimperf

Print[“Global optimal inspection times of IMPERFECT INSPECTIONS over and infinite planning horizon”];

maxvalimperfinf

reprimperfinf

ListPlot[{startsol, Table[Subscript[x, i], {$i, 1, n + 1$}]/.reprperf, Table[Subscript[x, i], {$i, 1, n + 1$}]/.reprimperf,

Table[Subscript[x, i], {$i, 1, n + 1$}]/.reprimperfinf}, AxesLabel $\rightarrow$ {“Inspection number”, “Inspection time”}]

Export[“hwcoutput.txt”, {n, α, β, maxvalnoinspect, reprperf0, maxvalunif, maxvalperf, Table[Subscript[x, i], {$i, 1, n + 1$}]/.reprperf,

maxvalimperf, Table[Subscript[x, i], {$i, 1, n + 1$}]/.reprimperf, maxvalimperfinf, Table[Subscript[x, i], {$i, 1, n + 1$}]/.reprimperf}];

A.2 Mathematica Program for Determining Optimum Inspection Permutation

$c_R = 20;$

$\alpha = 0.1;$

$R = \text{Table}\,[R_i, \{i, 1, 6\}]\,/.\,\{R_1 \to 100, R_2 \to 100, R_3 \to 0, R_4 \to 0, R_5 \to 100, R_6 \to 0\}\,;$

$Z = \text{Table}\,[Z_i, \{i, 1, 6\}]\,/.\,\{Z_1 \to 100, Z_2 \to 100, Z_3 \to 0, Z_4 \to 0, Z_5 \to 100, Z_6 \to 0\}\,;$

“Matrix ZZ of repair times”

ZZ = Permutations[{Z1, Z2, Z3, Z4, Z5, Z6}];

ZZ = ZZ/.{Z1 → 100, Z2 → 100, Z3 → 0, Z4 → 0, Z5 → 100, Z6 → 0};

ZZ//MatrixForm;

Dimensions[ZZ];

“Matrix RR of repair costs”

$\mathrm{RR} = \mathrm{Permutations}[\{\mathrm{R1}, \mathrm{R2}, \mathrm{R3}, \mathrm{R4}, \mathrm{R5}, \mathrm{R6}\}];$

$\mathrm{RR} = \mathrm{RR}/.\{\mathrm{R1} \to 100, \mathrm{R2} \to 100, \mathrm{R3} \to 0, \mathrm{R4} \to 0, \mathrm{R5} \to 100, \mathrm{R6} \to 0\};$

$\mathrm{RR}//\mathrm{MatrixForm};$

$\mathrm{Dimensions}[\mathrm{RR}];$

$\theta\theta = \mathrm{Permutations}[\{\theta 1, \theta 2, \theta 3, \theta 4, \theta 5, \theta 6\}];$

$\theta\theta = \theta\theta/.\{\theta 1 \to 1/100, \theta 2 \to 1/150, \theta 3 \to 1/200, \theta 4 \to 1/150, \theta 5 \to 1/50, \theta 6 \to 1/100\};$

$\theta\theta//\mathrm{MatrixForm};$

$\mathrm{Dimensions}[\theta\theta];$

$\mathrm{Clear}[\theta\mathrm{v}];$

$\theta\mathrm{v} = \mathrm{Table}\left[\theta_k, \{k, 1, 6\}\right] /. \left\{\theta_1 \to 1/100, \theta_2 \to 1/150, \theta_3 \to 1/200, \theta_4 \to 1/150, \theta_5 \to 1/50, \theta_6 \to 1/100\right\};$

$\theta\mathrm{v};$

$\theta = \mathrm{Total}[\theta\mathrm{v}];$

θ

“Parameters of exponential distribution lifetimes”

$P = \theta\mathrm{v}/\theta$;

PP = Permutations[{P1, P2, P3, P4, P5, P6}];

PP = PP/.{P1 → 6/35, P2 → 4/35, P3 → 3/35, P4 → 4/35, P5 → 12/35, P6 → 6/35};

PP//MatrixForm;

Dimensions[PP];

"Probabilities of machine failure caused by the different units"

"Parameters of exponential distribution lifetimes"

"Probabilities of machine failure caused by the different units"

$T = \mathrm{Table}\,[T_k, \{k, 1, 6\}] \,/. \,\{T_1 \to 3, T_2 \to 2, T_3 \to 4, T_4 \to 6, T_5 \to 3, T_6 \to 4\}$;

TT = Permutations[{T1, T2, T3, T4, T5, T6}];

TT = TT/.{T1 → 3, T2 → 2, T3 → 4, T4 → 6, T5 → 5, T6 → 4};

TT//MatrixForm;

Dimensions[TT];

$\mathrm{CC} = \mathrm{Permutations}[\{\mathrm{C1}, \mathrm{C2}, \mathrm{C3}, \mathrm{C4}, \mathrm{C5}, \mathrm{C6}\}];$

$\mathrm{CC} = \mathrm{CC}/.\{\mathrm{C1} \to 4, \mathrm{C2} \to 3, \mathrm{C3} \to 5, \mathrm{C4} \to 3, \mathrm{C5} \to 7, \mathrm{C6} \to 4\};$

CC//MatrixForm;

Dimensions[CC];

$\mathrm{AA} = \mathrm{Permutations}\left[\{a_1, a_2, a_3, a_4, a_5, a_6\}\right];$

AA//MatrixForm;

Dimensions[AA];

$\mathrm{Tcum} = \mathrm{Table}[0.0, \{720\}, \{6\}];$

$V = \mathrm{Table}[0.0, \{720\}, \{6\}];$

$\mathrm{For}\left[i = 1, i \leq 720, i{+}{+}, \mathrm{For}\left[j = 1, j \leq 6, j{+}{+}, V[[i,j]] = N\left[\sum_{k=1}^{j} \mathrm{CC}[[i,k]]\mathrm{TT}[[i,k]]\right]\right]\right];$

$\mathrm{Dimensions}[V];$

$V//\mathrm{MatrixForm};$

$\mathrm{For}\left[i = 1, i \leq 720, i{+}{+}, \mathrm{For}\left[j = 1, j \leq 6, j{+}{+}, \mathrm{Tcum}[[i,j]] = N\left[\sum_{k=1}^{j} \mathrm{TT}[[i,k]]\right]\right]\right];$

Tcum//MatrixForm;

$\mathbf{ECost} = \mathbf{Table}[0.0, \{720\}];$

$\mathbf{For}\left[i = 1, i \leq 720, i{+}{+}, \mathbf{ECost}[[i]] = N\left[\sum_{j=1}^{6} V[[i,j]]\mathbf{PP}[[i,j]]\right]\right];$

ECost//MatrixForm;

Dimensions[ECost];

Matrix ZZ of repair times

Matrix RR of repair costs

$\mathbf{ETime} = \mathbf{Table}[0.0, \{720\}];$

$\mathbf{For}\left[i = 1, i \leq 720, i{+}{+}, \mathbf{ETime}[[i]] = N\left[\sum_{j=1}^{6} \mathbf{Tcum}[[i,j]]\mathbf{PP}[[i,j]]\right]\right];$

ETime//MatrixForm;

"Long-run average net income under the inspection strategies"

$\mathbf{Clear}[\psi];$

$\psi = \mathbf{Table}[0.0, \{720\}, \{1\}];$

$\mathbf{For}\left[i = 1, i \leq 720, i{+}{+}, \psi[[i]] = N\left[\frac{\frac{c_R}{\theta} - \mathbf{ECost}[[i]] - \sum_{k=1}^{6} P[[k]]R[[k]]}{\frac{1}{\theta} + \mathbf{ETime}[[i]] + \sum_{k=1}^{6} P[[k]]Z[[k]]}\right]\right];$

ETime//MatrixForm;

ψ**//MatrixForm;**

$\text{Clear}[\text{H1}];$

$\text{H1} = \text{Table}[0.0, \{720\}, \{6\}];$

$\text{For}[i = 1, i \leq 720, i{+}{+}, \text{H1}[[i, 1]] = N[\text{CC}[[i, 1]](1 - \text{Exp}[-\alpha\text{TT}[[i, 1]]])]];$

$\text{H1}//\text{MatrixForm};$

$\text{Clear}[\text{H2}];$

$\text{H2} = \text{Table}[0.0, \{720\}, \{6\}];$

$\text{For}[i = 1, i \leq 720, i{+}{+},$

$\text{For}\left[j = 2, j \leq 6, j{+}{+}, \text{H2}[[i, j]] = N[\text{CC}[[i, 1]](1 - \text{Exp}[-\alpha\text{TT}[[i, 1]]])] + N\left[\sum_{k=2}^{j} \text{CC}[[i, k]]\text{Exp}[-\alpha\text{Tcum}[[i, k-1]]](1 - \text{Exp}[-\alpha\text{TT}[[i, k]]])\right]\right]\Big];$

$\text{H2}//\text{MatrixForm};$

$\text{H3} = \text{Table}[0.0, \{720\}, \{6\}];$

$\text{For}[i = 1, i \leq 720, i{+}{+}, \text{For}[j = 1, j \leq 6, j{+}{+}, \text{H3}[[i, j]] = N[\text{RR}[[i, j]]\text{Exp}[-\alpha\text{Tcum}[[i, j]]](1 - \text{Exp}[-\alpha\text{ZZ}[[i, j]]])]]];$

$\text{H3}//\text{MatrixForm};$

$H = N[\text{H1} + \text{H2} + \text{H3}];$

$H//\text{MatrixForm};$

Clear[η];

$\eta = \text{Table}[0.0, \{720\}, \{1\}]$;

For $\left[i = 1, i \leq 720, i{+}{+}, \eta[[i]] = N\left[\frac{c_R}{\alpha+\theta} - \frac{1}{\alpha(\alpha+\theta)} \sum_{j=1}^{6} \theta\theta[[i,j]]H[[i,j]]\right] / N\left[\frac{1}{\theta} + \sum_{j=1}^{6}(\text{PP}[[i,j]](\text{Tcum}[[i,j]] + \text{ZZ}[[i,j]]))\right]\right]$;

η//MatrixForm;

"Long-run average net income under the inspection strategies for the discounted case"

Clear[Optsorteddiscounted];

Optsoldiscounted = Transpose[Join[Transpose[AA], {η}]];

Optsoldiscounted//MatrixForm;

Optsorteddiscounted = SortBy[Optsoldiscounted, Last];

josd = Join[Optsorteddiscounted[[;;5]], Optsorteddiscounted[[716;;]]]//MatrixForm;

"Long-run average net income under the inspection strategies for the undiscounted case"

Clear[Optsortedundiscounted];

```
Optsolundiscounted = Transpose[Join[Transpose[AA], {ψ}]];

Optsolundiscounted//MatrixForm;

Optsortedundiscounted = SortBy[Optsolundiscounted, Last];

josu = Join[Optsortedundiscounted[[;;5]], Optsortedundiscounted[[716;;]]]//MatrixForm;


Row@{josd, josu}


(*Optimalsolutions = Join[Transpose[Optsortedundiscounted], Transpose[Optsorteddiscounted]];

Optimalsolutions//MatrixForm; *)
```

Appendix B

Optimal scheduling of imperfect inspections for different error sizes when system time to failure follows a uniform distribution

Table B.1: Inspection times (for imperfect inspections) when time to failure follows a uniform distribution: $\alpha = \beta = 0$

α, β	Planning horizon	n	G	Inspection times and planning horizon/cycle length
0.0, 0.0	Finite	0	34166.67	L = 83.33
		1	35500,00	x1 = 50.00, x2 = 100.00
		2	36068.80	x_1=38.125,x_2=66.25,x_3=94.375
		3	35833.33	x_1=36.67,x_2=63.33,x_3=80.,x_4=96.67
		4	35592.31	x_1=37.6923,x_2=65.3846,x_3=83.0769,x_4=90.7692,x_5=98.4615
		5	35500.00	x_1=40.00,x_2=70.00,x_3=90.00,x_4=10.00,x_5=100.00,x_6=100.00
		10	35500.00	$x_1 = 40.00, x_2 = 70.00, x_3 = 90.00, x_4 = 100.00, x_5 = 100.00, x_6 = 100.00, x_7 = 100.00, x_8 = 100.00, x_9 = 100.00, x_{10} = 100.00, x_{11} = 100.00$
		15	35500.00	$x_1 = 40.00, x_2 = 70.00, x_3 = 90.00, x_4 = 100.00, x_5 = 100.00, x_6 = 100.00, x_7 = 100.00, x_8 = 100.00, x_9 = 100.00, x_{10} = 100.00, x_{11} = 100.00, x_{12} = 100.00, x_{13} = 100.00, x_{14} = 100.00, x_{15} = 100.00, x_{16} = 100.00$
		20	35500.00	$x_1 = 40.00, x_2 = 70.00, x_3 = 90.00, x_4 = 100.00, x_5 = 100.00, x_6 = 100.00, x_7 = 100.00, x_8 = 100.00, x_9 = 100.00, x_{10} = 100.00, x_{11} = 100.00, x_{12} = 100.00, x_{13} = 100.00, x_{14} = 100.00, x_{15} = 100.00, x_{16} = 100.00, x_{17} = 100.00, x_{18} = 100.00, x_{19} = 100.00, x_{20} = 100.00, x_{21} = 100.00$
	Infinite	0	575.74	$L = 35.36$
		1		
		2	619.1260	$x_1 = 21.38, x_2 = 40.32, x_3 = 59.26$
		3	618.0719	$x_1 = 19.98, x_2 = 37.51, x_3 = 52.60, x_4 = 67.68$
		4	613.6636	$x_1 = 19.37, x_2 = 36.28, x_3 = 50.74, x_4 = 62.74, x_5 = 74.73$
		5	608.0905	$x_1 = 19.23, x_2 = 35.99, x_3 = 50.28, x_4 = 62.09, x_5 = 71.42, x_6 = 80.75$
		10	592.2637	$x_1 = 21.21, x_2 = 39.89, x_3 = 56.05, x_4 = 69.69, x_5 = 80.80, x_6 = 89.39, x_7 = 95.45, x_8 = 98.99, x_9 = 100.00, x_{10} = 100.00, x_{11} = 100.00$
		15	592.2637	$x_1 = 21.21, x_2 = 39.89, x_3 = 56.05, x_4 = 69.69, x_5 = 80.80, x_6 = 89.39, x_7 = 95.45, x_8 = 98.99, x_9 = 100.00, x_{10} = 100.00, x_{11} = 100.00, x_{12} = 100.00, x_{13} = 100.00, x_{14} = 100.00, x_{15} = 100.00, x_{16} = 100.00$
		20	592.2637	$x_1 = 21.21, x_2 = 39.89, x_3 = 56.05, x_4 = 69.69, x_5 = 81.00, x_6 = 89.39, x_7 = 95.45, x_8 = 98.99, x_9 = 100.00, x_{10} = 100.00, x_{11} = 100.00, x_{12} = 100.00, x_{13} = 100.00, x_{14} = 100.00, x_{15} = 100.00, x_{16} = 100.00, x_{17} = 100.00, x_{18} = 100.00, x_{19} = 100.00, x_{20} = 100.00, x_{21} = 100.00$

Table B.2: Inspection times (for imperfect inspections) when system time to failure follows a uniform distribution: $\alpha = 0, \beta = 0.2$

α,β		n	G	Inspection times and planning horizon/cycle length
0.0,0.2		0	34166.67	$L = 83.33$
		1		
		2	35128.07	$x_1 = 39.04, x_2 = 61.93, x_3 = 92.63$
		3	34833.33	$x_1 = 37.50, x_2 = 59.17, x_3 = 74.17, x_4 = 95.00$
		4	34502.30	$x_1 = 37.93, x_2 = 59.94, x_3 = 75.29, x_4 = 83.97, x_5 = 96.90$
		5	34254.90	$x_1 = 39.46, x_2 = 62.70, x_3 = 79.26, x_4 = 89.17, x_5 = 92.41, x_6 = 98.53$
		10	34156.07	$x_1 = 41.48, x_2 = 66.33, x_3 = 84.51, x_4 = 96.02, x_5 = 100.00, x_6 = 100.00, x_7 = 100.00, x_8 = 100.00, x_9 = 100.00, x_{10} = 100.00, x_{11} = 100.00$
		15	34156.06	$x_1 = 41.48, x_2 = 66.33, x_3 = 84.51, x_4 = 96.02, x_5 = 100.00, x_6 = 100.00, x_7 = 100.00, x_8 = 100.00, x_9 = 100.00, x_{10} = 100.00, x_{11} = 100.00, x_{12} = 100.00, x_{13} = 100.00, x_{14} = 100.00, x_{15} = 100.00, x_{16} = 100.00$
		20	34156.06	$x_1 = 41.48, x_2 = 66.33, x_3 = 84.51, x_4 = 96.02, x_5 = 100.00, x_6 = 100.00, x_7 = 100.00, x_8 = 100.00, x_9 = 100.00, x_{10} = 100.00, x_{11} = 100.00, x_{12} = 100.00, x_{13} = 100.00, x_{14} = 100.00, x_{15} = 100.00, x_{16} = 100.00, x_{17} = 100.00, x_{18} = 100.00, x_{19} = 100.00, x_{20} = 100.00, x_{21} = 100.00$
		0	575.74	$L = 35.36$
		1		
		2	591.1834	$x_1 = 2130, x_2 = 36.23, x_3 = 55.43$
		3	587.1581	$x_1 = 20.36, x_2 = 34.52, x_3 = 47.00, x_4 = 63.12$
		4	581.0141	$x_1 = 19.93, x_2 = 33.73, x_3 = 45.83, x_4 = 56.22, x_5 = 69.75$
		5	574.1031	$x_1 = 19.83, x_2 = 33.53, x_3 = 45.52, x_4 = 55.78, x_5 = 64.33, x_6 = 75.54$
		10	548.3949	$x_1 = 21.82, x_2 = 37.04, x_3 = 50.48, x_4 = 62.15, x_5 = 72.03, x_6 = 80.12, x_7 = 86.44, x_8 = 90.98, x_9 = 93.73, x_{10} = 94.70, x_{11} = 96.48$
		15	547.2990	$x_1 = 22.50, x_2 = 38.27, x_3 = 52.25, x_4 = 64.45, x_5 = 74.87, x_6 = 83.50, x_7 = 90.35, x_8 = 95.41, x_9 = 98.69, x_{10} = 100.00, x_{11} = 100.00, x_{12} = 100.00, x_{13} = 100.00, x_{14} = 100.00, x_{15} = 100.00, x_{16} = 100.00$
		20	547.2988	$x_1 = 2250, x_2 = 38.27, x_3 = 52.25, x_4 = 64.45, x_5 = 74.87, x_6 = 83.50, x_7 = 90.35, x_8 = 95.41, x_9 = 98.68894293036743, x_{10} = 100.00, x_{11} = 100.00, x_{12} = 100.00, x_{13} = 100.00, x_{14} = 100.00, x_{15} = 100.00, x_{16} = 100.00, x_{17} = 100.00, x_{18} = 100.00, x_{19} = 100.00, x_{20} = 100.00, x_{21} = 100.00$

Table B.3: Inspection times (for imperfect inspections) when system time to failure follows a uniform distribution: $\alpha = 0.2, \beta = 0.0$

α, β	Planning horizon	n	G	Inspection times and planning horizon/cycle length
0.2, 0.0	Finite	0	34166.67	$L = 83.33$
		1		
		2	35128.07	$x_1 = 39.04, x_2 = 61.93, x_3 = 92.63$
		3	34833.33	$x_1 = 37.50, x_2 = 59.17, x_3 = 74.17, x_4 = 95.00$
		4	34502.30	$x_1 = 37.93, x_2 = 59.94, x_3 = 75.29, x_4 = 83.97, x_5 = 96.90$
		5	34254.90	$x_1 = 39.46, x_2 = 62.70, x_3 = 79.26, x_4 = 89.17, x_5 = 92.40, x_6 = 98.53$
		10	34156.07	$x_1 = 41.48, x_2 = 66.33, x_3 = 84.51, x_4 = 96.02, x_5 = 100.00, x_6 = 100.00, x_7 = 100.00, x_8 = 100.00, x_9 = 100.00, x_{10} = 100.00, x_{11} = 100.00$
		15	34156.06	$x_1 = 41.48, x_2 = 66.33, x_3 = 84.51, x_4 = 96.02, x_5 = 100.00, x_6 = 100.00, x_7 = 100.00, x_8 = 100.00, x_9 = 100.00, x_{10} = 100.00, x_{11} = 100.00, x_{12} = 100.00, x_{13} = 100.00, x_{14} = 100.00, x_{15} = 100.00, x_{16} = 100.00$
		20		
	Infinite	0	575.7359	$L = 35.36$
		1		
		2	591.1834	$x_1 = 21.30, x_2 = 36.23, x_3 = 55.43$
		3	587.1581	$x_1 = 20.36, x_2 = 34.52, x_3 = 47.00, x_4 = 63.12$
		4	581.0141	$x_1 = 19.93, x_2 = 33.73, x_3 = 45.83, x_4 = 56.22, x_5 = 69.75$
		5	574.1031	$x_1 = 19.83, x_2 = 33.53, x_3 = 45.52, x_4 = 55.78, x_5 = 64.33, x_6 = 75.54$
		10	548.3949	$x_1 = 21.82, x_2 = 37.04, x_3 = 50.48, x_4 = 62.15, x_5 = 72.03, x_6 = 80.12, x_7 = 86.44, x_8 = 90.98, x_9 = 93.73, x_{10} = 94.70, x_{11} = 96.48$
		15	547.3000	$x_1 = 22.50, x_2 = 38.27, x_3 = 52.25, x_4 = 64.45, x_5 = 74.87, x_6 = 83.50, x_7 = 90.35, x_8 = 95.41, x_9 = 98.69, x_{10} = 100.00, x_{11} = 100.00, x_{12} = 100.00, x_{13} = 100.00, x_{14} = 100.00, x_{15} = 100.00, x_{16} = 100.00$
		20	547.2990	$x_1 = 22.50, x_2 = 38.27, x_3 = 52.25, x_4 = 64.45, x_5 = 74.87, x_6 = 83.50, x_7 = 90.35, x_8 = 95.41, x_9 = 98.69, x_{10} = 100.00, x_{11} = 100.00, x_{12} = 100.00, x_{13} = 100.00, x_{14} = 100.00, x_{15} = 100.00, x_{16} = 100.00, x_{17} = 100.00, x_{18} = 100.00, x_{19} = 100.00, x_{20} = 100.00, x_{21} = 100.00$

Table B.4: Inspection times (for imperfect inspections) when system time to failure follows a uniform distribution: $\alpha = \beta = 0.2$

α,β	Planning horizon	n	G	Inspection times and planning horizon/cycle length
0.2,0.2	Finite	0	34166.67	$L = 83.33$
		1	33166.67	x1 = 66.67, x2 = 100.00
		2	32972.88	$x_1 = 65.33, x_2 = 81.90, x_3 = 95.61253561246455$
		3	32702.04	$x_1 = 66.06, x_2 = 83.82, x_3 = 91.22, x_4 = 97.87$
		4	32591.38	$x_1 = 66.46, x_2 = 84.87, x_3 = 93.73, x_4 = 96.21, x_5 = 99.08$
		5	32556.91	$x_1 = 66.65, x_2 = 85.36, x_3 = 94.92, x_4 = 98.97, x_5 = 98.97, x_6 = 99.74$
		10	32550.45	$x_1 = 66.71, x_2 = 85.54, x_3 = 95.35, x_4 = 100.00, x_5 = 100.00, x_6 = 100.00, x_7 = 100.00, x_8 = 100.00, x_9 = 100.00, x_{10} = 100.00, x_{11} = 100.00$
		15	32550.44	$x_1 = 66.71, x_2 = 85.54, x_3 = 95.35, x_4 = 100.00, x_5 = 100.00, x_6 = 100.00, x_7 = 100.00, x_8 = 100.00, x_9 = 100.00, x_{10} = 100.00, x_{11} = 100.00, x_{12} = 100.00, x_{13} = 100.00, x_{14} = 100.00, x_{15} = 100.00, x_{16} = 100.00$
		20	32550.44	$x_1 = 66.71, x_2 = 85.54, x_3 = 95.35, x_4 = 100.00, x_5 = 100.00, x_6 = 100.00, x_7 = 100.00, x_8 = 100.00, x_9 = 100.00, x_{10} = 100.00, x_{11} = 100.00, x_{12} = 100.00, x_{13} = 100.00, x_{14} = 100.00, x_{15} = 100.00, x_{16} = 100.00, x_{17} = 100.00, x_{18} = 100.00, x_{19} = 100.00, x_{20} = 100.00, x_{21} = 100.00$
	Infinite	0	575.7359	$L = 35.36$
		1		
		2	562.5062	$x_1 = 26.52, x_2 = 41.63, x_3 = 59.46$
		3	555.0774	$x_1 = 26.74, x_2 = 41.83, x_3 = 53.92, x_4 = 68.24$
		4	549.1827	$x_1 = 27.06, x_2 = 42.29, x_3 = 54.48, x_4 = 63.99, x_5 = 75.34$
		5	544.8751	$x_1 = 27.36, x_2 = 42.77, x_3 = 55.14, x_4 = 64.87, x_5 = 72.20, x_6 = 81.05$
		10	537.6207	$x_1 = 28.07, x_2 = 43.98, x_3 = 56.94, x_4 = 67.47, x_5 = 75.96, x_6 = 82.72, x_7 = 87.95, x_8 = 91.78, x_9 = 94.23, x_{10} = 95.18, x_{11} = 96.74$
		15	537.3690	$x_1 = 28.13, x_2 = 44.09, x_3 = 57.13, x_4 = 67.77, x_5 = 76.42, x_6 = 83.41, x_7 = 89.01, x_8 = 93.40, x_9 = 96.71, x_{10} = 98.96, x_{11} = 100., x_{12} = 100., x_{13} = 100.00, x_{14} = 100.00, x_{15} = 100.00, x_{16} = 100.$
		20	537.3690	$x_1 = 28.13, x_2 = 44.09, x_3 = 57.13, x_4 = 67.77, x_5 = 76.42, x_6 = 83.41, x_7 = 89.01, x_8 = 93.40, x_9 = 96.71, x_{10} = 98.96, x_{11} = 100.00, x_{12} = 100.00, x_{13} = 100.00, x_{14} = 100.00, x_{15} = 100.00, x_{16} = 100.00, x_{17} = 100.00, x_{18} = 100.00, x_{19} = 100.00, x_{20} = 100.00, x_{21} = 100$

Appendix C

Optimal scheduling of imperfect inspections for different error sizes when system time to failure follows an exponential distribution

Table C.1: Inspection times (for imperfect inspections) when system time to failure follows an exponential distribution: $\alpha = \beta = 0$

α, β	Planning horizon	n	G	Inspection times and planning horizon/cycle length
0, 0	Finite	0	24582.41	$L = 89.59$
		5	30169.17	$x_1 = 40.40, x_2 = 81.33, x_3 = 123.45, x_4 = 168.29, x_5 = 219.62, x_6 = 309.21$
		10	30252.86	$x_1 = 39.99, x_2 = 79.98, x_3 = 120.00, x_4 = 160.06, x_5 = 200.23, x_6 = 240.63, x_7 = 281.56, x_8 = 323.68, x_9 = 368.52, x_{10} = 419.85, x_{11} = 509.44$
		15	30254.39	$x_1 = 39.98, x_2 = 79.96, x_3 = 119.93, x_4 = 159.91, x_5 = 199.90, x_6 = 239.88, x_7 = 279.88, x_8 = 319.89, x_9 = 359.96, x_{10} = 400.13, x_{11} = 440.53, x_{12} = 481.46, x_{13} = 523.58, x_{14} = 568.42, x_{15} = 619.75, x_{16} = 709.34$
		20	30254.42	$x_1 = 39.98, x_2 = 79.96, x_3 = 119.94, x_4 = 159.91, x_5 = 199.89, x_6 = 239.87, x_7 = 279.85, x_8 = 319.82, x_9 = 359.80, x_{10} = 399.79, x_{11} = 439.77, x_{12} = 479.77, x_{13} = 519.78, x_{14} = 559.85, x_{15} = 600.02, x_{16} = 640.42, x_{17} = 681.35, x_{18} = 723.47, x_{19} = 768.30, x_{20} = 819.64, x_{21} = 909.22$
	Infinite	0	452.42	$L = 30.47$
		5	456.69	$x_1 = 23.47, x_2 = 46.96, x_3 = 70.47, x_4 = 94.01, x_5 = 117.60, x_6 = 147.74$
		10	456.89	$x_1 = 23.45, x_2 = 46.91, x_3 = 70.36, x_4 = 93.82, x_5 = 117.28, x_6 = 140.75, x_7 = 164.23, x_8 = 187.73, x_9 = 211.27, x_{10} = 234.85, x_{11} = 264.97$
		15	456.91	$x_1 = 23.45, x_2 = 46.90, x_3 = 70.35, x_4 = 93.80, x_5 = 117.25, x_6 = 140.70, x_7 = 164.16, x_8 = 187.61, x_9 = 211.07, x_{10} = 234.53, x_{11} = 258.00, x_{12} = 281.48, x_{13} = 304.98, x_{14} = 328.51, x_{15} = 352.09, x_{16} = 382.22$
		20	456.91	$x_1 = 23.47, x_2 = 46.93, x_3 = 70.39, x_4 = 93.86, x_5 = 117.21, x_6 = 140.82, x_7 = 164.04, x_8 = 187.36, x_9 = 210.68, x_{10} = 234.08, x_{11} = 257.47, x_{12} = 280.80, x_{13} = 304.24, x_{14} = 328.24, x_{15} = 351.69, x_{16} = 375.41, x_{17} = 399.12, x_{18} = 422.44, x_{19} = 445.84, x_{20} = 468.85, x_{21} = 492.63$

Table C.2: Inspection times (for imperfect inspections) when system time to failure follows an exponential distribution: $\alpha = 0.0, \beta = 0.05$

α, β	Planning horizon	n	G	Inspection times and planning horizon/cycle length
0, 0.05	Finite	0	24582.4053	$L = 89.59$
		2	28619.6453	$x_1 = 44.56, x_2 = 90.91, x_3 = 176.41$
		3	29155.4019	$x_1 = 42.16, x_2 = 83.52, x_3 = 129.89, x_4 = 215.37$
		5	29520.3712	$x_1 = 40.53, x_2 = 78.55, x_3 = 117.62, x_4 = 158.98, x_5 = 205.35, x_6 = 290.83$
		10	29623.1492	$x_1 = 40.06, x_2 = 77.16, x_3 = 114.28, x_4 = 151.45, x_5 = 188.74, x_6 = 226.26, x_7 = 264.29, x_8 = 303.36, x_9 = 344.72, x_{10} = 391.09, x_{11} = 476.57$
		15	29625.66	$x_1 = 40.05, x_2 = 77.13, x_3 = 114.20, x_4 = 151.27, x_5 = 188.35, x_6 = 225.44, x_7 = 262.53, x_8 = 299.65, x_9 = 336.8275295718677, x_{10} = 374.11, x_{11} = 411.64, x_{12} = 449.66, x_{13} = 488.74, x_{14} = 530.09, x_{15} = 576.46, x_{16} = 661.95$
		20		$x_1 = 40.05, x_2 = 77.12, x_3 = 114.20, x_4 = 151.27, x_5 = 188.34, x_6 = 225.42, x_7 = 262.49, x_8 = 299.56, x_9 = 336.64, x_{10} = 373.72, x_{11} = 410.80, x_{12} = 447.90, x_{13} = 485.02, x_{14} = 522.12, x_{15} = 559.48, x_{16} = 597.00, x_{17} = 635.02, x_{18} = 674.10, x_{19} = 715.45, x_{20} = 761.82, x_{21} = 847.31$
	Infinite	0	452.4247	$L = 30.47$
		2	443.91	$x_1 = 23.44, x_2 = 45.25, x_3 = 74.92$
		3	442.70	$x_1 = 23.57, x_2 = 45.57, x_3 = 67.44, x_4 = 97.19$
		5	441.63	$x_1 = 23.68, x_2 = 45.84, x_3 = 67.95, x_4 = 89.99, x_5 = 111.90, x_6 = 141.74$
		10	441.06	$x_1 = 23.75, x_2 = 45.99, x_3 = 68.23, x_4 = 90.47, x_5 = 112.69, x_6 = 134.89, x_7 = 157.06, x_8 = 179.19, x_9 = 201.24, x_{10} = 223.18, x_{11} = 253.06$
		15	441.00	$x_1 = 23.75, x_2 = 46.01, x_3 = 68.26, x_4 = 90.52, x_5 = 112.77, x_6 = 135.02, x_7 = 157.27, x_8 = 179.51, x_9 = 201.74, x_{10} = 223.96, x_{11} = 246.17, x_{12} = 268.34, x_{13} = 290.47, x_{14} = 312.52, x_{15} = 334.47, x_{16} = 364.35$
		20	440.99	$x_1 = 23.75, x_2 = 46.01, x_3 = 68.27, x_4 = 90.52, x_5 = 112.78, x_6 = 135.03, x_7 = 157.29, x_8 = 179.54, x_9 = 201.80, x_{10} = 224.05, x_{11} = 246.30, x_{12} = 268.55, x_{13} = 290.79, x_{14} = 313.02, x_{15} = 335.24, x_{16} = 357.45, x_{17} = 379.62, x_{18} = 401.75, x_{19} = 423.81, x_{20} = 445.75, x_{21} = 475.63$

Table C.3: Inspection times (for imperfect inspections) when time to failure follows an exponential distribution: $\alpha = 0.05, \beta = 0.00$

α, β	Planning horizon	n	G	Inspection times and planning horizon/cycle length
0.05, 0.0	Finite	0	24582.41	$L = 89.59$
		5	28904.41	$x_1 = 46.73, x_2 = 93.69, x_3 = 141.30, x_4 = 190.63, x_5 = 244.75, x_6 = 334.34$
		10	28932.52	$x_1 = 46.59, x_2 = 93.18, x_3 = 139.77, x_4 = 186.38, x_5 = 233.02, x_6 = 279.74, x_7 = 326.71, x_8 = 374.31, x_9 = 423.64, x_{10} = 477.77, x_{11} = 567.36$
		15	28932.72	$x_1 = 46.59, x_2 = 93.17, x_3 = 139.76, x_4 = 186.35, x_5 = 232.93, x_6 = 279.52, x_7 = 326.11, x_8 = 372.70, x_9 = 419.31, x_{10} = 465.95, x_{11} = 512.68, x_{12} = 559.64, x_{13} = 607.24, x_{14} = 656.57, x_{15} = 710.70, x_{16} = 800.28$
		20	28932.73	$x_1 = 46.59, x_2 = 93.17, x_3 = 139.76, x_4 = 186.35, x_5 = 232.93, x_6 = 279.52, x_7 = 326.10, x_8 = 372.69, x_9 = 419.28, x_{10} = 465.86, x_{11} = 512.45, x_{12} = 559.04, x_{13} = 605.63, x_{14} = 652.24, x_{15} = 698.87, x_{16} = 745.58, x_{17} = 792.50, x_{18} = 839.97, x_{19} = 889.03, x_{20} = 943.12, x_{21} = 1031.04$
	Infinite	0	452.42	$L = 30.47$
		5	447.43	$x_1 = 24.53, x_2 = 49.04, x_3 = 73.53, x_4 = 97.99, x_5 = 122.38, x_6 = 153.24$
		10	447.25	$x_1 = 24.55, x_2 = 49.09, x_3 = 73.63, x_4 = 98.18, x_5 = 122.71, x_6 = 147.24, x_7 = 171.76, x_8 = 196.26, x_9 = 220.72, x_{10} = 245.18, x_{11} = 276.00$
		15	447.24	$x_1 = 24.55, x_2 = 49.09, x_3 = 73.64, x_4 = 98.19, x_5 = 122.73, x_6 = 147.28, x_7 = 171.83, x_8 = 196.37, x_9 = 220.91, x_{10} = 245.45, x_{11} = 269.98, x_{12} = 294.50, x_{13} = 319.00, x_{14} = 343.46, x_{15} = 367.87, x_{16} = 398.73$
		20	447.24	$x_1 = 24.55, x_2 = 49.09, x_3 = 73.64, x_4 = 98.19, x_5 = 122.74, x_6 = 147.28, x_7 = 171.83, x_8 = 196.38, x_9 = 220.92, x_{10} = 245.47, x_{11} = 270.02, x_{12} = 294.56, x_{13} = 319.11, x_{14} = 343.65, x_{15} = 368.19, x_{16} = 392.72, x_{17} = 417.24, x_{18} = 441.73, x_{19} = 466.20, x_{20} = 490.60, x_{21} = 521.47$

Table C.4: Inspection times (for imperfect inspections) when time to failure follows a exponential distribution: $\alpha = \beta = 0.05$

α, β	Planning horizon	n	G	Inspection times and planning horizon/cycle length
0.05,0.05	Finite	0	24582.4053	$L = 89.59$
		5	28225.88	$x_1 = 46.73, x_2 = 89.79, x_3 = 133.41, x_4 = 178.42, x_5 = 226.96, x_6 = 311.85$
		10	28261.95	$x_1 = 46.58, x_2 = 89.26, x_3 = 131.95, x_4 = 174.66, x_5 = 217.40, x_6 = 260.23, x_7 = 303.29, x_8 = 346.91, x_9 = 391.92, x_{10} = 440.46, x_{11} = 525.35$
		15	28262.34	$x_1 = 46.57, x_2 = 89.25, x_3 = 131.94, x_4 = 174.62, x_5 = 217.30, x_6 = 259.98, x_7 = 302.67, x_8 = 345.36, x_9 = 388.06, x_{10} = 430.81, x_{11} = 473.64, x_{12} = 516.70, x_{13} = 560.32, x_{14} = 605.33, x_{15} = 653.87, x_{16} = 738.76$
		20		$x_1 = 46.57, x_2 = 89.25, x_3 = 131.94, x_4 = 174.62, x_5 = 217.30, x_6 = 259.98, x_7 = 302.66, x_8 = 345.34, x_9 = 388.02, x_{10} = 430.70, x_{11} = 473.39, x_{12} = 516.07, x_{13} = 558.76, x_{14} = 601.47, x_{15} = 644.21, x_{16} = 687.04, x_{17} = 730.10, x_{18} = 773.72, x_{19} = 818.74, x_{20} = 867.28, x_{21} = 952.17$
	Infinite	0	452.4247	$L = 30.47$
		5	432.03	$x_1 = 24.77, x_2 = 47.84, x_3 = 70.82, x_4 = 93.66, x_5 = 116.27, x_6 = 146.71$
		10	431.20	$x_1 = 24.87, x_2 = 48.06, x_3 = 71.25, x_4 = 94.43, x_5 = 117.59, x_6 = 140.73, x_7 = 163.81, x_8 = 186.82, x_9 = 209.69, x_{10} = 232.34, x_{11} = 262.84$
		15	431.13	$x_1 = 24.87, x_2 = 48.08, x_3 = 71.28, x_4 = 94.49, x_5 = 117.69, x_6 = 140.89, x_7 = 164.09, x_8 = 187.28, x_9 = 210.46, x_{10} = 233.63, x_{11} = 256.76, x_{12} = 279.85, x_{13} = 302.85, x_{14} = 325.73, x_{15} = 348.38, x_{16} = 378.89$
		20	431.13	$x_1 = 24.87, x_2 = 48.08, x_3 = 71.29, x_4 = 94.49, x_5 = 117.70, x_6 = 140.90, x_7 = 164.11, x_8 = 187.32, x_9 = 210.52, x_{10} = 233.72, x_{11} = 256.92, x_{12} = 280.12, x_{13} = 303.31, x_{14} = 326.49, x_{15} = 349.66, x_{16} = 372.79, x_{17} = 395.88, x_{18} = 418.89, x_{19} = 441.76, x_{20} = 464.41, x_{21} = 494.92$