Gauge/Gravity Duality at Finite N

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A dissertation submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, in fulfilment of the requirements for the degree of Doctor of Philosophy.

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Declaration

I declare that all results presented in this thesis are original except where reference is made to the work of others. The following is the list of my original works discussed in this thesis.

1. R. d. M. Koch, B. A. E. Mohammed and S. Smith, "Nonplanar Integrability: Beyond the $SU(2)$ Sector," Int.J.Mod.Phy. A26 (2011) 4553-4583 [arXiv:1106.2483 [hep-th]].

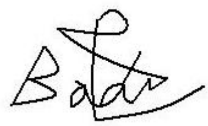
2. R. de Mello Koch, B. A. E. Mohammed, Jeff Murugan, Andrea Prinsloo, "Beyond the Planar limit in ABJM," JHEP 1210 (2012) 037 [arXiv:1202.4925 [hep-th]].

3. R. d. M. Koch, G. Kemp, B. A. E. Mohammed, and S. Smith, "Nonplanar integrability at two loops," JHEP 1210 (2012) 144 [arXiv:1206.0813v1 [hep-th]].

4. B. A. E. Mohammed, "Nonplanar Integrability and Parity in ABJ theory," Int.J.Mod.Phy. A-D-13-00031 [arXiv:1207.6948v2 [hep-th]].

5. Pawel Caputa and B. A. E. Mohammed, "From Schurs to Giants in ABJ(M)," JHEP 01 (2013) 055 [arXive:1210.7705 [hep-th]].

These papers are references [93], [94], [97], [98], [102] in the bibliography and appear in sections 4.2, 4.3, 3.5, 5.3, 6.1 respectively. This thesis is being submitted for the degree of Doctor of Philosophy to the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.



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Badr Mohammed

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Abstract

In the past decade, the gauge/gravity duality has been extensively explored in the large N limit. In particular, the spectrum of anomalous dimensions have been compared with the energy spectrum of the dual string theory showing remarkable agreement. In this limit, for operators with a bare dimension of order 1, planar diagrams give the leading contribution to the anomalous dimension. To obtain the anomalous dimensions, one needs to diagonalize the dilatation operator. One of the methods used to accomplish this is integrability. This allows an exact computation of the spectrum of the anomalous dimensions. There is by now a great deal of evidence that $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory and $\mathcal{N} = 6$ superconformal Chern Simons (ABJ(M)) theory are integrable in the planar limit.

In this thesis we probe the gauge gravity duality at finite N using novel tools developed from the representation theory of symmetric and unitary groups. We start by studying the action of the nonplanar dilatation operator of $\mathcal{N} = 4$ SYM theory and ABJ(M) theory. The gauge invariant operators we consider are the restricted Schur polynomials. In the case of $\mathcal{N} = 4$ SYM theory, we obtain the spectrum of the anomalous dimension beyond the $SU(2)$ sector at one loop, and in the $SU(2)$ sector at two loops. In both cases, we obtain the spectrum at arbitrary (finite) N. We then obtain the spectrum of anomalous dimensions in the $SU(2)$ sector of ABJ(M) theory at two loops. The class of gauge invariant operators we consider have classical dimension of order $O(N)$. In both theories, the spectrum of the anomalous dimensions reduces to a set of decoupled harmonic oscillators at large N. This indicates, for the first time, that $\mathcal{N} = 4$ SYM theory and ABJ(M) theory exhibit non-planar integrability. We expect to recover non-perturbative quantum gravity effects, from the gauge/gravity duality, when N is finite. The non-planar integrability we discover here may play an important role in finite N studies of the gauge/gravity duality, and hence may play an important role in understanding non-perturbative string stringy physics. In addition, we study various classes of correlators in ABJ(M) theory. In this context, we derive extremal n-point correlators in ABJ(M) theory and we probe the giant graviton dynamics in these theories.

Acknowledgements

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I thank the school of Physics for the excellent training I have received and for the opportunity to conduct in-depth physics research as a postgraduate student.

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To my parents, I am grateful for all the support and encouragement during the course of my PhD and leading up to it. I don't also forget to thank my wife and my son, their sacrifices and support to give me learning environment. I thank Allah who made all these possible.

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Chapter 1

Introduction

A very active field in theoretical physics in the past decade is given by research concerned with the AdS/CFT correspondence. The correspondence claims a duality between string theory and gauge theory, and it is for this reason that the AdS/CFT correspondence is sometimes refered to as the gauge/gravity duality. Our main interest in this thesis is in the gauge theory side where much progress has been made due to newly developed calculational tools. These tools allow novel ways to probe this duality. In this chapter we give an introduction to this duality and its applications in various fields of physics. We then describe the two-point function and its relation to the dilatation operator.

1.1 Introduction to the AdS/CFT Correspondence

The AdS/CFT correspondence is a conjectured duality between quantum gravity in D-dimensional, negatively curved Anti-de Sitter space-time (AdS) and a conformal field theory (CFT) living on the $D-1$ dimensional boundary of the AdS spacetime. The first example of such a duality was discovered by Maldacena in 1997 [1] where $\mathcal{N}=4$ supersymmetric Yang-Mills theory (SYM theory) in four dimensions was found to be equivalent to type II B superstring theory on the $AdS_5 \times S^5$ background.
In the past decade, the AdS/CFT correspondence has become an extremely promising field in theoretical physics. An extensive review can be found in [2].

1.2 The Conjecture

The AdS/CFT correspondence conjectures the complete equivalence of two theories

- $\mathcal{N}=4$ $SU(N)$ super Yang-Mills theory in 4-dimensional space-time.

- Type II B superstring theory on asymptotically $AdS_5 \times S^5$ spacetime.

The parameters of the two theories are related as follows

$$g_s = g_{YM}^2, \qquad\qquad R^2 = 4\pi g_s N \alpha'^2,$$

where g_s is the string coupling, g_{YM} is the Yang-Mills coupling, $\alpha' = l_s^2$ is the dimensionful parameter related to the string tension by $T = \frac{1}{2\pi\alpha'}$ where l_s is the string length, and R is the radius of curvature of both AdS_5 and S^5.

1.3 The Maldacena Limit

In this limit g_s and N are kept fixed and $\alpha' \to 0$. This is the mandatory limit to produce the duality. In the field theory, the 't Hooft coupling λ is related to g_s and N via $\lambda = g_s N$. Thus, keeping g_s and N fixed in the Maldacena limit corresponds to keeping the 't Hooft coupling fixed. We need to keep the 't Hooft coupling λ fixed as we will see in the next section. We will show how the Maldacena limit works using both the D-brane formalism and the black-brane formalism.

D-brane Formalism

The massless modes of open strings with both ends on the same D3-brane give a $U(1)$ gauge theory in 4-dimensions. For N D3-branes, we have a $U(1)^N$ gauge theory. We also have open strings stretched between different D3-branes. Thanks to these additional degrees of freedom, the $U(1)^N$ gauge theory is enhanced to a $U(N)$ gauge theory (see Figure 1.1).

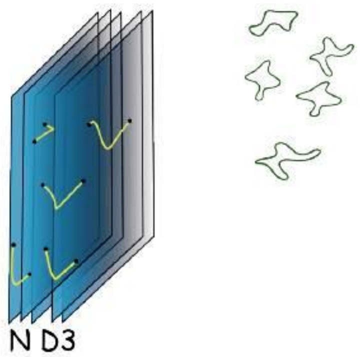


Figure 1.1: Open strings ending on a stack of N D3-branes (left), closed strings in the near-throat region of D3-branes (right).

The low-energy effective action of the massless modes of the theory takes the form

$$S = S_{\text{brane}} + S_{\text{bulk}} + S_{\text{int}}. \tag{1.1}$$

The open string modes are described by the action S_{brane} defined on the 4-dimensional brane world-volume. It is well-known that the D-brane theory reduces to $\mathcal{N}=4$ $SU(N)$ super Yang-Mills theory in the $\alpha \to 0$ limit [1]. This can be seen from the Dirac-Born-Infeld (DBI) action of the D3-brane [3]

$$S_{\text{brane}} = -T_{D3} \int d^4\sigma e^{-\Phi} \sqrt{-\det(\mathcal{P}[g]_{ab} + \mathcal{F}_{ab})} + \text{fermions}, \tag{1.2}$$

where $\mathcal{P}[g]_{ab}$ is the metric pullback, $\mathcal{F}_{ab} = B_{ab} + (2\pi\alpha')F_{ab}$ with B_{ab} NS-NS two form potentials (NS-NS is Neveu Schwarz-Neveu Schwarz) and F_{ab} world volume flux, $T_{D3} = (g_s(2\pi)^3\alpha'^2)^{-1}$ is the tension of D-brane. Defining $g_{YM}^2 = 2\pi g_s$ with $g_s = e^{\phi}$, the brane action in a flat target space can be expanded as

$$S_{\text{brane}} = \frac{1}{4g_{YM}^2} \int d^4\sigma F_{ab}F^{ab} + ... + \mathcal{O}(\alpha') = S_{\mathcal{N}=4} + \mathcal{O}(\alpha'). \tag{1.3}$$

In the above equation, ... comprises the scalars and fermions terms which we don't display explicitly for simplicity. Now consider the bulk action for 10 dimensional supergravity. Schematically, the Einstein-Hilbert term is given by

$$S_{\text{bulk}} = \frac{1}{2\kappa} \int \sqrt{-g}\mathcal{R} + \mathcal{O}(\mathcal{R}^2) \overset{r \gg R}{\to} \int (\partial h)^2 + \kappa(\partial h)^2 h + ..., \tag{1.4}$$

where the metric $g_{\mu\nu}$ is expanded around flat space as $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$. Moreover, S_{int} which describes the interaction between the supergravity modes propagating in the bulk and the modes localized on the brane, is proportional to $g_s\alpha'^2$. Thus, the limit $\alpha' \to 0$ decouples the bulk from the brane. The two decoupled descriptions we obtain are

1. $\mathcal{N}=4$ $SU(N)$ super-Yang-Mills theory (describing the open strings on the brane).

2. 10d supergravity in flat space (describing closed strings in the bulk space-time).

Black Brane Formalism

The metric sourced by a black 3-brane [4] is

$$ds^2 = H^{-1/2}(r)\,\eta_{\mu\nu}dx^\mu dx^\nu + H^{1/2}(r)\left(dr^2 + r^2 d\Omega_5^2\right), \quad \mu,\nu = 0,...,3,$$

$$H = 1 + \frac{R^4}{r^4}, \qquad R^4 = 4\pi g_s N\alpha'^2, \tag{1.5}$$

where $d\Omega_5^2$ is the metric element of the five dimensional sphere S^5. The metric (1.5) is the solution of supergravity theory with a dilaton field ϕ and Ramond-Ramond (R-R) five form field strength F_5 given by

$$e^{\phi} = g_s, \qquad F_{tj_1,j_2,j_3r} = \epsilon_{j_1j_2j_3}H^{-2}(r)\,\frac{Q}{r^5},$$

where Q is the charge of the black brane. The energy E_r of the D_3 brane measured at a radial distance r and the energy measured at infinite distance $(r \to \infty)$ E_∞ are related by

$$E_\infty = H^{-\frac{1}{4}}(r)\, E_r. \tag{1.6}$$

By performing the change of coordinates $u = R^2/r$ in (1.5), we find

$$ds^2 = \frac{R^2}{u^2}\left[\tilde{H}^{-1/2}(u)\,\eta_{\mu\nu}dx^\mu dx^\nu + \tilde{H}^{1/2}(u)\left(du^2 + u^2 d\Omega_5^2\right)\right] \equiv G_{MN}dx^M dx^N,$$

$$\tilde{H} = 1 + \frac{R^4}{u^4}, \qquad R^4 = 4\pi g_s N \alpha'^2, \qquad (M = 0, ..., 9)\,. \tag{1.7}$$

Taking the Maldacena limit of (1.7) leads to $R = (4\pi g_s N \alpha')^{1/4} \to 0$ from which, we find $\tilde{H}(u) \to 1$ for nonzero u. Thus, the metric G_{MN} reduces to the metric on $AdS_5 \times S^5$,

$$\tilde{G}_{MN}dx^M dx^N \overset{R \to 0}{=} \frac{1}{u^2}\left[\eta_{\mu\nu}dx^\mu dx^\nu + du^2 + u^2 d\Omega_5^2\right], \tag{1.8}$$

where $\tilde{G}_{MN}(x) = G_{MN}(x)/R^2$. This metric is an exact solution to superstring theory. It describes closed-strings propagating in the $AdS_5 \times S^5$ supergravity background.
From this analysis we see that the Maldacena limit decouples string theory in the $AdS_5 \times S^5$ near-horizon region from supergravity in the asymptotically flat space. The two decoupled theories are thus

1. Type II B superstring theory on $AdS_5 \times S^5$ (describing the dynamics in the geometry created by the D3 branes).

2. 10 dimensional supergravity in flat space (describing physics of the bulk spacetime).

In conclusion, we see that in the Maldacena limit, the brane formalism leads to two decoupled theories, $\mathcal{N} = 4$ $SU(N)$ super-Yang-Mills theory and 10d supergravity in flat space. However, the black brane formalism leads to superstring theory on $AdS_5 \times S^5$ and 10d supergravity in flat space. Since in both formalisms we have 10d supergravity in flat space, it is natural to conjecture that the $\mathcal{N} = 4$ $SU(N)$ super-Yang-Mills theory is equivalent (dual) to type II B superstring theory on $AdS_5 \times S^5$.

1.4 ’t Hooft Limit

In order to understand how string theory on the $AdS_5 \times S^5$ background might arise from $\mathcal{N} = 4$ super Yang-Mills theory, we take the ’t Hooft limit,

$$N \to \infty \qquad \text{with} \quad \lambda = g_{YM}^2 N = g_s N \quad \text{fixed.}$$

In this limit, if we focus on low energy excitations, the planar sector of the $\mathcal{N}=4$ SYM theory becomes relevant. Non-planar corrections are important at finite N which may describe quantum dynamics of strings or states of large energy. Although the 't Hooft limit suppresses the non-planar diagrams, they can still contribute to the computation of the large N anomalous dimension as we will see in chapters 4 and 5. To explain this limit further, consider the matrix field X^a_b in the adjoint representation of $SU(N)$. Then the propagators and vertices are represented as shown in Figure 1.2

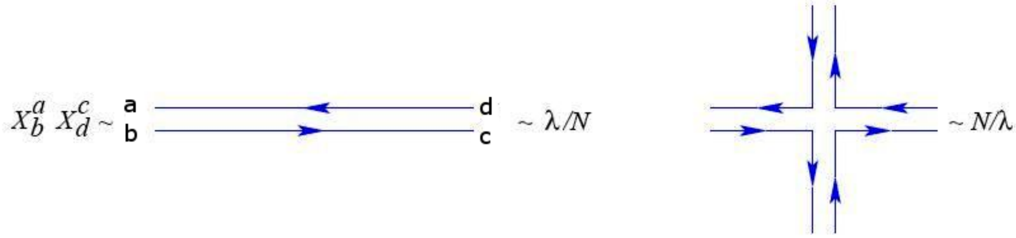


Figure 1.2: The propagator (left) is proportional to λ/N and the vertices (right) are proportional to N/λ.

In any Yang Mills theory including $\mathcal{N}=4$ SYM theory, the 3 point vertices of the matrix fields X^a_b are proportional to g_{YM} while the 4 point vertices are proportional to g^2_{YM}. The schematic Lagrangian is [2]

$$\mathcal{L} \sim \mathrm{Tr}\left(d\Phi_i d\Phi_i\right) + g_{YM} c^{ijk} \mathrm{Tr}\left(\Phi_i \Phi_j \Phi_k\right) + g^2_{YM} d^{ijkl} \mathrm{Tr}\left(\Phi_i \Phi_j \Phi_k \Phi_l\right), \qquad (1.9)$$

where c^{ijk} and d^{ijkl} are constants. The Lagrangian (1.9) has been written down for gauge group $SU(N)$. If we rescale the fields by $\tilde{\Phi}_i = g_{YM}\Phi_i$, then (1.9) becomes

$$\mathcal{L} \sim \frac{1}{g^2_{YM}} \left[\mathrm{Tr}\left(d\tilde{\Phi}_i d\tilde{\Phi}_i\right) + c^{ijk} \mathrm{Tr}\left(\tilde{\Phi}_i \tilde{\Phi}_j \tilde{\Phi}_k\right) + d^{ijkl} \mathrm{Tr}\left(\tilde{\Phi}_i \tilde{\Phi}_j \tilde{\Phi}_k \tilde{\Phi}_l\right)\right], \qquad (1.10)$$

where $1/g^2_{YM} = N/\lambda$.
In the 't Hooft limit, the coefficient in front of the Lagrangian diverges. However, the dimension of the matrix fields also goes to infinity, so that overall we have a well defined theory. To see this, note that each vertex in (1.10) is proportional to N/λ, and each propagator is proportional to λ/N. The sum over the loop indices give an additional factor of N for each loop. Thus, the vacuum to vacuum diagram with V vertices, E propagators and F index loops is proportional to

$$\left(\frac{N}{\lambda}\right)^V \left(\frac{\lambda}{N}\right)^E N^F = N^{V-E+F}\lambda^{E-V} = N^{2-2g}\lambda^{E-V}, \qquad (1.11)$$

where g is the genus of the surface which explains how a surface arises from a ribbon graph. This N dependence of N^{2-2g} arises for any matrix model, that

is for theories with an arbitrary number of fields and type of interaction.
Quite generally, in matrix models, there are two broad types of Feynman diagrams that we can draw. These are the planar diagrams and the non-planar diagrams. An example of each is given in Figure 1.3. A planar diagram can be drawn on a plane without any propagators crossing. If a non-planar diagram is drawn on a plane, some of the propagators necessarily cross. A diagram with N dependence N^{2-2g} must be drawn on a surface of genus g to avoid crossings.

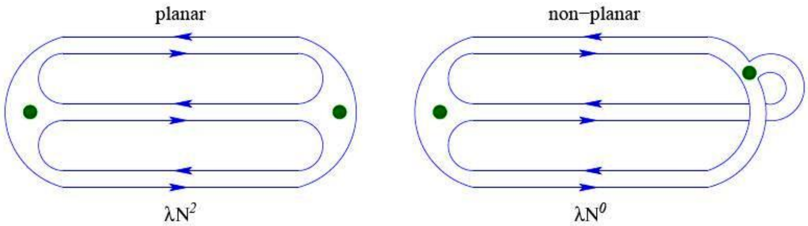


Figure 1.3: The planar diagram (left) (genus= 0) scales as λN^2 ($V = 2, E = 3, F = 3$). The non-planar diagram (right) (genus= 1) scales as λN^0 ($V = 2, E = 3, F = 1$).

Using (1.11) which holds in the 't Hooft limit, we see that the non-planar diagrams are suppressed by $\frac{1}{N^{2g}}$.
The extension of this method including vertex operators is discussed in [5]. Considering the 't Hooft limit in $\mathcal{N} = 4$ SYM corresponds to studying classical II B superstring theory on the $AdS_5 \times S^5$ geometry.

1.5 Large λ Limit

Taking $\lambda \to \infty$ together with $N \to \infty$ has interesting consequences on both the field theory side and for its string dual as we now explain.

i) Since in $\mathcal{N} = 4$ SYM theory the gauge coupling g^2_{YM} is related to λ and N through $\lambda = g^2_{YM}N$, taking λ to infinity takes the gauge theory to the strongly-coupled regime.

ii) Type II B superstring theory on $AdS_5 \times S^5$ reduces to supergravity at large λ. Since the Riemann tensor scales as $\mathcal{R} \sim \frac{1}{R^2} \sim \frac{\lambda^{-1/2}}{\alpha'}$, the superstring theory can be replaced by an effective theory described by supergravity. The effective Lagrangian becomes a power series expansion in $\lambda^{-1/2}$,

$$\begin{aligned} \mathcal{L} &= a_1\alpha'\mathcal{R} + a_2\alpha'^2\mathcal{R}^2 + a_3\alpha'^3\mathcal{R}^3 + ... \\ &= a_1\lambda^{-1/2} + a_2\lambda^{-1} + ... \end{aligned} \tag{1.12}$$

The higher-curvature corrections (which involve higher-derivatives) will be suppressed in the large λ limit. Therefore, the superstring theory reduces to su-

pergravity.
We conclude that the large λ limit corresponds to strongly-coupled CFT and that its string dual in AdS becomes weakly-curved. In contrast, taking small λ limit ($\lambda \to 0$) gives weakly-coupled CFT and gives rise to very large curvatures in the dual string theory. Therefore, we have weakly coupled QFT/highly curved quantum gravity or strongly coupled QFT/weakly curved quantum gravity depending on which λ limit we consider.
As a summary, Table 1.1 contains the three forms of the AdS/CFT conjecture.

Table 1.1: The left column contains the three forms of the AdS/CFT conjecture for each particular limit in the right column (M=Maldacena limit, H='t Hooft limit, L=large λ limit).

Strong form	Limits
$\mathcal{N}=4$ $SU(N)$ SYM $\Leftrightarrow$ full quantum type II B for all N, λ string theory on $AdS_5 \times S^5$	M
Planar limit	
$\mathcal{N}=4$ $SU(N)$ SYM $\Leftrightarrow$ classical type II B for $N \to \infty, \lambda$ fixed string theory on $AdS_5 \times S^5$	M+H
Weak form	
$\mathcal{N}=4$ $SU(N)$ SYM $\Leftrightarrow$ classical type II B for $N \to \infty, \lambda \gg 1$ supergravity on $AdS_5 \times S^5$	M+H+L

1.6 Evidence for the Conjecture

Although there is no rigorous mathematical proof for this conjectured duality between the field theory and the string theory, there is an exact matching of symmetries. The superconformal group of $\mathcal{N}=4$ super-Yang-Mills is $SU(2,2|4)$. The bosonic subgroup is

$$SU\left(2,2|4\right) \supset SO\left(2,4\right) \times SU\left(4\right)_R . \tag{1.13}$$

In type II B string theory, the isometry group of AdS_5 is $SO(2,4)$ and the isometry group of S^5 is $SO\left(6\right) \simeq SU\left(4\right)_R$ which is in agreement with (1.13). Furthermore, 16 out of 32 Poincare$'$ supersymmetries preserved by the D3-branes are supplemented by the 16 conformal supersymmetries in the AdS limit. This is in precise agreement with the superconformal symmetries of $\mathcal{N}=4$ SYM theory.

1.7 CFT Operators and Dual Scalar Fields in *AdS*

In this section we discuss the CFT operators and their relation to fields in supergravity on the *AdS* space. We will then compute the correlation functions of CFT operators using supergravity theory.

Scalar Field in *AdS* Space

The Klein-Gordon action for a scalar field ϕ in $d+1$ anti-de Sitter space, AdS_{d+1}, is

$$\begin{aligned} S &= -\frac{\eta}{2R^{d-1}} \int dz d^d x \sqrt{g} \left(g^{MN} \partial_M \phi \partial_N \phi + m^2 R^2 \phi^2 \right) \\ &= -\frac{\eta}{2} \int \frac{dz d^d x}{z^{d+1}} \left(z^2 \partial_z \phi \partial_z \phi + z^2 \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 R^2 \phi^2 \right), \end{aligned} \tag{1.14}$$

where η is a normalization constant. Rescaling ϕ as $\phi = z^{d/2}\psi$ and introducing the variable $y = -lnz$, (1.14) becomes

$$\begin{aligned} S = &- \frac{\eta}{2} \int dy d^d x \left(\partial_y \psi \partial_y \psi + e^{-2y} \eta^{\mu\nu} \partial_\mu \psi \partial_\nu \psi + \left[m^2 R^2 + \frac{1}{4} d^2 \right] \psi^2 \right) \\ &+ \eta \frac{d}{4} \int d^d x \psi^2 \Big|_{y=-\infty}^{y=\infty} . \end{aligned} \tag{1.15}$$

This action leads to a positive Hamiltonian as long the Breitenlohner-Freedman bound (BF)

$$m^2 R^2 \geq -\frac{d^2}{4}, \tag{1.16}$$

is obeyed. The equations of motion for the scalar field ϕ are

$$\frac{1}{\sqrt{g}} \partial_M \left(\sqrt{g} g^{MN} \partial_N \phi\left(z, x\right) \right) - m^2 R^2 \phi\left(z, x\right) = 0. \tag{1.17}$$

Using the plane wave ansatz, $\phi\left(z, x^\mu\right) = \phi\left(z\right) e^{ik\cdot x}$ in (1.17), we get

$$z^{d+1} \partial_z \left(z^{1-d} \partial_z \phi\left(z\right) \right) - \left(m^2 R^2 + k^2 z^2 \right) \phi\left(z\right) = 0. \tag{1.18}$$

In the region near the AdS_{d+1} boundary ($z \to 0$), the term $k^2 z^2 \to 0$ and the solution becomes

$$\lim_{z\to 0} \phi\left(z, x\right) = \phi_1\left(x\right) z^{\triangle_+} + \phi_0\left(x\right) z^{\triangle_-}, \quad \triangle_\pm = \frac{d}{2} \sqrt{\frac{d^2}{4} + m^2 R^2}, \tag{1.19}$$

where $\triangle$ solves

$$\triangle\left(\triangle - d\right) = m^2 R^2. \tag{1.20}$$

Inserting the solution (1.19) into (1.14) gives a normalizable action near the boundary ($z \to 0$) for $\triangle > \frac{d}{2}$. Thus, $z^{\triangle_+}$ is called the normalizable solution while $z^{\triangle_-} = z^{d-\triangle_+}$ is called the non-normalizable solution. The normalizable solution can be used to define a field

$$\phi_0 (x) = \lim_{z\to 0} z^{-\triangle_-} \phi (z, x) , \tag{1.21}$$

which does not vanish as $z \to 0$. The bulk mode ($\phi(z, x)$) with $\phi_0(x) \neq 0$ is dual to a source term $\int d^d x \phi_0 (x) \mathcal{O}$ in the conformal field theory. More precisely, the duality is

$$\begin{aligned} \phi (z, x) &\longleftrightarrow \mathcal{O} (x) \\ \phi_0 (x) &\longleftrightarrow J (x) , \end{aligned} \tag{1.22}$$

where $J(x)$ is the source for the field theory operator $\mathcal{O}(x)$

Correlators from Supergravity

Using (1.22), one can compute correlators in supergravity. Correlators in the field theory are obtained from the generating functional [6, 7]

$$e^{-\Gamma[J]} = \left\langle \exp \left(\int d^d x J (x) \mathcal{O} (x) \right) \right\rangle. \tag{1.23}$$

For example, the connected n-point function (*correlator*) is obtained from (1.23) by taking n-derivatives of $\Gamma[J]$ with respect to $J(x)$ evaluated at $J = 0$

$$\left\langle \mathcal{O} (x_1) ...\mathcal{O} (x_n) \right\rangle = \frac{\delta^n \Gamma [J]}{\delta J (x_1) ...\delta J (x_n)} \Big|_{J=0}. \tag{1.24}$$

Now, we can compute the same correlators from the gravity side using the map (1.22). The AdS/CFT correspondence states

$$e^{-\Gamma[\phi_0]} = \left\langle \exp \left(\int_{\partial AdS} \phi_0 \mathcal{O} (x) \right) \right\rangle = \int_{\phi_0} D\phi \exp (-S [\phi]) , \tag{1.25}$$

from which the connected n-point function is obtained by taking n-derivatives of the supergravity action S with respect to sources $\phi_0(x)$ evaluated at zero

$$\left\langle \mathcal{O} (x_1) ...\mathcal{O} (x_n) \right\rangle = \frac{\delta^n S [\phi]}{\delta \phi_0 (x_1) ...\delta \phi_0 (x_n)} \Big|_{\phi_0=0}. \tag{1.26}$$

1.8 Two-Point Correlators

A conformal field theory is completely characterized by its two-point and three-point functions [2]. The two-point correlators for the local gauge invariant operators in a certain basis take the form[1]

$$\langle O_A (x) O_B (y) \rangle = \frac{\delta_{AB}}{|x - y|^{2D_A}}$$

[1]This is for a scalar operators. It can be generalized for vector and spinor operators.

where O_A is a local gauge invariant operator with scaling dimension D_A. In the conformal field theory, the scaling dimension is given by the eigenvalues of the dilatation operator. The scaling dimension can be split into two parts

$$D_A = D_A^{(0)} + \delta D_A,$$

where $D_A^{(0)}$ is the classical dimension and δD_A is the anomalous dimension, which is a quantum correction to the classical dimension. The local gauge invariant operator O_A can be understood as an eigen-operator of D with an eigenvalue $D_A^{(0)} + \delta D_A$. Since the CFT has a large-N expansion, δD can be *topologically* expanded as

$$\delta D\left(\lambda, 1/N\right) = \sum_{g=0}^{\infty} \frac{1}{N^{2g}} \sum_{l=1} \lambda^l \delta D_{l,g}, \tag{1.27}$$

where λ is the 't Hooft coupling that we have seen in the previous sections. For small λ, δD can be computed perturbatively using (1.27). For large λ, one can use the AdS/CFT correspondence and compute the energy of the string state in the dual string theory, since as we have seen in section 1.5 it is a weakly coupled theory. Then, using the AdS/CFT dictionary we can find the corresponding scaling dimension through the following relation

$$D_A(\lambda, 1/N) = E_A(R^2/\alpha', g_s),$$

where E_A is the energy of the string state dual to the operator O_A [6, 7]. In (1.27) only the $g = 0$ term contributes in the large N limit. This is the planar limit. In this limit the coupling between the multi-trace operators is small enough so that they decouple from single-trace operators as we will see in the next chapter.

1.9 New Duality

Recently, a new example of the AdS/CFT correspondence has been discovered, relating superconformal Chern-Simons theories in 3-dimensions to type II A string theory on $AdS_4 \times CP^3$, where CP^3 is the complex projective space in 3-complex dimensions. Superconformal Chern-Simons theories were originally studied to better understand M-theory. M-theory has two fundamental objects, M2-branes and M5-branes [8]. A maximally supersymmetric Chern-Simons theory was discovered by Bagger, Lambert and Gustavsson (BLG) [9, 10, 11]. It was found that BLG-theory describes two interacting M2-branes [12, 13]. It is now known that BLG-theory is the only unitary and maximally supersymmetric Chern-Simons theory [14, 15, 16, 17, 18]. A generalization of the BLG-theory that can describe an arbitrary number of M2-branes has been proposed by Aharony, Bergman, Jafferis and Maldacena (ABJM) [19]. The

ABJM theory is an $\mathcal{N} = 6$ superconformal Chern-Simons matter theory with gauge group $U(N)_k \times U(N)_{-k}$, where k is the Chern-Simons level. The supersymmetry of ABJM theory is enhanced to $\mathcal{N} = 8$ for $k = 1, 2$ due to quantum effects [20, 21]. Unlike $\mathcal{N} = 4$ SYM theory, ABJM theory is dual to type II A string theory on $AdS_4 \times CP^3$ when the level k is large. At large N with $k \ll N \ll k^5$, ABJM theory is dual to M-theory on $AdS_4 \times S^7/Z_k$. Hence, ABJM theory is dual to two different theories depending on the parameters N and k. Similar to $\mathcal{N} = 4$ SYM theory, ABJM theory has a $1/N$ expansion at fixed 't Hooft parameter $\lambda = N/k$. The AdS/CFT dictionary that relates the gauge theory parameters to the string theory is

$$\lambda^{5/4} N = g_s, \quad \sqrt{\lambda} \sim R^2/\alpha'. \tag{1.28}$$

1.10 Application of the AdS/CFT Correspondence

The AdS/CFT correspondence has proved to be a powerful tool with which we can probe strongly-coupled field theories through the strong/weak duality that we have discussed. There are many research areas actively concerned with the AdS/CFT correspondence. An incomplete list is

i) AdS/QCD
Quantum Chromodynamics (QCD) is strongly-coupled in the infra-red (IR) limit and therefore not accessible by perturbative QFT methods.
A QCD like theory is obtained from the original AdS/CFT setup by deforming the standard AdS/CFT duality. This deformation breaks supersymmetry, introducing a "wall" or cut-off providing a scale that is dual to the QCD scale Λ_{QCD} [22].

ii) AdS/CMT
This is one of the most recent applications of the AdS/CFT correspondence. In this setting it is condensed matter theories (CMT) that are the strongly-coupled systems which cannot be described using a perturbative approach. An example of these systems are the high-temperature superconductors (Cuprates) at their critical temperature $T_c \geq 90K$. For a detailed review of the AdS/CMT applications see [23].

iii) Integrability
Integrability is a technique for probing various aspects of the AdS/CFT duality. In this thesis, integrability will be the main tool used to investigate the gauge/gravity duality for $\mathcal{N} = 4$ SYM theory and ABJM theory.

1.11 Integrability

Integrability is a promising set of ideas which may provide a better understanding of the gauge/gravity duality. In particular, it may serve as a tool to confirm and complete the dictionary of the AdS/CFT correspondence.
A system is integrable if it has a conserved quantity for each degree of freedom. In practice, integrability implies that the calculation of physical observables is reduced to a mathematical problem that can be solved exactly. Since quantum field theory contains an infinite number of degrees of freedom, an integrable quantum field theory must have an infinite number of independent symmetries. Integrability was first found in two-dimensional field theories. Later, higher dimensional models were found to be dual to integrable systems. For instance $\mathcal{N} = 4$ SYM theory in the planar sector is found to be dual to spin chain systems. Moreover, integrability of the ABJM theory is confirmed in the planar limit using the same spin chain method. These two examples will be discussed in more detail in chapter 2. As we will see in chapter 2, integrability does not necessarily mean that the theory can directly be solved in terms of its representative quantities. It is possible to find a physical equivalence (dual) model in which the observables in the field theory can be computed in the dual integrable model. In this way, complicated calculations in the field theory are mapped to a simpler solvable calculation in the dual system.
Although integrability has been used successfully in solving the spectral problem[2] of $\mathcal{N} = 4$ SYM theory and ABJM theory in the planar limit, we still cannot judge whether these theories are integrable in general. This is because the theory should be integrable at arbitrary N in order to be an integrable theory. Allowing for arbitrary dependence on N means one has to solve the spectral problem at finite N. One can see from (1.11) that the non-planar diagrams will no longer be suppressed when we compute the scaling dimension. An attempt to extend the spin chain method to the non-planar dilatation generator will be discussed in chapter 2.
If integrability persists for the spectral problem, even when non-planar diagrams are included, we talk about "non-planar integrability".
In chapters 3,4,5 and 6, we develop new techniques to probe gauge/gravity duality at finite N by computing the anomalous dimensions for $\mathcal{N} = 4$ SYM theory and ABJM theory in a large N but non-planar limit and by studying the extremal correlators in ABJ(M) theories. In chapter 3, the methods of the representation theory of symmetric and unitary groups developed in [24, 25, 27] are used to compute the two-point function of our gauge invariant operators which are Schur-polynomials. Then in chapter 4, we compute the spectrum of anomalous dimensions of $\mathcal{N} = 4$ SYM theory. The anomalous dimensions of ABJ(M) theory are computed in chapter 5. The calculations in chapter 4 and 5 signal non-planar integrability in the large N limit for both theories. This is in contrast to the method discussed in chapter 2 which does not reveal any

[2]The spectral problem is the problem of finding the eigenvalue of the scaling dimension (dilatation operator) D when acting on gauge invariant operators in the CFT.

sign of integrability beyond the planar limit. In chapter 6, we compute various classes of extremal correlators in ABJ(M) theories. We then probe the giant graviton dynamics at large and finite N.

Chapter 2

Integrability in the Planar Limit

2.1 Introduction

In this chapter, we present a review of integrability in the planar limit. Integrability in the planar limit was first discovered using spin chains, by associating the dilatation operator with Heisenberg's Hamiltonian. This Hamiltonian can be diagonalized using Bethe Ansatz methods. We then construct a dictionary between operators in AdS/CFT and spin-chains. For a more detailed discussion of integrability in the planar limit see [27].
Classical integrability exists[3] if a classical system with N-degrees of freedom described by a Hamiltonian H has N conserved charges Q_i which Poisson commute

$$\{Q_i,\, Q_j\} = 0. \tag{2.1}$$

The Hamiltonian H is one of the charges. For each of these charges there is a conservation law that can be solved (integrated) to fix all of the independent degrees of freedom.
At the quantum level, the system is integrable if H is one of the N conserved charges Q_i and they all commute with each other

$$[Q_i,\, Q_j] = 0. \tag{2.2}$$

In this case, all the charges can be diagonalized simultaneously. We will restrict ourselves to quantum integrable systems and will show how these systems can be solved using the Bethe Ansatz.

2.2 Introduction to the Bethe Ansatz

The energy eigenvalues of an integrable quantum spin-chain are determined by means of the Bethe ansatz. This is a technique that does not involve direct diagonalization of the Hamiltonian. The Bethe ansatz produces a set of algebraic equations whose solution leads directly to the energy eigenvalues of

[3] Liouville integrable

the Hamiltonian as well as the eigenvalues of the higher commuting charges. To show how the Bethe ansatz works, we will consider the simplest Heisenberg spin-chain $XXX_{1/2}$. This system is called the $SU(2)$ chain (for a detailed introduction, consult [28]).
In this chapter, we analyse the spin-chain methods for $\mathcal{N} = 4$ SYM theory and ABJM theory in the $SU(2)$ sector. In this planar limit we obtain a complete description of the dynamics by considering single trace operators. To establish the connection between the $\mathcal{N} = 4$ SYM and the Heisenberg chain, we employ a map between single trace operators and spin chain states. The number of operators in the trace is identified with the number of lattice sites of the spin chain. The field Z is identified with spin up and the field Y with spin down. Thus, the spin chain state shown in Figure 2.1 corresponds to the operator $\mathrm{Tr}(Z^2Y^2ZYZ^2)$. The dilatation operator is identified with the Hamiltonian of the spin chain so that the energy of the spin chain state gives the anomalous dimension of the operator. Since the BPS operators are characterized by the fact that their anomalous dimension vanishes, they correspond to zero energy states of the spin chain.

2.3 Hiesenberg Spin Chain

The spin-chain system on a circle consists of n sites labeled by $1, ..., n$ where $i = i + L$ (see Figure 2.1). Here L is equal to the volume of the space and is sometimes refered to as a fundamental domain.

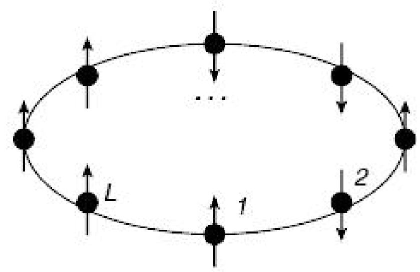


Figure 2.1: Periodic spin-chain with L sites.

It is convenient to describe the spin-spin interaction using L copies of Pauli's matrices

$$\sigma_n^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_n^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma_n^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.3}$$

The subscript n refers to the copy at site n. These matrices satisfy the $SU(2)$ algebra at each site

$$\left[\sigma_n^i,\ \sigma_m^j\right] = 2i\varepsilon^{ijk}\sigma_n^k\delta_{mn}. \tag{2.4}$$

Creation and annihilation operators can be constructed from the Pauli matrices as follows

$$\sigma_n^{\pm} = \frac{1}{2}\left(\sigma_n^1 \pm i\sigma_n^2\right), \tag{2.5}$$

from which the action of the creation operator σ^+ and the annihilation operator σ^- on the states $\Big|\downarrow\Big\rangle$ and $\Big|\uparrow\Big\rangle$ respectively is

$$\sigma^+\Big|\uparrow\Big\rangle = 0, \qquad \sigma^+\Big|\downarrow\Big\rangle = \Big|\uparrow\Big\rangle, \qquad \sigma^-\Big|\uparrow\Big\rangle = \Big|\downarrow\Big\rangle, \qquad \sigma^-\Big|\downarrow\Big\rangle = 0.$$

The Heisenberg Hamiltonian describes an interaction between nearest neighbors and is defined by

$$H_0 = \frac{1}{2}\sum_{n=1}^{L}\Big[1 - \bar{\sigma}_n \otimes \bar{\sigma}_{n+1}\Big], \tag{2.6}$$

where

$$\bar{\sigma}_n \otimes \bar{\sigma}_{n+1} = \sigma_n^1 \otimes \sigma_{n+1}^1 + \sigma_n^2 \otimes \sigma_{n+1}^2 + \sigma_n^3 \otimes \sigma_{n+1}^3,$$

is the tensor product of Pauli matrices at sites n and $n+1$. We will now show that the Heisenberg Hamiltonian H_0 is

$$H_0 = \frac{1}{2}\sum_{n=1}^{L}\left(1 - \bar{\sigma}_n \otimes \bar{\sigma}_{n+1}\right) = \sum_{n=1}^{L}\left(1 - \hat{P}_{n,n+1}\right), \tag{2.7}$$

where $\hat{P}_{n,n+1}$ is the permutation operator that swaps the spin at site n with the spin at the neighboring site $n+1$. To prove (2.7), recall the definition of the tensor product of 2 matrices

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \otimes \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix}. \tag{2.8}$$

To find the matrix representation of $\hat{P}_{n,n+1}$, consider the tensor product of $(x \otimes y)$, where x occupies site n and y occupies site $n+1$. Thus, the action of the permutation operator for

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

is

$$\hat{P}\left(x \otimes y\right) = \hat{P}\begin{pmatrix} x_1y_1 \\ x_1y_2 \\ x_2y_1 \\ x_2y_2 \end{pmatrix} = \begin{pmatrix} y_1x_1 \\ y_1x_2 \\ y_2x_1 \\ y_2x_2 \end{pmatrix}. \tag{2.9}$$

Clearly, the permutation operator $\hat{P}$ is a 4×4 matrix of the form

$$\hat{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.10}$$

It is easy to see that

$$1 - \hat{P} = \frac{1}{2}\left(1 - \bar{\sigma} \otimes \bar{\sigma}\right) \tag{2.11}$$

which demonstrates (2.7). Next, we show how the eigenvalues of the Hamiltonian in (2.7) can be obtained using the Bethe ansatz.

2.4 The Bethe Ansatz

Consider a spin-chain with length L. The state with all spins up or down is the vacuum state. The state with M spins opposite to the spins of the vacuum state is the state with M excitations[4]. The set of eigenvalues E^L_M of eigenstates $\left|\psi^L_M\right\rangle$ are obtained from the eigenequation

$$H_0\left|\psi^L_M\right\rangle = E^L_M\left|\psi^L_M\right\rangle. \tag{2.12}$$

An eigenstate can be written as

$$\left|\psi^L_M\right\rangle = \sum_{1 \leq n_1 \leq n_M}^{L} \psi\left(n_1, ..., n_M\right)\left|n_1, ..., n_M\right\rangle, \tag{2.13}$$

where n_i refers to the number of the site in the spin chain and therefore, represents the position of the excitation (magnon).
Since the spin chain is periodic, the wavefunction must satisfy

$$\psi\left(n_2, ..., n_M, n_1 + L\right) = \psi\left(n_1, n_2, ..., n_M\right). \tag{2.14}$$

Hans Bethe [29] proposed that the wavefunction ψ can be written as

$$\psi\left(n_1, ..., n_M\right) = \sum_{\pi \in S_M} A_\pi \text{exp}\left(i \sum_{j=1}^{M} p_{\pi(j)} n_j\right), \tag{2.15}$$

where the sum over $\pi \in S_M$ gives all possible $M!$ permutations of the M magnons, and the p's are the "psedu-momenta" of the excitation M. A_π is a constant that can only depend on the psedu-momenta.
For one excitation (a single magnon), the eigenstate is

$$\left|\psi^L_1\right\rangle = A \sum_{n_1=1}^{L} e^{ip_1 n_1}\left|n_1\right\rangle. \tag{2.16}$$

[4]In condensed matter literature, the spin excitation is usually called a magnon [29, 30].

Applying the Heisenberg Hamiltonian H_0 on (2.16) we get

$$H_0\Big|\psi_1^L\Big\rangle = L\Big|\psi_1^L\Big\rangle - (L-2)\Big|\psi_1^L\Big\rangle - \sum_{n_1=1}^{L} e^{ip_1n_1}\Big|n_1-1\Big\rangle - \sum_{n_1=1}^{L} e^{ip_1n_1}\Big|n_1+1\Big\rangle, \tag{2.17}$$

where the first term is from the identity part of the Hamiltonian, the second term is the result of the permutation part when acting on sites with the same spin, and the last two terms in (2.17) represent the result of the action of the permutation operator when it acts on nearest neighbors of the excitation. More precisely, the action of the permutation operators are

$$\hat{P}_{n_1-1,n_1}\Big|n_1\Big\rangle = \Big|n_1-1\Big\rangle, \quad \hat{P}_{n_1,n_1+1}\Big|n_1\Big\rangle = \Big|n_1+1\Big\rangle. \tag{2.18}$$

Since the sum in (2.17) is over the entire length L, we can shift n_1 to n_1+1 and n_1-1 respectively. Thus (2.17) becomes

$$H_0\Big|\psi_1^L\Big\rangle = \left(2 - e^{ip_1} - e^{-ip_1}\right)\Big|\psi_1^L\Big\rangle = E_1^L\Big|\psi_1^L\Big\rangle, \tag{2.19}$$

where

$$E_1^L = \left(2 - e^{ip_1} - e^{-ip_1}\right) = 2\left(1-\cos p_1\right) = 4\sin^2\frac{p_1}{2}. \tag{2.20}$$

The periodicity condition in (2.14) can be written as

$$e^{ip_1(n_1+L)} = e^{ip_1n_1}. \tag{2.21}$$

This condition quantizes the pseudo-momentum p_1

$$p_1 = \frac{2\pi k}{L}, \qquad k = 0, ..., L-1. \tag{2.22}$$

The quantization of momentum as a result of the periodicity condition (2.21) is a general feature of the Bethe ansatz.
In the case of two excitations (n_1 and n_2) the Bethe wavefunction is

$$\psi\left(n_1, n_2\right) = e^{ip_1n_1+ip_2n_2} + S\left(p_2, p_1\right) e^{ip_2n_1+ip_1n_2}, \tag{2.23}$$

where $S(p_2, p_1)$ is the S-matrix responsible for interchanging the two magnons. The two magnon eigenstate is given by

$$\big|\psi_2^L\big\rangle = A\sum_{n_1=1}^{L}\sum_{n_2=1}^{L}\psi(n_1,\, n_2)\big|n_1,\, n_2\big\rangle.$$

To find $S(p_2, p_1)$, consider a wavefunction $\psi(n_1, n_2)$ for two adjacent magnons ($n_2 = n_1+1$). In this case the action of the Hamiltonian H_0 is

$$\begin{aligned} H_0\big|\psi_2^L\big\rangle =& E_2^L\big|\psi_2^L\big\rangle \\ H_0\psi(n_1,\, n_2) =& 4\psi\left(n_1, n_2\right) - \psi\left(n_1-1, n_2\right) - \psi\left(n_1+1, n_2\right) \\ &- \psi(n_1,\, n_2-1) - \psi(n_1, n_2+1). \end{aligned} \tag{2.24}$$

Inserting the Bethe wavefunction (2.23) into (2.24) yields

$$\left(E_2^L - 4\sin^2\frac{p_1}{2} - 4\sin^2\frac{p_2}{2}\right)\psi\left(n_1, n_2\right) = 0. \tag{2.25}$$

We note that, (2.25) describes the energy of two magnons as the sum of the individual magnon energies

$$E_2^L(p_1, p_2) = 4\sin^2\frac{p_1}{2} + 4\sin^2\frac{p_2}{2} = E_1^L(p_1) + E_2^L(p_2). \tag{2.26}$$

This is a direct consequence of the integrability of the Hamiltonian (2.7). Substituting (2.23) and (2.26) into (2.24), the S-matrix reads

$$S\left(p_2, p_1\right) = -\frac{e^{i(p_1+p_2)} - 2e^{ip_2} + 1}{e^{i(p_1+p_2)} - 2e^{ip_1} + 1}. \tag{2.27}$$

We note that, interchanging p_1 and p_2 in (2.27) gives $S\left(p_1, p_2\right) = S\left(p_2, p_1\right)^{-1}$, which implies that

$$S\left(p_1, p_2\right) S\left(p_1, p_2\right)^{-1} = 1. \tag{2.28}$$

As a remarkable consequence of the integrability of the spin-chain system, one can generalize (2.26) for an arbitrary number of magnons. The energy in this case is the sum of the energies of the individual magnons, that is

$$E_M^L = \sum_{k=1}^{M} 4\sin^2\frac{p_k}{2}. \tag{2.29}$$

Consequently, the system of Bethe equations for M magnons is

$$e^{ip_kL} = \prod_{j=1, j\neq k}^{M} S\left(p_k, p_j\right). \tag{2.30}$$

2.5 Rapidity

The Bethe equations are a set of quantization rules for the magnons's psedumomenta in the chain. When the system contains more than one excitation, the Bethe equations become very complicated. However, changing to a new set of variables simplifies the equations, thus making them easier to solve. These new variables, called rapidities u, are defined as follows

$$u_k = \frac{1}{2}\cot\frac{p_k}{2}. \tag{2.31}$$

Using these variables for the case of two excitations, the S-matrix becomes

$$S\left(u_1, u_2\right) = \frac{u_1 - u_2 + i}{u_1 - u_2 - i}. \tag{2.32}$$

To translate the Bethe equations, it is useful to note that

$$e^{ip_k} = \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}. \tag{2.33}$$

For a spin-chain of length L, (2.30) reads

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^L = \prod_{j \neq k}^{M} \frac{u_k - u_j + i}{u_k - u_j - i}. \tag{2.34}$$

It is also simple to check that the energy of a simple excitation is

$$E\left(u_k\right) = \frac{i}{u_k + \frac{i}{2}} - \frac{i}{u_k - \frac{i}{2}} = \frac{1}{u_k^2 + \frac{1}{4}}, \tag{2.35}$$

and that the total energy is

$$E = \sum_{k}^{M} \frac{1}{u_k^2 + \frac{1}{4}}. \tag{2.36}$$

To apply the spin-chain method in the AdS/CFT setting, we need an additional constraint. Since the single trace operators have a symmetry under cyclic permutations, the translation operator

$$e^p \equiv \exp\left(\sum_{k=1}^{M} p_k\right) \tag{2.37}$$

leaves the trace invariant. This means we must have

$$e^p = 1,$$

which implies

$$p = \sum_{k=1}^{M} p_k = 0. \tag{2.38}$$

Using rapidity variables, (2.38) can be written as

$$e^{ip} = \prod_{k=1}^{M} \frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}} = 1. \tag{2.39}$$

The rapidity variables are called Bethe roots.

2.6 The Spin-Chain for $\mathcal{N}=4$ SYM Theory

In the $SU(2)$ sector of $\mathcal{N}=4$ SYM theory, the scalar fields Z and Y can be labeled as either spin up or down, reflecting the fact that the two fields are in a doublet of $SU(2)$. We choose the Z field as spin up ($\uparrow$) and the Y field as spin down ($\downarrow$). Then, when acting on single traces, the dilatation operator is

$$D=\frac{\lambda}{8\pi^2}\sum_{l=1}^{L}\left(1-P_{l,l+1}+\frac{1}{2}K_{l,l+1}\right), \tag{2.40}$$

where $P_{l,l+1}$ is the same permutation operator we have encountered above, and $K_{l,l+1}=K_{i_l,i_{l+1}}^{j_l,j_{l+1}}=\delta_{i_l,i_{l+1}}\delta^{j_l,j_{l+1}}$ is a trace operator. In this section, we have changed the index labeling the site on the spin-chain from n to l.
In the $SU(2)$ sector, it is easy to see that $K_{l,l+1}=0$ as a result of the fact that we have Z and Y fields in our single trace operators and not their conjugates. Thus, our Hamiltonian (2.40) reduces to

$$D_{SU(2)}=\frac{\lambda}{8\pi^2}\sum_{l=1}^{L}\left(1-P_{l,l+1}\right). \tag{2.41}$$

As we see, this Hamiltonian is similar to the Heisenberg spin-chain Hamiltonian (2.7). $D_{SU(2)}$ can also be written in terms of the spin operators, one copy at each of the L lattice sites

$$D_{SU(2)}=\frac{\lambda}{8\pi^2}\sum_{l=1}^{L}\left(\frac{1}{2}-2\vec{S}_l\cdot\vec{S}_{l+1}\right). \tag{2.42}$$

This Hamiltonian commutes with the total spin $\vec{S}=\sum_l\vec{S}_l$. Therefore, the spin eigenstates are simultaneously the energy eigenstates.
Applying the spin-chain method discussed in this chapter, it is easy to see that $D_{SU(2)}$ annihilates the ground state, which has all spins up or down i.e. $\Big|\uparrow\uparrow\uparrow\ldots\uparrow\uparrow\Big\rangle$ or $\Big|\downarrow\downarrow\downarrow\ldots\downarrow\downarrow\Big\rangle$. $D_{SU(2)}$ has a zero eigenvalue ($E_M^L=0$). The operators corresponding to these states are called chiral primary.
Let us now consider the states with one spin down. We will see that these do not describe excited states so that these states describe BPS operators. In this case, the action of $D_{SU(2)}$ gives

$$\begin{aligned} &D_{SU(2)}\Big|\uparrow\ldots\uparrow\overset{l}{\downarrow}\uparrow\ldots\uparrow\Big\rangle \\ &\quad=\frac{\lambda}{8\pi^2}\left(2\Big|\uparrow\ldots\uparrow\overset{l}{\downarrow}\uparrow\ldots\uparrow\Big\rangle-\Big|\uparrow\ldots\overset{l-1}{\downarrow}\uparrow\uparrow\ldots\uparrow\Big\rangle-\Big|\uparrow\ldots\uparrow\uparrow\overset{l+1}{\downarrow}\ldots\uparrow\Big\rangle\right), \end{aligned} \tag{2.43}$$

and the corresponding Bethe eigenstates are

$$\Big|p\Big\rangle\equiv\frac{1}{\sqrt{L}}\sum_{l=1}^{L}e^{ipl}\Big|\uparrow\ldots\uparrow\overset{l}{\downarrow}\uparrow\ldots\uparrow\Big\rangle, \tag{2.44}$$

where

$$D_{SU(2)}\Big|p\Big\rangle = E(p)\Big|p\Big\rangle, \qquad E(p) = \frac{\lambda}{2\pi^2}\sin\frac{p}{2}. \tag{2.45}$$

As we have seen, the periodicity condition quantizes the pseudo-momentum p i.e. $p = \frac{2\pi n}{L}$ and $\Big|p\Big\rangle$ is invariant under $l \to l+L$. However, in the $SU(2)$ sector of $\mathcal{N} = 4$ SYM theory, the dilatation operator acts on single trace operators that are invariant under the shift $l \to l+1$. This is only consistent with $p = 0$. Then, from (2.45) this corresponds to $E(p) = 0$, which implies that a state with only one spin down (one excitation) is a chiral primary state with a single Y field.

The simplest non-chiral primary states are the states with two magnons. These states can be constructed using the general argument of [31] which does not necessarily involve a closed spin chain of length L. They considered an infinite spin-chain ($L \to \infty$) with un-normalized two-magnon state

$$\Big|p_1, p_2\Big\rangle = \sum_{l_1<l_2} e^{ip_1l_1+ip_2l_2}\Big|\ldots \overset{l_1}{\downarrow} \ldots \overset{l_2}{\downarrow} \ldots\Big\rangle + e^{i\phi}\sum_{l_1>l_2} e^{ip_1l_1+ip_2l_2}\Big|\ldots \overset{l_2}{\downarrow} \ldots \overset{l_1}{\downarrow} \ldots\Big\rangle. \tag{2.46}$$

The S-matrix, which ensures that the two magnons are next to each other at sites l and $l+1$, is

$$S_{12} = -\frac{e^{ip_1+ip_2} - 2e^{ip_2} + 1}{e^{ip_1+ip_2} - 2e^{ip_2} - 1}. \tag{2.47}$$

Now, if we consider our chain as a cyclic chain of length L, the single trace condition requires $p_1 + p_2 = 0$. Furthermore, the momentum quantization condition does not necessarily imply $p_1 = p_2 = 0$ which will lead to chiral primary states. However, if we move one magnon around the spin-chain circle, our state remains invariant. In this process the first magnon will pass the second one which produces an extra phase $e^{i\phi}$. Then we have $e^{ip_1L}e^{i\phi} = 1$. If we choose the solution that does not involve a chiral primary state, i.e. $p_2 = -p_1$, then we readily obtain $e^{i\phi} = e^{-ip_1}$. Thus, the allowed values for p_1 are $p_1 = \frac{2\pi n}{(L-1)}$, which corresponds to energy eigenvalues γ equal to

$$\gamma = \frac{\lambda}{\pi^2}\sin^2\frac{\pi n}{L-1}. \tag{2.48}$$

Using the Bethe roots u defined in the previous section, the dispersion relation is then

$$\mathcal{E}(u) = \frac{\lambda}{8\pi^2}\frac{1}{u^2+1/4}. \tag{2.49}$$

For M magnons, the energy of the state is

$$\gamma = \sum_{j=1}^{M}\mathcal{E}(u_j), \tag{2.50}$$

where $\mathcal{E}(u_j)$ is given (2.49). Finally, the trace condition requires

$$\prod_{j=1}^{M} \frac{u_j + i/2}{u_j - i/2} = 1. \tag{2.51}$$

In conclusion, the planar dilatation operator of $\mathcal{N} = 4$ SYM theory in the $SU(2)$ sector is integrable. Further analysis can be found in [32, 33, 34]. The generalization to other sectors including the complete $PSU(2,2|4)$ description is discussed in [35, 36].

2.7 The Spin-Chain for ABJM Theory

The bifundamental fields in ABJM theory are in the fundamental/ antifundamental representation of the $U(N) \times U(N)$ gauge group. Therefore, the gauge invariant operators are constructed from pairs $AB^\dagger$ which are in the adjoint of the first $U(N)$ factor (for more details, see appendix B). Like in $\mathcal{N} = 4$ SYM theory, we will consider the simplest sector in the ABJM theory, that is the $SU(2) \times SU(2)$ sector where the gauge invariant operators are built from the pairs $AB^\dagger$. A more complete discussion can be found in [37].

2.7.1 Bethe ansatz in the $SU(2) \times SU(2)$ sector

In this sector $A_I \in (A_1, A_2)$ are called the odd sites of the spin-chain while $B_I^\dagger \in \left(B_1^\dagger, B_2^\dagger\right)$ are the even sites. The planar two-loop dilatation operator is given by

$$\delta D = \lambda^2 \sum_{i=1}^{L} \left(1_{2i-1,2i+1} - P_{2i-1,2i+1} + 1_{2i,2i+2} - P_{2i,2i+2}\right). \tag{2.52}$$

A natural extension of the analysis in section 2.6 for the ABJM theory in the $SU(2) \times SU(2)$ sector is accomplished by attaching two Bethe roots u and v for the psedu-momenta.
We will show that the planar ABJM dilatation operator at two-loops in this sector is integrable, by obtaining Bethe equations. These equations diagonalize the anomalous dimension. We then give simple solutions to the Bethe ansatz. The spin-chain in the $SU(2) \times SU(2)$ sector is built from operators of length $2L$ with K_u excitations on the odd sites and K_v excitations on the even sites. Thus the Bethe ansatz equations are

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{k \neq j}^{K_u} \frac{u_j - u_k + i}{u_j - u_k - i}, \tag{2.53}$$

$$\left(\frac{v_j + i/2}{v_j - i/2}\right)^L = \prod_{k \neq j}^{K_v} \frac{v_j - v_k + i}{v_j - v_k - i}, \tag{2.54}$$

$$\prod_{j=1}^{K_u}\left(\frac{u_j+i/2}{u_j-i/2}\right)\prod_{k=1}^{K_v}\left(\frac{v_k+i/2}{v_k-i/2}\right)=1, \tag{2.55}$$

where

$$u_j=\frac{1}{2}\cot\frac{p_j}{2},\quad v_k=\frac{1}{2}\cot\frac{p_k}{2}.$$

Using the Bethe equations, the anomalous dimension in terms of Bethe roots is

$$\delta D=\lambda^2\left(\sum_{j=1}^{K_u}\frac{1}{u_j^2+1/4}+\sum_{k=1}^{K_v}\frac{1}{v_k^2+1/4}\right). \tag{2.56}$$

To show how the Bethe ansatz equations diagonalize the anomalous dimension, let us add one excitation in the odd sites and another one in the even sites. We can think of the system as two Heisenberg spin chains with $L+1$ sites in each chain. In this case (2.53), (2.54) and (2.55) reduce to

$$\left(\frac{u+i/2}{u-i/2}\right)^{L+1}=\left(\frac{v+i/2}{v-i/2}\right)^{L+1}=\frac{u+i/2}{u-i/2}\frac{v+i/2}{v-i/2}=1. \tag{2.57}$$

If we consider the excitation in the odd sites with momentum p_1 and the even sites with momentum p_2, then the resulting Bethe roots are $u=\frac{1}{2}\cot p_1/2$ and $v=\frac{1}{2}\cot p_2/2$ respectively. The trace condition implies

$$e^{ip_1(L+1)}=e^{ip_2(L+1)}=e^{i(p_1+p_2)}=1. \tag{2.58}$$

The non-chiral solution is $p_1=-p_2=\frac{2\pi n}{L+1}$. Then (2.56) becomes

$$\delta D_{\text{two-magnons}}=\lambda^2\left(4\sin^2\frac{p_1}{2}+4\sin^2\frac{p_2}{2}\right)=8\lambda^2\sin^2\left(\frac{\pi n}{L+1}\right), \tag{2.59}$$

and the full scaling dimension (including the quantum correction) is given by

$$D=L+1+8\lambda^2\sin^2\left(\frac{\pi n}{L+1}\right)+O\left(\lambda^4\right). \tag{2.60}$$

In (2.60) the scaling dimension of the vacuum operator is L, and the energy of the two excitations is $1+8\lambda^2\sin^2\left(\frac{\pi n}{L+1}\right)$. Therefore, in the large L limit, the energy of the two excitations reduces to

$$E_{\text{two excitations}}=1+\frac{8\pi^2\lambda^2n^2}{L^2}+O\left(\frac{1}{L^6}\right). \tag{2.61}$$

It is easy to see that the energy of one excitation is

$$E_{\text{one excitation}}=\frac{1}{2}+\frac{4\pi^2\lambda^2n^2}{L^2}+O\left(\frac{1}{L^6}\right). \tag{2.62}$$

Let us generalize this example by adding L excitations to both odd and even sites of the spin-chain. In this case, the two-spins are identical to each other and we can set $u_j = v_j$. Thus, the Bethe ansatz equations reduce to

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^{2L} = \prod_{k \neq j}^{L} \frac{u_j - u_k + i}{u_j - u_k - i}, \tag{2.63}$$

$$\left(\prod_{j=1}^{L} \left(\frac{u_j + i/2}{u_j - i/2}\right)\right)^2 = 1. \tag{2.64}$$

The Bethe ansatz equation (2.64) can be written as

$$\sum_{j=1}^{L} \ln\left(\frac{u_j + i/2}{u_j - i/2}\right) = -\pi m i, \tag{2.65}$$

where m is an integer and corresponds to the winding number of the dual classical string. Using (2.63) and (2.64), the anomalous dimension in this case is

$$\delta D = 2\lambda^2 \sum_{j=1}^{L} \frac{1}{u_j^2 + 1/4}. \tag{2.66}$$

In the large L limit, the solution of the Bethe equation is given in [38, 39]. In particular, from section 3 of [39] one gets

$$\delta D = \left(\frac{\pi^2 \lambda^2 m^2}{L} + ...\right) + \frac{1}{L}\left(\frac{2a\pi^2\lambda^2}{L} + ...\right), \tag{2.67}$$

$$a = \frac{m^2}{4} + \sum_{n=1}^{\infty}\left(n\left(\sqrt{n^2 - m^2} - n\right) + \frac{m^2}{2}\right). \tag{2.68}$$

An interesting feature of this result is that (2.67) and (2.68) exactly match the one-loop energies of the classical string theory solutions for type II A string theory computed using the world-sheet approach and the algebraic curve formalism [39, 40, 41, 42, 43].

Chapter 3

Schur Polynomials and Correlators

In the previous chapter, we have seen that $\mathcal{N} = 4$ SYM theory and ABJM theory are integrable in the planar limit by mapping the dynamics to the dynamics of a spin-chain and employing the Bethe ansatz. In this chapter we furnish a basis that provides the tools needed to search for integrability beyond the planar limit. Our methods make good use of the representation theory of the symmetric and unitary groups [24, 25, 27]. We begin by defining the gauge invariant operators built from Schur polynomials. We then compute the two-point function for $\mathcal{N} = 4$ SYM theory and ABJM theory.

3.1 Schur polynomials

$\mathcal{N} = 4$ SYM theory has six hermitian scalar fields ϕ_i $i = 1, 2..., 6$. These scalar fields transform in the adjoint of a $U(N)$ gauge group. Schur polynomials are trace polynomials of a complex field $Z = \phi_1 + i\phi_2$. They are defined as follows

$$\chi_R(Z) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \operatorname{Tr}\left(\sigma Z^{\otimes n}\right), \tag{3.1}$$

$$\operatorname{Tr}\left(\sigma Z^{\otimes n}\right) = Z^{i_1}_{i\sigma(1)} Z^{i_2}_{i\sigma(2)} ... Z^{i_{n-1}}_{i\sigma(n-1)} Z^{i_n}_{i\sigma(n)},$$

where R is a Young diagram that contains n boxes, i.e. R labels an irreducible representation of the symmetric group S_n and $\chi_R(\sigma)$ is the character of the group element σ which is given by

$$\chi_R(\sigma) = \operatorname{Tr}_R(\Gamma_R(\sigma)). \tag{3.2}$$

The trace structure of the Schur polynomials (3.1) implies they are invariant under the local $U(N)$ gauge symmetry of the theory. It is then natural to think of Schur polynomials as a gauge invariant basis of local operators for a sector of $\mathcal{N} = 4$ SYM theory. In [24], Schur polynomials have been found to be in one-to-one correspondence to operators in the space of a $\frac{1}{2}$ BPS representation of $\mathcal{N} =$

4 SYM theory. Furthermore, they have shown that these Schur polynomials have diagonal two-point functions. To get some insight into the structure of the Schur polynomials, the following examples represent all of the possible Schur polynomials for $n = 1, 2, 3$. In this case for $n = 1$ we have one representation $R = \square$ and for $n = 2$ we have two possible representations $R = \square\square$ and $R = \begin{smallmatrix}\square\\\square\end{smallmatrix}$. Finally for $n = 3$ we have three possible representations $R = \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$, $R = \square\square\square$ and $R = \begin{smallmatrix}\square\square\\\square\;\;\end{smallmatrix}$. The corresponding Schur polynomials are

$$
\begin{aligned}
\chi_{\square}(Z) &= \mathrm{Tr}Z, \\
\chi_{\square\square}(Z) &= \frac{1}{2}\left((\mathrm{Tr}Z)^2 + \mathrm{Tr}Z^2\right), \\
\chi_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}(Z) &= \frac{1}{2}\left((\mathrm{Tr}Z)^2 - \mathrm{Tr}Z^2\right), \\
\chi_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}(Z) &= \frac{1}{6}\left((\mathrm{Tr}Z)^3 - 3\left(\mathrm{Tr}Z\right)\left(\mathrm{Tr}Z^2\right) + 2\mathrm{Tr}Z^3\right), \\
\chi_{\square\square\square}(\sigma) &= \frac{1}{6}\left((\mathrm{Tr}Z)^3 + 3\left(\mathrm{Tr}Z\right)\left(\mathrm{Tr}Z^2\right) + 2\mathrm{Tr}Z^3\right), \\
\chi_{\begin{smallmatrix}\square\square\\\square\;\;\end{smallmatrix}}(\sigma) &= \frac{1}{3}\left((\mathrm{Tr}Z)^3 - \mathrm{Tr}Z^3\right).
\end{aligned} \tag{3.3}
$$

In each of these examples, the character $\chi_R(\sigma)$ was computed for each group element of S_n in irreducible representation R using the graphical method of Strand diagrams introduced in [49]. Notice that the Schur polynomials exhibit a non-trivial multi-trace structure.

3.2 Restricted Schur Polynomials

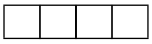
Schur polynomials can be generalized by adding matrix fields say Y, X...etc to the original field Z. The resulting polynomials are called restricted Schur polynomials. For example, restricted Schur polynomials with two matrix fields Z and Y are

$$
\chi_{R(r_1,r_2)\alpha\beta}(Z,Y) = \frac{1}{n!m!}\sum_{\sigma\in S_{n+m}} \chi_{R,(r_1,r_2)\alpha\beta}(\sigma)\,\mathrm{Tr}\left(\sigma Z^{\otimes n}Y^{\otimes m}\right), \tag{3.4}
$$

where

$$
\chi_{R,(r_1,r_2)\alpha\beta}(\sigma) = \mathrm{Tr}_{(r_1,r_2)\alpha\beta}\left(\Gamma_R(\sigma)\right), \tag{3.5}
$$

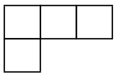
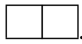
is the restricted character.

Here R is a Young diagram with $n+m$ boxes labeling an irreducible representation (irrep) of S_{n+m}. r_1 is a Young diagram consisting of n boxes labeling an irrep of S_n while r_2 is a Young diagram consisting of m boxes labeling an irrep of S_m. The trace $\mathrm{Tr}_{(r_1,r_2)}$ is a restricted trace over the elements of $S_n \times S_m$ subgroup of S_{n+m}. The indices α and β are multiplicity indices needed because (r,s) can be subduced more than once from R. We will focus on Young diagrams R with two rows. In this case the α, β indices can be dropped. Restricted Schur polynomials were first identified in [44], and they provide a complete set of gauge invariant operators. Furthermore, restricted Schur polynomials have diagonal two point functions which make them useful operators with which we can study the anomalous dimension. The following two examples construct restricted Schur polynomials built using two matrix scalar fields Z and Y. The first example is $R =$  with $r_1 = r_2 =$ . In this case $n = m = 2$. The restricted characters are

$$\chi_{R,(r_1,r_2)}(\sigma) = \mathrm{Tr}_{(r_1,r_2)}(\Gamma_R(\sigma)) = 1,$$

where we have used the graphical method of Strand diagram to compute these characters [49]. The restricted Schur polynomial for this example is

$$\begin{aligned}\chi_{R,(r_1,r_2)}(Z,Y) =& \frac{1}{4}\mathrm{Tr}(Z)^2\,\mathrm{Tr}(Y)^2 + \frac{1}{4}\mathrm{Tr}\left(Z^2\right)\mathrm{Tr}(Y)^2 + \frac{1}{4}\mathrm{Tr}(Z)^2\,\mathrm{Tr}\left(Y^2\right)\\ &+\mathrm{Tr}(ZY)\,\mathrm{Tr}(Z)\,\mathrm{Tr}(Y) + \mathrm{Tr}\left(Z^2Y\right)\mathrm{Tr}(Y) + \mathrm{Tr}\left(ZY^2\right)\mathrm{Tr}(Z)\\ &+\frac{1}{4}\mathrm{Tr}\left(Z^2\right)\mathrm{Tr}\left(Y^2\right) + \frac{1}{2}\mathrm{Tr}(ZY)^2 + \mathrm{Tr}\left(Z^2Y^2\right) + \frac{1}{2}\mathrm{Tr}(ZYZY).\end{aligned} \tag{3.6}$$

The second example is for $R =$  with $r_1 = r_2 =$ . The restricted characters are

$$\begin{aligned}&\chi_{R,(r_1,r_2)}((1)) = 1, \quad \chi_{R,(r_1,r_2)}((12)) = \chi_{R,(r_1,r_2)}((34)) = 1,\\ &\chi_{R,(r_1,r_2)}((24)) = \chi_{R,(r_1,r_2)}((23)) = \chi_{R,(r_1,r_2)}((14)) = \chi_{R,(r_1,r_2)}((13)) = 0,\\ &\chi_{R,(r_1,r_2)}((123)) = \chi_{R,(r_1,r_2)}((124)) = \chi_{R,(r_1,r_2)}((134)) = \chi_{R,(r_1,r_2)}((234))\\ &= \chi_{R,(r_1,r_2)}((132)) = \chi_{R,(r_1,r_2)}((142)) = \chi_{R,(r_1,r_2)}((143)) = \chi_{R,(r_1,r_2)}((243)) = 0,\\ &\chi_{R,(r_1,r_2)}((12)(34)) = 1, \quad \chi_{R,(r_1,r_2)}((13)(24)) = \chi_{R,(r_1,r_2)}((14)(23)) = -1,\\ &\chi_{R,(r_1,r_2)}((1234)) = \chi_{R,(r_1,r_2)}((1243)) = \chi_{R,(r_1,r_2)}((1342)) = \chi_{R,(r_1,r_2)}((1432)) = 0,\\ &\chi_{R,(r_1,r_2)}((1324)) = \chi_{R,(r_1,r_2)}((1423)) = -1.\end{aligned}$$

The corresponding restricted Schur polynomial is

$$\begin{aligned}\chi_{R,(r_1,r_2)}(Z,Y) =& \frac{1}{4}\mathrm{Tr}(Z)^2\,\mathrm{Tr}(Y)^2 + \frac{1}{4}\mathrm{Tr}\left(Z^2\right)\mathrm{Tr}(Y)^2 + \frac{1}{4}\mathrm{Tr}(Z)^2\,\mathrm{Tr}\left(Y^2\right)\\ &+\frac{1}{4}\mathrm{Tr}\left(Z^2\right)\mathrm{Tr}\left(Y^2\right) - \frac{1}{2}\mathrm{Tr}(ZY)^2 - \frac{1}{2}\mathrm{Tr}(ZYZY).\end{aligned} \tag{3.7}$$

3.2.1 Restricted Schur Polynomial for $\mathcal{N}=4$ SYM Theory

As we have seen in Chapter 2, $\mathcal{N}=4$ SYM theory in the $SU(2)$ sector is integrable, in the planar limit. It is natural to search for nonplanar integrability in this sector. Restricted Schur polynomials with two matrix fields Z and Y as in (3.4), represent a basis of gauge invariant operators in the $SU(2)$ sector. To proceed even further, one could also build gauge invariant operators that take us beyond the $SU(2)$ sector of the $\mathcal{N}=4$ SYM theory. The simplest example in this case is provided by the restricted Schur polynomials with three scalar matrix fields Z, Y and X

$$\chi_{R(r_1,r_2,r_3)\alpha\beta}(Z,Y,X) = \frac{1}{n!m!p!}\sum_{\sigma\in S_{n+m+p}} \mathrm{Tr}_{(r_1,r_2,r_3)\alpha\beta}(\Gamma_R(\sigma))\,\mathrm{Tr}\left(\sigma Z^{\otimes n}Y^{\otimes m}X^{\otimes p}\right). \tag{3.8}$$

3.3 Schur Polynomials for ABJM Theory

In this section we will build gauge invariant operators for ABJM theory using the Schur polynomials discussed in the previous section. As we see from appendix B, ABJM theory with the gauge group $U(N)\times U(N)$ contains gauge invariant operators built from traces of the products of $AB^\dagger$ where A and B are the bifundamental scalar fields in the ABJM Lagrangian. It is then straightforward to construct Schur polynomials for ABJM theory in analogy with Schur polynomials of $\mathcal{N}=4$ SYM theory as [50]

$$\chi_R\left(AB^\dagger\right) = \frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{Tr}_R(\Gamma_R(\sigma))\,\mathrm{Tr}\left(\sigma\left(AB^\dagger\right)^{\otimes n}\right), \tag{3.9}$$

where

$$\mathrm{Tr}\left(\sigma\left(AB^\dagger\right)^{\otimes n}\right) = \left(AB^\dagger\right)^{i_1}_{i\sigma(1)}\left(AB^\dagger\right)^{i_2}_{i\sigma(2)}\cdots\left(AB^\dagger\right)^{i_{n-1}}_{i\sigma(n-1)}\left(AB^\dagger\right)^{i_n}_{i\sigma(n)},$$

and

$$\left(AB^\dagger\right)^{i_a}_{i\sigma(a)} = A^{i_a}_{\alpha}B^{\dagger\alpha}_{i\sigma(a)} \quad i,\ \alpha = 1,2,...N.$$

3.3.1 Restricted Schur Polynomials for ABJM Theory

In the next chapter, we will compute the action of the dilatation operator in the $SU(2)\times SU(2)$ subsector of ABJM theory. We will consider the bifundamental scalar fields A_1 and $B_1^\dagger$ as background fields and the $B_2^\dagger$ as excitation fields. We will make heavy use of a restricted Schur polynomial basis for this particular subsector. Toward this end, in this subsection, we construct general restricted Schur polynomials valid for any sector. We will denote the fields of the theory that we use as A_1, A_2, $B_1^\dagger$ and $B_2^\dagger$. The number of A_is is n_i; the number of $B_i^\dagger$ s is m_i. Set $n = n_1 + n_2 = m_1 + m_2$. R is an irrep of S_n, i.e. $R \vdash n$. Introduce the notation

$$\phi_{11b}^a = A_{1\alpha}^a B_{1b}^{\dagger\alpha}, \qquad \phi_{12b}^a = A_{1\alpha}^a B_{2b}^{\dagger\alpha},$$

$$\phi_{21b}^a = A_{2\alpha}^a B_{1b}^{\dagger\alpha}, \qquad \phi_{22b}^a = A_{2\alpha}^a B_{2b}^{\dagger\alpha}.$$

The number of ϕ_{ij}s is n_{ij}. $r_{ij} \vdash n_{ij}$ is an irrep of $S_{n_{ij}}$. The collection, $(r_{11}, r_{12}, r_{21}, r_{22}) \equiv \{r\}$ is an irrep of $S_{n_{11}} \times S_{n_{12}} \times S_{n_{21}} \times S_{n_{22}} \subset S_n$. We will be exploiting the fact that general multi trace operators can be realized as a single trace over the larger space $V^{\otimes n}$. In $V^{\otimes n}$ permutations have matrix elements

$$\left\langle i_1, i_2, \cdots, i_n \middle| \tau \middle| j_1, j_2, \cdots, j_n \right\rangle = \delta^{i_1}_{j_{\tau(1)}} \delta^{i_2}_{j_{\tau(2)}} \cdots \delta^{i_n}_{j_{\tau(n)}}.$$

Consider the most general gauge invariant operator built using an arbitrary number of A_1s, A_2s, $B_1^\dagger$s and $B_2^\dagger$s. Due to the index structure of the fields, any single trace gauge invariant operator is given by an alternating sequence of pairs of A_is and $B_i^\dagger$s. The possible pairs are the ${\phi_{ij}}_b^a$ defined above. Any single trace gauge invariant operator is given by a unique (up to cyclic permutations) product of the $(\phi_{ij})_b^a$. The most general gauge invariant operator is given by a product of an arbitrary number of these single trace operators.
In this subsection we will provide a new basis for these operators. The basis we have constructed is given by a restricted Schur polynomial in the ϕ_{ij}

$$O_{R,\{r\}} = \frac{1}{n_{11}! n_{12}! n_{21}! n_{22}!} \sum_{\sigma \in S_n} \mathrm{Tr}_{\{r\}} \left(\Gamma_R \left(\sigma \right) \right) \mathrm{Tr} \left(\sigma \phi_{11}^{\otimes n_{11}} \phi_{12}^{\otimes n_{12}} \phi_{21}^{\otimes n_{21}} \phi_{22}^{\otimes n_{22}} \right). \tag{3.10}$$

The irrep R will in general be a reducible representation of the $S_{n_{11}} \times S_{n_{12}} \times S_{n_{21}} \times S_{n_{22}}$ subgroup of S_n. One of the $S_{n_{11}} \times S_{n_{12}} \times S_{n_{21}} \times S_{n_{22}}$ irreps that R subduces is $\{r\}$. In the above formula, $\mathrm{Tr}_{\{r\}}$ is an instruction to trace only over the $\{r\}$ subspace of the carrier space of R. A very convenient way to implement this trace is as

$$\mathrm{Tr}_{\{r\}} \left(\Gamma_R \left(\sigma \right) \right) = \mathrm{Tr} \left(P_{R,\{r\}} \Gamma_R \left(\sigma \right) \right) \equiv \chi_{R,\{r\}} \left(\sigma \right)$$

where $P_{R,\{r\}}$ is a projector which projects from the carrier space of R to the $\{r\}$ subspace.

3.3.2 A Complete Set of Operator

To prove that these operators form a basis, we simply need to show that they are complete. We will demonstrate completeness by showing that the most general gauge invariant operator built using an arbitrary number of A_1s, A_2s, $B_1^\dagger$s and $B_2^\dagger$s can be written as a linear combination of the operators (3.10). Further, we can argue that all the operators in this set are linearly independent. This follows immediately from the fact that the number of restricted Schur polynomials is equal to the number of gauge invariant operators (which are

linearly independent). This counting agreement was proved in [45] at both finite and infinite N.
Now for the demonstration: The most general gauge invariant operator that we are considering can be written as

$$o\left(\tau\right)=\mathrm{Tr}\left(\tau\phi_{11}^{\otimes n_{11}}\phi_{12}^{\otimes n_{12}}\phi_{21}^{\otimes n_{21}}\phi_{22}^{\otimes n_{22}}\right)$$

for a suitable choice of the permutation $\tau\in S_n$. The completeness of this basis now follows from the identity (which is derived in [46])

$$o\left(\tau\right)=\sum_{R,\{r\}}\frac{d_R n_{11}!n_{12}!n_{21}!n_{22}!}{d_{r_{11}}d_{r_{12}}d_{r_{21}}d_{r_{22}}n!}\chi_{R,\{r\}}\left(\tau\right)O_{R,\{r\}}$$

where the sum over R runs over all irreps of S_n and $\{r\}$ ranges over all irreps of $S_{n_{11}}\times S_{n_{12}}\times S_{n_{21}}\times S_{n_{22}}$. This completes the demonstration.
The operators given in (3.10) do not have simple two point functions and they are not orthogonal. For this reason we will need to generalize (3.10). The main insight we gained is the fact that the number of gauge invariant operators is equal to the number of distinct restricted Schur labels $R,\{r\}$.

3.4 Correlators for $\mathcal{N}=4$ SYM Theory

The two point function of Schur polynomials (3.1) in the free field theory was computed for the first time in [24]

$$\left\langle\chi_R\left(Z\right)\chi_S\left(Z\right)^{\dagger}\right\rangle=\delta_{RS}f_R,$$

where δ_{RS} is 1 if $R=S$ and zero otherwise, f_R is the product of the weights of the boxes in the Young diagram R. It is given by

$$f_R=\prod_{i,j}\left(N-i+j\right).$$

In this formula, i and j represent the box in the ith row and the jth column of the Young diagram R. Thus, the weight of this box is $(N-i+j)$. To illustrate this rule, an example is given in Figure 3.1.
A natural generalization for the two point function of Schur polynomials is the two point function of restricted Schur polynomials [46, 47]

$$\left\langle\chi_{R,(r_{\alpha1},r_{\alpha2})}\chi^{\dagger}_{S,\left(s_{\beta1},s_{\beta2}\right)}\right\rangle=\delta_{RS}\delta_{r_{\alpha1}s_{\beta1}}\delta_{r_{\alpha2}s_{\beta2}}\frac{(\mathrm{hooks})_R}{(\mathrm{hooks})_{R_\alpha}}f_R,\qquad(3.11)$$

where $(\mathrm{hooks})_R$ and $(\mathrm{hooks})_{R_\alpha}$ are the product of the hook length of all boxes comprising the Young diagram R and its subduced Young diagram R_α respectively. An example of how hooks are computed is given in Figure 3.2. In this example, the number of boxes passed by the lines of the elbows represent the

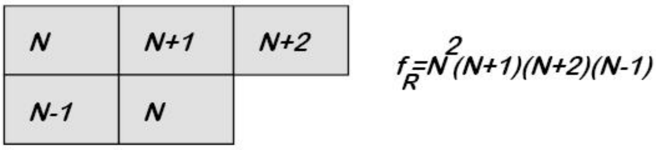


Figure 3.1: An example of the product of weights for a Young diagram R of 5 boxes.

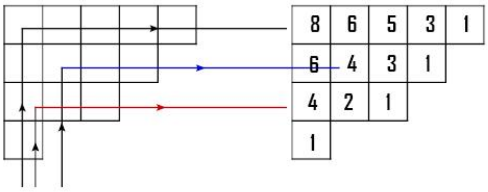


Figure 3.2: An example of how hook lengths are computed for a Young diagram R 13 boxes.

hook value of the box containing the apex of the elbow (see [110]).

For this particular example, we have $\text{hooks}_R = 8.6.6.5.4.4.3.3.2.1.1.1.1 = 414720$.
It is straightforward to find the two point functions of the restricted Schur polynomials (3.8) as a generalization of (3.11)

$$\left\langle \chi_{R,(r,s,t)}\chi^{\dagger}_{T,(u,v,w)} \right\rangle = \delta_{R,(r,s,t)T,(u,v,w)} \frac{\text{hooks}_R}{\text{hooks}_r \text{hooks}_s \text{hooks}_t} f_R. \tag{3.12}$$

3.5 Correlators for ABJM Theory

In this section we will study the two point correlation functions of the operators (3.10). These correlators provide an interesting generalization of the correlators considered for operators built from complex Higgs fields transforming in the adjoint of a $U(N)$ gauge theory. For correlators built from n Higgs fields transforming in the adjoint of a $U(N)$ gauge group one is able to reduce the computation of the correlator to the computation of a trace in $V^{\otimes n}$ where V is the carrier space of the fundamental representation of $U(N)$. In the present case, because we consider a theory with $U(N) \times U(N)$ gauge group, the computation of the correlator reduces to a product of two traces (one for each $U(N)$ factor in the gauge group) each of which run over $V^{\otimes n}$. We will explain how to explicitly compute this trace. The first result we obtain in this section is a general formula for the two point correlation function. We then consider the explicit evaluation of this general result in two special cases:

when $n_{12} = n_{21} = 0$ and when $n_2 = 0$. These special cases are simpler than the general result, and the case $n_2 = 0$ represents a class of operators that are closed under the action of the two loop dilatation operator. We will study the anomalous dimensions of these operators in a later section. The final result of this section is a general formula for two point functions. In appendix C we explain how this result generalizes to gauge groups with more factors.
As has become standard in computations of this type, we ignore spacetime dependence; it is uniquely determined in the final result by conformal invariance. Consequently, the two point functions of the Higgs fields that we use are

$$\left\langle A^a_{i\alpha} A^{\dagger\beta}_{jb} \right\rangle = \delta_{ij}\delta^a_b\delta^\beta_\alpha = \left\langle B^a_{i\alpha} B^{\dagger\beta}_{jb} \right\rangle.$$

In terms of these Higgs fields we can write our operator as

$$\begin{aligned} O_{R,\{r\}} &= \frac{1}{n_{11}!n_{12}!n_{21}!n_{22}!} \sum_{\sigma\in S_n} \mathrm{Tr}_{\{r\}}\left(\Gamma_R(\sigma)\right) \prod_{i=1}^{n_1} (A_1)^{a_i}_{\alpha_{\tau(i)}} \times \\ &\times \prod_{i=1+n_1}^{n} (A_2)^{a_j}_{\alpha_{\tau(j)}} \prod_{i=1}^{m_1} \left(B_1^\dagger\right)^{a_i}_{\alpha_{\tau(i)}} \prod_{j=1+m_1}^{n} \left(B_2^\dagger\right)^{a_j}_{\alpha_{\tau(j)}} \qquad (3.13) \\ &\equiv \mathrm{Tr}\left(P_{R,\{r\}} A_1^{\otimes n_{11}+n_{12}} A_2^{\otimes n_{21}+n_{22}} \tau \left(B_1^\dagger\right)^{\otimes n_{11}+n_{21}} \left(B_2^\dagger\right)^{\otimes n_{12}+n_{22}}\right), \end{aligned}$$

where

$$P_{R,\{r\}} = \frac{1}{n_{11}!n_{12}!n_{21}!n_{22}!} \sum_{\sigma\in S_n} \mathrm{Tr}_{\{r\}}\left(\Gamma_R(\sigma)\right).$$

In the last line of (3.13) we have switched to a trace within $V^{\otimes n}$. The above explicit formula spells out how we are filling the "slots" from 1 to n with the A_is and $B_j^\dagger$s. The operator τ dictates how the A_is and $B_j^\dagger$s are combined to produce ϕ_{ij}s. The specific τ we must choose to achieve a specific joining will not in general be unique. It would also be possible to replace τ by some more general element of the group algebra[5]. We will pursue this possibility below. The name "restricted Schur polynomial," regardless of the specific τ used in the construction, reflects that fact that for all of these operators the index structure associated with the $U(N)$ group on which the projector $P_{R,\{r\}}$ acts is organized using the symmetric group and its subgroups. It is now a simple exercise to show that

$$\left\langle O_{R,\{r\}} O^\dagger_{S,\{s\}} \right\rangle = \qquad (3.14)$$

$$\sum_{\psi\circ\lambda\in S_{n_1}\times S_{n_2}} \sum_{\mu\circ\nu\in S_{m_1}\times S_{m_2}} \mathrm{Tr}\left(P_{R,\{r\}}\psi\circ\lambda P_{S,\{s\}}\mu\circ\nu\right) \mathrm{Tr}\left(\tau^\dagger\psi^{-1}\circ\lambda^{-1}\tau\mu^{-1}\circ\nu^{-1}\right).$$

The sum over $\psi\circ\lambda$ sums all possible Wick contractions between the A_is and the sum over $\mu\circ\nu$ sums all possible Wick contractions between the B_is After making a convenient choice for τ we will show how to evaluate (3.14) in general.

[5]The n_{ij} continue to count the number of boxes in the Young diagrams r_{ij} , but no longer give the number of composite scalars $(\phi_{ij})^a_b$ from which the operator is built.

3.5.1 Number of A_is Equal Number of $B_i^\dagger$s; $n_{12} = n_{21}$=0

In this subsection we consider the case that $n_1 = m_1$, $n_2 = m_2$ and further that $n_{12} = 0 = n_{21}$. With this choice $\{r\} = \{r_{11}, r_{22}\}$. There are a number of nice simplifications that arise in this case. First, we may take τ to be the identity permutation. Secondly, both $P_{R,\{r\}}$ and $P_{S,\{s\}}$ commute with all elements of $S_{n_{11}} \times S_{n_{22}}$. Thus, the two point correlator becomes

$$\left\langle O_{R,\{r\}} O^\dagger_{S,\{s\}} \right\rangle =$$
$$\sum_{\psi\circ\lambda\in S_{n_{11}}\times S_{n_{22}}} \sum_{\mu\circ\nu\in S_{n_{11}}\times S_{n_{22}}} \mathrm{Tr}\left(P_{R,\{r\}}\psi\circ\lambda P_{S,\{s\}}\mu\circ\nu\right)\mathrm{Tr}\left(\psi^{-1}\circ\lambda^{-1}\mu^{-1}\circ\nu^{-1}\right)$$
$$= \sum_{\psi\circ\lambda\in S_{n_{11}}\times S_{n_{22}}} \sum_{\mu\circ\nu\in S_{n_{11}}\times S_{n_{22}}} \mathrm{Tr}\left(P_{R,\{r\}}P_{S,\{s\}}\psi\mu\circ\lambda\nu\right)\mathrm{Tr}\left((\psi\mu)^{-1}\circ(\lambda\nu)^{-1}\right)$$
$$= n_{11}!n_{22}! \sum_{\psi\circ\lambda\in S_{n_{11}}\times S_{n_{22}}} \mathrm{Tr}\left(P_{R,\{r\}}P_{S,\{s\}}\psi\circ\lambda\right)\mathrm{Tr}\left(\psi^{-1}\circ\lambda^{-1}\right).$$

Next we use the identity

$$P_{R,\{r\}}P_{S,\{s\}} = \delta_{RS}\delta_{\{r\}\{s\}}\frac{n!}{n_{11}!n_{22}!d_R}P_{R,\{r\}}$$

proved in [47], and the identity (in this next formula Tr_n denotes a trace over $V^{\otimes n}$, f_s is a product of the factors of Young diagram s and hooks_s is a product of the hook lengths of Young diagram s)

$$\mathrm{Tr}_n(\psi\circ\lambda) = \mathrm{Tr}_{n_{11}}(\psi)\,\mathrm{Tr}_{n_{22}}(\lambda) = \sum_{s\vdash n_{11}}\chi_s(\psi)\frac{f_s}{\mathrm{hooks}_s}\sum_{t\vdash n_{22}}\chi_t(\lambda)\frac{f_t}{\mathrm{hooks}_t}$$

which follows as a consequence of Schur-Weyl duality, to obtain

$$\left\langle O_{R,\{r\}} O^\dagger_{S,\{s\}} \right\rangle = \frac{n!}{d_R}\delta_{RS}\delta_{\{r\},\{s\}}\sum_{\psi\circ\lambda\in S_{n_{11}}\times S_{n_{22}}}\times$$
$$\times\sum_{u\vdash n_{11}}\chi_u(\psi)\frac{f_u}{\mathrm{hooks}_u}\sum_{t\vdash n_{22}}\chi_t(\lambda)\frac{f_t}{\mathrm{hooks}_t}\mathrm{Tr}\left(P_{R,\{r\}}\psi\circ\lambda\right).$$

To do this sum, note that

$$\sum_{\psi\in S_{n_{11}}}\chi_s(\psi)\,\psi = \frac{n_{11}!}{d_s}P_s \qquad \sum_{\lambda\in S_{n_{22}}}\chi_t(\lambda)\,\lambda = \frac{n_{22}!}{d_t}P_t$$

where P_s and P_t are correctly normalized projectors, projecting to the irrep s of $S_{n_{11}}$ and t of $S_{n_{22}}$ respectively. Thus,

$$\left\langle O_{R,\{r\}} O^\dagger_{S,\{s\}} \right\rangle = \sum_u\sum_t\frac{n!n_{11}!n_{22}!f_u f_t}{d_R d_u d_t \mathrm{hooks}_u \mathrm{hooks}_t}\delta_{RS}\delta_{\{r\},\{s\}}\mathrm{Tr}\left(P_{R,\{r\}}P_u P_t\right)$$
$$= \frac{n!n_{11}!n_{22}!f_{r_{11}}f_{r_{22}}}{d_R d_{r_{11}} d_{r_{22}}\mathrm{hooks}_{r_{11}}\mathrm{hooks}_{r_{22}}}\delta_{RS}\delta_{\{r\},\{s\}}\mathrm{Tr}\left(P_{R,\{r\}}\right)$$
$$= \delta_{RS}\delta_{\{r\},\{s\}}\frac{\mathrm{hooks}_R f_{r_{11}} f_{r_{22}} f_R}{\mathrm{hooks}_{r_{11}}\mathrm{hooks}_{r_{22}}}.$$

To obtain the final result we used the value of $\text{Tr}\left(P_{R,\{r\}}\right)$ which has been computed in [48]. The basic result of the subsection is

$$\left\langle O_{R,\{r\}} O^{\dagger}_{S,\{s\}} \right\rangle = \delta_{RS}\delta_{\{r\},\{s\}} \frac{\text{hooks}_R f_{r_{11}} f_{r_{22}} f_R}{\text{hooks}_{r_{11}} \text{hooks}_{r_{22}}}. \tag{3.15}$$

3.5.2 $n_2 = 0$, $n_1 = m_1 + m_2$

With this choice $\{r\} = \{r_{11}, r_{22}\}$. There are again a number of nice simplifications that arise in this case. First, we may again take τ to be the identity permutation. Secondly, both $P_{R,\{r\}}$ and $P_{S,\{s\}}$ commute with all elements of $S_{n_{11}} \times S_{n_{12}}$. Thus, the two point correlator becomes

$$\begin{aligned}
\left\langle O_{R,\{r\}} O^{\dagger}_{S,\{s\}} \right\rangle =& \sum_{\sigma\in S_n} \sum_{\rho \in S_{m_1}\times S_{m_2}} \text{Tr}\left(P_{R,\{r\}}\sigma P_{S,\{s\}}\rho\right) \text{Tr}\left(\sigma^{-1}\rho^{-1}\right) \\
=& m_1! m_2! \sum_{\sigma\in S_n} \text{Tr}\left(P_{R,\{r\}} P_{S,\{s\}}\sigma\right) \text{Tr}\left(\sigma^{-1}\right) \\
=& \frac{n!}{d_R}\delta_{RS}\delta_{\{r\},\{s\}} \sum_{\sigma\in S_n} \text{Tr}\left(P_{R,\{r\}}\sigma\right) \text{Tr}\left(\sigma^{-1}\right) \\
=& \frac{n!}{d_R}\delta_{RS}\delta_{\{r\},\{s\}} \sum_{T\vdash n} \frac{f_T}{\text{hooks}_T} \sum_{\sigma\in S_n} \text{Tr}\left(P_{R,\{r\}}\sigma\right) \chi_T\left(\sigma^{-1}\right) \\
=& \frac{n!}{d_R}\delta_{RS}\delta_{\{r\},\{s\}} \sum_{T\vdash n} \frac{f_T}{\text{hooks}_T} \frac{n!}{d_T} \text{Tr}\left(P_{R,\{r\}} P_T\right) \\
=& \frac{f_R n!}{d_R}\delta_{RS}\delta_{\{r\},\{s\}} \text{Tr}\left(P_{R,\{r\}}\right) \\
=& \frac{\left(f_R\right)^2 n!}{d_R \text{hooks}_{r_{11}} \text{hooks}_{r_{12}}} \delta_{RS}\delta_{\{r\},\{s\}}.
\end{aligned}$$

The basic result of the subsection is

$$\left\langle O_{R,\{r\}} O^{\dagger}_{S,\{s\}} \right\rangle = \delta_{RS}\delta_{\{r\},\{s\}} \frac{\text{hooks}_R \left(f_R\right)^2}{\text{hooks}_{r_{11}} \text{hooks}_{r_{12}}}. \tag{3.16}$$

3.5.3 General Case

In this section we will consider general n_{ij} . We will find it useful to allow τ to be a general element of the group algebra. We will find it convenient to distribute the Higgs fields in the slots as follows

$$\begin{aligned}
O_{R,\{r\}} = & \frac{1}{n_{11}! n_{22}! n_{12}! n_{21}!} \sum_{\sigma\in S_n} \text{Tr}_{\{r\}}\left(\Gamma_R(\sigma)\right) \prod_{i=1}^{n_1} \left(A_1\right)^{a_i}_{\alpha_i} \prod_{j=1+n_1}^{n} \left(A_2\right)^{a_j}_{\alpha_j} (\tau)^{\alpha_1\ldots\alpha_n}_{\beta_1\ldots\beta_n} \times \\
& \times \prod_{i=1}^{n_{11}} \left(B_1^{\dagger}\right)^{\beta_i}_{a_{\sigma(i)}} \prod_{i=1+n_{11}}^{n_1} \left(B_2^{\dagger}\right)^{\beta_i}_{a_{\sigma(i)}} \prod_{i=1+n_1}^{n_1+n_{21}} \left(B_1^{\dagger}\right)^{\beta_i}_{a_{\sigma(i)}} \prod_{i=1+n_1+n_{21}}^{n} \left(B_2^{\dagger}\right)^{\beta_i}_{a_{\sigma(i)}}.
\end{aligned} \tag{3.17}$$

Compare to (3.13) and notice that this is not the same distribution of the Higgs fields. We will summarize this as

$$\mathcal{O}_{R,\{r\}} = \mathrm{Tr}\left(P_{R,\{r\}} A^{\otimes n} \tau B^{\otimes n}\right)$$

where for simplicity, our notation does not spell out which fields inhabit which slots. Standard manipulations give (this assumes a Hermittian τ which is the case we consider below)

$$\left\langle \mathcal{O}_{R,\{r\}} \mathcal{O}^{\dagger}_{S,\{s\}} \right\rangle = \sum_{\rho \in S_{m_1} \times S_{m_2}} \sum_{\sigma \in S_{n_1} \times S_{n_2}} \mathrm{Tr}\left(P_{R,\{r\}} \sigma P_{S,\{s\}} \rho\right) \mathrm{Tr}\left(\tau \rho^{-1} \sigma^{-1}\right).$$

To see how the subgroups are embedded in S_n, note that $S_{n_1} \times S_{n_2}$ acts on slots occupied by the As and $S_{m_1} \times S_{m_2}$ acts on slots occupied by $B^{\dagger}$s. The formula (3.17) clearly states how the slots are populated. Note that on the right hand side there are two traces and the sums to be performed have one element of the symmetric group in one trace and the inverse of this in the second trace. The corresponding computation for gauge group $U(N)$ has one trace and both an element of the symmetric group and its inverse, in the same trace. This is a key observation that motivates what follows.

We could reduce the above result to the corresponding result obtained for a $U(N)$ gauge group if we choose τ so that

$$\mathrm{Tr}\left(\tau \rho^{-1} \sigma^{-1}\right) = \delta\left(\rho^{-1} \sigma^{-1}\right). \tag{3.18}$$

Define $(R \vdash n)$

$$\Phi_R = \frac{d_R}{n! f_R} \sum_{\sigma \in S_n} \chi_R\left(\sigma^{-1}\right) \sigma.$$

A rather straight forward computation now gives

$$\mathrm{Tr}\left(\Phi_R \psi\right) = \frac{d_R}{n!} \chi_R(\psi)$$

where the trace is over $V^{\otimes n}$. Recalling that the delta function on the symmetric group is

$$\delta(\sigma) = \frac{1}{n!} \sum_R d_R \chi_R(\sigma)$$

we have

$$\mathrm{Tr}\left(\sum_{R \vdash n} \Phi_R \psi\right) = \delta(\psi).$$

This motivates the choice

$$\tau = \sum_R \frac{d_R}{n! \sqrt{f_R}} \sum_{\sigma \in S_n} \chi_R\left(\sigma^{-1}\right) \sigma. \tag{3.19}$$

With this choice (3.18) holds so that

$$\left\langle \mathcal{O}_{R,\{r\}} \mathcal{O}^{\dagger}_{S,\{s\}} \right\rangle = \sum_{\sigma \in S_{n_1} \times S_{n_2} \cap S_{m_1} \times S_{m_2}} \mathrm{Tr}\left(P_{R,\{r\}} \sigma P_{S,\{s\}} \sigma^{-1} \right) .$$

Notice that $S_{n_1} \times S_{n_2} \cap S_{m_1} \times S_{m_2} = S_{n_{11}} \times S_{n_{12}} \times S_{n_{21}} \times S_{n_{22}}$. Thus, σ commutes with the projectors in the last equation. After summing over σ we have

$$\left\langle \mathcal{O}_{R,\{r\}} \mathcal{O}^{\dagger}_{S,\{s\}} \right\rangle = n_{11}! n_{12}! n_{21}! n_{22}! \mathrm{Tr}\left(P_{R,\{r\}} P_{S,\{s\}} \right) .$$

A straight forward application of the results of [47, 48] now gives

$$\left\langle \mathcal{O}_{R,\{r\}} \mathcal{O}^{\dagger}_{S,\{s\}} \right\rangle = \delta_{RS} \delta_{\{r\},\{s\}} \frac{\mathrm{hooks}_R f_R}{\mathrm{hooks}_{r_{11}} \mathrm{hooks}_{r_{12}} \mathrm{hooks}_{r_{21}} \mathrm{hooks}_{r_{22}}} . \qquad (3.20)$$

This clearly shows that our operators diagonalize the two point function in the subspace of operators with fixed n_{ij} . However, even after fixing n_i, m_i, we can still change the n_{ij}, by changing the way we populate the slots with the $B^{\dagger}$s which corresponds to changing the way that we embed $S_{m_1} \times S_{m_2}$ in S_n. Projectors corresponding to different n_{ij} will not in general be orthogonal. However, in this case (3.18) is never satisfied so that the operators continue to be orthogonal.

Chapter 4

Nonplanar Integrability in $\mathcal{N} = 4$ SYM Theory

In this chapter, we study the action of the nonplanar dilatation operator of $\mathcal{N} = 4$ SYM theory. The gauge invariant operators we consider were built from restricted Schur polynomials. We first study the $SU(2)$ sector of the theory where restricted Schur polynomials built using two scalar fields Z and Y are considered. We then investigate the spectrum of the dilatation operator beyond the $SU(2)$ sector [93] where restricted Schur polynomials are generalized to include three scalar fields Z, Y and X. We conclude this chapter by studying the two loop dilatation operator in the $SU(2)$ sector of the theory [94].

4.1 One Loop $SU(2)$ Sector

The one loop dilatation operator in the $SU(2)$ sector is [51]

$$D = -g_{YM}\mathrm{Tr}\,[Y,\, Z]\,[\partial_Y,\, \partial_Z]$$

We will compute the action of this operator on restricted Schur polynomials (3.4). We will focus on restricted Schur polynomials that have only 2 rows or columns. In this case, the multiplicity labels α, β of the restricted Schur polynomial $\chi_{R,(r,s)\alpha\beta}$ are not needed because (r, s) is subduced once from R. A simple calculation gives [52, 53]

$$\begin{aligned}
&D\chi_{R,(r,s)}\left(Z^{\otimes n}, Y^{\otimes m}\right) = \\
&\frac{g_{YM}^2}{(n-1)!\,(m-1)!}\sum_{\psi\in S_{n+m}}\mathrm{Tr}_{(r,s)}\left(\Gamma_R\left((n,n+1)\,\psi - \psi\,(n,n+1)\right)\right)\times \\
&\times Z^{i_1}_{i_{\psi(1)}}\cdots Z^{i_{n-1}}_{i_{\psi(n-1)}}\,(YZ-ZY)^{i_n}_{i_{\psi(n)}}\,\delta^{i_{n+1}}_{i_{\psi(n+1)}}\,Y^{i_{n+2}}_{i_{\psi(n+2)}}\cdots Y^{i_{n+m}}_{i_{\psi(n+m)}}.
\end{aligned} \tag{4.1}$$

Due to the delta $\delta^{i_{n+1}}_{i_{\psi(n+1)}}$, the sum runs only over permutations that obey $\psi(n+1) = (n+1)$. This sum is performed by writing the sum over S_{n+m} in terms of a sum over the S_{n+m-1} subgroup (defined by keeping permutations that satisfy $\psi(n+1) = (n+1)$) and its cosets. Using the reduction rule for Schur polynomials [55], we get

$$D\chi_{R,(r,s)} = \frac{g_{YM}^2}{(n-1)!\,(m-1)!}\times$$

$$\times \sum_{\psi\in S_{n+m-1}} \sum_{R'} c_{RR'} \mathrm{Tr}_{(r,s)} \left(\Gamma_R\left(n,n+1\right)\Gamma_{R'}\left(\psi\right) - \Gamma_{R'}\left(\psi\right)\Gamma_R\left(n,n+1\right)\right)\times$$
$$\times Z^{i_1}_{i_{\psi(1)}}...Z^{i_{n-1}}_{i_{\psi(n-1)}}\left(YZ-ZY\right)^{i_n}_{i_{\psi(n)}} Y^{i_{n+2}}_{i_{\psi(n+2)}}...Y^{i_{n+m}}_{i_{\psi(n+m)}},$$

where R' is subduced from R by pulling off one box. $c_{RR'}$ is the weight of the removed box[6]. Now one can write the restricted character introduced in (3.5) as

$$\chi_{R,(r,s)}\left(\sigma\right) = \mathrm{Tr}_{(r,s)}\left(\Gamma_R\left(\sigma\right)\right) = \mathrm{Tr}\left(P_{R\to(r,s)}\Gamma_R\left(\sigma\right)\right),$$

where $P_{R\to(r,s)}$ is a projector from the carrier space of R to the carrier space of (r,s). Using the identity

$$Z^{i_1}_{i_{\psi(1)}}...Z^{i_{n-1}}_{i_{\psi(n-1)}}\left(YZ-ZY\right)^{i_n}_{i_{\psi(n)}} Y^{i_{n+2}}_{i_{\psi(n+2)}}...Y^{i_{n+m}}_{i_{\psi(n+m)}} =$$
$$\mathrm{Tr}\left(\left(\left(n,n+1\right)\psi - \psi\left(n,n+1\right)\right)Z^{\otimes n}Y^{\otimes m}\right)$$

where

$$\mathrm{Tr}\left(\sigma Z^{\otimes n}Y^{\otimes m}\right) = Z^{i_1}_{i_{\sigma(1)}}...Z^{i_n}_{i_{\sigma(n)}}Y^{i_{n+1}}_{i_{\sigma(n+1)}}...Y^{i_{n+m}}_{i_{\sigma(n+m)}},$$

together with the identity (proved in [46])

$$\mathrm{Tr}\left(\sigma Z^{\otimes n}Y^{\otimes m}\right) = \sum_{T,(t,u)} \frac{d_T n! m!}{d_t d_u\left(n+m\right)!}\chi_{T,(t,u)}\left(\sigma^{-1}\right)\chi_{T,(t,u)}\left(Z,Y\right) \qquad (4.2)$$

we get

$$D\chi_{R,(r,s)}\left(Z,Y\right) = \sum_{T,(t,u)} M_{R,(r,s);T,(t,u)}\chi_{T,(t,u)}\left(Z,Y\right),$$

$$M_{R,(r,s);T,(t,u)} = g_{YM}^2 \sum_{\psi\in S_{n+m-1}}\sum_{R'} \frac{c_{RR'}d_T nm}{d_t d_u\left(n+m\right)!}\times$$
$$\times \mathrm{Tr}_{(r,s)}\left(\Gamma_R\left(n,n+1\right)\Gamma_{R'}\left(\psi\right) - \Gamma_{R'}\left(\psi\right)\Gamma_R\left(n,n+1\right)\right)\times$$
$$\times \chi_{T,(t,u)}\left(\left(n,n+1\right)\psi - \psi\left(n,n+1\right)\right).$$

From the fundamental orthogonality relation, the sum over ψ gives

$$M_{R,(r,s);T,(t,u)} = 2g_{YM}^2\sum_{R'}\frac{c_{RR'}d_T nm}{d_{R'}d_t d_u\left(n+m\right)}\times$$

[6] The box removed from row i and column j has a weight $N-i+j$.

$$\times \text{Tr}_{(r,s)} \left(\left[\Gamma_R (n, n+1) \,, \, P_{R \to (r,s)} \right] I_{R'T'} \left[\Gamma_T (n, n+1) \,, \, P_{T \to (t,u)} \right] I_{T'R'} \right) . \quad (4.3)$$

In this equation $I_{R'T'}$ and $I_{T'R'}$ are intertwiners defined in Appendix B of [52]. The job of the intertwiner is to glue the carrier space of irrep R' with the carrier space of irrep T'.

The result (4.3) is an exact expression for the one loop dilatation operator, i.e. it is correct to all orders in $\frac{1}{N}$.

This expression can be written in terms of the normalized operators, whose form is most easily obtained from the two point function of restricted Schur polynomials (3.11), that is

$$\chi_{R,(r,s)} (Z, Y) = \sqrt{\frac{f_R \text{hooks}_R}{\text{hooks}_r \text{hooks}_s}} O_{R,(r,s)} (Z, Y) \,.$$

The result is

$$DO_{R,(r,s)} (Z, Y) = \sum_{T,(t,u)} N_{R,(r,s);T,(t,u)} O_{T,(t,u)} (Z, Y)$$

$$N_{R,(r,s);T,(t,u)} = 2g_{YM}^2 \sum_{R'} \frac{c_{RR'} d_T nm}{d_{R'} d_t d_u (n+m)} \sqrt{\frac{f_T \text{hooks}_T \text{hooks}_r \text{hooks}_s}{f_R \text{hooks}_R \text{hooks}_t \text{hooks}_u}} \times$$

$$\times \text{Tr}_{(r,s)} \left(\left[\Gamma_R (n, n+1) \,, \, P_{R \to (r,s)} \right] I_{R'T'} \left[\Gamma_T (n, n+1) \,, \, P_{T \to (t,u)} \right] I_{T'R'} \right) .$$

The spectrum of the dilatation operator was studied numerically in [52]. In [53], the spectrum was studied analatically using the $SU(2)$ representation theory. Both these studies have proved that the spectrum of one loop dilatation operator in the $SU(2)$ sector at large N is reduced to the spectrum of decoupled harmonic oscillators. This signals integrability of $\mathcal{N} = 4$ SYM theory in the $SU(2)$ sector. In the next subsection, we give a detailed analysis of the spectrum of the one loop dilatation operator, beyond the $SU(2)$ sector of the theory.

4.2 One loop beyond the $SU(2)$ sector

4.2.1 Action of the Dilatation Operator

In this subsection we will study the action of the one loop dilatation operator on restricted Schur polynomials built using three complex adjoint scalars. The main result of this section, which generalizes results known for the $SU(2)$ sector[52], is the simple formula (4.4) for the action of the dilatation operator.

Our operators are built using the six scalar fields ϕ^i, which take values in the adjoint of $u(N)$ in $\mathcal{N} = 4$ super Yang Mills theory. Assemble these scalars into the three complex combinations

$$Z = \phi_1 + i\phi_2, \qquad Y = \phi_3 + i\phi_4, \qquad X = \phi_5 + i\phi_6 \,.$$

The operators we consider are built using $O(N)$ of these complex scalar fields. These operators have a large $\mathcal{R}$-charge and consequently, non-planar contributions to the correlation functions of these operators are not suppressed at large N[64]. The computation of the anomalous dimensions of these operators is then a problem of considerable complexity. This problem has been effectively handled by new methods which employ group representation theory[24, 58, 44, 48, 59, 25, 49, 27, 46, 60, 61, 62, 63] allowing one to sum all diagrams (planar and non-planar) contributing. Indeed, the two point function of restricted Schur polynomials[44, 48, 59, 49] can be evaluated exactly in the free field theory limit[47]. The restricted Schur polynomials provide a basis for the local operators[46] which diagonalize the free two point function and which have highly constrained mixing at the quantum level[59, 49, 65, 52, 53]. For the applications that we have in mind, this basis is clearly far superior to the trace basis. Mixing between operators in the trace basis with this large $\mathcal{R}$-charge is completely unconstrained even at the level of the free theory.

The restricted Schur polynomials are

$$\chi_{R,(r,s,t)}(Z^{\otimes n}, Y^{\otimes m}, X^{\otimes p}) = \frac{1}{n!m!p!} \sum_{\sigma \in S_{n+m+p}} \mathrm{Tr}_{(r,s,t)}(\Gamma_R(\sigma)) X^{i_1}_{i_{\sigma(1)}} \cdots X^{i_p}_{i_{\sigma(p)}} \times$$
$$\times Y^{i_{p+1}}_{i_{\sigma(p+1)}} \cdots Y^{i_{p+m}}_{i_{\sigma(p+m)}} Z^{i_{p+m+1}}_{i_{\sigma(p+m+1)}} \cdots Z^{i_{n+m+p}}_{i_{\sigma(n+m+p)}} .$$

We use n to denote the number of Zs, m to denote the number of Ys and p to denote the number of Xs. R is a Young diagram with $n+m+p$ boxes or equivalently an irreducible representation of S_{n+m+p}. r is a Young diagram with n boxes or equivalently an irreducible representation of S_n, s is a Young diagram with m boxes or equivalently an irreducible representation of S_m and t is a Young diagram with p boxes or equivalently an irreducible representation of S_p. The S_n subgroup acts on $m+p+1, m+p+2, ..., m+p+n$ and therefore permutes indices belonging to the Zs. The S_m subgroup acts on $p+1, p+2, ..., p+m$ and hence permutes indices belonging to the Ys. The S_p subgroup acts on $1, 2, ..., p$ and hence permutes indices belonging to the Xs. Taken together (r, s, t) specify an irreducible representation of $S_n \times S_m \times S_p$. $\mathrm{Tr}_{(r,s,t)}$ is an instruction to trace over the subspace carrying the irreducible representation[7] (r, s, t) of $S_n \times S_m \times S_p$ inside the carrier space for irreducible representation R of S_{n+m+p}. This trace is easily realized by including a projector $P_{R\to(r,s,t)}$ (from the carrier space of R to the carrier space of (r, s, t)) and tracing over all of R, i.e. $\mathrm{Tr}_{(r,s,t)}(\Gamma_R(\sigma)) = \mathrm{Tr}(P_{R\to(r,s,t)}\Gamma_R(\sigma))$.

The one loop dilatation operator, when acting on operators composed from the three complex scalars X, Y, Z, is [66, 67, 68, 69, 32, 70]

$$D = -g^2_{\mathrm{YM}} \mathrm{Tr}\left[Y, Z\right]\left[\partial_Y, \partial_Z\right] - g^2_{\mathrm{YM}} \mathrm{Tr}\left[X, Z\right]\left[\partial_X, \partial_Z\right] - g^2_{\mathrm{YM}} \mathrm{Tr}\left[Y, X\right]\left[\partial_Y, \partial_X\right].$$

[7] In general, because (r, s, t) can be subduced more than once, we should include a multiplicity index. We will not write or need this index here. We will, in the next section, restrict our attention to restricted Schur polynomials that are labeled by Young diagrams with two rows or columns. A huge simplification that results is that all possible representations (r, s, t) are subduced exactly once.

The action of the dilatation operator on the restricted Schur polynomials belonging to the $SU(2)$ sector has been worked out in [65, 52]. In what follows, we will work with operators normalized to give a unit two point function. The normalized operators $O_{R,(r,s,t)}(Z,Y)$ can be obtained from

$$\chi_{R,(r,s,t)}(Z,Y,X) = \sqrt{\frac{f_R \,\mathrm{hooks}_R}{\mathrm{hooks}_r \,\mathrm{hooks}_s \,\mathrm{hooks}_t}} O_{R,(r,s,t)}(Z,Y,X)\,.$$

The computation of the dilatation operator is a simple extension of the analysis presented in the previous section so that we will only quote the final result. In terms of the normalized operators

$$DO_{R,(r,s,t)}(Z,Y,X) = \sum_{T,(u,v,w)} N_{R,(r,s,t);T,(u,v,w)} O_{T,(u,v,w)}(Z,Y,X) \tag{4.4}$$

$$\begin{aligned}
N_{R,(r,s,t);T,(u,v,w)} = \\
&-\sum_{R'} \frac{2g_{YM}^2 c_{RR'} d_T n}{d_{R'} d_u d_v d_w \,(n+m+p)} \sqrt{\frac{f_T \mathrm{hooks}_T \mathrm{hooks}_r \mathrm{hooks}_s \mathrm{hooks}_t}{f_R \mathrm{hooks}_R \mathrm{hooks}_u \mathrm{hooks}_v \mathrm{hooks}_w}} \\
&\times \Big[nm \mathrm{Tr}\Big(\left[\Gamma_R\left((p+m+1,p+1)\right), P_{R\to(r,s,t)}\right] I_{R'T'} \times \\
&\times \left[\Gamma_T\left((p+m+1,p+1)\right), P_{T\to(r,s,t)}\right] I_{T'R'} \Big) \\
&+ np \mathrm{Tr}\Big(\left[\Gamma_R\left((1,p+m+1)\right), P_{R\to(r,s,t)}\right] I_{R'T'} \times \\
&\times \left[\Gamma_T\left((1,p+m+1)\right), P_{T\to(r,s,t)}\right] I_{T'R'} \Big) \\
&+ mp \mathrm{Tr}\Big(\left[\Gamma_R\left((1,p+1)\right), P_{R\to(r,s,t)}\right] I_{R'T'} \times \\
&\times \left[\Gamma_T\left((1,p+1)\right), P_{T\to(r,s,t)}\right] I_{T'R'} \Big) \Big].
\end{aligned} \tag{4.5}$$

$c_{RR'}$ is the factor of the corner box removed from Young diagram R to obtain diagram R', and similarly T' is a Young diagram obtained from T by removing a box. This factor arises after using the reduction rule of [71, 48]. The intertwiner I_{AB} is a map from the carrier space of irreducible representation A to the carrier space of irreducibe representation B. Consequently, by Schur's Lemma, A and B must be Young diagrams of the same shape. The intertwiner operators relevant for our study have been discussed in detail in [52].

4.2.2 Projection Operators

The goal of this subsection is to construct the projection operators needed to define the restricted Schur polynomials we study. This construction clearly defines the class of operators being considered. The approximations being employed in this construction are carefully considered.

The class of operators $\chi_{R,(r,s,t)}(Z,Y,X)$ we will study are labeled by Young diagrams that each have 2 rows or columns. We further take n to be order N and m,p to be αN with $\alpha \ll 1$. Thus, there are a lot more Z fields than there are Ys or Xs. The mixing of these operators with restricted Schur polynomials that have $n \neq 2$ rows or columns (or of even more general shape) is suppressed at least by a factor of order $\frac{1}{\sqrt{N}}$[8]. Thus, at large N the 2 row or column restricted Schur polynomials do not mix with other operators, which is a huge simplification. This is the analog of the statement that for operators with a dimension of $O(1)$, different trace structures do not mix at large N. The fact that the two column restricted Schur polynomials are a decoupled sector at large N is expected: these operators correspond to a well defined stable semi-classical object in spacetime (the two giant graviton system) [52].

Note that as a consequence of the fact that there are a lot more Zs than Ys and Xs, contributions to the dilatation operators coming from interactions between Zs and Ys or between Zs and Xs will over power the contribution coming from interactions between Xs and Ys. Consequently we can simplify the action of the dilatation operator to

$$\begin{aligned}
N_{R,(r,s,t);T,(u,v,w)} = \\
&-\sum_{R'} \frac{2g_{YM}^2 c_{RR'} d_T n}{d_{R'} d_u d_v d_w (n+m+p)} \sqrt{\frac{f_T \text{hooks}_T \text{hooks}_r \text{hooks}_s \text{hooks}_t}{f_R \text{hooks}_R \text{hooks}_u \text{hooks}_v \text{hooks}_w}} \\
&\times \Big[nm \text{Tr}\Big(\left[\Gamma_R\left((p+m+1,p+1)\right), P_{R\to(r,s,t)}\right] I_{R'T'} \times \\
&\times \left[\Gamma_T\left((p+m+1,p+1)\right), P_{T\to(r,s,t)}\right] I_{T'R'} \Big) \\
&+ np \text{Tr}\Big(\left[\Gamma_R\left((1,p+m+1)\right), P_{R\to(r,s,t)}\right] I_{R'T'} \times \\
&\times \left[\Gamma_T\left((1,p+m+1)\right), P_{T\to(r,s,t)}\right] I_{T'R'} \Big). \qquad (4.6)
\end{aligned}$$

Two Rows

We will make use of Young's orthogonal representation for the symmetric group. This representation is most easily defined by considering the action of adjacent permutations (permutations of the form $(i,i+1)$) on the Young-Yamonouchi states. The permutation $(i,i+1)$ when acting on any given Young-Yamonouchi state will produce a linear combination of the original state and the state obtained by swapping the positions of i and $i+1$ in the Young-Yamonouchi symbol. The precise rule is most easily written in terms of the axial distance between i and $i+1$. If i appears in row r_i and column c_i of the Young-Yamonouchi symbol and $i+1$ appears in row r_{i+1} and column c_{i+1} of the Young-Yamonouchi symbol, then the axial distance between i and $i+1$ is

$$d_{i,i+1} = c_i - r_i - (c_{i+1} - r_{i+1}).$$

[8]Here we are talking about mixing at the quantum level. There is no mixing in the free theory[47].

In terms of this axial distance, the action of $(i,i+1)$ is

$$(i,i+1)\,|\text{state}\rangle = \frac{1}{d_{i,i+1}}\,|\text{state}\rangle + \sqrt{1-\frac{1}{d_{i,i+1}^2}}\,|\text{swapped state}\rangle$$

where the Young-Yamonouchi symbol of $|\text{swapped state}\rangle$ state is obtained from the Young-Yamonouchi symbol of $|\text{state}\rangle$ by swapping the positions of i and $i+1$. See [72] for more details.

The reason why we use Young's orthogonal representation is that it simplifies dramatically for the operators we are interested in. To construct the projectors $P_{R\to(r,s,t)}$ we will imagine that we start by removing $m+p$ boxes from R to produce r. We label the boxes in the order that they are removed. Of course, after each box is removed we are left with a valid Young diagram; this is a nontrivial constraint on the allowed numberings. Thus, after labeling these boxes we have a total of 2^{m+p} partially labeled Young diagrams, each corresponding to a subspace r of the subgroup $S_n\times(S_1)^{m+p}$ of the original S_{n+m+p} group. We now need to take linear combinations of these subspaces in such a way that we obtain the correct irreducible representation (s,t) of the $S_m\times S_p$ subgroup that acts on the labeled boxes. For the class of operators that we consider, the number of boxes that we remove $(=m+p)$ is much less that the number of boxes in R $(=m+n+p\approx n)$. In the Figure below we show R and the boxes that must be removed from R to obtain r. It is clear that the axial distance $d_{i,i+1}$ is 1 if the boxes are in the same row so that

$$(i,i+1)\,|\text{state}\rangle = |\text{state}\rangle \qquad \text{for boxes in the same row}\,.$$

It is also clear that $d_{i,i+1}$ is $O(N)$ for boxes in different rows. At large N we can simply set $(d_{i,i+1})^{-1}=0$ so that

$$(i,i+1)\,|\text{state}\rangle = |\text{swapped state}\rangle \qquad \text{for boxes in different rows}\,.$$

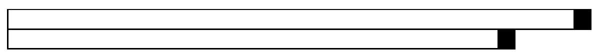
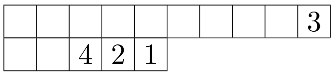
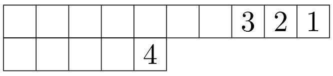


Figure 4.1: Shown above is the Young diagram R. The boxes that are to be removed from R to obtain r are colored black.

The representation that we have obtained is very similar to a representation which has already been studied in the mathematics literature [73]. Motivated by this background define a map from a labeled Young diagram to a monomial. Our Young diagram has $m+p$ boxes labeled and the labels are distributed between the upper and lower rows. Ignore the boxes that appear in the lower row. For boxes labeled i in the upper row include a factor of x_i in the monomial if $1 \le i \le p$ and a factor of y_i if $p+1 \le i \le p+m$. If none of the boxes in the first row are labeled, the Young diagram maps to 1. Thus, for example, when $m = 2$ and $p = 2$



$\leftrightarrow y_3$



$\leftrightarrow x_1 x_2 y_3$

The symmetric group acts by permuting the labels on the factors in the monomial. Thus, for example, $(12)x_1 y_3 = x_2 y_3$. This defines a reducible representation of the group $S_m \times S_p$. It is clear that the operators[9]

$$d_1 = \sum_{i=1}^{p} \frac{\partial}{\partial x_i} \quad \text{and} \quad d_2 = \sum_{i=p+1}^{p+m} \frac{\partial}{\partial y_i} \tag{4.7}$$

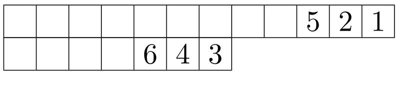
commute with the action of the $S_m \times S_p$ subgroup. These operators generalize closely related operators introduced by Dunkl in his study of intertwining functions [74]. They act on the monomials by producing the sum of terms that can be produced by dropping one x factor for d_1 or one y factor for d_2 at a time. For example

$$d_1(x_1 x_2 y_3) = x_2 y_3 + x_1 y_3, \qquad d_2(x_1 x_2 y_3) = x_1 x_2 \,.$$

The adjoint[10] produces the sum of monomials that can be obtained by appending a factor, without repeating any of the factors (this is written for $m = 2 = p$ impurities but the generalization to any m is obvious)

$$d_1^\dagger(y_3) = x_1 y_3 + x_2 y_3, \qquad d_1^\dagger(x_1 y_3) = x_1 x_2 y_3, \qquad d_2^\dagger(x_1 y_3) = x_1 y_3 y_4 \,.$$

The fact that d_1 and d_2 commute with all elements of $S_m \times S_p$, implies that $d_1^\dagger$ and $d_2^\dagger$ will too. Thus, $d_1^\dagger d_1$ and $d_2^\dagger d_2$ will also commute with all the elements of the $S_m \times S_p$ subgroup and consequently their eigenspaces will furnish representations of the subgroup. These eigenspaces are irreducible representations - consult [73] for useful details and results. This last fact implies that the problem of computing the projectors needed to define the restricted Schur polynomials can be replaced by the problem of constructing projectors onto the eigenspaces of $d_1^\dagger d_1$ and $d_2^\dagger d_2$. This amounts to solving for the eigenvectors and eigenvalues of $d_1^\dagger d_1$ and $d_2^\dagger d_2$. This problem is most easily solved by mapping the labeled Young diagrams into states of a spin chain. The spin at site i can be in state spin up $(+\frac{1}{2})$ or state spin down $(-\frac{1}{2})$. The spin chain has $m + p$ sites and the box labeled i tells us the state of site i. If box i appears in the first row, site i is in state $+\frac{1}{2}$; if it appears in the second row site i is in state $-\frac{1}{2}$. For example,



$$\leftrightarrow \left| \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right\rangle$$

[9]It may be helpful (and it is accurate) for the reader to associate the x_i, y_j of these operators with the $X^i_{\sigma(i)}, Y^j_{\sigma(j)}$ appearing in the definition of the restricted Schur polynomials.

[10]Consult Appendix E for details on the inner product on the space of monomials.

Both $d_1^\dagger d_1$ and $d_2^\dagger d_2$ have a very simple action on this spin chain: Introduce the states

$$\left|\frac{1}{2}\right\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \left|-\frac{1}{2}\right\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

for the possible states of each site and the operators

$$\sigma^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \sigma^- = (\sigma^+)^\dagger = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

which act on these states

$$\sigma^+ \left|-\frac{1}{2}\right\rangle = \left|\frac{1}{2}\right\rangle, \qquad \sigma^+ \left|\frac{1}{2}\right\rangle = 0, \qquad \sigma^- \left|\frac{1}{2}\right\rangle = \left|-\frac{1}{2}\right\rangle, \qquad \sigma^- \left|-\frac{1}{2}\right\rangle = 0\,.$$

We can write any of the states of the spin chain as a tensor product of the states $|\frac{1}{2}\rangle$ and $|-\frac{1}{2}\rangle$. For example

$$\left|-\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle = \left|-\frac{1}{2}\right\rangle \otimes \left|-\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}\right\rangle \otimes \left|-\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}\right\rangle$$

for a system with 6 lattice sites. Label the sites starting from the left, as site 1, then site 2 and so on till we get to the last site, which is site 6. The operator σ^- acting at the third site (for example) is

$$\sigma_3^- = 1 \otimes 1 \otimes \sigma^- \otimes 1 \otimes 1 \otimes 1\,.$$

We can then write

$$d_1^\dagger d_1 = \sum_{\alpha=1}^{p} \sum_{\beta=1}^{p} \sigma_\alpha^+ \sigma_\beta^-\,, \tag{4.8}$$

$$d_2^\dagger d_2 = \sum_{\alpha=p+1}^{p+m} \sum_{\beta=p+1}^{p+m} \sigma_\alpha^+ \sigma_\beta^- \tag{4.9}$$

This is a long ranged spin chain. In terms of the Pauli matrices

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

we define the following "total spins" of the system

$$J^1 = \sum_{\alpha=1}^{p} \frac{1}{2}\sigma_\alpha^1\,, \quad J^2 = \sum_{\alpha=1}^{p} \frac{1}{2}\sigma_\alpha^2\,, \quad J^3 = \sum_{\alpha=1}^{p} \frac{1}{2}\sigma_\alpha^3\,,$$

$$\mathbf{J}^2 = J^1J^1 + J^2J^2 + J^3J^3\,,$$

and

$$K^1 = \sum_{\alpha=p+1}^{p+m} \frac{1}{2}\sigma^1_\alpha\,, \quad K^2 = \sum_{\alpha=p+1}^{p+m} \frac{1}{2}\sigma^2_\alpha\,, \quad K^3 = \sum_{\alpha=p+1}^{p+m} \frac{1}{2}\sigma^3_\alpha\,,$$

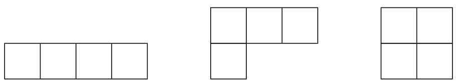
$$\mathbf{K}^2 = K^1K^1 + K^2K^2 + K^3K^3\,.$$

We use capital letters for operators and little letters for eigenvalues. In terms of these total spins we have

$$d_1^\dagger d_1 = \mathbf{J}^2 - J^3(J^3+1)\,, \qquad d_2^\dagger d_2 = \mathbf{K}^2 - K^3(K^3+1)\,.$$

Thus, eigenspaces of $d_1^\dagger d_1$ can be labeled by the eigenvalues of $\mathbf{J}^2$ and eigenvalues of J^3, and the eigenspaces of $d_2^\dagger d_2$ can be labeled by the eigenvalues of $\mathbf{K}^2$ and eigenvalues of K^3. Consequently, the labels $R,(r,s,t)$ of the restricted Schur polynomial can be traded for these eigenvalues. Indeed, consider the restricted Schur polynomial $\chi_{R,(r,s,t)}(Z,Y,X)$. The $\mathbf{K}^2 = k(k+1)$ quantum number tells you the shape of the Young diagram s that organizes the impurities: if there are N_1 boxes in the first row of s and N_2 boxes in the second, then $2k = N_1 - N_2$. The $\mathbf{J}^2 = j(j+1)$ quantum number tells you the shape of the Young diagram t that organizes the impurities: if there are N_1 boxes in the first row of t and N_2 boxes in the second, then $2j = N_1 - N_2$. The $J^3 + K^3$ eigenvalue of the state is always a good quantum number, both in the basis we start in where each spin has a sharp angular momentum or in the basis where the states have two sharp "total angular momenta". The $j^3 + k^3$ quantum number tells you how many boxes must be removed from each row of R to obtain r. Denote the number of boxes to be removed from the first row by n_1 and the number of boxes to be removed from the second row by n_2. We have $2j^3 + 2k_3 = n_1 - n_2$. This gives a complete construction of the projection operators we need.

To get some insight into how the construction works, lets count the states which appear for the example $m = p = 4$. There are three possible Young diagram shapes which appear



These correspond to a spin of $2, 1, 0$ respectively. As irreducible representations of S_4 they have a dimension of 1, 3 and 2 respectively. Coupling four spins we have

$$\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} = \mathbf{2} \oplus \mathbf{31} \oplus \mathbf{20}\,.$$

These results illustrate that each state of a definite spin labels an irreducible representation of the symmetric group and further that for our 8 spins we find the following organization of states

$S_m \times S_p$ irrep	**K** irrep	**J** irrep	dimension
$(\yng(4),\yng(4))$	$k=2$	$j=2$	25
$(\yng(4),\yng(3,1))$	$k=2$	$j=1$	45
$(\yng(4),\yng(2,2))$	$k=2$	$j=0$	10
$(\yng(3,1),\yng(4))$	$k=1$	$j=2$	45
$(\yng(3,1),\yng(3,1))$	$k=1$	$j=1$	81
$(\yng(3,1),\yng(2,2))$	$k=1$	$j=0$	18
$(\yng(2,2),\yng(4))$	$k=0$	$j=2$	10
$(\yng(2,2),\yng(3,1))$	$k=0$	$j=1$	18
$(\yng(2,2),\yng(2,2))$	$k=0$	$j=0$	4

The last column is obtained by taking a product of the dimension of the $S_m \times S_p$ irreducible representation by the dimension $(2k+1)(2j+1)$ of the associated spin multiplets. Summing the entries in the last column we obtain 256 which is indeed the number of states in the spin chain. For a detailed example of how the construction works see Appendix D.

Summary of the Approximations made:

- We have neglected mixing with restricted Schur polynomials that have $n \neq 2$ rows. These mixing terms are at most $O(\frac{1}{\sqrt{N}})$ so that this approximation is accurate at large N.
- The terms arising from an interaction between the Xs and Ys have been neglected. Since there are a lot more Zs than Xs and Ys the one loop dilatation operator will be dominated by terms arising from an interaction between Zs and Xs and between Zs and Ys.
- In simplifying Young's orthogonal representation for the symmetric group we have replaced certain factors $(d_{i,i+1})^{-1} = O(N^{-1})$ by $(d_{i,i+1})^{-1} = 0$. This is valid at large N. The fact that $d_{i,i+1} = O(N)$ is a consequence of the fact that we have Young diagrams with two rows, that we consider an operator whose bare dimension grows parametrically with N and that there are a lot more Zs than Xs and Ys. Thus boxes in different rows, corresponding to Xs and Ys, are always separated by a large axial distance at large N.

Two Columns

To treat the case of two columns, we need to account for the fact that Young's orthogonal representation simplifies to

$$(i, i+1)\,|\text{state}\rangle = -\,|\text{state}\rangle \qquad \text{for boxes in the same column}\,,$$

$$(i, i+1)\,|\text{state}\rangle = |\text{swapped state}\rangle \qquad \text{for boxes in different columns}\,.$$

Note the minus sign on the first line above. We can account for this sign, generalizing [53], by employing a description that uses Grassmann variables. To describe the first p boxes, introduce the $2p$ variables x_i^+, x_i^-, where $i = 1, 2, ..., p$. To describe the next m boxes, introduce the $2m$ variables y_j^-, y_j^+, where $j = p+1, p+2, ..., p+m$. Each labeled Young diagram continues to have an expression in terms of a monomial. Boxes in the right most column have a superscript $+$; boxes in the left most column have a superscript $-$. Each monomial is ordered with (i) xs to the left of ys and (ii) within each type (x or y) of variable, variables with a $-$ superscript to the left of variables with a $+$ superscript. Finally within a given type and a given superscript the variables are ordered so that the subscripts increase from left to right. Thus, for example, when $m = 3 = p$ we have

$$\begin{array}{|c|c|}\hline & \\ \hline & 5 \\ \hline & 4 \\ \hline & 1 \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} 6 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \;\leftrightarrow x_1^- x_2^+ x_3^+ y_4^- y_5^- y_6^+ \,.$$

If we now allow $S_m \times S_p$ to act on the monomials by acting on the subscripts of each variable without changing the order of the variables, we recover the correct action on the labeled Young diagrams.

It is a simple matter to show that

$$d_1 = \sum_{i=1}^{p} x_i^+ \frac{\partial}{\partial x_i^-}, \qquad d_2 = \sum_{i=p+1}^{p+m} y_i^+ \frac{\partial}{\partial y_i^-},$$

both commute with the symmetric group. It is again simple to show that[11]

$$d_1^\dagger = \sum_{i=1}^{p} x_i^- \frac{\partial}{\partial x_i^+}, \qquad d_2^\dagger = \sum_{i=p+1}^{p+m} y_i^- \frac{\partial}{\partial y_i^+}.$$

[11] Assuming we only consider monomials that are ordered as we described above, the inner product of two identical monomials is 1 and of two different monomials is 0.

We can again define two $S_m \times S_p$ Casimirs as $d_1^\dagger d_1$ and $d_2^\dagger d_2$. In terms of the spin variables

$$\tilde{\sigma}_n^i = (\sigma_n^3)^n \sigma_n^i (\sigma_n^3)^n$$

we have

$$d_1^\dagger d_1 = \tilde{\mathbf{J}}^2 - \tilde{J}^3(\tilde{J}^3 + 1)\,, \qquad d_2^\dagger d_2 = \tilde{\mathbf{K}}^2 - \tilde{K}^3(\tilde{K}^3 + 1)\,.$$

Thus, the eigenspaces of $d_1^\dagger d_1$ can be labeled by the eigenvalues of $\tilde{\mathbf{J}}^2$ and eigenvalues of $\tilde{J}^3$, and the eigenspaces of $d_2^\dagger d_2$ can be labeled by the eigenvalues of $\tilde{\mathbf{K}}^2$ and eigenvalues of $\tilde{K}^3$. Consequently, the labels $R, (r, s, t)$ of the restricted Schur polynomial can again be traded for these eigenvalues. The remaining discussion is now identical to that of two rows and is thus not repeated.

4.2.3 Evaluation of the Dilatation Operator

In this subsection we will argue that all of the factors in the dilatation operator have a natural interpretation as operators acting on the spin chain. This allows us to explicitly evaluate the action of the dilatation operator. Our final formula for the dilatation operator is given as the last formula in this section.

The bulk of the work involved in evaluating the dilatation operator comes from evaluating the traces

$$\mathrm{Tr}\left(\left[\Gamma_R((p{+}m{+}1,p{+}1)),P_{R\to(r,s,t)}\right]I_{R'\,T'}\left[\Gamma_T((p{+}m{+}1,p{+}1)),P_{T\to(u,v,w)}\right]I_{T'\,R'}\right),$$

and

$$\mathrm{Tr}\left(\left[\Gamma_R((1,p+m+1)),P_{R\to(r,s,t)}\right]I_{R'\,T'}\left[\Gamma_T((1,p+m+1)),P_{T\to(u,v,w)}\right]I_{T'\,R'}\right).$$

When we evaluate the second trace above, the intertwiners can be taken to act on the first site of the spin chain. This term corresponds to an interaction between a Z and X field. The first p sites of the spin chain correspond to X fields so that the intertwiner could have acted on any of the first p sites of the chain. When we evaluate the first trace above, the intertwiners can be taken to act on the $(p + 1)$th site of the spin chain. This term corresponds to an interaction between a Z and Y field. The last m sites of the spin chain correspond to Y fields so that the intertwiner could have acted on any of the last m sites of the chain. Consider an intertwiner which acts on the first site of the chain. If the box from row i is dropped from R and the box from row j is dropped from T, the intertwiner becomes

$$I_{R'T'} = E_{ij} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}\,, \qquad I_{T'R'} = E_{ji} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}\,,$$

where E_{ij} is a 2×2 matrix of zeroes except for a 1 in row i and column j. We will use a simpler notation according to which we suppress all factors of the

2×2 identity matrix and indicate which site a matrix acts on by a superscript. Thus, for example

$$I_{R'T'} = E^{(1)}_{ij}, \qquad I_{T'R'} = E^{(1)}_{ji}\,.$$

Next, consider $\Gamma_R((p+m+1,p+1))$ which acts on a slot occupied by a Z and a slot occupied by a Y and $\Gamma_R((1,p+m+1))$ which acts on a slot occupied by a Z and a slot occupied by an X. To allow an action on the Z slot, enlarge the spin chain by one extra site (the Z site). The projectors and intertwiners all have a trivial action on this $(m+p+1)$th site. $\Gamma_R((p+m+1,p+1))$ will swap the spin in the $(m+p+1)$th site with the spin in site $p+1$. Thus, we have

$$I_{R'T'}\Gamma_R((p+m+1,p+1)) = \sum_{k=1}^{2} E^{(p+1)}_{ij} E^{(m+p+1)}_{kk} \Gamma_R((p+m+1,p+1))$$

$$= \sum_{k=1}^{2} E^{(p+1)}_{ik} E^{(m+p+1)}_{kj},$$

$$\Gamma_R((p+m+1,p+1)) I_{R'T'} = \sum_{k=1}^{2} E^{(p+1)}_{kj} E^{(m+p+1)}_{ik},$$

$$\Gamma_R((p+m+1,p+1)) I_{R'T'} \Gamma_R((p+m+1,p+1)) = E^{(m+p+1)}_{ij}\,.$$

Since $\Gamma_R((1,p+m+1))$ will swap the spin in the $(m+p+1)$th site with the spin in site 1, very similar arguments give

$$I_{R'T'}\Gamma_R((1,p+m+1)) = \sum_{k=1}^{2} E^{(1)}_{ik} E^{(m+p+1)}_{kj},$$

$$\Gamma_R((1,p+m+1)) I_{R'T'} = \sum_{k=1}^{2} E^{(1)}_{kj} E^{(m+p+1)}_{ik},$$

$$\Gamma_R((1,p+m+1)) I_{R'T'} \Gamma_R((1,p+m+1)) = E^{(m+p+1)}_{ij}\,.$$

Our only task now is to evaluate traces of the form

$$\mathrm{Tr}\left(\Gamma_R((1,p+m+1)) P_{R\to(r,s,t)} I_{R'\,T'} \Gamma_T((1,p+m+1)) P_{T\to(u,v,w)} I_{T'\,R'}\right)$$

$$= \sum_{k,l=1}^{2} \mathrm{Tr}\left(E^{(1)}_{ik} E^{(m+p+1)}_{kj} P_{R\to(r,s,t)} E^{(1)}_{jl} E^{(m+p+1)}_{li} P_{T\to(u,v,w)}\right).$$

To perform this final trace, our strategy is always the same two steps. For the first step, evaluate the trace over the $(n+p+1)$th slot. It is clear that the trace over the $p+m+1$th slot factors out and further that

$$\mathrm{Tr}\,(E^{(m+p+1)}_{kj} E^{(m+p+1)}_{li}) = \delta_{jl}\delta_{ik}$$

so that we obtain

$$\mathrm{Tr}\left(E^{(1)}_{ii} P_{R\to(r,s,t)} E^{(1)}_{jj} P_{T\to(u,v,w)}\right)$$

To evaluate this final trace we will rewrite the projectors a little. Notice that $E^{(1)}_{kk}$ only has a nontrivial action on the first site of the spin chain. Thus, we rewrite the projector, separating out the first site. As an example, consider

$$P_{R\to(r,s,t)} = \sum_{\alpha=1}^{d_t} \left|j, j^3, \alpha\right\rangle \left\langle j, j^3, \alpha\right| \otimes \sum_{\beta=1}^{d_s} \left|k, k^3, \beta\right\rangle \left\langle k, k^3, \beta\right| \ .$$

To make sense of this formula recall that the labels j, k, j^3, k^3 can be traded for the r, s, t labels. In going from the LHS of this last equation to the RHS we have translated labels and we assure you that nothing is lost in translation [53]. In Figure 4.2 we remind the reader of how the translation is performed. We will refer to the Young diagram corresponding to spin j, built with p blocks as s^p_j in what follows.

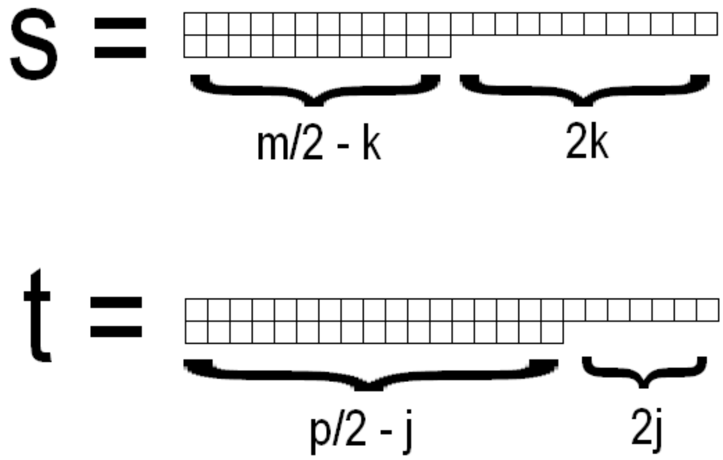


Figure 4.2: How to translate between the j, k and the s, t labels.

The piece of the projector that acts on the first p sites is

$$P_{\to t} \equiv \sum_{\alpha=1}^{d_t} \left|j, j^3, \alpha\right\rangle \left\langle j, j^3, \alpha\right| \ . \tag{4.10}$$

If we couple the spins at sites $2, 3, ..., p$ together, we obtain the states $|j \pm \frac{1}{2}, j^3 \pm \frac{1}{2}, \alpha\rangle$ with the degeneracy label α running from 1 to the dimension of the irreducible S_{p-1} representation associated to spin $j \pm \frac{1}{2}$. This irreducible representation is labeled by the Young diagram $s^{p-1}_{j\pm\frac{1}{2}}$. The Clebsch-Gordan coefficients

$$\left\langle j - \frac{1}{2}, j^3 - \frac{1}{2}; \frac{1}{2}, \frac{1}{2} | j, j^3 \right\rangle = \sqrt{\frac{j + j^3}{2j}} \, ,$$

$$\left\langle j+\frac{1}{2},j^3-\frac{1}{2};\frac{1}{2},\frac{1}{2}|j,j^3\right\rangle = -\sqrt{\frac{j-j^3+1}{2(j+1)}}\,,$$

$$\left\langle j-\frac{1}{2},j^3+\frac{1}{2};\frac{1}{2},-\frac{1}{2}|j,j^3\right\rangle = \sqrt{\frac{j-j^3}{2j}}\,,$$

$$\left\langle j+\frac{1}{2},j^3+\frac{1}{2};\frac{1}{2},-\frac{1}{2}|j,j^3\right\rangle = \sqrt{\frac{j+j^3+1}{2(j+1)}}\,,$$

tell us how to couple the first site with the remaining spins to obtain the projector (4.10). Thus, we finally have ($s1=s^{p-1}_{j-\frac{1}{2}}$, $s2=s^{p-1}_{j+\frac{1}{2}}$)

$$|\phi,\alpha\rangle = \sqrt{\frac{j+j^3}{2j}}\left|\frac{1}{2},\frac{1}{2};j-\frac{1}{2},j^3-\frac{1}{2},\alpha\right\rangle + \sqrt{\frac{j-j^3}{2j}}\left|\frac{1}{2},-\frac{1}{2};j-\frac{1}{2},j^3+\frac{1}{2},\alpha\right\rangle\,,$$

$$|\psi,\beta\rangle = -\sqrt{\frac{j-j^3+1}{2(j+1)}}\left|\frac{1}{2},\frac{1}{2};j+\frac{1}{2},j^3-\frac{1}{2},\beta\right\rangle + \sqrt{\frac{j+j^3+1}{2(j+1)}}\left|\frac{1}{2},-\frac{1}{2};j+\frac{1}{2},j^3+\frac{1}{2},\beta\right\rangle\,,$$

$$P_{\to t} = \sum_{\alpha=1}^{d_{s1}} |\phi,\alpha\rangle\langle\phi,\alpha| + \sum_{\beta=1}^{d_{s2}} |\psi,\beta\rangle\langle\psi,\beta|\,.$$

We could of course perform exactly the same manipulations on the projector $P_{\to s}$ that acts on the last m sites of the spin chain. Now, using the obvious identities

$$E^{(1)}_{11}|\phi,\alpha\rangle = \sqrt{\frac{j+j^3}{2j}}\left|\frac{1}{2},\frac{1}{2};j-\frac{1}{2},j^3-\frac{1}{2},\alpha\right\rangle\,,$$

$$E^{(1)}_{22}|\phi,\alpha\rangle = \sqrt{\frac{j-j^3}{2j}}\left|\frac{1}{2},-\frac{1}{2};j-\frac{1}{2},j^3+\frac{1}{2},\alpha\right\rangle\,,$$

$$E^{(1)}_{11}|\psi,\beta\rangle = -\sqrt{\frac{j-j^3+1}{2(j+1)}}\left|\frac{1}{2},\frac{1}{2};j+\frac{1}{2},j^3-\frac{1}{2},\beta\right\rangle\,,$$

$$E^{(1)}_{22}|\psi,\beta\rangle = \sqrt{\frac{j+j^3+1}{2(j+1)}}\left|\frac{1}{2},-\frac{1}{2};j+\frac{1}{2},j^3+\frac{1}{2},\beta\right\rangle\,,$$

it becomes a simple matter to evaluate the above traces.

Finally, in the limit that we consider, the coefficients of the traces appearing in the dilatation operator are easily evaluated using

$$\frac{c_{RR'}d_T d_{r'} n}{d_{R'}d_u d_v d_w(n+m+p)}\sqrt{\frac{f_T \mathrm{hooks}_T \mathrm{hooks}_r \mathrm{hooks}_s \mathrm{hooks}_t}{f_R \mathrm{hooks}_R \mathrm{hooks}_u \mathrm{hooks}_v \mathrm{hooks}_w}}$$

$$= \frac{\sqrt{c_{RR'}c_{TT'}}\sqrt{\text{hooks}_s\text{hooks}_t\text{hooks}_v\text{hooks}_w}}{m!p!}.$$

In the above expression, r' is obtained by removing a box from r. The box that must be removed from R to obtain R' and the box that must be removed from r to obtain r' are both removed from the same row. Putting things together we find

$$\begin{aligned}
DO_{j,j^3}(b_0,b_1) = g_{YM}^2 \Bigg[&-\frac{1}{2}\left(m - \frac{(m+2)(j^3)^2}{j(j+1)}\right)\Delta O_{j,j^3,k,k^3}(b_0,b_1) \\
&+\sqrt{\frac{(m+2j+4)(m-2j)}{(2j+1)(2j+3)}\frac{(j+j^3+1)(j-j^3+1)}{2(j+1)}}\Delta O_{j+1,j^3,k,k^3}(b_0,b_1) \\
&+\sqrt{\frac{(m+2j+2)(m-2j+2)}{(2j+1)(2j-1)}\frac{(j+j^3)(j-j^3)}{2j}}\Delta O_{j-1,j^3,k,k^3}(b_0,b_1) \\
&-\frac{1}{2}\left(p - \frac{(p+2)(k^3)^2}{k(k+1)}\right)\Delta O_{j,j^3,k,k^3}(b_0,b_1) \\
&+\sqrt{\frac{(p+2k+4)(p-2k)}{(2k+1)(2k+3)}\frac{(k+k^3+1)(k-k^3+1)}{2(k+1)}}\Delta O_{j,j^3,k+1,k^3}(b_0,b_1) \\
&+\sqrt{\frac{(p+2k+2)(p-2k+2)}{(2k+1)(2k-1)}\frac{(k+k^3)(k-k^3)}{2k}}\Delta O_{j,j^3,k-1,k^3}(b_0,b_1)\Bigg]
\end{aligned} \tag{4.11}$$

where

$$\begin{aligned}
\Delta O(b_0,b_1) &= \sqrt{(N+b_0)(N+b_0+b_1)}(O(b_0+1,b_1-2)+O(b_0-1,b_1+2)) \\
&\quad -(2N+2b_0+b_1)O(b_0,b_1).
\end{aligned} \tag{4.12}$$

Above, we have explicitly carried out the discussion for two long rows. To obtain the result for two long columns, replace

$$\sqrt{(N+b_0)(N+b_0+b_1)} \to \sqrt{(N-b_0)(N-b_0-b_1)},$$

$$(2N+2b_0+b_1) \to (2N-2b_0-b_1)$$

in the expression for $\Delta O(b_0,b_1)$. This completes our evaluation of the dilatation operator.

4.2.4 Diagonalization of the Dilatation Operator

In this subsection we reduce the eigenvalue problem for the dilatation operator to the problem of solving a five term recursion relation. The explicit solution of this recursion relation allows us to argue that the dilatation operator reduces to a set of decoupled oscillators. Thus, the problem we are studying is indeed integrable.

We make the following ansatz for the operators of good scaling dimension

$$\sum_{b_1} f(b_0,b_1)\, O_{pq,j^3,k^3}(b_0,b_1) \;= \sum_{j,k,b_1} C_{pq,j^3,k^3}(j,k)\, f(b_0,b_1) O_{j,j^3,k,k^3}(b_0,b_1)\,.$$

Inserting this ansatz into (4.11) we find that the $O_{pq,j^3,k^3}(b_0,b_1)$'s satisfy the recursion relation

$$\begin{aligned}
&-\alpha_{rq,j^3,k^3} C_{rq,j^3,k^3}(j,k) = \\
&\sqrt{\frac{(m+2j+4)(m-2j)}{(2j+1)(2j+3)}}\frac{(j+j^3+1)(j-j^3+1)}{2(j+1)} C_{rq,j^3,k^3}(j+1,k) \\
&+\sqrt{\frac{(m+2j+2)(m-2j+2)}{(2j+1)(2j-1)}}\frac{(j+j^3)(j-j^3)}{2j} C_{rq,j^3,k^3}(j-1,k) \\
&-\frac{1}{2}\left(m-\frac{(m+2)(j^3)^2}{j(j+1)}\right) C_{rq,j^3,k^3}(j,k) \\
&+\sqrt{\frac{(p+2k+4)(p-2k)}{(2k+1)(2k+3)}}\frac{(k+k^3+1)(k-k^3+1)}{2(k+1)} C_{rq,j^3,k^3}(j,k+1) \\
&+\sqrt{\frac{(p+2k+2)(p-2k+2)}{(2k+1)(2k-1)}}\frac{(k+k^3)(k-k^3)}{2k} C_{rq,j^3,k^3}(j,k-1) \\
&-\frac{1}{2}\left(p-\frac{(p+2)(k^3)^2}{k(k+1)}\right) C_{rq,j^3,k^3}(j,k)\,.
\end{aligned} \tag{4.13}$$

Exploiting the $j^3 \to -j^3$ and $k^3 \to -k^3$ symmetries of this equation, we need only solve for the $j^3 \geq 0$ and $k^3 \geq 0$ cases. The ranges for j and k are

$$0 \leq |j^3| \leq j \leq \frac{m}{2} \qquad 0 \leq |k^3| \leq k \leq \frac{p}{2}\,.$$

From the form of the recursion relation, it is natural to make the "separation of variables" ansatz

$$C_{rq,j^3,k^3}(j,k) = C_{r,j^3}(j) C_{q,k^3}(k)\,.$$

Our five term recurrence relation now reduces to two three term recurrence relations

$$\begin{aligned}
&-\alpha_{r,j^3} C_{p,j^3}(j,) = \\
&\sqrt{\frac{(m+2j+4)(m-2j)}{(2j+1)(2j+3)}}\frac{(j+j^3+1)(j-j^3+1)}{2(j+1)} C_{r,j^3}(j+1) \\
&+\sqrt{\frac{(m+2j+2)(m-2j+2)}{(2j+1)(2j-1)}}\frac{(j+j^3)(j-j^3)}{2j} C_{r,j^3}(j-1) \\
&-\frac{1}{2}\left(m-\frac{(m+2)(j^3)^2}{j(j+1)}\right) C_{r,j^3}(j)\,,
\end{aligned} \tag{4.14}$$

$$
\begin{aligned}
&-\alpha_{q,k^3} C_{q,k^3}(k) = \\
&\sqrt{\frac{(p+2k+4)(p-2k)}{(2k+1)(2k+3)}} \frac{(k+k^3+1)(k-k^3+1)}{2(k+1)} C_{q,k^3}(k+1) \\
&+\sqrt{\frac{(p+2k+2)(p-2k+2)}{(2k+1)(2k-1)}} \frac{(k+k^3)(k-k^3)}{2k} C_{q,k^3}(k-1) \\
&-\frac{1}{2}\left(p-\frac{(p+2)(k^3)^2}{k(k+1)}\right) C_{q,k^3}(k)\,.
\end{aligned} \tag{4.15}
$$

These are identical to the three term recursion relations that appear in [53]. To solve these recurrence relations, introduce the Hahn polynomial[75]

$$
Q_n(x;\alpha,\beta,N) \equiv {}_3F_2\left({-n,n+\alpha+\beta+1,-x \atop \alpha+1,-N}\Big| 1\right)
$$

From the recurrence relation obeyed by Hahn polynomials (see equation (1.5.3) in [75]) we have

$$
\begin{aligned}
& r\,{}_3F_2\left({|j^3|-j,j+1+|j^3|,-r \atop 1,|j^3|-\frac{m}{2}}\Big| 1\right) = \\
&\frac{(j+j^3+1)(j-j^3+1)(m-2j)}{2(j+1)(2j+1)}\,{}_3F_2\left({-1+|j^3|-j,j+2+|j^3|,-r \atop 1,|j^3|-\frac{m}{2}}\Big| 1\right) \\
&-\left(\frac{m}{2}-\frac{(m+2)(j^3)^2}{2j(j+1)}\right){}_3F_2\left({|j^3|-j,j+1+|j^3|,-r \atop 1,|j^3|-\frac{m}{2}}\Big| 1\right) \\
&+\frac{(j+j^3)(j-j^3)(m+2j+2)}{2j(2j+1)}\,{}_3F_2\left({1+|j^3|-j,j+|j^3|,-r \atop 1,|j^3|-\frac{m}{2}}\Big| 1\right)
\end{aligned}
$$

Consequently, our recursion relation is solved by

$$
\begin{aligned}
C_{r,j^3}(j) &= (-1)^{\frac{m}{2}-p}\left(\frac{m}{2}\right)!\sqrt{\frac{(2j+1)}{\left(\frac{m}{2}-j\right)!\left(\frac{m}{2}+j+1\right)!}}\,{}_3F_2\left({|j^3|-j,j+|j^3|+1,-r \atop |j^3|-\frac{m}{2},1}\Big| 1\right) \\
& |j^3| \le j \le \frac{m}{2}, \qquad 0 \le r \le \frac{m}{2}-|j^3|
\end{aligned} \tag{4.16}
$$

and

$$
\begin{aligned}
C_{q,k^3}(k) &= (-1)^{\frac{p}{2}-q}\left(\frac{p}{2}\right)!\sqrt{\frac{(2k+1)}{\left(\frac{p}{2}-k\right)!\left(\frac{p}{2}+k+1\right)!}}\,{}_3F_2\left({|k^3|-k,k+|k^3|+1,-q \atop |k^3|-\frac{p}{2},1}\Big| 1\right) \\
& |k^3| \le k \le \frac{p}{2}, \qquad 0 \le q \le \frac{p}{2}-|k^3|\,.
\end{aligned} \tag{4.17}
$$

The associated eigenvalues are

$$
-\alpha_{rq,j^3,k^3} = -2(r+q) = 0,-2,-4,...,-(m-2|j^3|+p-2|k^3|)\,.
$$

Our eigenfunctions are essentially the Hahn polynomials. It is a well known fact that the Hahn polynomials are closely related to the Clebsch-Gordan coefficients of $SU(2)$ [76].

The eigenproblem of the dilatation operator now reduces to solving

$$\lambda \sum_{b_1} f(b_0,b_1)\, O_{rq,j^3,k^3}(b_0,b_1) = -\alpha_{rq,j^3,k^3} \sum_{b_1} f(b_0,b_1)\, \Delta O_{rq,j^3,k^3}(b_0,b_1)\,.$$

This eigenproblem implies $f(b_0,b_1)$ satisfy the recursion relation

$$-\alpha_{rq,j^3,k^3} g_{YM}^2 [\sqrt{(N+b_0)(N+b_0+b_1)}(f(b_0-1,b_1+2)+f(b_0+1,b_1-2)) \\ -(2N+2b_0+b_1)f(b_0,b_1)] = \lambda f(b_0,b_1)\,. \tag{4.18}$$

Since we work at large N, we can replace (4.18) by

$$\lambda f(b_0,b_1) = -\alpha_{rq,j^3,k^3} g_{YM}^2 [\sqrt{(N+b_0)(N+b_0+b_1+1)} f(b_0-1,b_1+2) \\ +\sqrt{(N+b_0+1)(N+b_0+b_1)} f(b_0+1,b_1-2) - (2N+2b_0+b_1) f(b_1,b_1)]\,.$$

This recursion relation is precisely the recursion relation of the finite oscillator [87]! In the continuum limit (which corresponds to the large N limit) we recover the usual description of the harmonic oscillator, demonstrating rather explicitly that the eigenproblem of the dilatation operator reduces to solving a set of decoupled harmonic oscillators. The solution to (4.18) is [87]

$$f(b_0,b_1) = (-1)^n (\frac{1}{2})^{N+b_0+\frac{b_1}{2}} \sqrt{\binom{2N+2b_0+b_1}{N+b_0+b_1}\binom{2N+2b_0+b_1}{n}}\, {}_2F_1\Big(\begin{smallmatrix} -n,-(N+b_0+b_1) \\ -(2N+2b_0+b_1)\end{smallmatrix}\Big|2\Big)\,. \tag{4.19}$$

These solutions are closely related to the symmetric Kravchuk polynomial $K_n(x,1/q,p)$ defined by

$${}_2F_1\left(\begin{smallmatrix} -n,-x \\ -p \end{smallmatrix}; q\right) = K_n(x,1/q,p)\,.$$

The corresponding eigenvalue is $\lambda = 2n\alpha_{rq,j^3,k^3} g_{YM}^2$. Recall that $b_1 \ge 0$ so that only half of the wavefunctions are selected (those that vanish when $b_1 = 0$) and consequently the eigenvalue λ level spacing is $4\alpha_{rq,j^3,k^3} g_{YM}^2 = 8(p+q) g_{YM}^2$.

4.2.5 Discussion

We have studied the action of the dilatation operator on restricted Schur polynomials $\chi_{R,(r,s,t)}(Z,Y,X)$, built from three complex scalars X, Y and Z and labeled by Young diagrams with at most two rows or two columns. The operators have $O(N)$ fields of each of the three flavors, but there are many many more Zs than Xs or Ys. Our main result is that the dilatation operator reduces to a set of decoupled oscillators and is hence an integrable system. If we have m Ys and p Xs with p,m both even, we obtain a set of oscillators with frequency ω_{ij} and degeneracy d_{ij} given by

$$\omega_{ij} = 8(i+j) g_{YM}^2, \qquad d_{ij} = (2(m-i)+1)\,(2(p-j)+1)\,,$$

$$i = 0,1,...,m, \qquad j = 0,1,...,p\,.$$

If p is even and m is odd we have

$$\omega_{ij} = 8(i+j)g_{YM}^2, \qquad d_{ij} = 2\,(m-i+1)\,(2(p-j)+1)\,,$$
$$i = 0,1,...,m, \qquad j = 0,1,...,p\,.$$

If m is even and p is odd we have

$$\omega_{ij} = 8(i+j)g_{YM}^2, \qquad d_{ij} = 2\,(2(m-i)+1)\,(p-j+1)\,,$$
$$i = 0,1,...,m, \qquad j = 0,1,...,p\,.$$

If both p and m are odd we have

$$\omega_{ij} = 8(i+j)g_{YM}^2, \qquad d_{ij} = 4\,(m-i+1)\,(p-j+1)\,,$$
$$i = 0,1,...,m, \qquad j = 0,1,...,p\,.$$

The oscillators corresponding to a zero frequency are BPS operators built using three complex scalars X, Y and Z.

The form of the dilatation operator (4.11) is intriguing: it looks like the sum of two of the dilatation operators computed in [53], with one acting on the Ys (with quantum numbers k, k^3) and one acting on the Xs (with quantum numbers j, j^3). With the benefit of hindsight, could we have anticipated this structure? The bulk of our effort involved evaluating traces like this one

$$\mathrm{Tr}\left(\Big[\Gamma_R((p+m+1,p+1)),P_{R\to(r,s,t)}\Big]I_{R'\,T'} \times \Big[\Gamma_T((p+m+1,p+1)),P_{T\to(u,v,w)}\Big]I_{T'\,R'}\right).$$

Notice that both $\Gamma_R((p+m+1,p+1))$ and $I_{R'\,T'}$ do not act on the first p sites of the spin chain. Further, our projector factorizes into a projector acting on the first p sites times a projector acting on the remaining m sites. Consequently, the trace over the first p sites gives $\delta_{tw}d_w$. The trace that remains is exactly of the form considered in [53], explaining our final answer (4.11). An important new feature we have found here, described in detail in Appendix C, is that before making the approximations described in section 3.1, the spectrum of the dilatation operator is not equivalent to a collection of harmonic oscillators. This is similar to what one finds in the sector of operators with a bare dimension of order $O(1)$: in the large N limit (which in this case is the planar limit) one obtains an integrable system. Adding $1/N$ corrections seems to spoil the integrability [84, 85].

Apart from computing the spectrum of the dilatation operator, we have managed to compute the associated eigenstates. These states are given in terms of Kravchuk polynomials and Hahn polynomials. The Hahn polynomials are closely related to the wave functions of the one dimensional harmonic oscillator[87] while the Hahn polynomials are closely related to the wave functions of the 2d radial oscillator[53]. The argument of these polynomials are given by j, k or b_1, which have a direct link to the Young diagrams labeling the operators, as summarized for example in Figure 4.2[12]. Thus, the "space" on which the wave functions are defined comes from the Young diagram itself. Based on our experience with the half BPS sector, it is natural to associate each one of the rows of the Young diagram with each one of the giant gravitons. Recalling that $Y = \phi_3 + i\phi_4$ we know that the number of Ys in each operator tells us the angular momentum of the operator in the 3-4 plane. Similarly, the number of Xs in each operator tells us the angular momentum of the operator in the 5-6 plane and the number of Xs in each operator tells us the angular momentum of the operator in the 1-2 plane. Giving an angular momentum to the giant gravitons will cause them to expand as a consequence of the Myers effect[78]. Thus, for example, the separation between the two gravitons in the 3-4 plane will be related to the difference in angular momenta of the two giants. Consequently, the quantum number k is acting like a coordinate for the radial separation between the two giants in the 3-4 plane. Thus, we see very concretely the emergence of local physics from the system of Young diagrams labeling the restricted Schur polynomial. This is strongly reminiscent of the 1/2 BPS case where the Schur polynomials provide wave functions for fermions in a harmonic oscillator potential and further, these wave functions very naturally reproduce features of the geometries and the phase space [88].

For the matrix model we are studying here it is not true that the matrices Z,Y,X commute, we can't simultaneously diagonalize them and there is no analog of the eigenvalue basis that is so useful for the large N dynamics of single matrix models. For the subsystem describing the BPS states however [79] has argued that the matrices might commute in the interacting theory and hence there may be a description in terms of eigenvalues. The argument uses the fact that the weak coupling and strong coupling limits of the BPS sector agree and the fact that at strong coupling we can be confident that the matrices commute. If this is the case, the eigenvalue dynamics should be the dynamics in an oscillator potential with repulsions preventing the collision of eigenvalues. We have described a part of the BPS sector (as well as non-BPS operators) among the operators we have studied. We do indeed find the dynamics of harmonic oscillators. In the case of a single matrix it is possible to associate the rows of the Young diagram labeling a Schur polynomial with the eigenvalues of the matrix [80]. This provides a connection between the eigenvalue description and the Schur polynomial description for single matrix models. Our results suggest this might have a generalization to multimatrix models.

The operators we have considered are dual to giant gravitons. A connection between the geometry of giant gravitons and harmonic oscillators was already uncovered in [81, 82, 83]. This work quantizes the moduli space of Mikhailov's giant gravitons so that one is capturing a huge space of states. It is this huge

[12]The Young diagram r is not shown in Figure 4.2. The number of columns with a single box is given by b_1.

space of states that connects to harmonic oscillators. Our study is focused on a two giant system. Consequently, the oscillators that we have found are associated to this two giant system and excitations of it. It is natural to think that our oscillators arise from the quantization of the possible excitation modes of a giant graviton.

4.3 Two Loop $SU(2)$ Sector

In this section we study the action of two loop dilatation operator of $\mathcal{N} = 4$ SYM theory in the $SU(2)$ sector. Our discussion from here on is for a general Young diagram. For this reason we need to reinstate the multiplicity labels.

4.3.1 Two Loop Dilatation Operator

Our goal is to evaluate the action of the two loop dilatation operator in the $SU(2)$ sector [51]

$$
\begin{aligned}
D_4 = & - 2g^2 : \mathrm{Tr}\left(\left[[Y,Z]\,,\frac{\partial}{\partial Z}\right]\left[\left[\frac{\partial}{\partial Y},\frac{\partial}{\partial Z}\right],Z\right]\right): \\
& - 2g^2 : \mathrm{Tr}\left(\left[[Y,Z]\,,\frac{\partial}{\partial Y}\right]\left[\left[\frac{\partial}{\partial Y},\frac{\partial}{\partial Z}\right],Y\right]\right): \\
& - 2g^2 : \mathrm{Tr}\left([[Y,Z]\,,T^a]\left[\left[\frac{\partial}{\partial Y},\frac{\partial}{\partial Z}\right],T^a\right]\right):, \qquad (4.20)
\end{aligned}
$$

$$
g = \frac{g_{YM}^2}{16\pi^2}, \qquad (4.21)
$$

on restricted Schur polynomials. The normal ordering symbols here indicate that derivatives within the normal ordering symbols do not act on fields inside the normal ordering symbols. For the operators we study, $n \gg m$ so that only the first term in D_4 will contribute. We have in mind a systematic expansion in two parameters: $\frac{1}{N}$ and $\frac{m}{n}$. In Appenidx H we show that keeping only the first term in D_4 corresponds to the computation of the leading term in this double expansion. The evaluation of the action of the one loop dilatation operator was carried out in [52]. The two loop computation uses many of the same techniques but there are a number of subtle points that must be treated correctly. The computation can be split into the evaluation of two types of terms, one having all derivatives adjacent to each other (for example $\mathrm{Tr}(ZYZ\partial_Z\partial_Y\partial_Z)$) and one in which only two of the derivatives are adjacent (for example $\mathrm{Tr}(YZ\partial_Z Z\partial_Y\partial_Z)$). We will deal with an example of each term paying special attention to points that must be treated with care.

First Term: Start by allowing the derivatives to act on the restricted Schur polynomial

$$\mathrm{Tr}(\mathrm{ZYZ}\partial_{\mathrm{Z}}\partial_{\mathrm{Y}}\partial_{\mathrm{Z}})\chi_{R,(r,s)\alpha\beta}(Z,Y) = \frac{mn(n-1)}{n!m!}\sum_{\psi\in S_{n+m}}\mathrm{Tr}_{(r,s)\alpha\beta}(\Gamma^{(R)}((1,m+2)\psi(m+1,m+2))) \times \delta^{i_1}_{i_{\psi(1)}}Y^{i_2}_{i_{\psi(2)}}\cdots Y^{i_m}_{i_{\psi(m)}}(ZYZ)^{i_{m+1}}_{i_{\psi(m+1)}}\delta^{i_{m+2}}_{i_{\psi(m+2)}}Z^{i_{m+3}}_{i_{\psi(m+3)}}\cdots Z^{i_{m+n}}_{i_{\psi(m+n)}}\,. \tag{4.22}$$

The two delta functions will reduce the sum over S_{n+m} to a sum over an S_{n+m-2} subgroup. This sum is most easily evaluated using the reduction rule of [71, 48]. The reduction rule rewrites the sum over S_{n+m} as a sum over S_{n+m-2} and its cosets. This is most easily done by making use of Jucys-Murphy elements whose action is easily evaluated. To employ the same strategy in the current computation, the action of the Jucys-Murphy element will only be the simple one if we swap the delta function from slot $m+2$ to slot 2. This gives

$$\frac{mn(n-1)}{n!m!}\times \sum_{\psi\in S_{n+m-2}}\mathrm{Tr}_{(r,s)\alpha\beta}(\Gamma^{(R)}((1,m+2)(2,m+2)\psi(2,m+2)\hat{C}(m+1,m+2))) \times Y^{i_3}_{i_{\psi(3)}}\cdots Y^{i_m}_{i_{\psi(m)}}(ZYZ)^{i_{m+1}}_{i_{\psi(m+1)}}Y^{i_{m+2}}_{i_{\psi(m+2)}}Z^{i_{m+3}}_{i_{\psi(m+3)}}\cdots Z^{i_{m+n}}_{i_{\psi(m+n)}} \tag{4.23}$$

where $\hat{C}=(N+J_2)(N+J_3)$ with J_i a Jucys-Murphy element

$$J_i=\sum_{k=i}^{n+m}(i-1,k)\,. \tag{4.24}$$

Since we sum over the S_{n+m-2} subgroup, we can decompose $R\vdash m+n$ into a direct sum of terms which involve the irreps $R''\vdash m+n-2$ of the subgroup[13]. As usual [71, 48], for each term in the sum, $\hat{C}$ is equal to the product of the factors of the boxes that must be removed from R to obtain R''. To rewrite the result in terms of restricted Schur polynomials, note that

$$\begin{aligned} Y^{i_3}_{i_{\psi(3)}} & \cdots Y^{i_m}_{i_{\psi(m)}}(ZYZ)^{i_{m+1}}_{i_{\psi(m+1)}}Y^{i_{m+2}}_{i_{\psi(m+2)}}Z^{i_{m+3}}_{i_{\psi(m+3)}}\cdots Z^{i_{m+n}}_{i_{\psi(m+n)}} \\ &= \mathrm{Tr}\left(\psi(2,m+1,1)Y\otimes Z\otimes Y^{\otimes m-2}\otimes Z\otimes Y\otimes Z^{\otimes n-2}\right) \\ &= \mathrm{Tr}\left((2,m+2)\psi(2,m+1,1)(2,m+2)Y^{\otimes m}\otimes Z^{\otimes n}\right) \end{aligned} \tag{4.25}$$

and make use of the identity(4.2). After this rewriting the sum over S_{n+m-2} can be carried out using the fundamental orthogonality relation. The result is

$$\sum_{T,(t,u)\gamma\delta}\sum_{R'',T''}\frac{d_T n(n-1)m}{d_t d_u d_{R''}(n+m)(n+m-1)}c_{RR'}c_{R'R''}\ \chi_{T,(t,u)\gamma\delta}(Z,Y)$$

[13]In general if R denotes a Young diagram, then R' denotes a Young diagram that can be obtained from R by removing one box, R'' denotes a Young diagram that can be obtained from R by removing two boxes etc.

$$\times \mathrm{Tr}(I_{T''R''}(2,m+2,m+1)P_{R,(r,s)\alpha\beta}(1,m+2,2)I_{R''T''}(2,m+2)$$
$$\times P_{T,(t,u)\delta\gamma}(m+2,2,1,m+1))$$

The intertwiner $I_{R''T''}$ is a map (see Appendix D of [54] for details on its properties) from irrep R'' to irrep T''. It is only non-zero if R'' and T'' have the same shape. Thus, to get a non-zero result R and T must differ at most, by the placement of two boxes. We make further comments relevant for this trace before equation (4.26) below.

Second Term: Evaluation of the second term is very similar. In this case however, taking the derivatives produces a single delta function, which will reduce the sum over S_{n+m} to a sum over S_{n+m-1}. The delta function should be in slot 1. The reader wanting to check an example may find it useful to verify that

$$: \mathrm{Tr}(YZ\partial_Z Z\partial_Y\partial_Z) : \chi_{R,(r,s)\alpha\beta}(Z,Y) = \sum_{T,(t,u)\gamma\delta}\sum_{R',T'} \frac{d_T n(n-1)m}{d_t d_u d_{R'}(n+m)} c_{RR'}$$
$$\times \mathrm{Tr}(I_{T'R'}(1,m+2,m+1)P_{R,(r,s)\alpha\beta}I_{R'T'}(1,m+1)P_{T,(t,u)\delta\gamma})\chi_{T,(t,u)\gamma\delta}(Z,Y)$$

The intertwiner $I_{R'T'}$ is a map from irrep R' to irrep T'. It is only non-zero if R' and T' have the same shape. Thus, to get a non-zero result R and T must differ at most, by the placement of a single box. It is perhaps useful to spell out explicitely the meaning of the trace above. The above trace is taken over the reducible S_{n+m} representation $R \oplus T$. In addition, the projectors within the trace allow us to rewrite the permutations appearing in the trace as

$$\mathrm{Tr}\left(I_{T'R'}\Gamma^{(R)}\Big((1,m+2,m+1)\Big)P_{R,(r,s)\alpha\beta}I_{R'T'}\Gamma^{(T)}\Big((1,m+1)\Big)P_{T,(t,u)\delta\gamma}\right) \quad (4.26)$$

The final result for the action of the dilatation operator is (this includes only the first term in (4.20) since $n \gg m$)

$$D_4\chi_{R,(r,s)\alpha\beta}(Z,Y) =$$
$$-2g^2\sum_{T,(t,u)\gamma\delta}\sum_{R'\,T'}\frac{d_T n(n-1)mc_{RR'}}{d_t d_u d_{R'}(n+m)}M^{(b)}_{R,(r,s)\alpha\beta\ T,(t,u)\gamma\delta}\chi_{T,(t,u)\delta\gamma}(Z,Y)$$
$$-2g^2\sum_{T,(t,u)\gamma\delta}\sum_{R''\,T''}\frac{d_T n(n-1)mc_{RR'}c_{R'R''}}{d_t d_u d_{R''}(n+m)(n+m-1)}M^{(a)}_{R,(r,s)\alpha\beta\ T,(t,u)\gamma\delta}\chi_{T,(t,u)\delta\gamma}(Z,Y)$$

where

$$M^{(a)}_{R,(r,s)\alpha\beta\ T,(t,u)\gamma\delta} = \mathrm{Tr}\left(I_{T''R''}(2,m+2)P_{R,(r,s)\alpha\beta}C_1 I_{R''T''}(2,m+2)P_{T,(t,u)\gamma\delta}C_1\right)$$
$$+\mathrm{Tr}\left(I_{T''R''}C_2 P_{R,(r,s)\alpha\beta}(2,m+2)I_{R''T''}C_2 P_{T,(t,u)\gamma\delta}(2,m+2)\right) \quad (4.27)$$

$$C_1 = \big[(m+2,2,1),(1,m+1)\big],\ C_2 = -C_1^T = \big[(m+2,1,2),(1,m+1)\big] \quad (4.28)$$

and

$$M^{(b)}_{R,(r,s)\alpha\beta\ T,(t,u)\gamma\delta} = \mathrm{Tr}\left(I_{T'R'}C_3 I_{R'T'}\left[(1,m+1),P_{T,(t,u)\gamma\delta}\right]\right) + \mathrm{Tr}\left(I_{T'R'}C_4 I_{R'T'}\left[(1,m+1),P_{T,(t,u)\gamma\delta}\right]\right) \quad (4.29)$$

$$C_3 = \left[(1,m+2,m+1),P_{R,(r,s)\alpha\beta}\right],\ C_4 = \left[(1,m+1,m+2),P_{R,(r,s)\alpha\beta}\right]. \quad (4.30)$$

This formula is correct to all orders in $1/N$. Denote the number of rows in the Young diagram R labeling the restricted Schur polynomial by p. This implies that, since R subduces $S_n \times S_m$ representation (r,s) and $n \gg m$ that r has p rows and s has at most p rows. Now we will make use of the displaced corners approximation. To see how this works, recall that to subduce $r \vdash n$ from $R \vdash m+n$ we remove m boxes from R. Each removed box is associated with a vector in a p dimensional vector space V_p. Thus, the m removed boxes associated with the Ys thus define a vector in $V_p^{\otimes m}$. In the displaced corners approximation, the trace over $R \oplus T$ factorizes into a trace over $r \oplus t$ and a trace over $V_p^{\otimes m}$. The structure of the projector

$$P_{R,(r,s)\alpha\beta} = \mathbb{I}_r \otimes \sum_r \big|s,\ \alpha; a\big\rangle\big\langle s,\ \beta; b\big|,$$

makes it clear that the bulk of the work is in evaluating the trace over $V_p^{\otimes m}$. This trace can be evaluated using the methods developed in [54]. Introduce a basis for the fundamental representation of the Lie algebra u(p) given by $(E_{ij})_{ab} = \delta_{ia}\delta_{jb}$. Recall the product rule

$$E_{ij}E_{kl} = \delta_{jk}E_{il} \quad (4.31)$$

which we use extensively below. If a box is removed from row i it is associated to a vector v_i which is an eigenstate of E_{ii} with eigenvalue 1. The intertwining maps can be written in terms of the E_{ij}. For example, if we remove two boxes from row i of R and two boxes from row j of T, assuming that R'' and T'' have the same shape, we have

$$I_{T''R''} = E^{(1)}_{ji}E^{(2)}_{ji}\,. \quad (4.32)$$

A big advantage of realizing the intertwiners in this way is that it is simple to evaluate the product of symmetric group elements with the intertwiners. For example, using the identification (for background, see for example [57])

$$(1,2,m+1) = \mathrm{Tr}\big(E^{(1)}E^{(2)}E^{(m+1)}\big), \quad (4.33)$$

we easily find

$$(1,2,m+1)I_{T''R''} = E^{(1)}_{kl}E^{(2)}_{lm}E^{(m+1)}_{mk}E^{(1)}_{ji}E^{(2)}_{ji} = E^{(1)}_{ki}E^{(2)}_{ji}E^{(m+1)}_{jk} \quad (4.34)$$

This is now enough to evaluate the traces appearing in (4.27) and (4.29). The normalized operators are given by

$$\chi_{R,(r,s)}(Z,Y) = \sqrt{\frac{f_R \,\mathrm{hooks}_R}{\mathrm{hooks}_r \,\mathrm{hooks}_s}} O_{R,(r,s)}(Z,Y)\,.$$

The components m_i of the vector $\vec{m}(R)$ record the number of boxes removed from row i of R to produce r. In the SU(2) sector, both the one loop dilatation operator[54] and the two loop dilatation operator conserve $\vec{m}(R)$, recorded in the factor $\delta_{\vec{m}(R)\vec{m}(T)}$ in (4.35) below. In terms of these normalized operators the dilatation operator takes the form

$$\begin{aligned} &D_4 O_{R,(r,s)\mu_1\mu_2} = \\ &-2g^2 \sum_{u\,\nu_1\,\nu_2} \delta_{\vec{m}(R)\vec{m}(T)} M^{(ij)}_{s\mu_1\mu_2\,;\,u\nu_1\nu_2} \left(\Delta^{(1)}_{ij} + \Delta^{(2)}_{ij}\right) O_{R,(r,u)\nu_1\nu_2}, \end{aligned} \quad (4.35)$$

where

$$\begin{aligned} &M^{(ij)}_{s\mu_1\mu_2\,;\,u\nu_1\nu_2} = \\ &\frac{m}{\sqrt{d_u d_s}} \Big(\Big\langle \vec{m}, s, \mu_2\,;\, a | E^{(1)}_{ii} | \vec{m}, u, \nu_2\,;\, b \Big\rangle \Big\langle \vec{m}, u, \nu_1\,;\, b | E^{(1)}_{jj} | \vec{m}, s, \mu_2\,;\, a \Big\rangle \\ &+ \Big\langle \vec{m}, s, \mu_2\,;\, a | E^{(1)}_{jj} | \vec{m}, u, \nu_2\,;\, b \Big\rangle \Big\langle \vec{m}, u, \nu_1\,;\, b | E^{(1)}_{ii} | \vec{m}, s, \mu_2\,;\, a \Big\rangle \Big)\,. \end{aligned}$$

To spell out the action of the operators $\Delta^{(1)}_{ij}$ and $\Delta^{(2)}_{ij}$ we will need a little more notation. Denote the row lengths of r by r_i. The Young diagram r^+_{ij} is obtained by deleting a box from row j and adding it to row i. The Young diagram r^-_{ij} is obtained by deleting a box from row i and adding it to row j. In terms of these Young diagrams define

$$\Delta^0_{ij} O_{R,(r,s)\mu_1\mu_2} = -(2N + r_i + r_j) O_{R,(r,s)\mu_1\mu_2}, \quad (4.36)$$

$$\Delta^+_{ij} O_{R,(r,s)\mu_1\mu_2} = \sqrt{(N+r_i)(N+r_j)}\, O_{R^+_{ij},(r^+_{ij},s)\mu_1\mu_2}, \quad (4.37)$$

$$\Delta^-_{ij} O_{R,(r,s)\mu_1\mu_2} = \sqrt{(N+r_i)(N+r_j)}\, O_{R^-_{ij},(r^-_{ij},s)\mu_1\mu_2}. \quad (4.38)$$

We can now write

$$\Delta^{(1)}_{ij} = n(\Delta^+_{ij} + \Delta^0_{ij} + \Delta^-_{ij}), \quad (4.39)$$

$$\Delta^{(2)}_{ij} = (\Delta^+_{ij})^2 + \Delta^0_{ij}\Delta^+_{ij} + 2\Delta^+_{ij}\Delta^-_{ij} + \Delta^0_{ij}\Delta^-_{ij} + (\Delta^-_{ij})^2\,. \quad (4.40)$$

This completes the evaluation of the dilatation operator.

Our result for $\Delta^{(2)}_{ij}$ deserves a comment. The intertwiners $I_{T''R''}$ appearing in (4.27) only force the shapes of T and R to agree when two boxes have been removed from each. One might imagine removing a box from rows i, j of R to obtain R'' and from rows k, l of T to obtain T'', implying that in total four rows could participate. We see from $\Delta^{(2)}_{ij}$ that this is not the case - the mixing is much more constrained with only two rows participating. We discuss this point further in Appendix I.

4.3.2 Spectrum

An interesting feature of the result (4.35) is that the action of the dilatation operator has factored into the product of two actions: $\Delta_{ij}^{(1)}+\Delta_{ij}^{(2)}$ acts only on Young diagram r i.e. on the Zs, while $M_{s\mu_1\mu_2\,;\,u\nu_1\nu_2}^{(ij)}$ acts only on Young diagram s, i.e. on the Ys. This factored form, which also arises at one loop, implies that we can diagonalize on the $s\mu_1\mu_2;u\nu_1\nu_2$ and the $R,r;T,t$ labels separately. The diagonalization on the $s\mu_1\mu_2;u\nu_1\nu_2$ labels is identical to the diagonalization problem which arises at one loop. The solution was obtained analytically for two rows in [53] and then in general in [89]. Each possible open string configuation consistent with the Gauss Law constraint can be identified with an element of a double coset. A very natural basis of functions, constructed from representation theory, is suggested by Fourier transformation applied to this double coset. In this way [89] constructed an explicit formula for the wavefunction which solves the $s\mu_1\mu_2;u\nu_1\nu_2$ diagonalization. The resulting Gauss graph operators are labeled by elements of the double coset. The explicit solution obtained in [89] is

$$O_{R,r}(\sigma)=\frac{|H|}{\sqrt{m!}}\sum_{j,k}\sum_{s\vdash m}\sum_{\mu_1,\mu_2}\sqrt{d_s}\Gamma^{(s)}_{jk}(\sigma)B^{s\to 1_H}_{j\mu_1}B^{s\to 1_H}_{k\mu_2}O_{R,(r,s)\mu_1\mu_2}, \qquad (4.41)$$

where the group $H=S_{m_1}\times S_{m_2}\times\cdots\times S_{m_p}$ and the branching coefficients $B^{s\to 1_H}_{j\mu_1}$ provide a resolution of the projector from irrep s of S_m onto the trivial representation of H

$$\frac{1}{|H|}\sum_{\sigma\in H}\Gamma^{(s)}_{ik}(\sigma)=\sum_{\mu}B^{s\to 1_H}_{i\mu}B^{s\to 1_H}_{k\mu}\,. \qquad (4.42)$$

The action of the dilatation operator on the Gauss graph operator is

$$D_4 O_{R,r}(\sigma)=-2g^2\sum_{i<j}n_{ij}(\sigma)\left(\Delta_{ij}^{(1)}+\Delta_{ij}^{(2)}\right)O_{R,r}(\sigma). \qquad (4.43)$$

The numbers $n_{ij}(\sigma)$ can be read off of the element of the double coset σ. Each possible Gauss operator is given by a set of m open strings stretched between p different giant graviton branes. As an example, consider $p=4$ with $m=5$. Two possible configurations are shown in Figure 4.3. Label the open strings with integers from 1 to $m=5$ for our example. The double coset element can then be read straight from the open string configuration by recording how the open strings are ordered as closed circuits in the graph are traversed. For the graphs shown, (a) corresponds to $\sigma=(1245)(3)$ and (b) corresponds to $\sigma=(12)(34)(5)$. The numbers $n_{ij}(\sigma)$ tell us how many strings stetch between branes i and j. The branes themselves are numbered with integers from 1 to p, as shown in Figure 4.4 for our example. Thus, for (a) the non-zero n_{ij} are $n_{12}=1$, $n_{23}=1$, $n_{34}=1$, and $n_{14}=1$. Notice that we don't record strings that emanate and terminate on the same brane - string 3 in (a) or string 5 in

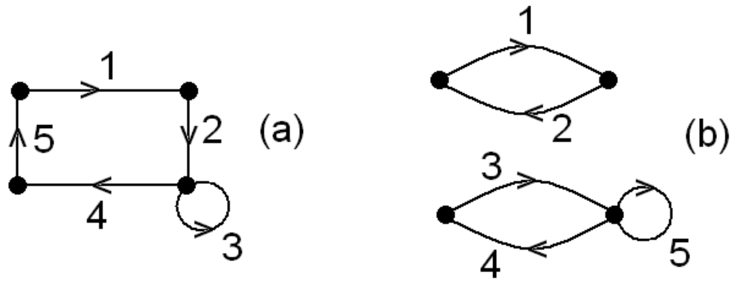


Figure 4.3: Two possible configurations for operators with $p = 4$ and $m = 5$.

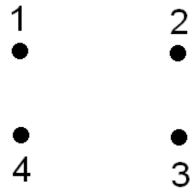


Figure 4.4: Labeling of the giant graviton branes.

(b), in this example. For (b) the non-zero n_{ij} are $n_{12} = 2$ and $n_{34} = 2$. For the details, see [89].

To obtain the anomalous dimensions, inspection of (4.43) shows that we now have to solve the eigenproblem of $\Delta_{ij}^{(1)}$ and $\Delta_{ij}^{(2)}$. The operator $\Delta_{ij}^{(1)}$ is simply a scaled version of the operator which plays a role in the one loop dilatation operator. The corresponding operator which participates at one loop was identified as an element of u(p) [99]. It is related to a system of p particles in a line with 2-body harmonic oscillator interactions[99]. The operator $\Delta_{ij}^{(2)}$ is new. Following [99], a useful approach is to study the continuum limit of $\Delta_{ij}^{(1)}$ and $\Delta_{ij}^{(2)}$. Towards this end, introduce the variables

$$y_j = \frac{r_{j+1} - r_1}{\sqrt{N + r_1}}, \qquad j = 1, 2, 3, ..., p - 1 \tag{4.44}$$

which become continuous variables in the large N limit. We have numbered rows so that $r_1 < r_2 < \cdots < r_p$. In the continuum limit our Gauss graph operators become functions of y_i

$$O_{R,r}(\sigma) \equiv O^{\vec{m}(R)}(\sigma, r_1, r_2, \cdots, r_p) \to O^{\vec{m}(R)}(r_1, y_1, \cdots, y_{p-1}). \tag{4.45}$$

Using the expansions

$$\sqrt{(N + r_i)(N + r_j)} = N + r_1 + \frac{y_i + y_j}{2}\sqrt{N + r_1} - \frac{(y_i - y_j)^2}{8} + O\left(\frac{1}{\sqrt{N + r_1}}\right), \tag{4.46}$$

and

$$O^{\vec{m}(R)}(r_1, y_1, \cdots, y_i + \frac{1}{\sqrt{N+r_1}}, \cdots, y_j - \frac{1}{\sqrt{N+r_1}}, \cdots, y_{p-1})$$
$$= O^{\vec{m}(R)}(r_1, y_1, \cdots, y_{p-1}) + \frac{1}{\sqrt{N+r_1}} \frac{\partial}{\partial y_i} O^{\vec{m}(R)}(r_1, y_1, \cdots, y_{p-1})$$
$$- \frac{1}{\sqrt{N+r_1}} \frac{\partial}{\partial y_j} O^{\vec{m}(R)}(r_1, y_1, \cdots, y_{p-1})$$
$$+ \frac{1}{2(N+r_1)} \left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_j} \right)^2 O^{\vec{m}(R)}(r_1, y_1, \cdots, y_{p-1}), \tag{4.47}$$

we find that in the continuum limit

$$\Delta^{(1)}_{i+1\, j+1} O_{R,r}(\sigma) \to n \left[\left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_j} \right)^2 - \frac{(y_i - y_j)^2}{4} \right] \times O^{\vec{m}(R)}(r_1, y_1, \cdots, y_{p-1}), \tag{4.48}$$

$$\Delta^{(1)}_{1\, i+1} O_{R,r}(\sigma) \to n \left[\left(2\frac{\partial}{\partial y_i} + \sum_{j \neq i} \frac{\partial}{\partial y_j} \right)^2 - \frac{y_i^2}{4} \right] O^{\vec{m}(R)}(r_1, y_1, \cdots, y_{p-1}), \tag{4.49}$$

and

$$\Delta^{(2)}_{i+1\, j+1} O_{R,r}(\sigma) \to 2(N+r_1) \left[\left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_j} \right)^2 - \frac{(y_i - y_j)^2}{4} \right] \times O^{\vec{m}(R)}(r_1, y_1, \cdots, y_{p-1}), \tag{4.50}$$

$$\Delta^{(2)}_{1\, i+1} O_{R,r}(\sigma) \to 2(N+r_1) \left[\left(2\frac{\partial}{\partial y_i} + \sum_{j \neq i} \frac{\partial}{\partial y_j} \right)^2 - \frac{y_i^2}{4} \right] \times O^{\vec{m}(R)}(r_1, y_1, \cdots, y_{p-1}). \tag{4.51}$$

Remarkably, in the continuum limit both $\Delta^{(1)}_{ij}$ and $\Delta^{(2)}_{ij}$ have reduced to scaled versions of exactly the same operator that appears in the one loop problem. In the Appendix G we argue for the same conclusion without taking a continuum limit. This implies that the operators that have a good scaling dimension at one loop are uncorrected at two loops.

It is now straight forward to obtain the two loop anomalous dimension for any operator of interest. An instructive and simple example is provided by $p = 2$ with[14] $n_{12} = n_{12}^+ + n_{12}^- \neq 0$. In this case, the anomalous dimension $\gamma(g^2)$ which is the eigenvalue of

$$D = D_2 + D_4, \tag{4.52}$$

[14]The number n_{12}^+ counts the number of open strings stretching from giant graviton 1 to giant graviton 2; the number n_{12}^- counts the number of open strings stretching from giant graviton 2 to giant graviton 1. The Gauss Law constraint forces $n_{12}^+ = n_{12}^-$. See [89] for more details.

with[15]

$$D_2 = -2g : \mathrm{Tr}\left([Y,Z]\left[\frac{\partial}{\partial Y},\frac{\partial}{\partial Z}\right]\right) :, \tag{4.53}$$

and D_4 given in (4.20), is

$$\gamma = 16qn_{12}^{+}\big(g + (2N + 2r_1 + n)g^2\big), \tag{4.54}$$

$$q = 0,1,2,..,M \quad n_{12} = 0,1,2,... \tag{4.55}$$

where the upper cut off M is itself a number of order N. Clearly, if the g^2 term is to be a small correction to the leading term, we must hold $\lambda_g \equiv gN$ fixed, which corresponds to the usual 't Hooft limit. The fact that the usual 't Hooft scaling leads to a sensible perturbative expansion in this sector of the theory was already understood in [86]. We then find

$$\gamma = \frac{16qn_{12}}{N}\big(\lambda_g + (2 + 2\frac{r_1}{N} + \frac{n}{N})\lambda_g^2\big). \tag{4.56}$$

For a given open string plus giant system (i.e. a given n_{12}), in the large N limit, $x = \frac{q}{N}$ varies continuously from 0 to $x = \frac{M}{N}$ implying that the spectrum of anomalous dimensions

$$\gamma = 16xn_{12}\big(\lambda_g + (2 + 2\frac{r_1}{N} + \frac{n}{N})\lambda_g^2\big), \tag{4.57}$$

is itself continuous. At finite N this spectrum is discrete. Notice that since both n and r_1 are of order N, all three terms multiplying λ_g^2 in (4.57) are of the same size. Note that the value for γ (4.57) will recieve both $\frac{1}{N}$ corrections and $\frac{m}{n}$ corrections.

4.3.3 Discussion

We conclude this section by collecting our results as a set of questions and their answers.

1 **Is the dilatation operator integrable in the large N displaced corners approximation at higher loops?**

We don't know. We have however been able to argue that the dilatation operator is integrable in the large N displaced corners approximation at two loops. This requires both sending $N \to \infty$ and keeping $m \ll n$ to ensure the validity of the displaced corners approximation. At large N with $m \sim n$ we do not know how to compute the action of the dilatation operator and hence integrability in this situation is an interesting open problem. It seems reasonable to hope that integrability will persist in the large N displaced corners approximation at higher loops.

[15]The normalization for both D_2 and D_4 follows [51]. This normalization for D_2 is a factor of 2 larger than the normalization used in [65, 52, 53, 56, 93, 54, 89, 99].

2 **Do the $O_{R,r}(\sigma)$ of [89] continue to solve the Y eigenproblem at higher loops?**

Yes, the Gauss graph operators do indeed solve the Y eigenproblem at two loops. The Y eigenproblem at two loops is identical to the Y eigenproblem at one loop, so that even the eigenvalues (given by $n_{ij}(\sigma)$ in (4.43)) are unchanged. The fact that the Gauss operators continue to solve the Y eigenproblem does not depend sensitively on the coefficients of the individual terms in the two loop dilatation operator (see Appendix C).

3 **Can the two loop Z eigenproblem be mapped to a system of p particles, again using the Lie algebra of $\mathbf{U}(p)$?**

We have indeed managed to map the Z eigenproblem to the dynamics of p particles (in the center of mass frame). The two loop problem again has a very natural phrasing in terms of the Lie algebra of U(p). The one loop and two loop problems are different: they share the same eigenstates but have different eigenvalues. The fact that the eigenstates are the same does depend sensitively on the coefficients of the individual terms in the two loop dilatation operator (see Appendix C).

4 **Does the two loop correction to the anomalous dimension determine the precise limit that should be taken to get a sensible perturbative expansion?**

Yes - requiring that the two loop correction in (4.56) is small compared to the one loop term clearly implies that we should be taking the standard 't Hooft limit. Our result then has an interesting consequence: at large N, $x = q/N$ becomes a continuous parameter and we recover a continuous energy spectrum. This is clearly related to [90]. At any finite N the spectrum is discrete.

Our discussion has been developed for operators with a label R that has p long rows, which are dual to giant gravitons wrapping an $S^3 \subset AdS_5$. Operators labeled by an R that has p long columns are dual to giant gravitons wrapping an $S^3 \subset S^5$. The anomalous dimensions for these operators are easily obtained from our results in this section (see section D.6 of [54] for a discussion of this connnection). The $\Delta_{ij}^{(1)}$ for this case is obtained by replacing the $r_i \to -r_i$ and $r_j \to -r_j$ in (4.48) and (4.49), while $\Delta_{ij}^{(2)}$ for this case is obtained by replacing the $r_i \to -r_i$ and $r_j \to -r_j$ in (4.50) and (4.51). The result (4.43) is unchanged when written in terms of the new $\Delta_{ij}^{(1)}$ and $\Delta_{ij}^{(2)}$.

Finally, the fact that our operators are not corrected at two loops is remarkable. It is natural now to conjecture that they are in fact exact and will not be corrected at any higher loop. This is somewhat reminiscent of the BMN operators[91]. In that case it is possible to determine the exact anomalous dimensions as a function of the 't Hooft coupling λ_g[92].

Chapter 5

Nonplanar Integrability in ABJ(M) Theory

In this chapter we study the action of two loop dilatation operator of ABJM and ABJ theories. For brevity, we refer to both of them by ABJ(M) theory. In the case of ABJM theory, we study the spectrum of the nonplanar dilatation operator at two loops in the $SU(2) \times SU(2)$ subsector using the restricted Schur polynomials introduced in chapter 3. As we have seen in chapter 3, these restricted Schur polynomials provide a basis for the local gauge invariant operators for theories with gauge group $U(N) \times U(N)$. We then generalize our study to Chern-Simons theory with gauge group $U(M) \times U(N)$. As we will see, this class of theories exhibit parity violation. A connection between parity in the field theory and the holonomy of the string theory dual is confirmed. This chapter is based on [97] and [98].

5.1 Action of the Dilatation Operator

The two loop dilatation generator in the sector with $n_2 = 0$ is [95]

$$D = -\left(\frac{4\pi}{k}\right)^2 : \mathrm{Tr}\Big[\left(B_2^\dagger A_1 B_1^\dagger - B_1^\dagger A_1 B_2^\dagger\right) \left(\frac{\partial}{\partial B_2^\dagger}\frac{\partial}{\partial A_1}\frac{\partial}{\partial B_1^\dagger} - \frac{\partial}{\partial B_1^\dagger}\frac{\partial}{\partial A_1}\frac{\partial}{\partial B_2^\dagger}\right) \Big] : \,.$$

We will compute the action of the dilatation operator on operators normalized so that

$$\langle \hat{O}_{R,\{r\}} \hat{O}^\dagger_{S,\{s\}} \rangle = f_R \delta_{RS} \delta_{\{r\}\{s\}} \,.$$

This choice of normalization makes the present problem look as similar as possible to that of [52]. The relation of these normalized operators (indicated with a hat) to the operators of subsection 3.5.2 is

$$O_{R,\{r\}} = \sqrt{\frac{\mathrm{hooks}_R\, f_R}{\mathrm{hooks}_{r_{11}}\, \mathrm{hooks}_{r_{12}}}} \hat{O}_{R,\{r\}} \,.$$

Using the methods of [65, 52], it is straight forward to obtain

$$D\hat{O}_{R,\{r\}} = \sum_{S,\{s\}} M_{R,\{r\},S,\{s\}} \hat{O}_{S,\{s\}}$$

where

$$M_{R,\{r\},S,\{s\}} = \sqrt{\frac{\text{hooks}_S\, f_S\, \text{hooks}_{r_{11}}\, \text{hooks}_{r_{12}}}{\text{hooks}_R\, f_R\, \text{hooks}_{s_{11}}\, \text{hooks}_{s_{12}}}} \sum_{R'} \frac{m_1\, m_2\, d_S\, c_{RR'}}{d_{s_{11}}\, d_{s_{12}}\, n\, d_{R'}} \times$$

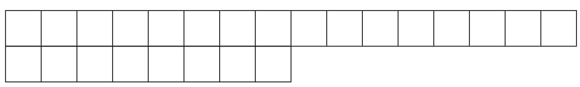
$$\times \Big[N \text{Tr}\big(I_{S'R'}\left[\Gamma_R\big((1, m_2+1)\big), P_{R,\{r\}}\right] I_{R'S'} C\big) +$$
$$+ \text{Tr}\big(I_{S'R'}\left[\Gamma_R\big((1, m_2+1)\big) P_{R,\{r\}} \Gamma_R\big((1, m_2+1)\big) - P_{R,\{r\}}\right] I_{R'S'} C\big) +$$
$$+ (m_1 - 1) \text{Tr}\big(I_{S'R'}\big[\Gamma_R\big((1, m_2+1)\big) P_{R,\{r\}} \Gamma_R\big((1, m_2+2)\big)$$
$$- \Gamma_S\big((m_2+2, 1, m_2+1)\big) P_{R,\{r\}}\big] I_{R'S'} C\big) + (m_2 - 1) \times$$
$$\text{Tr}\big(I_{S'R'}\big[\Gamma_R\big((1, m_2+1)\big) P_{R,\{r\}} \Gamma_R\big((1, 2)\big)$$
$$- P_{R,\{r\}} \Gamma_S\big((2, 1, m_2+1)\big)\big] I_{R'S'} C\big) \Big]$$

and

$$C = \left[P_{S,\{s\}}, \Gamma_S\left((1, m_2+1)\right)\right].$$

To obtain this result we have used the first m_2 slots for the ϕ_{12}s and the next m_1 slots for the ϕ_{11}s.

We will study the spectrum of anomalous dimensions for operators whose labels $R, \{r\}$ are all Young diagrams with two long rows. In this case we can use $U(2)$ group theory to construct the projectors as explained in [53, 54]. The extension to operators whose labels $R, \{r\}$ are all Young diagrams with p long rows is also possible; in this case $U(p)$ group theory is used[54]. To set up the two long rows problem, we will employ a more convenient labeling for our operators. Notice that m_1 and m_2 are fixed. We study the limit in which both m_1 and m_2 are $O(N)$, but $m_2 \ll m_1$. Denote the number of boxes in row 1 of r_{12} minus the number of boxes in row 2 by $2j$. The number of boxes in row 2 is thus $\frac{m_2-2j}{2}$. For $m_2 = 24$ and $j = 4$ we have

$r_{12} =$ 

In this way, we trade r_{12} for an integer j. Next, imagine that to obtain r_{11} from R we need to pull ν_1 boxes from the first row of R and ν_2 boxes from the second row of R. Since we know that $\nu_1 + \nu_2 = m_2$ it is enough to specify $\nu_1 - \nu_2 \equiv 2j^3$. Finally, we will trade r_{11} for the two integers b_0 and b_1. b_1 is the number of columns with a single box while b_0 is the number of columns containing two boxes. Note that this notation is redundant because $2b_0 + b_1 = m_1$. Thus, we trade the three Young diagrams R, r_{11}, r_{12} for the integers b_0, b_1, j, j^3. See

Figure 5.1 for a summary. Using the ideas developed in [53, 54] we find after a straight forward but tedious computation

$$
\begin{aligned}
D\hat{O}_{j,j^3}(b_0,b_1) \quad &= \left(\frac{4\pi}{k}\right)^2 \Bigg[\Bigg(-\frac{N}{2}\left(m_2 - \frac{(m_2+2)(j^3)^2}{j(j+1)}\right) \\
&\quad -\frac{m_2^2}{4} + m_2 + j_3^2 - j(j+1) - \frac{j_3^2(4-m_2^2)}{4j(j+1)}\Bigg)\Delta\hat{O}_{j,j^3}(b_0,b_1)
\end{aligned} \tag{5.1}
$$

$$
\begin{aligned}
&+N\sqrt{\frac{(m_2+2j+4)(m_2-2j)}{(2j+1)(2j+3)}\frac{(j+j^3+1)(j-j^3+1)}{2(j+1)}}\left(1+\frac{m_2-2j-4}{2N}\right)\times \\
&\quad \Delta\hat{O}_{j+1,j^3}(b_0,b_1) + \sqrt{\frac{(m_2+2j+2)(m_2-2j+2)}{(2j+1)(2j-1)}\frac{(j+j^3)(j-j^3)}{2j}}\times \\
&\quad \left(1+\frac{m_2-2j-2}{2N}\right)\Delta\hat{O}_{j-1,j^3}(b_0,b_1)\Bigg]
\end{aligned}
$$

where

$$
\begin{aligned}
&\Delta\hat{O}_{j,j^3}(b_0,b_1) = \\
&\quad \sqrt{(N+b_0)(N+b_0+b_1)}(\hat{O}_{j,j^3}(b_0+1,b_1-2) + \hat{O}_{j,j^3}(b_0-1,b_1+2)) \\
&\quad -(2N+2b_0+b_1)\hat{O}_{j,j^3}(b_0,b_1).
\end{aligned} \tag{5.2}
$$

This is remarkably similar to the result obtained for two row restricted Schurs in the $SU(2)$ sector of $\mathcal{N}=4$ super Yang Mills theory[53]. In particular, the fact that only the combination $\Delta O_{j,j^3}(b_0,b_1)$ appears implies that after we have diagonalized on the j label, the problem of diagonalizing on the b_0, b_1 labels again reduces to diagonalizing a set of decoupled harmonic oscillators.

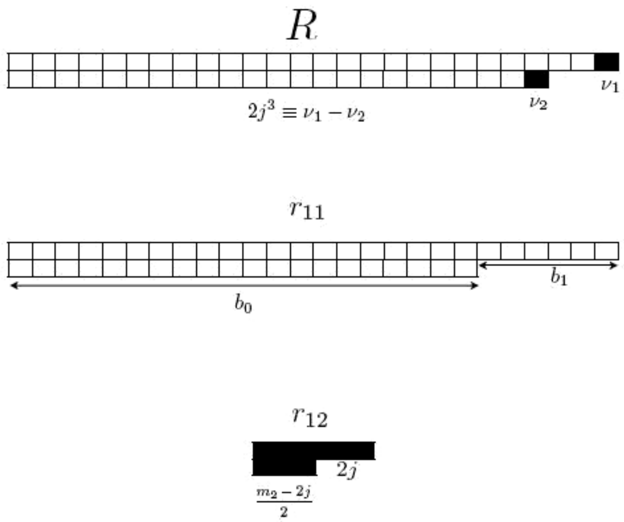


Figure 5.1: A summary of the $U(2)$ labeling.

5.2 Spectrum of Anomalous Dimensions

For the $SU(2)$ sector of $\mathcal{N}=4$ super Yang-Mills theory [99] have proved that studying the dilatation operator in a continuum limit reproduces the spectrum obtained by solving the original discrete anomalous dimension eigenvalue problem. In this section we will consider a continuum limit of the diltation operator that reduces to the problem studied in [53].

Consider first the problem of diagonalizing on the b_0, b_1 labels. We introduce $x = 2\frac{b_1}{\sqrt{N+b_0}}$. For any finite arbitrarily large x we have $b_1 \sim \sqrt{N}$. In this limit [53] show that the operator Δ defined in (5.2) reduces to the harmonic oscillator Hamiltonian which is easily diagonalized. Consider next the problem of diagonalizing on the j, j_3 labels. To solve this problem we will consider the double scaling limit defined by taking $m_2 \to \infty$, $b_1 \to \infty$ holding $\frac{m}{b_1} \sim \gamma \ll 1$ fixed. In this case $\sqrt{\frac{2}{m}}j$ becomes a continuous variable. It is straight forward to see that in this continuum limit, the action found for the dilatation operator in the previous section reduces to the continuum limit of the action of the dilatation operator studied in [53]. From the results of that work we know that if $m_2 = 2n$ we obtain a set of oscillators with frequency ω_i and degeneracy d_i given by

$$\omega_i = 8iN\left(\frac{4\pi}{k}\right)^2, \qquad d_i = 2(n-i)+1, \qquad i = 0, 1, ..., n\,.$$

and if $m_2 = 2n+1$ we obtain a set of oscillators with frequency ω_i and degeneracy d_i given by

$$\omega_i = 8iN\left(\frac{4\pi}{k}\right)^2, \qquad d_i = 2(n-i+1), \qquad i = 0, 1, ..., n\,.$$

5.3 ABJ Dilatation Operator

ABJ theory is a three-dimensional $\mathcal{N}=6$ superconformal Chern-Simons-matter theory with gauge group $U(M)_k \times \overline{U(N)}_{-k}$ and R-symmetry group $SU(4)$. In this case the bifundamental scalar fields A and B transform as

$$A \to A' = U(M)\,A\overline{U(N)}^{\dagger},$$

$$B \to B' = U(M)\,B\overline{U(N)}^{\dagger}.$$

Therefore all traces constructed from the pairs $AB^{\dagger}$ are invariant under $U(M)_k \times \overline{U(N)}_{-k}$ gauge transformations.

The ABJ dilatation operator is closely related to the ABJM one, since both of them come from the F-terms of the bosonic potential. The key difference is in their coupling constants. In ABJM theory the coupling constant is $\left(\frac{\lambda}{N}\right)^2$

while in ABJ theory it is $\left(\frac{\lambda}{N}\right)\left(\frac{\hat{\lambda}}{M}\right)$ where $\lambda=\frac{4\pi N}{k}$ and $\hat{\lambda}=\frac{4\pi M}{k}$. Therefore in ABJ theory we have a double 't Hooft limit that is given by

$$N, M\to\infty,\quad k\to\infty,\quad \lambda,\hat{\lambda}\,\text{fixed}.$$

The dilatation operator of ABJ theory has a closed action on the $SU(2)\times SU(2)$ subsector built with one type of excitation field B_2 [96]. On this sector the dilatation operator is

$$D=\left(V_F^{bos}\right)^{eff}= \\ -\frac{\lambda}{N}\frac{\hat{\lambda}}{M}:\mathrm{Tr}\left[\left(\mathrm{B}_2^\dagger\mathrm{A}_1\mathrm{B}_1^\dagger-\mathrm{B}_1^\dagger\mathrm{A}_1\mathrm{B}_2^\dagger\right)\left(\frac{\partial}{\partial\mathrm{B}_2^\dagger}\frac{\partial}{\partial\mathrm{A}_1}\frac{\partial}{\partial\mathrm{B}_1^\dagger}-\frac{\partial}{\partial\mathrm{B}_1^\dagger}\frac{\partial}{\partial\mathrm{A}_1}\frac{\partial}{\partial\mathrm{B}_2^\dagger}\right)\right]:,\tag{5.3}$$

where A_1 and $B_1^\dagger$ are the "background" fields, and :: means that the fields in (5.3) should not be self-contracted.
Following our study of the ABJM case, we will consider the action of the dilatation operator (5.3) on the gauge invariant operators built from restricted Schur polynomials for the gauge group $U(M)_k\times\overline{U(N)_{-k}}$. With one type of "excitation" $B_2^\dagger$ these polynomials are

$$O_{R,\{r\}}=\frac{1}{\prod_{ij}n_{ij}!}\sum_{\sigma\in S_n}\mathrm{Tr}_{\{r\}}\left(\Gamma_R(\sigma)\right)\prod_{j=1}^{m_2}\left(A_1B_2^\dagger\right)^{a_j}_{a_{\sigma(j)}}\prod_{i=m_2+1}^{n}\left(A_1B_1^\dagger\right)^{a_i}_{a_{\sigma(i)}},\tag{5.4}$$

where the label R specifies an irreducible representation of the symmetric group S_n and $\{r\}\equiv\{r_{12},r_{11}\}$ is an irreducible representation of $S_{n_{12}}\times S_{n_{11}}\subset S_n$. We have $n_{12}+n_{11}=n$.
In this expression $m_2=n_{12}$ is the number of $A_1B_2^\dagger$ pairs and $m_1=n_{11}$ is the number of $A_1B_1^\dagger$ pairs. Since the number of background fields is much larger than the number of excitation fields we have $m_2\ll m_1$.
As in the case of ABJM, we are interested in computing the action of ABJ dilatation operator on normalized operators $\widehat{O}_{R,\{r\}}$ using the two point function obtained in subsection 3.5.2. Before we study the action of the dilatation operator on (5.4), it is important to point out that in ABJ theory both A and B are matrix fields, and hence we have the following index structure

$$\left(A_kB_l^\dagger\right)^{a_i}_{a\sigma(i)}=\left(A_k\right)^{a_i}_{\alpha}\left(B_l^\dagger\right)^{\alpha}_{a\sigma(i)},\qquad \alpha=1,2,...,M,\tag{5.5}$$

where $k,\ l\in\{1,2\}$.
The result of acting with (5.3) on (5.4) is similar to the result obtained in [97]. The only difference is in the contraction δ^α_α which gives M instead of N . The result is thus

$$D\hat{O}_{R,\{r\}}=\sum_{S,\{s\}}M_{R,\{r\},S,\{s\}}\hat{O}_{S,\{s\}}\tag{5.6}$$

where

$$M_{R,\{r\},S,\{s\}} = \sqrt{\frac{\text{hooks}_S\, f_S\, \text{hooks}_{r_{11}}\, \text{hooks}_{r_{12}}}{\text{hooks}_R\, f_R\, \text{hooks}_{s_{11}}\, \text{hooks}_{s_{12}}}} \sum_{R'} \frac{m_1\, m_2\, d_S\, c_{RR'}}{d_{s_{11}}\, d_{s_{12}}\, n\, d_{R'}} \times$$

$$\times \Big[M \text{Tr}\big(I_{S'R'}\big[\Gamma_R\big((1,m_2+1)\big), P_{R,\{r\}}\big] I_{R'S'} C\big) +$$
$$+ \text{Tr}\big(I_{S'R'}\big[\Gamma_R\big((1,m_2+1)\big) P_{R,\{r\}} \Gamma_R\big((1,m_2+1)\big) - P_{R,\{r\}}\big] I_{R'S'} C\big) +$$
$$+(m_1-1)\text{Tr}\big(I_{S'R'}\big[\Gamma_R\big((1,m_2+1)\big) P_{R,\{r\}} \Gamma_R\big((1,m_2+2)\big)$$
$$-\Gamma_S\big((m_2+2,1,m_2+1)\big) P_{R,\{r\}}\big] I_{R'S'} C\big) + (m_2-1)\times$$
$$\text{Tr}\big(I_{S'R'}\big[\Gamma_R\big((1,m_2+1)\big) P_{R,\{r\}} \Gamma_R\big((1,2)\big)$$
$$-P_{R,\{r\}} \Gamma_S\big((2,1,m_2+1)\big)\big] I_{R'S'} C\big)\Big]$$

and

$$C = \left[P_{S,\{s\}}, \Gamma_S\left((1,m_2+1)\right)\right].$$

5.4 Parity Operation

The action of the parity operator introduced in [96], on the trace of a product of fields, inverts the order of the fields inside each of its traces. For example, for a single trace operator we have:

$$\hat{P}\ :\ \text{Tr}\left(A^{a_1} B_{b_1} ... A^{a_l} B_{b_l}\right) \rightarrow \text{Tr}\left(B_{b_l} A^{a_l} ... B_{b_1} A^{a_1}\right),$$

where $a_i, b_i \in 1,2$. In this way, acting with the parity operation on the restricted Schur polynomial (5.4) leads to

$$\hat{P} O_{R,\{r\}} = \frac{1}{\prod_{ij} n_{ij}!} \sum_{\sigma\in S_n} \text{Tr}_{\{r\}}\left(\Gamma_R\left(\sigma^{-1}\right)\right) \prod_{j=1}^{m_2} \left(A_1 B_2^\dagger\right)^{a_j}_{a_{\sigma(j)}} \prod_{i=m_2+1}^{n} \left(A_1 B_1^\dagger\right)^{a_i}_{a_{\sigma(i)}}. \tag{5.7}$$

Therefore, parity changes σ to σ^{-1}.

The action of three-dimensional parity P takes the $U(N+l)_k \times U(N)_{-k}$ superconformal theory with $M-N=l$ to the $U(N)_k \times U(N+l)_{-k}$ theory [101]. This means that parity flips the levels of the Chern-Simons terms, and consequently produces a different theory. In this case, the action of (5.3) on a $U(N)_k \times U(N+l)_{-k}$ gauge invariant operator of the form (5.4) produces the result (5.6) except that M changes to N.

5.5 Spectrum of Anomalous Dimensions

In ABJ theory, the spectrum of anomalous dimensions is similar to the ABJM case [97]. The only difference is that the sum over the dummy indices gives M. The irrep r_{11} is obtained from R by removing ν_1 (or ν_2) boxes from the first (or the second) row of R respectively. Since we know that the removed boxes ν_1 and ν_2 form the irrep r_{12} with $\nu_1 + \nu_2 = m_2$, it is enough to specify $\nu_1 - \nu_2 \equiv 2j^3$. Specify the irrep r_{11} by b_0 and b_1 with $2b_0 + b_1 = m_1$ and b_1 is the number of boxes in the short row. Denote the number of boxes in the first row of r_{12} minus the number of boxes in the second row by $2j$. The number of boxes in the second row of r_{12} is thus $\frac{m_2-2j}{2}$. The explicit evaluation of (5.6) gives

$$\begin{aligned} D\hat{O}_{j,j^3}(b_0,b_1) \quad &= \left(\frac{4\pi}{k}\right)^2 \Bigg[\Bigg(-\frac{M}{2}\left(m_2 - \frac{(m_2+2)(j^3)^2}{j(j+1)} \right) \qquad (5.8) \\ &-\frac{m_2^2}{4} + m_2 + j_3^2 - j(j+1) - \frac{j_3^2(4-m_2^2)}{4j(j+1)} \Bigg) \Delta\hat{O}_{j,j^3}(b_0,b_1) \end{aligned}$$

$$\begin{aligned} &+M\sqrt{\frac{(m_2+2j+4)(m_2-2j)}{(2j+1)(2j+3)}}\frac{(j+j^3+1)(j-j^3+1)}{2(j+1)}\left(1+\frac{m_2-2j-4}{2N}\right)\times \\ &\Delta\hat{O}_{j+1,j^3}(b_0,b_1) + M\sqrt{\frac{(m_2+2j+2)(m_2-2j+2)}{(2j+1)(2j-1)}}\frac{(j+j^3)(j-j^3)}{2j}\times \\ &\left(1+\frac{m_2-2j-2}{2N}\right)\Delta\hat{O}_{j-1,j^3}(b_0,b_1)\Bigg]. \qquad (5.9) \end{aligned}$$

This differs from the result given [97] since certain factor of N are replaced by M. In the above expression

$$\begin{aligned} &\Delta\hat{O}_{j,j^3}(b_0,b_1) = \\ &\quad \sqrt{(M+b_0)(M+b_0+b_1)}(\hat{O}_{j,j^3}(b_0+1,b_1-2) + \hat{O}_{j,j^3}(b_0-1,b_1+2)) \\ &\quad -(2M+2b_0+b_1)\hat{O}_{j,j^3}(b_0,b_1). \qquad (5.10) \end{aligned}$$

The combination $\triangle O_{j,j^3}(b_0,b_1)$ in (5.10) is identical to the one obtained in the case of $\mathcal{N}=4$ SYM theory. Furthermore, the label j in (5.10) implies the possibility of diagonalising (5.9). We note that the structure of the dilatation operator in (5.9) is different from the result obtained in [53]. It would be interesting to obtain a direct diagonalisation of (5.9). In the double scaling limit

$$M,\, N \to \infty, \qquad \frac{m_2}{M}, \frac{m_2}{N} \ll 1,$$

(5.9) reduces to

$$\begin{aligned} D\hat{O}_{j,j^3}(b_0,b_1) &= \left(\frac{4\pi}{k}\right)^2 \Bigg[\left(-\frac{M}{2}\left(m_2 - \frac{(m_2+2)(j^3)^2}{j(j+1)} \right)\right) \triangle\hat{O}_{j,j^3}(b_0,b_1) \\ &+M\sqrt{\frac{(m_2+2j+4)(m_2-2j)}{(2j+1)(2j+3)}}\frac{(j+j^3+1)(j-j^3+1)}{2(j+1)}\triangle\hat{O}_{j+1,j^3}(b_0,b_1) \end{aligned}$$

$$+M\sqrt{\frac{(m_2+2j+2)\,(m_2-2j+2)}{(2j+1)\,(2j+3)}}\frac{(j+j^3)\,(j-j^3)}{2j}\triangle\hat{O}_{j-1,j^3}\,(b_0,b_1)\Bigg],\quad (5.11)$$

which is similar to the result of $\mathcal{N}=4$ SYM theory differing only because it is multiplied by M. In this limit the continuum limit in ABJ theory therefore reduces to the action of the dilatation operator studied in [97], from which we obtain a set of oscillators with frequency ω_i and degeneracy d_i given by the following two cases

1 If $m_2 = 2n$

The theory with gauge group $U(M)_k \times U(N)_{-k}$ has a spectrum of anomalous dimensions ω_i with degeneracy d_i given by

$$\omega_i = 8iM\left(\frac{4\pi}{k}\right)^2,\quad d_i = 2\,(n-i)+1,\quad i=0,1,...,n.$$

The theory with gauge group $U(N)_k \times U(M)_{-k}$ has the spectrum

$$\omega_i = 8iN\left(\frac{4\pi}{k}\right)^2,\quad d_i = 2\,(n-i)+1,\quad i=0,1,...,n.$$

2 If $m_2 = 2n+1$

The theory with gauge group $U(M)_k \times U(N)_{-k}$ has the spectrum

$$\omega_i = 8iM\left(\frac{4\pi}{k}\right)^2,\quad d_i = 2\,(n-i+1)\,,\quad i=0,1,...,n.$$

The theory with gauge group $U(N)_k \times U(M)_{-k}$ has the spectrum

$$\omega_i = 8iN\left(\frac{4\pi}{k}\right)^2,\quad d_i = 2\,(n-i+1)\,,\quad i=0,1,...,n.$$

Since $U(M)_k \times U(N)_{-k}$ and $U(N)_k \times U(M)_{-k}$ Chern-Simons theories are related by a parity transformation, from the spectrum of the dilatation operator of these theories, we observe the following

$$\left[D^{ABJ}_{non-planar},\, P\right] \propto (M-N)\,. \quad (5.12)$$

Recall that since $M-N=l$, we have

$$U\,(M)_k \times \overline{U\,(N)}_{-k} = U\,(N+l)_k \times \overline{U\,(N)}_{-k},$$

$$\left[D^{ABJ}_{non-planar},\, P\right] \propto l.$$

A similar analysis for the theory with gauge group $U\,(N)_k \times \overline{U\,(N+k-l)}_{-k}$ gives

$$\left[D^{ABJ}_{non-planar},\, P\right] \propto (k-l)\,. \quad (5.13)$$

From type IIA string theory [101], the theory with holonomy $b_2 = l/k$ is related to the theory with holonomy $b_2 = (k-l)/k$ by a parity transformation, this corresponds on the field theory side to the fact that the theory with gauge group $U(N+l)_k \times \overline{U(N)}_{-k}$ is equivalent to the $U(N)_k \times \overline{U(N+k-l)}_{-k}$ theory. Indeed (5.12) and (5.13) imply that this is the case provided that $\left[D^{ABJ}_{non-planar}, P\right] \propto \mathcal{B}_2$.

5.6 Discussion

We have studied the action of the nonplanar dilatation operator for ABJ theory at two loops on operators built from restricted Schur polynomials. The spectrum of the anomalous dimensions signals nonplanar integrability. Our analysis shows that the ABJ theory breaks parity invariance. This is in contrast to the planar two loop dilatation generator which was found to be parity invariant. When ABJ theory reduces to ABJM theory, parity invariance is recovered as expected. In this analysis we note that parity breaking does not destroy integrability. Furthermore, in the field theory, we have found that $(M-N)$ is related to the holonomy $\mathcal{B}_2$ of the equivalent string theory description. We have considered the case where the irreducible representation R of the gauge invariant operator (5.4) is a Young diagram with two long rows. In this case our operators have a dimension of order $O(N)$. We have solved (5.11) within the continuum limit approximation.
The exact solution of (5.6) in other limits may provide further insight into whether the ABJ theory enjoys integrability or not.

Chapter 6

From Schurs to Giants in ABJ(M)

In this chapter we study various classes of ABJ(M) correlators in the free field theory limit. We then use them to study various correlators that probe the geometry of giant gravitons and dual giant gravitons in $AdS_4 \times \mathbb{CP}^3$. This chapter is based on [102].

6.1 ABJ(M) correlators from Schurs

This section contains our main technical result. We compute two, three and four-point extremal correlators of single trace half-BPS operators in ABJ(M) in the free field theory limit using Schur polynomial technology. Based on these formulas we write the general form of the n-point correlator. Analogous results in $\mathcal{N} = 4$ SYM have been known for a long time [66] and played important roles in matching dual observables.
As shown in [50], ABJ(M) Schurs form an orthogonal basis and their two point functions are diagonal. For two Young diagrams R_1 and R_2 the two point function of corresponding Schur polynomial operators in $AB^\dagger$ is given by

$$\langle \chi_{R_1}(AB^\dagger)\chi_{R_2}(BA^\dagger)\rangle = \delta_{R_1R_2}\, f_{R_2}(N) f_{R_2}(M), \tag{6.1}$$

where $f_R(N)$ and $f_R(M)$ are products over the weights of each box in R_i.

The n-point extremal correlators of the half-BPS chiral primary operators in ABJ(M) that we compute here are defined as

$$C_n^{J_1,\dots,J_{n-1}} \equiv \langle \mathrm{Tr}\,((AB^\dagger)^{J_1})\dots \mathrm{Tr}\,((AB^\dagger)^{J_{n-1}})\mathrm{Tr}\,((BA^\dagger)^{J_n})\rangle, \qquad \sum_{i=1}^{n-1} J_i = J_n. \tag{6.2}$$

Our main tool will be the formula for the extremal n-point correlation functions in terms of the two point correlator of Schurs labelled by hooks. Such expression can be derived in a general class of gauge theories where the

Schur basis can be constructed in some matrix X and the single trace operators are linear combinations of Schurs labeled by hooks[16]. The formula reads

$$C_n^{J_1,\ldots,J_{n-1}} = \left(\prod_{l=1}^{n-1}\sum_{k_l=1}^{J_l}\right)\sum_{i=0}^{n-2}(-1)^i \binom{n-2}{i} f_{h_{J_n}^{k_n-i}} \tag{6.3}$$

where $J_n = \sum_{i=1}^{n-1} J_i$, $k_n = \sum_{l=1}^{n-1} k_l$, and $f_{h_J^k}$ is the value of the two point correlator of Schur polynomials in X labeled by hooks

$$\langle \chi_{h_J^l}(X)\chi_{h_J^k}(\bar{X})\rangle = \delta_{l,k}\, f_{h_J^k}, \tag{6.4}$$

where h_J^k is the hook of length J with k boxes in the first row.
A short outline of the proof of this result follows. Single trace operators in the extremal n-point correlator can be expressed in a basis of appropriate Schur polynomials and the coefficients of this expansion are non-zero only when diagrams that label Schurs are hooks. The sums over possible Young diagrams become sums over the number of boxes k_i in the first row of each hook, $i = 1, ..., n$. Moreover, by applying the Littlewood-Richardson fusion rule enough times, the extremal correlator of Schurs can always be written as a linear combination of the Schur's two point functions. The crucial observation is then that all of the Littlewood-Richardson coefficients must be evaluated on hooks and for two given hooks $h_{J_1}^{k_1}$ and $h_2^{k_2}$ the coefficient $g(h_{J_1}^{k_1}, h_2^{k_2}; h_{J_1+J_2}^l)$ is non-zero only for $l = \{k_1 + k_2, k_1 + k_2 - 1\}$. Finally in every term we get rid of the coefficients by solving for k_n and we arrive at (6.3). More pedagogical details as well as the constructive proof of (6.3) in the $\mathcal{N} = 4$ SYM context are given in appendix J.

The only input for (6.3) is then the two point function of the ABJ Schurs labeled by h_J^k. From (6.1) we can easily find that all we need is

$$\begin{aligned} f_{h_J^k}(N, M) =& \prod_{i=1}^{k}(N-1+i)(M-1+i)\prod_{j=1}^{J-k}(N-j)(M-j) \\ =& \frac{\Gamma(N+k)}{\Gamma(N-J+k)}\frac{\Gamma(M+k)}{\Gamma(M-J+k)}, \end{aligned} \tag{6.5}$$

where Γ is the Euler Gamma function. It is clear that (6.5) is just the product of the two $\mathcal{N} = 4$ results (J.11) for N and M. However, as we will see in following sections, this "squaring" does not carry to the level of the observables (as one could naively expect). The only important message from this structure is that

[16] In ABJ(M) we have

$$\mathrm{Tr}\left((AB^\dagger)^J\right) = \sum_{k=1}^{J}(-1)^{J-k}\chi_{h_J^k}(AB^\dagger)$$

.

in ABJ we are formally dealing with two Young diagrams that constrain each other. This will become clear in the details of the correlators.

Formulas (6.3)[17] and (6.5) provide the expansion of (6.3) for arbitrary n at tree level but to all orders in N and M. In addition, it is also possible to rewrite these answers in a similar form to [66] and below we do that in ABJ for $n = 2, 3, 4$ and provide the general n expression. Results for ABJM model are obtained by setting $N = M$.

6.1.1 Two-Point Functions

For $n = 2$ our formula yields

$$C_2^J = \sum_{k=1}^{J} \frac{\Gamma(N+k)\,\Gamma(M+k)}{\Gamma(N-J+k)\,\Gamma(M-J+k)}. \tag{6.6}$$

The sum can be formally performed using Mathematica and the result is

$$C_2^J = G(N, M; J) - G(N+J, M+J; J), \tag{6.7}$$

where by $G(a, b; c)$ we denote a special case of the Meijer G-function[18] $G^{1,3}_{3,3}$ that is expressed in terms of the generalized hypergeometric function ${}_3F_2$ as

$$G(a, b; c) \equiv \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a-c+1)\Gamma(b-c+1)}\,{}_3F_2\left(\begin{matrix} 1 & a+1 & b+1 \\ & a-c+1 & b-c+1 \end{matrix}; 1\right). \tag{6.8}$$

For large N and M we can expand the two point function and reproduce the known, leading result and first sub-leading corrections

$$C_2^J \sim J(NM)^J \left(1 + \frac{J^2(J^2-1)}{12N\,M} + \frac{J(J^2-1)(J-2)}{24N^2} + \frac{J(J^2-1)(J-2)}{24M^2} + \ldots\right), \tag{6.9}$$

where ellipses stand for terms of order $J^8N^{-3}M^{-1}$, $J^8N^{-2}M^{-2}$ and $J^8N^{-1}M^{-3}$ etc. Notice that the structure is more involved than in $\mathcal{N} = 4$ SYM, but again, it clearly shows that if J is of order $\sqrt{N}$ or $\sqrt{M}$, the perturbative $\frac{1}{N}$ expansion breaks down.
For ABJM the three subleading contributions collapse into one of order $O(N^{-2})$ so that a perturbative $\frac{1}{N}$ expansion is sensible only for J smaller than $O(\sqrt{N})$.

[17] It is actually easier to use slightly more explicit version (J.24).

[18] A more general definition and further details on Meijer G-functions can be found in [103].

6.1.2 Three-Point Functions

Similarly, setting $n = 3$ in our formula (or its more explicit form (J.24)) gives

$$C_3^{J_1,J_2} = \left(\sum_{k=J_2+1}^{J_1+J_2} - \sum_{k=1}^{J_1} \right) \frac{\Gamma(N+k)\,\Gamma(M+k)}{\Gamma(N-J_3+k)\,\Gamma(M-J_3+k)}, \tag{6.10}$$

where $J_3 = J_1 + J_2$. Mathematica can formally sum it into a combination of the Meijer G-functions

$$\begin{aligned} C_3^{J_1,J_2} = G(N+J_1, M+J_1; J_3) + G(N+J_2, M+J_2; J_3) \\ -G(N, M; J_3) - G(N+J_3, M+J_3; J_3). \end{aligned} \tag{6.11}$$

To the leading order in N and M and the answer is

$$C_3^{J_1,J_2} = J_1\, J_2\, J_3\, N^{J_1+J_2-1} M^{J_1+J_2} + \; J_1\, J_2\, J_3\, N^{J_1+J_2} M^{J_1+J_2-1} + ... \tag{6.12}$$

Using this result we can compute the normalized three point functions

$$\frac{C_3^{J_1,J_2}}{\sqrt{J_1\, J_2\, J_3}\,(NM)^{J_1+J_2+J_3}} = \frac{\sqrt{J_1\, J_2\, J_3}}{N} + \frac{\sqrt{J_1\, J_2\, J_3}}{M} + ... \tag{6.13}$$

This leading contribution was also computed in [104] and it is equal to the sum of two leading $\mathcal{N} = 4$ three point functions in N and M.

Three point correlators in ABJM are again obtained by setting $N = M$. The leading answer for the normalized three point functions in ABJM is then twice the $\mathcal{N} = 4$ SYM counterpart.

6.1.3 Four-Point Functions

The tree level four point functions in ABJ are obtained by setting $n = 4$ in (J.24) and we have

$$C_4^{J_1,J_2,J_3} = \left(\sum_{k=1}^{J_1} - \sum_{k=J_2+1}^{J_1+J_2} - \sum_{k=J_3+1}^{J_1+J_3} + \sum_{k=J_2+J_3+1}^{J_4} \right) \frac{\Gamma(N+k)\,\Gamma(M+k)}{\Gamma(N-J_4+k)\,\Gamma(M-J_4+k)}. \tag{6.14}$$

Performing the sums we obtain

$$\begin{aligned} C_4^{J_1,J_2,J_3} = G(M, N; J_4) - G(M+J_4, N+J_4; J_4) \\ -G(M+J_1, N+J_1; J_4) - G(M+J_2, N+J_2, J_4) \\ -G(M+J_3, N+J_3, J_4) + G(M+J_1+J_2, N+J_1+J_2; J_4) \\ +G(M+J_1+J_3, N+J_1+J_3; J_4) + G(M+J_2+J_3, N+J_2+J_3; J_4). \end{aligned} \tag{6.15}$$

Expanding the answer to the leading order in N and M yields

$$\frac{C_4^{J_1,J_2,J_3}}{\sqrt{J_1 J_2 J_3 J_4}(NM)^{J_4}} = \sqrt{J_1 J_2 J_3 J_4} \left(\frac{J_4 - 1}{N^2} + \frac{2J_4}{NM} + \frac{J_4 - 1}{M^2} \right), \tag{6.16}$$

where $J_4 = J_1 + J_2 + J_3$.

6.1.4 n-Point Correlators

Clearly the above correlation functions exhibit an interesting structure that can be naturally generalized to n-point functions. Namely, the tree level n-point correlator of the half-BPS chiral primary operators in ABJ can be formally written in terms of Meijer G-functions as

$$\begin{aligned} C_n^{J_1,\dots,J_{n-1}} = (-1)^n \Bigg[& G(M,N;J_n) - \sum_{i=1}^{n-1} G(M+J_i, N+J_i; J_n) \\ & + \sum_{1\le i_1 \le i_2 \le n-1} G(M+J_{i_1}+J_{i_2}, N+J_{i_1}+J_{i_2}; J_n) - \dots \\ & - \sum_{1\le i_1 \le i_2 \le n-1} G(M+J_n, N+J_n; J_n) \Bigg] \end{aligned} \quad (6.17)$$

where $J_n = \sum_{i=1}^{n-1} J_i$, and the ellipsis denote possible terms where arguments of G contain a sum of three, four, etc. J's with appropriate sign. Analogous results for ABJM are obtained by setting $N = M$ in the above formula.

A relevant comment is in order at this point. In $\mathcal{N} = 4$ SYM n-point extremal correlators of chiral primary operators (CPOs) are protected (for a recent proof see [105]) and the tree level answer is exact. On the contrary, in ABJ(M) they depend on the 't Hooft coupling(s) [106]. Determining this coupling dependence is beyond the scope of this study but we hope that (6.17) will serve as a good starting point for understanding the higher loop structure.

In the remaining part of this chapter, we use these formulas to evaluate various correlators of giants with open and closed strings from the perspective of the ABJ(M) gauge theories.

6.2 Excited Giants from ABJ(M)

In this section we consider radiation of closed strings from giant gravitons and dual giants in $AdS_4 \times \mathbb{CP}^3$, as well as joining and splitting of open strings attached to giants from the gauge theory perspective. This will allow for probing and constraining the giant's geometry. We closely follow [48][19] that performed this analysis for $\mathcal{N} = 4$ SYM.

Based on the experience from AdS_5/CFT_4, it is natural to propose ABJ(M) operators dual to excited giants in the type IIA background as Schur polynomials with strings attached

$$\begin{aligned} & \chi^{(k)}_{R,R_1}(A_1B_1^\dagger, W^{(1)}, \dots, W^{(k)}) \\ & = \frac{1}{(n-k)!} \sum_{\sigma\in S_n} \mathrm{Tr}_{R_1}(\Gamma_R(\sigma)) \mathrm{Tr}\,(\sigma(A_1B_1^\dagger)^{\otimes n-k} W^{(1)} \dots W^{(k)}), \end{aligned} \quad (6.18)$$

[19]See also [44, 107]

where $\mathrm{Tr}_{R_1}(\Gamma_R(\sigma))$ is the trace over the subspace of representation Γ_R and the trace with strings is defined as

$$\begin{aligned}&\mathrm{Tr}\,(\sigma(A_1B_1^\dagger)^{\otimes n-k}W^{(1)}...W^{(k)})\\&=(A_1B_1^\dagger)^{i_1}_{i_{\sigma(1)}}...(A_1B_1^\dagger)^{i_{n-k}}_{i_{\sigma(n-k)}}(W^{(1)})^{i_{n-k+1}}_{i_{\sigma(n-k+1)}}...(W^{(1)})^{i_n}_{i_{\sigma(n)}},\end{aligned} \tag{6.19}$$

and the strings are represented by words of $A_2B_2^\dagger$ of an arbitrary length J

$$(W^{(i)})^i_j=\Big((A_2B_2^\dagger)^J\Big)^i_j. \tag{6.20}$$

Notice that because of the two different ranks N and M we have two different families of strings. In other words, the building blocks of our operators can be arranged in two distinct orders that form matrices of a different size

$$(AB^\dagger)_{N\times N},\qquad (B^\dagger A)_{M\times M}, \tag{6.21}$$

we call the first type "N-strings" and the second "M-strings". A consequence of the gauge invariance is that N-strings can only be attached to the $N\times N$ and M-strings to the $M\times M$ product of A_1s and $B_1^\dagger$s respectively.

6.2.1 Emission of Closed Strings from Giant Gravitons

In this subsection we study the emission of closed strings from the giant gravitons ($\subset\mathbb{CP}^3$). This is done by evaluating the leading answer to the two point correlator between a Schur with one open string attached and the bound state operator of a Schur and a closed string.

The operators dual to excited giant gravitons in ABJ theory are given by Schur polynomials with $O(N)$ rows and $O(1)$ columns with open string attached. Namely, the operator dual to a giant graviton with momentum p with one string of momentum J attached is

$$\chi^{(1)}_{h^1_{p+1},\,h^1_p}(A_1B_1^\dagger,\,A_2B_2^\dagger)=\frac{1}{(n-1)!}\sum_{\sigma\in S_n}\mathrm{Tr}_{h^1_p}(\Gamma_{h^1_{p+1}}(\sigma))\mathrm{Tr}((A_1B_1^\dagger)^{\otimes n-1}(A_2B_2^\dagger)^J), \tag{6.22}$$

where the superscript (1) refers to one open string attached.
The operator dual to a D-brane (M-brane) with closed string emitted is given by

$$\mathrm{Tr}((A_2B_2^\dagger)^J)\chi_{h^1_p}(A_1B_1^\dagger).$$

The amplitude $\mathcal{A}$ that describes the interaction of a D-brane and the giant graviton is thus

$$\mathcal{A}_{h^1_{p+1},h^1_p}=\frac{\big\langle\mathrm{Tr}((B_2A_2^\dagger)^J)\chi^\dagger_{h^1_p}(B_1A_1^\dagger)\chi^{(1)}_{h^1_{p+1},\,h^1_p}(A_1B_1^\dagger,\,A_2B_2^\dagger)\big\rangle}{\big\|\mathrm{Tr}((A_2B_2^\dagger)^J)\big\|\big\|\chi_{h^1_p}\big\|\big\|\chi^{(1)}_{h^1_{p+1},\,h^1_p}\big\|}. \tag{6.23}$$

In order to evaluate all the ingredients of this amplitude we can repeat the analysis of [48] with (6.5) for a single column. Using (6.9) and the correlators of open strings that we derived in appendix K, gives the following result for the two point correlator

$$\left\langle \mathrm{Tr}((B_2A_2^\dagger)^J)\chi^\dagger_{h_p^1}(B_1A_1^\dagger)\chi^{(1)}_{h^1_{p+1},\,h^1_p}(A_1B_1^\dagger,\,A_2B_2^\dagger)\right\rangle = JM^JN^{J-1}f_{h^1_{p+1}}(N,M), \tag{6.24}$$

and the norm of the excited giant

$$\|\chi^{(1)}_{h^1_{p+1},\,h^1_p}\| = \left(\frac{p+1}{M}+(J-1)\left(1-\frac{p}{N}\right)\left(1-\frac{p}{M}\right)\right)M^{J+1}N^{J-1}f_{h^1_{p+1}}(N,M). \tag{6.25}$$

This norm differs significantly from the counterpart in [48] and now the leading contribution comes from the second term[20].
Therefore, the leading contribution to the amplitude for emission of a closed string from the giant graviton is given by

$$\mathcal{A}_{h^1_{p+1},h^1_p} \sim \sqrt{\frac{J(1-\frac{p}{N})(1-\frac{p}{M})}{\frac{p+1}{M}+(J-1)(1-\frac{p}{N})(1-\frac{p}{M})}}\,. \tag{6.26}$$

Similarly to $\mathcal{N}=4$ SYM the amplitude is of order unity for small momenta p. However, in ABJ, it only exists for p smaller than M. This is a manifestation of the the bound from the number of boxes in a single column$p \leq M$. Namely, completely antisymmetric Schur polynomials only exist when the number of boxes does not exceed $min(N,M)$. Moreover for maximal giants ($p=M$) the amplitude vanishes. This can be understood as the consequence of the fact that we cannot excite maximally excited giant by attaching more strings to it.

Notice also, that the amplitude is not invariant under the exchange of $M \leftrightarrow N$. This is another manifestation of the parity breaking by subleading corrections that is a known subtlety of the ABJ model (see also [96, 97]).

6.2.2 Emission of Closed Strings from Dual Giants

The AdS giant graviton of ABJ theory can be obtained in a similar way. In this case the representation R of ABJ Schur polynomials is the symmetric representation, we denote it by hook h_p^p. The amplitude is

$$\mathcal{A}_{h^{p+1}_{p+1},h^p_p} \sim \sqrt{\frac{J(1+\frac{p}{N})(1+\frac{p}{M})}{\frac{p+1}{M}+(J-1)(1+\frac{p}{N})(1+\frac{p}{M})}}. \tag{6.27}$$

The AdS giant amplitude agrees with the sphere giant amplitude for small p as expected. However it is non zero for the maximal case $p=M$ and slowly decreases for large p.

[20] In [48] this F_0 contribution was always subleading

6.2.3 String Splitting and Joining

To study the splitting and joining of open strings, we need to compute the amplitude of two ABJ Schur polynomials one with string attached and the other with two strings attached. The relevant amplitude describing this process is

$$\mathcal{A} = \frac{\left\langle (\chi^{(1)}_{S,S'})^\dagger \chi^{(2)}_{R,R''} \right\rangle}{\|\chi^{(1)}_{S,S'}\| \|\chi^{(2)}_{R,R''}\|}, \tag{6.28}$$

where $\chi^{(2)}_{R,R''}$ is the ABJ Schur with two strings attached given by

$$W^{(1)} = (A_2 B_2^\dagger)^{J_1}, \qquad W^{(2)} = (A_2 B_2^\dagger)^{J_2}.$$

$\chi^{(1)}_{S,S'}$ has one string attached and it is given by

$$W = (A_2 B_2^\dagger)^{J_1+J_2}.$$

Labels R and S denote Young diagrams labeling Schurs dual to giants and S' and R'' stand for diagram S with one box removed and diagram R with two boxes removed.
The amplitude can be computed using appendix H of [48] for ABJ operators together with the results obtained in our appendix D, we consider the case where both R and S have rows of $O(N)$ and columns of $O(1)$ (the other case follows directly by changing to the symmetric representation). We consider the first column of R has a length $b_1 + b_2$ where b_1 is the length of the second column and it is $O(N)$, b_2 is $O(1)$ (see Figure 6.1).

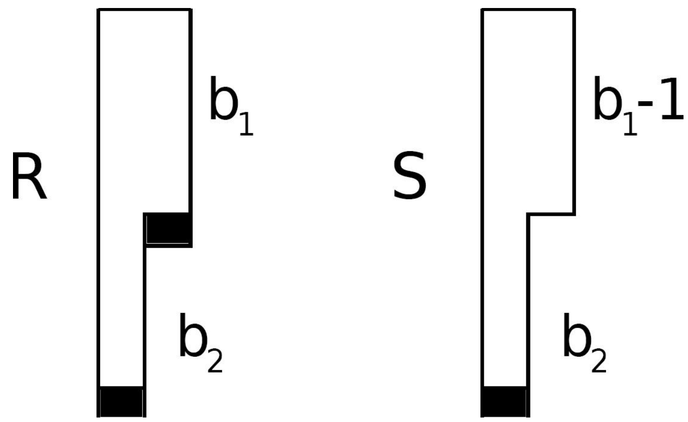


Figure 6.1: Young diagrams used in the computation of the string joining amplitude

The leading contribution to the numerator comes from the terms that contain C_5 and C_6 in appendix D. The result is

$$\mathcal{A} = 2\sqrt{\frac{(N-b_1)(M-b_1)}{b_1 NM}} \frac{1}{(b_2+1)} \left(1 + O(J^8 (NM)^{-2})\right). \tag{6.29}$$

We note that this amplitude is independent of the angular momentum of the open strings. Moreover, this leading answer is invariant under the exchange of $M \leftrightarrow N$.

6.2.4 Predictions for Dual Probes

Analyzing the above amplitudes one can propose the following qualitative picture for the dual probe branes on $\mathbb{CP}^3$ with NS Bfield B_2 field. Naturally, the geometry has to generalize the dual ABJM giant graviton found in [108] that is described by two $D4$-branes wrapping separate $\mathbb{CP}^2$s. In ABJ the two different ranks should correspond to two different radius sizes of the $\mathbb{CP}^2$s. If we then want to attach strings into the $D4$s, N-strings end on the space with radius N and M-strings on the space with radius M (see Figure 6.2). Moreover, from gauge invariance, no open strings can stretch between $D4$-branes on the two different spaces. This constraint disappears when we set $M = N$ for ABJM.

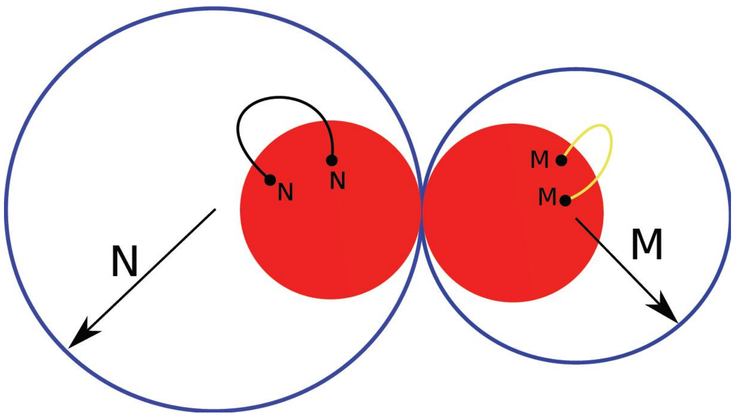


Figure 6.2: Possible geometrical dual of the antisymmetric ABJ Schurs. Two different ranks in the ABJ gauge group might be interpreted as two different radii of the two $\mathbb{CP}^2$s. A cut off on the number of boxes of the Young diagrams labeling the gauge theory operator ($k \leq M$) is then naturally realized in the dual geometry. Gauge invariance requires that end points of strings must be attached to the part in space with radius N (N-strings) or M (M-strings), and no strings can stretch between the two separate parts.

Chapter 7

Discussion

The gauge/gravity duality has been extensively studied in the past decade. The importance of this duality come from the fact that it links non gravitational physics represented by the gauge theory to gravitational physics represented by the string theory.

Although this duality lacks a rigorous mathematical proof, it has been successfully tested in the large N limit. In this limit for operators with a bare dimension of order 1, the planar diagrams contribute significantly and one can suppress the non planar diagrams. Although this agreement is encouraging, to establish the gauge/gravity duality one needs to extend existing studies to finite N.

One of the methods used to test the duality is by computing the anomalous dimension in the gauge theory side and comparing it to the energy spectrum of the dual string theory. The duality predicts these two quantities must match. The importance of this particular comparison is that this spectral problem can be solved exactly because the dilatation operator can be identified as the Hamiltonian of an integrable system. A key result of this approach is the proof that $\mathcal{N} = 4$ SYM theory and ABJ(M) theory are integrable in the planar limit. Attempts have been made to explore the non planar integrability using the existing tools. However, integrability is spoiled when non planar diagrams are considered.

In this thesis we have developed novel tools introduced in chapter 3 to probe non planar integrability. In section 3.5 we provide, for the first time, a complete set of operators built from scalar fields that are in the bi fundamental of the $U(N) \times U(N)$ gauge group of ABJM theory and which we generalized to the $U(M) \times U(N)$ gauge group of ABJ theory. Our operators diagonalize the two point function of the free field theory at all orders in $1/N$.

The action of the non planar dilatation operator of $\mathcal{N} = 4$ SYM theory and of ABJ(M) theory were studied in chapters 4 and 5 respectively. In chapter 4, we have studied the non planar dilatation operator on restricted Schur polynomials with bare dimension of order $O(N)$, that are gauge invariant. We have first presented the results of [52, 53] where the one loop dilatation operator in the $SU(2)$ sector has been studied. This sector is spanned by restricted Schur

polynomials with two complex scalar matrix fields Z and Y. In [52, 53] in a large N but non planar limit, the spectrum of the anomalous dimensions was reduced to a set of harmonic oscillator and hence the integrability in this limit was confirmed. Motivated by this result, we considered the action of the non planar dilatation operator beyond the $SU(2)$ sector, where our gauge invariant operators are restricted Schur polynomials of order $O(N)$ with three complex matrix fields Z, Y and X. Once again, in a large N but non planar limit, the spectrum of the anomalous dimensions reduces to a set of harmonic oscillators. We then considered the action of the two loop dilatation operator in the $SU(2)$ sector on restricted Schur polynomials with a bare dimension of order $O(N)$, in the displaced corners approximation. In this non-planar large N limit, operators that diagonalize the one loop dilatation operator are not corrected at two loops. The resulting spectrum of anomalous dimensions is related to a set of decoupled harmonic oscillators, indicating integrability in this sector of the theory at two loops. The anomalous dimensions are a non-trivial function of the 't Hooft coupling, with a spectrum that is continuous and starting at zero at large N, but discrete at finite N.
In chapter 5, the spectrum of the anomalous dimensions of the ABJ(M) theory in the continuum limit is reduced to the continuum limit of the action of the dilatation operator studied in [53]. This indicates that ABJM theory and ABJ theory are integrable in the large N and large N, M double scaling limits respectively.
In the case of ABJ theory, the spectrum of the anomalous dimensions breaks parity invariance. Indeed, our results show that $\left[D^{ABJ}_{non-planar}, P\right] \propto (M-N)$. This is in contrast to the planar two loop dilatation generator which was found to be parity invariant. When ABJ theory reduces to ABJM theory, parity invariance is recovered as expected. In this analysis we note that parity breaking does not destroy integrability. Furthermore, in the field theory, we have found that $(M-N)$ is related to the holonomy $\mathcal{B}_2$ of the string theory side.
Although in this thesis we have obtained an analytical expression for the action of dilatation operator in $\mathcal{N}=4$ SYM theory and ABJ(M) theory at finite N, the solution of the spectral problem for both theories is obtained in the large N limit. The problem of diagonalizing this finite N dilatation operator is still open. However, these initial results are promising indications that we can probe the gauge/gravity duality at finite N.
In chapter 6, we have studied various classes of extremal correlators and we have found that the extremal n-point correlators in ABJ(M), expressed in terms of the Meijer G-function, can be written in a similar form as their $\mathcal{N}=4$ SYM counterparts. We then studied the giant graviton dynamics. In particular, we have found that the antisymmetric Schur polynomials, as well as the amplitudes for radiation and string joining, only exist when the number of boxes or equivalently the giants momentum is smaller than the smaller rank of ABJ(M) gauge groups. In the case of ABJ theory, the leading order amplitudes are parity invariant but subleading corrections break parity. Moreover, we have found that only probes dual to Schurs with hooks have a non-zero

overlap with pointlike gravitons.
In this thesis we have studied restricted Schur polynomials that have a bare dimension of order N. The operators are labeled by Young diagrams R that have $O(1)$ long rows or columns. An important direction in which to extend this work, is to operators with a bare dimension of order N^2. This requires our representation R to be Young diagram with long rows and long columns. This direction will enrich the AdS/CFT dictionary. Our result do not immediately generalize to this case. Studying the action of dilatation operators of both $\mathcal{N} = 4$ SYM theory and ABJ(M) theories for operators of order $O(N^2)$ remains an interesting open challenge.

Appendix A

Review of $\mathcal{N} = 4$ SYM Theory

$\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory is the maximally supersymmetric gauge theory in 4-dimension spacetime. $\mathcal{N}$ is the number of supersymmetry generators.
The field content of $\mathcal{N} = 4$ SYM theory is one gauge field A_μ, 4 complex Weyl spinors λ, $(A = 1, ..., 4)$ and six scalars X^I $(I = 1, ..., 6)$.
The Lagrangian of the $\mathcal{N} = 4$ SYM theory can be written in $\mathcal{N} = 1$ superspace language as follows

$$\mathcal{L} = -2\mathrm{Tr}\Big\{\frac{1}{g^2}(\int d^2\theta W^\alpha W_\alpha + h.c.) + (\int d^2\theta d^2\bar{\theta} e^{-gV}\bar{\Phi}^i e^{gV}\Phi^i) - \frac{g}{3\sqrt{2}}(\int d^2\theta \varepsilon_{ijk}\Phi^i\left[\Phi^j, \Phi^k\right] + h.c.)\Big\}, \tag{A.1}$$

where the first two terms are the kinetic terms and the last term is the superpotential needed for $\mathcal{N} = 4$ supersymmetry. These fields are in the adjoint representation of the gauge group

$$\Phi^i \to e^{-i\Lambda}\Phi^i e^{i\bar{\Lambda}},\ e^{\pm gV} \to e^{i\bar{\Lambda}}e^{\pm gV}e^{i\Lambda},\ W^\alpha \to e^{-i\Lambda}W^\alpha e^{i\Lambda},$$

where $\Lambda = t^a_{ij}\Lambda_a$ and t^a_{ij} $(a = 1, ..., \text{rank G})$ are the gauge group generators. The superpotential in (A.1) is gauge invariant if the tensor product of three adjoint representations $R_i \times R_j \times R_k$ of G contains a singlet. This indeed the case for $G = SU(N)$.
Integrating out θ and the auxiliary field in (A.1), we get

$$\mathcal{L}_{\mathcal{N}=4} = \mathrm{Tr}\Big\{\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} - \sum_A i\bar{\lambda}\bar{\sigma}^\mu D_\mu\lambda_A - \sum_I D_\mu X^I D^\mu X^I + \sum_{A,B,I} g\varepsilon^{ABI}\lambda_A\left[X_I, \lambda_B\right] + \sum_{A,B,I} g\varepsilon_{ABI}\bar{\lambda}^A\left[X^I, \bar{\lambda}^B\right] + \sum_{I,J}\frac{g^2}{2}\left[X^I, X^J\right]\left[X_I, X_J\right]\Big\}, \tag{A.2}$$

$I,\ J = 1, ..., 6$ and $A, B = 1, ..., 4$.
This Lagrangian is invariant under $\mathcal{N} = 4$ Poincare$'$ supersymmetry and it is scale invariant. These symmetries combine into the superconformal symmetry $SU(2, 2|4)$.

Appendix B

Review of ABJ(M) Theory

The ABJM theory [100] is a three-dimensional $\mathcal{N} = 6$ superconformal Chern-Simons theory with gauge group $U(N) \times U(N)$. The generalization of this theory to gauge group $U(M) \times U(N)$ is known as ABJ theory [101]. The field content of these theories are the four scalars Φ^I, four Dirac spinors ψ_I and the gauge fields A_μ and $\hat{A}_\mu$. Here I is the index of an $SU(4)$ R-symmetry. The Lagrangian of ABJ(M) theory is

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_{CS} + \mathcal{L}_4 + \mathcal{L}_6,$$

$$\mathcal{L}_2 = \mathrm{Tr}(D_\mu \phi^I D_\mu \phi_I + i\bar{\psi}_I D_\mu \psi^I),$$

$$\mathcal{L}_{CS} = \varepsilon^{\mu\nu\lambda}\mathrm{Tr}(\frac{1}{2}A_\mu \partial_\nu A_\lambda + \frac{i}{3} g A_\mu A_\nu A_\lambda - \frac{1}{2}\hat{A}_\mu \partial_\nu \hat{A}_\lambda - \frac{i}{3} g \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda),$$

$$\begin{aligned}\mathcal{L}_4 =& ig^2 \varepsilon^{IJKL}\mathrm{Tr}(\bar{\psi}_I \bar{\phi}_J \bar{\psi}_K \bar{\phi}_L) - ig^2 \varepsilon_{IJKL}\mathrm{Tr}(\bar{\psi}^I \bar{\phi}^J \bar{\psi}^K \bar{\phi}^L) \\ &+ ig^2 \mathrm{Tr}(\bar{\psi}^I \bar{\psi}_I \bar{\phi}_J \bar{\phi}^J) - 2ig^2 \mathrm{Tr}(\bar{\psi}^J \bar{\psi}_I \bar{\phi}_J \bar{\phi}^I) \\ &- ig^2 \mathrm{Tr}(\bar{\psi}_I \bar{\psi}^I \bar{\phi}^J \bar{\phi}_J) + 2ig^2 \mathrm{Tr}(\bar{\psi}_I \bar{\psi}^J \bar{\phi}^I \bar{\phi}_J),\end{aligned}$$

$$\begin{aligned}\mathcal{L}_6 =& \frac{g^4}{3}\mathrm{Tr}\Big[\phi^I \phi_I \phi^J \phi_J \phi^K \phi_K + \phi_I \phi^I \phi_J \phi^J \phi_K \phi^K \\ &+ 4\phi_I \phi^J \phi_K \phi^I \phi_J \phi^K - 6\phi^I \phi_J \phi^J \phi_I \phi^K \phi_K\Big],\end{aligned}$$

where $g = \left(\frac{2\pi}{k}\right)^{\frac{1}{2}}$ and k is an integer known as the Chern-Simons level. The scalars ϕ^I $I = 1, ..., 4$ are complex $N \times N$ matrices in the case of ABJM theory and $M \times N$ matrices in the case of ABJ theory

$$\phi^I = \left\{A_1, A_2, B_1^\dagger, B_2^\dagger\right\}.$$

The gauge transformations of the bifundamental scalar are, in the case of ABJM theory

$$A_i' \to U(N) A_i \overline{U(N)}^\dagger$$

$$B_i' \to U(N) B_i \overline{U(N)}^\dagger$$

In the case of ABJ theory we have

$$A_i' \to U(M) A_i \overline{U(N)}^\dagger$$

$$B_i' \to U(M) B_i \overline{U(N)}^\dagger$$

with $i = 1, 2$.
The Feynman graphs contributing to the two loop dilatation operator of ABJ(M) theory are represented in Figure B.1

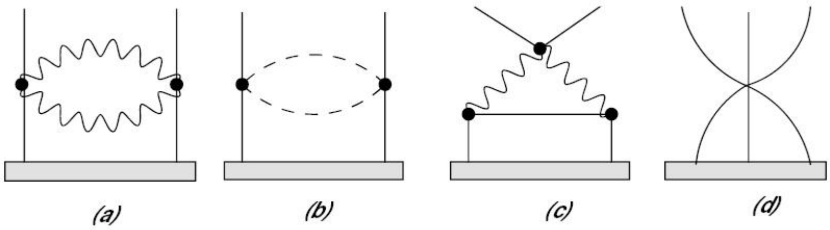


Figure B.1: Feynman diagrams contributing to the two loop dilatation operator. In the $SU(2) \times SU(2)$ sector, diagram (c) does not contribute.

Appendix C

More General Gauge Group

The operators we have constructed in section 3.5 also give a complete basis for gauge groups with an arbitrary number of factors $U(N) \times U(N) \times \cdots \times U(N)$. We will illustrate the example of three factors $U(N) \times U(N) \times U(N)$. Denote the indices associated with the gauge groups by a, α, A. The theory is assumed to have three sets of fields, all transforming in different bi fundamental representations of the factors

$$(A_i)^a_\alpha\,, \qquad (B_i)^\alpha_A\,, \qquad (C_i)^A_a\,.$$

The most general operator in the theory can be written as a product of traces of the operators

$$(\phi_{ijk})^a_b \equiv (A_i B_j C_k)^a_b\,.$$

We will assume that i runs from 1 to n_A, j from 1 to n_B and k from 1 to n_C. The number of ϕ_{ijk} fields will be denoted by n_{ijk}. Repeating the arguments of subsection 3.3.2, it is clear that the restricted Schur polynomials are constructed by subducing $S_{n_{111}} \times S_{n_{112}} \times \cdots \times S_{n_A n_B n_C}$ irreps from S_n irreps where $n = \sum_{ijk} n_{ijk}$. Thus, $\{r\}$ is a set of $n_A n_B n_C$ Young diagrams. Our operators (τ is chosen as in (3.19))

$$O_{R,\{r\}} = \mathrm{Tr}(P_{R,\{r\}} A^{\otimes n} \tau B^{\otimes n} \tau C^{\otimes n})$$

have two point function

$$\langle O_{R,\{r\}} O^\dagger_{S,\{s\}} \rangle = \delta_{RS} \delta_{\{r\},\{s\}} \frac{\mathrm{hooks}_R f_R}{\mathrm{hooks}_{\{r\}}} \tag{C.1}$$

where $\mathrm{hooks}_{\{r\}}$ is a product of hook factors, one for each of the $n_A n_B n_C$ Young diagrams appearing in $\{r\}$.

Appendix D

Example Projector

In the section we will consider the case that $m = p = 3$. Towards this end, we couple the states of 3 spin $\frac{1}{2}$-particles to obtain

$$\left|-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\rangle = \left|\frac{3}{2},-\frac{3}{2}\right\rangle ,$$

$$\left|-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle^{A} + \frac{1}{\sqrt{6}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle^{B} + \frac{1}{\sqrt{3}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle ,$$

$$\left|-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle = -\frac{1}{\sqrt{2}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle^{A} + \frac{1}{\sqrt{6}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle^{B} + \frac{1}{\sqrt{3}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle ,$$

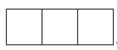
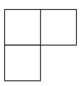
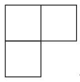
$$\left|-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle = +\sqrt{\frac{2}{3}}\left|\frac{1}{2},\frac{1}{2}\right\rangle^{B} + \frac{1}{\sqrt{3}}\left|\frac{3}{2},\frac{1}{2}\right\rangle ,$$

$$\left|\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right\rangle = -\sqrt{\frac{2}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle^{B} + \frac{1}{\sqrt{3}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle ,$$

$$\left|\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}\left|\frac{1}{2},\frac{1}{2}\right\rangle^{A} - \frac{1}{\sqrt{6}}\left|\frac{1}{2},\frac{1}{2}\right\rangle^{B} + \frac{1}{\sqrt{3}}\left|\frac{3}{2},\frac{1}{2}\right\rangle ,$$

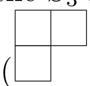
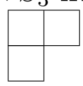
$$\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle = -\frac{1}{\sqrt{2}}\left|\frac{1}{2},\frac{1}{2}\right\rangle^{A} - \frac{1}{\sqrt{6}}\left|\frac{1}{2},\frac{1}{2}\right\rangle^{B} + \frac{1}{\sqrt{3}}\left|\frac{3}{2},\frac{1}{2}\right\rangle ,$$

$$\left|\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle = \left|\frac{3}{2},\frac{3}{2}\right\rangle$$

The spin $\frac{3}{2}$ representation is organized by S_3 irreducible representation , which is one dimensional, so that the spin $\frac{3}{2}$ multiplet is not degenerate. The spin $\frac{1}{2}$ representation is organized by S_3 irreducible representation  which is two dimensional. Consequently, the spin $\frac{1}{2}$ occurs with degeneracy 2. A and B label the two multiplets. Thus, picking a particular state, A and B should label the two states in the S_3 irreducible representation which is labeled by the Young diagram  . From the results above we easily find

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle^A = \frac{1}{\sqrt{2}}\left|\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle - \frac{1}{\sqrt{2}}\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle$$

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle^B = -\frac{1}{\sqrt{6}}\left|\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle - \frac{1}{\sqrt{6}}\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle$$

Taking the direct product with another such multiplet arising from coupling a further three spins, we should obtain the four states of the $S_3 \times S_3$ irreducible representation labeled by the pair of Young diagrams ( , ). These four states are easily constructed

$$|1,1\rangle = \frac{1}{2}\left|\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle - \frac{1}{2}\left|\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle$$
$$-\frac{1}{2}\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle + \frac{1}{2}\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle$$

$$|1,2\rangle = -\frac{1}{2\sqrt{3}}\left|\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle - \frac{1}{2\sqrt{3}}\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle$$
$$+\frac{1}{\sqrt{3}}\left|\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle + \frac{1}{2\sqrt{3}}\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle$$
$$+\frac{1}{2\sqrt{3}}\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle - \frac{1}{\sqrt{3}}\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle$$
$$|2,1\rangle = -\frac{1}{2\sqrt{3}}\left|\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle - \frac{1}{2\sqrt{3}}\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle$$
$$+\frac{1}{\sqrt{3}}\left|-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle + \frac{1}{2\sqrt{3}}\left|\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle$$
$$+\frac{1}{2\sqrt{3}}\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle - \frac{1}{\sqrt{3}}\left|-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle$$
$$|2,2\rangle = \frac{1}{6}\left|\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle + \frac{1}{6}\left|\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle$$
$$-\frac{1}{3}\left|\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle + \frac{1}{6}\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle$$
$$+\frac{1}{6}\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle - \frac{1}{3}\left|\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle$$

$$-\frac{1}{3}\left|-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right\rangle-\frac{1}{3}\left|-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle$$
$$+\frac{2}{3}\left|-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle$$

It is rather simple to check that these four states do indeed span the carrier space of the $S_3 \times S_3$ representation labeled by $(\yng(2,1),\yng(2,1))$. As an example, (12) has a matrix representation

$$\Gamma\,(12)=\begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & \\ 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}\otimes\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}$$
$$=\Gamma_{\yng(2,1)}((12))\otimes\Gamma_{\yng(2,1)}(\mathbf{1})\;.$$

Given a basis of the required carrier space, it is now trivial to construct the associated projector.

Appendix E

The Space $L(\Omega_{m,p})$

In this Appendix we discuss the representation theory relevant for chapter 4. We highly recommend the article [73] for related background material. Consider the group $S_p \times S_m$. Define

$$\Omega_{k,l} = (S_p/S_{p-l} \times S_l) \times (S_m/S_{m-k} \times S_k)$$

to be the space of all pairs of k, l subsets, where the k subsets are subsets of $\{1, 2, ..., p\}$ and the l subsets are subsets of $\{p+1, p+2, ..., p+m\}$. If $p = 2$ and $m = 2$ then $\Omega_{1,1} = \{\{1; 3\}, \{1; 4\}, \{2; 3\}, \{2; 4\}\}$ and $\Omega_{2,2} = \{\{1, 2; 3, 4\}\}$ etc. You can identify a k, l subset with a monomial. For example, we'd identify $\{1; 3\}$ with $x_1 y_3$ and $\{1, 2; 4\}$ with $x_1 x_2 y_4$. Thus, we can consider $\Omega_{k,l}$ to be the space of distinct monomials in two types of variables (x_i and y_i) with $k + l$ factors and no factor repeats. Ordering of the factors is not important so that $x_1 x_2 y_4$ and $y_4 x_1 x_2$ are exactly the same element of $\Omega_{2,1}$. Our main interest is in $L(\Omega_{k,l})$ which is the space of complex valued functions on $\Omega_{k,l}$. The group $S_p \times S_m$ has a very natural action on $L(\Omega_{k,l})$: we can define this action by defining it on each monomial. The symmetric group $S_m \subset S_p \times S_m$ acts by permuting the labels on the x_i factors in the monomial and the symmetric group $S_p \subset S_p \times S_m$ acts by permuting the labels on the y_i factors in the monomial. Thus, for example, for $m = 3 = p$

$$(12)x_1 x_2 y_4 = x_1 x_2 y_4 \qquad (45)x_1 x_2 y_4 = x_1 x_2 y_5 \,.$$

There is a natural inner product under which the monomials are orthonormal, so that, for example

$$\langle x_1 x_2 y_4, x_1 x_2 y_4 \rangle = 1, \qquad \langle x_1 x_2 y_4, x_1 x_3 y_4 \rangle = 0 = \langle x_1 x_2 y_4, x_1 x_2 y_5 \rangle \,.$$

$L(\Omega_{k,l})$ furnishes a reducible representation of the group $S_m \times S_p$. The relevance of $L(\Omega_{k,l})$ for us here is that the projectors acting in $L(\Omega_{k,l})$ projecting onto an irreducible representation of $S_p \times S_m$ are precisely the projectors we need to define the restricted Schur polynomials. Consider the operator

$$d_1 = \sum_{i=1}^{p} \frac{\partial}{\partial x_i} \,. \tag{E.1}$$

It maps from $L(\Omega_{k,l})$ to $L(\Omega_{k-1,l})$. Further, it commutes with the action of $S_p \times S_m$. Because of this, elements of the kernel of d_1 form an invariant $S_p \times S_m$ subspace. Similarly,

$$d_2 = \sum_{i=p+1}^{p+m} \frac{\partial}{\partial y_i}\,, \tag{E.2}$$

maps $L(\Omega_{k,l})$ to $L(\Omega_{k,l-1})$ and it also commutes with the action of $S_p \times S_m$. Thus, the elements of the kernel of d_2 will also form an invariant $S_p \times S_m$ subspace. Using results from [73] it follows that the intersection of the kernel of d_1, the kernel of d_2 and $L(\Omega_{k,l})$ is an irreducible representation of $S_p \times S_m$.

An example will help to make this discussion concrete. For $m = 3 = p$ the intersection of the kernel of d_1, the kernel of d_2 and $L(\Omega_{1,1})$ is clearly spanned by the polynomials

$$\phi_1 = \frac{x_1 - x_2}{\sqrt{2}} \frac{y_4 - y_5}{\sqrt{2}}, \qquad \phi_2 = \frac{x_1 - x_2}{\sqrt{2}} \frac{y_4 + y_5 - 2y_6}{\sqrt{6}},$$

$$\phi_3 = \frac{x_1 + x_2 - 2x_3}{\sqrt{6}} \frac{y_4 - y_5}{\sqrt{2}}, \qquad \phi_4 = \frac{x_1 + x_2 - 2x_3}{\sqrt{6}} \frac{y_4 + y_5 - 2y_6}{\sqrt{6}}\,.$$

It is easy to check that

$$(12)\phi_1 = -\phi_1, \qquad (12)\phi_2 = -\phi_2, \qquad (12)\phi_3 = \phi_3, \qquad (12)\phi_4 = \phi_4,$$

$$(23)\phi_1 = \frac{1}{2}\phi_1 + \frac{\sqrt{3}}{2}\phi_3, \qquad (23)\phi_2 = \frac{1}{2}\phi_2 + \frac{\sqrt{3}}{2}\phi_4,$$

$$(23)\phi_3 = -\frac{1}{2}\phi_3 + \frac{\sqrt{3}}{2}\phi_1, \qquad (23)\phi_4 = -\frac{1}{2}\phi_4 + \frac{\sqrt{3}}{2}\phi_2,$$

$$(45)\phi_1 = -\phi_1, \qquad (45)\phi_2 = \phi_2, \qquad (45)\phi_3 = -\phi_3, \qquad (45)\phi_4 = \phi_4,$$

$$(56)\phi_1 = \frac{1}{2}\phi_1 + \frac{\sqrt{3}}{2}\phi_2, \qquad (56)\phi_2 = -\frac{1}{2}\phi_2 + \frac{\sqrt{3}}{2}\phi_1,$$

$$(56)\phi_3 = \frac{1}{2}\phi_3 + \frac{\sqrt{3}}{2}\phi_4, \qquad (56)\phi_4 = -\frac{1}{2}\phi_4 + \frac{\sqrt{3}}{2}\phi_3,$$

Thus, we have the following group elements

$$\Gamma\left((12)\right) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\Gamma\left((23)\right) = \begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\Gamma\left((45)\right) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

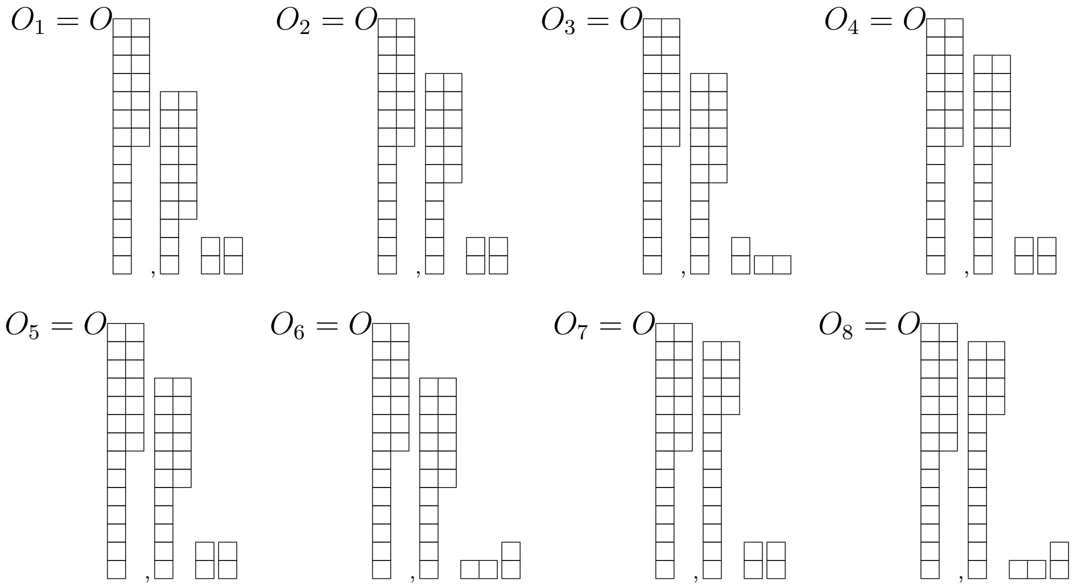
$$\Gamma\left((56)\right) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.$$

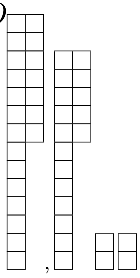
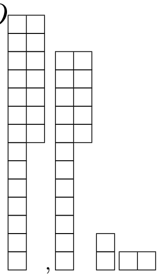
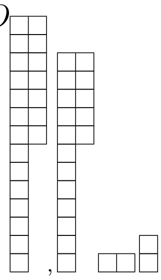
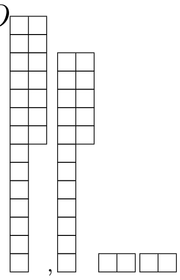
Using these matrices it is possible to compute all elements of the group now, and then to compute characters. In this way, it is a simple matter to identify this as the $(\ydiagram{2,1}, \ydiagram{2,1})$ irreducible representation of $S_3 \times S_3$.

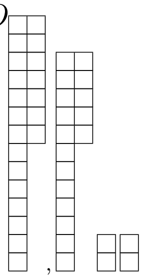
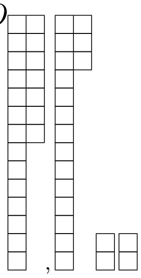
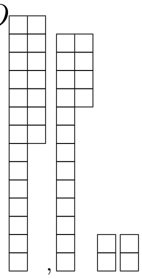
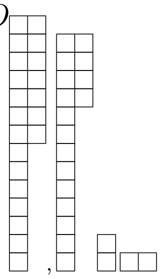
Appendix F

Explicit Evaluation of the Dilatation Operator for $m = p = 2$ and Numerical Spectrum

We have explicitly evaluated the dilatation operator (4.6) for the case $m = p = 2$. There are a total of 16 operators that can be defined. Our notation for these operators is $O_{R,(r,s,t)} = O_i(b_0, b_1)$. The labels b_0 and b_1 specifies the second label of the restricted Schur polynomial: r has b_0 rows with two boxes and b_1 rows with a single box. The label $i = 1, ..., 16$ tells you what the labels s, t are and it tells you how the boxes are removed from R to obtain r. These labels are defined as



$O_9 = O$  $O_{10} = O$  $O_{11} = O$  $O_{12} = O$ 

$O_{13} = O$  $O_{14} = O$  $O_{15} = O$  $O_{16} = O$  .

When computing the dilatation operator, we assume that $b_1 \ll b_0$, $b_0 = O(N)$ and $b_1 = O(N)$. The spectrum of the dilatation operator that we obtain, when diagonalized numerically, does not reproduce the spectrum of a set of decoupled oscillators. We do obtain a set of energy levels that is very well approximated by a linear spectrum $E_n = \omega n$ with ω given by the average (over n) of $E_{n+1} - E_n$. However, $E_{n+1} - E_n$ is not exactly equal to $8g_{YM}^2$ - it fluctuates around this value. We have also numerically verified that after invoking the approximations spelled out at the end of section 3.1, we do indeed obtain equation (4.11) and hence with these approximations the spectrum of the dilatation operator is again reproduced by a collection of decoupled oscillators. Thus, it is only after invoking the approximations of section 3.1 that we definitely obtain an integrable system.

The same conclusion is reached by studying the simpler system $m = 2$, $p = 1$, which involves 8 operators.

Appendix G

$\Delta_{ij}^{(2)}$ as an Element of U(p)

In this appendix we will argue that, at large N, the eigenstates of $\Delta_{ij}^{(1)}$ are also eigenstates of $\Delta_{ij}^{(2)}$. We focus on the case that $p = 2$. Towards this end we will review relevant background from [99]. Recall that in the fundamental representation of $u(N)$ the generators can be taken as

$$(E_{kl})_{ab} = \delta_{ak}\delta_{bl} \qquad k, l, a, b = 1, 2, ..., N \tag{G.1}$$

Introduce the operators (the labeling is such that $i > j$ i.e. Q_{ij} is not defined if $i < j$)

$$Q_{ij} = \frac{E_{ii} - E_{jj}}{2} \qquad Q_{ij}^{+} = E_{ij}, \qquad Q_{ij}^{-} = E_{ji} \tag{G.2}$$

which obey the familiar algebra of angular momentum raising and lowering operators

$$\left[Q_{ij}, Q_{ij}^{+}\right] = Q_{ij}^{+} \qquad \left[Q_{ij}, Q_{ij}^{-}\right] = -Q_{ij}^{-} \qquad \left[Q_{ij}^{+}, Q_{ij}^{-}\right] = 2Q_{ij} \tag{G.3}$$

Irreps of these su(2) subalgebras can be labeled with the eigenvalue of

$$L_{ij}^2 \equiv Q_{ij}^{-}Q_{ij}^{+} + Q_{ij}^2 + Q_{ij} = Q_{ij}^{+}Q_{ij}^{-} + Q_{ij}^2 - Q_{ij} \tag{G.4}$$

and states in the representation are labeled by the eigenvalue of Q_{ij}

$$Q_{ij}|\lambda, \Lambda\rangle = \lambda|\lambda, \Lambda\rangle \qquad L_{ij}^2|\lambda, \Lambda\rangle = (\Lambda^2 + \Lambda)|\lambda, \Lambda\rangle \qquad -\Lambda \le \lambda \le \Lambda \tag{G.5}$$

The restricted Schur polynomials can be identified with particular states in a definite irrep. The reader may consult [99] for the details. Identifying the restricted Schur polynomials with states of a $U(p)$ representation allows us to write $\Delta_{ij}^{(1)}$ as a $u(p)$ valued operator

$$\Delta_{ij}^{(1)} = n\left(-\frac{1}{2}(E_{ii} + E_{jj}) + Q_{ij}^{-} + Q_{ij}^{+}\right) \equiv n\Delta_{ij} \tag{G.6}$$

Note that

$$\mathcal{C} = E_{ii} + E_{jj} \tag{G.7}$$

commutes with all elements (G.2) of the su(2) algebra and hence defines a Casimir of this algebra. It is simply a constant times the identity in a given u(p) irrep. It is not difficult to check [99] that $\Delta_{ij}^{(1)}$ defines a discrete oscillator with creation operator given by

$$A^\dagger = \frac{1}{2}(E_{ii} - E_{jj}) + \frac{1}{2}E_{ij} - \frac{1}{2}E_{ji} \qquad \left[\Delta_{ij}, A^\dagger\right] = -2A^\dagger \tag{G.8}$$

As pointed out in [99], a correctly normalized creation operator is given by $a^\dagger$ with $A^\dagger = \sqrt{M}a^\dagger$, where M is introduced in (4.55). It is straight forward to verify that $\Delta_{ij}^{(2)}$ is given by

$$\Delta_{ij}^{(2)} = (Q^+)^2 - \frac{\mathcal{C}}{2}Q^+ + 2Q^+Q^- - \frac{\mathcal{C}}{2}Q^- + (Q^-)^2 \tag{G.9}$$

and hence that

$$\left[\Delta_{ij}^{(2)}, A^\dagger\right] = -4(\Delta_{ij} + \frac{\mathcal{C}}{4})A^\dagger - 4Q^+ - 4Q \tag{G.10}$$

In terms of a correctly normalized operator at large N we have (the last two terms in (G.10) can be dropped in the limit)

$$\left[\Delta_{ij}^{(2)}, a^\dagger\right] = -4(\Delta_{ij} + \frac{\mathcal{C}}{4})a^\dagger \tag{G.11}$$

There are two things worth noting at this point. First, when acting in the basis of energy eigenstates, it is clear that $a^\dagger$ is indeed a creation operator but, due to the appearance of Δ_{ij}, with a "state dependent frequency". Said differently, $a^\dagger$ continues to move us to higher eigenstates but the energies of these states are not equally spaced. Second, we can show that this result is in perfect agreement with section 4.3.2. To make a comparison with section 4.3.2 we need to restrict attention to states for which the eigenvalue of Δ_{ij} is finite, so that on this subspace we can replace $\Delta_{ij} + \frac{\mathcal{C}}{4} \to \frac{\mathcal{C}}{4}$. Using the value for $\mathcal{C}$ computed in [99], for any state of finite energy, we have

$$\left[\Delta_{ij}^{(2)}, a^\dagger\right] = -2\left(2N + 2r_1\right)a^\dagger \tag{G.12}$$

in perfect agreement with section 4.3.2.

Appendix H

Simplifications of the $m \ll n$ Limit

In this Appendix we will explain why keeping the first term in (4.20) corresponds to computing the leading term in a systematic expansion of the anomalous dimension in a series expansion in $\frac{1}{N}$ and $\frac{m}{n}$. Notice that the first term in (4.20) contains two derivatives with respect to Z and one derivative with respect to Y, whilst the second term contains one derivative with respect to Z and two derivatives with respect to Y. Since the number of Zs (given by n) is much greater than the number of Ys (given by m) we should expect the leading contribution to come from the first term in (4.20). In this Appendix we will demonstrate that this is indeed the case.

It is simplest to consider the expression (4.35). The factor $M^{(ij)}_{s\mu_1\mu_2\,;\,u\nu_1\nu_2}$ includes

$$\left\langle \vec{m}, s, \mu_2\,;\, a|E^{(1)}_{ii}|\vec{m}, u, \nu_2\,;\, b\right\rangle \left\langle \vec{m}, u, \nu_1\,;\, b|E^{(1)}_{jj}|\vec{m}, s, \mu_2\,;\, a\right\rangle$$
$$+ \left\langle \vec{m}, s, \mu_2\,;\, a|E^{(1)}_{jj}|\vec{m}, u, \nu_2\,;\, b\right\rangle \left\langle \vec{m}, u, \nu_1\,;\, b|E^{(1)}_{ii}|\vec{m}, s, \mu_2\,;\, a\right\rangle \quad \text{(H.1)}$$

which involves traces over interwiners acting in $V^{\otimes m}$. It has no dependence on the representation r of the Zs and hence, has no dependence on n. Thus, all n dependence comes from the coefficient multiplying the above term (H.1). We will therefore study the coefficient of this term. As a consequence of the fact that the first term in (4.20) contains two derivatives with respect to Z and one derivative with respect to Y, this term will have a coefficient which includes the factor

$$\frac{d_T n(n-1) m d_{r''}}{d_t d_u d_{R''}(n+m)(n+m-1)} \quad \text{(H.2)}$$

Recall that r'' is obtained by removing two boxes from r. The factor of $d_{r''}$ is produced when we take two derivatives with respect to Z. In the limit that $m \ll n$ we now find

$$\frac{d_T n(n-1) m d_{r''}}{d_t d_u d_{R''}(n+m)(n+m-1)} = \frac{m}{d_u}\left[1 + O\left(\frac{m}{n}\right)\right] \quad \text{(H.3)}$$

For the second term in (4.20), the corresponding factor is now

$$\frac{d_T m(m-1) n d_{r'}}{d_t d_u d_{R''}(n+m)(n+m-1)} \tag{H.4}$$

The Young diagram r' is obtained by removing one box from r. The factor of $d_{r'}$ is produced when we take a single derivative with respect to Z. In the limit that $m \ll n$ we now find

$$\frac{d_T m(m-1) n d_{r'}}{d_t d_u d_{R''}(n+m)(n+m-1)} = \frac{m(m-1)}{n d_u}\left[1 + O\left(\frac{m}{n}\right)\right] \tag{H.5}$$

Notice that (H.5) is smaller than (H.4) by a factor of $\frac{m}{n}$ as we expected. The second term in (4.20) will thus contribute at higher order in a systematic $\frac{m}{n}$ expansion.

Finally, performing the sum over the Lie algebra index in the third term in (4.20) gives a term that is identical to the one loop dilatation operator, except that it is supressed by a power of N. Thus, it does not contribute to the leading order in a large N expansion.

Thus, to summarize, keeping only the first term in D_4 in (4.20) corresponds to the computation of the leading term in the double expansion in the parameters $\frac{1}{N}$ and $\frac{m}{n}$.

Appendix I

On the Action of the Dilatation Operator

In this Appendix we want to discuss how sensitively integrability depends on the coefficients of the individual terms appearing in D_4. We will start by making a few comments on the structure of $\Delta^{(2)}_{ij}$ that we obtained in (4.40).

Recall that we argued

$$\begin{aligned}
&\mathrm{Tr}(ZYZ\partial_Z\partial_Y\partial_Z)\chi_{R,(r,s)\alpha\beta}(Z,Y) = \\
&\sum_{T,(t,u)\gamma\delta}\sum_{R'',T''}\frac{d_T n(n-1)m}{d_t d_u d_{R''}(n+m)(n+m-1)}c_{RR'}c_{R'R''}\ \ \chi_{T,(t,u)\gamma\delta}(Z,Y) \\
&\times\mathrm{Tr}(I_{T''R''}(2,m+2,m+1)P_{R,(r,s)\alpha\beta}(1,m+2,2)I_{R''T''}(2,m+2) \\
&\times P_{T,(t,u)\delta\gamma}(m+2,2,1,m+1))
\end{aligned}$$

in section 4.3.1. Focus on the trace appearing in the second line above. Assume that we obtain R' from R by dropping a box from row i and that we obtain R'' from R' by dropping a box from row j. Further, assume that we obtain T' from T by dropping a box from row k and that we obtain T'' from T' by dropping a box from row l. Clearly then, we are allowing four rows of the Young diagram to participate when the dilatation operator acts. With these assumptions, we easily find (see (4.32), (4.33) and (4.34) as well as the discussion around these equations)

$$I_{R''T''} = E^{(1)}_{ik}E^{(2)}_{jl} \qquad I_{T''R''} = E^{(1)}_{ki}E^{(2)}_{lj} \tag{I.1}$$

and

$$(m+2,2,1,m+1)I_{T''R''}(2,m+2,m+1) = E^{(1)}_{li}E^{(m+1)}_{kj} \tag{I.2}$$

$$(1,m+2,2)I_{R''T''}(2,m+2) = E^{(1)}_{jk}E^{(m+2)}_{il} \tag{I.3}$$

In obtaining these results we have made heavy use of the simplifications in the action of the symmetric group that arise in the displaced corners approximation. It is now a simple matter to find

$$\mathrm{Tr}(I_{T''R''}(2,m+2,m+1)P_{R,(r,s)\alpha\beta}(1,m+2,2)I_{R''T''}(2,m+2)P_{T,(t,u)\delta\gamma}(m+2,2,1,m+1))$$

$$= \mathrm{Tr}(E^{(1)}_{li} E^{(m+1)}_{kj} P_{R,(r,s)\alpha\beta} E^{(1)}_{jk} E^{(m+2)}_{il} P_{T,(t,u)\delta\gamma}) \tag{I.4}$$

Since the projectors $P_{R,(r,s)\alpha\beta}$ and $P_{T,(t,u)\delta\gamma}$ have a trivial action on slots $m+1$ and $m+2$, the above result is only non-zero when $i = l$ and $k = j$ - so that only two rows participate.

This reduction from four possible rows participating to two rows participating is determined by (I.2) and (I.3). These equations are corrected when going beyond the displaced corners approximation and, in that case, all four rows do indeed enter. For all of the terms appearing in the first line of D_4, we find this reduction to two rows for each term separately. Further, we find that each trace is individually proportional to $M^{(ij)}_{s\mu_1\mu_2\,;\,u\nu_1\nu_2}$ defined in (4.36). This implies that the answer to question 2 that we posed in the introduction is completely insensitive to the precise coefficients of the terms appearing in D_4[21].

At this point it is natural to ask if the reduction of the dilatation operator to a set of decoupled oscillators (and thus the observed integrability) is likewise also insensitive to the detailed coefficients. We will see that this is not the case - the emergence of an oscillator does depend sensitively on the precise values of the coefficients of the terms appearing in D_4.

Consider equation (4.40). Individual terms appearing in (4.40) can be traced back to particular terms appearing in D_4. For example, the terms proportional to $(\Delta^+_{ij})^2$ and $(\Delta^-_{ij})^2$ come from the terms $\mathrm{Tr}(ZZY\partial_Z\partial_Z\partial_Y)$ and $\mathrm{Tr}(YZZ\partial_Y\partial_Z\partial_Z)$. Notice that these two terms are related by daggering. Similarly, the terms $\Delta^0_{ij}\Delta^+_{ij}$ and $\Delta^0_{ij}\Delta^-_{ij}$ come from the terms $\mathrm{Tr}(ZYZ\partial_Z\partial_Z\partial_Y)$ $\mathrm{Tr}(ZZY\partial_Z\partial_Y\partial_Z)$, $\mathrm{Tr}(ZYZ\partial_Y\partial_Z\partial_Z)$ and $\mathrm{Tr}(YZZ\partial_Z\partial_Y\partial_Z)$ which are again related by daggering. Changing the relative weights of terms appearing in D_4 will change the relative weight of terms appearing in (4.40).

To explore the effect of these changed coefficients on integrability, imagine we assign coefficient α to the terms $\mathrm{Tr}(ZZY\partial_Z\partial_Z\partial_Y)$ and $\mathrm{Tr}(YZZ\partial_Y\partial_Z\partial_Z)$ in D_4. We now find $\Delta^{(2)}_{ij}$ is replaced by

$$\Delta^{\alpha(2)}_{ij} = \alpha(\Delta^+_{ij})^2 + \Delta^0_{ij}\Delta^+_{ij} + 2\Delta^+_{ij}\Delta^-_{ij} + \Delta^0_{ij}\Delta^-_{ij} + \alpha(\Delta^-_{ij})^2 \tag{I.5}$$

It is straight forward to check, using the approach of [99] that this operator does not admit creation and annihilation operators and hence does not define an oscillator. A very instructive way to get some insight into what is going on, is to consider the continuum limit of section 4.3.2. We find

$$\Delta^{\alpha(2)}_{ij} O_{R,r}(\sigma) \to 2N^2(\alpha-1)O_{R,r}(\sigma) + 2(r_i+r_j)N(\alpha-1)O_{R,r}(\sigma) + O(N) \tag{I.6}$$

Compare this to (4.50) and (4.51). Even the scaling with N of the eigenvalues of $\Delta^{\alpha(2)}_{ij}$ and $\Delta^{(2)}_{ij}$ disagree. Indeed, with $\alpha = 1$ we have a delicate cancelation

[21] If one includes the remaining (subleading) terms in D_4 that we have discarded in the $m \ll n$ limit, the dilatation operator starts to mix different Gauss graph operators. This suggests that the integrability we study here is a property of the large N limit and of the displaced corners approximation (i.e. $m << n$) and may not survive when subleading corrections are included.

of the leading order terms - as we clearly see in (I.6). It is the subleading terms that combine to produce an oscillator. Note that all of the terms in (4.40) contribute at the leading order. Thus, the sensitive dependence we see on the coefficient of the terms $\mathrm{Tr}(ZZY\partial_Z\partial_Z\partial_Y)$ and $\mathrm{Tr}(YZZ\partial_Y\partial_Z\partial_Z)$ extends to the other terms in D_4 too.

This last point deserves explanation. The terms in $\Delta_{ij}^{(2)}$ can be collected into three groups which are each hermittian: $(\Delta_{ij}^{+})^2+(\Delta_{ij}^{-})^2$, $\Delta_{ij}^{0}\Delta_{ij}^{+}+\Delta_{ij}^{0}\Delta_{ij}^{-}$ and finally $2\Delta_{ij}^{+}\Delta_{ij}^{-}$. The relative coefficients of the terms producing these pieces is fixed by hermitticity. For example $\mathrm{Tr}(ZZY\partial_Z\partial_Z\partial_Y)+\beta\mathrm{Tr}(YZZ\partial_Y\partial_Z\partial_Z)$ is only hermittian if $\beta=1$ and in this case the terms sum to $(\Delta_{ij}^{+})^2+(\Delta_{ij}^{-})^2$. The particular coefficients of the terms that appear in $\Delta_{ij}^{(2)}$ ensure that when we take the continuum limit (i) the terms proportional to N^2 cancel, (ii) the terms proportional to $(r_i+r_j)N$ cancel and (iii) the surviving terms sum to produce an operator that admits exactly the same creation and annihilation operators as the one loop dilatation operator does. The integrability we have studied here depends on a careful fine tuning of the terms appearing in D_4.

Appendix J

Extremal Correlators from Schurs

In this Appendix we review and provide a detailed derivation of extremal two, three, and four point correlators of chiral primary operators (CPO) in $\mathcal{N}=4$ SYM with $U(N)$ gauge group using Schurs. They were first obtained using both, Schur polynomial technology (two and three point) and the matrix model in [66] (see also [24],[58]). A general expression for the n-point correlator was conjectured in [66, 109]. Here we give a constructive proof of these formulas and express them in terms of weights of hook diagrams. This makes it easily extendable to a larger class of gauge theories (including ABJ(M) models) in which single trace chiral primary operators can be expressed in a basis of Schur polynomials.

Recall that in $\mathcal{N}=4$ SYM chiral primary operators are single-trace symmetrized and traceless products of the six scalar fields

$$O_I^{CPO} = \frac{1}{\sqrt{JN^J}}\, C_I^{i_1\ldots i_J}\,\mathrm{Tr}\,(\phi_{i_1}\ldots\phi_{i_J}), \tag{J.1}$$

where $C_I^{i_1\ldots i_J}$ are symmetric traceless tensors of $SO(6)$.

A special class of these operators, the so-called BMN-type chiral primaries, are the highest-weight states in the $[0,J,0]$ representation of $SO(6)$ and are expressed in terms of one complex scalar

$$O_J = \frac{1}{\sqrt{JN^J}}\,\mathrm{Tr}\,(Z^J), \qquad Z=\phi_1+i\,\phi_2. \tag{J.2}$$

These are the operators that we will be concerned with in this part, and in particular their n-point extremal correlators

$$C_n^{J_1,\ldots,J_{n-1}} \equiv \langle \mathrm{Tr}\,(Z^{J_1})\ldots\mathrm{Tr}\,(Z^{J_{n-1}})\mathrm{Tr}\,(\bar{Z}^{J_n})\rangle, \qquad \sum_{i=1}^{n-1} J_i = J_n. \tag{J.3}$$

For convenience we drop the normalization factors that can be easily recovered at any stage.

Below we evaluate these correlators for general n using the technology of Schur polynomials. Three tools are sufficient for this task:

- CPOs can be expanded in a basis of Schur polynomials[22]

$$\mathrm{Tr}\,(Z^J) = \sum_R \chi_R(\sigma_J)\,\chi_R(Z), \tag{J.4}$$

where the sum is over all possible Young diagrams with J boxes, $\chi_R(\sigma_J)$ is a character of the J-cycle permutation in representation R and $\chi_R(Z)$ is the Schur polynomial in matrix Z that transforms in the adjoint representation of $U(N)$ (for $SU(N)$ see [58, 71]). It is also a known fact that characters of the J-cycle permutations are non-vanishing only for hook diagrams [110], and for a hook with k boxes in the first row we have

$$\chi_{h^k_J}(\sigma_J) = (-1)^{J-k}, \tag{J.5}$$

where we denote the hook of length J with k boxes in the first row by h^k_J. This way the sum over R can be written as sum over the k, and we have

$$\mathrm{Tr}\,(Z^J) = \sum_k (-1)^{J-k}\,\chi_{h^k_J}(Z). \tag{J.6}$$

- The two point correlator of Schurs is given by[58]

$$\langle \chi_{R_1}(Z)\chi_{R_2}(\bar{Z})\rangle = \delta_{R_1R_2}\,f_{R_2}(N), \tag{J.7}$$

where $f_R(N)$ is the product of the weights of the Young diagram.

- The Littlewood-Richardson rule [110] states that the product of two Schur polynomials with J_1 and J_2 boxes can be expressed as a linear combination of Schurs with $J_1 + J_2$ boxes

$$\chi_{R_1}(Z)\chi_{R_2}(Z) = \sum_{T_3} g(R_1, R_2; T_3)\chi_{T_3}(Z), \tag{J.8}$$

where the Littlewood-Richardson coefficients $g(R_1, R_2; T_3)$ give the multiplicity of the representation T_3 in the tensor product of representations R_1 and R_2.

The constructive proof of the general form of (J.3) can then be obtained by applying the following **Algorithm**:

- Start by expressing all the CPOs in the basis of Schurs using (J.6)

- Next, use the Littlewood-Richardson rule enough times that the answer is expressed as a linear combination of the results for the two point function of Schurs labeled by hooks with J_n boxes. Coefficients of this linear combination will be $g(R_1, R_2; T_3)$ with all entries given by hooks (this is valid only for extremal correlators).

[22]see [24],[58] for more details on the Schur polynomial basis.

- Finally, somewhat simple and easy to check but a crucial observation is that there are only two possible hooks that can appear in a tensor product of two hook diagrams (other diagrams appearing in the product are not hooks). Namely for hooks with k_1 and k_2 boxes the two hooks in the direct product have $k_1 + k_2$ and $k_1 + k_2 - 1$ boxes in the first row.
- Use this fact to get rid of the Littlewood coefficients obtaining the elegant answer only in terms of $f_{h^k_{J_n}}$.

Below we demonstrate how the algorithm works in practise and how it yields the n-point correlators conjectured in [66, 109].

Let us start with the two point correlator. We first express the two point function of the CPO's in a basis of Schurs using (J.6)

$$C_2^J = \langle \mathrm{Tr}\left(Z^J\right) \mathrm{Tr}\left(\bar{Z}^J\right)\rangle = \sum_{k_1,k_2=1}^{J} (-1)^{2J-k_1-k_2} \langle \chi_{h^{k_1}_J}(Z) \chi_{h^{k_2}_J}(\bar{Z})\rangle. \tag{J.9}$$

For hooks δ_{R_1,R_2} becomes δ_{k_1,k_2} and we have

$$C_2^J = \sum_{k=1}^{J} f_{h^k_J}(N), \tag{J.10}$$

hence the correlator is just the sum over weights of all the possible hook diagrams with J boxes. In $\mathcal{N} = 4$ SYM with $U(N)$ gauge group, the product of the weights of the hook is given by

$$f_{h^k_J}(N) = \prod_{i=1}^{k}(N-1+i) \prod_{m=1}^{J-k}(N-m) = \frac{\Gamma(N+k)}{\Gamma(N-J+k)}, \tag{J.11}$$

where Γ is the Euler Gamma function. Inserting this to (J.10) reproduces the two point function of the BMN-type CPO's [66]

$$C_2^J = \frac{1}{J+1}\left(\frac{\Gamma(N+J+1)}{\Gamma(N)} - \frac{\Gamma(N+1)}{\Gamma(N-J)}\right). \tag{J.12}$$

This baby example and the result can be obtained alternatively using Ginibre's method [111].

Similarly we can proceed with extremal three point functions. First we express the CPOs in terms of Schurs

$$C_3^{J_1,J_2} = \sum_{k_1,k_2,k_3=1}^{J_1,J_2,J_3} \prod_{i=1}^{3} (-1)^{J_i-k_i} \langle \chi_{h^{k_1}_{J_1}}(Z) \chi_{h^{k_2}_{J_2}}(Z) \chi_{h^{k_3}_{J_3}}(\bar{Z})\rangle, \tag{J.13}$$

where $J_3 = J_1 + J_2$. Next, using (J.7) brings us to the sum

$$C_3^{J_1,J_2} = \sum_{k_1,k_2,k_3=1}^{J_1,J_2,J_3} \prod_{i=1}^{3} (-1)^{J_i-k_i} g(h^{k_1}_{J_1}, h^{k_2}_{J_2}; h^{k_3}_{J_3}) f_{h^{k_3}_{J_3}}, \tag{J.14}$$

where the crucial ingredient is the Littlewood-Richardson coefficient for hooks. We "kill" it by replacing the sum over k_3 by the only two possible terms, $k_3 = k_1 + k_2$ and $k_3 = k_1 + k_2 - 1$. This brings us to

$$C_3^{J_1,J_2} = \sum_{k_1=1}^{J_1}\sum_{k_2=1}^{J_2}\left(f_{h_{J_3}^{k_1+k_2}} - f_{h_{J_3}^{k_1+k_2-1}}\right) = \left(\sum_{k=J_2+1}^{J_1+J_2} - \sum_{k=1}^{J_1}\right) f_{h_{J_3}^{k}}, \qquad (J.15)$$

where in the second equality we took into account the mutual cancellations between the terms. Finally, plugging (J.11) gives the exact three point function of CPOs in $\mathcal{N} = 4$ SYM [66]

$$C_3^{J_1,J_2} = \frac{1}{J_1+J_2+1}\left(\frac{\Gamma(N+J_1+J_2+1)}{\Gamma(N)} + \frac{\Gamma(N+1)}{\Gamma(N-J_1-J_2)} - \frac{\Gamma(N+J_1+1)}{\Gamma(N-J_2)} - \frac{\Gamma(N+J_2+1)}{\Gamma(N-J_1)}\right). \qquad (J.16)$$

Following the algorithm for four points, we expand CPOs in Schurs and apply the Littlewood-Richardson rule twice to obtain[23]

$$C_4^{J_1,J_2,J_3} = \left(\prod_{i=1}^{4}\sum_{k_i=1}^{J_i}(-1)^{k_i}\right)\sum_T g(h_{J_1}^{k_1}, h_{J2}^{k_2}; T)\, g(T, h_{J_3}^{k_3}; h_{J_4}^{k_4})\, f_{h_{J_4}^{k_4}}. \qquad (J.17)$$

By carefully analyzing the Littlewood-Richardson coefficients we can see that there are only two possibilities for T in the first coefficient, namely: $T \in \{h_{J_1+J_2}^{k_1+k_2}, h_{J_1+J_2}^{k_1+k_2-1}\}$. These two cases inserted to the second coefficient reduce the sum over k_4 into four terms. The first one with $k_4 = k_1 + k_2 + k_3$ with a plus sign, then we have twice $k_4 = k_1 + k_2 + k_3 - 1$ with a minus sign and $k_4 = k_1 + k_2 + k_3 - 2$ with plus again. This way the four point correlator becomes

$$C_4^{J_1,J_2,J_3} = \sum_{k_1=1}^{J_1}\sum_{k_2=1}^{J_2}\sum_{k_3=1}^{J_3}\left(f_{h_{J_4}^{k_1+k_2+k_3}} - 2f_{h_{J_4}^{k_1+k_2+k_3-1}} + f_{h_{J_4}^{k_1+k_2+k_3-2}}\right). \qquad (J.18)$$

Clearly, most of the terms in these sums mutually cancel and it is easy to check that the only ones left can be written as

$$C_4^{J_1,J_2,J_3} = \left(\sum_{k=1}^{J_1} - \sum_{k=J_2+1}^{J_1+J_2} - \sum_{k=J_3+1}^{J_1+J_3} + \sum_{k=J_2+J_3+1}^{J_4}\right) f_{h_{J_4}^{k}} \qquad (J.19)$$

Inserting (J.11), we can easily perform the sums in Mathematica and a tree level answer for the four point correlator of BMN-type chiral primaries in

[23] Note that we use $(-1)^{\sum_{i=1}^{n-1} 2J_i - \sum_{i=1}^{n} k_1} = (-1)^{\sum_{i=1}^{n} k_i}$

$\mathcal{N} = 4$ SYM is

$$C_4^{J_1,J_2,J_3} = \frac{1}{J_4+1}\left\{\frac{\Gamma(N+J_4+1)}{\Gamma(N)} - \frac{\Gamma(N+J_4-J_1+1)}{\Gamma(N-J_1)} - \frac{\Gamma(N+J_4-J_2)}{\Gamma(N-J_2)} - \frac{\Gamma(N+J_4-J_3)}{\Gamma(N-J_3)} + \frac{\Gamma(N+J_4-J_1-J_2)}{\Gamma(N-J_1-J_2)} + \frac{\Gamma(N+J_4-J_2-J_3)}{\Gamma(N-J_2-J_3)} + \frac{\Gamma(N+J_4-J_1-J_3)}{\Gamma(N-J_1-J_3)} - \frac{\Gamma(N+1)}{\Gamma(N-J_4)}\right\}. \quad \text{(J.20)}$$

It is straightforward to write down an expression for the n-point extremal correlator of half-BPS, single trace CPOs. In fact we can do it for a general class of gauge theories where a basis of Schur polynomials in some unitary matrix X can be constructed and CPOs written as linear combinations of Schurs labeled by hooks. Namely, following our algorithm the crucial step becomes the evaluation of the correlator of Schurs

$$\langle \prod_{i=1}^{n-1} \chi_{h_{J_i}^{k_i}}(X)\, \chi_{h_{J_i}^{k_n}}(\bar{X})\rangle = \left(\prod_{i=1}^{n-3}\sum_{T_i}\right) g(h_{J_1}^{k_1}, h_{J_2}^{k_2}; T_1) \left(\prod_{j=1}^{n-4} g(T_j, h_{J_{j+2}}^{k_{j+2}}; T_{j+1})\right) \times \times\ g(T_{n-3}, k_{n-1}; k_n) f_{h_{J_n}^{k_n}}. \quad \text{(J.21)}$$

Then by taking into account the fact that each sum over T_i contains only two possible hooks and finally kiling the sum over k_n we end up with our master formula

$$C_n^{J_1,\ldots,J_{n-1}} = \left(\prod_{l=1}^{n-1}\sum_{k_l=1}^{J_l}\right)\sum_{i=0}^{n-2}(-1)^i \begin{pmatrix} n-2 \\ i \end{pmatrix} f_{h_{J_n}^{k_n-i}} \quad \text{(J.22)}$$

where $J_n = \sum_{i=1}^{n-1} J_i$, $k_n = \sum_{l=1}^{n-1} k_l$, and $f_{h_J^k}$ is the value of the two point correlator of Schur polynomials of X labeled by hooks of length J with k boxes in the first row

$$\langle \chi_{h_J^l}(X)\chi_{h_J^k}(\bar{X})\rangle = \delta_{l,k}\, f_{h_J^k}. \quad \text{(J.23)}$$

If we take into account the cancellations between terms in (J.22), the answer can be written as

$$C_n^{J_1,\ldots,J_{n-1}} = (-1)^n\left(\sum_{k=1}^{J_1} - \sum_{k=J_2+1}^{J_1+J_2} - \ldots - \sum_{k=J_{n-1}+1}^{J_1+J_{n-1}} + \sum_{k=J_2+J_3+1}^{J_1+J_2+J_3} + \ldots + \sum_{k=J_{n-2}+J_{n-1}+1}^{J_{n-2}+J_{n-1}+J_1} - \sum_{k=J_2+J_3+J_4+1}^{J_1+J_2+J_3+J_4} - \ldots + \sum_{k=J_2+..+J_{n-1}}^{J_n}\right) f_{h_{J_n}^k}. \quad \text{(J.24)}$$

This formula can be easily proved by induction.

For $\mathcal{N} = 4$ SYM our formula precisely gives the n-point correlators conjectured in [66, 109] which are

$$C_n^{J_1,\ldots,J_{n-1}} = \frac{1}{J_n+1}\left\{\frac{\Gamma(N+J_n+1)}{\Gamma(N)} - \sum_{i=1}^{n-1}\frac{\Gamma(N+J_n-J_i+1)}{\Gamma(N-J_i)}\right.$$
$$\left. + \sum_{1\leq i_1\leq i_2\leq n-1}\frac{\Gamma(N+J_n-J_{i_1}-J_{i_2}+1)}{\Gamma(N-J_{i_1}-J_{i_2})} - \ldots - \frac{\Gamma(N+1)}{\Gamma(N-J_n)}\right\} \tag{J.25}$$

where $J_n = \sum_{i=1}^{n-1} J_i$, and ellipsis stand for terms where we subtract more of the available J's inside the argument of the Γ in the numerator with appropriate sign for even and odd numbers.

Appendix K

Open String Correlators

Let us start with the two point correlator of open strings. Notice that there are two possible correlators that we can write in ABJ theory. Ones in terms of $(AB^\dagger)_{N\times N}$ and others in terms of $(A^\dagger B)_{M\times M}$. We call the former N-strings and the later M-strings. From the conservation of $U(N)(U(M))$ charge they are both constrained to the form

$$\langle\left((AB^\dagger)^J\right)^i_j\left((BA^\dagger)^J\right)^l_k\rangle = C_1\delta^i_j\delta^l_k + C_2\delta^i_k\delta^l_j \tag{K.1}$$

$$\langle\left((A^\dagger B)^J\right)^\alpha_\beta\left((B^\dagger A)^J\right)^\lambda_\kappa\rangle = D_1\delta^\alpha_\beta\delta^\lambda_\kappa + D_2\delta^\alpha_\kappa\delta^\lambda_\beta, \tag{K.2}$$

where we distinguish latin indices $i, j = 1, ..., N$ from greek $\alpha, \beta = 1, ..., M$, and constants C_1, C_2 and D_1, D_2 are to be determined. We begin with the correlator of N-strings (K.1). There are two ways to contract the indices hence we have

$$\langle\mathrm{Tr}\,((AB^\dagger)^J)\mathrm{Tr}\,((BA^\dagger)^J)\rangle = C_1\,N^2 + C_2\,N \tag{K.3}$$

$$\langle\mathrm{Tr}\,((AB^\dagger)^J(BA^\dagger)^J)\rangle = C_1\,N + C_2\,N^2 \tag{K.4}$$

To the leading order in $N(M)$ the second correlator can be expressed in terms of the two point function of the single trace operators. Notice that to the leading order in N and M, only contractions between pairs $AB^\dagger$ matter. This is because

$$(\check{A}\check{B}^\dagger)^i_j(B^\dagger A)^l_k = \check{A}^i_\alpha(\check{B}^\dagger)^\alpha_j(B^\dagger)^l_\beta A^\beta_k = \delta^i_k\delta^\beta_\alpha\delta^\alpha_\beta\delta^l_j = M\,\delta^i_k\delta^l_j \tag{K.5}$$

Hence if we perform just a single contraction in the two point correlator of $J+1$ pairs, the leading answer is

$$\langle\mathrm{Tr}\,((AB^\dagger)^{J+1})\mathrm{Tr}\,((BA^\dagger)^{J+1})\rangle \sim M(J+1)\langle\mathrm{Tr}\,((AB^\dagger)^J(BA^\dagger)^J)\rangle. \tag{K.6}$$

Finally we arrive at two equations

$$\langle J\rangle \equiv \langle\mathrm{Tr}\,((AB^\dagger)^J)\mathrm{Tr}\,((BA^\dagger)^J)\rangle = C_1\,N^2 + C_2\,N \tag{K.7}$$

$$\langle J+1\rangle \equiv \langle \mathrm{Tr}\,((AB^\dagger)^{J+1})\mathrm{Tr}\,((BA^\dagger)^{J+1})\rangle$$
$$\sim M(J+1)\left(C_1\,N + C_2\,N^2\right), \qquad \text{(K.8)}$$

that are solved to

$$C_1 \;\sim\; \frac{1}{N^3-N}\left(N\,\langle J\rangle - \frac{\langle J+1\rangle}{M(J+1)}\right) \sim (J-1)M^J N^{J-2} \qquad \text{(K.9)}$$

$$C_2 \;\sim\; \frac{1}{N^3-N}\left(\frac{N\langle J+1\rangle}{M(J+1)} - \langle J\rangle\right) \sim M^J N^{J-1} \qquad \text{(K.10)}$$

The three point functions are important for studying string dynamics such as splitting and joining of open strings, which is a known phenomenon in the case of giant graviton dynamics. The three point function of this type is

$$\langle((AB^\dagger)^{J_1})^j_i((AB^\dagger)^{J_2})^l_k((BA^\dagger)^{J_1+J_2})^q_p\rangle = \delta^j_i\delta^l_k\delta^q_p C_1 + \delta^l_i\delta^j_k\delta^q_p C_2 + \delta^j_i\delta^q_k\delta^l_p C_3$$
$$+\, \delta^q_i\delta^l_k\delta^j_p C_4 + \delta^l_i\delta^q_k\delta^j_p C_5 + \delta^q_i\delta^j_k\delta^l_p C_6 \qquad \text{(K.11)}$$

There are six possible contractions which lead to the following three point functions

$$A \equiv \langle\mathrm{Tr}((AB^\dagger)^{J_1})\mathrm{Tr}((AB^\dagger)^{J_2})\mathrm{Tr}((BA^\dagger)^{J_1+J_2})\rangle = N^3C_1 + N^2(C_2+C_3+C_4) + N(C_5+C_6),$$
$$B \equiv \langle\mathrm{Tr}((AB^\dagger)^{J_1+J_2})\mathrm{Tr}((BA^\dagger)^{J_1+J_2})\rangle = N^3C_2 + N^2(C_1+C_5+C_6) + N(C_3+C_4),$$
$$C \equiv \langle\mathrm{Tr}((AB^\dagger)^{J_1})\mathrm{Tr}((AB^\dagger)^{J_2}(BA^\dagger)^{J_1+J_2})\rangle = N^3C_3 + N^2(C_1+C_5+C_6) + N(C_2+C_4),$$
$$D \equiv \langle\mathrm{Tr}((AB^\dagger)^{J_2})\mathrm{Tr}((AB^\dagger)^{J_1}(BA^\dagger)^{J_1+J_2})\rangle = N^3C_4 + N^2(C_1+C_5+C_6) + N(C_2+C_3),$$
$$E \equiv \langle\mathrm{Tr}((AB^\dagger)^{J_1+J_2}(BA^\dagger)^{J_1+J_2})\rangle = N^3C_5 + N^2(C_2+C_3+C_4) + N(C_1+C_6),$$
$$F \equiv \langle\mathrm{Tr}((AB^\dagger)^{J_1+J_2}(BA^\dagger)^{J_1+J_2})\rangle = N^3C_6 + N^2(C_2+C_3+C_4) + N(C_1+C_5).$$

It is easy to see that in the above set $E = F$. Now solving these equations, we find

$$
\begin{aligned}
C_1 =& \frac{1}{(N^2-1)(N^2-4)}\Big[\frac{N^2-2}{N}A - B - C - D + \frac{4}{N}E\Big],\\
C_2 =& \frac{1}{(N^2-1)(N^2-4)}\Big[-A + \frac{N^2-2}{N}B + \frac{2}{N}C + \frac{2}{N}D - 2E\Big],\\
C_3 =& \frac{1}{(N^2-1)(N^2-4)}\Big[-A + \frac{2}{N}B + \frac{N^2-2}{N}C + \frac{2}{N}D - 2E\Big],\\
C_4 =& \frac{1}{(N^2-1)(N^2-4)}\Big[-A + \frac{2}{N}B + \frac{2}{N}C + \frac{N^2-2}{N}D - 2E\Big],\\
C_5 =& C_6 = \frac{1}{(N^2-1)(N^2-4)}\Big[\frac{2}{N}A - B - C - D + NE\Big].
\end{aligned}
\tag{K.12}
$$

We note that, there are only two new correlators C and D that need to be computed. The other correlators can directly be computed from results we have already obtained. To find the new correlators, we consider the leading terms of the three point function. That is

$$
\begin{aligned}
&\langle \mathrm{Tr}((AB^\dagger)^{J_1})\mathrm{Tr}((AB^\dagger)^{J_2})\mathrm{Tr}((BA^\dagger)^{J_1+J_2})\rangle\\
\sim& MJ_1\langle \mathrm{Tr}((AB^\dagger)^{J_2})\mathrm{Tr}((AB^\dagger)^{J_1-1}(BA^\dagger)^{J_1+J_2-1})\rangle\\
+& MJ_2\langle \mathrm{Tr}((AB^\dagger)^{J_1})\mathrm{Tr}((AB^\dagger)^{J_2-1}(BA^\dagger)^{J_1+J_2-1})\rangle
\end{aligned}
$$

Upon using equation (6.12), we get

$$
\begin{aligned}
&\langle \mathrm{Tr}((AB^\dagger)^{J_2})\mathrm{Tr}((AB^\dagger)^{J_1-1}(BA^\dagger)^{J_1+J_2-1})\rangle\\
&\qquad = J_1J_2N^{J_1+J_2-1}M^{J_1+J_2-1} + J_1J_2M^{J_1+J_2}N^{J_1+J_2}
\end{aligned}
$$

Shifting J_1 to J_1+1, we arrive at

$$
\begin{aligned}
&\langle \mathrm{Tr}((AB^\dagger)^{J_2})\mathrm{Tr}((AB^\dagger)^{J_1}(BA^\dagger)^{J_1+J_2})\rangle\\
&\qquad = J_2(J_1+1)\left(N^{J_1+J_2}M^{J_1+J_2} + M^{J_1+J_2+1}N^{J_1+J_2+1}\right)
\end{aligned}
\tag{K.13}
$$

Similarly

$$
\begin{aligned}
&\langle \mathrm{Tr}((AB^\dagger)^{J_1})\mathrm{Tr}((AB^\dagger)^{J_2}(BA^\dagger)^{J_1+J_2})\rangle\\
&\qquad = J_1(J_2+1)\left(N^{J_1+J_2}M^{J_1+J_2} + M^{J_1+J_2+1}N^{J_1+J_2+1}\right)
\end{aligned}
\tag{K.14}
$$

Recall the result of open string correlators in $\mathcal{N}=4$ SYM theory [48]

$$
\left\langle \mathrm{Tr}(Y^{J_1})\mathrm{Tr}(Y^{J_2}(Y^\dagger)^{J_1+J_2})\right\rangle = J_1(J_2+1)N^{J_1+J_2} \tag{K.15}
$$

We note that the ABJ answer is not simply a product of two open string correlators. The solution for C_1, C_2, C_3, C_4 and C_5 in the large N, M limit is

$$\begin{aligned}
C_1 =& [J_1 J_2 (J_1 + J_2) + 4] M^{J_1+J_2} N^{J_1+J_2-4} - 2 J_1 J_2 M^{J_1+J_2+1} N^{J_1+J_2-3}, \\
C_2 =& (J_1 + J_2 - 2) M^{J_1+J_2} N^{J_1+J_2-3}, \\
C_3 =& (J_1 J_2 + J_1) M^{J_1+J_2+1} N^{J_1+J_2-2} - 2 M^{J_1+J_2} N^{J_1+J_2-3}, \\
C_4 =& (J_1 J_2 + J_2) M^{J_1+J_2+1} N^{J_1+J_2-2} - 2 M^{J_1+J_2} N^{J_1+J_2-3}, \\
C_5 =& C_6 = M^{J_1+J_2} N^{J_1+J_2-2}.
\end{aligned} \tag{K.16}$$

It is now easy to compute the amplitude of closed string propagating between two excited D-brane states, since we need to compute the leading contribution of the correlator $\left\langle \mathrm{Tr}(AB^{\dagger J_1})(AB^{\dagger J_2})^i_j (BA^{\dagger (J_1+J_2)})^l_k \right\rangle$. From the above result, we get

$$\left\langle \mathrm{Tr}(AB^{\dagger J_1})(AB^{\dagger J_2})^i_j (BA^{\dagger (J_1+J_2)})^l_k \right\rangle = M^{J_1+J_2} N^{J_1+J_2-2} \delta^i_k \delta^l_j. \tag{K.17}$$

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