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Masters dissertation

Symmetry Reductions of Some Non-Linear 1+1 D and 2+1 D Black-Scholes Models

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Declaration

I declare that this project is my own, unaided work. It is being submitted as a fulfillment of the Degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before any degree or examination in any other university.

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Abstract

In this dissertation, we consider a number of modified Black-Scholes equations being either non-linear or given in higher dimensions. In particular we focus on the non-linear Black-Scholes equation describing option pricing with hedging strategies in one case, and two dimensional models in the other. Classical Lie point symmetry techniques are employed in an attempt to construct exact solutions. Some large symmetry algebras are admitted. We proceeded by determining the one dimensional optimal systems of sub-algebras for the admitted Lie algebras. The elements of the optimal systems are used to reduce the number of variables by one. In some cases, exact solutions are constructed. For the cases for which exact solutions are difficult to construct, we employed the numerical solutions. Some simulations are observed and interpreted.

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Chapter 1

Introduction

1.1 Literature review

Lie symmetry methods have been recently applied in finance, see [2], [13], and [14]. These methods are used as an alternative tool to solve Partial Differential Equations (PDEs), arising in mathematical models describing financial markets. It is easy to solve linear PDEs using Lie symmetry, and solutions to bond pricing equations have been derived using Lie symmetry. The problem becomes more interesting if we look at non-linear PDEs.

Some scholars elaborate on methodology of the one factor interest-rate bond-pricing problem consisting of the PDE and the final condition. For example, Lesniewski [1] uses a one factor model to price a zero coupon bond. The general assumption is that short term interest rates follow a random walk, making interest rate changes to be sensitive to the level of the risk less rate. Lie symmetry methods solves one factor models and leads to variable reductions and exact solutions to the bond pricing equation. Short rates models use the instantaneous spot rate $r(t)$ as the basic state variable. It is a one factor model because this is the only stochastic driver.

The intrinsic value of bonds depends on interest rates. That short term interest rate follows a stochastic differential equation and the price of the bond also satisfies a deterministic model given as differential equation. Many models for the short rate have been proposed in the past, such as [7, 8, 9, 10]. It is found that the most successful models in capturing the movements of the short rate were those that allowed the volatility of interest rate changes to be highly sensitive to the level of the interest rate.

Goard [6] uses the symmetries admitted by the bond pricing equation to find the group invariant solutions. These exact solutions must satisfy the final condition on a bond price that is equal to one. The author then also compares the performances of the Lie symmetry models with that of the Chan-Karolyi-Longstaff-Sanders (CKLS) type and Ahn and Gao models. The Ahn and Gao models are based on state variable with inverted square root diffusions. These models represent the United States (US) 1-month T-Bill yields from the year $1946-1994$. Simulation of US interest rates seem to be more exact when using Lie symmetry methods.

The authors in [5] consider the bond pricing equation and employs the Lie point symmetries. It turned out that only trivial Lie point symmetries are obtained. However, the authors identify drift functions for which there exist non trivial symmetries, not just time translations or multiplication by scalars. Here, the assumption is that the drift function is a solution to one of Riccati's equations, and this yielded the non trivial symmetries. Then the Lie algebra is spanned vector fields, and there is an infinite-dimensional subalgebra spanned by vector fields.

Craddock, Konstandatos and Lennox [4] also calculate Lie point symmetries for the Cox Ingersoll Ross (CIR) model for interest rates. Again they assume that the drift is a solution for a Ricatti equation. The Lie algebra of symmetries is then given by four vector fields, v_i. To get to a group transformation of the vector fields, they exponentiate the infinitesimal symmetries $v_1, ..., v_4$

Craddock and Lennox [5] also show an application that the Lie algebra of symmetries of the Black-Scholes equation is spanned by vector fields $v_1, ..., v_i$. To get to a fundamental solution of the Black-Scholes equation they use one of the vectors v_i. They end up performing the Mellin inversion by converting the equation to a corresponding Fourier inversion. The final expression is easily recognised as the transition probability density function for Geometrical Brownian Motion (GBM). This is expected for the fundamental solution for the Black-Scholes PDE, as the underlying asset price dynamics is driven by GBM in the classic Black-Scholes case. To summarise, the authors Craddock and Lennox [5] say that whenever a second order, linear, parabolic PDE in one spatial variable x has a non trivial symmetry group, then for at least one of the vector fields in the Lie algebra, an integral equation can be identified as a classical integral transform.

A more interesting problem however, is that of solving non-linear PDEs in finance. For example, Qiu and Lorenz [11] modify the Black-Scholes equation by assuming that the volatility is not constant. They assume that volatility is a function of the value of the option and the price of the underlying asset. A simple choice of volatility such that we have estimates for the maximal and minimal values of the volatility results in a non-linear PDE. They consider the case where the volatility is a smooth function, and presents a basic existence and uniqueness result.

Bordag [13] looks at a non-linear pricing model for a derivative security. The price of the security depends on a demand function. They study the demand function, still looking at a non-linear Black-Scholes equation. The diffusion coefficient is dependent on various variables like u_s, u_{ss}, and α which affects the interest rate. Depending on whether the interest rate r is or isn't zero, they find different Lie algebras admitted by the equation. They find optimal systems of sub-algebras in both cases. The optimal systems of sub-algebras allow to describe the set of independent reductions of these non-linear PDEs to different ordinary differential equations (ODEs). Some explicit solutions to these equations were derived.

Liu [14] studies symmetry classifications and exact solutions to bond pricing equations using Lie symmetry analysis and the power series method, hence the result being exponentiated solutions

Chapter 2

Symmetry methods for differential equations

2.1 Introduction

In this chapter we give a brief summary of the work by Sophus Lie on symmetry methods. Sophus Lie developed a transformation, currently known as Lie group of transformation, which map a given differential equation to itself. The differential equations remain invariant under some continuous group of transformations, usually known as symmetries of a differential equation. The theory of Lie Groups may be found in texts such as Bluman and Kumei [18], Bluman and Anco [19], Arrigo [20] and Olver [21].

2.2 Symmetry

Symmetry is a change or a transformation that leave an object invariant. For example, the area of a triangle is an invariant with respect to isometries of the Euclidean plane. An identity equation is unchanged for all values of its variables. Angles and ratios of distances are invariant under scalings, rotations, translations and reflections. Symmetry groups are invariant transformations when applied to an equation, they do not change the structural form of the equation under investigation.

Groups of transformations, otherwise referred to as Lie symmetries, are used to lower the order of the Ordinary Differential Equation (ODE). In the case of PDEs, Lie groups of transformations are used to reduce the number of independent variables by one. This may lead to construction of invariant solutions. In this dissertation, group of transformations will be utilized to determine exact (group invariant) solutions.

2.2.1 Infinitesimal transformations

The Taylor series expansion of a function $f(x)$ about a point a, is given by

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \tag{2.1}$$

Consider a one parameter group of transformations

$$\bar{x} = X(x, \epsilon) \tag{2.2}$$

with identity $\epsilon = 0$ and a composition law ψ. Expanding (2.1) about $\epsilon = 0$ we obtain

$$\begin{aligned} \bar{x} &= x + \epsilon(\frac{\partial X}{\partial \epsilon}(x,\epsilon))\bigg|_{\epsilon=0} + \frac{\epsilon^2}{2}(\frac{\partial^2 X}{\partial \epsilon^2}(x,\epsilon))\bigg|_{\epsilon=0} + \cdots \\ &= x + \epsilon(\frac{\partial X}{\partial \epsilon})(x,\epsilon)\bigg|_{\epsilon=0} + o(\epsilon^2) \end{aligned} \tag{2.3}$$

Let

$$\xi(x) = \frac{\partial X}{\partial \epsilon}(x,\epsilon)\bigg|_{\epsilon=0} \tag{2.4}$$

The transformation $\bar{x} \approx x + \epsilon\xi(x)$ is called an infinitesimal transformation of the Lie group of transformations. The components of $\xi(x)$ are called infinitesimals of (2.1)

2.2.2 Infinitesimal generators

According to Lies theory, the construction of one parameter group G is equivalent to the determination of the corresponding infinitesimal transformation:

$$\bar{\underline{x}} = \underline{x} + \varepsilon\xi(\underline{x}) + O(\varepsilon^2) \tag{2.5}$$

an ODE of order n. Here $\underline{x} = (x, y)$.

The infinitesimal generator of the one parameter Lie Group of transformations is the operator

$$X = \sum_{i=1}^{n} \xi^i(x) \frac{\partial}{\partial x^i} \tag{2.6}$$

We usually write

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial \eta^\alpha} \tag{2.7}$$

Theorem 2.2.1.

$$\zeta_k(x, y, y_1, ..., y_k) = D_x(\zeta_{k-1}) - y_k D_x(\xi(x, y))) \qquad k = 1, 2, ...,$$

where

$$\zeta_0 = \eta(x, y).$$

and $y_1 = y', y_2 = y'', y_3 = y'''$ and so on. Explicit formulas for ζ_k follow immediately

$$\begin{aligned}
\zeta_1 &= D_x(\eta) - y' D_x(\xi) \\
&= \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2, \\
\zeta_2 &= D_x(\zeta_1) - y'' D_x(\xi) \\
&= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 + (\eta_y - 2\xi_x - 3\xi_y y')y'', \\
\zeta_3 &= D_x(\zeta_2) - y''' D_x(\xi) \\
&= \eta_{xxx} + (3\eta_{xxy} - \xi_{xxx})y' + 3(\eta_{xyy} - \xi_{xxy})y'^2 + (\eta_{yyy} - 3\xi_{xyy})y'^3 - \xi_{yyy}y'^4 \\
&\quad +3[\eta_{xy} - 2\xi_{xx} + (\eta_{yy} - 3\xi_{xy})y' - 2\xi - yyy'^2]y'' - 3\xi_y y''^2 + \\
&\quad (\eta_y - 3\xi_x - 4\xi_y y')y''.
\end{aligned}$$

Infinitesimal transformations can be applied to ODEs, one independent and one dependent variable. They can also be applied to PDEs with one independent and n dependent variables.

If an ODE is of the k^{th} order, then the infinitesimal transformation needs to be extended to the same order as well.

Theorem 2.2.2. An infinitesimal criterion for invariance of an n^{th} order ODE. Let

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$$

be an infinitesimal generator of the one parameter Lie group of point transformation

$$\bar{x} \approx x + \xi\epsilon(x) + O(\epsilon^2) \tag{2.8}$$

$$\bar{y} \approx y + \eta\epsilon(x) + O(\epsilon^2) \tag{2.9}$$

Let

$$X^{[n]} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \zeta_1(x,y,y') + \cdots + \zeta_k(x,y,y',y'',\cdots,y^n)\frac{\partial}{\partial y^n}$$

Then a Lie point symmetry is admitted by an n^{th} order ODE if and only if

$$X^{[n]}(y^n - f(x, y, y', y'', , y^{n-1})) = 0, \quad \text{when} \quad y^n = f$$

The transformers (2.8), (2.9) leave the PDE invariant. The transformations are generated by

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta(x, y)\frac{\partial}{\partial u} \tag{2.10}$$

The corresponding (k^{th} extended) infinitesimal generator for PDEs is given by the following

$$X^{[k]} = \xi^i\frac{\partial}{\partial x^i} + \eta\frac{\partial}{\partial u} + \zeta_i\frac{\partial}{\partial u_i} + \cdots + \zeta_{i_1,i_2,\cdots i_k,}\frac{\partial}{\partial u_{i_1,\cdots,i_k,}}, k \geq 1$$

Then a Lie point symmetry is admitted by a k^{th} order PDE if and only if

$$X^{[k]}(\Delta) = 0, \quad when \quad \Delta = 0$$

Here

$$\zeta_i = D_i(\eta) - u_j D_i(\xi^j) \tag{2.11}$$

$$\zeta_{i_1,i_2,\ldots,i_k} = D_{ik}(\zeta_{i_1,i_2,\ldots,i_{k-1}}) - u_{i_1,i_2,\ldots,i_{k-1}} D_{i_k}(\xi^j) \tag{2.12}$$

2.2.3 An example

Consider the Burgers' equation

$$u_t = u_{xx} + uu_x. \tag{2.13}$$

The infinitesimal criterion for invariance is given by

$$X^{[2]}(u_t - u_{xx} - uu_x)\Big|_{(2.13)} = 0 \tag{2.14}$$

$$\zeta_t - \zeta_{xx} - u_x\eta - u\zeta_x = 0 \tag{2.15}$$

This gives the following determining equations

$$u_{tx}u_x : \xi^1_u = 0 \tag{2.16}$$

$$u_{tx} : \xi^1_x = 0 \tag{2.17}$$

$$u_t u_x : 2\xi^2_u + 2\xi^1_{xu} = 0 \tag{2.18}$$

$$u_{x^3} : \xi^2_u = 0 \tag{2.19}$$

$$u^2_x : 2\xi^2_{xu} - \eta_{uu} - 2\xi^2_u u = 0 \tag{2.20}$$

$$u_x : -\xi^2_t - 2\eta_{xu} + 2\xi^2_{xx} - \xi^2_x u - \eta = 0 \tag{2.21}$$

$$u_t : -\xi^1_t + \xi^1_{xx} + 2\xi^2_x + \xi^1_x u = 0 \tag{2.22}$$

$$u : \eta_t - \eta_{xx} - \eta_x u = 0 \tag{2.23}$$

The solution of the overdetermined system of equations $(2.16 - 2.23)$ is given by

$$\begin{aligned}
\xi^1 &= -c_1 t^2 - 2c_2 t + c_5, \\
\xi^2 &= (c_1 t + c_2)x + (-c_3 t + c_4), \\
\eta &= (c_1 t + c_2)u + (c_1 x + c_3), \\
X &= (-c_1 t^2 - 2c_2 t + c_5)\frac{\partial}{\partial t} + [(c_1 t + c_2)x + (-c_3 t + c_4)]\frac{\partial}{\partial x} \\
&\quad + [(c_1 t + c_2)u + (c_1 x + c_3)]\frac{\partial}{\partial u}.
\end{aligned}$$

Hence, the generators are as follows

$$X_1 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + (tu + x)\frac{\partial}{\partial u} \tag{2.24}$$

$$X_2 = -2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u} \tag{2.25}$$

$$X_3 = -t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} \tag{2.26}$$

$$X_4 = \frac{\partial}{\partial x} \tag{2.27}$$

$$X_5 = \frac{\partial}{\partial t} \tag{2.28}$$

This implies that the Burgers' equation admits a 5-dimensional Lie algebra which corresponds to a five parameter Lie Group of transformation, see [18], [19], and [21].

2.3 Concluding remarks

We have gone through fundamental theory behind symmetry analysis of differential equations. Various applications of symmetry and operator methods are used in the literature referencing the works of Lie. There are recent applications to the space of finance, particularly banking. We show in the next chapter the main approach on computation of symmetry groups, for models of finance.

Consider the non-linear Black-Scholes equation, so our focus is on $1 + 1$ D Black-Scholes equations, and invoke an element of volatility. We also look at the hedging strategies in finance and trading environment, applied onto a different non-linear equation. Also, we consider higher dimensional models of relevance in mathematics of finance.

Chapter 3

Mathematical description of non-linear 1+1 D Black-Scholes equations

3.1 Introduction

In this chapter we will look at derivation methods of non-linear equations in finance.

3.2 Option pricing theory

Black and Scholes derived a linear partial differential equation of diffusion type which can be applied to the pricing of financial instruments, like options.

An option is an agreement that gives the holder a right, not an obligation, to buy from or sell to, the seller of the option a certain amount of an underlying asset at a specified price at a future time. The specified price is the strike price and the future time is the expiration date. The value v of an option is a function of s, the price of the underlying asset, t, current time, μ, the drift of s, σ, the volatility of s, E is the strike price and T is the expiration date of the option, r is the risk free interest rate.

The classical Black-Scholes follows the next assumptions

- r, risk free interest rate is a known constant for the life of the option
- the price s follows a lognormal random walk, drift μ and volatility σ are constants known in advance
- transaction costs associated with buying or selling the underlying asset are ignored
- no dividends paid from the underlying asset
- there is continuous hedging
- the price of the underlying asset is divisible so that we can trade any fractional share of the asset
- arbitrage-free market.

Let π denote the value of a portfolio with long position in the option and short position in some quantity Δ of the underlying asset,

$$\chi = \pi = v(s,t) - \delta s \tag{3.1}$$

A short position is the sale (also known as *writing*) of an options contract. A long position is the buying of an options contract. We assume that the price s of the underlying asset follows a log-normal random walk

$$ds = \mu s dt + \sigma s d\chi \tag{3.2}$$

where χ is Brownian motion. As time changes from t to $t + dt$, the value of the portfolio will also change. This is due to the change of the underlying assets, and hence the change in the value of the option.

$$d\pi = dv - \Delta ds \tag{3.3}$$

By Itôs formula, we have

$$dv = (v_t + \frac{1}{2}\sigma^2 s^2 v_{ss})dt + v_s ds \tag{3.4}$$

Combining the last two equations gives

$$d\pi = (v_t + \frac{1}{2}\sigma^2 s^2 v_{ss})dt + (v_s - \Delta)ds \tag{3.5}$$

Delta hedging aims to reduce (hedge) the risk associated with price movements in the underlying asset by offsetting long and short positions. For example, a long call position may be delta hedged by shorting the underlying stock. Short selling is the sale of a stock that is not owned by the seller. This is done to speculation on price declining, enabling profit due to the price difference. The Greek letter Δ is used to denote the quantity of stock we sell short for hedging. Simply put, the hedge ratio can be described as follows:

$$\Delta = \frac{\text{Change in Option Value}}{\text{Change in Stock Value}} \tag{3.6}$$

This strategy is based on the change in premium (price of option) caused by a change in the price of the underlying security.

If we use a delta hedging strategy, and choose $\Delta = v_s$ we obtain

$$d\pi = (v_t + \frac{1}{2}\sigma^2 s^2 v_{ss})dt \tag{3.7}$$

By the assumption of an arbitrage-free market (same assets or with identical cash flows trade with the same price in different markets), the change $d\pi$ equals the growth of π in a risk-free interest-bearing account,

$$d\pi = r\pi dt = r(v - \Delta s)dt. \tag{3.8}$$

Therefore,

$$r(v - \Delta s)dt = (v_t + \frac{1}{2}\sigma^2 s^2 v_{ss})dt \tag{3.9}$$

Substituting $\Delta = v_s$, we arrive at the Black-Scholes equation

$$v_t + rsv_s + \frac{1}{2}\sigma^2 s^2 v_{ss} - rv = 0 \qquad \text{for } 0 \leq t \leq T \tag{3.10}$$

This is a 1 dimensional Black-Scholes equation for the pricing of a European option.

The equation has an end-condition at the expiration time T,

$$v(s,t) = \begin{cases} max(s-E,0) & \text{for a call option,} \\ max(E-s,0) & \text{for a put option,} \\ H(s-E,0) & \text{for a binary call option, H is the Heaviside function.} \end{cases} \tag{3.11}$$

This problem can easily be transformed into the heat equation, the statement known as

$$\frac{\partial u}{\partial t} - \alpha \bigtriangledown^2 u = 0. \tag{3.12}$$

Denote the right hand side in formula (3.9) by $v_0(s)$. If one uses the transformation

$$\tau = T - t,$$
$$x = ln(s) + (r - \frac{1}{2}\sigma^2)(T - t),$$
$$w(x, \tau) = e^{r(T-t)v(s,t)}$$

then the equation (3.8) transforms into the heat equation.

$$w_\tau = \frac{1}{2}\sigma^2 w_{xx} \tag{3.13}$$

To transform the Black-Scholes equation derived above into a non-linear PDE, we can make certain assumption. This follows from [11]. Starting with volatility, we will assume that it is known in the following range $0 < \sigma^- \leq \sigma \leq \sigma^+$.

$\sigma^+$and σ^- are estimate maximal and minimal values of σ

We then have

$$min_{\sigma^- \leq \sigma \leq \sigma^+} \frac{1}{2}\sigma^2 s^2 v_{ss} = \begin{cases} \frac{1}{2}{\sigma^+}^2 s^2 v_{ss} & \text{if } v_{ss} < 0, \\ \frac{1}{2}{\sigma^-}^2 s^2 v_{ss} & \text{if } v_{ss} \geq 0. \end{cases} \tag{3.14}$$

Hence,

$$\sigma_d(v_{ss}) = \begin{cases} \sigma^+ & \text{if } v_{ss} < 0, \\ \sigma^- & \text{if } v_{ss} \geq 0. \end{cases} \tag{3.15}$$

Under delta hedging, $\Delta = v_s$ we have

$$d\pi = (v_t + \frac{1}{2}{\sigma^+}^2 s^2 v_{ss})dt \tag{3.16}$$

Assume the minimum return on the portfolio with volatility σ varying over the range $\sigma^- \leq \sigma \leq \sigma^+$ equals the risk-free return $r\pi dt$. We then obtain

$$v_t + \frac{1}{2}{\sigma_d}^2(v_{ss})s^2 v_{ss} dt = r\pi dt = r(v - sv_s)dt \tag{3.17}$$

with $\sigma_d(v_{ss})$ given by (3.13). We then get to the non-linear PDE

$$v_t + rsv_s + \frac{1}{2}{\sigma_d}^2(v_{ss})s^2 v_{ss} - rv = 0 \tag{3.18}$$

if we use the transformation

$$\tau = T - t,$$

$$x = s,$$

$$u(x,\tau) = e^{r\tau v(s,t)}$$

then

$$u_\tau = \frac{1}{2}{\sigma_d}^2(u_{xx})x^2 u_{xx} + rxu_x, \qquad x > 0$$

$$u(x,0) = v(s,T) \tag{3.19}$$

This is the Black-Scholes equation with variable volatility.

Taking this further, to a different type of non-linear PDE, Cimpoiasu and Constantinescu [15] assumes the European call options on a basket of two assets x, y with mean tendencies (or expected rates of returns) , $\mu_i, i = 1, 2$, volatilities σ_i and correlation ρ. We assume that x, y are governed by stochastic processes of the form:

$$dx = \mu_1 x dt + \sigma_1 x dW^1$$

$$dy = \mu_2 y dt + \sigma_2 y dW^2$$

$$\rho = (dW^1, dW^2)$$

The option μ with payoff $u_T(x, y)$ at maturity T will satisfy a two dimensional Black-Scholes PDE:

$$u_t + \mu_1 x u_x + \mu_2 y u_y + \frac{1}{2}{\sigma_1}^2 x^2 u_{2x} + \frac{1}{2}{\sigma_2}^2 y^2 u_{2y} + \rho\sigma_1\sigma_2 xyu_{xy} - ku = 0,$$

k is a constant

$$u(x, y, T) = u_T(x, y) \tag{3.20}$$

This is a $1 + 1$ D Black-Scholes equation.

3.3 Option pricing theory: Non-linear Black-Scholes

The market structure is important to risk managers because it impacts in liquidity. A good market brings together sellers and buyers, and reduces search and transaction costs.

Trading of financial securities, derivatives, and so on, takes place either on an over-the-counter (OTC) platform, or a formal financial exchange. The best prices for both buyers and sellers are found when we have high trading volumes, which are affected by having more buyers and sellers, narrowing the bid - ask price. Liquidity, loosely interpreted is being able to find the best price with the least effort, and this attracts buyers and sellers.

There is a relationship between pricing and liquidity. The amount of order flow is significant when assessing liquidity. The greater the order flow, then chances are that large trades will not have an adverse price impact. For illiquid stocks with little order flows, there is a large buy - sell spread.

What happens when you have an illiquid market, where large trades of an asset affect the price. Frey [38] discusses a model where a hedging strategy affects the price of the underlying instrument. He derives a formula for the feedback effect of dynamic hedging on market volatility. Dynamic hedging is a technique that involves adjusting a hedge as the underlier moves often several times a day, hence dynamic.

Dealers at times hold large numbers of short option positions on an underlier which they want to offset by purchasing long options, but can't find long options. The delta $\triangle$, of an option is the sensitivity of the option to the underlying. It's the rate of change of value with respect to the underlying. The trader will create a delta hedge of a non-linear position (i.e. exotic option, vanilla options, exotic derivatives and bonds with embedded options) to reduce exposure with a linear position (i.e. spot trade, forward position and futures). The deltas of the linear and non-linear positions offset. As the value of the underlier changes the trader will have to take out new linear positions to offset the changing non-linear delta. A non linear version of the Black-Scholes equation is used to capture perfect hedging strategies. The hedging of derivatives via dynamic trading strategies in markets that are not perfectly liquid is looked at.

The model looks at two assets being traded, a riskless one, typically a bond, and a risky one, like a stock. They assume that the market for the bond is perfectly liquid, meaning that investors can buy and sell large quantities without impact on the price. Money markets are usually more liquid than stock markets. A money market is where financial instruments with high liquidity and very short maturities are traded. The money market is used as a means for borrowing and lending in the short term, from several days to just under a year. The stock market is a market in which shares of publicly held companies are issued and traded. Also known as the equity market, the stock market provides companies with access to capital in exchange for ownership in the company.

The price of the stock, is modelled as a stochastic process $(S_t)_t$. The major assumption is that there exists a group of N agents hedging OTC derivatives on the stock, with a common maturity date T and a path-independent payoff. Individual hedgers are assumed to be price takers. Looking at an individual hedger, with a derivative contract on the stock, with maturity T and payoff $h(S_T)$ using a dynamic trading strategy in stock and bond. The trading s-trategy is represented by a pair $(\alpha_t, \beta_t)_t$ where α_t is the number of shares of the stock, and β_t, the number of bonds at time t. The implementation of the traders' hedging strategy has a feedback effect on the price of the stock. That means that stock prices may fall(rise) if the hedger sell(buys) additional shares of the stock.

The following restriction are inferred on the stock trading strategies permissable for the trader:

(i) The stockholdings $(\alpha_t)_t$ *are left-continuous (i.e.* $\alpha_t = lim_{s \to t} < \alpha_s$*)*

(ii) The right-continuous process α^+ *with* $\alpha^+ = lim_{s \to t} < \alpha_s$*) is a semi martingale.*

(iii) The downward-jumps of the strategy are bounded: $\Delta\alpha_t^+ := \alpha_t^+ - \alpha_t > -1/\overline{\rho}$ *for some* $\overline{\rho} > 0$.

The model is also described as the perturbation of the Black-Scholes Model. ρ measures the market liquidity, and also describes the perturbation. If ρ is 0, and if there is no trade, $\alpha_t = 0$, then the asset follows the normal Black-Scholes equation.

$S_t(\rho, \alpha)$ denotes the asset price affected by trading strategy and liquidity. It can be described by the following stochastic differential equation:

(iv) *Given a Brownian motion W two constants $\alpha >$ and $\rho > 0$ and a continuous function λ such that $\rho\lambda(S) \leq \overline{\rho}$ for all $S \geq 0$. Suppose that the large trader uses a stock-trading strategy $(\alpha_t)_t$ satisfying assumptions i and ii. Then the asset price process solves the following stochastic differential equation*

$$dS_t = \sigma S_{t^-} dW_t + \rho\lambda(S_{t^-})S_{t^-} d\alpha_t^+; \tag{3.21}$$

where S_{t^-} denotes the left limit $lim_{s \to t} S_t$.

λ is normalized by assuming that $\lambda(S_0) = 1$. $\frac{1}{\rho\lambda(S_{t^-})S_{t^-}}$ measures the size of the change in the large traders stock position which causes the price to move.

3.3.1 Dynamic hedging

Frey [38] continues on the subject of dynamic hedging. The value of the traders' position has to be determined, this is normally the mark to market value. It is given by $V_M^t = \alpha_t S_t(\rho, \alpha) + \beta_t$. Assuming that the strategy is given by $\xi = (\alpha, \beta)$. The gains from the stock trading strategy α are given by $G_t = \int_0^t \alpha_s dS_s(\rho, \alpha)$. The strategy is self financing if $V_t^M = V_0^M + G_t$ for all $0 \leq t \leq T$. The strategy follows $(i), (ii), (iii)$ and an initial investment V_0. The tracking error of the strategies is then looked at. This is the difference in the value of a self financing hedging strategy and the payoff of the derivative at maturity. The tracking error of a self financing strategy α, initial value V_0, payoff $h(S_t)$ is

$$e_t^M = h(S_t(\rho, \alpha)) - V_M^t = h(S_t(\rho, \alpha)) - \left(V_0 + \int_0^t \alpha_s dS_s(\rho, \alpha)\right) \tag{3.22}$$

A positive (negative) e_t^M indicates that a profit (loss) has been made.

3.3.2 Dynamic hedging with volatility

We now look at dynamic hedging with volatility as studied by Frey [38]. Assuming that the trader's stock strategy is given by a function ϕ, the dynamics of the asset price and volatility are analysed. ϕ is assumed to follow:

(v) *The function* $\phi : [0;T] \times \Re^{+} \rightarrow \Re$ *is of class* $C^{1,2}([0;T] \times \Re^{+})$. *Moreover,* $\rho\lambda(S)\phi_S(t,S) < 1$ *for all* $(t,S)\epsilon[0;T] \times \Re^{+}$.

If we suppose that a large trader uses a stock-trading-strategy of the form $\alpha_t = \phi_S(t, S_t)$ for a function ϕ satisfying assumptions (v), and that the stock price process $S_t = S_t(\rho, \alpha)$ follows an Itô process of the form

$$dS_t = v(t, S_t)S_t dW_t + b(t, S_t)S_t dt \tag{3.23}$$

for the two functions v and b, then under assumption (iii)

$$v(t,S) = \frac{\sigma}{1 - \rho\lambda(S)S\phi_S(t,S)}, \text{and} \tag{3.24}$$

$$b(t,S) = \frac{\rho}{1 - \rho\lambda(S)S\phi_S(t,S)}\left(\phi_t(t,S) + \frac{\sigma^2 S^2 \phi_{SS}(t,S)}{2(1-\rho\lambda(S)S\phi_S(t,S))^2}\right) \tag{3.25}$$

Proof 3.3.2.1. Itô's formula and assumption (v) imply that stockholding α are semimartingale. By Itô's formula, we get that

$$d\alpha_t = \phi_S(t, S_t)dS_t + \left(\phi_t(s, S_s) + \frac{1}{2}\phi ss(t, S_t)v^2(t, S_t)(S_s)^2\right)ds$$

Assumption (iv) together with equation (3.20) give the following for the equilibrium stock price process S:

$$dS_t = \sigma S_t dW_t + \rho S_t \phi_S(t, S_t)dS_t + \rho S_t\left(\phi_t(t, S_t)dt + \frac{1}{2}\phi_{ss}(t, S_t)d\langle S\rangle_t\right)$$

equivalently

$$(1-\rho S_t\phi_S(t,S_t))dS_t = \sigma S_t dW_t + \rho S_t\bigg(\phi_t(t,S_t)dt + \frac{1}{2}\phi_{ss}(t,S_t)d\langle S\rangle_t\bigg)$$

$(1-\rho S_t\phi_S(t,S_t))dS_t$ is strictly positive.

Integrating $(1-\rho S_t\phi_S(t,S_t))dS_t$ over both sides yields the following for the stock price dynamics

$$dS_t = \frac{\sigma}{1-\rho S_t\phi_S(t,S_t)}S_t dW_t + \frac{\rho S_t}{1-\rho S_t\phi_S(t,S_t)}\bigg(\phi_t(t,S) + \frac{\sigma^2 S_t^2}{(1-\rho S_t\phi_S(t,S_t))^2}\bigg) \tag{3.26}$$

The feedback effect from dynamic hedging on volatility is described above. The trading-activity of the large investor with the constant volatility σ is transformed into the time and price dependent volatility $v(t,S)$ in (3.22). $v(t,S) > \sigma$ if $\phi(t,S) > 0$, meaning if the trader uses a positive feedback strategy which calls for additional buying if the stock price rises. This is common in hedging strategies for derivatives with a convex terminal payoff such as European call or put options. If the trader uses $v(t,S) < \sigma$ if $\phi(t,S) < 0$, then volatility is decreased.

From this work, [17] Frey and Patie looked at the feedback effect of the option replication strategy of the large trader on the asset price process. The purpose is to find out if the hedger will reproduce the payoff of derivative contracts by dynamic trading, even though the hedging affects the asset prices. This perfect hedging strategy is represented by a non-linear Black-Scholes equation.

Proposition 3.3.2.1. Assume that there is a solution $u \in C^{1,2}([0;T]\times\Re^+)$ of the following non-linear Black-Scholes equation

$$u_t(t,S) + \frac{1}{2}\frac{\sigma^2}{(1-\rho\lambda(S)Su_{SS}(t,S))^2}S^2u_{SS}(t,S) = 0, \quad u(T,S) = h(S) \tag{3.27}$$

whose space derivative $u_S(t, S) = \frac{\partial u(t,S)}{\partial S}$ satisfies (v). Then a strategy with $\alpha_t = u_S(t, S_t)$ and the value process $V_t = u(t, S_t), \quad 0 \leq t \leq T$ is a strategy that makes the tracking error value equal to 0, so it works well for a derivatives with payoff $h(S_T)$.

Proof 3.3.2.1. Using a strategy $\alpha_t = u_S(t, S_t)$, volatility of the asset price is:

$$\sigma_u(t, S) := \frac{\sigma}{1 - \rho\lambda(S)Su_{SS}(t, S)}. \tag{3.28}$$

Applying Itô's formula to u we get

$$\begin{aligned} h(S_T) &= u(T, S_T) && (3.29)\\ &= u(0, S_0) + \int_0^T u_S(t, S_t)dS_T + \int_0^T u_S(t, S_t) \\ &+ \frac{1}{2}u_{SS}(t, S_t)\sigma_u^2(t, S_t)S_t^2 dt && (3.30) \end{aligned}$$

where S stands for $S(\rho, \alpha)$. The right hand side falls away. Hence, $h(S_T(\rho, \alpha)) = V_0 \int_0^T \alpha_t dS_t(\rho, \alpha)$, proving that the tracking error is zero.

The values of ρ and $\lambda(S)$ can be estimated from the observed option prices and depend on the payoff $h(S)$. Remember, S is the price of the underlying asset, $u(S, t)$ is the hedge-cost of the claim with a payoff $h(S)$. t is the time variable, σ defines the volatility of the underlying asset. (3.25) can be generalized as

$$u(t, S) + \frac{1}{2}(v(t, S, u_{SS}))^2 u_{SS}(t, S) = 0; \tag{3.31}$$

The above equation emphasises the dependence on volatility

3.3.3 Hedging strategy for a perfect replication of derivatives

[38] Frey derives an expression that uses a non-linear Black-Scholes PDE to hedge a derivative with a payoff $h(S_T)$ in an illiquid market.

Proposition 3.3.3.1. Assume that there is a solution $u \in C^{1,2}([0;T] \times \Re^+)$ of the following non-linear Black-Scholes equation

$$u_t(t, s) + \frac{1}{2}\frac{\sigma^2}{(1 - \rho Su_{SS}(t, S))^2}S^2 u_{SS}(t, S) = 0, \quad u(T, S) = h(S) \tag{3.32}$$

whose space derivative $u_S(t,S) = \frac{\partial u(t,S)}{\partial S}$ satisfies (v). Then a strategy with $\alpha_t = u_S(t,S_t)$ and the value process $V_t = u(t,S_t)$, $0 \le t \le T$ is a strategy that makes the tracking error value equal to 0, so it works well for a derivatives with payoff $h(S_T)$.

Proof 3.3.3.1. Using a strategy $\alpha_t = u_S(t,S_t)$, volatility of the asset price is $v(t,S,u) = \frac{\sigma}{1-\rho S u_{SS}(t,S)}$. Applying Itô's formula to u we get

$$\begin{aligned} h(S_T) &= u(T,S_T) && (3.33) \\ &= u(0,S_0) + \int_0^T u_S(t,S_t)dS_T + \int_0^T u_S(t,S_t) \\ &\quad + \frac{1}{2}u_{SS}(t,S_t)v^2(t,S_t,u)S_t^2 dt && (3.34) \end{aligned}$$

where S stands for $S(\rho,\alpha)$. The right hand side falls away. Hence, $h(S_T(\rho,\alpha)) = V_0 \int_0^T \alpha_T dS_T(\rho,\alpha)$, proving that the tracking error is zero.

3.4 Concluding remarks

In this chapter, a detailed account of mathematical description for non-linear Black-Scholes equations is given. Furthermore, we discussed the applications of these equations to financial markets.

Chapter 4

Symmetry reductions of 1+1 D non-linear Black-Scholes equation

4.1 Introduction

In this chapter, we will continue with elaboration on [38]. We look at the model where the hedging strategy affects the price of the reference entity. The model on feedback effects of dynamic hedging is reviewed.

4.2 Hedging strategy for a perfect replication of derivatives

Proposition 4.2.1. Assume that there is a solution $u \in C^{1,2}([0;T] \times \Re^+)$ of the following non-linear Black-Scholes equation

$$u_t(t,S) + \frac{1}{2}\frac{\sigma^2}{(1-\rho\lambda(S)Su_{SS}(t,S))^2}S^2u_{SS}(t,S) = 0, \quad u(T,S) = h(S) \quad (4.1)$$

whose space derivative $u_S(t,S) = \dfrac{\partial u(t,S)}{\partial S}$ satisfies (v). Then a strategy with $\alpha_t = u_S(t,S_t)$ and the value process $V_t = u(t,S_t), \quad 0 \leq t \leq T$ is a strategy that makes the tracking error value equal to 0, so it works well for a derivatives with payoff $h(S_T)$.

4.3 Determining system of equations

The infinitesimal symmetry operator is of the form:

$$\tilde{X} = \xi^1(x,y,t,u)\frac{\partial}{\partial x} + \xi^2(x,y,t,u)\frac{\partial}{\partial y} + \eta(x,y,t,u)\frac{\partial}{\partial y} \tag{4.2}$$

The following consideration is made $\varphi = c_0$, then the determining system of equations looks as follows:

$$
\begin{aligned}
\rho {\xi_t}^2 &= 0 \\
\rho {\eta_t}^2 &= 0 \\
\sigma {\xi_s}^1 &= 0 \\
\sigma {\xi_{uu}}^2 &= 0 \\
\sigma {\xi_u}^1 &= 0 \\
\rho\sigma {\xi_s}^1 &= 0 \\
\rho\sigma {\xi_{uu}}^2 &= 0 \\
\rho\sigma {\xi_u}^1 &= 0 \\
\sigma(\eta_{uu} - 2{\xi_{su}}^2) &= 0 \\
\rho\sigma(\eta_{uu} - 2{\xi_{su}}^2) &= 0 \\
\rho(s\sigma^2{\xi_u}^2 + 3\rho(\xi_t)^2) &= 0 \\
2s^2\sigma^2\eta_{ss} + 4\eta_t &= 0 \\
\sigma(-4\rho\eta_{uu} + s\sigma^2\xi^1_{uu} + 8\rho\xi^2_{su}) &= 0 \\
\rho\sigma(-4\rho\eta_{uu} + s\sigma^2\xi^1_{uu} + 8\rho\xi^2_{su}) &= 0 \\
\rho(s^2\sigma^4\xi^1_u + s\sigma^2\xi^2_u - 15\rho\xi^2_t) &= 0 \\
4s^2\sigma^2\eta_{su} - 2s^2\sigma^2\xi^2_{ss} - 4\xi^2_t &= 0
\end{aligned}
$$

$$\rho(2s\sigma^2\eta_u - 2s\sigma^2\xi^2_s + 12\rho\eta_t - s\sigma^2\xi^1_t) = 0$$

$$\rho(10s\sigma^2\xi^2_u + \rho(2s^2\sigma^2\eta_{su} - s^2\sigma^2\xi^2_{ss} + 30\xi^2_t)) = 0$$

$$\rho(-6s\sigma^2\xi^2_u - 4s^2\rho\sigma^2\eta_{su} + s^3\sigma^4\xi^1_{su} + 2s^2\rho\sigma^2\xi^2_{ss} - 40\rho\xi^2_t) = 0$$

$$\rho(-3\sigma^2(\xi^2)[t,s,u] + s\sigma^2\eta_u + s^2\sigma^4\xi^1_s + 2s\sigma^2\xi^2_s + 15\rho\eta_t - 2s\sigma^2\xi^1_t) = 0$$

$$2s^2\sigma^4\xi^1_u - 2s\sigma^2\xi^2_u - 4s^2\rho\sigma^2\eta_{su} + s^3\sigma^4\xi^1_{su} + 2s^2\rho\sigma^2\xi^2_{ss} + 12\rho\xi^2_t] = 0$$

$$4\sigma^2(\xi^2)[t,s,u] + 4s^2\sigma^4\xi^1_s - 4s\sigma^2\xi^2_s - 4s^2\rho\sigma^2\eta_{ss} + s^3\sigma^4\xi^1_{ss} - 24\rho\eta_t + 2s\sigma^2\xi^1_t = 0$$

$$\rho(2\rho)\sigma^2(\xi^2)[t,s,u] - 6s\rho\sigma^2\eta_u + s^2\sigma^4\xi^1_u + 4s\rho\sigma^2\xi^2_s + s^2\rho^2\sigma^2\eta_{ss} - 30\rho^2\eta_t + 4s\rho\sigma^2\xi^1_t) = 0$$

$$\rho(12\rho\sigma^2(\xi^2)[t,s,u] - 12s\rho\sigma^2\eta_u + 2s^2\sigma^4\xi^1_u + 4s^2\rho^2\sigma^2\eta_{ss} - s^3\rho\sigma^4\xi^1_{ss} - 80\rho^2\eta_t + 12s\rho\sigma^2\xi^1_t) = 0$$

This can be easily verified using software "SYM".

A solution has to be found for one of the functions ξ^2, η, ξ^1

Case 1: $\rho = 0$**,** $\sigma = 0$

The coefficients of the Infinitesimal Generator are:

$$\eta[t,s,u] = F_1[s,u] \tag{4.3}$$

$$\xi^2[t,s,u] = F_2[s,u] \tag{4.4}$$

The solution is spanned by the following:

$$\xi^1[t,s,u]\frac{\partial}{\partial t} + F_1[s,u]\frac{\partial}{\partial u} \tag{4.5}$$

$$F_2[s,u]\frac{\partial}{\partial s} + \xi^1[t,s,u]\frac{\partial}{\partial t} \tag{4.6}$$

Case 2: $\rho \neq 0$**,** $\sigma = 0$

The coefficients of the Infinitesimal Generator are:

$$\xi^2[t,s,u] = F_1[s,u] \tag{4.7}$$

$$\eta[t,s,u] = F_2[s,u] \tag{4.8}$$

The solution is spanned by the following:

$$\xi^1[t,s,u]\frac{\partial}{\partial t} + F_2[s,u]\frac{\partial}{\partial u} \tag{4.9}$$

$$F_1[s,u]\frac{\partial}{\partial s} + \xi^1[t,s,u]\frac{\partial}{\partial t} \tag{4.10}$$

Case 3: $\rho \neq 0$**,** $\sigma \neq 0$

The coefficients of the Infinitesimal Generator are:

$$\xi^2[t,s,u] = sc_3 \tag{4.11}$$

$$\eta[t,s,u] = c_1 + sc_2 + uc_3 \tag{4.12}$$

$$\xi^1[t,s,u] = c_4 \tag{4.13}$$

The solution is spanned by the following:

$$X_1 = \frac{\partial}{\partial u}$$
$$X_2 = \frac{\partial}{\partial t}$$
$$X_3 = s\frac{\partial}{\partial u}$$
$$X_4 = s\frac{\partial}{\partial s} + u\frac{\partial}{\partial u}$$

Now we want to capture the results in a commutator table.

Definition 4.3.1. The commutator of any two infinitesimal generators of an r-parameter Lie Group of transformations is also an infinitesimal generator. In particular:

$$[X_i, X_j] = \sum_{k=1}^{r} C^k_{i,j} X_k \tag{4.14}$$

where the coefficients $C^k_{i,j}$ are the constants called structure constants, $i, j, k = 1, 2, ...r$.

Suppose one is given the generators

$$X_1 = \xi^i_1(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha_1(x,u)\frac{\partial}{\partial u^\alpha},$$
$$X_2 = \xi^i_2(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha_2(x,u)\frac{\partial}{\partial u^\alpha}.$$

X_1 and X_2 span the vector space (Lie Algebra) L if the commutator

$$\begin{aligned}[X_1, X_2] &= (X_1(\xi^i_2) - X_2(\xi^i_1))\frac{\partial}{\partial x^i} + (X_1(\eta^\alpha_2) - X_2(\eta^\alpha_1))\frac{\partial}{\partial u^\alpha} \\ &= X_1X_2 - X_2X_1 \in L\end{aligned}$$

The axiom of skew symmetry also holds, that means that if $X_1, X_2 \in L$, then $[X_1, X_2] = -[X_2, X_1]$. These axioms will be very helpful as we look at the commutator table. All of the above can be summarised as shown in the commutator Table 4.1.

Table 4.1: Commutator table of the sub-algebras using Black-Scholes and hedging strategies

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	0	0	X_1
X_2	0	0	0	0
X_3	0	0	0	sX_1
X_4	$-X_1$	0	$-X_3$	0

Using the commutator above and the formula

$$Ad(e^{\epsilon X_i})X_j = X_j - \epsilon[X_i, X_j] + \frac{1}{2!}\epsilon^2[X_i, [X_i, X_j]] - ..., \tag{4.15}$$

where i and j take on values from 1 to 4, we can find the adjoint representation of X_i.

Definition 4.3.1. [21] Let G be a Lie group. An optimal system of s-parameter subgroups is a list of conjugacy inequivalent s-parameter subgroups with the property that any other subgroup is conjugate to precisely one subgroup in the list. Similarly, a list of s-parameter sub-algebras forms an optimal system if every s-parameter subalgebra of g is equivalent to a unique member of the list under some element of the adjoint representation: $\breve{h} = Ad[g(h)], g \in G$.

The adjoint representation can be summarised by the Table.4.2 below

Table 4.2: Adjoint table of sub-algebras

$[Adj.]$	X_1	X_2	X_3	X_4
X_1	X_1	X_2	X_3	$X_4 - \epsilon X_1$
X_2	X_1	X_2	X_3	X_4
X_3	X_1	X_2	X_3	$X_4 - s\epsilon X_1$
X_4	$X_1 e^{\epsilon}$	X_2	$X_3 e^{\epsilon}$	X_4

From this it is possible to calculate the one-dimensional optimal system of the symmetry subgroups. Using the operator below that is also non- zero

$$X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4, \tag{4.16}$$

we have to simplify the coefficients of a_i by the application of the adjoint maps to the operator X. a_i are arbitrary constants

Let $a_4 = 1$, then

$$X = a_1X_1 + a_2X_2 + a_3X_3 + X_4,$$

With reference to the table above we act on X by $Ad(e^{a_1/sX_3})X$ to eliminate a_1X_1. This gives

$$X^I = a_2X_2 + a_3X_3 + X_4 \tag{4.17}$$

this cannot be simplified further.

$$\boxed{X^I = X_4 + \alpha X_2 + \beta X_3}$$

Next we assume $a_4 = 0, a_3 \neq 0, a_3 = 1$, then

$$X = a_1X_1 + a_2X_2 + X_3,$$

$$X = a_1X_1 + a_2X_2 + X_3,$$

Next we assume $a_4 = 1, a_2 = 0$, then

$$Ad(e^{\beta X_4})X = a_1X_1 + a_2X_2 + e^{\epsilon}X_3,$$

depending on the sign of a_1, a_2, we have

$$\boxed{X_3 \pm X_1 \pm X_2, X_3, X_3 \pm X_2, X_3 \pm X_1}$$

Now, $a_3 = 0, a_2 = 1$

$$X = a_1X_1 + X_2,$$

Now, $a_2 = 0$, then

$$\boxed{X_2 + \alpha X_1}$$

The one dimensional optimal system of sub-algebras is

$$\{X_4 + \alpha X_2 + \beta X_3, X_3 \pm X_1 \pm X_2, X_3, X_3 \pm X_2, X_3 \pm X_1, X_2 + \alpha X_1, X_1\}$$

$\boxed{X_1}$

$$\alpha, \beta \in \Re \tag{4.18}$$

Now, the work to provide a complete list of invariant solutions has been completed. As per our optimal systems of one dimensional sub-algebras of the full symmetry algebra, we need to find group-invariant solutions for the one parameter subgroups. For example, the solutions, which corresponds to the symmetry group generated by

$$\alpha(X_2 + X_3) = \frac{\partial}{\partial t} + s\frac{\partial}{\partial u}, \qquad \alpha = 1$$

can be recovered from the stationary solutions $u = f(x)$. Solving the above and substituting it back to our original equation

$$u_t(t,S) + \frac{1}{2}\frac{\sigma^2}{(1-\rho\lambda(S)Su_{SS}(t,S))^2}S^2u_{SS}(t,S) = 0, \quad u(T,S) = h(S) \tag{4.19}$$

we see that $f(x)$ is a function satisfying

$$2\rho\lambda s^2 F''^2 + (4\rho\lambda s - \sigma^2 s)F'' + 2 = 0 \tag{4.20}$$

The particular solution of this ODE is as follows

$$F_1(s) = c_1 + sc_2 + \frac{(4\rho - \sigma^2 + \sqrt{-8\rho\sigma^2 + \sigma^4})(s + s\ln\frac{1}{s})}{4\rho^2} \tag{4.21}$$

$$F_2(s) = c_1 + sc_2 - \frac{(4\rho - \sigma^2 + \sqrt{-8\rho\sigma^2 + \sigma^4})(s + s\ln\frac{1}{s})}{4\rho^2} \tag{4.22}$$

We can also get to other forms of the solutions by making certain assumptions.

Case 1: $\rho = \dfrac{\sigma^2}{4}$

Then the ODE we have to solve becomes

$$F''^2 = \frac{-16}{\sigma^4 s^2} \tag{4.23}$$

The solution to this ODE is as follows

$$\begin{aligned} F_1(s) &= c_1 + sc_2 - \frac{4i(-s + s\ln s)}{\sigma^2} && (4.24) \\ F_2(s) &= c_1 + sc_2 + \frac{4i(-s + s\ln s)}{\sigma^2} && (4.25) \end{aligned}$$

Case 2: $\rho = 0$

Then the ODE we have to solve becomes

$$F'' = \frac{2}{\sigma^2 s} \tag{4.26}$$

The solution to this ODE is as follows

$$F(s) = c_1 + sc_2 - \frac{2lns}{\sigma} \tag{4.27}$$

The rest of the symmetry reductions and invariant solutions are shown in the tables below. We show the second subalgebra is $X_1 + X_3$, and find solutions for the arising equations. Lastly, we find solutions using subalgebra $X_1 + X_2$.

Table 4.3: Summary of symmetry reductions and invariant solutions using sub-algebra $X_1 + X_3$

Symmetry	Reduced Equation	Invariant solution in terms of original variables
	$X_1 + X_3$	
Case 1: ODE as is	$2\rho\lambda s^2 F''^2 + (4\rho\lambda s - \sigma^2 s)F'' + 2 = 0$	$F_1(s) = c_1 + sc_2 + \frac{(4\rho-\sigma^2+\sqrt{-8\rho\sigma^2+\sigma^4})(s+s\ln\frac{1}{s})}{4\rho^2}$ $F_2(s) = c_1 + sc_2 - \frac{(4\rho-\sigma^2+\sqrt{-8\rho\sigma^2+\sigma^4})(s+s\ln\frac{1}{s})}{4\rho^2}$
Case 2: $\rho = \frac{\sigma^2}{4}$	$F''^2 = \frac{-4}{\sigma^2 s^2}$	$F_1(s) = c_1 + sc_2 - \frac{4i(-s+s\ln s)}{\sigma^2}$ $F_2(s) = c_1 + sc_2 + \frac{4i(-s+s\ln s)}{\sigma^2}$
Case 3: $\rho = 0$	$F'' = \frac{2}{\sigma^2 s}$	$F(s) = c_1 + sc_2 + \frac{2(-s+s\ln s)}{\sigma^2}$

Case 1: ODE as is

To get to a particular solution, we will make the following initial assumptions.

$$\rho \in \{0.1, 0.5\}$$

$$\sigma \in \{0.4, 0.6, 0.7, 0.8\}$$

$$\lambda = 1$$

Then our hedging profile can be represented in the figures 4.1 and 4.2. The plot shows two sets of results, the first solution lies on the x axis, and the second

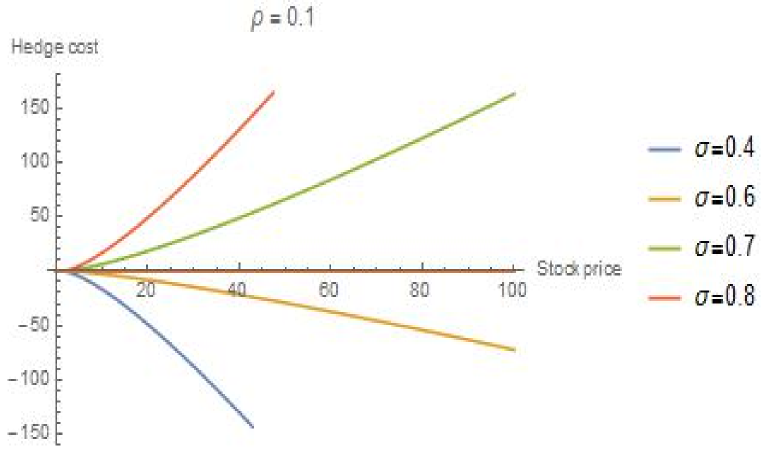


Figure 4.1: Varying levels of volatility, but very low liquidity

one, depending on the volatility level, lies on the positive or negative y axis. The interesting result with this plot seems to be that the higher the volatility, the higher the cost of hedging. These factors seems to have a proportionally increasing relationship. We look at a similar exercise, but in this case the market liquidity ρ parameter is increased. The results are what we expect, in

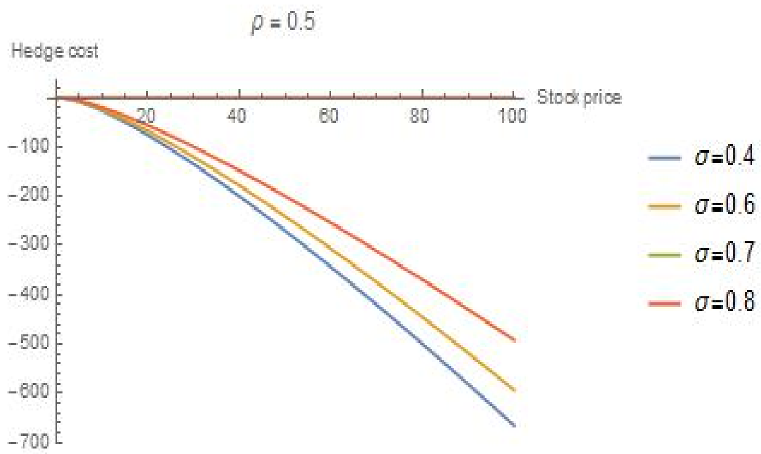


Figure 4.2: Varying levels of volatility, but a slight increase in liquidity

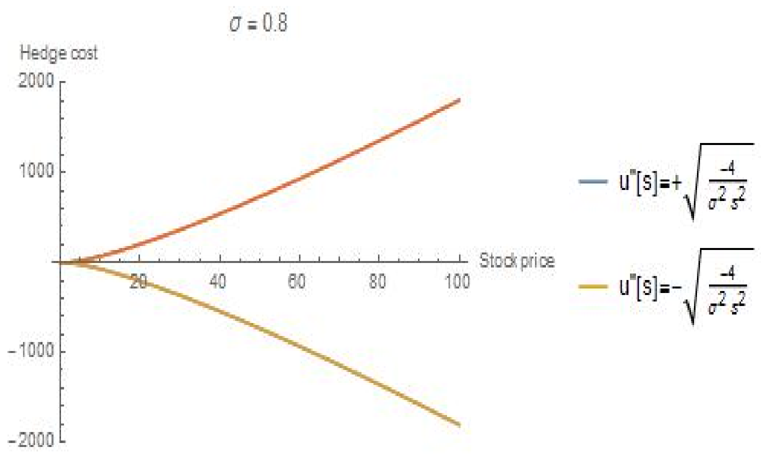
that in a market with high liquidity, the hedging cost should decrease. But how much impact does an increase in volatility have on the hedging cost when the asset is volatile. We see in the plots that not a lot, the increase in volatility leads to a negligible increase.

Case 2: $\rho = \dfrac{\sigma^2}{4}$**,** $\lambda = 1$

λ represents the level dependent liquidity profile. If we assume that $\lambda = 1$, then that means there is a constant liquidity profile. Remember that ρ represents the market liquidity parameter. We will use varying values of ρ to see the impact on the hedging cost. We want to see $\frac{1}{\rho\lambda s}$ (depth of the market), how the size of the change in large traders stock positions will be affected by liquidity availability or scarcity. To get to a particular solution, we will make the following initial assumptions.

$$\begin{aligned} \rho &= 0.1 \qquad \text{or} \\ &= 0.2 \\ \sigma &= 0.4 \\ \lambda &= 1 \\ s &\in [0, 100] \end{aligned}$$

Then our hedging profile can be represented as shown in the following plots



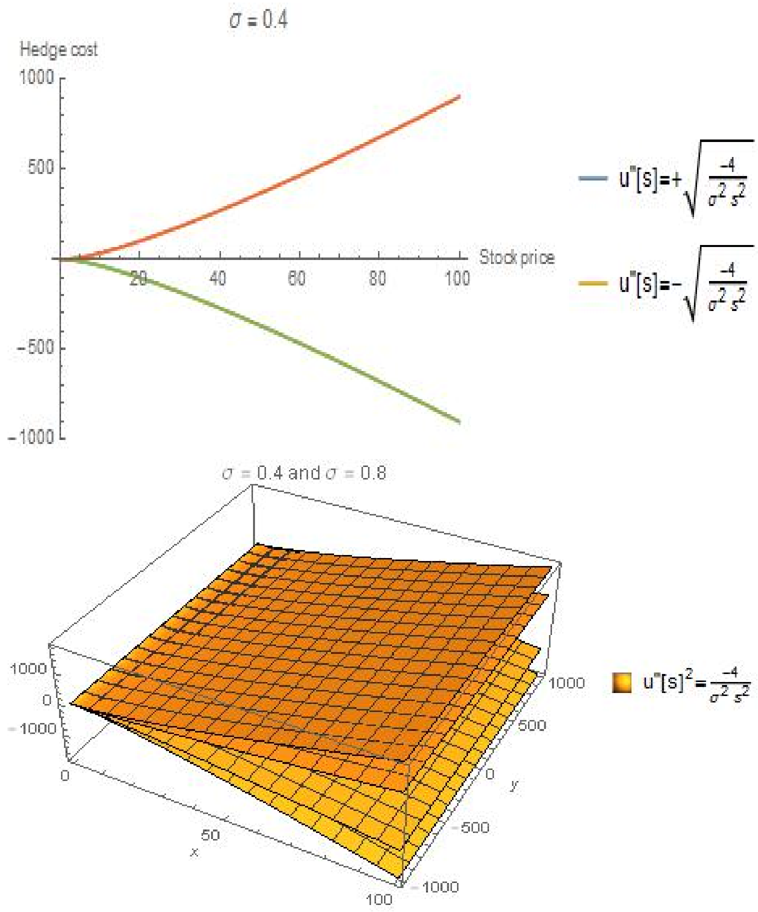


Figure 4.3: Option with fixed payoff values of u

The solutions are interesting in that they yield a complex number solution. As before, we look at scenarios with two levels of volatility and show the drastic effect that this has on the hedging cost. The volatility of the stock also has a proportional influence on the stocks' liquidity. We have assumed a constant liquidity profile.

Case 3: $\rho = 0$

Remember that ρ represents the market liquidity parameter. The model used is a perturbation of the Black-Scholes model, and it is controlled by the parameter ρ. If we assume the $\rho = 0$, then the resultant is that the asset simply follows

Black-Scholes with volatility σ. To get to a particular solution, we will make the following initial assumptions.

$$\begin{aligned} \rho &= 0 \\ \sigma &= 0.4 \\ s &\in [0, 100] \end{aligned}$$

Then our hedging profile can be represented as shown in the figure below.

This is a very interesting profile because the graph seems to be polynomial

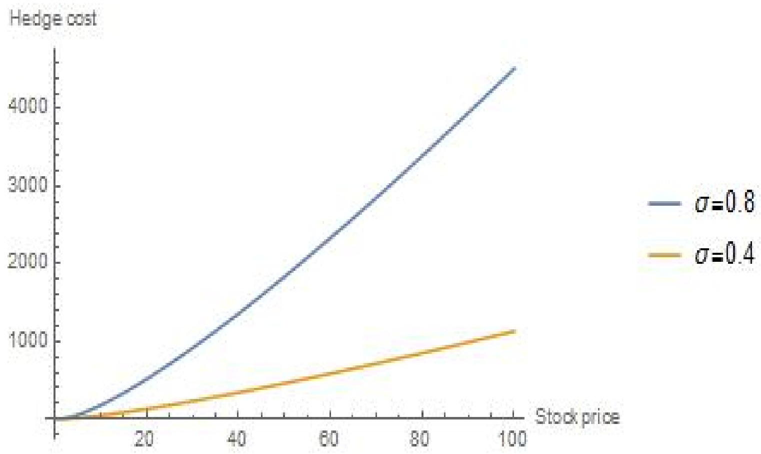


Figure 4.4: Hedge cost u, assuming no market liquidity $\rho = o$

plot, meaning an increase in the stock price means for a direct increase in the hedging cost to a certain degree. This is a direct impact arising from market illiquidity, and high probability of failure to sell off at a profitable margin. The play on volatility further exacerbates the problem caused by illiquidity, such that an increase in the volatility of the stock results also in a substantial increase of the hedging cost. This of course is what we would intuitively expect.

Liquidity risk is an important risk faced by the financial industry. It is important to look at liquidity costs, the extra price one pays over the theoretical price of a tradable asset, due to finite liquidity.

Subalgebra : $X_1 + X_2$

Similar methodology in solving equations in the subalgebra and plots is followed.

Table 4.4: Summary of symmetry reductions and invariant solutions using sub-algebra $X_1 + X_2$

Symmetry	**Reduced Equation**	**Invariant solution in terms of original variables**
	$X_1 + X_2$	
Case 1: ODE as is	$2\rho^2 s^2 F''^2 + (\sigma^2 s^2 - 4\rho s)F'' + 2 = 0$	$F_1(s) = c_1 + sc_2 - \frac{1}{4\rho^2}\left(4\rho s + \frac{s^2\sigma^2}{2} + \frac{4\rho\sqrt{s}\sqrt{-8\rho+s\sigma^2}}{\sigma} + \sigma\sqrt{-8\rho+s\sigma^2}(\frac{s^{\frac{3}{2}}}{2} - \frac{2\rho\sqrt{s}}{\sigma^2}) - 4\rho s\ln s - 8\rho s\ln(\sigma\sqrt{s}\sqrt{-8\rho+s\sigma^2}) + \frac{32\rho^2\ln(\sigma\sqrt{s}+\sqrt{-8\rho+s\sigma^2})}{\sigma^2} - \frac{16\rho^2\ln(\sigma\sqrt{s}+\sqrt{-8\rho+s\sigma^2})}{\sigma^2}\right)$ $F_2(s) = c_1 + sc_2 + \frac{1}{4\rho^2}\left(4\rho s + \frac{s^2\sigma^2}{2} + \frac{4\rho\sqrt{s}\sqrt{-8\rho+s\sigma^2}}{\sigma} + \sigma\sqrt{-8\rho+s\sigma^2}(\frac{s^{\frac{3}{2}}}{2} - \frac{2\rho\sqrt{s}}{\sigma^2}) - 4\rho s\ln s - 8\rho s\ln(\sigma\sqrt{s}\sqrt{-8\rho+s\sigma^2}) + \frac{32\rho^2\ln(\sigma\sqrt{s}+\sqrt{-8\rho+s\sigma^2})}{\sigma^2} - \frac{16\rho^2\ln(\sigma\sqrt{s}+\sqrt{-8\rho+s\sigma^2})}{\sigma^2}\right)$
Case 2: $\rho = \frac{\sigma^2 s}{4}$	$F''^2 = \frac{-16}{\sigma^4 s^4}$	$F_1(s) = c_1 + sc_2 - \frac{4i\ln s}{\sigma^2}$ $F_2(s) = c_1 + sc_2 + \frac{4i\ln s}{\sigma^2}$
Case 3: $\rho = 0$	$F'' = -\frac{2}{\sigma^2 s^2}$	$F(s) = c_1 + sc_2 + \frac{2lns}{\sigma^2})$

Case 1: ODE as is

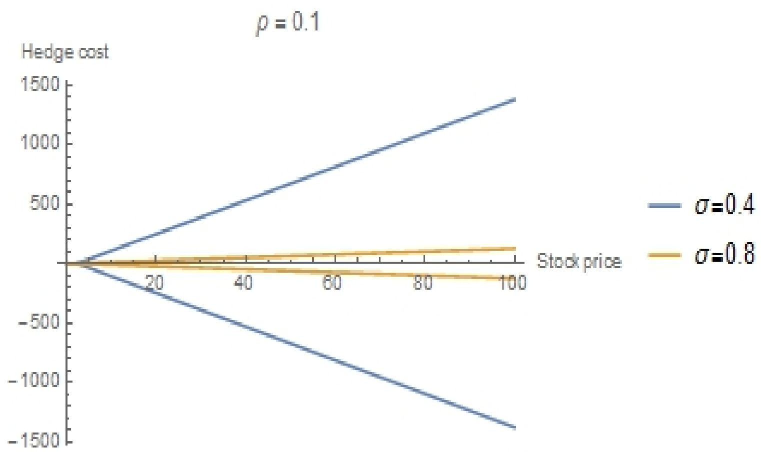
To get to a particular solution, we will make the following initial assumptions.

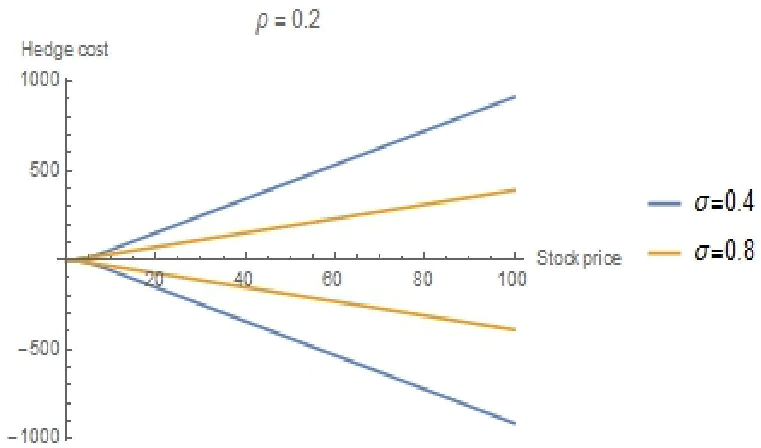
$$\rho \in \{0.1, 0.2, 0.3, 0.5\}$$

$$\sigma \in \{0.4, 0.8\}$$

$$\lambda = 1$$

Then our hedging profile can be represented as shown in the following plots below





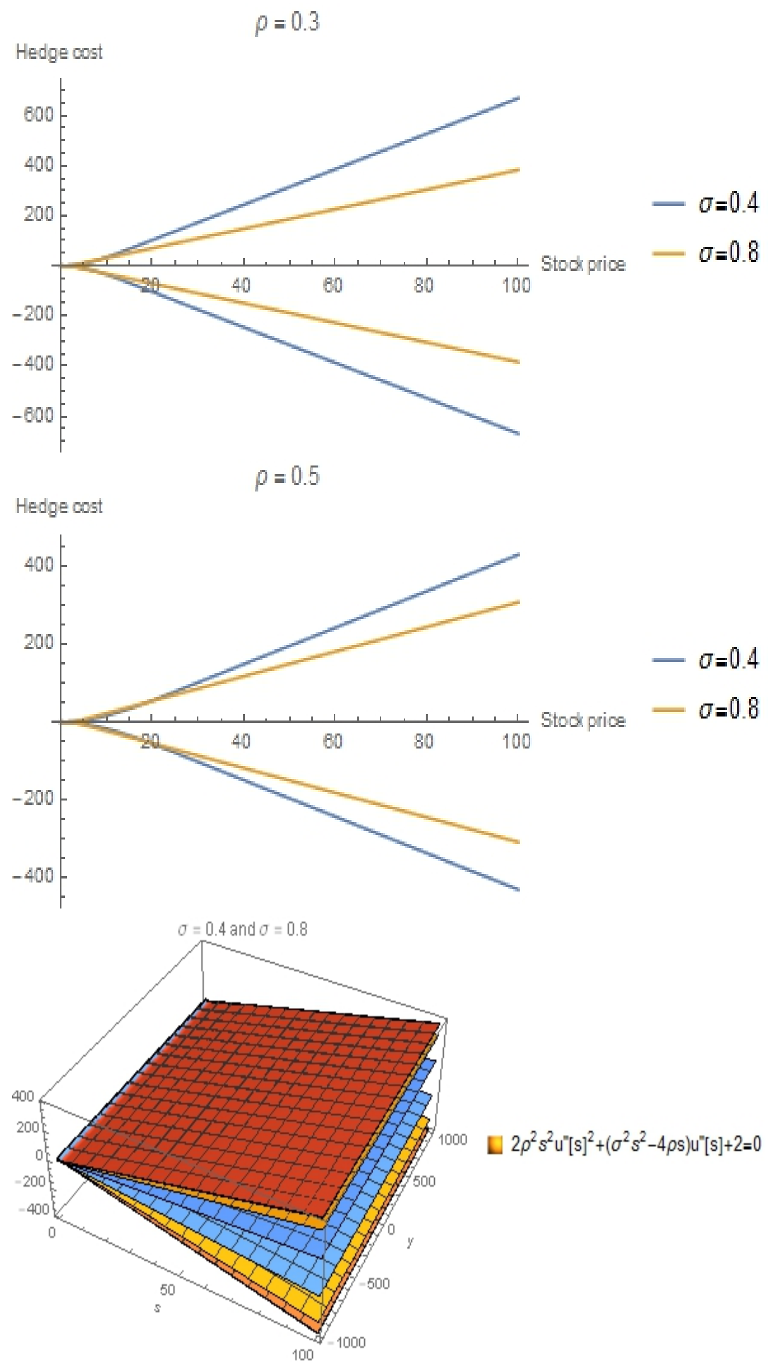


Figure 4.5: Hedge cost u, different levels of liquidity ρ and varying volatilities σ

We first try to understand the relationship that this profile shows between liquidity and the stock price. What happens to the hedging cost when we increase liquidity in the market. If the assumption is that we have constant volatility, then the hedge cost should decrease as we increase liquidity. This is only when volatility is assumed to be very low. However, if we change the position of volatility and increase it significantly higher, then the results on liquidity are different. For very volatile stocks, this shows that when there is little liquidity, then the hedging cost is rather inexpensive. As liquidity grows, the hedging cost increases by a small margin for a while, but eventually tapers off.

Case 2: $\rho = \dfrac{\sigma^2 s}{4}$, $\lambda = 1$

To get to a particular solution, we will make the following initial assumptions.

$$\sigma \in \{0.4, 0.65, 0.8\}$$

$$\lambda = 1$$

We show firstly the impact when the volatility is low at 0.4. We vary values of s to see clearer plots of the profile, and to see how quickly the hedging costs change as we increase the stock cost. For the sake of a clearer graph, we show

$$x = [0, 1)$$

$$x = [1, 60]$$

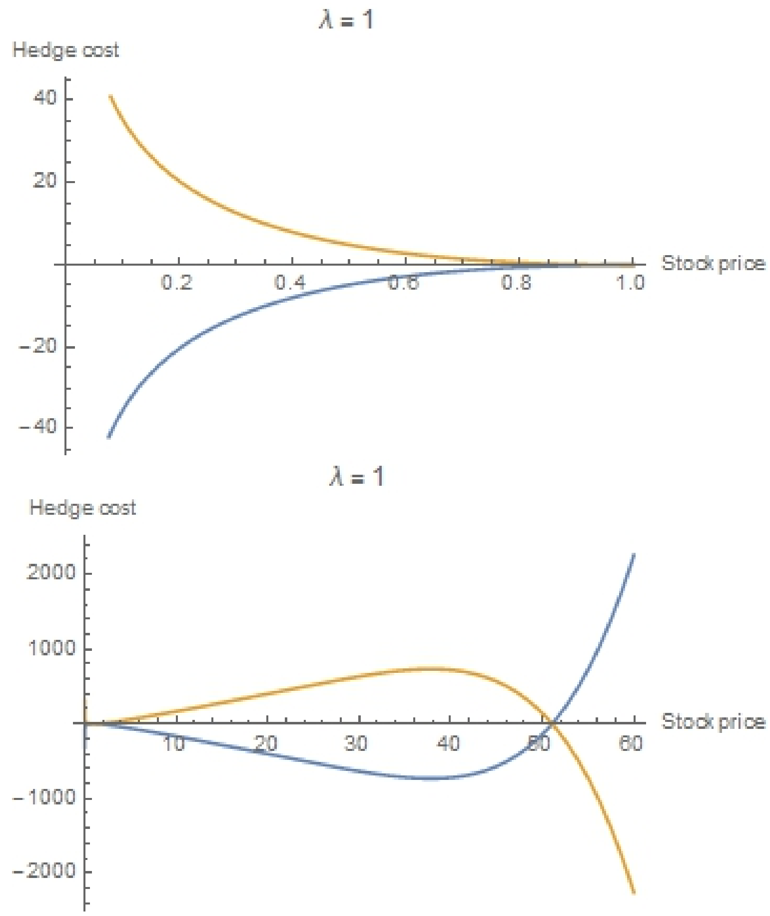


Figure 4.6: Hedge cost u, volatility $\sigma = 0.4$

This is a polynomial, in the beginning the hedge cost is very low, then tends to zero, increases, and at some point of the stock cost starts decreasing again, no a negative level. Here we can assume that the cost is close to nothing. Let us see what happens if we increase volatility slightly.

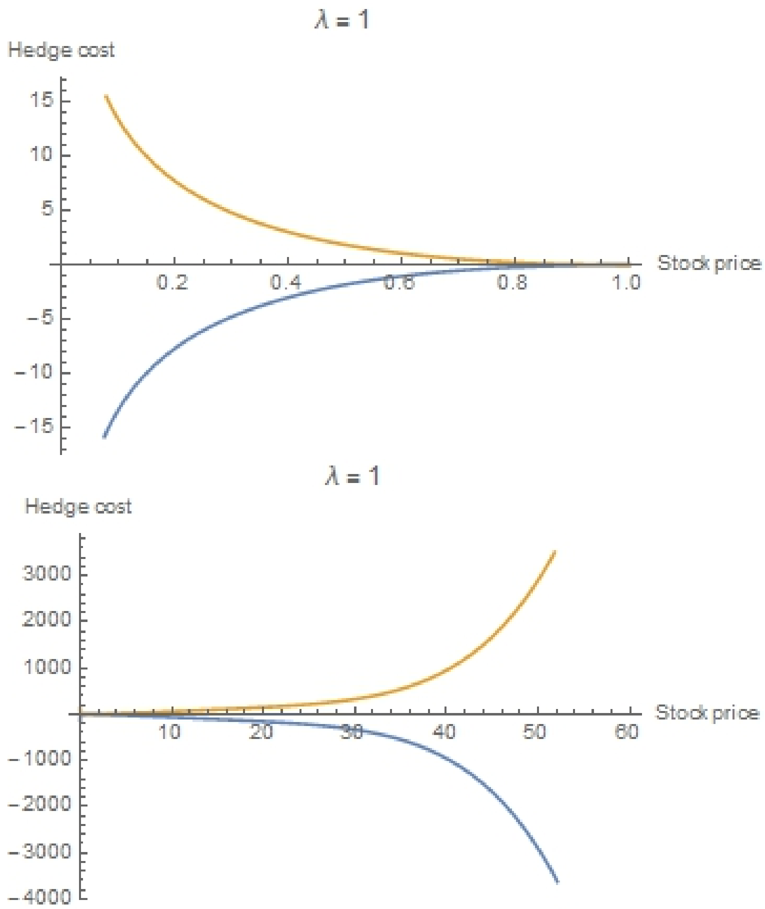


Figure 4.7: Hedge cost u, volatility $\sigma = 0.65$

Very interesting that at very low stock prices, the hedge cost is slightly higher than the previous example, but then afterwards increases almost exponentially. This is due to a slight increase in volatility $\sigma = 0.65$
We then look at the third scenario, further increasing volatility. The behaviour of the plot is similar to when $\sigma = 0.4$. However, in this case, the hedge cost is much more cheaper than at a lower volatility. This does not fulfil our natural expectations, we expect the behavior to be opposite.

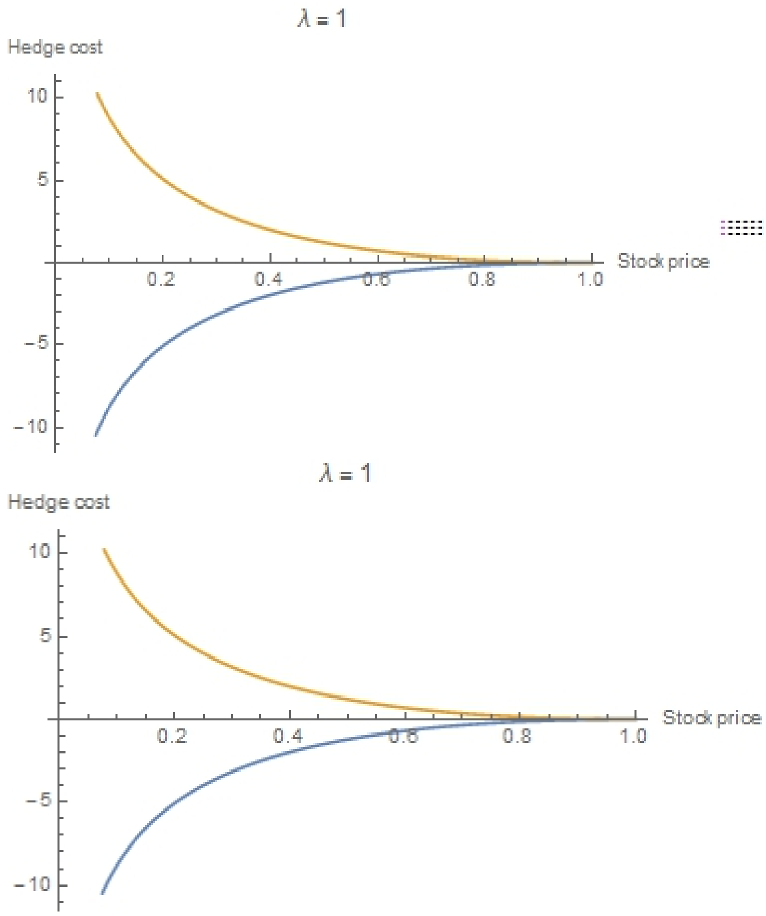


Figure 4.8: Hedge cost u, volatility $\sigma = 0.8$

Case 2: $\rho = 0$**,** $\lambda = 1$

To get to a particular solution, we will make the following initial assumptions.

$$\sigma \in \{0.4, 0.65, 0.8\}$$

$$\lambda = 1$$

We show firstly the impact when the volatility is low at 0.4. We vary values of s to see clearer plots of the profile, and to see how quickly the hedging costs change as we increase the stock cost. For the sake of a clearer graph, we show

$$x = [0, 10)$$

$$x = [0, 100]$$

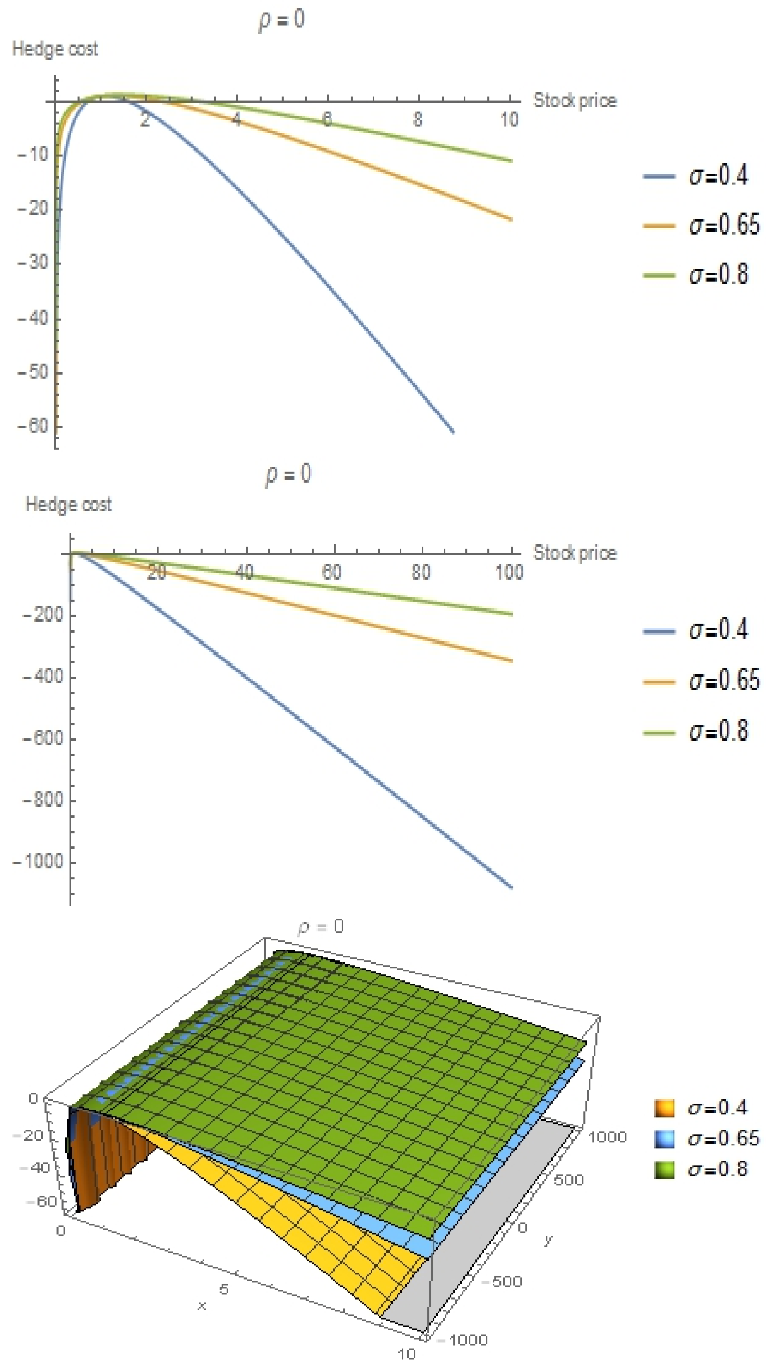


Figure 4.9: Hedge cost u, with varying values of volatility σ

Subalgebra : $X_3 + X_4$

Table 4.5: Summary of symmetry reductions and invariant solutions using sub-algebra $X_3 + X_4$

Symmetry	**Reduced Equation**	**Invariant solution in terms of original variables**
$X_3 + X_4$		
Case 1: ODE as is	$F' = 0$	$F(s) = c_1$

We follow a similar methodology in solving equations in the subalgebra as we did previously. We also plot the results as shown before. This can simply be plotted as a straight line, dependant of what values are assigned to the constant c.

4.4 Concluding remarks

In this chapter, classical Lie point symmetry methods are applied to the non-linear Black-Scholes equations. Some large Lie algebras are admitted. We determined the one dimensional optimal systems and constructed some exact solutions. Wherever necessary, we employ numerical methods to determine approximate solutions.

Chapter 5

Black-Scholes equation with variable volatility

In this chapter we consider the non-linear equation

$$u_\tau = \frac{1}{2}{\sigma_d}^2(u_{xx})x^2u_{xx} + rxu_x, \qquad x > 0$$

$$u(x,0) = v(s,T) \tag{5.1}$$

The difficulty in solving for this equation lies with the following terms: ${\sigma_d}^2(u_{xx})u_{xx}$ because this is where the non linearity comes into play.

5.1 Determining system for Lie symmetries

The equation has the form

$$u_t = A(x,y)u_{xx}^2 + B(x,y)u_x \tag{5.2}$$

The infinitesimal symmetry operator is given by

$$\tilde{X} = \varphi(x,u,\tau)\frac{\partial}{\partial x} + \xi(x,u,\tau)\frac{\partial}{\partial u} + \eta(x,u,\tau)\frac{\partial}{\partial \tau} \tag{5.3}$$

The solution for the determining equations is spanned by the following operators:

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial t} \\
X_2 &= \frac{1}{2re^{2rt}}[\frac{\partial}{\partial t} - xr\frac{\partial}{\partial x}] \\
X_3 &= -xe^{tr}\frac{\partial}{\partial u} \\
X_4 &= \frac{\partial}{\partial u} \\
X_5 &= x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}
\end{aligned}
$$

The commutator table is given by

Table 5.1: Commutator table of the sub-algebras for Black-Scholes with variable volatility

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	$-2rX_2$	$-rX_3$	0	0
X_2	$2rX_2$	0	0	0	0
X_3	rX_3	0	0	0	X_3
X_4	0	0	0	0	$2X_4$
X_5	0	0	$-X_3$	$-2X_4$	0

Using the commutator above and the formula

$$Ad(e^{\epsilon X_i})X_j = X_j - \epsilon[X_i, X_j] + \frac{1}{2!}\epsilon^2[X_i, [X_i, X_j]] - ..., \tag{5.4}$$

where i and j take on values from 1 to 5, we can find the adjoint representation of X_i. This is shown in the examples below:

for *i=1, j=2*

$$\begin{aligned} Ad(e^{\epsilon X_1})X_2 &= X_2 - \epsilon[X_1, X_2] + \frac{1}{2!}\epsilon^2[X_1, [X_1, X_2]] - ..., \\ &= X_2 - \epsilon(-2rX_2) + \frac{\epsilon^2}{2!}[X_1, -2rX_2] \\ &= X_2(1 + 2r\epsilon - \frac{(2r)^2\epsilon^2}{2!} + \frac{(2r)^3\epsilon^3}{3!} - ...) \\ &= X_2 e^{2r\epsilon} \end{aligned}$$

for *i=2, j=1*

$$\begin{aligned} Ad(e^{\epsilon X_2})X_1 &= X_1 - \epsilon[X_2, X_1] + \frac{1}{2!}\epsilon^2[X_2, [X_2, X_1]] - ..., \\ &= X_1 - 2r\epsilon X_2 + \frac{\epsilon^2}{2!}[X_2, 2rX_2] \\ &= X_1 - 2r\epsilon X_2 \end{aligned}$$

The rest of the adjoint representations are summarised in the table as below

Table 5.2: Adjoint table of sub-algebras

$[Ad.]$	X_1	X_2	X_3	X_4	X_5
X_1	X_1	$X_2e^{2r\epsilon}$	$X_3e^{\epsilon r}$	X_4	X_5
X_2	$X_1 - 2r\epsilon X_2$	X_2	X_3	X_4	X_5
X_3	$X_1 - \epsilon r X_3$	X_2	X_3	X_4	$X_5 - \epsilon X_3$
X_4	X_1	X_2	X_3	X_4	$X_5 - 2\epsilon X_4$
X_5	X_1	X_2	X_3e^{ϵ}	$X_4e^{2\epsilon}$	X_5

From the definition on adjoints above, it is possible to calculate the one-dimensional optimal system of the symmetry subgroups. Using the operator below that is also non- zero

$$X = a_1X_1 + a_2X_2 + ... + a_5X_5, \tag{5.5}$$

we have to simplify the coefficients of a_i by the application of the adjoint maps to the operator X. a_i are arbitrary constants.

To get to a one dimensional optimal system of sub-algebras we look at the equation below. Let

$$a_5 \neq 0, \qquad a_5 = 1$$

$$\begin{aligned} Ad(e^{\beta X_2})X &= a_1X_1 - 2a_1r\beta X_2 + a_2X_2 + a_3X_3 + a_4X_4 + X_5 \\ &= a_1X_1 + (a_2 - 2a_1\beta r)X_2 + a_3X_3 + a_4X_4 + X_5 \end{aligned}$$

$$\beta = \frac{a_2}{2a_1r}, \qquad a_1 \neq 0$$

The X operator then converts to X^*

$$X^I = a_1X_1 + a_3X_3 + a_4X_4 + X_5$$

$$Ad(e^{\beta X_3})X^I = a_1(X_1 - \beta rX_3) + a_3X_3 + a_4X_4 + X_5 - \beta X_3$$

choosing $\beta = \frac{a_3}{a_1r+1}$, then

$$X^{II} = a_1X_1 + a_4X_4 + X_5$$

$$Ad(e^{\beta X_4})X^{II} = a_1 + a_4X_4 + X_5 - 2\beta\alpha_4$$

choosing $\beta = \frac{a_4}{2}$, then we obtain

$$\boxed{a_1X_1 + X_5}$$

No further simplification can be made so $X_5 + a_1X_1$ is the first element of the one dimensional optimal system.

Next, we let

$$\boxed{a_5 = 0, a_4 = 1}$$

$$\begin{aligned} X &= a_1X_1 + a_2X_2 + a_3X_3 + X_4 \\ Ad(e^{\beta X_2})X &= a_1(X_1 - 2\gamma\beta X_2) + a_2X_2 + a_3X_3 + X_4 \end{aligned}$$

Choose $\beta = \frac{a_2}{2a_1r}$, then

$$\begin{aligned} X^I &= a_1X_1 + a_3X_3 + a_4X_4 \\ Ad(e^{\beta X_3})X^I &= a_1(X_1 - \beta rX_3) + a_3X_3 + X_4 \end{aligned}$$

choosing $\beta = \frac{a_3}{a_1r}$ we obtain

$$\boxed{X^{II} = a_1X_1 + X_4}$$

$$\begin{aligned} Ad(e^{\beta X_5})X^{II} &= a_1X_1 + e^{2\beta}X_4 \\ &= X_4 + e^{-2\beta}a_1X_1 \end{aligned}$$

depending on the sign of a_1, then we have now, $a_5 = a_4 = 0, a_3 = 1$,

$$\boxed{X_4 \pm X_1, X_4}$$

$$\begin{aligned} X &= a_1X_1 + a_2X_2 + X_3 \\ Ad(e^{\beta X_2})X &= a_1(X_1 - 2r\beta X_2) + a_2X_2 + X_3 \\ X^I &= a_1X_1 + X_3 \qquad \text{following elimination of } X_2 \\ Ad(e^{\beta X_5})X^I &= a_1X_1 + e^{\beta}X_3 \\ &= X_3 + a_1e^{-\beta}X_1 \end{aligned}$$

$$\boxed{X_3 \pm X_1, X_3}$$

depending on the sign of a_1, we obtain

now, $a_5 = a_4 = a_3 = 0, a_1 = 1$,

$$\begin{aligned} X &= X_1 + a_2 X_2 \\ Ad(e^{\beta X_2})X &= X_1 - 2r\beta X_2 + a_2 X_2 \end{aligned}$$

Let $\beta = \frac{a_2}{2r}$,then if $a_5 = a_4 = a_3 = a_1 = 0$,then

$$\boxed{X_1}$$

$$\boxed{X_2}$$

The complete one-dimensional optimal system of sub-algebras is therefore

$$\{X_5 + \alpha X_1, X_4 \pm X_1, X_4, X_3 \pm X_1, X_3, X_1, X_2\}, \tag{5.6}$$

for all $\alpha \in \Re$

5.1.1 Symmetry reductions and invariant solutions

Elements of the optimal system obtained above are used for symmetry reductions. We must find group invariant solutions for the one parameter subgroups generated. The results using the subalgebra $\alpha X_1 + X_4$ are shown below.

$$\alpha X_1 + X_4 = \frac{\partial}{\partial t} + \frac{\partial}{\partial u}$$

$$dt = du$$

$$t = u + I_2$$

but remember that

$$I_1 = x$$

$$I_2 = F(I_1)$$

$$I_2 = F(x)$$

Our primary equation becomes

$$u = t - F(x) \tag{5.7}$$

Substituting into our original PDE,

$$u_\tau = \frac{1}{2}{\sigma_d}^2(u_{xx})x^2u_{xx} + rxu_x, \qquad x > 0$$

$$u(x, 0) = v(s, T) \tag{5.8}$$

we obtain the ODE

$$\sigma_d^2 x^2 F''^2 - 2rxF' - 2 = 0 \tag{5.9}$$

We were unable to come about with a solution to the ODE above as is, so we made the following assumptions to get to a particular solution.

Case 1: $r = 0$

then

$$F''^2 = \frac{2}{\sigma_d^2 x^2} \tag{5.10}$$

Remember that r represents the risk free interest rate. Academics and practitioners have used government security rates as risk free rates, though at times there is confusion arising regarding using long term of short term rates. Realistically, to work out the interest rate, a risk free bond issued by the government is chosen, where the risks of default are negligible.

In particular, a solution can be written in terms of

$$F_1(x) = c_1 + \frac{xc_2 - \sqrt{2}(-x + x\ln x)}{\sqrt{\sigma_d}} \tag{5.11}$$

$$F_2(x) = c_1 + \frac{xc_2 + \sqrt{2}(-x + x\ln x)}{\sqrt{\sigma_d}} \quad (5.12)$$

In terms of the original variables, we have

$$u = t - [c_1 + \frac{xc_2 \pm \sqrt{2}(-x + x\ln x)}{\sqrt{\sigma_d}}] \quad (5.13)$$

Case 2: $r = 1$

This case yields an ode for which we are unable to find a solution to.

Case 3: $\sigma_d = 0$

σ_d represents volatility. Zero volatility is theoretically possible, but not observed in financial markets. This is because option prices cannot be calculated, infinity, and the Black-Scholes equation explodes. This causes the asset to be degenerate. Zero volatility means that Brownian motion is no longer stochastic and will move in the direction of the drift.
Given a zero volatility, equation (5.9) has a solution.

$$F' = \frac{1}{-rx} \quad (5.14)$$

The solution can be written as

$$F(x) = c_1 - \frac{lnx}{r} \quad (5.15)$$

in terms of the original variables we obtain

$$u = t - \big(c_1 - \frac{lnx}{r}\big) \quad (5.16)$$

The rest of the symmetry reductions and invariant solutions are shown in the tables below.

Table 5.3: Summary of symmetry reductions and invariant solutions using sub-algebra $X_1 + X_4$

Symmetry	Reduced Equation	Invariant solution in terms of original variables
	$X_1 + X_4$	
Case 1: ODE as is	$\sigma_d x^2 F''^2 - 2rxF' - 2 = 0$	No solution
Case 2: r=0	$F''^2 = \frac{2}{\sigma_d x^2}$	$F_1[x] = c_1 + xc_2 - \frac{\sqrt{2}(-x+x\ln x)}{\sqrt{\sigma_d}}$ $F_2[x] = c_1 + xc_2 + \frac{\sqrt{2}(-x+x\ln x)}{\sqrt{\sigma_d}}$
Case 3: $\sigma_d = 0$	$F' = \frac{1}{-rx}$	$F[x] = c_1 - \frac{\ln x}{r}$

Case 1: ODE as is

We attempt solving (5.9) ODE using numerical methods, and in this case Explicit Euler method. We make certain assumption on initial boundary values as seen below. We also use hypothetical values for volatility σ_d, and risk free interest rate r.

$$\begin{aligned} \sigma_d x^2 F''^2 - 2rxF' - 2 &= 0 \\ y'[1] &= 1 \\ y[1] &= 1 \\ \sigma_d &= 0.47 \\ r &= 0.04 \\ x \in & [0, 40] \end{aligned}$$

We show the second subalgebra is $X_1 + X_2$, and find solutions for the arising equations. Lastly, we find solutions using subalgebra $X_1 + X_3$. We have used the Explicit Runga-Kutta method, and the plot of the extrapolation of the function based on initial conditions stated above. We get a solution with two graphs. We plot third order Hermitian graphs. We also evaluate the behaviour based on varying values of the risk free rate to see the impact on the hedging cost. It is very interesting to see that slightly reducing the risk free rate causes

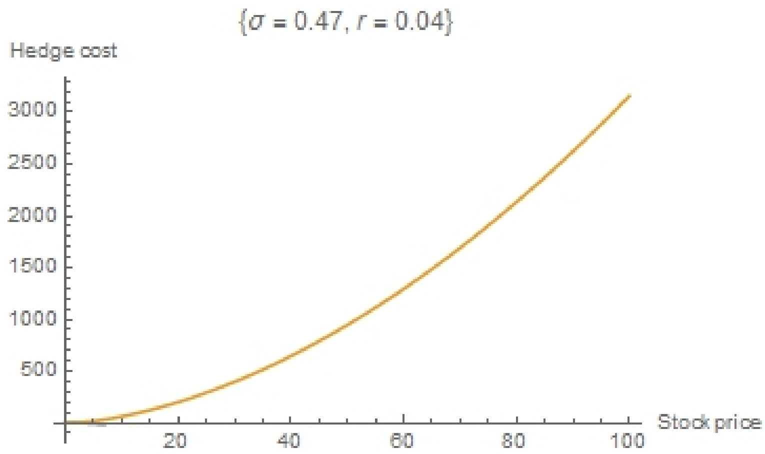


Figure 5.1: Option with fixed payoff values of u, and variable interest rates r

our solution to be chaotic. The plot becomes extremely volatile and the hedge cost fluctuates from one range to the other. So very low adjustments of the interest rate r causes results that are not negligible.

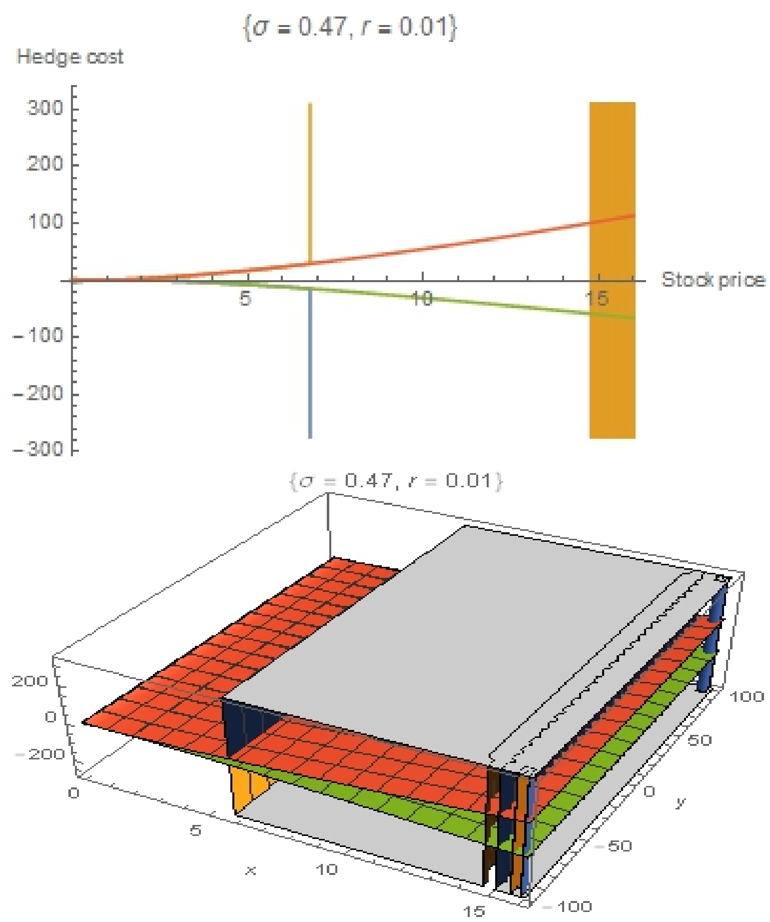


Figure 5.2: Comparison on impact to hedge cost when interest rates are greatly reduced

Case 2: $r = 0$

Looking at this case in the table above, we try to solve for constants c_1, and c_2. To get to a particular solution, we invoke an initial condition to solve for the arbitrary constants.

$$
\begin{aligned}
x_0(\text{price}) &= 100 \\
\sigma_d(\text{volatility}) &\in [0.32, 0.84] \\
F(x_0)(\text{u}) &= 50
\end{aligned}
$$

From this we are able to conclude that

$$c_1 = -100c_2 + 961.29$$

If we suppose that at some time $t_1 = 3$, we have $x_1 = 105$, then we can invoke boundary condition

$$\left.\frac{\partial F}{\partial x}\right|_{x=105} = 0$$

This gives a solution for our constant.

$$c_2 = 11.64$$

Hence, our solution can we written as follows

$$F_1(x) = -1164 + 11.64x - \frac{\sqrt{2}(-x + x\ln x)}{\sqrt{0.32}} \quad (5.17)$$

$$F_2(x) = 1164 - 11.64x + \frac{\sqrt{2}(-x + x\ln x)}{\sqrt{0.32}} \quad (5.18)$$

The profile for both these solutions are then plotted as shown below The graph shows that when volatility is high, then from very low costs of a stock, the hedge becomes costly. At very low stock prices, the impact of volatility on the hedge is negligible. If the volatility is increased however, then the cost continues to decrease at an increasing rate, as compared to when volatility is low.

This is interesting, because when we solve the problem using a numerical method like explicit Euler or Adams method, then we get very a different profile, yet intuitive. The hedge cost increases as the stock price increases.

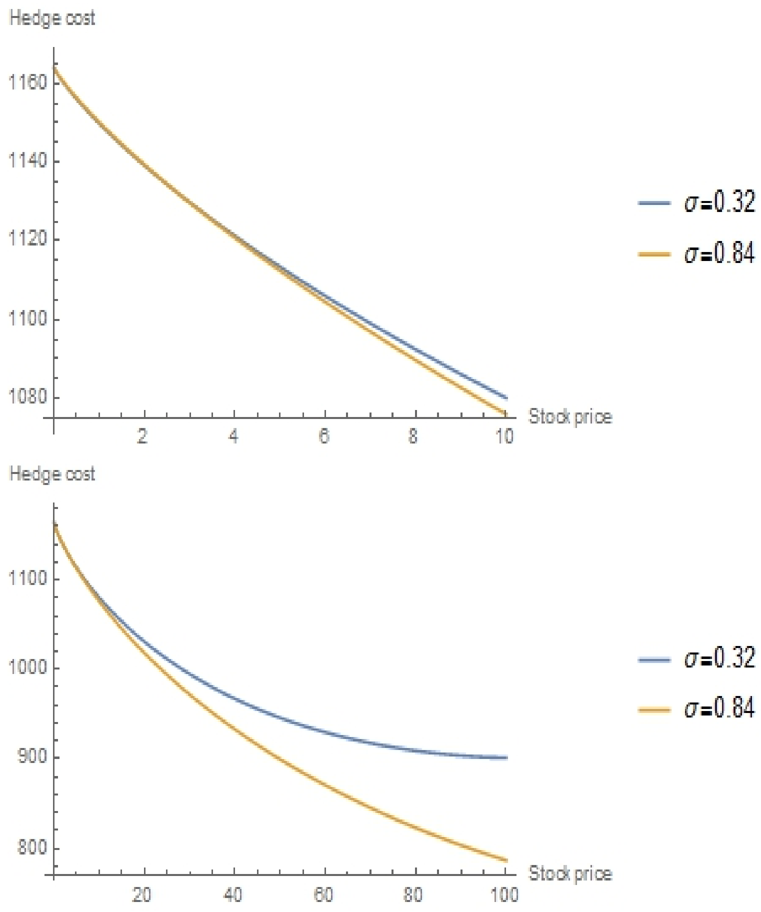


Figure 5.3: Impact of adjusting various points of volatility σ_d

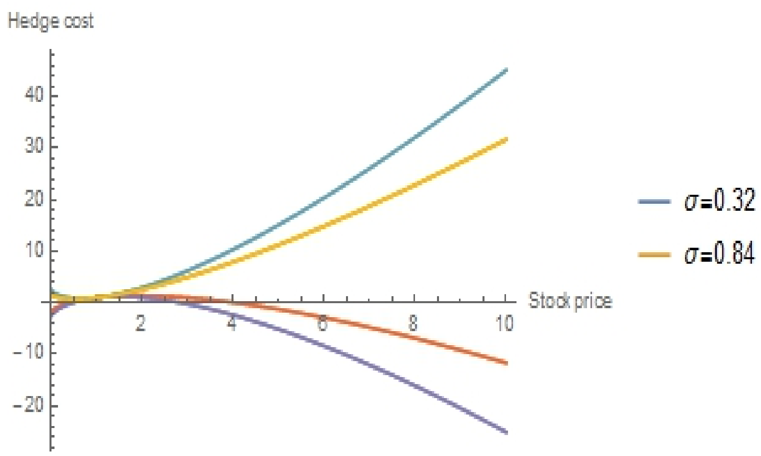


Figure 5.4: Profile using numerical methods Euler and Adams

Case 3: $\sigma_d = 0$

To get to a particular solution, we will make the following initial assumptions.

$$\begin{aligned} x_0(\text{price}) &= 100 \\ r(\text{risk free interest rate}) &\in [0.01, 0.05] \\ F(x_0)(\text{u}) &= 60 \end{aligned}$$

Depending on the varying constants chosen for r, then we are able to find profiles for the solution of u. The graph below shows impact of interest rates of volatile values of x for the solution. We do a similar construct to solve

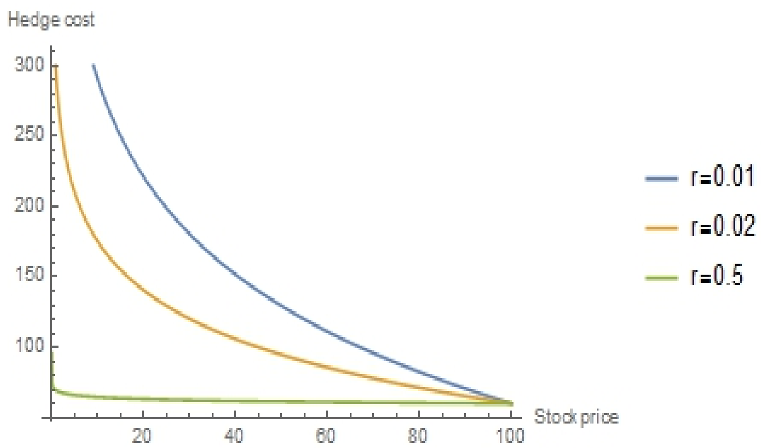
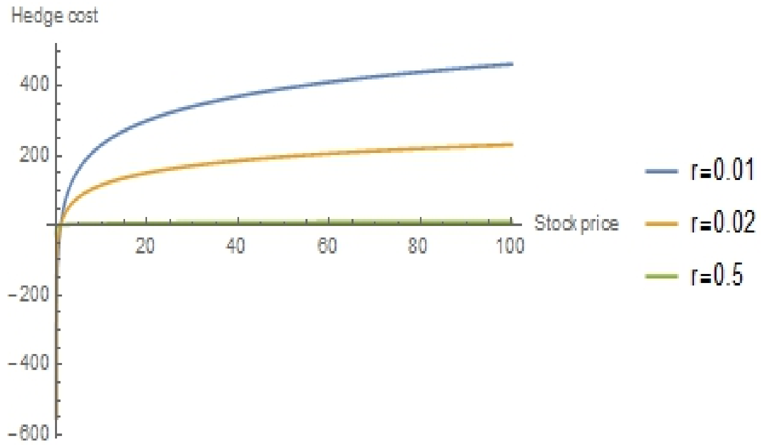


Figure 5.5: Assumed variable interest rates, and no volatility

the ODE. Here Runga-Kutta methods are employed to construct approximate solutions. We make the following assumptions

$$\begin{aligned} x &\in [0, 100] \\ y(1) &= 1 \end{aligned}$$

Then the plot we get differs from the solution previously attained. So we try



using similar initial conditions. We assume that

$$x \in [0, 300]$$
$$y(100) = 60$$

We observe from the plots that at very high values of the interest rate r, the

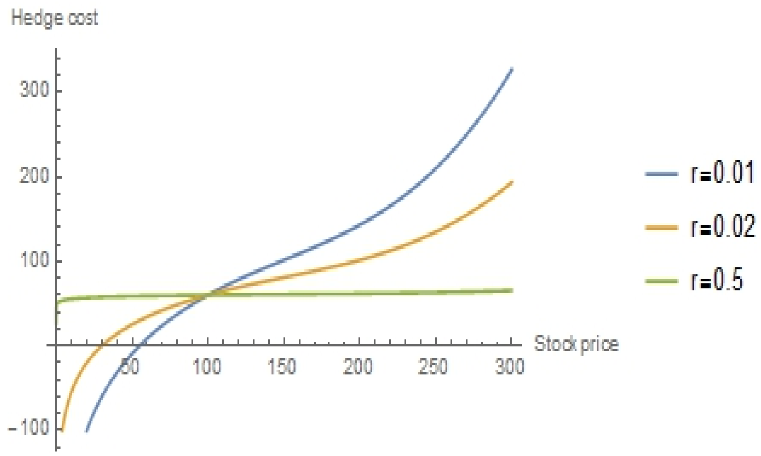


Figure 5.6: Option with payoff u, plotted using R-K methods

hedging cost greatly reduces and almost plateaus. As the interest rate increases, the hedging cost decreases.

We elaborate further here on implied volatility. For a call option, Black-Scholes takes as input the strike, interest rate, together with the volatility to output the price. Volatility is very hard to measure and we don't really know what value to insert in the formula. The following information on a call option, with five months until expiry and a strike of 98 is trading at 6.51 with the underlying at 99.5 and a short-term interest rate of 0.02 can be made available to a trader.

How do we infer the relationship between volatility and an option price. We see the price where the option trades, and can get to the market price by making an assumption on volatility. This is the implied volatility. The implied volatility is the volatility of the underlying, and when inserted into the BlackScholes formula, gives a theoretical price equal to the market price. Simply put, it is how the market sees volatility over the life of the option. Volatility is not constant, and typically has a smile shape, characteristic of the market. Empirical evidence shows that the market does not price European

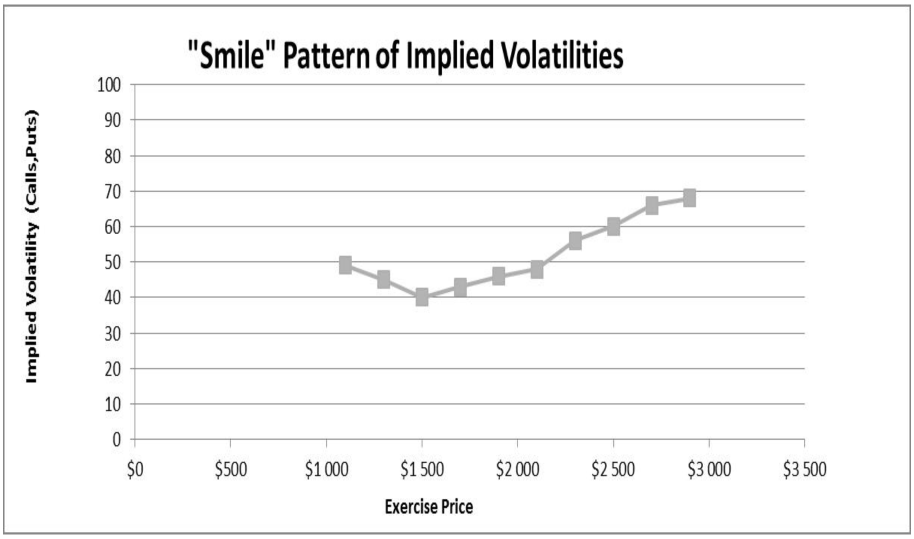


Figure 5.7: Implied Volatility with skew shape

call options according to the Black-Scholes model, for the volatility σ to be constant isn't plausible. This derives models where volatility is not constant but dependent on time.

Subalgebra : $X_1 + X_2$

Here we look at the next sub-algebra and follow a similar process as above to get to solutions. The symmetry reductions and invariant solutions are shown in the table below. This is a very intricate problem to find a solution to as the resultant gives complex numbers.

Table 5.4: Summary of symmetry reductions and invariant solutions using sub-algebra $X_1 + X_2$

Symmetry	**Reduced Equation**	**Invariant solution in terms of original variables**
$X_1 + X_2$		
Case 1: ODE as is $z = \frac{xe^{rt}}{1+2re^{rt}}$	$\sigma_d^2 F''^2 + \frac{4r^2}{z^2} F' = 0$	$F_1[z] = \frac{-2r^2z}{\sigma^2} + \frac{\mathbf{i}rzc_1}{\sigma} + \frac{zc_1^2}{4} + c_2 + \frac{2r^2 z\ln z}{\sigma^2} - \frac{\mathbf{i}rzc_1 \ln z}{\sigma} - \frac{r^2 z \ln z^2}{\sigma^2}$ $F_2[z] = \frac{-2r^2z}{\sigma^2} - \frac{\mathbf{i}rzc_1}{\sigma} + \frac{zc_1^2}{4} + c_2 + \frac{2r^2 z\ln z}{\sigma^2} + \frac{\mathbf{i}rzc_1 \ln z}{\sigma} - \frac{r^2 z \ln z^2}{\sigma^2}$

Subalgebra : $X_1 + X_5$

The symmetry reductions and invariant solutions are shown in the table below. This problem is similar to the previous one in that the resultant solution is in the form of complex numbers.

Table 5.5: Summary of symmetry reductions and invariant solutions using sub-algebra $X_1 + X_5$

Symmetry	**Reduced Equation**	**Invariant solution in terms of original variables**
$X_1 + X_5$		
Case 1: ODE as is $z = xe^{-t}$	$\frac{1}{2}\sigma_d^2 F''^2 + \frac{r}{z}F' + \frac{1}{z}F' - \frac{2}{z^2}F = 0$	$F_1[z] = z\big[c_2 + \frac{1}{4}(\frac{-e^{2c_1}}{z} - z - 2\mathbf{i}e^{c_1}\ln z)\big]$ $F_2[z] = z\big[c_2 + \frac{1}{4}(\frac{-e^{2c_1}}{z} - z + 2\mathbf{i}e^{c_1}\ln z)\big]$

The interesting result from this is we can clearly see that the costs on the subalgebra $X_1 + X_2$ increase much faster.

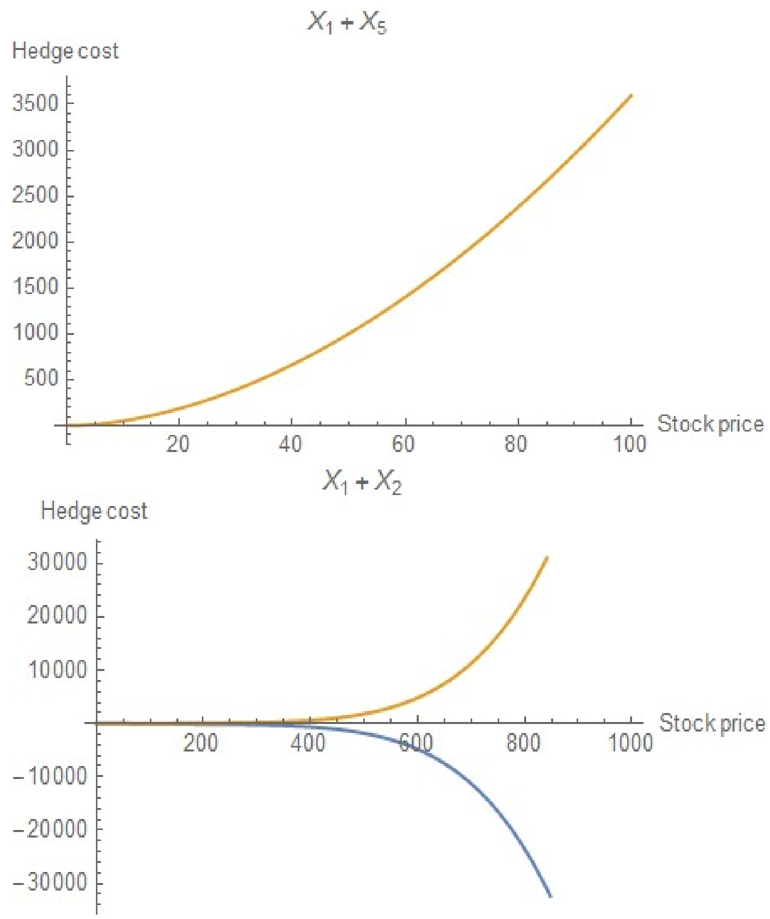


Figure 5.8: Cost plots arising from the subalgebra $X_1 + X_5$ and $X_1 + X_2$

Subalgebra : $X_1 + X_3$

For this problem, this is the last subalgebra that we will look at. Assumption on the risk free interest rate are made, as well as on volatility. Though the assumptions may not be practical in the real financial world, they are made to come to the solution of the mathematical problem.

Table 5.6: Summary of symmetry reductions and invariant solutions using sub-algebra $X_1 + X_3$

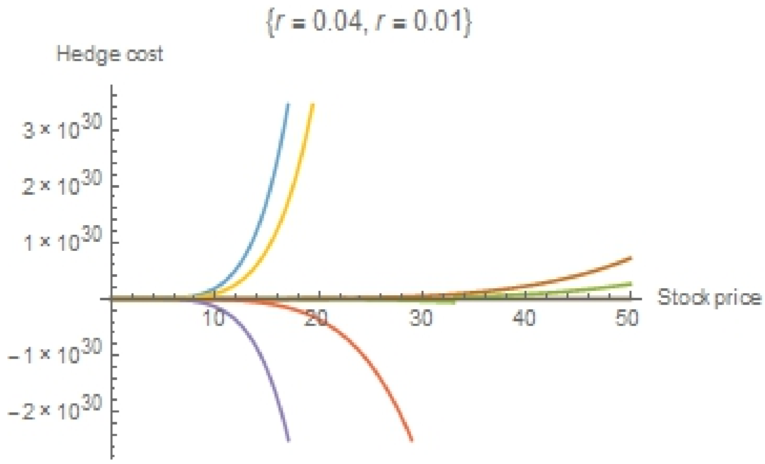
Symmetry	**Reduced Equation**	**Invariant solution in terms of original variables**
	$X_1 + X_3$	
Case 1: ODE as is	$\sigma_d(2F'+xF'')^2+2rx^2F'+2rxF=0$	No solution
Case 2: r=0	$\sigma_d(2F'+xF'')^2=0$	$F[x]=\frac{-c_1}{x}+c_2$
Case 3: $\sigma_d=0$	$xF'+F=0$	$F[x]=\frac{c_1}{x}$

Case 1: ODE as is

We attempt solving ODE in table (5.5) using numerical methods, and in this case Explicit Euler method. We make certain assumption on initial boundary values as seen below. We also use hypothetical values for volatility σ_d, and risk free interest rate r.

$$\begin{aligned}
\sigma_d(2F'+xF'')^2+2rx^2F'+2rxF &= 0\\
F'[1] &= 1\\
F[1] &= 1\\
\sigma_d &= 0.47\\
r &\in [0.01, 0.04]\\
x &\in [0, 100]
\end{aligned}$$

We have used the Explicit Runga-Kutta method, and the plot of the extrapolation of the function based on initial conditions stated above. We get a solution with two graphs. We plot third order Hermitian graphs. This is really just a



chaotic plot that we can draw no real conclusion from.

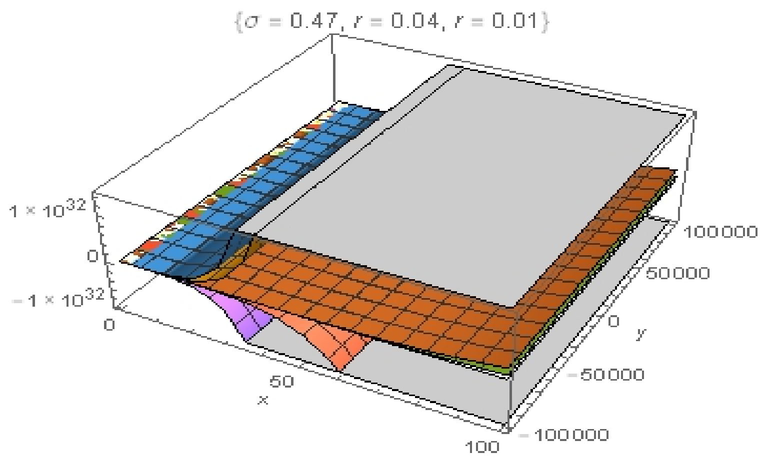


Figure 5.9: Option with fixed payoff values of u

Case 2: $r = 0$

To get to a particular solution, we will make the following initial assumptions.

$$\begin{aligned} x_0(\text{price}) &= 100 \\ F(x_0)(\text{u}) &= 120 \end{aligned}$$

From this we are able to conclude that

$$c_1 = -12000c_2 + 100c_2$$

If we suppose that at some time $t_1 = 5$, we have $x_1 = 105$, then we can invoke boundary condition

$$\left.\frac{\partial F}{\partial x}\right|_{x=105} = 0$$

This gives a solution for our constant.

$$c_2 = 120$$

Making $c_1 = 0$. Hence, our solution can we written as follows

$$F(x) = 120 \tag{5.19}$$

This can simply be plotted as a straight line as shown below.

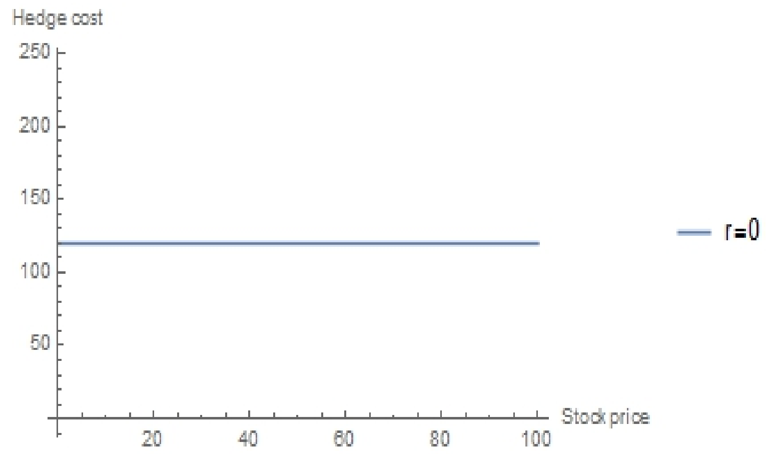


Figure 5.10: Option with fixed payoff values of u

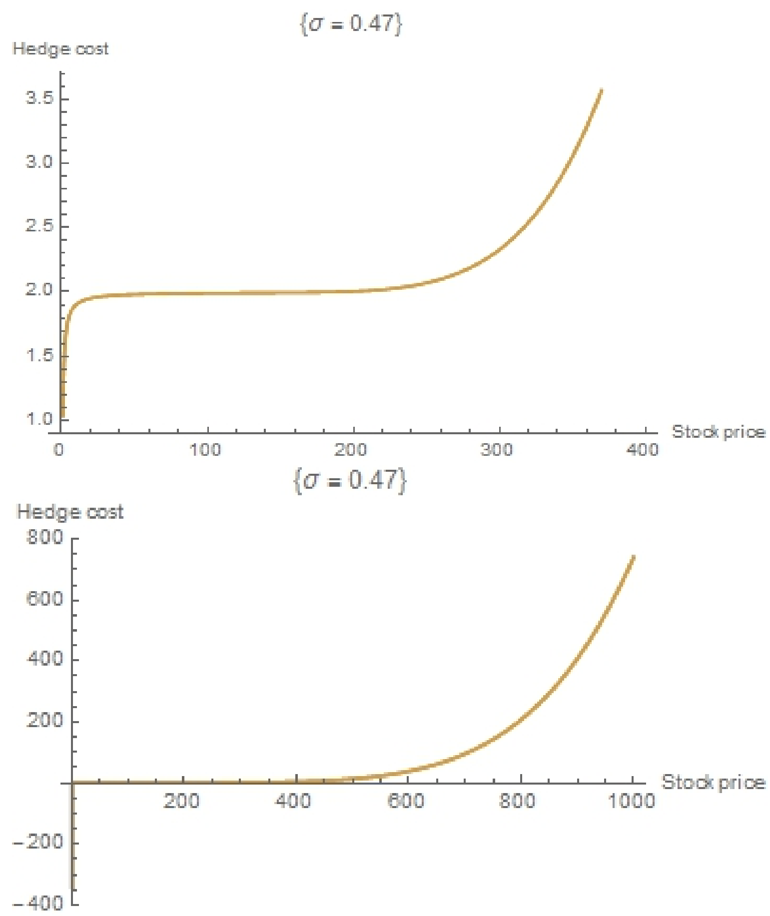


Figure 5.11: Risk free rate is zero

Case 3: $\sigma_d = 0$

This case brings about a similar solution as when we assume that $r = 0$. This is simply a plot of a straight line as seen in Figure 4.4, so we will not plot again as it is of a similar fashion.

One of the first and simplest methods for solving initial value problems was proposed by Euler. Euler's method is not very accurate. Local accuracy is measured by how high terms are matched with the Taylor expansion of the solution. Euler's method is first-order accurate, so that errors occur one order higher starting at powers of h^2. The ODE has been solved using Mathematica as follows

$$\begin{aligned} xy'[x] + y[x] &= 0 \\ y[1] &= 1 \\ x \in & \; [0, 500] \end{aligned}$$

We have used the Explicit Runga-Kutta method, and the plot of the extrapolation of the function based on initial conditions stated above. With no volatility, the hedge cost is negligible as expected.

A number of questions arise

1. Is this a true reflection of the markets?
2. What impact does this have on the stock price?
3. What is the impact on the costs of insurance?
4. How do we model the costs to be reflective of the disruption caused by volatility?

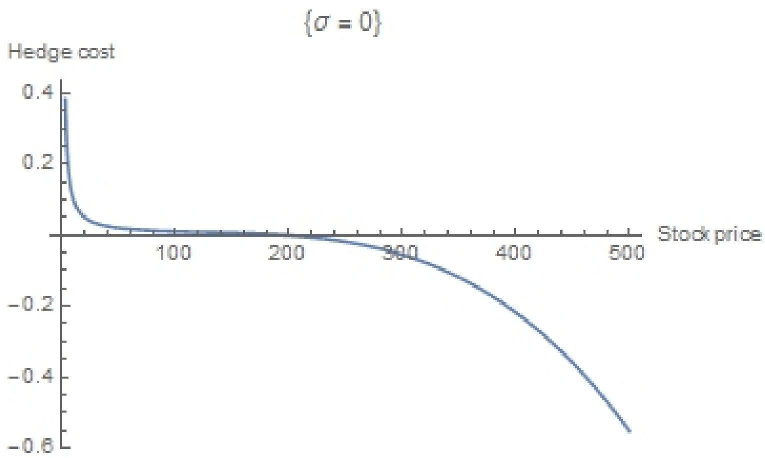


Figure 5.12: Option with fixed payoff values of u

5. Does this have a knock on effect on the market liquidity?

We look at work done on the effects of constant volatility and implied volatility in the next section. We try answer these questions in sections to follow.

We look at work done on illiquid markets and pricing of derivatives and give examples to give clarify. The graphs and solutions we have represented above don't always give a real life solution because we are solving a mathematical function and certain hypotheses don't always hold in reality.

5.2 Hedging a call option

Suppose that we have sold a call option with strike price K and maturity time T for an amount c_0 at time 0, and want to hedge this position. The assumption is that volatility is some constant σ, but the true volatility is given by the stochastic process $\beta(t,\omega)$. Thus, we believe that the price of the option at time t, which we denote by P_t, is given by $P_t = F(t, S_t)$, where F solves the Black-Scholes equation

$$\begin{aligned} \frac{\partial F}{\partial u} + rx\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2} &= rF \\ F(T,x) &= (x-k)^+ \end{aligned}$$

We can hedge the position as follows

$$\delta_t = \frac{\partial F}{\partial x}(t, S_t) \tag{5.20}$$

and

$$h_t^B = \frac{F(t,S_t) - \frac{\partial F(t,S_t)}{\partial x} S_t}{B_t} \tag{5.21}$$

If σ represents the value of volatility, then this delta hedge means that the value of our portfolio perfectly matches the value of the option at any time $t \in [0,T]$. This portfolio is not self financing. P is given by

$$\begin{aligned} dP_t &= \delta_t dS_t + h_t^B dB_t + S_t d\delta_t + B_t dh_t^B \\ &= \delta_t dS_t + H_t^B dB_t - dC_t \end{aligned}$$

Ito's formula gives the following

$$\begin{aligned} dP_t &= dF(t,S_t) \\ &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial x}dS_t + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}d < S_t > \\ &= [\frac{\partial F}{\partial t} + \frac{1}{2}\beta_t^2 S_t^2 \frac{\partial^2 F}{\partial x^2}]dt + \frac{\partial F}{\partial x}dS_t \end{aligned}$$

The resultant cost strategy arrived at using δ_t and h_t^B is shown below

$$\begin{aligned} -dC_t &= dP_t - \delta_t dS_t - h_t^B dB_t \\ &= [\frac{\partial F}{\partial t} + \frac{1}{2}\beta_t^2 S_t^2 \frac{\partial^2 F}{\partial x^2}]dt + \frac{\partial F}{\partial x}dS_t - \frac{\partial F}{\partial x}dS_t + rFdt - r\frac{\partial F}{\partial x}S_t dt \\ &= [\frac{\partial F}{\partial t}{}_r S_t \frac{\partial F}{\partial x} - rF + \frac{1}{2}\beta_t^2 S_t^2 \frac{\partial^2 F}{\partial x^2}]dt \end{aligned}$$

F solves the Black-Scholes equation, so substituting $-\frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial x^2}$ gives

$$-dC_t = \frac{1}{2}S_t^2 \frac{\partial^2 F}{\partial x^2}(t,S_t)(\beta_t^2 - \sigma^2)dt$$

integration from $0 - T$ gives

$$C_0 = \frac{1}{2}\int_0^T S_t^2 \frac{\partial^2 F}{\partial x^2}(t,S_t)(\beta^2(t,\omega) - \sigma^2)dt \qquad (5.22)$$

since $c_T = 0$. For a European call option, $\sigma \geq \beta_t$ for all $t \in [0,T]$, since $\frac{\partial^2 F}{\partial x^2}$ is strictly positive. The cost is then non positive. The work of [37] is also used for consistency in a similar fashion, and the example below follows from their cost function.

An Example

$$
\begin{aligned}
\text{Strike Price} &= 90 \\
\text{Term} &= 3 \text{ months} \\
\text{Current Price} &= 94 \\
\text{Risk Free Interest Rate} &= 0.045 \\
\text{Volatility} &= 0.35 \\
&\not> 0.7 \text{ for the next three months}
\end{aligned}
$$

We then want to solve for the hedging cost, using hypothetical values of volatility.

$c(\sigma)$ = Price of European Call Option in Black-Scholes model if σ is constant

$$
\begin{aligned}
c(0.35) &= 9.19 \\
c(0.70) &= 15.37
\end{aligned}
$$

The assumption is that the buyer doesn't want to pay more than 12.00 for example, then different assumptions on volatility must be made and the hedge cost must be different. Money might be made or lost.

As volatility increases, it becomes time and price dependant. This effect is also impacted by the share of total demand that is due to hedging. The heterogeneity of the distribution of the hedged payoffs is also greatly impacted. This particular outcome is further elaborated in the section below.

5.3 Hedge demand generated by Black-Scholes strategies

Here we look at the paper of [38], he shows the feedback effect of portfolio hedging can be alleviated if distribution of strike prices and times to maturity is heterogeneous. To get to this, the problem of replicating payoff of a call option with the following characteristics is considered

$$\begin{aligned} K &= \text{Strike Price} \\ T &= \text{Maturity} \\ \sigma &= \text{Volatility} \\ t &= \text{time} \end{aligned}$$

The assumption that the underlying follows GBM with constant volatility σ is made, and the price at any time t is given by a solution $c(t, x_t)$ of the problem

$$(\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2})c(t,x) = 0 \qquad c(T,x) = [x-K]^+, \tag{5.23}$$

The strategy is then represented by $\frac{\partial c}{\partial x}(t, x_t)$. The following represent price and strategy functions, $c(\sigma, k, T-t, x)$ and $\varphi(\sigma, K, T-t, x)$. The strategy function is given by $\varphi(\sigma, K, \tau, x) = \aleph(\frac{\log x - \log K}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau})$ where

$$\begin{aligned} \aleph &= \text{Standard Normal Distribution} \\ \tau &= T - t, \text{ Time to Maturity} \\ \sigma &= \text{Volatility used for computation of Hedging Strategies} \end{aligned}$$

[38] shows the traders demand as $\rho\phi(\sigma, x)$, ρ the market weight. He then represents

$$\phi(\sigma, x) = a + \int_{R^2_{\oplus}} \varphi(\sigma, K, \tau, x)\nu_{-}(dK \otimes d\tau), \tag{5.24}$$

where

$$a = \text{Static position in underlying}$$

$$\nu_- = \text{Measure on } R^2_\oplus \text{ describing the distribution of strike prices } K \text{ and times to maturity } \tau \text{ in portfolio}$$

They then show $(\sigma, x) = \frac{\partial \phi}{\partial x}(\sigma, x)$. [38] goes on to conclude on the following

Assumption 5.3.1. ν_- has a smooth density with respect to the Lebesgue measure, ν_- is of the form $\nu_-(dK \otimes d\tau) = g(K, \tau)dK \otimes d\tau$ where $g.\Re_+ \times [0, \infty) \to \Re_+$ is a smooth density function having compact support in $\Re_+ \times [0, \infty)$.

There is distinct equilibrium in the economy when ρ is very small, and this is verified. On a single contract, function $x\frac{\partial \varphi}{\partial x}(\sigma, K, \tau, x)$ blows up when $x \to K$ and $\tau \to 0$. This is supported by the fact that when the option is close to maturity and at the money, hedging strategies need large movements of the hedge portfolio. For the collective, the problem is resolved if distribution ν_- is non singular. This implies that bounds on (x, σ) can be arrived at that rely on the heterogeneity of the distribution of ν_-.

To conclude on this, the following proposition is laid out by [38]

Proposition 5.3.1. Suppose $\sigma > \eta$ for some $\eta > 0$. We have the estimation for all $x \in \Re_+$

1.
$$|(\sigma, x)| \leq \int_0^\infty \int_0^\infty |\frac{\partial}{\partial K}(Kg(k, \tau))|dKd\tau \tag{5.25}$$

2.
$$|\frac{\partial}{\partial \sigma}, (\sigma, x)| \leq \frac{2}{\eta} \int_0^\infty \int_0^\infty |\frac{\partial^2}{\partial \tau \partial K}(\tau Kg(K, \tau))|dKd\tau \tag{5.26}$$

From the proposition, [38] achieves the assumption above without restriction on the distribution ν_{-}, if there is a requirement that the portfolio insurance weight ρ not to be too large. The feedback effect of dynamic hedging on market volatility mainly manifests itself through the appearance of $[\sigma, x)$ in the denominator of $v([\sigma, x))$. Hence by (i) we see that this disturbance is controlled by the degree of heterogeneity of ν. From this [38] comes to the following corollary

$$\nu(0, x) = \frac{1 - \rho\phi(\sigma, x)}{1 - \rho\phi(\sigma, x) - \rho(\sigma, x)} \leq \frac{1 - \rho}{1 - \rho - \rho sup(\sigma, x)} \tag{5.27}$$

They conclude by saying that even maximal increases in volatility is controlled by the degree of heterogeneity of ν.

5.4 Concluding remarks

In this chapter we considered the Black-Scholes equation with variable volatility. It turns out that the equation admits a five dimensional Lie algebra. Some symmetry reductions are performed using the elements of the optimal system, and group invariant (exact) solutions are constructed. We also determine approximate solutions.

Chapter 6

Symmetry reductions of 2+1 D Black-Scholes equation

6.1 Introduction

In this chapter we attempt constructing group invariant solutions for the $2+1$ D Black-Scholes equation.

6.2 2+1 D Black-Scholes equation

Cimpoiasu and Constantinescu [15] derive the following PDE

$$u_t + \mu_1 x u_x + \mu_2 y u_y + \frac{1}{2}{\sigma_1}^2 x^2 u_{2x} + \frac{1}{2}{\sigma_2}^2 y^2 u_{2y} + \rho\sigma_1\sigma_2 xy u_{xy} - ku = 0,$$

k is a constant

$$u(x, y, T) = u_T(x, y), \tag{6.1}$$

where (x, y) is a basket of two assets, and the returns by u_i.

They compute the Lie symmetry generators, and Lie Algebra. In this dissertation, we extend to this work by constructing the one dimensional optimal system of sub-algebras of the Lie algebra admitted by (6.1)

6.2.1 Determination of Lie symmetries for 2+1 D Black-Scholes equation

The $2 + 1$ D Black-Scholes equation has the form

$$\begin{aligned} u_t &= A(x,y)u_{xy} + B(x,y)u_x u_y + C(x,y)u_{2x} + D(x,y)u_{2y} + E(x,y)u_y \\ &+ F(x,y)u_x + G(x,y,u). \end{aligned} \tag{6.2}$$

We seek the infinitesimal symmetry operator of the form

$$\tilde{X} = \varphi(x,y,t,u)\frac{\partial}{\partial t} + \xi(x,y,t,u)\frac{\partial}{\partial x} + \eta(x,y,t,u)\frac{\partial}{\partial y} + \phi(x,y,t,u)\frac{\partial}{\partial u} \tag{6.3}$$

If the following consideration is made $\varphi = c_0$, then the determining system of equations is given by

$$B\xi_y - C\phi_{2u} = 0$$

$$B\eta_x - D\phi_{2u} = 0$$

$$A\eta_y - \eta A_y + A\xi_x - \xi A_x + 2D\xi_y + 2C\eta_x = 0$$

$$A\xi_y + 2C\xi_x - \xi C_x - \eta C_y = 0$$

$$A\eta_x + 2D\eta_y - \eta D_y - \xi D_x = 0$$

$$-A\phi_{2u} + B\xi_x - B\phi_u + B\eta_y - B_x\xi - B_y\eta = 0$$

$$-\eta_t + F\eta_x - B\phi_x + E\eta_y - E_x\xi - E_y\eta + A\eta_{xy} - A\phi_{xu} + C\eta_{2x} + D\eta_{2y} - 2D\phi_{yu} = 0$$

$$\xi_t - B\phi_y + F\xi_x + E\xi_y - F_x\xi - F_y\eta + A\xi_{xy} - A\phi_{yu} + C\xi_{2x} + D\xi_{2y} - 2C\phi_{xu} = 0$$

$$\phi_t + G\phi_u - F\phi_x - E\phi_y - G_x\xi - G_y\eta - G_u\phi - A\phi_{xy} - C\phi 2x - D\phi 2y = 0$$

To solve the 2+1 D Black-Scholes equation one may consider the following [15]

$$A = \rho\sigma_1\sigma_2 xy;$$
$$B = 0;$$
$$C = -\frac{1}{2}\sigma_1^2 x^2;$$
$$D = -\frac{1}{2}\sigma_2^2 y^2;$$
$$E = \mu_2 y;$$
$$F = -\mu_1 x;$$
$$G = ku.$$

Choosing $c_0 = 1$, yields the solution [15]

$$\xi = \frac{c_3 x}{\rho\sigma_2}[\rho\sigma_2 lnx - \sigma_1 lny] + x(c_1 t + c2) \tag{6.4}$$
$$\eta = -\frac{c_3 y}{\rho\sigma_1}[\rho\sigma_1 lny - \sigma_2 lnx] + y(c_4 t + c5) \tag{6.5}$$
$$\phi = \omega + \beta u \tag{6.6}$$

where $c_i, i = 1, ...6$ and ω is any solution of (6.1).

$$\beta = \frac{1}{\rho\sigma_1^2\sigma_2^2(1-\rho^2)} * \Bigg\{ \Bigg[-c_1\frac{\rho^2\sigma_1\sigma_2}{2} + c_3\frac{\sigma_1\sigma_2}{2}[\sigma_1^2 - 2\mu_1](\rho^2 - 1) + c_2\rho\sigma_1^2 \Bigg] \ln y + \Bigg[c_1\rho\sigma_2^2 - c_3\frac{\sigma_1\sigma_2}{2}[\sigma_2^2 - 2\mu_2](\rho^2 - 1) - c_4\rho^2\sigma_1\sigma_2 \Bigg] \ln x + \gamma \Bigg\} \tag{6.7}$$

$$\gamma = t\Bigg\{ c_1\Bigg[\frac{\rho\sigma_2^2}{2}[\sigma_1^2 - 2\mu_1] - \frac{\rho^2\sigma_1\sigma_2}{2}[\sigma_2^2 - 2\mu_2] \Bigg] + c_4\Bigg[\frac{\rho\sigma_1^2}{2}[\sigma_2^2 - 2\mu_2] - \frac{\rho^2\sigma_2\sigma_1}{2}[\sigma_1^2 - 2\mu_1] \Bigg] \Bigg\} + c_6\rho\sigma_1^2\sigma_2^2(1-\rho^2) \tag{6.8}$$

The Lie operator is decomposed as

$$\tilde{X} = X + X_w \tag{6.9}$$

This can be interpreted as

$$X = \frac{\partial}{\partial t} + \left\{ \frac{c_3 x}{\rho\sigma_2}[\rho\sigma_2 lnx - \sigma_1 lny] + x(c_1 t + c_2) \right\} \frac{\partial}{\partial x} + \left\{ \frac{c_3 y}{\rho\sigma_1}[\rho\sigma_1 lny - \sigma_2 lnx] + y(c_4 t + c_5) \right\} \frac{\partial}{\partial y} + \beta(x,y,t) u \frac{\partial}{\partial u} \tag{6.10}$$

$$X_w = w\frac{\partial}{\partial u} \tag{6.11}$$

For the purpose of our exercise, we will not look at the solutions in ω.

The solution for the determining equations is spanned by the following operators

$$X_1 = \frac{\partial}{\partial t}$$

$$X_2 = xt\frac{\partial}{\partial x} + \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)}\left\{ \sigma_2^2 \ln x - \rho\sigma_1\sigma_2 \ln y + \frac{t}{2}[\sigma_2^2(\sigma_1^2 - 2\mu_1) - \rho\sigma_1\sigma_2(\sigma_2^2 - 2\mu_2)] \right\} u\frac{\partial}{\partial u}$$

$$X_3 = x\frac{\partial}{\partial x}$$

$$X_4 = x\left(\frac{-\sigma_1}{\rho\sigma_2} x \ln y + x \ln x \right)\frac{\partial}{\partial x} + \left(\frac{\sigma_2}{\rho\sigma_1} y \ln x + y \ln y \right)\frac{\partial}{\partial y} + \left\{ \frac{(2\mu_1 - \sigma_1^2)\ln y - (2\mu_2 - \sigma_2^2)\ln x}{2\rho\sigma_1\sigma_2} u \right\}\frac{\partial}{\partial u}$$

$$X_5 = yt\frac{\partial}{\partial y} + \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)}\left\{ \sigma_1^2 \ln y - \rho\sigma_1\sigma_2 \ln x + \frac{t}{2}[\sigma_1^2(\sigma_2^2 - 2\mu_2) - \rho\sigma_1\sigma_2(\sigma_1^2 - 2\mu_1)] \right\} u\frac{\partial}{\partial u}$$

$$X_6 = y\frac{\partial}{\partial y}$$

$$X_7 = u\frac{\partial}{\partial u}$$

It is easy to verify these solutions by software algebra such as Reduce, SYM and Yalie.

6.2.2 Lie algebras

In the case of the example above, the Lie algebra of the admitted symmetry algebra has 7 representatives, ranging from X_1 to X_7. The ideal structure of the table for $i, j = 1$ to n, should be as shown in table (6.1) We show the results for only $i = 1$, and the rest will be displayed in a commutator table.

$$[X_1, X_1] = 0$$

$$[X_1, X_2] = X_3 + \frac{1}{2\sigma_1^2\sigma_2^2(1-\rho^2)}\left[\sigma_2^2(\sigma_1^2 - 2\mu_1) - \rho\sigma_1\sigma_2(\sigma_2^2 - 2\mu_2)\right]X_7$$

$$[X_1, X_3] = 0$$

$$[X_1, X_4] = 0$$

$$[X_1, X_5] = X_6 + \frac{u}{2\sigma_1^2\sigma_2^2(1-\rho^2)}\left[\sigma_1^2(\sigma_2^2 - 2\mu_2) - \rho\sigma_1\sigma_2(\sigma_1^2 - 2\mu_1)\right]X_7$$

$$[X_1, X_6] = 0$$

$$[X_1, X_7] = 0$$

Set the parameters as follows,

$$
\begin{aligned}
z_1 &= \frac{1}{\sigma_1^2(1-\rho^2)} \\
z_2 &= \frac{-\rho}{\sigma_1\sigma_2(1-\rho^2)} \\
z_3 &= \frac{z_1(\sigma_1^2 - 2\mu_1)}{2} \\
z_4 &= \frac{z_2(\sigma_2^2 - 2\mu_2}{2} \\
z_5 &= \frac{1}{\sigma_2^2(1-\rho^2)} \\
z_6 &= \frac{z_4 z_4}{z_2} \\
z_7 &= \frac{z_2 z_3}{z_1} \\
z_8 &= \frac{-\sigma_1}{\rho\sigma_2} \\
z_9 &= \frac{-z_1}{z_2} \\
z_{10} &= \frac{z_4}{z_2\rho\sigma_1\sigma_2} \\
z_{11} &= \frac{-z_3}{z_1\rho\sigma_1\sigma_2}
\end{aligned}
$$

The above workings similar to that of [15] has been verified using "SYM" software.

Substituting into the commutators gives the following

Table 6.1: Commutator table of the sub-algebras for 2D Black-Scholes equation

$[\mathbf{X_i}, \mathbf{X_j}]$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$	$\mathbf{X_6}$	$\mathbf{X_7}$
$\mathbf{X_1}$	0	$X_3 + (z_3 + z_4)X_7$	0	0	$X_6 + (z_6 + z_7)X_7$	0	0
$\mathbf{X_2}$	$-X_3 - (z_3 + z_4)X_7$	0	$-z_1X_7$	$X_2 + z_9X_5$	0	$-z_2X_7$	0
$\mathbf{X_3}$	0	z_1X_7	0	$X_3 + z_9X_6 + x_{10}X_7$	z_2X_7	0	0
$\mathbf{X_4}$	0	$-X_2 - z_9X_5$	$-X_3 - z_9X_6 - z_10X_7$	0	$X_5 - z_8X_2$	$-z_8X_3 + X_6 - z_{11}X_7$	0
$\mathbf{X_5}$	$-X_6 - (z_6 + z_9)X_7$	0	$-z_2X_7$	$-X_5 + z_8X_2$	0	$-z_5X_70$	0
$\mathbf{X_6}$	0	z_2X_7	0	$z_8X_3 - X_6 + z_{11}X_7$	z_5X_7	0	0
$\mathbf{X_7}$	0	0	0	0	0	0	0

6.2.3 Verification of results

We will add onto the work of Cimpoiasu and Constantinescu [15] by considering the $3D$ Lie algebra which we verified using Yalie software.

We consider the PDE

$$u_t + \mu_1 x u_x + \mu_2 y u_y + \frac{1}{2}{\sigma_1}^2 x^2 u_{2x} + \frac{1}{2}{\sigma_2}^2 y^2 u_{2y} + \rho\sigma_1\sigma_2 xy u_{xy} - ku = 0, \tag{6.12}$$

k is a constant. We focus on a $3D$ sub-algebra

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t} \\ X_2 &= -x\frac{\partial}{\partial x} \\ X_3 &= y\frac{\partial}{\partial y} + \frac{\rho\sigma_1}{\sigma_2}x\frac{\partial}{\partial x} \end{aligned} \tag{6.13}$$

admitted by equation (6.15). The commutators are given in table (6.2)

Table 6.2: Commutator table of the sub-algebra (6.15)

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	0
X_2	0	0	0
X_3	0	0	0

Commutations results in zero, which means that L_3 is a subalgebra. It is well known that the linear combination of these symmetries is also a symmetry of the original equation. The linear combination of these symmetries is given by

$$X = c_1\frac{\partial}{\partial t} + c_3 y\frac{\partial}{\partial y} + [\frac{c_3\rho\sigma_1}{\sigma_2} - c_2]x\frac{\partial}{\partial x} \tag{6.14}$$

To determine the basis for invariants, ie. $XI = 0$, implies solving the following characteristic equations

$$\frac{dt}{c_1} = \frac{dy}{c_3 y} = \frac{dx}{(\frac{c_3\rho\sigma_1}{\sigma_2} - c_2)x} = \frac{du}{0} \tag{6.15}$$

This gives the following solutions:

$$I_1 = u \tag{6.16}$$

$$\begin{aligned}\frac{c_3}{c_1}dt &= \frac{dy}{y}\\ \ln I_2 + \frac{c_3}{c_1}t &= \ln y\\ I_2 &= ye^{-\frac{c_3}{c_1}t}\end{aligned} \tag{6.17}$$

also

$$\begin{aligned}\frac{dy}{y} &= \frac{c_3}{(\frac{c_3\rho\sigma_1}{\sigma_2} - c_2)}\frac{dx}{x}\\ \ln I_3 + \ln y &= \frac{c_3}{(\frac{c_3\rho\sigma_1}{\sigma_2} - c_2)}\ln x\\ I_3 &= \frac{x^B}{y}, \qquad B = \frac{c_3}{(\frac{c_3\rho\sigma_1}{\sigma_2} - c_2)}\end{aligned} \tag{6.18}$$

Let

$$I_1 = u, \qquad I_2 = \gamma, \qquad I_3 = \psi \tag{6.19}$$

In most cases we would determine the one dimensional optimal system to determine reductions that are not characterised by any point transformation. So, writing I_1 as a function of the other two invariants we obtain

$$I_1 = F(I_2, I_3) \tag{6.20}$$

or equivalently

$$u = F(\gamma, \psi) \tag{6.21}$$

We then substitute this into the original equation, being

$$u_t + \mu_1 x u_x + \mu_2 y u_y + \frac{1}{2}{\sigma_1}^2 x^2 u_{2x} + \frac{1}{2}{\sigma_2}^2 y^2 u_{2y} + \rho\sigma_1\sigma_2 xy u_{xy} - ku = 0, \tag{6.22}$$

k is a constant

so,

$$\begin{aligned} u_t &= F_\gamma(\frac{\partial\gamma}{\partial t}) \\ &= \frac{-c_3}{c_1} y e^{-\frac{c_3}{c_1}t} F_\gamma \\ &= \frac{-c_3}{c_1}\gamma F_\gamma \end{aligned} \tag{6.23}$$

$$\begin{aligned} u_y &= F_\gamma(\frac{\partial\gamma}{\partial y}) + F_\psi(\frac{\partial\psi}{\partial y}) \\ &= e^{-\frac{c_3}{c_1}t} F_\gamma - \frac{x^B}{y^2} F_\psi \\ &= \frac{\gamma}{y} F_\gamma - \frac{\psi}{y} F_\psi \end{aligned} \tag{6.24}$$

$$\begin{aligned} u_x &= F_\gamma(\frac{\partial\gamma}{\partial x}) + F_\psi(\frac{\partial\psi}{\partial x}) \\ &= \frac{B}{x}\frac{x^B}{y} F_\psi \\ &= \frac{B}{x}\psi F_\psi \end{aligned} \tag{6.25}$$

$$\begin{aligned} u_{xx} &= \frac{B}{x}\psi F_{\psi\psi}\frac{\partial\psi}{\partial x} + \frac{B}{x}F_\psi\frac{\partial\psi}{\partial x} + B\psi F_\psi \frac{\partial}{\partial x}(\frac{1}{x}) \\ &= \frac{B^2}{x^2}\psi^2 F_{\psi\psi} + \frac{B^2}{x^2}\psi F_\psi - \frac{B}{x^2}\psi F_\psi \end{aligned} \tag{6.26}$$

$$\begin{aligned} u_{yy} &= F_{\gamma\gamma}(\frac{\partial^2\gamma}{\partial x^2}) e^{-\frac{c_3}{c_1}t} \\ &= 0 \end{aligned} \tag{6.27}$$

$$\begin{aligned} u_{xy} &= \frac{B}{x}\psi F_{\psi\psi}\frac{\partial}{\partial y}(\frac{x^B}{y}) + \frac{B}{x}F_\psi\frac{\partial}{\partial y}(\psi) \\ &= \frac{-B}{xy}\psi^2 F_{\psi\psi} - \frac{B}{xy}\psi F_\psi \end{aligned} \tag{6.28}$$

substituting the above into 4.22 we get the following

$$\begin{aligned} &(\mu_2 - \frac{c_3}{c_1})\gamma F_\gamma + (\frac{1}{2}\sigma_1^2 B^2 - \rho\sigma_1\sigma_2)\psi^2 F\psi\psi + \\ &(\frac{1}{2}\sigma_1^2 B^2 - \frac{1}{2}\sigma_1^2 B - \rho\sigma_1\sigma_2)\psi F_\psi - kF = 0 \end{aligned} \tag{6.29}$$

where

$$\begin{aligned} A &= \mu_2 - \frac{c_3}{c_1} \\ B^* &= \frac{1}{2}\sigma_1^2 B^2 - \rho\sigma_1\sigma_2 \\ C &= \frac{1}{2}\sigma_1^2 B^2 - \frac{1}{2}\sigma_1^2 B - \rho\sigma_1\sigma_2 \end{aligned}$$

then the equation becomes

$$A\gamma F_\gamma + B^*\psi^2 F_{\psi\psi} + C\psi F_\psi - kF = 0 \tag{6.30}$$

Let us explore the following scenarios to try get to a solution

Case 1: $A = 0$

we get

$$B^*\psi^2 F_{\psi\psi} + C\psi F_\psi - kF = 0 \tag{6.31}$$

This is an Euler equation that we can get a solution for. We assume that $\psi > 0$ and that solutions are of the form

$$F(\gamma, \psi) = \psi^r \tag{6.32}$$

substituting (4.32) into (4.31) we get the following

$$F_{\psi} = r\psi^{r-1} \tag{6.33}$$

$$F_{\psi\psi} = r(r-1)\psi^{r-2} \tag{6.34}$$

This then brings forth the following equation

$$[B^*r(r-1) + Cr - k]\psi^r = 0 \tag{6.35}$$

we said that $\psi > 0$, meaning that the equation above will only be zero if

$$[B^*r(r-1) + Cr - k] = 0$$

The equation is a quadratic in r, so we can have real distinct roots, double roots or complex roots. The general solution will be of the form

$$F(\gamma, \psi) = c_4\psi^{r_1} + c_5\psi^{r_2} \tag{6.36}$$

$$r_1 = \frac{B^* - C + \sqrt{(C - B^*)^2 + 4B^*k}}{2B^*} \tag{6.37}$$

$$r_2 = \frac{B^* - C - \sqrt{(C - B^*)^2 + 4B^*k}}{2B^*} \tag{6.38}$$

We recall that

$$B^* = \frac{1}{2}\sigma_1^2 B^2 - \rho\sigma_1\sigma_2$$

$$C = \frac{1}{2}\sigma_1^2 B^2 - \frac{1}{2}\sigma_1^2 B - \rho\sigma_1\sigma_2$$

and assume

Subcase 1.1 k=0

then

$$r_1 = 0 \tag{6.39}$$

$$r_2 = \frac{\sigma_1^2 B}{\sigma_1^2 B^2 - 2\rho\sigma_1\sigma_2} \tag{6.40}$$

and since $B = \frac{c_3}{(\frac{c_3\rho\sigma_1}{\sigma_2} - c_2)}$,the we can conclude that

$$r_2 = \frac{\sigma_1^2\sigma_2 c_3}{\sigma_1^2\sigma_2 c_3 - 2\rho^2\sigma_1^2\sigma_2 c_3 + 2\rho\sigma_1\sigma_2^2 c_2} \tag{6.41}$$

The general solution then becomes

$$\begin{aligned} F(\gamma, \psi) &= c_4\psi^{r_1} + c_5\psi^{r_2} \\ &= c_4 + c_5\psi^{\frac{\sigma_1^2\sigma_2 c_3}{\sigma_1^2\sigma_2 c_3 - 2\rho^2\sigma_1^2\sigma_2 c_3 + 2\rho\sigma_1\sigma_2^2 c_2}} \end{aligned} \tag{6.42}$$

The complete solution may be constructed subject to the relevant boundary conditions. This task will be completed elsewhere. In terms of the original variable, we have

$$u = c_4 c_5 (\frac{x^B}{y})^{r_2}. \tag{6.43}$$

Case 2: $B^* = C = 0$

we get

$$A_\gamma F_\gamma - kF = 0, \tag{6.44}$$

Subcase k=1

$$\begin{aligned} A_\gamma F_\gamma - F &= 0 \\ F = e^{\frac{x}{Ax}} c_1 \end{aligned} \tag{6.45}$$

Substituting this into equation (6.13) gives the following solution

$$\frac{\mu_1}{A} e^{\frac{x}{Ax}} C_1 + \frac{1}{2A}\sigma_1^2 e^{\frac{x}{Ax}} C_1 - k e^{\frac{x}{Ax}} C_1 \tag{6.46}$$

Subcase k=0

$$\begin{aligned} A_\gamma F_\gamma &= 0 \\ F = c_1 \end{aligned} \tag{6.47}$$

6.3 Derivative pricing in illiquid markets

[37] gives an interesting input regarding the parametrisation of liquidity. He builds a model that relies on observable or attainable inputs. He builds a model where trading in assets takes place in times $t_0, t_1, ..., t_n = T$. The state of the economy is given by a finite set $\Omega = \omega_1, ..., \omega_m$, and the true state is represented by a sequence of sub-algebras $(\mathcal{F}_1)_t \in t_0, ..., T$.
The initial set of states is assumed to be $\mathcal{F}_{t_0} = \Omega$, and the eventual state of the economy is $\mathcal{F}_T = \omega_j, \qquad \forall \omega_y \in \Omega$.

The following are defined as the stock $S_t(\omega)$, and a riskless bond B_t. [37] uses the binomial model [CCR], as a measure of randomness due to trading in stock due to arrival of information. There is an assumed increase in risky assets by $u-1$, with probability ρ, or a decrease by $1-d$, with a corresponding probability of $1-\rho$ in 1 time step.

The following is assumed

$$S_{t_{i+1}} = \begin{cases} uS_{t_i} & \text{if } \omega_j = \omega_u, \\ dS_{t_i} & \text{if } \omega_j = \omega_d. \end{cases} \tag{6.48}$$

where $u > d$. The bond yields a riskless return r defined by

$$B_{t+1} = (1+r)B_{t_i} \tag{6.49}$$

If $S_{t_0} = S$, and $B_{t_0} = 1$, then the assumption made is that

- No arbitrage given that $0 < d < 1 + r < u$
- For a chosen u, d, ρ,

The models first two moments can be fitted into GBF

$$dS_t = rS_t dt + rS_t dX_t \tag{6.50}$$

where

$$\begin{aligned} \sigma &= \text{Volatility} \\ dX_t &= \text{Increments of the Standard Brownian Motion} \end{aligned}$$

A controlled process representing the effect of a large or influential trader on the market is put together.

$$\begin{aligned} \text{Holding process in stock} &= (H_t(\omega))_{\forall t,\omega} \\ \text{Holding process in bond} &= (\hat{H}_t(\omega))_{t\in\forall t,\omega} \end{aligned}$$

adapted to filtration $(\digamma_t)_{t\in t_0,\ldots,T}$.

If S = mid-market price, then the best price is above and below S for buyer and seller respectively, and $\bar{S}$, the average transaction price, is a monotonically increasing function $f\odot$ of current spot S_t, liquidity λ and trade size $(H_{t_{i+1}} - H_{t_i})$. [37] states that the trade reaction/price impact function must have the following properties

$$\begin{aligned} \lim_{H_{t_{i+1}}-H_{t_i}\downarrow-\infty} f &= 0 \\ \lim_{H_{t_{i+1}}-H_{t_i}\uparrow\infty} f &= \infty \\ f(H_{t_{i+1}} - H_{t_i} = 0) &= S_{t_i} \end{aligned}$$

[38] suggests the following

$$\begin{aligned} \bar{S_{t_i}} &= S_{t_i} \exp^{\lambda(H_{t_{i+1}}-H_{t_i})} \qquad \text{where } \lambda \geq 0 \qquad \text{trade reaction} \\ \text{Cash Flows} &= (H_{t_{i+1}} - H_{t_i})\bar{S_{t_i}} \\ \text{Transaction Costs} &= -(H_{t_{i+1}} - H_{t_i})(\bar{S_{t_i}} - S_{t_i}) \end{aligned}$$

Then we can conclude that the price is as follows

$$S_{t_{i+1}}(\omega_j) = \begin{cases} uS_t^\alpha \bar{S_{t_i}}^{1-\alpha} & \text{if } \omega_j = \omega_u, \\ dS_{t_i}^\alpha \bar{S_{t_i}}^{1-\alpha} & \text{if } \omega_j = \omega_d. \qquad \text{permanent slippage} \end{cases} \tag{6.51}$$

If $0 \leq \alpha \leq 1$ and constant, then S_{t_i} is a convex combination. Combining the trade reaction and permanent slippage using a binomial representation, they get to the following model

$$\begin{aligned} S_{t_i} &\rightarrow \bar{S_{t_i}} \\ &= S_{t_i} \exp^{\lambda(H_{t_{i+1}} - H_{t_i})} \\ &\rightarrow \begin{cases} uS_{t_i}\alpha\bar{S_{t_i}}^{1-\alpha} = uS_{t_i} \exp^{\lambda(1-\alpha)(H_{t_{i+1}} - H_{t_i})}, \\ dS_{t_i}^\alpha \bar{S_{t_i}}^{1-\alpha} = dS_{t_i} \exp^{\lambda(1-\alpha)(H_{t_{i+1}} - H_{t_i})}. \end{cases} \end{aligned}$$

The model can be applied to the following

1. Portfolio trading
2. Liquidity options
3. Exotic options in illiquid markets
4. Strike detection

For further elaboration on this, see [37].

6.4 Concluding remarks

In this chapter, higher dimension Black-Scholes equation is considered. This is a review of the work in [15]. However, in this chapter we extended the work in [15] by constructing the one dimensional optimal system of algebras. We also considered a $3D$ symmetry sub-algebra and constructed some exact solutions.

Chapter 7

Conclusion.

In this dissertation we focused on models arising in Mathematics of finance. In particular we considered the non-linear Black-Sholes equation in 1+1 and 2+1 dimensions. We employed the classical Lie symmetry methods in an attempt to construct the group-invariant (exact) solutions. In Chapter one, we revisited work done by scholars in the area of application of Lie symmetry methods to PDEs arising in modelling the pricing of options, hedging and volatility.

An account on algebraic techniques for symmetry reduction is provided in chapter two. There are a number of excellent texts on this topic in the public domain. We restrict our analysis using the classical Lie point symmetries throughout this dissertation. In the determination of Lie point symmetries the infinitesimal criterion for invariance results in a system of overdetermined linear equations, known as determining equations. The solutions of this system of equations are algorithmic to obtain, however one may use software algebra such as YaLie, SYM, Maple and Reduce. The Lie point symmetries leaving the equation in question invariant are then used to reduce the number of variables of PDEs by one.

We first considered the classical Black-Sholes equation and study the transformation that maps it to a linear diffusion equation. Perhaps this observation points to the fact that classical Black-Sholes equation admits a six dimensional Lie algebra. Notably it is claimed in [39], that this is not the case when we construct conservation laws. The classical Black-Sholes models is used for the pricing of options. The intricacy comes in when we assume that volatility is not constant. As such, the equations are rendered non-linear. Furthermore, in this dissertation we focussed on the derivation of Black-Sholes equation with variable volatility. Here we considered both the 1+1 D and 2+1 D Black-Sholes equations. In the subsequent chapters, we obtained the classical Lie point symmetries which span some large Lie symmetry algebras. We then determined the one-dimensional optimal systems of these Lie algebras. An attempt is undertaken to construct group-invariant (exact) solutions for 1+1 D and 2+1 D Black-Sholes models. Where such construction was difficult, numerical schemes were employed to determined approximate solutions. The effects of market liquidity and volatility are deduced and summarised.

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