RIGHT DISTRIBUTIVELY GENERATED NEAR-RINGS AND THEIR LEFT/RIGHT REPRESENTATIONS

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Declaration

I declare that this thesis is my own, unaided work. It is being submitted for the degree of Doctor of Philosophy in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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_____ day of _______________ 2006

Abstract

For right near-rings the left representation has always been considered the natural one. A study of right representation for right distributively generated (d.g.) near-rings was initiated by Rahbari and this work is extended here to introduce radical-like objects in the near-ring R using right R-groups. The right radicals ${}^rJ_0(R)$, ${}^rJ_{1/2}(R)$ and ${}^rJ_2(R)$ are defined as counterparts of the left radicals $J_0(R)$, $J_{1/2}(R)$ and $J_2(R)$ respectively, and their properties are discussed. Of particular interest are the relationships between the left and right radicals. It is shown for example that for all finite d.g. near-rings R with identity, $J_2(R) = {}^rJ_0(R) = {}^rJ_{1/2}(R) = {}^rJ_2(R)$. A right anti-radical, ${}^rSoi(R)$, is defined for d.g. near-rings with identity, using a construction that is analogous to that of the (left) socle-ideal, $Soi(R)$. In particular, it is shown that for finite d.g. near-rings with identity, an ideal A is contained in ${}^rSoi(R)$ if and only if $A \cap J_2(R) = (0)$. The relationship between the left and right socle-ideals is investigated, and it is established that ${}^rSoi(R) \subseteq Soi(R)$ for d.g. near-rings with identity and satisfying the descending chain condition for left R-subgroups.

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Contents

Chapter 1

Introduction

If R is a ring with identity, then several characterizations of the Jacobson radical of R, $J(R)$, exist. Amongst others, each of the following subsets of R is equal to $J(R)$:

- $\mathcal{J}_1$: the intersection of all maximal left ideals of R
- $\mathcal{J}_1^*$: the intersection of all maximal right ideals of R
- $\mathcal{J}_2$: the intersection of annihilators of all irreducible left R-modules
- $\mathcal{J}_2^*$: the intersection of annihilators of all irreducible right R-modules
- $\mathcal{J}_3$: a left quasi-regular left ideal that contains every left quasi-regular left ideal of R
- $\mathcal{J}_3^*$: a right quasi-regular right ideal that contains every right quasi-regular right ideal of R.

A notable feature of the Jacobson radical is the left-right symmetry of its characterizations. In the case of a (non-ring) right near-ring R, the left characterizations of $J(R)$ can result in a number of distinct Jacobson-type radicals. In right near-rings with identity, the sets $\mathcal{J}_1$ and $\mathcal{J}_3$ both represent the left ideal $J_{1/2}(R)$. As addition in near-rings is not commutative, there is a distinction between R-subgroups and R-submodules, resulting in near-ring modules of type-0 and those of type-2. Hence the set $\mathcal{J}_2$ can characterize either $J_0(R)$ or $J_2(R)$. The properties of and relationships between these left radicals are well-known, but the sets $\mathcal{J}_1^*$, $\mathcal{J}_2^*$ and $\mathcal{J}_3^*$ have not been extensively investigated as they involve the right representation for right near-rings which is considered the unnatural one. In this thesis we examine the sets $\mathcal{J}_1^*$, $\mathcal{J}_2^*$ and $\mathcal{J}_3^*$ in the case of right distributively generated (d.g.) near-rings and seek to find relationships between these sets and their left counterparts. In order to do so, we first consider the formation of a right representation theory for such near-rings. There is a small body of work that is related to right representation for right near-rings. Fröhlich [4], Laxton [13], [14], Rahbari [21], Tharmaratinam [25], [26], and Grainger [6] have considered right R-groups for such near-rings under various special conditions. It is the right representation theory for right d.g. near-rings initiated by Rahbari that we use and extend here.

All near-rings R in this thesis are assumed to be right distributive and distributively generated by the semi-group S. If R satisfies the descending chain condition for left ideals (right ideals), we will say R satisfies the DCCL (respectively, DCCR). If R satisfies the descending chain condition for left R-subgroups, we will say that R satisfies the DCCLS.

The next chapter builds on Rahbari's definitions and results to develop a representation theory that parallels, where possible, the already well-established left representation theory for right d.g. near-rings. The use of a distributively generated right R-group, defined by Rahbari [21], is pivotal to the establishment of a right representation theory and, in particular, a right radical theory. Concepts of right primitive ideals, right modular right ideals and right quasi-regularity are introduced in this chapter in an analogous way to the notions from left representation theory, and many of the standard results for left R-groups are modified and proved for right R-groups. Various types of right R-groups are defined, the annihilators of which will be used to construct the right Jacobson-type radicals.

The third chapter is devoted to right radical theory. The right radical ${}^{r}J_0(R)$ was defined in [10] in the case of d.g. near-rings R with identity as an analogue of the left radical $J_0(R)$, and was shown to be the largest two-sided ideal contained in ${}^{r}J_{1/2}(R)$, the intersection of all maximal right ideals of R. In this chapter we define the right 2-radical for arbitrary d.g. near-rings by using annihilators of certain right R-groups. At the same time, we extend the definitions of ${}^{r}J_0(R)$ and ${}^{r}J_{1/2}(R)$ to include d.g. near-rings R without identity. We explore some of the properties of the right radicals; of particular interest are the relationships between the left and right radicals. In ring theory, the Jacobson radical that is defined using left R-modules coincides with the radical that is defined using right R-modules, and we investigate the possibility of the same being true in the near-ring case. It was shown in [23] for d.g. near-rings with identity and satisfying the DCCLS that $J_0(R) \subseteq {}^{r}J_0(R)$ and the relationship was improved in [10] where it was shown under the same assumptions that $J_2(R) = {}^{r}J_0(R)$. In this chapter we show in the case of finite d.g. near-rings with identity that all the right radicals coincide and are equal to the left radical $J_2(R)$.

Examples of right R-groups for right near-rings are presented in Chapter 4, where the structure of the d.g. near-ring T considered by Laxton [15] is examined in some detail. We further investigate the relationship between T and the matrix near-ring, $\mathbb{M}_n(R)$, defined by Meldrum and van der Walt [19], and show that $\mathbb{M}_n(R)$ can be expressed as a direct sum of normal right $\mathbb{M}_n(R)$-groups. Chapter 4 ends in showing that T is isomorphic to a generalized matrix near-ring.

Hartney [8] constructed an antiradical, the (left) socle-ideal, and in Chapter 5 we adapt this construction in order to define the right socle-ideal. The chapter is concerned with the antiradical properties of the right socle-ideal and with the connection between the left and right socle-ideals.

Some future problems relevant to the work in this thesis are discussed in the concluding chapter.

Chapter 2

Preliminaries of right representation theory

All near-rings R in this thesis are assumed to be right distributive and distributively generated by the semi-group S. If R satisfies the descending chain condition for left ideals (right ideals), we will say R satisfies the DCCL (respectively, DCCR). If R satisfies the descending chain condition for left R-subgroups, we will say that R satisfies the DCCLS.

Rings have both left and right modules, and for each result on right modules there is a corresponding left result. In particular, the same Jacobson radical arises whether left or right modules are used. Modules over a near-ring R have similarly been defined, but, in contrast to ring theory where either left or right modules can be used, only one type of module for a near-ring is usually considered. Due to the fact that one distributive law is missing, left R-groups are the natural choice for right near-rings, and for left near-rings right R-groups are used. To date, little work on the alternative 'unnatural'

R-group has been done. Rahbari [21] initiated a right representation theory for right distributively generated (d.g.) near-rings and these definitions and results were extended in [10] and [23] with the aim to develop a right radical theory. Only right d.g. near-rings with identity were considered in [10] and [23]. In this work we include right d.g. near-rings without identity. The distributively generated right R-group is central to the development of the theory and so we start with its definition.

2.1 Right R-groups

Definition 2.1 An additive group Ω is a **right R-group** if there is a mapping $\Omega \times R \to \Omega$ such that for all $\omega, \omega_1, \omega_2 \in \Omega,\ x, y \in R$,

(i) $\omega(xy) = (\omega x)y$

(ii) $(\omega_1 + \omega_2)x = \omega_1 x + \omega_2 x$.

Note that this is the corresponding definition for a right R-module from ring theory with the exception of the axiom $\omega(x+y) = \omega x + \omega y$ for all $\omega \in \Omega$ and $x, y \in R$. Also note that this definition does not depend on the semi-group S of the d.g. near-ring R, and could therefore apply to a general near-ring. However, with the view to generating a radical theory later on, we require additional conditions to be fulfilled as a right R-group defined above will not meet our requirements. Although the above ring-theoretic condition may not necessarily be satisfied by all the elements of Ω, we require that it be satisfied by certain elements of Ω. We also require that R be a d.g. near-ring. Thus we have

Definition 2.2 [21] Let R be a d.g. near-ring. A right R-group Ω is said to be **distributively generated (d.g.)** if it is additively generated by $\Delta \subseteq \Omega$ where

(i) $\delta(x+y) = \delta x + \delta y$ for all $\delta \in \Delta$ and $x, y \in R$, and

(ii) $\Delta S \subseteq \Delta$.

If R has identity 1, then we assume that $\omega 1 = \omega$ for all $\omega \in \Omega$. That is, Ω is unital.

The existence of a set Δ is crucial to right representation theory. We call Δ a **distributive basis** of Ω.

It is clear that R^+ is a d.g. right R-group with distributive basis S and that a right ideal is a right R-group but is not necessarily distributively generated.

The earliest work relating to right representation for right near-rings was done by H. Neumann [20] who constructed a right near-ring R by writing endomorphisms on the right of a reduced free group Ω. The details of the construction are taken from [18].

Example 2.3 Let Ω be a group with a relatively free basis Δ. Thus every map from Δ into Ω can be extended uniquely to an endomorphism of Ω. Define multiplication on $\mathrm{End}(\Omega)$ as mapping composition and define addition by the rule:

given $\alpha, \beta \in \mathrm{End}(\Omega)$ and $x \in \Delta$, then $\alpha + \beta$ is the endomorphism of Ω obtained by extending the map

$$x(\alpha + \beta) = x\alpha + x\beta$$

from Δ into Ω.

This addition is sometimes referred to as **addition of the second type** [3]. With these definitions of addition and multiplication $R := \mathrm{End}(\Omega)$ is a right near-ring, called the **Neumann near-ring on Ω with respect to Δ.** If, in particular, $\Delta = \{x_1, \ldots, x_n\}$ is finite, then R is a d.g. near-ring with distributive semi-group $S := \{\varepsilon^i_j \mid 1 \leq i, j \leq n\}$, where ε^i_j is defined by

$$x_k \varepsilon^i_j = \begin{cases} 0 & \text{if } k \neq i \\ x_j & \text{if } k = i \end{cases}.$$

Finally, if we define a map $\Omega \times R \to \Omega$ using the above action of ε^i_j on the elements of Δ, then it can be seen that Ω is a d.g. right R-group with distributive basis Δ.

Example 2.4 Let $R = \{n_0, \ldots, n_7\}$, where $+$ and $\cdot$ are given in the following tables.

$+$	n_0	n_1	n_2	n_3	n_4	n_5	n_6	n_7
n_0	n_0	n_1	n_2	n_3	n_4	n_5	n_6	n_7
n_1	n_1	n_0	n_3	n_2	n_5	n_4	n_7	n_6
n_2	n_2	n_6	n_0	n_4	n_3	n_7	n_1	n_5
n_3	n_3	n_7	n_1	n_5	n_2	n_6	n_0	n_4
n_4	n_4	n_5	n_6	n_7	n_0	n_1	n_2	n_3
n_5	n_5	n_4	n_7	n_6	n_1	n_0	n_3	n_2
n_6	n_6	n_2	n_4	n_0	n_7	n_3	n_5	n_1
n_7	n_7	n_3	n_5	n_1	n_6	n_2	n_4	n_0

$\cdot$	n_0	n_1	n_2	n_3	n_4	n_5	n_6	n_7
n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0
n_1	n_0	n_0	n_1	n_1	n_0	n_0	n_1	n_1
n_2	n_0	n_0	n_2	n_2	n_0	n_0	n_2	n_2
n_3	n_0	n_0	n_3	n_3	n_0	n_0	n_3	n_3
n_4	n_0	n_0	n_4	n_4	n_0	n_0	n_4	n_4
n_5	n_0	n_0	n_5	n_5	n_0	n_0	n_5	n_5
n_6	n_0	n_0	n_6	n_6	n_0	n_0	n_6	n_6
n_7	n_0	n_0	n_7	n_7	n_0	n_0	n_7	n_7

Note that $(R, +)$ is the octic group and $R = \text{LibraryNearRing}(\text{GTW8\_4}, 765)$ in [1]. The near-ring R is d.g. with $S = \{n_0, n_1, n_2, n_4, n_5, n_7\}$, and does not possess an identity element. It is not left distributive as $n_3(n_2 + n_2) \neq n_3n_2 + n_3n_2$. The group $\{n_0, n_2\}$ is a d.g. right R-group with distributive basis $\{n_0, n_2\}$. Notice that $\{n_0, n_2\}$ is not a left R-group. The right ideal $\{n_0, n_3, n_5, n_6\}$ is a right R-group but is neither a d.g. right R-group nor a left R-group. The ideal $\{n_0, n_1, n_4, n_5\}$ is a left R-group and a d.g. right R-

group with distributive basis $\{n_0, n_1, n_4, n_5\}$. The set $\{n_0, n_1, n_4\}$ also forms a distributive basis since $n_1 + n_4 = n_5$.

Definition 2.5 Let R be a d.g. near-ring.

(i) An element ω of a right R-group Ω is said to be **distributive** if $\omega(x + y) = \omega x + \omega y$ for all $x, y \in R$.

(ii) A d.g. right R-group Ω is **monogenic** if there is a distributive element ω of Ω such that $\Omega = \omega R$.

Note that if ω is not distributive, then ωR need not be a subgroup of Ω as we see in the following example.

Example 2.6 Consider the d.g. near-ring R in Example 2.4, page 8. The d.g. right R-group $\{n_0, n_5\} = n_5 R$ is monogenic, but is not monogenic as a left R-group because $Rn_5 = \{n_0\}$. Notice that the element n_3 is not distributive and that the set $\{n_0, n_3\} = n_3 R$ is closed under multiplication by R on the right, but is not a subgroup of R.

Definition 2.7 Let R be a d.g. near-ring. A subgroup Γ of a right R-group Ω is called a **right R-subgroup** of Ω if $\Gamma R \subseteq \Gamma$. If, in addition, Γ has a distributive basis then Γ is a **d.g. right R-subgroup of** Ω.

Homomorphisms between right R-groups are defined in the obvious way and the homomorphism theorems correspond to those for left R-groups.

Definition 2.8 Let R be a d.g. near-ring and let Ω and Ω' be right R-groups.

A group homomorphism $\phi : \Omega \to \Omega'$ is called a **right R-homomorphism** if

$$\phi(\omega r) = (\phi(\omega))r \quad \text{for all } \omega \in \Omega \text{ and } r \in R.$$

2.2 Constructing new right R-groups from old

There are a number of ways of constructing right R-groups. We discuss these using distributive monogenic generators, factor groups and direct sums.

If ω is an element of a right R-group and r an element of R then it is easily verified that $(-\omega)r = -(\omega r)$. However, due to the absence of the left distributive law, it may not be true that $\omega(-r) = -(\omega r)$. In particular, we have the following.

Lemma 2.9 Let R be a d.g. near-ring and Ω be a right R-group. If $\omega \in \Omega$ is distributive and $r \in R$ then

$$\omega(-r) = -(\omega r).$$

Proof

$$\omega(-r) + \omega r = \omega(-r + r) = 0.$$

■

We know that the set ωR is a d.g. right R-group if ω is distributive. More generally we have

Proposition 2.10 Let R be a d.g. near-ring, K a (d.g.) right R-subgroup of

R^+ and ω a distributive element of a right R-group Ω. Then ωK is a (d.g.) right R-subgroup of Ω. If, in particular, K is a right ideal of R and $\Omega = \omega R$ is a monogenic right R-group then ωK is a normal right R-subgroup of Ω.

Proof If $\omega k_1, \omega k_2 \in \omega K$ where $k_1, k_2 \in K$, then, using Lemma 2.9, $\omega k_1 - \omega k_2 = \omega k_1 + \omega(-k_2) = \omega(k_1 - k_2) \in \omega K$. Clearly $(\omega K)R \subseteq \omega K$, so ωK is a right R-subgroup of Ω. Next suppose that K is a d.g. right R-subgroup of R^+ with distributive basis $\Delta = \{\delta_i\}_{i \in I}$. For $\omega k \in \omega K,\ k \in K$ we have

$$\begin{aligned} \omega k &= \omega(\Sigma_{j\in J}\ \epsilon_j \delta_j) && \text{for some } J \subseteq I \text{ and } \epsilon_j = \pm 1 \\ &= \Sigma_{j \in J}\ \omega(\epsilon_j \delta_j) && \text{since } \omega \text{ is distributive} \\ &= \Sigma_{j \in J}\ \epsilon_j(\omega \delta_j) && \text{using Lemma 2.9.} \end{aligned}$$

Hence $\omega\Delta = \{\omega\delta_i \mid \delta_i \in \Delta\}$ is a distributive basis for ωK. Finally, to prove normality, let $\omega k \in \omega K$ and $\omega r \in \Omega$ where $k \in K$ and $r \in R$. Then

$$-(\omega r) + \omega k + \omega r = \omega(-r) + \omega k + \omega r = \omega(-r + k + r) \in \omega K,$$

where we have again used Lemma 2.9. ∎

It is seen from this that every monogenic right R-group is a d.g. right R-group.

The following is a generalization of a result proved in [23].

Proposition 2.11 Let R be a d.g. near-ring, Ω a right R-group and Γ a normal right R-subgroup of Ω. Then Ω/Γ is a right R-group, called the **right factor R-group of Ω by** Γ, where we define $(\omega+\Gamma)r = \omega r + \Gamma$ for $\omega + \Gamma \in \Omega/\Gamma$ and $r \in R$. Moreover, if Ω is a d.g. right R-group with distributive basis Δ then Ω/Γ is a d.g. right R-group with distributive basis $\{\delta + \Gamma \mid \delta \in \Delta\}$.

Example 2.12 Let R be the d.g. near-ring in Example 2.4, page 8. The set $I = \{n_0, n_5\}$ is an ideal of R and $R/I = \{I, n_1 + I, n_2 + I, n_3 + I\}$ is a d.g. right R-group with distributive basis $S/I = \{I, n_1 + I, n_2 + I\}$. Notice that $R/I = R(n_2 + I)$ is a monogenic left R-group, but is not monogenic as a right R-group.

Proposition 2.13 Let R be a d.g. near-ring and $\{\Omega_i\}_{i \in I}$ be a set of normal right R-subgroups of a right R-group Γ. If

$$\Omega = \sum_{i \in I} \Omega_i \quad \text{and} \quad \Omega_k \bigcap \sum_{\substack{i \in I \\ i \neq k}} \Omega_i = (0) \text{ for each } k \in I,$$

then Ω is a right R-subgroup of Γ. We call Ω a **direct sum** of the Ω_i and write $\Omega = \bigoplus_{i \in I} \Omega_i$. If, in addition, each Ω_i is a d.g. right R-group with distributive basis Δ_i, then Ω is a d.g. right R-group with distributive basis $\cup_{i \in I} \Delta_i$.

Lemma 2.14 Let R be a d.g. near-ring. Let Ω be a right R-group and $\Omega_1, \ldots, \Omega_k$ be normal right R-subgroups of Ω such that $\Omega = \Omega_1 \oplus \cdots \oplus \Omega_k$. If $\omega \in \Omega$ where $\omega = \omega_1 + \cdots + \omega_k,\ \omega_i \in \Omega_i$, then ω is distributive if and only if ω_i is distributive for $i = 1, \ldots, k$.

Proof Firstly, if $\omega = \omega_1 + \cdots + \omega_k$ is distributive and $x, y \in R$ then

$$
\begin{aligned}
&\omega_1(x+y) + \omega_2(x+y) + \cdots + \omega_k(x+y)\\
&= (\omega_1 + \omega_2 + \cdots + \omega_k)(x+y)\\
&= \omega(x+y)\\
&= \omega x + \omega y\\
&= (\omega_1 + \omega_2 + \cdots + \omega_k)x + (\omega_1 + \omega_2 + \cdots + \omega_k)y\\
&= \omega_1 x + \omega_2 x + \cdots + \omega_k x + \omega_1 y + \omega_2 y + \cdots + \omega_k y\\
&= (\omega_1 x + \omega_1 y) + (\omega_2 x + \omega_2 y) + \cdots + (\omega_k x + \omega_k y),
\end{aligned}
$$

since $\omega_i x + \omega_j y = \omega_j y + \omega_i x$ if $i \neq j$. In addition, $\omega_i(x+y) \in \Omega_i$ for each i and by the uniqueness of expression as a sum of elements from different Ω_i, we conclude that $\omega_i(x+y) = \omega_i x + \omega_i y$, so each ω_i is distributive. For the converse, if each ω_i is distributive then

$$
\begin{aligned}
&\omega(x+y)\\
&= (\omega_1 + \omega_2 + \cdots + \omega_k)(x+y)\\
&= \omega_1(x+y) + \omega_2(x+y) + \cdots + \omega_k(x+y)\\
&= \omega_1 x + \omega_1 y + \omega_2 x + \omega_2 y + \cdots + \omega_k x + \omega_k y\\
&= (\omega_1 x + \omega_2 x + \cdots + \omega_k x) + (\omega_1 y + \omega_2 y + \cdots + \omega_k y)\\
&= (\omega_1 + \omega_2 + \cdots + \omega_k)x + (\omega_1 + \omega_2 + \cdots + \omega_k)y\\
&= \omega x + \omega y,
\end{aligned}
$$

where again we have used the fact that $\omega_i x, \omega_i y \in \Omega_i$ and that addition of elements from distinct Ω_i is commutative. Thus ω is distributive. ■

2.3 Right annihilators

Definition 2.15 Let R be a d.g. near-ring, Ω be a right R-group and X and Y be non-empty subsets of Ω. We define

$$(X : Y) = \{r \in R \mid Xr \subseteq Y\}.$$

The case $Y = (0)$ is of particular importance. For this situation we write $(X : 0)$; similarly we write $(x : Y)$ if $X = \{x\}$ is a singleton. The following lemma is easily verified.

Lemma 2.16 Let R be a d.g. near-ring. If ω is a distributive element of a right R-group Ω then $(\omega : 0)$ is a right ideal of R. Moreover, if Ω is a d.g. right R-group then $(\Omega : 0)$ is a two-sided ideal of R and $(\Omega : 0) = (\Delta : 0)$ for any distributive basis Δ of Ω.

Definition 2.17 Let R be a d.g. near-ring and let ω be a distributive element of a d.g. right R-group Ω. We call

(i) $(\omega : 0)$ the **annihilating right ideal of** ω

(ii) $(\Omega : 0)$ the **annihilating ideal of** Ω

(iii) Ω **faithful** if $(\Omega : 0) = (0)$.

One reason why a distributive basis for a d.g. right R-group is essential to right representation theory is that if Ω is a right R-group that is not d.g., then $(\Omega : 0)$ need not be an ideal of R. We see in the next chapter how annihilating ideals of certain d.g. right R-groups play an important role in right radical theory.

Proposition 2.18 If Y is a right ideal of a d.g. near-ring R then $(R : Y)$ is an ideal of R, and if I is an ideal of R contained in Y then $I \subseteq (R : Y)$.

Proof We have $(R : Y) = (R/Y : 0)$ which is an ideal of R because R/Y is a d.g. right R-group by Proposition 2.11. Furthermore, $RI \subseteq I \subseteq Y$, giving $I \subseteq (R : Y)$. ∎

Proposition 2.19 Let R be a d.g. near-ring and let ω be a distributive element of a right R-group Ω. Then $R/(\omega : 0) \cong \omega R$.

Proof The mapping $r \mapsto \omega r$ for all $r \in R$ is a right R-homomorphism of R^+ onto ωR with kernel $(\omega : 0)$. Hence $R/(\omega : 0) \cong \omega R$. ∎

2.4 Right primitivity

We shall use two classes of monogenic right R-groups when defining the right radicals, namely type-0 and type-2. An analogue of a left R-group of type-0 was given in [10] for d.g. near-rings with identity.

Definition 2.20 Let R be a d.g. near-ring. A monogenic right R-group $\Omega \neq (0)$ is said to be **(right) type-**0 if Ω does not possess any proper, non-zero normal right R-subgroups.

This definition is motivated by the fact that for a zero-symmetric near-ring R with identity, a unital left R-group Ω is of type-0 if and only if $\Omega \cong R/M$

where M is a maximal left ideal of R. For the case of right representation, if X is a maximal right ideal of R then the right factor R-group R/X has no proper, non-zero normal right R-subgroups.

In order to define the concept of right type-2, we note the following in connection with a unital left R-group Ω of type-2 of a zero-symmetric near-ring R with identity: for any $\omega \in \Omega,\ \omega \neq 0$, we have $R\omega = \Omega$. This idea is used in the definition of a right R-group of type-2, Γ, as follows. If γ is a non-zero distributive element in Γ, we should like γR to be equal to Γ. But γR is a d.g. right R-subgroup of Γ and this suggests that (0) and Γ are the only d.g. right R-subgroups of Γ. Thus we have the following.

Definition 2.21 [11] Let R be a d.g. near-ring. A monogenic right R-group $\Omega \neq (0)$ is said to be **(right) type-**2 if

(i) Ω does not possess any proper, non-zero normal right R-subgroups, and

(ii) Ω does not possess any proper, non-zero d.g. right R-subgroups.

Remark 2.22 (a) Property (i) in the above definition ensures that every right R-group of type-2 is also of type-0.

(b) A right R-group of type-2 is equivalent to a strongly minimal right R-group in [21].

(c) In [21], a d.g. right R-group Ω is irreducible if Ω does not possess any proper, non-zero normal d.g. right R-subgroups. Clearly a right R-group of type-0 is irreducible, but an irreducible d.g. right R-group may not be right type-0. Rahbari also defined a minimal d.g. right

R-group to have no proper, non-zero d.g. right R-subgroups.

We note that a right R-group Ω of type-ν ($\nu = 0, 2$) is monogenic and therefore is distributively generated. Hence $(\Omega : 0)$ is an ideal of R.

Definition 2.23 [10] For $\nu = 0, 2$, an ideal of a d.g. near-ring R is **right ν-primitive** if it is the annihilating ideal of a right R-group of type-ν. The d.g. near-ring R is **right ν-primitive** if there is a faithful right R-group of type-ν.

The following proposition was proved for d.g. near-rings with identity [23]. Using a proof that mirrors that for left representation [15], the result also holds for d.g. near-rings without identity.

Proposition 2.24 Let R be a d.g. near-ring. Every right 0-primitive ideal of R is a prime ideal.

The following lemma was proved by Hartney [8] for the left case; the proof carries over almost verbatim for right representation.

Lemma 2.25 Let R be a d.g. near-ring. Let $\Omega = \oplus_{i \in I} \Omega_i$ be a d.g. right R-group where each Ω_i is a normal right R-subgroup of Ω of type-0. Then any proper, non-zero normal right R-subgroup of Ω is a direct summand of Ω and is right R-isomorphic to a direct sum of some of the Ω_i.

Proof Let Δ be a proper, non-zero normal right R-subgroup of Ω and let X be the collection of all normal right R-subgroups Γ of Ω which are of the form

$$\Gamma = \Delta \oplus (\oplus_{k \in K} \Omega_k) \quad \text{where} \quad K \subseteq I.$$

Partially order X by

$$\Gamma_1 = \Delta \oplus (\oplus_{k\in K_1}\Omega_k) \leq \Gamma_2 = \Delta \oplus (\oplus_{k\in K_2}\Omega_k)$$

if and only if $K_1 \subseteq K_2$. Apply Zorn's Lemma to obtain a maximal element of X, say $\Gamma^* = \Delta \oplus (\oplus_{j\in J}\Omega_j)$. If $\Gamma^* \neq \Omega$, then there exists Ω_i such that $\Omega_i \not\subseteq \Gamma^*$. So $\Omega_i \cap \Gamma^* = (0)$, since $\Omega_i \cap \Gamma^*$ is a proper normal right R-subgroup of Ω_i and Ω_i is right type-0. But then we would have $\Gamma^* < \Delta \oplus (\oplus_{j\in J}\Omega_j) \oplus \Omega_i \in X$, contradicting the maximality of Γ^*. Thus $\Gamma^* = \Omega$. Therefore Δ is a direct summand of Ω and $\Delta \cong \Omega / \oplus_{j\in J} \Omega_j \cong \oplus_{i\notin J}\Omega_i$. ■

We conclude this section by using an example to illustrate some of the concepts defined thus far.

Example 2.26 Let R be the d.g. near-ring in Example 2.4, page 8. We begin by considering the right structures of R. The following is a complete list of the proper, non-zero right R-subgroups of R^+:

$$\begin{aligned} &\Omega_1 = n_5R = \{n_0, n_5\}, && \Omega_2 = n_1R = \{n_0, n_1\}, \\ &\Omega_3 = n_4R = \{n_0, n_4\}, && \Omega_4 = n_2R = \{n_0, n_2\}, \\ &\Omega_5 = n_7R = \{n_0, n_7\}, && \Omega_6 = \{n_0, n_1, n_4, n_5\}, \\ &\Omega_7 = \{n_0, n_3, n_5, n_6\}, && \Omega_8 = \{n_0, n_2, n_5, n_7\}. \end{aligned}$$

Of these, Ω_1 and Ω_6 are ideals of R, and Ω_7 and Ω_8 are right ideals. Therefore the right factor R-groups are

$$R/\Omega_1 = \{\Omega_1, n_1 + \Omega_1, n_2 + \Omega_1, n_3 + \Omega_1\},$$
$$R/\Omega_6 = (n_2 + \Omega_6)R = \{\Omega_6, n_2 + \Omega_6\},$$
$$R/\Omega_7 = (n_1 + \Omega_7)R = \{\Omega_7, n_1 + \Omega_7\},$$
$$R/\Omega_8 = (n_1 + \Omega_8)R = \{\Omega_8, n_1 + \Omega_8\}.$$

With the exception of Ω_7, all of the right R-groups are distributively generated. All of the monogenic right R-groups are of type-0, in fact of type-2, and the ideal Ω_6 is the annihilating ideal of each. Therefore Ω_6 is the only right 0-primitive ideal of R and the only right 2-primitive ideal of R. Furthermore, the only monogenic left R-group is R/Ω_6 and it is of type-2. The left annihilator of R/Ω_6 is the ideal Ω_6. Therefore Ω_6 is both a left 2-primitive ideal and a right 2-primitive ideal of R.

2.5 Right modular right ideals

If X is a maximal right ideal of R then the right factor R-group R/X does not possess any proper, non-zero normal right R-subgroups. If, in addition, R has identity 1 then $R/X = (1 + X)R$ is monogenic and therefore is of right type-0. However, if R does not have an identity then R/X need not be monogenic and, if not, is not of type-0. Thus we introduce the concept of modularity. The remaining definitions and results of this chapter appear in [11].

Definition 2.27 Let R be a d.g. near-ring. A right ideal X of R is called **right modular** if there exists $e \in R$ where $-er + r \in X$ for all $r \in R$. In this case we call e a **left identity modulo** X.

If R has identity 1, then every right ideal X of R is right modular with 1 a left identity modulo X.

Lemma 2.28 Let X be a right modular right ideal of a d.g. near-ring R. Then R/X is a monogenic right R-group.

Proof If $e \in R$ is a left identity modulo X, then it is easily verified that $R/X = (e+X)R$. To show that $e+X$ is distributive, let $r_1, r_2 \in R$. Then

$$\begin{aligned}(e+X)(r_1+r_2) &= e(r_1+r_2)+X \\ &= (r_1+r_2)-(r_1+r_2)+e(r_1+r_2)+X \\ &= r_1+r_2+X,\end{aligned}$$

as $-r+er \in X$ for all $r \in R$. Similarly,

$$\begin{aligned}(e+X)r_1+(e+X)r_2 &= (er_1+er_2)+X \\ &= (r_1-r_1)+er_1+(r_2-r_2)+er_2+X \\ &= r_1+(-r_1+er_1)+r_2+(-r_2+er_2)+X \\ &= r_1+r_2+X.\end{aligned}$$

■

Lemma 2.29 Let R be a d.g. near-ring and let Ω be a monogenic right R-group. Then $\Omega \cong R/X$ where X is a right modular right ideal of R.

Proof Write $\Omega = \omega R$ with ω distributive. Then $\Omega \cong R/(\omega : 0)$, where $r \mapsto \omega r$ by Proposition 2.19. We show that the right ideal $(\omega : 0)$ is right modular. Now $\omega \in \Omega$, so there exists $e \in R$ such that $e \mapsto \omega$. Also $e \mapsto \omega e$, so $\omega = \omega e$. Then for all $r \in R$, $\omega r = (\omega e)r$, giving $\omega(-er+r) = 0$ and

$-er + r \in (\omega : 0)$. Consequently, e is a left identity modulo $(\omega : 0)$. ■

As seen in Proposition 2.18, for any right ideal X of R, $(R : X)$ is an ideal of R that contains every two-sided ideal contained in X. However, if R does not have identity then $(R : X)$ itself need not be contained in X. We show that if X is a right modular right ideal of R then $(R : X) \subseteq X$, and in this case we can speak of $(R : X)$ as being the unique largest ideal of R contained in the right ideal X.

Lemma 2.30 Let X be a right modular right ideal of a d.g. near-ring R. Then $(R : X) \subseteq X$.

PROOF Let $e \in R$ be a left identity modulo X and let $r \in (R : X)$. Then $Rr \subseteq X$, so in particular $er \in X$. Also $-er + r \in X$, which gives $r \in X$. ■

Definition 2.31 For $\nu = 0, 2$, a right ideal X of a d.g. near-ring R is **right ν-modular** if X is right modular and R/X is a right R-group of type-ν.

Clearly every right 0-modular right ideal is a maximal right ideal of R and if R has identity then every maximal right ideal is right 0-modular. Likewise if R has identity, then a right ideal X is right 2-modular if and only if X is a maximal right ideal and R/X has no proper, non-zero d.g. right R-subgroups. We emphasise that if R does not have an identity element then these statements need not be true, for example, consider the following near-ring.

Example 2.32 Let $R = \{n_0, \ldots, n_7\}$, where $+$ and $\cdot$ are given in the following tables.

$+$	n_0	n_1	n_2	n_3	n_4	n_5	n_6	n_7
n_0	n_0	n_1	n_2	n_3	n_4	n_5	n_6	n_7
n_1	n_1	n_0	n_3	n_2	n_5	n_4	n_7	n_6
n_2	n_2	n_6	n_0	n_4	n_3	n_7	n_1	n_5
n_3	n_3	n_7	n_1	n_5	n_2	n_6	n_0	n_4
n_4	n_4	n_5	n_6	n_7	n_0	n_1	n_2	n_3
n_5	n_5	n_4	n_7	n_6	n_1	n_0	n_3	n_2
n_6	n_6	n_2	n_4	n_0	n_7	n_3	n_5	n_1
n_7	n_7	n_3	n_5	n_1	n_6	n_2	n_4	n_0

$\cdot$	n_0	n_1	n_2	n_3	n_4	n_5	n_6	n_7
n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0
n_1	n_0	n_0	n_5	n_5	n_0	n_0	n_5	n_5
n_2	n_0	n_5	n_1	n_4	n_5	n_0	n_4	n_1
n_3	n_0	n_5	n_4	n_1	n_5	n_0	n_1	n_4
n_4	n_0	n_0	n_5	n_5	n_0	n_0	n_5	n_5
n_5	n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0
n_6	n_0	n_5	n_4	n_1	n_5	n_0	n_1	n_4
n_7	n_0	n_5	n_1	n_4	n_5	n_0	n_4	n_1

The near-ring R is LibraryNearRing(GTW8_4, 1059) in [1] and is d.g. with $S = R$ and with no identity. The ideal $X = \{n_0, n_1, n_4, n_5\}$ is a maximal right ideal of R. However, as $-n_in_2 + n_2 \notin X$ for $i = 2, 3, 6, 7$, it is clear that there is no left identity modulo X and therefore X is not right modular. The d.g. right R-factor group $R/X = \{X, n_2 + X\}$ has no proper, non-zero right R-subgroups, but is not monogenic as $(n_2 + X)R = X$. Hence R/X is not of type-ν, $\nu = 0, 2$, and X is not right ν-modular.

Proposition 2.33 For $\nu = 0, 2$, an ideal I of a d.g. near-ring R is right ν-primitive if and only if $I = (R : X)$ for some right ν-modular right ideal X of R.

Proof Firstly, let $I = (\Omega : 0)$ where $\Omega = \omega R$ is a right R-group of type-ν. By Lemma 2.29, $\Omega \cong R/X$ where X is a right ν-modular right ideal of R. It is routinely shown that $I = (\Omega : 0) = (R/X : 0) = (R : X)$. Conversely, if $I = (R : X)$ and X is a right ν-modular right ideal of R, then R/X is a right R-group of type-ν and $(R/X : 0) = (R : X) = I$. Consequently I is a right ν-primitive ideal of R. ■

Example 2.34 Let R be the d.g. near-ring in Example 2.4, page 8, and I the ideal $\{n_0, n_1, n_4, n_5\}$. It is routinely verified that $-n_2 r + r \in I$ for all $r \in R$, so that I is right modular with n_2 a left identity modulo I. Notice that the element n_7 is also a left identity modulo I. Furthermore, as R/I is a right R-group of type-2, the ideal I is right 2-modular. Lastly, I is left modular with n_2, n_3, n_6 and n_7 each right identities modulo I, and as R/I is a left R-group of type-2, I is also a left 2-modular ideal of R.

2.6 Right quasi-regularity

There is a close connection between quasi-regularity and the Jacobson radical in rings. The concept of left quasi-regularity for right near-rings is well-known but right quasi-regularity has not been extensively investigated. In this, the final section of this chapter, we define the concept of right quasi-regularity for right d.g. near-rings.

Definition 2.35 Let R be a d.g. near-ring.

(i) An element z of R is **right quasi-regular** if every right ideal that contains the set $\{-zr + r \mid r \in R\}$ also contains z.

(ii) A subset X of R is **right quasi-regular** if every element of X is right quasi-regular.

As for left quasi-regularity, right quasi-regularity generalizes nilpotence.

Proposition 2.36 Every nilpotent element of a d.g. near-ring R is right quasi-regular.

PROOF Let $z \in R$ be nilpotent, say $z^n = 0$ for some $n \in \mathbb{N}$, and let X be a right ideal of R containing $\{-zr + r \mid r \in R\}$. We have $-z(z^j) + z^j = -z^{j+1} + z^j \in X$ for $j = 1, \ldots, n-1$. Therefore $(-z^n + z^{n-1}) + (-z^{n-1} + z^{n-2}) + \cdots + (-z^2 + z) = -z^n + z = z \in X$. ■

It is well-known that in near-rings satisfying the DCCLS, the notions of left quasi-regularity and nilpotence are equivalent. This leads to the following connection between left quasi-regularity and right quasi-regularity.

Corollary 2.37 Let R be a d.g. near-ring. If R satisfies the DCCLS then every left quasi-regular left R-subgroup of R^+ is right quasi-regular.

We shall see in the next chapter that the converse of the above corollary does not hold.

Lemma 2.38 If X is a proper right modular right ideal of a d.g. near-ring R with e a left identity modulo X, then e is not right quasi-regular.

PROOF We have $-er + r \in X$ for all $r \in R$. If e is also right quasi-regular then $e \in X$, which gives the contradiction $X = R$. ∎

Chapter 3

Right radical theory

The right radical ${}^{r}J_0(R)$ was defined in [23] for the case of a d.g. near-ring with identity, and was shown to be the largest two-sided ideal contained in ${}^{r}J_{1/2}(R)$, the intersection of all maximal right ideals of R. It was also shown in right d.g. near-rings with identity and satisfying the DCCLS that $J_0(R) \subseteq {}^{r}J_0(R)$. This relationship was refined in [10] where it was shown under the same assumptions that $J_2(R) = {}^{r}J_0(R)$. We extend the definitions of ${}^{r}J_0(R)$ and ${}^{r}J_{1/2}(R)$ to general right d.g. near-rings and define the right 2-radical for such near-rings. In addition, we investigate the properties of the right radicals and the relationships between the left and right radicals, improving the relationships presented in [10] and [23].

3.1 Right radicals

Definition 3.1 Let R be a d.g. near-ring. For $\nu = 0, 2$,

$$^{r}J_{\nu}(R) = \cap\{P \mid P \text{ is a right } \nu\text{-primitive ideal of } R\}.$$

If there are no right ν-primitive ideals of R, then define $^{r}J_{\nu}(R) = R$.

Example 3.2 Let R be the d.g. near-ring in Example 2.32, page 22. The proper, non-zero right R-groups are $\Omega_1 = \{n_0, n_5\}$, $\Omega_2 = \{n_0, n_1, n_4, n_5\}$, $R/\Omega_1 = \{\Omega_1, n_1 + \Omega_1, n_2 + \Omega_1, n_3 + \Omega_1\}$ and $R/\Omega_2 = \{\Omega_2, n_2 + \Omega_2\}$. As none of these right R-groups are monogenic, there are no right R-groups of type-ν for $\nu = 0, 2$, and therefore no right ν-primitive ideals of R. Hence $^{r}J_{\nu}(R) = R$. Notice that R also has no left ν-primitive ideals, so $J_{\nu}(R) = R$ for $\nu = 0, 2$.

It was proved in [10] that $R \to {}^{r}J_0(R)$ is a radical map in the case of right d.g. near-rings with identity. We use a similar proof for arbitrary right d.g. near-rings and include $^{r}J_2(R)$. For the proof we need the following lemma.

Proposition 3.3 [11] Let I be an ideal of a d.g. near-ring R and Ω a group. The group Ω is a d.g. right R-group with $I \subseteq (\Omega : 0)$ if and only if Ω is a d.g. right R/I-group. Furthermore, Ω is right type-ν as a right R-group if and only if Ω is right type-ν as a right R/I-group ($\nu = 0, 2$).

Proof Let Ω be a d.g. right R-group with $I \subseteq (\Omega : 0)$. Define the action of R/I on Ω by

$$\omega(r + I) := \omega r \text{ for all } \omega \in \Omega,\ r + I \in R/I.$$

To verify that this action is well-defined, suppose that $r_1 + I = r_2 + I$ and let Δ be a distributive basis of Ω. Then $r_1 - r_2 \in I \subseteq (\Omega : 0) = (\Delta : 0)$ so $\delta(r_1 - r_2) = 0$ for all $\delta \in \Delta$ and, as each δ is distributive, $\delta r_1 = \delta r_2$. Now if $\omega \in \Omega$ we can write $\omega = \Sigma_i \epsilon_i \delta_i$, where $\epsilon_i = \pm 1$ and $\delta_i \in \Delta$. Then $\omega r_1 = (\Sigma_i \epsilon_i \delta_i) r_1 = \Sigma_i \epsilon_i (\delta_i r_1) = \Sigma_i \epsilon_i (\delta_i r_2) = (\Sigma_i \epsilon_i \delta_i) r_2 = \omega r_2$. To verify that Ω is a d.g. right R/I-group, let $\omega, \omega_1, \omega_2 \in \Omega,\ x + I, y + I \in R/I$ and $\delta \in \Delta$. Then:

(i) $\omega[(x + I)(y + I)] = \omega(xy + I) = \omega(xy) = (\omega x)y = \omega x(y + I) = [\omega(x + I)](y + I)$.

(ii) $(\omega_1 + \omega_2)(x + I) = (\omega_1 + \omega_2)x = \omega_1 x + \omega_2 x = \omega_1(x + I) + \omega_2(x + I)$.

(iii) $\delta[(x+I)+(y+I)] = \delta((x+y)+I) = \delta(x+y) = \delta x+\delta y = \delta(x+I)+\delta(y+I)$.

(iv) A distributive semi-group for the factor near-ring R/I is $S + I/I$. If $s + I \in S + I/I$, then $\delta(s + I) = \delta s \in \Delta$ since $\Delta S \subseteq \Delta$.

(v) If R/I has identity $1 + I$, then $\omega(1 + I) = \omega 1 = \omega$.

Conversely, if Ω is a d.g. right R/I-group, then define the action of R on Ω by

$$\omega r := \omega(r + I) \text{ for all } \omega \in \Omega,\ r \in R.$$

Let Δ be a distributive basis for Ω. Then for $\omega, \omega_1, \omega_2 \in \Omega,\ x, y \in R$ and $\delta \in \Delta$:

(i) $\omega(xy) = \omega(xy+I) = \omega[(x+I)(y+I)] = [\omega(x+I)](y+I) = (\omega x)(y+I) = (\omega x)y$.

(ii) $(\omega_1+\omega_2)x = (\omega_1+\omega_2)(x+I) = \omega_1(x+I)+\omega_2(x+I) = \omega_1 x+\omega_2 x$.

(iii) $\delta(x+y) = \delta((x+y)+I) = \delta[(x+I)+(y+I)] = \delta(x+I)+\delta(y+I) = \delta x+\delta y$.

(iv) If $s \in S$, then $\delta s = \delta(s+I) \in \Delta$ since $\Delta(S/I) \subseteq \Delta$.

(v) If R has identity 1 then $\omega 1 = \omega(1+I) = \omega$.

Thus Ω is a d.g. right R-group. Clearly $\Omega I = (0)$ so $I \subseteq (\Omega : 0)_R$. To show that Ω is type-ν, we see that Γ is a normal right R/I-subgroup of Ω if and only if Γ is a normal right R-subgroup of Ω, by defining the action of R or R/I on Γ via that on Ω. Similarly, Γ is a d.g. right R/I-subgroup of Ω if and only if Γ is a d.g. right R-subgroup of Ω. Thus Ω is a right R-group of type-ν if and only if it is a right R/I-group of type-ν. ■

One of the properties for a map $R \to \mathcal{R}(R)$ to be a radical map in the sense of Hoehnke is that $\theta(\mathcal{R}(R)) \subseteq \mathcal{R}(\theta(R))$ where $\theta : R \to R'$ is any near-ring homomorphism. We note that $\theta(R) \cong R/\ker\theta$, and $\theta(R)$ is thus a d.g. near-ring.

Theorem 3.4 [11] Let R be a d.g. near-ring. For $\nu = 0, 2,\ R \to {}^rJ_\nu(R)$ is a radical map in the sense of Hoehnke. That is:

(i) ${}^rJ_\nu(R/{}^rJ_\nu(R)) = (0)$.

(ii) If $\theta : R \to R'$ is a near-ring homomorphism then $\theta({}^rJ_\nu(R)) \subseteq {}^rJ_\nu(\theta(R))$.

Proof (i) Let $r + {}^rJ_\nu(R) \in {}^rJ_\nu(R/{}^rJ_\nu(R))$ and let Ω be a right R-group of type-ν. By Proposition 3.3, since ${}^rJ_\nu(R) \subseteq (\Omega : 0)$, Ω is a right $R/{}^rJ_\nu(R)$-group of type-ν and is therefore annihilated by

${}^rJ_\nu(R/{}^rJ_\nu(R))$. Hence $(0) = \Omega(r + {}^rJ_\nu(R)) = \Omega r$ and so $r \in (\Omega : 0)$. Thus $r \in {}^rJ_\nu(R)$ and $r + {}^rJ_\nu(R) = {}^rJ_\nu(R)$, i.e., ${}^rJ_\nu(R/{}^rJ_\nu(R)) = (0)$.

(ii) Let $\theta(r) \in \theta({}^rJ_\nu(R))$, $r \in {}^rJ_\nu(R)$ and let Ω be a right $\theta(R)$-group of type-ν. By Proposition 3.3, since $\theta(R) \cong R/\ker\theta$, Ω can be considered as a right R-group of type-ν with $\ker\theta \subseteq (\Omega : 0)_R$. Then, as $r \in {}^rJ_\nu(R)$, $\Omega r = (0)$. Now $\Omega\theta(r) = \Omega(r + \ker\theta) = \Omega r = (0)$, so $\theta(r) \in (\Omega : 0)_{\theta(R)}$, and $\theta(r) \in {}^rJ_\nu(\theta(R))$. ■

Hence the right radicals are so-called Hoehnke radicals. On the other hand, it would be interesting to know whether these radicals are Kurosh-Amitsur radicals. In other words, do the radicals satisfy the additional idempotent property ${}^rJ_\nu(R) = {}^rJ_\nu({}^rJ_\nu(R))$? In order to prove this, it is sufficient to prove the hereditary property, namely ${}^rJ_\nu(I) = {}^rJ_\nu(R) \cap I$ for any ideal I of R. However, to study ${}^rJ_\nu(I)$ it will be necessary to define a right representation theory for the more general class of zero-symmetric near-rings, since if we regard the ideal I as a near-ring of which we wish to find the right ν-radical, then the near-ring is not necessarily distributively generated. Grainger [6] defines right R-groups for general right near-rings R, but this work does not extend to radical theory. Although a conclusion regarding the Kurosh-Amitsur property cannot be reached here, we can prove containment for a specific type of ideal.

Definition 3.5 [21] An ideal of a d.g. near-ring R is said to be a **distributive ideal** if it is a d.g. right R-subgroup of R^+.

Proposition 3.6 [11] Let R be a d.g. near-ring. Let I be a distributive ideal which is a direct summand of R. Then ${}^rJ_\nu(I) \supseteq {}^rJ_\nu(R) \cap I$ for $\nu = 0, 2$.

PROOF Let π be the natural projection of R onto I. Now ${}^{r}J_{\nu}(R) \cap I \subseteq {}^{r}J_{\nu}(R)$ so $\pi({}^{r}J_{\nu}(R) \cap I) \subseteq \pi({}^{r}J_{\nu}(R))$. As π is the identity map on I and using Theorem 3.4(ii) we have

$$ {}^{r}J_{\nu}(R) \cap I = \pi({}^{r}J_{\nu}(R) \cap I) \subseteq \pi({}^{r}J_{\nu}(R)) \subseteq {}^{r}J_{\nu}(\pi(R)) = {}^{r}J_{\nu}(I). $$

The following example from [1] shows that it is possible for equality to hold, but whether or not equality holds in general is unknown.

Example 3.7 Consider the d.g. near-ring $(R, +, \cdot)$ where $R = \{n_0, \ldots, n_{15}\}$ and $+$ and $\cdot$ are given in the following tables.

$+$	n_0	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	n_{10}	n_{11}	n_{12}	n_{13}	n_{14}	n_{15}
n_0	n_0	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	n_{10}	n_{11}	n_{12}	n_{13}	n_{14}	n_{15}
n_1	n_1	n_0	n_3	n_2	n_5	n_4	n_7	n_6	n_9	n_8	n_{11}	n_{10}	n_{13}	n_{12}	n_{15}	n_{14}
n_2	n_2	n_6	n_0	n_4	n_3	n_7	n_1	n_5	n_{10}	n_{14}	n_8	n_{12}	n_{11}	n_{15}	n_9	n_{13}
n_3	n_3	n_7	n_1	n_5	n_2	n_6	n_0	n_4	n_{11}	n_{15}	n_9	n_{13}	n_{10}	n_{14}	n_8	n_{12}
n_4	n_4	n_5	n_6	n_7	n_0	n_1	n_2	n_3	n_{12}	n_{13}	n_{14}	n_{15}	n_8	n_9	n_{10}	n_{11}
n_5	n_5	n_4	n_7	n_6	n_1	n_0	n_3	n_2	n_{13}	n_{12}	n_{15}	n_{14}	n_9	n_8	n_{11}	n_{10}
n_6	n_6	n_2	n_4	n_0	n_7	n_3	n_5	n_1	n_{14}	n_{10}	n_{12}	n_8	n_{15}	n_{11}	n_{13}	n_9
n_7	n_7	n_3	n_5	n_1	n_6	n_2	n_4	n_0	n_{15}	n_{11}	n_{13}	n_9	n_{14}	n_{10}	n_{12}	n_8
n_8	n_8	n_9	n_{10}	n_{11}	n_{12}	n_{13}	n_{14}	n_{15}	n_0	n_1	n_2	n_3	n_4	n_5	n_6	n_7
n_9	n_9	n_8	n_{11}	n_{10}	n_{13}	n_{12}	n_{15}	n_{14}	n_1	n_0	n_3	n_2	n_5	n_4	n_7	n_6
n_{10}	n_{10}	n_{14}	n_8	n_{12}	n_{11}	n_{15}	n_9	n_{13}	n_2	n_6	n_0	n_4	n_3	n_7	n_1	n_5
n_{11}	n_{11}	n_{15}	n_9	n_{13}	n_{10}	n_{14}	n_8	n_{12}	n_3	n_7	n_1	n_5	n_2	n_6	n_0	n_4
n_{12}	n_{12}	n_{13}	n_{14}	n_{15}	n_8	n_9	n_{10}	n_{11}	n_4	n_5	n_6	n_7	n_0	n_1	n_2	n_3
n_{13}	n_{13}	n_{12}	n_{15}	n_{14}	n_9	n_8	n_{11}	n_{10}	n_5	n_4	n_7	n_6	n_1	n_0	n_3	n_2
n_{14}	n_{14}	n_{10}	n_{12}	n_8	n_{15}	n_{11}	n_{13}	n_9	n_6	n_2	n_4	n_0	n_7	n_3	n_5	n_1
n_{15}	n_{15}	n_{11}	n_{13}	n_9	n_{14}	n_{10}	n_{12}	n_8	n_7	n_3	n_5	n_1	n_6	n_2	n_4	n_0

$\cdot$	n_0	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	n_{10}	n_{11}	n_{12}	n_{13}	n_{14}	n_{15}
n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0
n_1	n_0	n_1	n_0	n_1	n_1	n_0	n_1	n_0	n_0	n_1	n_0	n_1	n_1	n_0	n_1	n_0
n_2	n_0	n_0	n_2	n_2	n_5	n_5	n_7	n_7	n_0	n_0	n_2	n_2	n_5	n_5	n_7	n_7
n_3	n_0	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_0	n_1	n_2	n_3	n_4	n_5	n_6	n_7
n_4	n_0	n_1	n_0	n_4	n_1	n_0	n_4	n_0	n_0	n_1	n_0	n_4	n_1	n_0	n_4	n_0
n_5	n_0	n_0	n_0	n_5	n_0	n_0	n_5	n_0	n_0	n_0	n_0	n_5	n_0	n_0	n_5	n_0
n_6	n_0	n_1	n_2	n_6	n_4	n_5	n_3	n_7	n_0	n_1	n_2	n_6	n_4	n_5	n_3	n_7
n_7	n_0	n_0	n_2	n_7	n_5	n_5	n_2	n_7	n_0	n_0	n_2	n_7	n_5	n_5	n_2	n_7
n_8	n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_8	n_8	n_8	n_8	n_8	n_8	n_8	n_8
n_9	n_0	n_1	n_0	n_1	n_1	n_0	n_1	n_0	n_8	n_9	n_8	n_9	n_9	n_8	n_9	n_8
n_{10}	n_0	n_0	n_2	n_2	n_5	n_5	n_7	n_7	n_8	n_8	n_{10}	n_{10}	n_{13}	n_{13}	n_{15}	n_{15}
n_{11}	n_0	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	n_{10}	n_{11}	n_{12}	n_{13}	n_{14}	n_{15}
n_{12}	n_0	n_1	n_0	n_4	n_1	n_0	n_4	n_0	n_8	n_9	n_8	n_{12}	n_9	n_8	n_{12}	n_8
n_{13}	n_0	n_0	n_0	n_5	n_0	n_0	n_5	n_0	n_8	n_8	n_8	n_{13}	n_8	n_8	n_{13}	n_8
n_{14}	n_0	n_1	n_2	n_6	n_4	n_5	n_3	n_7	n_8	n_9	n_{10}	n_{14}	n_{12}	n_{13}	n_{11}	n_{15}
n_{15}	n_0	n_0	n_2	n_7	n_5	n_5	n_2	n_7	n_8	n_8	n_{10}	n_{15}	n_{13}	n_{13}	n_{10}	n_{15}

In [1], $R = \text{LibraryNearRingWithOne(GTW16\_6}, 416)$, and R is a d.g. near-ring with $S = \{n_0, n_1, n_3, n_8, n_9, n_{11}\}$ and identity n_{11}. Let $I_1 := \{n_0, n_8\}$ and $I_2 := \{n_0, n_1, n_2, n_3, n_4, n_5, n_6, n_7\}$. It can be seen that I_1 and I_2 are distributive ideals with distributive bases $\{n_0, n_8\}$ and $\{n_0, n_1, n_3\}$ respectively, and $R = I_1 \oplus I_2$. We show that ${}^rJ_\nu(I_2) = {}^rJ_\nu(R) \cap I_2$ for $\nu = 0, 2$. For the right 0-radical, as R possesses an identity element, we use the result that Ω is a right R-group of type-0 if and only if $\Omega \cong R/X$ for some maximal right ideal X of R. The maximal right ideals of R are I_2, $I_3 := \{n_0, n_2, n_5, n_7, n_8, n_{10}, n_{13}, n_{15}\}$ and $I_4 := \{n_0, n_1, n_4, n_5, n_8, n_9, n_{12}, n_{13}\}$. Each is a two-sided ideal. The right R-groups of type-0 are $R/I_2 = (n_8 + I_2)R$, $R/I_3 = (n_1 + I_3)R$ and $R/I_4 = (n_3 + I_4)R$. As I_2, I_3 and I_4 are ideals of R, we have $(R/I_2 : 0) = I_2$, $(R/I_3 : 0) = I_3$ and $(R/I_4 : 0) = I_4$. This gives ${}^rJ_0(R) = I_2 \cap I_3 \cap I_4 = \{n_0, n_5\}$ and ${}^rJ_0(R) \cap I_2 = \{n_0, n_5\}$. We use a similar method to compute ${}^rJ_0(I_2)$, since I_2 has identity n_3. The maximal right ideals of I_2 are $\{n_0, n_2, n_5, n_7\}$ and $\{n_0, n_1, n_4, n_5\}$, both of which are two-sided ideals. Therefore ${}^rJ_0(I_2) =$

$\{n_0, n_2, n_5, n_7\} \cap \{n_0, n_1, n_4, n_5\} = \{n_0, n_5\} = {}^rJ_0(R) \cap I_2$. Next, turning to the case of the right 2-radical, we make use of the result that as R has an identity, Ω is a right R-group of type-2 if and only if $\Omega \cong R/X$ where X is some maximal right ideal of R and R/X has no proper, non-zero d.g. right R-subgroups. As R/I_2, R/I_3 and R/I_4 are of order 2, it follows that they are right R-groups of type-2. Therefore ${}^rJ_2(R) = I_2 \cap I_3 \cap I_4 = \{n_0, n_5\}$, and a similar argument gives ${}^rJ_2(I_2) = \{n_0, n_5\}$, so that ${}^rJ_2(I_2) = {}^rJ_2(R) \cap I_2$. Finally, note that $J_2(R) = \{n_0, n_5\}$ from [1], giving $J_2(R) = {}^rJ_0(R) = {}^rJ_2(R)$. We will see later that this is no coincidence but a property enjoyed by all finite d.g. near-rings with identity.

Next we find an alternative characterisation for the radical ${}^rJ_2(R)$ using right modular right ideals.

Theorem 3.8 [11] Let R be a d.g. near-ring.

$$ {}^rJ_2(R) = \cap\{X \mid X \text{ is a right 2-modular right ideal of } R\}. $$

Proof Firstly, ${}^rJ_2(R) = \cap\{(R : X) | X \text{ is a right 2-modular right ideal of } R\} \subseteq \cap\{X \mid X \text{ is a right 2-modular right ideal of } R\}$ by Proposition 2.33 and Lemma 2.30. On the other hand, let Ω be a right R-group of type-2 with distributive basis Δ. Then $(\Omega : 0) = (\Delta : 0) = \cap_{\delta \in \Delta}(\delta : 0) = \cap_{\delta' R \neq (0)}(\delta' : 0)$. Now if $\delta' R \neq (0)$ then, as Ω is of right type-2, we have $R/(\delta' : 0) \cong \delta' R = \Omega$. Thus $(\delta' : 0)$ is a right 2-modular right ideal of R and $(\Omega : 0)$ is therefore an intersection of right 2-modular right ideals. Accordingly, ${}^rJ_2(R) \supseteq \cap\{X \mid X \text{ is a right 2-modular right ideal of } R\}$. ■

Similarly, using Proposition 2.33 and Lemma 2.30 we can show

$$^{r}J_0(R) \subseteq \cap\{X \mid X \text{ is a right 0-modular right ideal of } R\},$$

but it is not known whether equality holds. Instead, we define

Definition 3.9 Let R be a d.g. near-ring.

$$^{r}J_{1/2}(R) = \cap\{X \mid X \text{ is a right 0-modular right ideal of } R\}.$$

Both $^{r}J_0(R)$ and $^{r}J_2(R)$ are two-sided ideals of R, but as the intersection of an arbitrary collection of right ideals is a right ideal, $^{r}J_{1/2}(R)$ may, in general, only be a right ideal of R. However it will be seen that $^{r}J_{1/2}(R)$ is a two-sided ideal in the case of finite d.g. near-rings with identity. We note also that if R has identity then this definition of $^{r}J_{1/2}(R)$ coincides with the definition given in [10] as the intersection of all maximal right ideals of R.

In Example 3.7, page 32, it is easily seen that $^{r}J_{1/2}(I_2) = {}^{r}J_{1/2}(R) \cap I_2$. We prove that although $^{r}J_{1/2}(R)$ is not a radical map, Proposition 3.6 also holds for the right half-radical. We first need two preliminary lemmas, the proofs of which use the following properties of a direct sum of ideals, $R = I \oplus K$:

1 Left distribution: $r(i+k) = ri + rk$ for all $r \in R,\ i \in I$ and $k \in K$.

2 Commutativity of addition: $i + k = k + i$ for all $i \in I,\ k \in K$.

Lemma 3.10 [11] Let R be a d.g. near-ring and let $R = I \oplus K$ where I is a distributive ideal and K is an ideal of R. Let X be a right modular right ideal of I. Then $XR \subseteq X$ and I/X is a d.g. right R-group.

Proof As $R = I \oplus K$, each $r \in R$ can be uniquely written as $r = i + k$ where $i \in I$ and $k \in K$. If $x \in X$ and $r = i + k \in R$, then $xr = x(i+k) = xi + xk = xi \in X$ because $x \in X \subseteq I$ and $xk \in I \cap K = (0)$. Thus $XR \subseteq X$. Define $(i + X)r = ir + X \in I/X$ for all $i + X \in I/X$ and $r \in R$. Since $XR \subseteq X$, this product is well-defined and I/X is a right R-group. By Proposition 2.11, I/X is a d.g. right I-group with distributive basis $\{\delta_t + X \mid t \in T\}$, say. For all $i, i' \in I,\ t \in T$,

$$\begin{aligned}(\delta_t + X)(i + i') &= (\delta_t + X)i + (\delta_t + X)i' \\ &= (\delta_t i + \delta_t i') + X.\end{aligned}$$

We show that each $\delta_t + X$ distributes over sums of elements from R, from which it will follow that I/X is a d.g. right R-group. Let $r_1 = i_1 + k_1$, $r_2 = i_2 + k_2 \in R,\ i_i, i_2 \in I$ and $k_1, k_2 \in K$. Then

$$\begin{aligned}(\delta_t + X)(r_1 + r_2) &= \delta_t(r_1 + r_2) + X \\ &= \delta_t(i_1 + k_1 + i_2 + k_2) + X \\ &= \delta_t(i_1 + i_2 + k_1 + k_2) + X \\ &= (\delta_t(i_1 + i_2) + \delta_t(k_1 + k_2)) + X \\ &= \delta_t(i_1 + i_2) + X,\ \text{since } \delta_t(k_1 + k_2) \in I \cap K = (0) \\ &= (\delta_t + X)(i_1 + i_2) \\ &= (\delta_t i_1 + \delta_t i_2) + X,\ \text{by the above.}\end{aligned}$$

On the other hand,

$$\begin{aligned}(\delta_t + X)r_1 + (\delta_t + X)r_2 &= (\delta_t r_1 + \delta_t r_2) + X \\ &= (\delta_t(i_1 + k_1) + \delta_t(i_2 + k_2)) + X \\ &= (\delta_t i_1 + \delta_t k_1 + \delta_t i_2 + \delta_t k_2) + X \\ &= (\delta_t i_1 + \delta_t i_2) + X,\end{aligned}$$

since $\delta_t k_1,\ \delta_t k_2 \in I \cap K = (0)$. ■

Lemma 3.11 [11] Let R be a d.g. near-ring and $R = I \oplus K$ where I is a distributive ideal of R and K is an ideal of R. Let X be a right ν-modular right ideal of I for $\nu = 0, 2$. Then there exists a right ν-modular right ideal Y of R such that $Y \cap I = X$.

Proof Write $R = I \oplus K$, where I and K are ideals of R and let X be a right ν-modular right ideal of I. Define $Y = X + K$, then $Y \cap I = X$. Next let $y = x + k \in Y$ and $r = i + k' \in R$ where $x \in X$, $k, k' \in K$ and $i \in I$. Then $yr = (x + k)r = xr + kr = xr + k(i + k') = xr + ki + kk' = xr + kk'$, since $ki \in I \cap K = (0)$. By Lemma 3.10, $xr \in X$, so $yr \in X + K = Y$ and Y is a right R-subgroup of R^+. Furthermore,

$$\begin{aligned}
-r + y + r &= -k' - i + x + k + i + k' \\
&= -k' - i + x + i + k + k' \\
&= -k' + x' + k + k' \quad \text{for some } x' \in X \text{ since } X \text{ is normal in } I \\
&= x' - k' + k + k' \quad \text{since } x' \in X \subseteq I \text{ and } -k' \in K \\
&\in X + K = Y,
\end{aligned}$$

so Y is a right ideal of R. In order to show that Y is right ν-modular, we first have I/X is a d.g. right R-group of type-ν by Lemma 3.10. Next, note that every coset $r + Y$ in R/Y can be written in the form $i + Y$ for some $i \in I$ because $K \subseteq Y$. Hence define $\theta : R/Y \to I/X$ by $\theta(r+Y) = \theta(i+Y) = i+X$. To show that θ is well-defined, let $r_1+Y = r_2+Y$, then $i_1+Y = i_2+Y$ and so $-i_2+i_1 \in Y \cap I = X$, giving $i_1+X = i_2+X$. It is straightforward to prove that θ is a right R-isomorphism, therefore $R/Y \cong I/X$ and R/Y is a right R-group of type-ν. Lastly, suppose that e is a left identity modulo X. Then $-ei+i \in X$ for all $i \in I$, We show that e is also a left identity modulo Y. Let $r = i+k \in R$, then $r + Y = i + Y$ and $er + Y = e(i + k) + Y = ei + ek + Y = ei + Y$, since $ek \in K \subseteq Y$. Now $-ei + i \in X$ and $X = Y \cap I$, so $-ei + i \in Y$. Hence $r + Y = er + Y$, giving $-er + r \in Y$ for all $r \in R$. ∎

Proposition 3.12 [11] Let R be a d.g. near-ring and let I be a distributive ideal which is a direct summand of R. Then ${}^rJ_{1/2}(I) \supseteq {}^rJ_{1/2}(R) \cap I$.

Proof By the previous lemma, for each right 0-modular right ideal X of I there exists a right 0-modular right ideal Y of R such that $Y \cap I = X$. Let K be the set of all such right 0-modular right ideals of R. Then

$$\begin{aligned} {}^rJ_{1/2}(I) &= \cap\{X \mid X \text{ is a right 0-modular right ideal of } I\} \\ &= \cap\{Y \cap I \mid Y \in K\} \\ &= I \cap (\cap\{Y \mid Y \in K\}) \\ &\supseteq I \cap {}^rJ_{1/2}(R). \end{aligned}$$

∎

We now examine the relationships between the right radicals.

Proposition 3.13 Let R be a d.g. near-ring. Then

(i) ${}^rJ_0(R) = (R : {}^rJ_{1/2}(R))$

(ii) ${}^rJ_0(R)$ is the largest ideal of R contained in ${}^rJ_{1/2}(R)$.

Proof (i) As ${}^rJ_0(R)$ is a two-sided ideal contained in ${}^rJ_{1/2}(R)$, it follows that $R({}^rJ_0(R)) \subseteq {}^rJ_{1/2}(R)$ and so ${}^rJ_0(R) \subseteq (R : {}^rJ_{1/2}(R))$. Next, if $\Omega = \omega R$ is a right R-group of type-0 then $(\Omega : 0) = (R/Y : 0) = (R : Y)$ where $Y = (\omega : 0)$ is a right 0-modular right ideal of R. Now

${}^{r}J_{1/2}(R) \subseteq Y$, so $(R : {}^{r}J_{1/2}(R)) \subseteq (R : Y) = (\Omega : 0)$. As Ω is arbitrary, $(R : {}^{r}J_{1/2}(R)) \subseteq {}^{r}J_0(R)$.

(ii) Follows from (i) and Proposition 2.18. ■

Using the previous proposition together with Theorem 3.8 shows that the relationship between ${}^{r}J_0(R)$, ${}^{r}J_{1/2}(R)$ and ${}^{r}J_2(R)$ is similar to that between the corresponding left radicals, namely

$${}^{r}J_0(R) \subseteq {}^{r}J_{1/2}(R) \subseteq {}^{r}J_2(R).$$

3.2 Right semisimplicity

The following result was adapted from [15]. It shows that factoring out ${}^{r}J_{1/2}(R)$ from a near-ring satisfying the DCCR gives a direct sum of right R-groups of type-0 and factoring out ${}^{r}J_2(R)$ results in a direct sum of right R-groups of type-2. A proof of (i) appears in [23] for d.g. near-rings with identity, but the result does not depend on the presence of an identity.

Theorem 3.14 [11] If R is a d.g. near-ring that satisfies the DCCR then

(i) $R/{}^{r}J_{1/2}(R) = \Delta_1 \oplus \cdots \oplus \Delta_n$, where the Δ_i are right R-groups of type-0.

(ii) $R/{}^{r}J_2(R) = \Delta_1 \oplus \cdots \oplus \Delta_n$, where the Δ_i are right R-groups of type-2.

Proof Let $\nu \in \{\frac{1}{2}, 2\}$. If $\nu = 2$, then since R satisfies the DCCR, there exists a finite number of distinct right 2-modular (for $\nu = \frac{1}{2}$, 0-modular) right ideals $Y_1, \ldots, Y_n$ such that

$$\bigcap_{i=1}^{n} Y_i = {}^{r}J_\nu(R) \text{ and } X_k := \bigcap_{\substack{i=1 \\ i\neq k}}^{n} Y_i \neq {}^{r}J_\nu(R),\ k = 1, \ldots, n.$$

Hence

$$Y_i \supseteq X_k \text{ for all } i \neq k$$

and

$$Y_k \cap X_k = {}^{r}J_\nu(R),\ k = 1, \ldots, n.$$

Under the canonical right R-homomorphism of R^+ onto $R/{}^{r}J_\nu(R)$, let Y_k be mapped onto Ω_k and X_k onto Δ_k, $k = 1, \ldots, n$. It follows that $\bigcap_{i=1}^{n} \Omega_i = (0)$. Also

$$\Omega_i \supseteq \Delta_k \text{ for all } i \neq k$$

and

$$\Omega_k \cap \Delta_k = (0),\ k = 1, \ldots, n.$$

As $X_k \not\subseteq Y_k$ and Y_k is a maximal right ideal, we have $R = X_k + Y_k$. Thus

$$R/{}^{r}J_\nu(R) = \Delta_k \oplus \Omega_k \text{ for each } k = 1, \ldots, n.$$

Now

$$\begin{aligned}\Omega_1 &= \Omega_1 \cap (R/{}^{r}J_\nu(R)) \\ &= \Omega_1 \cap (\Delta_2 \oplus \Omega_2) \\ &= \Delta_2 \oplus (\Omega_1 \cap \Omega_2)\end{aligned}$$

by the modular law, so

$$\begin{aligned}R/{}^{r}J_\nu(R) &= \Delta_1 \oplus \Omega_1 \\ &= \Delta_1 \oplus \Delta_2 \oplus (\Omega_1 \cap \Omega_2).\end{aligned}$$

But

$$\begin{aligned}\Omega_1 \cap \Omega_2 &= \Omega_1 \cap \Omega_2 \cap (R/{}^rJ_\nu(R)) \\ &= \Omega_1 \cap \Omega_2 \cap (\Delta_3 \oplus \Omega_3) \\ &= \Delta_3 \oplus (\Omega_1 \cap \Omega_2 \cap \Omega_3),\end{aligned}$$

giving

$$R/{}^rJ_\nu(R) = \Delta_1 \oplus \Delta_2 \oplus \Delta_3 \oplus (\Omega_1 \cap \Omega_2 \cap \Omega_3).$$

Repeating this procedure and using the fact that $\bigcap\limits_{i=1}^{n} \Omega_i = (0)$, we obtain

$$R/{}^rJ_\nu(R) = \Delta_1 \oplus \cdots \oplus \Delta_n.$$

Since $R/{}^rJ_\nu(R) = \Delta_k \oplus \Omega_k$ for $k = 1, \ldots, n$,

$$\Delta_k \cong (R/{}^rJ_\nu(R))/\Omega_k \cong R/Y_k.$$

Consequently, each Δ_k is a right R-group of type-2 (for $\nu = \frac{1}{2}$, type-0). ■

Definition 3.15 Let R be a d.g. near-ring. If ${}^rJ_\nu(R) = (0)$, then the near-ring R is called **right ν-semisimple** for $\nu = 0, 2$.

Hence we have

Corollary 3.16 Let the d.g. near-ring R satisfy the DCCR and be right 2-semisimple. Then $R = \Delta_1 \oplus \cdots \oplus \Delta_n$, where the Δ_i are right ideals of R and right R-groups of type-2.

Remark 3.17 One can define $R/{}^rJ_{1/2}(R)$ to be right half-semisimple and extend the notion to d.g. right R-groups. For the d.g. right R-group R^+, ${}^rJ_{1/2}(R) = (0)$ implies that ${}^rJ_0(R) = (0)$. One can define the right half radical of a d.g. right R-group to be the intersection of all normal maximal right R-subgroups. If this intersection is zero, then we say the d.g. right R-group is right half-semisimple. We will not pursue the matter any further in this thesis.

3.3 Relationships between the left and right radicals in d.g. near-rings with identity

In ring theory the Jacobson radical that is defined using left R-modules is equal to the radical that is defined using right R-modules. A question arises of whether this characteristic of the radical depends on the presence of commutativity of addition and both distributive laws. Thus in this section we try to ascertain if the same property that holds for rings holds for any or all of the near-ring radicals. We restrict ourselves in this section to d.g. near-rings with identity and, in particular, have results for those d.g. near-rings with identity satisfying certain descending chain conditions. Note that as we are assuming that R has identity, we can use the alternative characterisation for ${}^{r}J_{1/2}(R)$, namely

$$ {}^{r}J_{1/2}(R) = \cap\{X \mid X \text{ is a maximal right ideal of } R\}. $$

A key result is stated, but not proved, in [14]. We present a proof here.

Theorem 3.18 [10] Let R be a d.g. near-ring with identity. The right half-radical ${}^{r}J_{1/2}(R)$ contains all the nilpotent left R-subgroups of R^+.

Proof Let Ω be a nilpotent left R-subgroup of R^+, say $\Omega^p = (0)$. We show that Ω is contained in every maximal right ideal of R. Let

$$Y := \Big\{ \sum_{i=1}^{n}(x_i + \omega_i' r_i - x_i) \;\Big|\; r_i, x_i \in R,\ \omega_i' \in \Omega \Big\}.$$

Then Y is a right ideal of R containing Ω. Suppose that there exists a maximal right ideal B of R such that $\Omega \not\subseteq B$. Then the right ideal Y is not contained in B so $R = Y + B$ and the identity can be written as $1 = y + b$, with $y \in Y$ and $b \in B$. Let $\omega_1, \omega_2, \ldots, \omega_{p-1} \in \Omega$. Now

$$\begin{aligned}
&\omega_1\omega_2\ldots\omega_{p-1}\\
=\;& (y+b)\omega_1\omega_2\ldots\omega_{p-1}\\
=\;& y\omega_1\omega_2\ldots\omega_{p-1} + b\omega_1\omega_2\ldots\omega_{p-1}\\
=\;& [\Sigma_{i=1}^{n}(x_i + \omega_i' r_i - x_i)]\omega_1\omega_2\ldots\omega_{p-1} + b' \quad \text{for some } \ b' \in B\\
=\;& \Sigma_{i=1}^{n}(x_i\omega_1\omega_2\ldots\omega_{p-1} + \omega_i' r_i\omega_1\omega_2\ldots\omega_{p-1} - x_i\omega_1\omega_2\ldots\omega_{p-1}) + b'\\
=\;& b',
\end{aligned}$$

since $r_i\omega_1 \in \Omega$ and so $\omega_i' r_i\omega_1\omega_2\ldots\omega_{p-1} \in \Omega^p = (0)$. Hence $\Omega^{p-1} \subseteq B$. A similar argument gives

$$\begin{aligned}
&\omega_1\omega_2\ldots\omega_{p-2}\\
=\;& (y+b)\omega_1\omega_2\ldots\omega_{p-2}\\
=\;& y\omega_1\omega_2\ldots\omega_{p-2} + b\omega_1\omega_2\ldots\omega_{p-2}\\
=\;& [\Sigma_{i=1}^{n}(x_i + \omega_i' r_i - x_i)]\omega_1\omega_2\ldots\omega_{p-2} + b'' \quad \text{for some } \ b'' \in B\\
=\;& \Sigma_{i=1}^{n}(x_i\omega_1\omega_2\ldots\omega_{p-2} + \omega_i' r_i\omega_1\omega_2\ldots\omega_{p-2} - x_i\omega_1\omega_2\ldots\omega_{p-2}) + b''\\
\in\;& B,
\end{aligned}$$

since $r_i\omega_1 \in \Omega$ and so $\omega_i' r_i\omega_1\omega_2\ldots\omega_{p-2} \in \Omega^{p-1} \subseteq B$. Hence $\Omega^{p-2} \subseteq B$. Repeating this procedure we finally arrive at the contradiction $\Omega \subseteq B$. ∎

We remark that the right ideal Y in the proof of Theorem 3.18 is actually a two-sided ideal. As this result is important in the sequel, we state it more formally as follows.

Proposition 3.19 [10] Let R be a d.g. near-ring with identity. Let X be a right ideal of R and let Ω be a left R-subgroup of R^+ contained in X. Then $Y = \{\Sigma_{i=1}^{n}(x_i + \omega_i r_i - x_i) \mid r_i, x_i \in R,\ \omega_i \in \Omega\}$ is an ideal of R contained in X and containing Ω.

Proof Now Y is a right ideal of R containing Ω from Theorem 3.18. Moreover, Y is the smallest right ideal containing Ω, so $Y \subseteq X$. To show that Y is also a left ideal of R, write $r \in R$ as $r = \Sigma_j \epsilon_j s_j$ where $\epsilon_j = \pm 1$ and $s_j \in S$. If $y = \Sigma_{i=1}^{n}(x_i + \omega_i r_i - x_i) \in Y$, it follows that for each j, $s_j y = \Sigma_{i=1}^{n}(s_j x_i + s_j \omega_i r_i - s_j x_i) \in Y$ since Ω is a left R-group, and so $ry \in Y$. ■

Theorem 3.20 [10] Let R be a d.g. near-ring with identity. The right radical ${}^{r}J_0(R)$ contains all nilpotent left R-subgroups of R^+.

Proof Let Ω be a nilpotent left R-subgroup of R^+, then by Theorem 3.18 $\Omega \subseteq {}^{r}J_{1/2}(R)$. Let Y be the ideal defined in Proposition 3.19. Then we have $Y \subseteq {}^{r}J_{1/2}(R)$. As ${}^{r}J_0(R)$ is the largest ideal of R contained in ${}^{r}J_{1/2}(R)$, $Y \subseteq {}^{r}J_0(R)$ and therefore $\Omega \subseteq {}^{r}J_0(R)$. ■

For a d.g. near-ring R with identity that satisfies the DCCLS, $J_2(R)$ is the smallest ideal that contains all nilpotent left R-subgroups of R^+ [14]. Hence we have the following.

Lemma 3.21 [10] Let R be a d.g. near-ring with identity that satisfies the

DCCLS. Then

$$J_2(R) \subseteq {}^rJ_0(R).$$

In order to prove the reverse inclusion we need the following lemma.

Lemma 3.22 [10] Let R be a d.g. near-ring with identity. Every maximal ideal of R is right 0-primitive.

Proof Let M be a maximal ideal of R. If M is also maximal as a right ideal, then R/M is a right R-group of type-0. Since M is an ideal of R and R has identity, we have $M = (R/M : 0)$, so M is a right 0-primitive ideal. If M is not maximal as a right ideal, then by Zorn's Lemma there exists a maximal right ideal, say X, containing M, and R/X is a right R-group of type-0. Now we see that $M \subseteq (R/X : 0)$ since $RM \subseteq M \subseteq X$. But M is a maximal ideal of R, so $(R/X : 0) = M$ and M is right 0-primitive. ∎

If R satisfies the DCCLS then $J_2(R) = \cap\{I \mid I \text{ is a maximal ideal of } R\}$ [14], and we can conclude that

$${}^rJ_0(R) \subseteq J_2(R).$$

Combining this with Lemma 3.21, we have

Theorem 3.23 [10] Let R be a d.g. near-ring with identity and satisfying the DCCLS. Then

$$J_0(R) \subseteq J_{1/2}(R) \subseteq J_2(R) = {}^rJ_0(R) \subseteq {}^rJ_{1/2}(R) \subseteq {}^rJ_2(R).$$

By Proposition 3.13(ii), it follows that

Corollary 3.24 Let R be a d.g. near-ring with identity that satisfies the DCCLS. Then $J_2(R)$ is the largest ideal of R contained in ${}^rJ_{1/2}(R)$.

Laxton states without proof in [14] that the intersection of all the maximal right ideals of R contains all the nilpotent right R-subgroups of R^+. He does however prove that in the case of DCCLS $J_2(R)$ contains all the nilpotent right R-subgroups. For the sake of completeness, we prove that ${}^rJ_{1/2}(R)$ contains all nilpotent right R-subgroups, even though this proof uses an argument similar to that given in Theorem 3.18.

Theorem 3.25 [10] Let R be a d.g. near-ring with identity. The right half-radical ${}^rJ_{1/2}(R)$ contains all nilpotent right R-subgroups of R^+.

Proof Let Ω be a nilpotent right R-subgroup of R^+, say $\Omega^p = (0)$. Define

$$Y = \Big\{ \sum_{i=1}^{n} (x_i + \omega_i' - x_i) \;\Big|\; x_i \in R,\ \omega_i' \in \Omega \Big\}.$$

Then Y is a right ideal of R containing Ω. Suppose that there exists a maximal right ideal B of R such that $\Omega \not\subseteq B$. Then $Y \not\subseteq B$ so $R = Y + B$ and the identity can be written as $1 = y + b$, with $y \in Y$ and $b \in B$. Let

$\omega_1, \omega_2, \ldots, \omega_{p-1} \in \Omega$. Then

$$\begin{aligned}
&\omega_1\omega_2\ldots\omega_{p-1} \\
&= (y+b)\omega_1\omega_2\ldots\omega_{p-1} \\
&= y\omega_1\omega_2\ldots\omega_{p-1} + b\omega_1\omega_2\ldots\omega_{p-1} \\
&= [\Sigma_{i=1}^{n}(x_i + \omega_i' - x_i)]\omega_1\omega_2\ldots\omega_{p-1} + b' \quad \text{for some} \;\; b' \in B \\
&= \Sigma_{i=1}^{n}(x_i\omega_1\omega_2\ldots\omega_{p-1} + \omega_i'\omega_1\omega_2\ldots\omega_{p-1} - x_i\omega_1\omega_2\ldots\omega_{p-1}) + b' \\
&= b',
\end{aligned}$$

since $\omega_i'\omega_1\omega_2\ldots\omega_{p-1} \in \Omega^p = (0)$. Hence $\Omega^{p-1} \subseteq B$.
Similarly,

$$\begin{aligned}
&\omega_1\omega_2\ldots\omega_{p-2} \\
&= (y+b)\omega_1\omega_2\ldots\omega_{p-2} \\
&= y\omega_1\omega_2\ldots\omega_{p-2} + b\omega_1\omega_2\ldots\omega_{p-2} \\
&= [\Sigma_{i=1}^{n}(x_i + \omega_i' - x_i)]\omega_1\omega_2\ldots\omega_{p-2} + b'' \quad \text{for some} \;\; b'' \in B \\
&= \Sigma_{i=1}^{n}(x_i\omega_1\omega_2\ldots\omega_{p-2} + \omega_i'\omega_1\omega_2\ldots\omega_{p-2} - x_i\omega_1\omega_2\ldots\omega_{p-2}) + b'' \\
&\in B,
\end{aligned}$$

since $\omega_i'\omega_1\omega_2\ldots\omega_{p-2} \in \Omega^{p-1} \subseteq B$. Hence $\Omega^{p-2} \subseteq B$.
Repeating this procedure we finally arrive at the contradiction $\Omega \subseteq B$. ■

We next proceed with the relationship between the left and right radicals in the case of finite d.g. near-rings with identity.

Lemma 3.26 [21] Every finite simple d.g. near-ring R with identity is right 2-primitive.

Proof As R^+ is a finite d.g. right R-group, let Ω denote a non-zero minimal d.g. right R-subgroup of R^+ and let Δ be a distributive basis for Ω. Since Ω

is minimal as a d.g. right R-subgroup, $\delta R = \Omega$ for all non-zero $\delta \in \Delta$. If Ω is not right type-2, then Ω has a proper, non-zero normal right R-subgroup. Since R is finite, let Ω' denote a maximal proper normal right R-subgroup of Ω. Thus Ω/Ω' is a right R-group of type-2. Since R is simple and has identity, $(\Omega/\Omega' : 0) = (0)$. ∎

Thus we have the following improvement on Lemma 3.22 in the finite case.

Corollary 3.27 [11] Let R be a finite d.g. near-ring with identity. Then every maximal ideal of R is right 2-primitive.

Corollary 3.28 [11] Let R be a finite d.g. near-ring with identity. Then

$$^{r}J_2(R) \subseteq J_2(R).$$

Proof From the previous Corollary,

$$^{r}J_2(R) \subseteq \cap\{M \mid M \text{ is a maximal ideal of } R\},$$

and as R is finite, $J_2(R) = \cap\{M \mid M \text{ is a maximal ideal of } R\}$ [14]. ∎

This, together with Theorem 3.23, proves

Theorem 3.29 [11] If R is a finite d.g. near-ring with identity, then

$$J_2(R) = {}^{r}J_0(R) = {}^{r}J_{1/2}(R) = {}^{r}J_2(R).$$

Corollary 3.30 [11] Let R be a finite d.g. near-ring with identity. Then R is left 2-semisimple if and only if R is right 2-semisimple.

Remark 3.31 1 As the mapping $R \to {}^rJ_2(R)$ is known to be Kurosh-Amitsur on the universal class of all zero-symmetric near-rings [12], Theorem 3.29 verifies that in the case of finite d.g. near-rings with identity the right radicals are indeed Kurosh-Amitsur, but whether this is true in general is not known.

2 In the finite case the same radical, $J_2(R)$, can be reached using left or right representation, however this is not the case for $J_0(R)$ or $J_{1/2}(R)$ as examples of finite d.g. near-rings exist for which $J_0(R) \subseteq J_{1/2}(R) \neq J_2(R)$.

3 Note that ${}^rJ_{1/2}(R)$ is a two-sided ideal in the case of finite d.g. near-rings with identity.

4 In [14] Laxton asks whether for right d.g. near-rings with identity $J_2(R)$ is the intersection of all maximal right ideals. Theorem 3.29 gives the affirmative answer for finite d.g. near-rings with identity but the problem remains unsolved for infinite near-rings.

For the case of finite d.g. near-rings with identity, we have $J_2(R) = {}^rJ_{1/2}(R)$ or, in other words,

$$\cap\{X \mid X \text{ a maximal ideal of } R\} = \cap\{X \mid X \text{ a maximal right ideal of } R\}.$$

The question arises whether every maximal right ideal of a finite d.g. near-ring R with identity is a two-sided ideal. The following example using [1] shows that this is not the case, even for rings.

Example 3.32 Consider the finite near-ring R where $R = \{n_0, \ldots, n_{15}\}$ and

$+$ and $\cdot$ are given in the following tables.

$+$	n_0	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	n_{10}	n_{11}	n_{12}	n_{13}	n_{14}	n_{15}
n_0	n_0	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	n_{10}	n_{11}	n_{12}	n_{13}	n_{14}	n_{15}
n_1	n_1	n_0	n_3	n_2	n_5	n_4	n_7	n_6	n_9	n_8	n_{11}	n_{10}	n_{13}	n_{12}	n_{15}	n_{14}
n_2	n_2	n_3	n_0	n_1	n_6	n_7	n_4	n_5	n_{10}	n_{11}	n_8	n_9	n_{14}	n_{15}	n_{12}	n_{13}
n_3	n_3	n_2	n_1	n_0	n_7	n_6	n_5	n_4	n_{11}	n_{10}	n_9	n_8	n_{15}	n_{14}	n_{13}	n_{12}
n_4	n_4	n_5	n_6	n_7	n_0	n_1	n_2	n_3	n_{12}	n_{13}	n_{14}	n_{15}	n_8	n_9	n_{10}	n_{11}
n_5	n_5	n_4	n_7	n_6	n_1	n_0	n_3	n_2	n_{13}	n_{12}	n_{15}	n_{14}	n_9	n_8	n_{11}	n_{10}
n_6	n_6	n_7	n_4	n_5	n_2	n_3	n_0	n_1	n_{14}	n_{15}	n_{12}	n_{13}	n_{10}	n_{11}	n_8	n_9
n_7	n_7	n_6	n_5	n_4	n_3	n_2	n_1	n_0	n_{15}	n_{14}	n_{13}	n_{12}	n_{11}	n_{10}	n_9	n_8
n_8	n_8	n_9	n_{10}	n_{11}	n_{12}	n_{13}	n_{14}	n_{15}	n_0	n_1	n_2	n_3	n_4	n_5	n_6	n_7
n_9	n_9	n_8	n_{11}	n_{10}	n_{13}	n_{12}	n_{15}	n_{14}	n_1	n_0	n_3	n_2	n_5	n_4	n_7	n_6
n_{10}	n_{10}	n_{11}	n_8	n_9	n_{14}	n_{15}	n_{12}	n_{13}	n_2	n_3	n_0	n_1	n_6	n_7	n_4	n_5
n_{11}	n_{11}	n_{10}	n_9	n_8	n_{15}	n_{14}	n_{13}	n_{12}	n_3	n_2	n_1	n_0	n_7	n_6	n_5	n_4
n_{12}	n_{12}	n_{13}	n_{14}	n_{15}	n_8	n_9	n_{10}	n_{11}	n_4	n_5	n_6	n_7	n_0	n_1	n_2	n_3
n_{13}	n_{13}	n_{12}	n_{15}	n_{14}	n_9	n_8	n_{11}	n_{10}	n_5	n_4	n_7	n_6	n_1	n_0	n_3	n_2
n_{14}	n_{14}	n_{15}	n_{12}	n_{13}	n_{10}	n_{11}	n_8	n_9	n_6	n_7	n_4	n_5	n_2	n_3	n_0	n_1
n_{15}	n_{15}	n_{14}	n_{13}	n_{12}	n_{11}	n_{10}	n_9	n_8	n_7	n_6	n_5	n_4	n_3	n_2	n_1	n_0

$\cdot$	n_0	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	n_{10}	n_{11}	n_{12}	n_{13}	n_{14}	n_{15}
n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0	n_0
n_1	n_0	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	n_{10}	n_{11}	n_{12}	n_{13}	n_{14}	n_{15}
n_2	n_0	n_2	n_2	n_0	n_{11}	n_9	n_9	n_{11}	n_{11}	n_9	n_9	n_{11}	n_0	n_2	n_2	n_0
n_3	n_0	n_3	n_0	n_3	n_{15}	n_{12}	n_{15}	n_{12}	n_3	n_0	n_3	n_0	n_{12}	n_{15}	n_{12}	n_{15}
n_4	n_0	n_4	n_{15}	n_{11}	n_1	n_5	n_{14}	n_{10}	n_8	n_{12}	n_7	n_3	n_9	n_{13}	n_6	n_2
n_5	n_0	n_5	n_{13}	n_8	n_5	n_0	n_8	n_{13}	n_0	n_5	n_{13}	n_8	n_5	n_0	n_8	n_{13}
n_6	n_0	n_6	n_{13}	n_{11}	n_{10}	n_{12}	n_7	n_1	n_3	n_5	n_{14}	n_8	n_9	n_{15}	n_4	n_2
n_7	n_0	n_7	n_{15}	n_8	n_{14}	n_9	n_1	n_6	n_{11}	n_{12}	n_4	n_3	n_5	n_2	n_{10}	n_{13}
n_8	n_0	n_8	n_0	n_8	n_{13}	n_5	n_{13}	n_5	n_8	n_0	n_8	n_0	n_5	n_{13}	n_5	n_{13}
n_9	n_0	n_9	n_2	n_{11}	n_9	n_0	n_{11}	n_2	n_0	n_9	n_2	n_{11}	n_9	n_0	n_{11}	n_2
n_{10}	n_0	n_{10}	n_2	n_8	n_6	n_{12}	n_4	n_{14}	n_3	n_9	n_1	n_{11}	n_5	n_{15}	n_7	n_{13}
n_{11}	n_0	n_{11}	n_0	n_{11}	n_2	n_9	n_2	n_9	n_{11}	n_0	n_{11}	n_0	n_9	n_2	n_9	n_2
n_{12}	n_0	n_{12}	n_{15}	n_3	n_{12}	n_0	n_3	n_{15}	n_0	n_{12}	n_{15}	n_3	n_{12}	n_0	n_3	n_{15}
n_{13}	n_0	n_{13}	n_{13}	n_0	n_8	n_5	n_5	n_8	n_8	n_5	n_5	n_8	n_0	n_{13}	n_{13}	n_0
n_{14}	n_0	n_{14}	n_{13}	n_3	n_7	n_9	n_{10}	n_4	n_{11}	n_5	n_6	n_8	n_{12}	n_2	n_1	n_{15}
n_{15}	n_0	n_{15}	n_{15}	n_0	n_3	n_{12}	n_{12}	n_3	n_3	n_{12}	n_{12}	n_3	n_0	n_{15}	n_{15}	n_0

In [1], $R = \text{LibraryNearRingWithOne}(\text{GTW16\_5}, 23)$ and is d.g. with identity n_1. In fact, R is a ring. The following sets are maximal right ideals of R: $\{n_0, n_2, n_9, n_{11}\}$, $\{n_0, n_3, n_{12}, n_{15}\}$ and $\{n_0, n_5, n_8, n_{13}\}$, but they are not two-sided ideals as R is simple. Clearly ${}^rJ_{1/2}(R)$, the intersection of all maximal right ideals of R, equals (n_0), and as R is simple and finite,

$$J_0(R) = J_{1/2}(R) = J_2(R) = {}^rJ_0(R) = {}^rJ_{1/2}(R) = {}^rJ_2(R) = (n_0).$$

Also note that the sets $\{n_0, n_2, n_{13}, n_{15}\}$, $\{n_0, n_5, n_9, n_{12}\}$ and $\{n_0, n_3, n_8, n_{11}\}$ are the maximal left ideals of R.

The next result shows that right R-groups of type-0 and those of type-2 are equivalent in finite d.g. near-rings with identity.

Proposition 3.33 If R is a finite d.g. near-ring with identity and Ω is a d.g. right R-group, then Ω is right type-0 if and only if Ω is right type-2.

Proof If Ω is right type-0, then let $\Omega \cong R/Y$ where Y is a maximal right ideal of R. Using Theorem 3.29,

$${}^rJ_2(R) = {}^rJ_0(R) \subseteq (\Omega : 0) = (R/Y : 0) = (R : Y) \subseteq Y.$$

Let $\bar{Y} := Y/{}^rJ_2(R)$. By Theorem 3.14(ii),

$$\bar{R} := R/{}^rJ_2(R) = \Delta_1 \oplus \cdots \oplus \Delta_n,$$

where each Δ_i is a right R-group of type-2. By Lemma 2.25, as $\bar{Y}$ is a normal right R-subgroup of $\bar{R}$, $\bar{R} = (\bigoplus_{i \in I} \Delta_i) \bigoplus \bar{Y}$ for some $I \subset \{1, \ldots, n\}$. Then

$$\Omega \cong R/Y \cong \bar{R}/\bar{Y} \cong \bigoplus_{i \in I} \Delta_i.$$

Now, as Ω is a right R-group of type-0 and has no proper, non-zero normal right R-subgroups, the index set I must be a singleton so that $\Omega \cong \Delta_i$ for some $1 \leq i \leq n$. Accordingly, Ω is a right R-group of type-2. ■

3.4 Examples

We consider four examples of d.g. near-rings, concentrating on the relationships between the left and right radicals. The first two examples are finite d.g. near-rings with identity and thus by Theorem 3.29 we know that $J_2(R) = {}^rJ_0(R) = {}^rJ_{1/2}(R) = {}^rJ_2(R)$. The last two near-rings do not meet all of the conditions of Theorem 3.29, so we use these examples to explore the possible relationships that exist between the left and right radicals in other classes of near-rings.

Example 3.34 [2] Let $G := S_n$, the symmetric group of degree n, for $n \geq 5$. Let R be the d.g. near-ring generated by $\mathrm{End}(G)$ in $M_0(G)$. The following results regarding R can be found in [2]:

(i) There exists an ideal N of R such that $N^2 = (0)$.

(ii) Let $H := A_n = \{0 = g_0, \ldots, g_m\}$ and $G - H = \{g_{m+1}, \ldots, g_{2m+1}\}$, where $2(m+1) = n!$. The near-ring R contains an subnear-ring which is isomorphic to $M_0(H)$. We can write $R = M_0(H) + N + \mathbb{Z}_2$, where

$\mathbb{Z}_2 = \{0, \theta\}$ and θ maps $G - H$ to $(1, 2)$ and H to zero.

(iii) $M_0(H) = \sum_{i=1}^{m} M_0(H)\epsilon_i$, where ϵ_i maps g_i to itself and $H - \{g_i\}$ to zero.

(iv) $N = \sum_{i=1}^{m+1} M_0(H)\delta_i$, where δ_i maps g_{i+m} to g_1 and $G - \{g_{i+m}\}$ to zero.

(v) The following is a complete list of left ideals of R:

$$\sum_{k\in K} M_0(H)\delta_k + \sum_{i\in I} M_0(H)\epsilon_i, \quad \sum_{i\in I} M_0(H)\epsilon_i + N + \mathbb{Z}_2,$$

where $K \subseteq \{1, \ldots, m+1\}$ and $I \subseteq \{1, \ldots, m\}$.

(vi) The following is a complete list of right ideals of R:

$$(0), \quad N, \quad N + \mathbb{Z}_2, \quad M_0(H) + N \;\text{ and }\; R.$$

(vii) All right ideals of R are two-sided ideals.

We use the above results to determine the left and right radicals of R. Consider the left ideals listed in (v). Firstly, when $K = \{1, \ldots, m+1\}$ and $I = \{1, \ldots, m\}$, then by (iii) and (iv),

$$\sum_{k\in K} M_0(H)\delta_k + \sum_{i\in I} M_0(H)\epsilon_i = N + M_0(H),$$

which by (ii) is not equal to R, and thus is a maximal left ideal. Next, when $I = \{1, \ldots, m\}$, then by (ii) and (iii), the left ideal $\sum_{i\in I} M_0(H)\epsilon_i + N + \mathbb{Z}_2$ is equal to R. Therefore, maximal left ideals of R will arise from $\sum_{i\in I} M_0(H)\epsilon_i + N + \mathbb{Z}_2$ only when I is a subset of $\{1, \ldots, m\}$ with cardinality $m - 1$. Taking the intersection of all maximal left ideals, we have $J_{1/2}(R) = N$. Hence $J_{1/2}(R)$ is an ideal of R and therefore coincides with $J_0(R)$. By (vii), $J_2(R)$, the intersection of all maximal ideals of R, coincides with the intersection of all maximal right ideals, i.e., with ${}^rJ_{1/2}(R)$. Using (vi), the only right ideals of R that are maximal are $N + \mathbb{Z}_2$ and $M_0(H) + N$ and their intersection is N. Consequently,

$$J_0(R) = J_{1/2}(R) = J_2(R) = {}^rJ_0(R) = {}^rJ_{1/2}(R) = {}^rJ_2(R) = N.$$

The next example focuses on finding the right 0-primitive ideals. A similar example was used in [7] to illustrate that it is possible for $J_0(R) \neq J_{1/2}(R) \neq J_s(R) \neq J_2(R)$ to hold.

Example 3.35 Consider the set $X = \{1, 2, \ldots, 12\}$ and let $X_1 = \{1, 2, 3, 4, 5\}$ and $X_2 = \{6, 7, 8, 9, 10, 11\}$. Denote the alternating group on X_1 by A_5 and the alternating group on X_2 by A_6. We regard A_5 and A_6 as subgroups of A_{12}, the alternating group on X. We use $+$ as the group operation in A_{12} and denote the permutation groups on X_1 and X_2 by S_5 and S_6, respectively. For each $x \in S_5$ or S_6, define a map $\phi_x : A_{12} \to A_{12}$ by

$$\phi_x : a \mapsto x + a - x$$

for all $a \in A_{12}$. Let Φ denote the set of all these mappings. Clearly Φ is a set of automorphisms of A_{12} which induce automorphisms on A_5 and A_6. We note that if $x \in S_5$, then ϕ_x is the identity map on A_6 and if $x \in S_6$, then ϕ_x is the identity map on A_5. Let T be the d.g. near-ring generated by Φ. Then T has identity and A_{12} is a faithful left T-group. We show that A_{12} is of type-0 by finding a monogenic generator. For this purpose we characterize the T-subgroups of A_{12}.

Let (a,b,c) be a fixed 3-cycle in A_5 and let e and d be the remaining two elements from X_1. Now $\phi_{(a,d)}(a,b,c) = (a,d)+(a,b,c)+(a,d) = (d,b,c) \in T(a,b,c)$. Similarly $(a,b)+(a,b,c)+(a,b) = (a,c,b) \in T(a,b,c)$ and $(a,e)+(a,c,b)+(a,e) = (e,c,b) \in T(a,b,c)$. We see that $(d,b,c)+(e,c,b) = (e,d,b) \in T(a,b,c)$. It readily follows that the monogenic left T-group $T(a,b,c)$ contains all 20 3-cycles so that $T(a,b,c) = A_5$.

The left T-group $T((a,b)+(c,d))$ contains $(a,e)+(a,b)+(c,d)+(a,e) = (e,b)+(c,d)$ and so contains $(a,b)+(c,d)+(e,b)+(c,d) = (a,b,e)$. It follows, as above, that $T((a,b)+(c,d))$ contains all 3-cycles and so coincides with A_5.

Now $T(a,b,c,d,e)$ contains $(a,d)+(a,b,c,d,e)+(a,d) = (a,e,d,b,c)$, and so contains $(a,b,c,d,e)+(a,e,d,b,c) = (b,d,c)$. Again one concludes that $T(a,b,c,d,e) = A_5$.

Since every non-zero element of A_5 is a monogenic generator, it is of type-2. Similarly one shows that A_6 is a left T-group of type-2.

Consider now the left T-subgroup $T(a,b,12)$ of A_{12}, where $a,b \in X_1$. Let $c \in X_1,\ c \neq a,\ c \neq b$. From $(a,c)+(a,b,12)+(a,c) = (c,b,12)$ and $(a,b,12)+(c,12,b) = (a,b,c) \in T(a,b,12)$, we see that $T(a,b,12)$ properly contains A_5. Indeed, A_5 is the only proper, non-zero left T-subgroup of $T(a,b,12)$ and as $(1,12,3)+(1,2,3)+(1,3,12) = (1,12,2) \notin A_5$ shows that A_5 is not normal in $T(a,b,12)$, we see that $K_1 = T(a,b,12)$ is of type-s. Similarly, $K_2 = T(a,b,12),\ a,b \in X_2$, is of type-$s$ with A_6 the only non-zero left T-subgroup. It is clear that $J_2(T) \neq J_s(T)$.

Consider the monogenic left T-subgroup H of A_{12} generated by $(a,b,c)+(e,d,12)$, where $a,b \in X_1$ and $c,d,e \in X_2$. From $(d,e)+(a,b,c)+(d,e,12)+(d,e) = (a,b,c)+(e,d,12) \in H$, we see that $(a,b,c)+(d,e,12)+(a,b,c)+(e,d,12) = (a,b,c) \in H$. Thus H contains the alternating group on the set of symbols $X_1 \cup X_2$. But $(a,b)+(a,b,c)+(d,e,12)+(a,b) = (a,c,b)+(d,e,12) \in$

H. This gives $(a,b,c)+(d,e,12)+(a,b,c)+(d,e,12)=(d,12,e)$. It readily follows that H is all of A_{12} and so is of type-0.

Let $R = T/J_s(T)$. Then by Lemma 2.14 in [7]

$$R = L_1 \oplus \cdots \oplus L_k \oplus L_{k+1} \oplus \cdots \oplus L_r \oplus L_{r+1} \oplus \cdots \oplus L_s \oplus L_{s+1} \oplus \cdots \oplus L_t,$$

where

$$\begin{aligned} L_i &\cong K_1, \quad i = 1,\ldots,k, \\ L_i &\cong K_2, \quad i = k+1,\ldots,r, \\ L_i &\cong A_5, \quad i = r+1,\ldots,s, \\ L_i &\cong A_6, \quad i = s+1,\ldots,t. \end{aligned}$$

We have the maximal ideals

$$M_1 = L_1 \oplus \cdots \oplus L_s \quad \text{and} \quad M_2 = L_1 \oplus \cdots \oplus L_r \oplus L_{s+1} \oplus \cdots \oplus L_t.$$

If Y_i is a maximal right ideal containing M_i, $i = 1,2$, then M_i is the annihilating ideal of the d.g. right R-group R/Y_i of type-0. Thus the M_i are right 0-primitive ideals and their intersection is the right 0-radical, ${}^rJ_0(R)$. We note that this right 0-radical is also $J_2(R) = L_1 \oplus \cdots \oplus L_r$.

Although the right radicals are defined for all d.g. near-rings, the relationships that exist between the left and right radicals were studied only for those d.g. near-rings with identity. In the following example we consider the left and right radicals of a d.g. near-ring without identity.

Example 3.36 Let R be the d.g. near-ring without identity in Example 2.4, page 8. We noted in Example 2.26, page 19, that $I = \{n_0, n_1, n_4, n_5\}$ is the only right ν-primitive ideal of R for $\nu = 0,2$, so that ${}^rJ_0(R) = {}^rJ_{1/2}(R) = {}^rJ_2(R) = I$. Furthermore, I is the only left ν-primitive ideal of R for $\nu = 0,2$.

We conclude that

$$I = J_0(R) = J_{1/2}(R) = J_2(R) = {}^rJ_0(R) = {}^rJ_{1/2}(R) = {}^rJ_2(R).$$

Thus the relationship between the right radicals and the left 2-radical in this finite d.g. near-ring without identity is identical to that in all finite d.g. near-rings with identity. It is not known if d.g. near-rings without identity exist in which the right radicals do not coincide. Furthermore, whether or not any relationship exists between the left and right radicals in infinite d.g. near-rings either with or without identity is an open question. We present an example of an infinite d.g. near-ring that does not satisfy the DCCLS in order to illustrate that a situation similar to the finite case can occur.

Example 3.37 [17] Let D denote the infinite dihedral group. We write

$$\begin{aligned} D &= \mathrm{Gp}\langle a, b \mid a+b+a+b=0,\ 2b=0\rangle \\ &= \{na+mb \mid n \in \mathbb{Z},\ m=0,1\}. \end{aligned}$$

The list of non-zero endomorphisms of D is

$$\operatorname{End} D := \{\alpha_{r,s},\ \beta_t,\ \gamma_t,\ \delta_t \mid r \neq 0,\ r,s,t \in \mathbb{Z}\},$$

where

$$\begin{array}{llll} \alpha_{r,s}a^{2n} = a^{2nr}, & \beta_t a^{2n} = 1, & \gamma_t a^{2n} = 1, & \delta_t a^{2n} = 1, \\ \alpha_{r,s}a^{2n+1} = a^{(2n+1)r}, & \beta_t a^{2n+1} = a^t b, & \gamma_t a^{2n+1} = a^t b, & \delta_t a^{2n+1} = 1, \\ \alpha_{r,s}a^{2n}b = a^{2nr+s}b, & \beta_t a^{2n}b = a^t b, & \gamma_t a^{2n}b = 1, & \delta_t a^{2n}b = a^t b, \\ \alpha_{r,s}a^{2n+1}b = a^{(2n+1)r+s}b, & \beta_t a^{2n+1}b = 1, & \gamma_t a^{2n+1}b = a^t b, & \delta_t a^{2n+1}b = a^t b. \end{array}$$

Let R be the right d.g. near-ring generated by the endomorphisms of D. To show that R does not satisfy the DCCLS, let p be any prime and consider the descending chain of left R-subgroups of R^+

$$R\alpha_{p,0} \supseteq R\alpha_{p,0}^2 \supseteq R\alpha_{p,0}^3 \supseteq \dots .$$

We show that $R\alpha_{p,0}^k \neq R\alpha_{p,0}^{k+1}$ for $k \in \mathbb{N}$, thereby concluding that the above chain is strictly descending. For this purpose, we look at how the generators of $R\alpha_{p,0}^k$ act on the group D. Now $\alpha_{p,0}^k = \alpha_{p^k,0}$, and for $r, s, t \in \mathbb{Z},\ r \neq 0$, we have

$$\begin{aligned}
\alpha_{r,s}\alpha_{p^k,0} &= \alpha_{p^k r,s}, \\
\beta_t\alpha_{p^k,0} &= \begin{cases} \delta_t & \text{if } p^k \text{ is even} \\ \beta_t & \text{if } p^k \text{ is odd} \end{cases}, \\
\gamma_t\alpha_{p^k,0} &= \begin{cases} 0 & \text{if } p^k \text{ is even} \\ \gamma_t & \text{if } p^k \text{ is odd} \end{cases}, \\
\delta_t\alpha_{p^k,0} &= \delta_t.
\end{aligned}$$

The action of the endomorphisms $\beta_t,\ \gamma_t$ and δ_t is given above, and that of $\alpha_{p^k r,s}$ by

$$\begin{aligned}
\alpha_{p^k r,s}a^{2n} &= a^{2np^k r}, & \alpha_{p^k r,s}a^{2n+1} &= a^{(2n+1)p^k r}, \\
\alpha_{p^k r,s}ba^{2n} &= a^{2np^k r+s}b, & \alpha_{p^k r,s}ba^{2n+1} &= a^{(2n+1)p^k r+s}b.
\end{aligned}$$

Accordingly, the images all lie in $\mathrm{Gp}\langle a^{p^k}, a^r b\rangle$. As $a^{p^k} \notin \mathrm{Gp}\langle a^{p^{k+1}}, a^r b\rangle$, it follows that $\mathrm{Gp}\langle a^{p^k}, a^r b\rangle \neq \mathrm{Gp}\langle a^{p^{k+1}}, a^r b\rangle$, and this gives an infinite descending chain.

Next we consider the relationship between the left and right radicals in this infinite near-ring. We have

$$J_0(R) = J_{1/2}(R) = J_2(R) = [R, R],$$

where $[R, R] = \mathrm{Gp}\,\langle [r, s] \mid r \in R, s \in R\rangle$ and $[r, s] = r + s - r - s$. As $[R, R]$ is nilpotent, by Theorem 3.20

$$J_0(R) = J_{1/2}(R) = J_2(R) \subseteq {}^{r}J_0(R).$$

Also, $\bar{R} = R/[R, R]$ is a ring, so

$$\begin{aligned}
&\text{Jacobson radical of } \bar{R} \\
&= \cap\{\bar{M} \mid \bar{M} \text{ is a maximal left ideal of } \bar{R}\} \\
&= \cap\{\bar{M} \mid \bar{M} \text{ is a maximal right ideal of } \bar{R}\}.
\end{aligned}$$

Using the one-to-one correspondence between the left ideals (right ideals) of R containing $[R, R]$, and the left ideals (respectively, right ideals) in $\bar{R}$, we have

$$\cap\{M \mid M \text{ maximal left ideal of } R\} = \cap\{M \mid M \text{ maximal right ideal of } R\}.$$

Therefore

$$J_{1/2}(R) = {}^{r}J_{1/2}(R).$$

So

$$J_0(R) = J_{1/2}(R) = J_2(R) = {}^{r}J_0(R) = {}^{r}J_{1/2}(R).$$

The relationship between these radicals and ${}^{r}J_2(R)$ is unknown.

3.5 Right radicals and right quasi-regularity

It is well-known that both $J_0(R)$ and $J_{1/2}(R)$ are left quasi-regular but $J_2(R)$ need not be. We now look at the connection between right quasi-regularity and the right radicals. It was shown in Section 3.3 that for d.g. near-rings with identity, ${}^rJ_0(R)$ contains all the nilpotent left R-subgroups of R^+ and ${}^rJ_{1/2}(R)$ all the nilpotent right R-subgroups of R^+. We attempt to generalize these results to right quasi-regularity. The results in this section all appear in [11].

Proposition 3.38 Let R be a d.g. near-ring. The right ideal ${}^rJ_{1/2}(R)$ is right quasi-regular.

Proof Let $z \in {}^rJ_{1/2}(R)$ and suppose that X is a right ideal of R containing $\{-zr+r \mid r \in R\}$. If $z \notin X$ then by Zorn's Lemma we can choose a right ideal Y of R such that $X \subseteq Y$ and $z \notin Y$, and Y is maximal with respect to these properties. We show that Y is right 0-modular. Firstly, $-zr + r \in X \subseteq Y$ for all $r \in R$, so Y is right modular with z a left identity modulo Y. By Lemma 2.28, R/Y is a monogenic right R-group. Now suppose that K is a right ideal of R such that $Y \subset K \subseteq R$. Then $-zr + r \in K$ for all $r \in R$, and we must have $z \in K$ since Y is maximal with respect to the properties $-zr + r \in Y$ for all $r \in R$ and $z \notin Y$. Thus $K = R$ and Y is a maximal right ideal of R. Therefore the right factor R-group R/Y possesses no proper, non-zero normal right R-subgroups. Hence R/Y is a right R-group of type-0, so Y is right 0-modular and ${}^rJ_{1/2}(R) \subseteq Y$. This gives the contradiction $z \in Y$. We must have $z \in X$ and it follows that ${}^rJ_{1/2}(R)$ is right quasi-regular. ∎

A d.g. near-ring satisfying the DCCLS has nilpotent left half-radical [14]. On the other hand, for d.g. near-rings with identity and satisfying the DCCLS, if ${}^rJ_{1/2}(R)$ is nilpotent then so is $J_2(R)$ by Theorem 3.23. As there are many examples of finite d.g. near-rings with non-nilpotent $J_2(R)$, we conclude that ${}^rJ_{1/2}(R)$ need not be nilpotent, even in the finite case. So although the left half-radical is nilpotent in the case of the DCCLS, the symmetric result does not hold for right representation. However, $J_{1/2}(R)$ contains all left quasi-regular left ideals of R, and we prove the corresponding result for the right case. We first need the following lemma.

Lemma 3.39 Let R be a d.g. near-ring. Let X be a right modular right ideal of R with e a left identity modulo X. If e can be expressed as $e = q + x$ where $q \in R$ and $x \in X$, then q is also a left identity modulo X.

Proof As $e + X = q + X$ it follows that e and q have the same properties in the right R-group R/X. ■

Proposition 3.40 Every right quasi-regular right ideal of a d.g. near-ring R is contained in ${}^rJ_{1/2}(R)$.

Proof Let Q be a right quasi-regular right ideal of R and suppose that $Q \nsubseteq {}^rJ_{1/2}(R)$. Hence there exists a right 0-modular right ideal X of R such that $Q \nsubseteq X$. Now $Q+X$ is a right ideal of R properly containing the maximal right ideal X, so $Q + X = R$. Let e be a left identity modulo X, then there exists $q \in Q,\ x \in X$ such that $e = q + x$. By Lemma 3.39, q is also a left identity modulo X which, using the fact that $q \in Q$ is right quasi-regular, contradicts Lemma 2.38. Thus $Q \subseteq {}^rJ_{1/2}(R)$. ■

It follows from Proposition 3.38 that ${}^{r}J_0(R)$ is right quasi-regular. Furthermore, by Propositions 3.13(ii) and 3.40 we have the following.

Corollary 3.41 Let R be a d.g. near-ring. The largest right quasi-regular two-sided ideal of R is ${}^{r}J_0(R)$.

Remark 3.42 Using Theorem 3.23 we see that $J_2(R)$ is right quasi-regular in the case of d.g. near-rings R with identity that satisfy the DCCLS. However, examples of finite d.g. near-rings exist for which $J_2(R) \neq J_{1/2}(R)$ so $J_2(R)$ is not necessarily left quasi-regular. From this we can conclude that the converse of Corollary 2.37 is not true.

The left 2-radical contains all left quasi-regular left R-subgroups of R^+ [22]. For the corresponding right representation result we restrict ourselves to near-rings with identity and to d.g. right R-subgroups of R^+.

Proposition 3.43 Let R be a d.g. near-ring with identity. Every right quasi-regular d.g. right R-subgroup of R^+ is contained in ${}^{r}J_2(R)$.

PROOF Let Ω be a right quasi-regular d.g. right R-subgroup of R^+ with Δ a distributive basis, and suppose that $\Omega \not\subseteq {}^{r}J_2(R)$. By Theorem 3.8 there exists a right 2-modular right ideal X of R such that $\Omega \not\subseteq X$, and so there exists $\delta \in \Delta$ such that $\delta \notin X$. Now δR is a d.g. right R-group and $\delta R \not\subseteq X$. It is easily seen that $(\delta R + X)/X$ is a right R-subgroup of R/X, and since

$$(\delta R + X)/X \cong \delta R/(\delta R \cap X),$$

it follows that $(\delta R + X)/X$ is a d.g. right R-subgroup of R/X. As R/X is a right R-group of type-2, either $\delta R + X = R$ or $\delta R + X = X$, and as

$\delta R \nsubseteq X$, it follows that $\delta R + X = R$. Let e be a left identity modulo X, then there exists $\delta r \in \delta R,\ x \in X$ such that $\delta r + x = e$. By Lemma 3.39, δr is a left identity modulo X. Also $\delta r \in \Omega$ is right quasi-regular which contradicts Lemma 2.38. Thus $\Omega \subseteq {}^{r}J_2(R)$. ■

To summarize, for a d.g. near-ring R with identity we have the following results where $\surd^{*}$ denotes true in the case of the DCCLS.

	$J_0(R)$	$J_{1/2}(R)$	$J_2(R)$	${}^rJ_0(R)$	${}^rJ_{1/2}(R)$	${}^rJ_2(R)$
left quasi-regular	$\surd$	$\surd$	$\times$	$\times$	$\times$	$\times$
contains all l.q.r. left R-subgroups	$\times$	$\times$	$\surd$	$\surd^*$	$\surd^*$	$\surd^*$
contains all l.q.r. left ideals	$\times$	$\surd$	$\surd$	$\surd^*$	$\surd^*$	$\surd^*$
contains all l.q.r. ideals	$\surd$	$\surd$	$\surd$	$\surd^*$	$\surd^*$	$\surd^*$
nil	$\surd^*$	$\surd^*$	$\times$	$\times$	$\times$	$\times$
contains all nil left R-subgroups	$\times$	$\times$	$\surd$	$\surd^*$	$\surd^*$	$\surd^*$
contains all nil left ideals	$\times$	$\surd$	$\surd$	$\surd^*$	$\surd^*$	$\surd^*$
contains all nil ideals	$\surd$	$\surd$	$\surd$	$\surd^*$	$\surd^*$	$\surd^*$
nilpotent	$\surd^*$	$\surd^*$	$\times$	$\times$	$\times$	$\times$
contains all nilpotent left R-subgroups	$\times$	$\times$	$\surd$	$\surd$	$\surd$	$\surd$
contains all nilpotent left ideals	$\times$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
contains all nilpotent ideals	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
contains all nilpotent right R-subgroups			$\surd^*$	$\surd^*$	$\surd$	$\surd$
contains all nilpotent right ideals			$\surd^*$	$\surd^*$	$\surd$	$\surd$
right quasi-regular	$\surd^*$	$\surd^*$	$\surd^*$	$\surd$	$\surd$	
contains all r.q.r. d.g. right R-subgroups						$\surd$
contains all r.q.r. right ideals					$\surd$	$\surd$
contains all r.q.r. ideals			$\surd^*$	$\surd$	$\surd$	$\surd$

Chapter 4

An example

4.1 The near-ring T

We use the example of the d.g. near-ring T from Laxton [15] to illustrate that the notions introduced in Chapters 2 and 3 occur quite naturally.

Consider the d.g. near-ring P generated by all inner automorphisms of a finite non-abelian simple group G. This near-ring was studied by Fröhlich [5] and the following results will be required:

1 (i) $(0) = J_0(P) = J_{1/2}(P) = J_2(P) = {}^rJ_0(P) = {}^rJ_{1/2}(P) = {}^rJ_2(P)$.

(ii) $P = L_1 \oplus L_2 \oplus \cdots \oplus L_n$, where each left ideal L_i is a P-group of type-2 that is P-isomorphic to G. The number of summands n is equal to $|G| - 1$.

(iii) For each $i = 1, \ldots, n$, there exists $e_i \in L_i$ such that $Pe_i = L_i$,

$1 = \sum_{i=1}^{n} e_i$ and $\{e_i\}_{i=1}^{n}$ is an orthogonal idempotent set.

(iv) There is a one-to-one correspondence between the right P-subgroups of P^+ and the subgroups of G.

Now let T be the d.g. near-ring generated by the endomorphisms $\{\phi_x\}_{x \in P}$ of P, where

$$\phi_x \cdot y = yx \quad \text{for all } y \in P.$$

The following results concerning the left structure of the near-ring T can be found in [15]:

2 (i) T is a subnear-ring of $M_0(P^+)$ with P^+ a faithful left T-group.

(ii) The right P-subgroups of P^+ are precisely the left T-subgroups of P^+.

(iii) Under the correspondence mentioned in 1(iv), normal subgroups of G correspond to right ideals of P and so P^+ is a faithful left T-group of type-0.

(iv) The finite non-abelian simple group G can be chosen such that $(0) = J_0(T) \neq J_{1/2}(T) \neq J_2(T)$.

We now look more closely at the near-ring T, focusing on its right structure and on finding connections that exist between the structures of P and T.

Firstly, as T is finite, we have the following.

Lemma 4.1 The right radicals of T coincide with $J_2(T)$ and are non-zero.

Next we express T as a direct sum that is similar to the direct sum decomposition of P in 1(ii) and (iii).

Proposition 4.2 [10]

(i) The identity of T is of the form

$$1_T = \phi_{e_1} + \phi_{e_2} + \cdots + \phi_{e_n} = \phi_1,$$

and $\{\phi_{e_i}\}_{i=1}^n$ is an orthogonal idempotent set. Here the e_i are the orthogonal idempotent components of $1 \in P$.

(ii) $T = \bigoplus_{i=1}^n \phi_{e_i} T$ where each $\phi_{e_i} T$ is a right ideal of T and a d.g. right T-group, and $\phi_{e_i} T \cong \phi_{e_j} T,\ i, j = 1, \ldots, n$.

PROOF (i) Firstly, if $p \in P$ then

$$\begin{aligned}(\phi_{e_1} + \phi_{e_2} + \cdots + \phi_{e_n}) \cdot p &= \phi_{e_1} \cdot p + \phi_{e_2} \cdot p + \cdots + \phi_{e_n} \cdot p \\ &= pe_1 + pe_2 + \cdots + pe_n \\ &= p(e_1 + e_2 + \cdots + e_n) \\ &= p,\end{aligned}$$

where we have used left distribution over a direct sum of left ideals, and

$$\phi_1 \cdot p = p1 = p,$$

giving $\phi_{e_1} + \phi_{e_2} + \cdots + \phi_{e_n} = \phi_1$. Next, let $t \in T$ and $p \in P$ then $t \cdot p = p'$

for some $p' \in P$. Now

$$\begin{aligned}(\phi_1 t) \cdot p &= \phi_1 \cdot (t \cdot p) \quad \text{since } P \text{ is a left } T\text{-group} \\ &= \phi_1 \cdot p' \\ &= p' \\ &= t \cdot p,\end{aligned}$$

so $\phi_1 t = t$. It is similarly shown that ϕ_1 is a right identity and hence a two-sided identity of T. If $p \in P$ then

$$\begin{aligned}(\phi_{e_i}\phi_{e_j}) \cdot p &= \phi_{e_i} \cdot (\phi_{e_j} \cdot p) \\ &= \phi_{e_i} \cdot (pe_j) \\ &= (pe_j)e_i \\ &= p(e_j e_i) \\ &= \phi_{e_j e_i} \cdot p \\ &= \begin{cases} \phi_{e_i} \cdot p & \text{if } i = j \\ \phi_0 \cdot p & \text{if } i \neq j \end{cases}\end{aligned}$$

by the orthogonal idempotence of the e_i in P.

(ii) That $\phi_{e_i} T$ is a d.g. right T-group with distributive basis

$$\{\phi_{e_i}\phi_x \mid x \in P\} = \{\phi_{xe_i} \mid x \in P\}$$

follows from Proposition 2.10. To prove normality, let $t, t' \in T$ and

$p \in P$. Then

$$\begin{aligned}
(-t' + \phi_{e_i} t + t') \cdot p &= -t' \cdot p + (\phi_{e_i} t) \cdot p + t' \cdot p \\
&= -t' \cdot p + \phi_{e_i} \cdot (t \cdot p) + t' \cdot p \\
&= -t' \cdot p + (t \cdot p)e_i + t' \cdot p \\
&= xe_i \ \text{ for some } x \in P, \text{ as } Pe_i \text{ is normal in } P \\
&= (xe_i)e_i \ \text{ since } e_i^2 = e_i \\
&= (-t' \cdot p + (t \cdot p)e_i + t' \cdot p)e_i \\
&= (-t' \cdot p + t \cdot p + t' \cdot p)e_i \\
&= \phi_{e_i} \cdot (-t' \cdot p + t \cdot p + t' \cdot p) \\
&= (\phi_{e_i}(-t' + t + t')) \cdot p\,.
\end{aligned}$$

Hence $-t'+\phi_{e_i}t+t' = \phi_{e_i}(-t'+t+t') \in \phi_{e_i}T$. To show that the $\phi_{e_i}T$ are right T-isomorphic, we have by 1(ii) and (iii) that in the decomposition of $P = \bigoplus_{i=1}^n Pe_i$ there is the (left) P-isomorphism $Pe_k \cong Pe_j$. Let $e_k \mapsto pe_j$ under this isomorphism. Now define a map $\Psi : \phi_{e_k}T \to \phi_{e_j}T$ by

$$\Psi(\phi_{e_k} t) = \phi_{pe_j} t = \phi_{e_j}(\phi_p t).$$

If $\phi_{e_k}t = 0$, then $0 = (\phi_{e_k}t) \cdot s = \phi_{e_k} \cdot (t \cdot s) = (t \cdot s)e_k$ for all $s \in P$. Hence $(t \cdot s)pe_j = 0$ by the left P-isomorphism, and $(\phi_{pe_j}t) \cdot s = 0$ for all $s \in P$. As P^+ is a faithful left T-group, $\phi_{pe_j}t = 0$ and it follows that Ψ is well defined. Using the fact that the ϕ_x are distributive elements of T it is routinely verified that Ψ is a right T-isomorphism. Lastly, let $t \in T$ then $t = (\sum_{i=1}^n \phi_{e_i})t = \sum_{i=1}^n (\phi_{e_i}t)$, which gives $T = \sum_{i=1}^n \phi_{e_i}T$. If $x \in \phi_{e_i}T \cap \sum_{j \neq i} \phi_{e_j}T$, then $x = \phi_{e_i}t = \sum_{j \neq i} \phi_{e_j}t_j$ so that $x = \phi_{e_i}\phi_{e_i}t = \sum_{j \neq i} \phi_{e_i}\phi_{e_j}t = 0$ by the orthogonal idempotence of the the set $\{\phi_{e_i}\}$. We have established that $T = \bigoplus_{i=1}^n \phi_{e_i}T$. ∎

We next look at the relationship between the normal subgroups of P and the right ideals of T.

Proposition 4.3 [10]

(i) If Δ is a normal subgroup of P, then

$$X_\Delta := \{t \in T \mid t \cdot p \in \Delta \text{ for all } p \in P\}$$

is a right ideal of T.

(ii) If X is a right ideal of T and ω a distributive element of P, then

$$X \cdot \omega = \{x \cdot \omega \mid x \in X\}$$

is a normal subgroup of ωP. In particular, $X \cdot 1 = X \cdot P := \{x \cdot p \mid x \in X,\ p \in P\}$ is a normal subgroup of P.

(iii) If I is a two-sided ideal of T, then $I \cdot 1 = I \cdot P = P$ or $I \cdot 1 = I \cdot P = (0)$, and $I \cdot 1 = I \cdot P = (0)$ if and only if $I = (0)$.

Proof It is straight forward to show that X_Δ is a right ideal of T and that $X \cdot \omega$ is a subgroup of ωP. To prove the normality of $X \cdot \omega$, let $\omega p \in \omega P,\ p \in P$ and $x \cdot \omega \in X \cdot \omega,\ x \in X$. Then

$$-\omega p + x \cdot \omega + \omega p = -\phi_p \cdot \omega + x \cdot \omega + \phi_p \cdot \omega = (-\phi_p + x + \phi_p) \cdot \omega \in X \cdot \omega,$$

as X is normal in T. Clearly, with $\omega = 1$, $X \cdot 1 \subseteq X \cdot P$ and if $x \cdot p \in X \cdot P,\ x \in X,\ p \in P$ then $x \cdot p = x \cdot (\phi_p \cdot 1) = (x\phi_p) \cdot 1 \in X \cdot 1$. For (iii) suppose that I is a two-sided ideal of T, then $I \cdot 1$ is a normal subgroup of P and if $p \in P$

and $i \cdot 1 \in I \cdot 1$, we have $(i \cdot 1)p = \phi_p \cdot (i \cdot 1) = (\phi_p i) \cdot 1 \in I \cdot 1$. Thus $I \cdot 1$ is a right ideal of P. But by 2(iii), P has no proper, non-zero right ideals, so it follows that either $I \cdot 1 = P$ or $I \cdot 1 = (0)$. Finally, as P is a faithful T-group, $I \cdot 1 = I \cdot P = (0)$ if and only if I is the zero ideal. ■

Proposition 4.4 [10] Let $\Omega_i = L_1 \oplus \cdots \oplus L_{i-1} \oplus L_{i+1} \oplus \cdots \oplus L_n$ and define $X_i := X_{\Omega_i}$. Then $X_i = \bigoplus_{j \neq i} \phi_{e_j} T$.

Proof It is not hard to see from the definition that $\phi_{e_j} \in X_i$ for all $j \neq i$ so that $\phi_{e_j} T \subseteq X_i$ for $j \neq i$ as X_i is a right ideal of T. Using the modular law on the direct sum in Proposition 4.2, we have $X_i = \bigoplus_{j \neq i} \phi_{e_j} T \bigoplus (X_i \cap \phi_{e_i} T)$. Now $\phi_{e_i} t \in X_i \cap \phi_{e_i} T$ implies that $(\phi_{e_i} t) \cdot p = (t \cdot p) e_i \in P e_i \cap \Omega_i = (0)$ by the definition of X_i. Thus $(\phi_{e_i} t) \cdot p = 0$ for all $p \in P$, hence $\phi_{e_i} t = 0$. It follows that $X_i = \bigoplus_{j \neq i} \phi_{e_j} T$. ■

Proposition 4.5 [10] If X is a maximal right ideal of T then for some i, $1 \leq i \leq n$, either $\phi_{e_i} X = \phi_{e_i} T \cap X$ or $\phi_{e_i} X = \phi_{e_i} T$.

Proof If $\phi_{e_j} T \subseteq X$ for each $j = 1, \ldots, n$, then $X = T$. Hence there exists i such that $\phi_{e_i} T \not\subseteq X$ and, as X is a maximal right ideal of T, we have $T = \phi_{e_i} T + X$. Let $Y := \{t \in T \mid \phi_{e_i} t \in X\}$. Then Y is a right ideal of T and for $j \neq i$ we have $\phi_{e_j} \in Y$. Let X_i be the right ideal of T defined in Proposition 4.4. Then $X_i \subseteq Y$. It is readily shown that $\phi_{e_i} T \cap Y = \phi_{e_i} T \cap X$ and, using the modular law on $T = \phi_{e_i} T \oplus X_i$, that $Y = (\phi_{e_i} T \cap Y) \oplus X_i = (\phi_{e_i} T \cap X) \oplus X_i$. As X is a maximal right ideal of T, T/X is a right T-group of type-0 and so, from the right T-isomorphism

$$T/X = (\phi_{e_i} T + X)/X \cong \phi_{e_i} T / (\phi_{e_i} T \cap X),$$

$\phi_{e_i}T/(\phi_{e_i}T \cap X)$ is a right T-group of type-0. As $\phi_{e_i}X/(\phi_{e_i}T \cap X)$ is a normal right T-subgroup of $\phi_{e_i}T/(\phi_{e_i}T \cap X)$ we have either $\phi_{e_i}X = \phi_{e_i}T \cap X$ or $\phi_{e_i}X = \phi_{e_i}T$. ∎

If Y is the right ideal of T defined in the above proof, then it can be shown that Y is a maximal right ideal of T so that T/Y is a right T-group of type-0. Furthermore, as T is finite, T/Y is a right T-group of type-2.

4.2 Similarities between T and $\mathbb{M}_n(R)$

Let R be a d.g. near-ring with identity. Multiplication of the endomorphisms ϕ_{e_i} in T is similar to multiplication of the matrices f_{ij}^r in $\mathbb{M}_n(R)$ [19], since in T we have

$$\phi_{e_i}\phi_{e_j} = \begin{cases} \phi_0 & \text{if } i \neq j \\ \phi_{e_i} & \text{if } i = j \end{cases},$$

whilst in $\mathbb{M}_n(R)$

$$f_{ij}^a f_{kl}^b = \begin{cases} f_{il}^0 & \text{if } j \neq k \\ f_{il}^{ab} & \text{if } j = k \end{cases}.$$

Furthermore, the identity of T is $\phi_{e_1} + \phi_{e_2} + \cdots + \phi_{e_n}$ and in $\mathbb{M}_n(R)$ the identity is $f_{11}^1 + f_{22}^1 + \cdots + f_{nn}^1$. In this section we seek to find additional similarities in the structures of T and $\mathbb{M}_n(R)$. At the same time, we obtain another example of a near-ring in which d.g. right R-groups naturally occur. Recall the alternative notation $[r; i, j]$ for f_{ij}^r.

Lemma 4.6 Let R be a d.g. near-ring with identity. For $i = 1, \ldots, n$, $f_{ii}^1\,\mathbb{M}_n(R)$ is a faithful d.g. right $\mathbb{M}_n(R)$-subgroup of $\mathbb{M}_n(R)$ and a right ideal of $\mathbb{M}_n(R)$.

PROOF Each f^1_{ii} is a distributive element of $\mathbb{M}_n(R)$ so it follows from Proposition 2.10 that each $f^1_{ii}\,\mathbb{M}_n(R)$ is a d.g. right $\mathbb{M}_n(R)$-group with distributive basis $\{f^1_{ii}f^s_{ik} \mid s \in S \text{ and } 1 \leq k \leq n\} = \{f^s_{ik} \mid s \in S \text{ and } 1 \leq k \leq n\}$. We first show that $f^1_{11}\,\mathbb{M}_n(R)$ is normal in $\mathbb{M}_n(R)$. Let $U, V \in \mathbb{M}_n(R)$, then

$$
\begin{aligned}
&-U + f^1_{11}V + U \\
&= -(f^1_{11} + f^1_{22} + \cdots + f^1_{nn})U + f^1_{11}V + (f^1_{11} + f^1_{22} + \cdots + f^1_{nn})U \\
&= -f^1_{nn}U - \cdots - f^1_{22}U - f^1_{11}U + f^1_{11}V + f^1_{11}U + f^1_{22}U + \cdots + f^1_{nn}U \\
&= -f^1_{nn}U - \cdots - f^1_{22}U + f^1_{11}(-U + V + U) + f^1_{22}U + \cdots + f^1_{nn}U \\
&\quad \text{(by Lemma 2.9, since } f^1_{11} \text{is distributive)} \\
&= -f^1_{nn}U - \cdots - f^1_{22}U + f^1_{11}W + f^1_{22}U + \cdots + f^1_{nn}U, \qquad (*)
\end{aligned}
$$

where $W = -U + V + U \in \mathbb{M}_n(R)$. Write $W = \Sigma_{j,k}\epsilon_{jk}[s_{jk}; j, k]$ and $U = \Sigma_{p,q}\epsilon_{pq}[s'_{pq}; p, q]$ where $s_{jk}, s'_{pq} \in S$ and $\epsilon_{jk}, \epsilon_{pq} \in \{-1, 1\}$. Then

$$
\begin{aligned}
&-f^1_{22}U + f^1_{11}W + f^1_{22}U \\
= \;& -f^1_{22}\Sigma_{p,q}\epsilon_{pq}[s'_{pq}; p, q] + f^1_{11}\Sigma_{j,k}\epsilon_{jk}[s_{jk}; j, k] + f^1_{22}\Sigma_{p,q}\epsilon_{pq}[s'_{pq}; p, q] \\
= \;& -\Sigma_{p,q}\epsilon_{pq}f^1_{22}[s'_{pq}; p, q] + \Sigma_{j,k}\epsilon_{jk}f^1_{11}[s_{jk}; j, k] + \Sigma_{p,q}\epsilon_{pq}f^1_{22}[s'_{pq}; p, q] \\
= \;& -\Sigma_r\epsilon_{2r}[s'_{2r}; 2, r] + \Sigma_l\epsilon_{1l}[s_{1l}; 1, l] + \Sigma_r\epsilon_{2r}[s'_{2r}; 2, r] \\
& \text{(where if } q \in \{1, \ldots, g\} \text{ and } k \in \{1, \ldots, h\}, \text{ then } r \in Q \subseteq \{1, \ldots, g\} \text{ and} \\
& l \in L \subseteq \{1, \ldots, h\}; \text{ some of the terms in the sums may be 0, hence the} \\
& \text{subsets } Q \text{ and } L) \\
= \;& -\Sigma_r\epsilon_{2r}[s'_{2r}; 2, r] + \Sigma_r\epsilon_{2r}[s'_{2r}; 2, r] + \Sigma_l\epsilon_{1l}[s_{1l}; 1, l] \\
& \text{(using the result that } [w; u, v] + [z; x, y] = [z; x, y] + [w; u, v] \text{ for } u \neq x) \\
= \;& \Sigma_l\epsilon_{1l}[s_{1l}; 1, l] \\
= \;& \Sigma_{j,k}\epsilon_{jk}f^1_{11}[s_{jk}; j, k] \\
= \;& f^1_{11}\Sigma_{j,k}\epsilon_{jk}[s_{jk}; j, k] \\
= \;& f^1_{11}W.
\end{aligned}
$$

From equation $(*)$ we have

$$-U + f^1_{11}V + U = -f^1_{nn}U - \cdots - f^1_{33}U + f^1_{11}W + f^1_{33}U + \cdots + f^1_{nn}U,$$

and after repeating the above argument, we see that

$$-U + f^1_{11}V + U = f^1_{11}W \in f^1_{11}\mathbb{M}_n(R).$$

Thus $f^1_{11}\mathbb{M}_n(R)$ is normal in $\mathbb{M}_n(R)$. Similarly, we can write

$$-U+f^1_{22}V+U = -f^1_{nn}U-\cdots-f^1_{22}U-f^1_{11}U+f^1_{22}V+f^1_{11}U+f^1_{22}U+\cdots+f^1_{nn}U,$$

and if $V = \Sigma_{c,d}\epsilon_{cd}[s''_{cd}; c, d]$ where $\epsilon_{cd} = \pm 1$, $s''_{cd} \in S$ and $d \in \{1, \ldots, t\}$, then

$$\begin{aligned}
&-f^1_{11}U + f^1_{22}V + f^1_{11}U \\
= \;& -f^1_{11}\Sigma_{p,q}\epsilon_{pq}[s'_{pq}; p, q] + f^1_{22}\Sigma_{c,d}\epsilon_{cd}[s''_{cd}; c, d] + f^1_{11}\Sigma_{p,q}\epsilon_{pq}[s'_{pq}; p, q] \\
= \;& -\Sigma_a\epsilon_{1a}[s'_{1a}; 1, a] + \Sigma_b\epsilon_{2b}[s''_{2b}; 2, b] + \Sigma_a\epsilon_{1a}[s'_{1a}; 1, a], \\
& (\text{where } a \in A \subseteq \{1, \ldots, g\} \text{ and } b \in B \subseteq \{1, \ldots, t\}) \\
= \;& -\Sigma_a\epsilon_{1a}[s'_{1a}; 1, a] + \Sigma_a\epsilon_{1a}[s'_{1a}; 1, a] + \Sigma_b\epsilon_{2b}[s''_{2b}; 2, b] \\
= \;& \Sigma_b\epsilon_{2b}[s''_{2b}; 2, b] \\
= \;& \Sigma_{c,d}\epsilon_{cd}f^1_{22}[s''_{cd}; c, d] \\
= \;& f^1_{22}\Sigma_{c,d}\epsilon_{cd}[s''_{cd}; c, d] \\
= \;& f^1_{22}V. \qquad (**)
\end{aligned}$$

Therefore

$$\begin{aligned}
&-U + f^1_{22}V + U \\
= \;& -f^1_{nn}U - \cdots - f^1_{22}U + f^1_{22}V + f^1_{22}U + \cdots + f^1_{nn}U \\
= \;& -f^1_{nn}U - \cdots - f^1_{33}U + f^1_{22}W + f^1_{33}U + \cdots + f^1_{nn}U,
\end{aligned}$$

giving a situation similar to that in equation $(*)$. Proceeding in an analogous way we have $f^1_{22}\mathbb{M}_n(R)$ is normal in $\mathbb{M}_n(R)$. For $i = 3, \ldots, n$, after repeating the arguments used for equation $(**)$, we have

$$
\begin{aligned}
&-U + f^1_{ii}V + U \\
&= -f^1_{nn}U - \cdots - f^1_{ii}U - \cdots - f^1_{11}U + f^1_{ii}V + f^1_{11}U + \cdots + f^1_{ii}U + \cdots + f^1_{nn}U \\
&= -f^1_{nn}U - \cdots - f^1_{(i+1)(i+1)}U - f^1_{ii}U + f^1_{ii}V + f^1_{ii}U + f^1_{(i+1)(i+1)}U + \cdots + f^1_{nn}U \\
&= -f^1_{nn}U - \cdots - f^1_{(i+1)(i+1)}U + f^1_{ii}W + f^1_{(i+1)(i+1)}U + \cdots + f^1_{nn}U,
\end{aligned}
$$

which is of a similar form to equation $(*)$. Hence we can show that $f^1_{ii}\mathbb{M}_n(R)$ is normal in $\mathbb{M}_n(R)$ for $i = 3, \ldots, n$. Finally, we show that each $f^1_{ii}\mathbb{M}_n(R)$ is faithful. Suppose there exists $U \in \mathbb{M}_n(R)$ such that $f^1_{ii}\,\mathbb{M}_n(R)U = 0$. Then, in particular, for $j = 1, \ldots, n$ we have $f^1_{ii}f^1_{ij}U = 0$, so that $f^1_{ij}U = 0$. Hence $f^1_{jj}U = (f^1_{ji}f^1_{ij})U = f^1_{ji}(f^1_{ij}U) = 0$ and $U = (f^1_{11} + f^1_{22} + \cdots + f^1_{nn})U = f^1_{11}U + f^1_{22}U + \cdots + f^1_{nn}U = 0$. ■

Next we show how $\mathbb{M}_n(R)$ can be decomposed into a direct sum of normal d.g. right $\mathbb{M}_n(R)$-groups. This decomposition resembles that of the near-ring T in Proposition 4.2.

Proposition 4.7 Let R be a d.g. near-ring with identity. The matrix near-ring $\mathbb{M}_n(R)$ can be expressed as a direct sum of normal d.g. right $\mathbb{M}_n(R)$-groups, specifically,

$$\mathbb{M}_n(R) = \bigoplus_{i=1}^{n} f^1_{ii}\,\mathbb{M}_n(R).$$

Proof If $U \in \mathbb{M}_n(R)$, then $U = (f^1_{11} + f^1_{22} + \cdots + f^1_{nn})U = f^1_{11}U + f^1_{22}U + \cdots + f^1_{nn}U$, so that $\mathbb{M}_n(R) = f^1_{11}\,\mathbb{M}_n(R) + f^1_{22}\,\mathbb{M}_n(R) + \cdots + f^1_{nn}\,\mathbb{M}_n(R)$. Next, let $x \in f^1_{ii}\,\mathbb{M}_n(R) \cap \Sigma_{j\neq i} f^1_{jj}\,\mathbb{M}_n(R)$. Then $x = f^1_{ii}V$ for some $V \in \mathbb{M}_n(R)$ and $f^1_{ii}x = f^1_{ii}(f^1_{ii}V) = f^1_{ii}V = x$. We can also write $x = \Sigma_{j\neq i} f^1_{jj}U_j$ for some

$U_j \in \mathbb{M}_n(R)$, which gives $x = f^1_{ii}x = f^1_{ii}(\Sigma_{j\neq i} f^1_{jj}U_j) = \Sigma_{j\neq i}(f^1_{ii}f^1_{jj}U_j) = 0$. Thus $\mathbb{M}_n(R) = f^1_{11}\,\mathbb{M}_n(R) \oplus f^1_{22}\,\mathbb{M}_n(R) \oplus \cdots \oplus f^1_{nn}\,\mathbb{M}_n(R)$. ■

4.3 T and generalized matrix near-rings

In this section we relate the near-ring T with a generalized matrix near-ring [24].

Let G be a finite group and A a group of automorphisms of G. Consider $N = M_A(G)$, the centralizer near-ring determined by A and G, where $N = \{f \in M_0(G) \mid f(\alpha g) = \alpha f(g)$ for all $\alpha \in A$ and $g \in G\}$. Let $\mathbf{R}$ denote the right near-ring of N, i.e., the d.g. near-ring generated by the set $\{\rho_a \mid a \in N\}$, where each ρ_a is an endomorphism of N defined by $\rho_a(n) = na$ for all $n \in N$. It is shown [24] that $\mathbf{R}$ is isomorphic to the generalized matrix near-ring $\mathrm{Mat}_k(S; G)$, where k is the number of non-zero A-orbits of G, and S is the subnear-ring of $M_0(G)$ generated by A.

Now consider the case where G is a finite non-abelian simple group and $A = \{1\}$, the identity automorphism of G. It is easily seen that $N = M_0(G)$. As in Section 4.1, let R be the d.g. near-ring generated by the inner automorphisms of G and T the d.g. near-ring generated by the endomorphisms ϕ_x of R. It is shown (2.4 in [5]) that if f is a mapping from G into itself such that $f(0) = 0$, then $f \in R$, hence $R = M_0(G)$ and we have $R = N$. Clearly the actions of ϕ_x from T and ρ_a from $\mathbf{R}$ are identical, and so $T = \mathbf{R}$. With the specific choice of $A = \{1\}$, we see that $k = |G| - 1$, and conclude that $T \cong \mathrm{Mat}_{|G|-1}(S; G)$.

Chapter 5

The right socle-ideal

5.1 Construction of the right socle-ideal

The (left) socle-ideal of R, $Soi(R)$, was constructed by Hartney [8] and is an antiradical in the sense that it annihilates $J_{1/2}(R)$. We follow these results and adapt them to right representation in order to form the analogous right socle-ideal. In this chapter all near-rings are assumed to have identity.

Definition 5.1 Let R be a d.g. near-ring with identity. Let the symbol $\mathcal{F}$ denote the collection of all ideals A of R which are of the form $A = \bigoplus_{i \in I} e_i R$ such that

(i) each e_i is a distributive element of R

(ii) each $e_i R$ is a right R-group of type-0 and a right ideal of R

(iii) $e_i^2 = e_i$ for all $i \in I$ and $e_i e_j = 0$ if $i < j$ for some total ordering on I.

In this case we say that $\bigoplus_{i\in I} e_iR$ is an $\mathcal{F}$-**decomposition** of A.

Lemma 5.2 Let R be a d.g. near-ring with identity. Let $A, B \in \mathcal{F}$ and suppose that $A = \oplus_{i\in I} e_iR$ and $B = \oplus_{k\in K} f_kR$ are $\mathcal{F}$-decompositions of A and B respectively. If $A \subseteq B$, then there exists an index set T such that $I \subseteq T$ and $B = \oplus_{t\in T} e_tR$ is an $\mathcal{F}$-decomposition for B.

PROOF Using Lemma 2.25, we can write $B = A \oplus (\oplus_{k\in K'} f_kR) = (\oplus_{i\in I} e_iR) \oplus (\oplus_{k\in K'} f_kR)$ for some $K' \subseteq K$, where, without loss of generality, we assume that I and K' are disjoint. Consequently, $(f_kR)A \subseteq f_kR \cap A = (0)$ and so $f_ke_i = 0$ for all $k \in K'$ and $i \in I$. Therefore the e_i and the f_k are distinct. Put $T = I \cup K'$ and order T as follows: The orderings on I and K' are as in the above $\mathcal{F}$-decompositions for A and B, and $k < i$ for all $k \in K'$ and $i \in I$. It is now clear that $B = \oplus_{t\in T} e_tR$ is an $\mathcal{F}$-decomposition for B with $e_t = f_t$ for all $t \in K'$. ■

Theorem 5.3 Let R be a d.g. near-ring with identity.

(i) If $A, B \in \mathcal{F}$, then $A + B \in \mathcal{F}$.

(ii) If $\mathcal{F}$ is not empty, then there exists a unique maximal element in $\mathcal{F}$.

PROOF (i) Suppose $A = \oplus_{i\in I} e_iR$ and $B = \oplus_{k\in K} f_kR$ are $\mathcal{F}$-decompositions of A and B respectively. For each $k \in K$, $A \cap f_kR = (0)$ or $f_kR \subseteq A$ since f_kR is a right R-group of type-0. By dropping those $f_kR \subseteq A$, if any, we may write $A + B = (\oplus_{i\in I} e_iR) + (\oplus_{k\in K'} f_kR) = A + X$, where $K' \subseteq K$ and $X = \oplus_{k\in K'} f_kR$ is a right ideal of R. Using the fact that $A \cap f_kR = (0)$ for $k \in K'$, it is straightforward to show that $A \cap X = (0)$ so that $A + B = A \oplus X$. Now for all $i \in I, k \in K'$, $f_ke_i \in XA \subseteq X \cap A =$

(0), and so by extending the orderings of I and K' to $I \cup K'$ (with $k < i$ for $i \in I, k \in K'$) we have $A + B \in \mathcal{F}$.

(ii) Partially order $\mathcal{F}$ by $A \leq B$ if and only if $A \subseteq B$. Consider a chain of ideals $A_1 < A_2 < \ldots < A_s < \ldots$ in $\mathcal{F}$. We first prove that $A = \cup_s A_s$ is an element of $\mathcal{F}$. By Lemma 5.2, we have a chain of ordered sets $I_1 \subset I_2 \subset \ldots \subset I_s \subset \ldots$ with $A_s = \oplus_{i \in I_s} e_i R$ an $\mathcal{F}$-decomposition for A_s. Furthermore, for each r and s, $r < s$, the ordering in I_r induces that in I_s in the sense that for $j \in I_s \backslash I_r$, $k \in I_r$, $j < k$. These orderings induce an ordering in $I = \cup_s I_s$. Thus $A = \oplus_{k \in I} e_k R$ is an $\mathcal{F}$-decomposition for A and hence A is an upper bound for the chain. By Zorn's Lemma, $\mathcal{F}$ has a maximal element, M, say. If $B \in \mathcal{F}$, then by (i), $B + M \in \mathcal{F}$. Consequently, $B \leq M$ and M is the unique maximal element of $\mathcal{F}$. ∎

Definition 5.4 Let R be a d.g. near-ring with identity. The unique maximal element of Theorem 5.3 is called the **right socle-ideal** of R, and we denote it by ${}^r Soi(R)$. If $\mathcal{F}$ is empty, then we define ${}^r Soi(R)$ to be the zero ideal.

By Propositions 2.10 and 2.13 we see that ${}^r Soi(R)$ is a d.g. right R-subgroup of R^+, which gives us another natural example of a d.g. right R-group.

Note that if ${}^r Soi(R) = \oplus_{i \in I} e_i R$ is an $\mathcal{F}$-decomposition of ${}^r Soi(R)$, then $(e_i : 0)$ is a maximal right ideal of R for each $i \in I$. Hence, as R has identity, ${}^r J_{1/2}(R) \subseteq \cap_{i \in I} (e_i : 0)$.

Lemma 5.5 Let R be a d.g. near-ring with identity. Then

$$ {}^r J_{1/2}(R) \cap {}^r Soi(R) = (0). $$

In particular, if ${}^rSoi(R) = R$ then ${}^rJ_{1/2}(R) = (0)$.

PROOF If $x \in {}^rJ_{1/2}(R) \cap {}^rSoi(R) \subseteq \cap_{i\in I}(e_i : 0) \cap (\oplus_{i\in I} e_iR)$, then $x = e_{i_1}r_{i_1} + e_{i_2}r_{i_2} + \cdots + e_{i_k}r_{i_k}$, $i_j \in I$, where we assume that $i_1 < i_2 < \ldots < i_k$. As e_{i_1} is distributive, $e_{i_1}x = e_{i_1}r_{i_1}$, and since $x \in (e_{i_1} : 0)$, we have $e_{i_1}r_{i_1} = 0$. Thus $x = e_{i_2}r_{i_2} + \cdots + e_{i_k}r_{i_k}$. Repeating this argument, we obtain $x = 0$. ∎

The above lemma implies that ${}^rJ_{1/2}(R)\, {}^rSoi(R) \subseteq {}^rJ_{1/2}(R) \cap {}^rSoi(R) = (0)$ and so also ${}^rJ_0(R)\, {}^rSoi(R) = (0)$. Thus ${}^rSoi(R)$ is antiradical in the sense that it annihilates at least one of the radicals of R. Furthermore, if R satisfies the DCCLS then by Theorem 3.23

$$\begin{aligned} & J_0(R)\, {}^rSoi(R) \\ \subseteq\ & J_{1/2}(R)\, {}^rSoi(R) \\ \subseteq\ & J_2(R)\, {}^rSoi(R) \\ =\ & {}^rJ_0(R)\, {}^rSoi(R) \\ \subseteq\ & {}^rJ_{1/2}(R)\, {}^rSoi(R) \\ =\ & (0), \end{aligned}$$

and in this case ${}^rSoi(R)$ is a right annihilator of both left and right radicals.

One-sided orthogonality can be replaced by the more familiar two-sided orthogonality if the index set is finite with the natural ordering, as the following theorem shows.

Theorem 5.6 Let R be a d.g. near-ring with identity. If ${}^rSoi(R)$ has an $\mathcal{F}$-decomposition which is a finite sum ${}^rSoi(R) = \oplus_{i=1}^k e_iR$, then ${}^rSoi(R) = \oplus_{i=1}^k f_iR$ where $f_iR = e_iR$, $f_i^2 = f_i$ and $f_if_j = 0$ if $i \neq j$ for $i, j = 1, \ldots, k$.

Proof Let $e := e_1 + \cdots + e_k$. We first show that ${}^{r}Soi(R) = e({}^{r}Soi(R))$. If $e_k r \in e_k R$ then $e_k r = (e_1 + \cdots + e_k)e_k r$, since $e_i e_k = 0$ for $1 \leq i < k$ and $e_k^2 = e_k$. Consequently, $e_k R \subseteq e({}^{r}Soi(R))$. Similarly, if $r \in R$ then

$$\begin{aligned} e_{k-1}r &= (e_1 + \cdots + e_k)e_{k-1}r - e_k e_{k-1}r \\ &= e(e_{k-1}r) - e_k e_{k-1}r \\ &\in e({}^{r}Soi(R)) \end{aligned}$$

since $e(e_{k-1}r) \in e({}^{r}Soi(R))$ and $e_k e_{k-1} r \in e_k R \subseteq e({}^{r}Soi(R))$. Thus $e_{k-1}R \subseteq e({}^{r}Soi(R))$ and by repeating this argument we obtain ${}^{r}Soi(R) = \oplus_{i=1}^{k} e_i R \subseteq e({}^{r}Soi(R))$. As $e({}^{r}Soi(R)) \subseteq {}^{r}Soi(R)$, we have ${}^{r}Soi(R) = e({}^{r}Soi(R))$. Next we show that the mapping $\psi : {}^{r}Soi(R) \to e({}^{r}Soi(R))$ given by $\psi(r) = er$ for all $r \in {}^{r}Soi(R)$ is a right R-automorphism. Using the fact that e is distributive by Lemma 2.14, we can show that ψ is a right R-epimorphism. If $\psi(r) = 0$, where $r = e_1 r_1 + \cdots + e_k r_k$, then

$$\begin{aligned} 0 &= \psi(r) \\ &= er \\ &= (e_1 + \cdots + e_k)(e_1 r_1 + \cdots + e_k r_k) \\ &= e_1(e_1 r_1 + \cdots + e_k r_k) + \cdots + e_k(e_1 r_1 + \cdots + e_k r_k) \\ &= (e_1 r_1) + (e_2 e_1 r_1 + e_2 r_2) + \cdots + (e_{k-1} e_1 r_1 + \cdots + e_{k-1} r_{k-1}) \\ &\quad +(e_k e_1 r_1 + \cdots + e_k r_k), \end{aligned}$$

since $e_i e_j = 0$ for $i < j$. Now each $e_i e_j r_j \in e_i R$ so that $e_1 r_1$ is the only term on the right hand side which is in $e_1 R$, so $e_1 r_1 = 0$. Hence also $e_i e_1 r_1 = 0$ for $i = 2, \ldots, k$. Thus we have

$$0 = (e_2 r_2) + (e_3 e_2 r_2 + e_3 r_3) + \cdots + (e_{k-1} e_2 r_2 + \cdots + e_{k-1} r_{k-1}) + (e_k e_2 r_2 + \cdots + e_k r_k).$$

Similarly, $e_2r_2 = 0$. Repeating this argument, we deduce that $r = 0$ and ψ is one-to-one. Now $e \in {}^rSoi(R) = e({}^rSoi(R))$ and so there exists $f \in {}^rSoi(R)$ such that $\psi(f) = e$. To show that f is a distributive element of R, let $a, b \in R$. Then

$$\begin{aligned}\psi(f(a+b)) &= (\psi(f))(a+b)\\ &= e(a+b)\\ &= ea+eb\\ &= (\psi(f))a+(\psi(f))b\\ &= \psi(fa+fb),\end{aligned}$$

which gives $f(a+b) = fa+fb$ for all $a, b \in R$ because ψ is one-to-one. Write $f = f_1+\cdots+f_k$ where $f_i \in e_iR$. By Lemma 2.14, each f_i is distributive. Next, if $x \in {}^rSoi(R)$ then $\psi(x) = ex = (\psi(f))x = \psi(fx)$, giving $x = fx \in fR$. Since also $fR \subseteq {}^rSoi(R)$, we have $fR = {}^rSoi(R)$. It is readily shown that $fR = \oplus_{i=1}^k f_iR$ and that $f_iR = e_iR$ for each i. Finally, since $fx = x$ for all $x \in {}^rSoi(R)$, we have $ff_i = f_i$ for each i, therefore $(f_1+f_2+\cdots+f_k)f_i = f_i$. This gives $f_i^2 = f_i$ and $f_jf_i = 0$ if $j \neq i$, by the uniqueness of the expression of an element in a direct sum. ■

Corollary 5.7 Let R be a d.g. near-ring with identity. If R satisfies the DCCR then ${}^rSoi(R)$ has an $\mathcal{F}$-decomposition ${}^rSoi(R) = \bigoplus_{i=1}^{k} f_iR$, where $\{f_i\}_{i=1}^k$ is an orthogonal idempotent set.

The next result shows that if R satisfies the DCCR then ${}^rSoi(R)$ is a direct summand of R.

Theorem 5.8 Let R be a d.g. near-ring with identity. If R satisfies the DCCR then $R = {}^rSoi(R) \oplus X$, where ${}^rSoi(R)$ has $\mathcal{F}$-decomposition ${}^rSoi(R) =$

$\oplus_{i=1}^k e_iR$ as in Corollary 5.7 and $X := \cap_{i=1}^k (e_i : 0)$ is a right ideal of R and a d.g. right R-subgroup of R^+.

PROOF Write $r \in R$ as $r = (e_1r + \cdots + e_kr) + (-e_kr - \cdots - e_1r + r)$. Clearly $e_1r + \cdots + e_kr \in {}^rSoi(R)$. Using Lemma 2.9 we obtain $e_i(-e_kr - \cdots - e_1r + r) = -e_ir + e_ir = 0$ for each i, giving $-e_kr - \cdots - e_1r + r \in X$. Therefore $R = {}^rSoi(R) + X$. If $x \in {}^rSoi(R) \cap X$ then $x = e_1r_1 + \cdots + e_kr_k$ and $x \in (e_i : 0)$ for $i = 1, \ldots, k$. So for each i, $0 = e_ix = e_i(e_1r_1 + \cdots + e_kr_k) = e_ir_i$, giving $x = 0$. Hence

$$R = {}^rSoi(R) \oplus X.$$

As an intersection of annihilating right ideals, X is a right ideal of R. To show that it is also a d.g. right R-group, write 1 as $1 = p + x$ where $p \in {}^rSoi(R)$ and $x \in X$. By Lemma 2.14, p and x are both distributive elements as 1 is. We will show that $X = xR$ and then apply Proposition 2.10. As X is a right ideal of R and $x \in X$, we have $xR \subseteq X$. Let $y \in X$. Then $y = 1y = (p + x)y = py + xy$ and so $y - xy = py \in X \cap {}^rSoi(R) = (0)$. Hence $y = xy \in xR$ and $X = xR$. ∎

In Lemma 5.5 it was seen that ${}^rJ_{1/2}(R) \cap {}^rSoi(R) = (0)$. The following theorem shows in the case of the DCCR that ${}^rSoi(R)$ is the largest ideal of R having zero intersection with ${}^rJ_{1/2}(R)$. The corresponding result holds in the case of DCCL for the left socle-ideal [9].

Theorem 5.9 Let R be a d.g. near-ring with identity that satisfies the DCCR and let A be an ideal of R. Then $A \cap {}^rJ_{1/2}(R) = (0)$ if and only if $A \subseteq {}^rSoi(R)$.

PROOF Firstly, if A is an ideal of R such that $A \subseteq {}^rSoi(R)$, then $A \cap$

${}^{r}J_{1/2}(R) = (0)$ since ${}^{r}Soi(R) \cap {}^{r}J_{1/2}(R) = (0)$ by Lemma 5.5. Conversely, put $J = {}^{r}J_{1/2}(R)$ and suppose that $A \cap J = (0)$. We show that A has an $\mathcal{F}$-decomposition. Firstly, since R satisfies the DCCR, by Theorem 3.14(i) we can write

$$R/J = (X_1/J) \oplus (X_2/J) \oplus \cdots \oplus (X_n/J),$$

where the X_i/J are right R-groups of type-0 and the X_i are right ideals of R. Since $(A \oplus J)/J$ is a normal right R-subgroup of R/J, it is a direct summand by Lemma 2.25. So, re-indexing if necessary,

$$R/J = [(A \oplus J)/J] \oplus (X_k/J) \oplus \cdots \oplus (X_n/J).$$

Let $1 + J = (a + J) + (e_k + J) + \cdots + (e_n + J)$ where $a \in A$ and $e_i \in X_i$ for $i = k, \ldots, n$. Then for any $b \in A$, $b + J = (1 + J)b = (ab + e_k b + \cdots + e_n b) + J$. We have

$$(-ab + b) + J = (e_k b + \cdots + e_n b) + J \in [(A \oplus J)/J] \cap \sum_{j=k}^{n} (X_j/J) = 0 + J,$$

so that $(-ab + b) \in J \cap A = (0)$. Hence $b = ab$ for all $b \in A$ and a acts as a left identity of A. Next, by the direct sum decomposition, for $i = k, \ldots, n$, $[(A \oplus J)/J] \cap (X_i/J) = J$, which gives $(A \oplus J) \cap X_i = J$. Then $X_i A \subseteq X_i \cap A \subseteq J \cap A = (0)$. So if $1 + J = (f_1 + \cdots + f_n) + J$ where $f_i \in X_i$, then for any $a' \in A$ we have

$$\begin{aligned} a' + J &= (f_1 a' + \cdots + f_n a') + J \\ &= (f_1 a' + \cdots + f_{k-1} a') + J, \end{aligned}$$

since $f_i a' \in X_i A = (0)$ for $i = k, \ldots, n$. Thus $-(f_1 a' + \cdots + f_{k-1} a') + a' \in$

$J \cap A = (0)$, giving for all $a' \in A$

$$a' = f_1 a' + \cdots + f_{k-1} a' \in (X_1 \cap A) \oplus \cdots \oplus (X_{k-1} \cap A).$$

Since also

$$A \supseteq (X_1 \cap A) \oplus \cdots \oplus (X_{k-1} \cap A),$$

we have

$$A = (X_1 \cap A) \oplus \cdots \oplus (X_{k-1} \cap A).$$

We show that this is an $\mathcal{F}$-decomposition of A. Let the decomposition of the left identity a of A be $a = a_1 + \cdots + a_{k-1},\ a_i \in X_i \cap A$. Then

$$\begin{aligned} a_i &= a a_i \\ &= (a_1 + \cdots + a_{k-1}) a_i \\ &= a_1 a_i + \cdots + a_i^2 + \cdots + a_{k-1} a_i. \end{aligned}$$

Therefore

$$\begin{aligned} -a_i^2 + a_i &= a_1 a_i + \cdots + a_{i-1} a_i + a_{i+1} a_i + \cdots + a_{k-1} a_i \\ &\in (X_i \cap A) \cap \sum_{\substack{j=1 \\ j \neq i}}^{k-1} (X_j \cap A) = (0). \end{aligned}$$

Hence $a_i = a_i^2$ for $i = 1, \ldots, k-1$ and $a_i a_j = 0$ for $i \neq j,\ i, j = 1, \ldots, k-1$. Next for $i = 1, \ldots, k-1,\ a_i R$ is contained in the right ideal $X_i \cap A$, and if $x \in X_i \cap A$, then

$$\begin{aligned} x &= ax \\ &= (a_1 + \cdots + a_i + \cdots + a_{k-1}) x \\ &= a_1 x + \cdots + a_i x + \cdots + a_{k-1} x \\ &\in (X_1 \cap A) \oplus \cdots \oplus (X_i \cap A) \oplus \cdots \oplus (X_{k-1} \cap A). \end{aligned}$$

So by uniqueness of expression in the direct sum, we have $a_j x = 0$ for $j \neq i$ and $x = a_i x \in a_i R$. Thus $X_i \cap A = a_i R$. In order to show that each a_i is distributive, we show that a is distributive and then invoke Lemma 2.14. The identity $1 + J = (a + J) + (e_k + J) + \cdots + (e_n + J)$ is distributive in R/J so, using Lemma 2.14, $a + J$ in particular is distributive. Hence for all $x, y \in R$, $(a + J)[(x + J) + (y + J)] = (a + J)(x + J) + (a + J)(y + J)$. That is, $a(x + y) + J = (ax + ay) + J$, and so $-(ax + ay) + a(x + y) \in J \cap A = (0)$. We have $a(x + y) = ax + ay$ for all $x, y \in R$. Finally, if $X_i \cap A \neq (0)$, then $(X_i \cap A) + J$ is a normal right R-subgroup of X_i properly containing J. Since X_i/J is right type-0, this implies that

$$X_i/J = [(X_i \cap A) + J]/J \cong (X_i \cap A)/[(X_i \cap A) \cap J] \cong X_i \cap A,$$

giving $X_i \cap A = a_i R$ is a right R-group of type-0. Thus A has an $\mathcal{F}$-decomposition, and by the maximality of ${}^r Soi(R)$, we have $A \subseteq {}^r Soi(R)$. ■

Corollary 5.10 Let R be a finite d.g. near-ring with identity and A be an ideal of R. Then $A \subseteq {}^r Soi(R)$ if and only if $A \cap J_2(R) = (0)$.

The following result is an immediate consequence of Lemma 5.5 and Theorem 5.9.

Corollary 5.11 Let R be a d.g. near-ring with identity that satisfies the DCCR. Then ${}^r Soi(R) = R$ if and only if ${}^r J_{1/2}(R) = (0)$.

5.2 The relationship between the left and right socle-ideals

As for the left and right radicals, we seek to find a relationship between the left and right socle-ideals.

Definition 5.12 [8] Let R be a zero-symmetric near-ring. The **(left) socle-ideal**, $Soi(R)$, is the maximal ideal of R of the form $\bigoplus_{i \in I} Re_i$ such that

(i) each Re_i is an R-group of type-0 and a left ideal of R

(ii) $e_i^2 = e_i$ for all $i \in I$ and $e_i e_j = 0$ if $i > j$ for some ordering on I.

The following results concerning the left socle-ideal are needed.

Theorem 5.13 [9] Let R be a zero-symmetric near-ring with DCCL and let A be an ideal of R. Then $A \cap J_{1/2}(R) = (0)$ if and only if $A \subseteq Soi(R)$.

Corollary 5.14 Let R be a zero-symmetric near-ring that satisfies the DCCL. Then $Soi(R) = R$ if and only if $J_{1/2}(R) = (0)$.

We can now explore the relationships that exist between the left and the right socle-ideals in near-rings with suitable chain conditions.

Proposition 5.15 Let R be a d.g. near-ring with identity that satisfies the DCCLS. Then ${}^rSoi(R) \subseteq Soi(R)$.

Proof By Lemma 5.5, ${}^rJ_{1/2}(R) \cap {}^rSoi(R) = (0)$ and since $J_{1/2}(R) \subseteq {}^rJ_{1/2}(R)$ for the DCCLS case, we have $J_{1/2}(R) \cap {}^rSoi(R) = (0)$. Using

Theorem 5.13, ${}^rSoi(R) \subseteq Soi(R)$. ■

We shall show by means of examples in the following section that either equality or strict containment can occur. A necessary and sufficient condition for the left and right socle-ideals to coincide in a finite near-ring is given in the next proposition.

Proposition 5.16 Let R be a finite d.g. near-ring with identity. Then $Soi(R) = {}^rSoi(R)$ if and only if $J_2(R) \cap Soi(R) = (0)$.

PROOF If $Soi(R) = {}^rSoi(R)$ then it is immediate from Lemma 5.5 and Theorem 3.29 that $J_2(R) \cap Soi(R) = (0)$. For the converse, using Theorem 3.29 gives ${}^rJ_{1/2}(R) \cap Soi(R) = (0)$. From Theorem 5.9 and Proposition 5.15 we conclude that $Soi(R) = {}^rSoi(R)$. ■

The following result gives equivalent conditions for $J_2(R) \cap Soi(R) = (0)$. Hence in the finite case, any of these conditions is necessary and sufficient for the left and right socle-ideals to coincide.

Proposition 5.17 Let R be a d.g. near-ring with identity and let R satisfy the DCCL. Write $R = Soi(R) \oplus L$, where $Soi(R) = \bigoplus_{i=1}^k Re_i$ and $L = \cap_{i=1}^k (0 : e_i)$. The following are equivalent:

(i) $J_2(R) \cap Soi(R) = (0)$

(ii) $J_2(R) \subseteq L$

(iii) Re_i is a left R-group of type-2 for $i = 1, \ldots, k$.

PROOF We first prove that (i) is equivalent to (ii).

$\underline{\Rightarrow}$ If $J_2(R) \cap Soi(R) = (0)$, then $J_2(R)Soi(R) = (0)$. So for all $r \in J_2(R)$ and $i = 1, \ldots, k$, we have $re_i = 0$. Hence $J_2(R) \subseteq \cap_{i=1}^{k}(0 : e_i) = L$.

$\underline{\Leftarrow}$ If $J_2(R) \subseteq L$, then, as $L \cap Soi(R) = (0)$, it follows that $J_2(R) \cap Soi(R) = (0)$.

Next, we prove that (ii) and (iii) are equivalent.

$\underline{\Rightarrow}$ Assume that $J_2(R) \subseteq L$. As R satisfies the DCCL, we can write [13]

$$\bar{R} := R/J_2(R) = \Delta_1 \oplus \cdots \oplus \Delta_n,$$

where each Δ_i is a left R-submodule of R^+ of type-2. Now $J_2(R) \subseteq L \subseteq (0 : e_i)$ for $i = 1, \ldots, k$, so define $\bar{Y}_i := (0 : e_i)/J_2(R)$. As $\bar{Y}_i$ is a normal left $\bar{R}$-subgroup of $\bar{R}$, it follows by Lemma 2.3 in [7] that

$$\bar{R} = \bigoplus_{p \in P} \Delta_p \oplus \bar{Y}_i \text{ for some } P \subseteq \{1, \ldots, n\}.$$

Then $\bar{R}/\bar{Y}_i \cong \bigoplus_{p \in P} \Delta_p$ and we also have $\bar{R}/\bar{Y}_i \cong R/(0 : e_i) \cong Re_i$. Hence $Re_i \cong \bigoplus_{p \in P} \Delta_p$, but as Re_i is of type-0, P must be a singleton and so $Re_i \cong \Delta_p$. Therefore Re_i is a left R-group of type-2.

$\underline{\Leftarrow}$ If each Re_i is of type-2, then $J_2(R) \subseteq \cap_{i=1}^{k}(0 : Re_i)$. As R has identity, it is easily seen that $(0 : Re_i) \subseteq (0 : e_i)$ for each i, which gives $J_2(R) \subseteq L$. ∎

5.3 Examples

We complete this chapter by considering examples to demonstrate the various relationships that exist between the left and right socle-ideals.

In view of Theorem 5.13 and Proposition 5.16, any finite d.g. near-ring with

$J_{1/2}(R) = J_2(R)$ has equal left and right socle-ideals. The following example from [1] gives us an instance of such a near-ring.

Example 5.18 Consider the finite d.g. near-ring R given in Example 3.7, page 32. It is seen from the multiplication table that the set of idempotents of R is $\{n_0, n_1, n_2, n_3, n_7, n_8, n_9, n_{10}, n_{11}, n_{15}\}$, and of these elements, n_0, n_1, n_3, n_8, n_9 and n_{11} are distributive. We see that n_8 is the only distributive idempotent which generates a right ideal that is also a right R-group of type-0, and as $n_8R = \{n_0, n_8\}$ is in fact a two-sided ideal of R, we conclude that ${}^rSoi(R) = n_8R$. Similarly, when considering left R-groups, n_8 is the only idempotent of R that generates a left ideal that is a left R-group of type-0, and so $Soi(R) = Rn_8 = \{n_0, n_8\}$. Therefore ${}^rSoi(R) = Soi(R)$, which illustrates Proposition 5.16 as $J_2(R) = \{n_0, n_5\}$, so that $J_2(R) \cap Soi(R) = (n_0)$. Furthermore, the right ideal X defined in Theorem 5.8 is equal to the annihilating right ideal of n_8 which, using the multiplication table of R, is the ideal $\{n_0, n_1, n_2, n_3, n_4, n_5, n_6, n_7\}$. This gives the direct sum decomposition

$$R = \{n_0, n_8\} \oplus \{n_0, n_1, n_2, n_3, n_4, n_5, n_6, n_7\}.$$

Example 5.19 Let R be the d.g. near-ring generated by the endomorphisms of S_n, the symmetric group of degree n, for $n \geq 5$. The left and right radicals of R were discussed in Example 3.34, page 52, now we determine the left and right socle-ideals of R. Firstly, by Lemma 5.5, $Soi(R)$ is an ideal of R having zero intersection with $J_{1/2}(R)$ and it was shown in Example 3.34 that $J_{1/2}(R) = N$. Using the list of ideals of R in Example 3.34 (vi) and (vii), we conclude that $Soi(R) = (0)$. Moreover, ${}^rSoi(R) \subseteq Soi(R)$ as R is finite, so

$${}^rSoi(R) = Soi(R) = (0).$$

Our next example illustrates that the left and right socle-ideals can be distinct.

Example 5.20 Let R be the right d.g. near-ring in Example 3.35, page 54. We have $(0) = J_0(R) = J_{1/2}(R) \subset J_2(R)$ and by Corollary 5.14 we conclude that $Soi(R) = R$. Hence $J_2(R) \cap Soi(R) = J_2(R) \neq (0)$ and using Proposition 5.16, it follows that

$$^{r}Soi(R) \subsetneq Soi(R).$$

We conclude by considering the left and right socle-ideals of the near-rings R and T discussed in Chapter 4. We will need the following lemma.

Lemma 5.21 For any d.g. near-ring R with identity that satisfies the DCCL, $J_2(R) \cap Soi(R) = J_2(R)Soi(R)$.

Proof Certainly $J_2(R)Soi(R) \subseteq J_2(R) \cap Soi(R)$ as $J_2(R)$ and $Soi(R)$ are ideals of R. If $x \in J_2(R) \cap Soi(R)$, and $Soi(R)$ has $\mathcal{F}$-decomposition $Soi(R) = \bigoplus_{i=1}^{k} Rf_i$, we can write $x = r_1 f_1 + r_2 f_2 + \cdots + r_k f_k$. For $i = 1, \ldots, k$, by the orthogonal idempotence of the f_i, we have $xf_i = (r_1 f_1 + r_2 f_2 + \cdots + r_k f_k) f_i = r_i f_i$. Thus $x = xf_1 + xf_2 + \cdots + xf_k = x(f_1 + f_2 + \cdots + f_k) \in J_2(R)Soi(R)$, where the left distributive law is valid over a direct sum of left ideals. ∎

Example 5.22 Let P and T be the d.g. near-rings from Chapter 4. Firstly, as $J_{1/2}(P) = {}^{r}J_{1/2}(P) = (0)$, we conclude that ${}^{r}Soi(P) = Soi(P) = P$.
To study the relationships between the socle-ideals in the near-ring T, we will use a result from [16]. The critical ideal is defined in [16] specifically for d.g. near-rings with identity satisfying the DCCL. This ideal coincides with the left

socle-ideal [8]. Theorem 2 in [16] shows that if R is left 0-semisimple then the critical ideal is not zero. That is, in d.g. near-rings with identity and DCCL, $J_0(R) = (0)$ implies that $Soi(R) \neq (0)$. As T is finite and $J_0(T) = (0)$, so $Soi(T) \neq (0)$. We will show that ${}^{r}Soi(T) \subsetneq Soi(T)$ by proving that $J_2(T) \cap Soi(T) \neq (0)$ and then applying Proposition 5.16. Firstly, both $J_2(T)$ and $Soi(T)$ are non-zero ideals of T, therefore using Proposition 4.3(iii) we have

$$Soi(T) \cdot 1 = P \quad \text{and} \quad J_2(T) \cdot P = P.$$

Next, using Lemma 5.21,

$$\begin{aligned}
(J_2(T) \cap Soi(T)) \cdot 1 &= (J_2(T)Soi(T)) \cdot 1 \\
&= J_2(T) \cdot (Soi(T) \cdot 1) \\
&= J_2(T) \cdot P \\
&= P.
\end{aligned}$$

That is, $(J_2(T) \cap Soi(T)) \cdot 1 \neq (0)$, and again by Proposition 4.3(iii) we conclude that $J_2(T) \cap Soi(T) \neq (0)$.

Chapter 6

Conclusion

The work in [10], [21] and [23] has been extended to develop a right representation theory for right d.g. near-rings. Right radicals and a right antiradical are defined using the corresponding definitions from left representation as a guide. The properties of these right structures are investigated, and particular significance is placed on examining how the left and right structures are connected. It was shown for d.g. near-rings R with identity and satisfying the DCCLS that $J_2(R) = {}^rJ_0(R)$, with the stronger result $J_2(R) = {}^rJ_2(R)$ holding in the finite case. Further investigation along these lines is still required. If a descending chain of right ideals or of right R-subgroups of R^+ is imposed on the d.g. near-ring R, then does the same relationship hold between $J_2(R)$ and ${}^rJ_0(R)$ as in d.g. near-rings with the DCCLS? An example of an infinite d.g. near-ring R without the DCCLS was given in which $J_2(R) = {}^rJ_0(R) = {}^rJ_{1/2}(R)$, but it is not known whether it is possible for the right radicals to exist independently of $J_2(R)$ in general infinite d.g. near-rings, or whether the right radicals always coincide with each other. Particular d.g. near-rings without an identity are given in Examples 3.2 and 3.36 for which the right radicals coincide with each other and with the left 2-radical, but what can be said about the relationships in general d.g. near-rings without identity? A related question is whether or not ${}^{r}J_2(R)$ is the smallest two-sided ideal of R containing ${}^{r}J_{1/2}(R)$. For left representation, Hartney [7] proved that in zero-symmetric near-rings R with DCCL, $J_s(R)$ is the smallest two-sided ideal of R that contains $J_{1/2}(R)$. Is there a distinct right radical that corresponds to $J_s(R)$?

As mentioned before, whether or not the right radicals are Kurosh-Amitsur radicals cannot yet be determined. In order to determine whether the right radicals satisfy the idempotent property ${}^{r}J_\nu(R) = {}^{r}J_\nu({}^{r}J_\nu(R))$ for $\nu = 0, \frac{1}{2}, 2$, it is sufficient to prove the hereditary property ${}^{r}J_\nu(I) = I \cap {}^{r}J_\nu(R)$ for I an ideal of R. However, it will first be necessary to define these right radicals for near-rings that are not distributively generated. Containment was proved for the particular case of distributive ideals in this thesis, and equality was shown possible by example. If the right radicals can be defined for non-d.g. near-rings then, even if the right radicals do not prove to be Kurosh-Amitsur, it may be worthwhile investigating other properties of the right radicals. For example, is the completeness property satisfied, i.e., if ${}^{r}J_\nu(I) = I$ for an ideal I of R, then does it follow that $I \subseteq {}^{r}J_\nu(R)$?

A concept of right quasi-regularity was defined in Chapter 2, and in the case of right d.g. near-rings with identity and DCCLS, $J_2(R)$ was shown to be the largest right quasi-regular ideal of R. However, $J_2(R)$ is not necessarily left quasi-regular, thus showing that left quasi-regularity and right quasi-regularity are distinct in the case of non-rings. For the case of DCCLS, every left quasi-regular left R-subgroup of R^+ is right quasi-regular, but whether or not this is true in general is not known. The concept of right quasi-regularity might be useful for further investigation into the relationships between the left and right radicals. For example, can it be proved that ${}^{r}J_2(R)$ is right quasi-regular for general d.g. near-rings R? If so, this will imply that ${}^{r}J_0(R) = {}^{r}J_2(R)$ by Corollary 3.41. Likewise, if it can be proved that $J_2(R)$ is right quasi-regular for arbitrary d.g. near-rings R, then it will follow that $J_2(R) \subseteq {}^{r}J_0(R)$.

No attempt was made to develop a right representation theory for the Meldrum-van der Walt matrix near-ring $\mathbb{M}_n(R)$. It would be interesting to find if there are any connections that exist between the right structures of R and $\mathbb{M}_n(R)$. For instance, consider that R^+ is both a left R-group and a d.g. right R-group for any d.g. near-ring R. As R^n is a left $\mathbb{M}_n(R)$-group we might expect that it is also a d.g. right $\mathbb{M}_n(R)$-group. To explore this possibility, we need to define a map $R^n \times \mathbb{M}_n(R) \to R^n$. Now the additive group of $\mathbb{M}_n(R)$ is generated by the f_{ij}^s, where $s \in S$, so as a first attempt we might consider the action of the f_{ij}^s on the set of n-tuples S^n as

$$\langle s_1, \ldots, s_n \rangle f_{ij}^s := \langle 0, \ldots, 0, s_j s, 0, \ldots, 0 \rangle, \text{with } s_j s \text{ in the } i\text{-th position.}$$

However, with this definition the multiplication identity

$$f_{ij}^r f_{kl}^s = \begin{cases} f_{il}^{rs} & \text{if } j = k \\ f_{il}^0 & \text{if } j \neq k \end{cases}$$

is not satisfied, as instead we have

$$f_{ij}^r f_{kl}^s = \begin{cases} f_{kj}^{rs} & \text{if } i = l \\ f_{kj}^0 & \text{if } i \neq l \end{cases} .$$

To avoid this contradiction, one might instead define the action of f_{ij}^s on S^n

by

$$\langle s_1, \ldots, s_n \rangle f_{ij}^s := \langle 0, \ldots, 0, s_i s, 0, \ldots, 0 \rangle, \text{with } s_i s \text{ in the } j\text{-th position.}$$

Using this definition, the multiplication identity is satisfied, but the addition identity $f_{ij}^r + f_{kl}^s = f_{ij}^r + f_{kl}^s$ for $i \neq k$ is not. It is therefore not clear how to define the action of $\mathbb{M}_n(R)$ on the right of elements of R^n. However, if a right representation theory for $\mathbb{M}_n(R)$ can be developed, it might be interesting to investigate the connections between ${}^rJ_\nu(R)^*$, ${}^rJ_\nu(R)^+$ and ${}^rJ_\nu(\mathbb{M}_n(R))$ for $\nu = 0, 2$.

Finally, turning to the relationship between the left and right antiradicals, it was shown that ${}^rSoi(R) \subseteq Soi(R)$ for right d.g. near-rings R with identity and satisfying the DCCLS, and examples were given in order to illustrate that it is possible for equality or proper containment to occur in the finite case. The relationship in general d.g. near-rings with identity has however not been established.

Index to examples

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