The Tomita relation\(^{(54)}\) used for the calculation of the friction factor is given by:

\[
\frac{1}{\sqrt{f_{BT}}} = 4 \log (Re_{BT} \sqrt{f_{BT}}) - 0.4 \quad (5.65)
\]

where \(Re_{BT} = \frac{\nu \rho_p}{n} \left[ (1-x)(\frac{x^4 - 4x + 3}{3}) \right] \)

\[
= Re_B \left[ (1-x)(\frac{x^4 - 4x + 3}{3}) \right] \quad (5.66)
\]

and for \(f_{BT} = \frac{D \Delta p}{2pLV^2} (1-x) \)

\[
= f/(1-x) \quad (5.67)
\]

where \(x = \Gamma_y/\Gamma_w\) and calculated as if the flow was laminar. These equations have been confirmed for \(2000 < Re_{BT} < 100000\).

The Torrance\(^{(57)}\) relation is given by:

\[
\frac{1}{\sqrt{f}} = 4.53 \log(1-x) + 4.53 \log(Re_B \sqrt{f}) - 2.3 \quad (5.68)
\]

The Thomas\(^{(64)}\) correlation is given by:

\[
f = B \left( \frac{D \nu_p}{\eta} \right)^{-b} \quad (5.69)
\]

where \(B = 0.079 \left[ (\frac{\mu}{\eta})^{0.48} + (p\Gamma_y K^2)^2 \right] \quad (5.70)

\[
and \ b = 0.25 \left[ (\frac{\mu}{\eta})^{0.15} + (p\Gamma_y K^2)^2 \right] \quad (5.71)
\]

where \(K = 6 \times 10^{-6} \text{ft}\)

Hanks and Dadia\(^{(59)}\) express the parameter \(\varphi/B\) as:
\[ \phi B = \frac{Re f - Re_c f_c}{2\sqrt{2B}} \] (5.72)

where \( B = B (H e) \) and is determined experimentally to be
\[ B = 157.5 \ H e^{-0.151} \] (5.73)

and is applicable in the range:
\[ 10^3 \leq H e \leq 2 \times 10^7 \]

An attempt at presenting a generalised Bingham flow model in the laminar and turbulent flow region is given by Round et al\(^{65}\). His correlation, although presented here is also valid for pseudoplastic, dilatant and yield pseudoplastic flow.

The laminar flow region analysis is based on:

\[ \Gamma_r = \Gamma_0 + KS^n \] (5.74)

where \( S = \left[ \frac{\Gamma - \Gamma_0}{K} \right]^{1/n} \) (5.75)

which reduces to zero for \( 0 \leq \Gamma_r \leq \Gamma_0 \) and
\[ 0 = 2\pi \int_0^{0^{1/2}} U_r dr \] (5.76)

which after manipulation results in:

\[ \frac{8Q}{\pi D^3} = n \left( \frac{\Gamma_r - \Gamma_0}{K} \right)^{1/n} \left[ \frac{a^2(1-a)}{1+n} + \frac{2a(1-a)^2}{1+2n} + \frac{(1-a)^3}{1+3n} \right] \] (5.77)

where \( a = \frac{\Gamma_r}{\Gamma_w} \)

For \( n = 1 \) and \( K = n e \) equation (5.77) reduces to the standard Buckingham equation and for Newtonian fluids becomes
\[ Q = \frac{\pi D^4 P}{128 \mu} \quad \text{for } n=1, \ K=\mu \text{ and } \Gamma_0 = 0 \quad (5.78) \]

A dimensionless analysis of the variables involved in equation 5.77, \( \Gamma_0, P, D, L, p, K, n \) and \( V \) results in:

\[ \frac{p \, d^{n+1}}{L K V^n} = 4 \left( \frac{\Gamma_0 D^n}{K V^n} \right) \quad (5.79) \]

Equation 5.77 is manipulated using equation 5.79 to get:

\[ \left[ 16C_1 \left( \frac{\epsilon}{6} \right)^2 + 4C_2 \left( \frac{\epsilon}{6} \right) + 1 \right]^n \left[ 1 - 4 \left( \frac{\epsilon}{6} \right) \right]^{n+1} = 32C_3 \quad (5.80) \]

where

\[ C_1 = \frac{2n^2}{(1+n)(1+2n)} \]

\[ C_2 = \frac{2n}{(1+2n)} \]

\[ C_3 = \frac{4n}{(1+3n)^3} \]

\[ \epsilon = \frac{\Gamma_0 D^n}{K V^n} \]

\[ \delta = \frac{4p \, d^{n+1}}{L K V^n} \]

Equation 5.80 is a function of \( \eta, \epsilon \) and \( \delta \) holds for all values of \( n \).

For a Bingham plastic and \( n = 1 \)

\[ C_1 = \frac{1}{3} \quad \text{and} \quad C_3 = \frac{1}{16} \]

which results in equation 5.80 being presented as

\[ \left[ \frac{16}{3} \left( \frac{\epsilon}{6} \right)^2 + \frac{8}{3} \left( \frac{\epsilon}{6} \right) + 1 \right] \left[ 1 - 4 \left( \frac{\epsilon}{6} \right) \right]^{n+1} \delta = 2 \quad (5.81) \]

The parameters \( \epsilon \) and \( \delta \) are plotted in Figure 5.14 and reveal the \( f(\epsilon) \) but a weak function of \( n \) in the range \( 0 < n < 2 \).
Figure 5.14 Relationship between $\delta$ and $\varepsilon$ for Newtonian, power law and generalized Bingham plastic fluids

Equation 5.79 can thus be rewritten as:

$$\frac{\Delta P D^{n+1}}{LK V^n} = f\left[ \frac{\Gamma_0 D^n}{KV^n} \right] \quad (5.82)$$

This is a universal relationship between the frictional $\Delta P/L$ and the flowrate through a pipe for fluids in the categories of pseudoplastic, dilatant, yield pseudoplastic and Bingham. It also assumes rheological properties which are constant over a wide range of shear stress.

The generalised Re is presented as

$$Re^1 = \left( \frac{V^{2-n} D^n p}{K^1 8^{n-1}} \right) \quad (5.83)$$

where $K^1 = K \left( \frac{1+3n}{n} \right)$

also $f = \varnothing \left[ \frac{V^{2-n} D^n p}{K^1 8^{n-1}}, \frac{\Gamma_0 D^n}{K^1 8^{n-1} V^n} \right] \quad (5.84)$

Here $(\Gamma_0 D^n)/(K^1 8^{n-1} V^n)$ has the form of a generalised plasticity no. For an ideal Bingham, the Hedström number is
defined as:

\[
\frac{\frac{D \rho^2 p}{\eta^2}}{\eta} = \left( \frac{\frac{D \rho}{\eta}}{\eta} \right) \left( \frac{\rho}{\eta^2} \right)
\]

\[\text{He} = \text{Re}_p \ast \text{P}_L\]  

Therefore \(\text{He}^1 = \text{Re}^1 \ast \text{P}^1_L = (\frac{\rho D^2 \eta^2 - 2 \eta p}{\eta^4}) / (\text{K}^1 \eta^{n-1})^2\)  

(5.85)

and \(f = \phi^1 (\text{Re}^1, \text{He}^1)\)  

(5.86)

and \(f = \phi^{11} (\text{Re}^1, P^1_L)\)  

(5.87)

Equations 5.87 and 5.88 are represented by a family of curves illustrated in Figure 5.15

Figure 5.15 Generalised Bingham friction factors

Both \(\phi^1\) and \(\phi^{11}\) are represented analytically in terms of:

\[\frac{64}{\text{Re}^1} = f(1 - S)^n \left( C_1 S^2 + C_2 S + 1 \right)^n\]  

(5.89)

where \(S = \frac{\mu_c}{f(\text{Re}^1)^2}\) for \(\phi\)

and \(S = \frac{8 P^1_L}{f(\text{Re}^1)}\) for \(\phi^{11}\)
Turbulent flow estimates from experimental data\(^{(65)}\) show that the following equation can be used to determine the friction factor:

\[ f = 0.1357 (Re^1)^{-0.174} \quad (5.90) \]

Further adaptations of the Bingham equations occur where researchers have fitted experimental data to the equations.

An example of this is given in the work presented by Duckworth\(^{(15)}\) where coarse material is added to a carrier fluid having Bingham properties. Duckworth states that the addition of coarse material has the effect of increasing the yield stress and the plastic viscosity. These pseudo properties are given as the apparent yield stress \( \Gamma_{ya} \) and apparent viscosity \( \eta_a \). It would appear that this is due to the lower concentration resulting in the carrier material occupying the lower portion of the pipe cross-section. The work has resulted in the following generalised terms:

\[(\Gamma^{11}-1) = \varphi_1 [(S^*-1), \ M^*, \ d_y/d, \ d_{50}/d, \ d/D, \ \mu_s, \ S_s] \]
\[(\eta^{11}-1) = \varphi_2 [(S^*-1), \ M^*, \ d_y/d, \ d_{50}/d, \ d/D, \ \mu_s, \ S_s] \]
\[(5.91)\]
\[(5.92)\]

where \( \Gamma^* = (\Gamma_{ya}/\Gamma_y) \)
\[ S^* = \left(S_m-1\right)/(S_{mca}-1) \]
\[ d_y = \text{yieldno} = 10\Gamma_y/P_nS(S_s-S_{mca}) \]
\[(5.93)\]
\[(5.94)\]

as determined from the Buckingham equation

\[ 8 \ N_{ya} = \left[1 - u_s/3 + c_a^4/3\right] / c_a \]
\[(5.95)\]
\[ N_{ya} = \text{yieldno} = \Gamma_{ya} D/n_a V_m \]  
and \[ a_a = \Gamma_{ya}/\Gamma_o \]  

(5.96)

The transition from laminar to turbulent flow was predicted by the Hanks method\(^{(61)}\).

\[ D_s S_s \Gamma_{ya} D^2/n_a = 16800 \rho_a/(1-\alpha_c)^3 \]  

(5.97)

In the turbulent flow region the wall shear stress was found to follow the Colebrook-White equation\(^{(66)}\) and in simplified explicit form, the Jain\(^{(67)}\) equation, by

\[ 1/f_{m}^{1/2} = 1.14 - 2 \log[K/D + 21.25 \text{ Re}^{-0.9}] \]  

(5.98)

where \( f_m = \Gamma_0/p_m V_m^2/8 \)  
and \( \text{Re} = p_m V_m D/n_a \)  

(5.99) \hspace{1cm} (5.100)

5.3.3 Yield-Pseudoplastic Fluids

Yield pseudo plastic fluids are represented by

\[ \Gamma = \Gamma_y + K \alpha^n \]  

(5.26)

which is represented in graphical form in Figure 5.16.

![Graphical representation of yield pseudoplastic fluid rheogram](image)

**Figure 5.16** Yield pseudoplastic fluid rheogram
Integrating equation 5.26 with \( u = 0 \) where \( r = R \) and for the volumetric discharge results in the following:

\[
\frac{2V}{D} = \left(\frac{1}{K}\right)^{1/n} \frac{D \Delta P}{\left(\frac{4L}{4L}\right)^3} \left[ \frac{1+n}{n} \right] \left[ \frac{D \Delta P}{4L} - \Gamma_y \right]^{\frac{1+n}{n}} \left[ \frac{(\frac{D \Delta P}{4L} - \Gamma_y)^2}{1+3n} \right] \frac{1+n}{n} + \right.

\[
\leq \frac{2 \Gamma_y}{\left(\frac{4L}{4L}\right)^{1/n}} + \frac{\Gamma_y^2}{\frac{1+n}{n}} \right]
\]

(5.101)

For turbulent flow of a yield pseudoplastic the Torrance relation can be utilised\(^{57}\)

\[
\sqrt{\frac{2}{f}} = (A - 1.5 \: nB) + B \cdot 2.303 \: \log(1-x) + \]

\[
B \cdot 2.303 \: \log(Re_{PLC} \: \sqrt{f^{2-n}}) + 0.347(5n-8)B \quad (5.102)
\]

\( A = 3.8/n \) and \( B = 2.78/n \) which enables the above equation to be rewritten as:

\[
\frac{1}{\sqrt{f}} = \left[ \frac{2.69}{n} - 2.95 \right] + \frac{4.53}{n} \left[ \log(1-x) + \log(Re_{PLC} \: \sqrt{f^{2-n}}) \right] + \]

\[
\frac{0.68}{n} \: (5n-8) \quad (5.103)
\]

which reduces to equation 5.47 or equation 5.68 with the appropriate values of \( n, K \) and \( \Gamma_y \) for pseudoplastic and Bingham fluids respectively.
5.4 The Flow of Time-Dependent Non-Newtonian Fluids in Pipes

In general, time-dependent fluids approach time-independent behaviour. It is however necessary to establish how long a time-dependent fluid must flow in a pipe to reach the condition of approximate time-independent behaviour and how the pressure gradient varies with time and distance travelled.

Two types of fluids exist where changes in rheological properties occur, namely thixotropic and rheoplectic.

5.4.1 Thixotropic Fluids

Thixotropic fluids are categorised by a decreasing shear stress with duration of shear. Strictly speaking, the classical definition of thixotropy pertains to those fluids that exhibit reversible structural changes. The term "falsebody" behaviour is then frequently applied to those fluids in which the changes are or appear to be reversible. In these discussions, all fluids that experience structural decay with time under constant sustained shear will be considered as thixotropic.

A thixotropic fluid is often considered to be a yield pseudoplastic whose properties change with the time application of shear, ultimately approaching equilibrium or limiting values.

Many investigators have attempted to formulate quantitative relationships between the rheological behaviour and time. These include Goodeve\(^{(68)}\), Storey and Merrill\(^{(69)}\),
Billington\textsuperscript{(70)}, Hahn Reè & Eyring\textsuperscript{(71)}, Moore\textsuperscript{(72)}, Denny and Brodky\textsuperscript{(73)}, Fredrikson\textsuperscript{(74)}, Ritter and Govier\textsuperscript{(75)} and Ritter and Batycky\textsuperscript{(76)}.  

The Ritter and Govier model\textsuperscript{(75)} is more readily applied than others of comparable general suitability since it is expressed directly in terms of $\Gamma$, $S$ and $a$. The model assumes that formulation of structures, networks or agglomerates of particles is analogous to a second order chemical reaction and the breakdown of the structure is analogous in a series of consecutive first order reactions by:

\begin{equation}
\Gamma_y = \Gamma - \Gamma_u \tag{5.104}
\end{equation}

where $\Gamma$ = shearing stress on the fluid

$\Gamma_u = \mu s$ is the shearing stress caused by the so-called Newtonian component of the fluid

$\mu =$ Newtonian viscosity of the fluid under conditions where all structures may be assumed broken down

Their equation\textsuperscript{(75)} is given by

\begin{equation}
\frac{\log(\Gamma_s - \Gamma_{sa})}{(\frac{\Gamma_{so}}{\Gamma_{sa}} - \Gamma_s)} = -K_0 \left(\frac{\Gamma_{so} + \Gamma_{sa}}{\Gamma_{so} - \Gamma_{sa}}\right) \log \phi - \log K_{OR} \tag{5.105}
\end{equation}

where $\Gamma_{so}$, $\Gamma_{sa}$ = structural stresses at a given shear rate after zero and infinite duration of shear

$\Gamma_{so} = \Gamma_0 - \mu s$ \hspace{1cm} (5.106)

and $\Gamma_{sa} = \Gamma_0 - \mu s$ \hspace{1cm} (5.107)

$\phi = $ duration of shear

$K_0 =$ constant characteristic of the liquid

$K_{OR} =$ dimensionless measure of the interaction
between the network and structure decay and the re-establishment process

\[
\Gamma_s = \frac{\Gamma_{so}^2 - \Gamma_{sl} \Gamma_{sa}}{\Gamma_{sl} \Gamma_{sa} - \Gamma_{sa}^2} \tag{5.108}
\]

where \(\Gamma_{sl}\) is the structural stress at \(\phi = 1\) min

The rheological behaviour and shape of \(\Gamma-s-\phi\) curve may and usually do depend on the previous shear and chemical history of the fluid.

The Govier and Ritter method\(^{(75)}\) is applicable to a fluid that does not exhibit a significant yield value, or if it does to conditions after any yield stress is exceeded.

Their solution is based on an approximate average time shear at \(L < L_c\) as

\[
\bar{\varepsilon} = \varepsilon_e + L/V \tag{5.109}
\]

where \(\varepsilon_e\) = rheological age of entering fluid

\(V\) = cross-sectional average velocity

\(\bar{\varepsilon}\) values between 0 and \(\alpha\) are chosen from the applicable \(\Gamma-S\) curve chosen from the \(\Gamma-\phi-S\) set and evaluating the relationship between \(\Gamma\) and \(S\) with a power law fit to the curve.

Values of \(k\) and \(n\) to fit the \(\Gamma-S\) curve at \(S\) values corresponding to \((du/dr)_w\) are evaluated where:

\[
- \frac{du}{dr} = \frac{8V}{D} \frac{1+3n}{4n} \tag{5.110}
\]

and \(\Gamma_w = DAP/4L\)

Trial and error is then used to find values of \(n\) and \(k\) as for a time-independent power law fluid. \(R_{\text{pr}}\) is calculated and from it the friction factor and pressure
gradient values at a chosen $\phi_{av}$. The whole procedure is then repeated at other values of $\phi_{av}$.

Another option is offered by Ritter and Batycky\(^{76}\) who base the data in the form of an interpolation polynomial.

To apply the equation $\Gamma = f \left(-\frac{du}{dr}, \phi\right)$ it is necessary to maintain a complete description of the shear duration of all the fluid elements in the flow system. Their analysis\(^{76}\) is based on constant flow streamlines in a radial and longitudinal direction.

$\phi_j$, the duration of shear at the leading edge of the $j$th longitudinal increment at a particular streamline within the radial increment is defined as:

$$\phi_j = \frac{2(i_j - i_{j-1})}{(U_j - U_{j-1})} + \phi_{j-1} \quad (5.111)$$

where $i_j - i_{j-1}$ is the longitudinal increment length and $\frac{1}{2}(U_j + U_{j-1})$ is the average velocity on the particular streamline within the $j$th radial increment and $\phi_{j-1}$ is the duration of shear at the leading edge of the previous increment.

For the solution at any duration of flow after startup, equation 5.111 is applied along each streamline until the duration of flow has been reached at which no further ageing occurs along the streamline.

To illustrate the time, length and previous history influence, the flow curve of a thixotropic oil is shown in Figure 5.17.
A thixotropic fluid has been shown to behave over time as a series of rheologically different fluids so that an element of fluid may display different pseudoplastic or yield pseudoplastic behaviour at different points in a pipeline. As a consequence even though laminar flow may prevail immediately following the entrance to a pipe, transition from laminar to turbulent flow may occur at a distance along the pipe as $K$ decreases and $Re$ increases.

5.4.2 Rheopectic Fluids

Rheopectic fluids are not often encountered. It appears however that after adequate shear duration, the fluid
behaves as a pseudoplastic at the shear rate below which rheopexy occurs (Figure 5.18).

![Graph showing shear stress vs. shear rate](image)

**Figure 5.18** Rheopetic fluid stress-strain response

### 5.4.3 Viscoelastic Fluids

The term viscoelastic is applied to fluids which exhibit the effect of partial elastic recovery and whose viscous properties effects may be important in sudden changes in flowrate (start and stop), in rapid oscillatory flows and in flows at high shear rate. However in most normal steady state flow conditions, most materials behave essentially as purely viscous fluids and can be modelled by the relevant viscous flow models.

This effect has been shown by Sylvester and Rosen\textsuperscript{(77)} for stabilised laminar flow.

### 5.5 The Flow of Mixed Regime Slurries

The problem of formulating correlations for the prediction of the friction loss gradients for dense phase settling
slurries is complex when compared to single phase flow.

In the case of single phase flow, the pipeline design depends on the regime of operation, while the friction head is a function of five variables, rheology of the system counting as a single variable.

For two-phase flow the number of independent variables is increased so that friction loss becomes a function of pipe diameter and roughness, slope, fluid density and rheology, particle density, representative size, size distribution shape, shape distribution, mean solids mass flowrate, mean fluid mass flowrate and the gravitational constant.

Several researchers have attempted to correlate data from experimental tests with some of the earlier and more recent results discussed in this section.

One of the first methodical experimental studies leading to empirical correlations was published by Durand and Condolios in 1952\(^{(3)}\) who found that for a given solid size in a given pipe, the excess head loss caused by the presence of solids was directly proportional to the volume concentration. This observation led to the use of the excess head loss parameter, which is still in use today in some form or another.

\[\phi = KN^{-n}\]  \hspace{2cm} (5.112)

The research conducted by Durand was carried out with single sized particles and no account was taken of the possible effect of an increased density and viscosity of the fine particle carrier medium present in the widely graded distribution slurries transported throughout the
Durand's work was followed up by amongst others Worster\(^{(78)}\), Newitt, Richardson, Abbot and Turtle\(^{(79)}\), Wilson and Warren\(^{(78)}\), Rose and Duckworth\(^{(80)}\), Zandi and Govatos\(^{(81)}\) who all postulated solutions for determining the headlosses of pseudo-homogeneous and heterogeneous slurries.

The last two decades have however seen a number of prediction methods evolving that included the effect of relatively "thickened" carrier mediums, made up of the carrier liquid and the finest particles conveyed.

Some of these methods used for the headloss predictions of dilute and medium phase mixed slurries are listed below.

5.5.1 Uhlmann\(^{(82)}\)

Uhlman's approach can be summarised as follows: For any given transport velocity the particle size applying at the transition from homogeneous to heterogeneous flow can be found. All particles with a size smaller than this determined particle size will then form the homogeneous part of the carrying fluid, while the remaining particles above the crucial size will be transported in a homogeneous fluid with calculated properties. Furthermore, measured settling velocities are used to determine equivalent \(C_0\)'s and diameters and the headloss is then obtained from the following

\[
\frac{J_m/S_f-J_w}{x C_v J_w} = K \left[ \frac{V_m^2 V C_0}{g D (S/S_f-1)} \right]^n
\]  

(5.113)
where \( x \) = the proportion of particles coarser than the critical particle diameter (i.e. portion in heterogeneous zone)

\[
C_0 = \text{weighted mean particle } C_0 \text{ for } x
\]

Further a modified Zandi \( I = \text{number is used and is represented as:} \)

\[
N_t = \frac{\nu^2 \sqrt{C_0}}{C_v D g (S/S_t-1)} \quad (5.114)
\]

5.5.2 \text{Wasp}^{(1)}

One of the more recent approaches that of Wasp and colleagues, presents what they claim to be a practical approach to the design of slurry facilities.

Wasp's method employs the concept of a "two phase vehicle". According to this concept, it is reasoned that the concentration and size distribution of the particles occurring at the top of the pipe can be considered as existing at all other points in the pipe, i.e. they represent the homogeneous or "vehicle" portion of the slurry. The remainder of the solids is considered as a heterogeneous suspension conveyed by this vehicle. The total friction loss is the sum of the vehicle friction loss and the over pressure due to the heterogeneity of the remainder of the solids.

One of the central features of the Wasp correlation is the determination of the split between the vehicle and heterogeneous portions of the slurry. This is done by:

\[
\log \frac{C}{C_A} = -1.8 (w/Bxu^*) \quad (5.115)
\]
which ensures that the procedure has a built-in means of allowing for the fact that as the amount of fine material in the vehicle increases, so does the carrying capacity for coarse solids.

The vehicle component friction factor is determined from the Moody diagram, using the vehicle viscosity and density in compilation of the Reynolds's no. The heterogeneous component friction factor is based on

\[
fm = f_l \left[ 1 + k \left( \frac{gD(P_s - P_{veh})}{V^2 P_{veh} VC_0} \right)^{3/2} \right] C_v/100
\]  \hspace{1cm} (5.116)

The critical deposition velocity is based on:

\[
V_D = 1.87(d/D)^{1/6} \left( 2gD \frac{(P_s - P_{veh})}{P_{veh}} \right)^{0.5}
\] \hspace{1cm} (5.117)

or

\[
V_D = 1.323 C_v^{0.186} (d/D)^{1/6} \left( 2gD \frac{(P_s - P_{veh})}{P_{veh}} \right)^{0.5}
\] \hspace{1cm} (5.118)

5.5.3 Lazarus and Sive\(^{83}\)

A generalised mechanistic model was developed for the hydraulic transport of mixed regime slurries in pipelines. The model consists of three components; a vehicle portion, a suspended portion and a bed load portion. In this method each component of the particle size distribution is analysed to determine its contribution to the fraction of vehicle load, suspended load and bed load.

The vehicle portion consists of water plus those non-settling and slow settling particles which affect the rheology of the vehicle.
For all particles larger than $d_{\text{max}}$, each size fraction $d(I)$ is divided into a suspended portion and a bed load portion using the mean mixture velocity at the threshold of turbulent suspension as the criterion. This value is obtained from:

$$V_{\text{mt}}(I) = 0.6 \, V_t^1 \, \sqrt{2/f} \, \exp \left( \frac{45 \, d(I)}{D} \right)$$  \hspace{1cm} (5.119)$$

where $V_t^1 = \text{hindered settling velocity in the vehicle portion for the size fraction } d(I)$.

The bed load portion is determined by means of successive approximations to determine the half angle ($B$) designating the bed load surface.

The vehicle and suspended load components are combined to determine the wall shear above the bed load interface. This wall shear is converted to a force and added to the bed load resisting force to determine the total friction head loss gradient along the pipe.

5.5.4 Weber$^{(84)}$

Weber adjusted the applicability of Durand's equation by applying suitable parameters representing the grain size distribution. The major component of the analysis is the division of the grain size distribution into homogeneous and heterogeneous components.

The limiting grain size for the transition from pseudo-homogeneous to heterogeneous depends on different influences but for general cases is given by:

$$d_s = \sqrt{\frac{3}{4} \cdot \frac{28}{g} \cdot \frac{0.0056}{(s-1)} \cdot \bar{u}_f \cdot V}$$ \hspace{1cm} (5.120)$$
With the fine part $f$, the coarse part $(1-f)$ and the total delivered concentration, the delivered concentration of the homogeneous and heterogeneous components are described by:

$$C_{\text{hom}} = f \cdot C_T$$  \hspace{1cm} (5.121)

$$C_{\text{het}} = (1-f)C_T$$  \hspace{1cm} (5.122)

The density of the enriched carrier fluid is given by:

$$P_{fs} = f_c \cdot P_s + (1-f_c)P_f$$  \hspace{1cm} (5.123)

Incorporating the above into the general Durand equation, the equation describing the flow of poly disperse heterogeneous Newtonian mixtures is given by:

$$\Delta P = \left\{83\cdot \frac{Dg}{V_m^2} \cdot \frac{(P_s-P_{fs})/P_{fs}/\sqrt{C_0}}{(1-f)C_T+1} \right\} f_f \cdot \frac{P_{fs}/2V_m^2 \cdot L}{U}$$  \hspace{1cm} (5.124)

Non-Newtonian behaviour of the carrier is accounted for in the case of a Bingham fluid by defining a Non-Newtonian Reynold's number

For laminar flow \hspace{1cm} \text{Re}_B = \frac{V_m^2 P_{fs}}{(\Gamma_0 + \mu_p \cdot V_m/d)} \hspace{1cm} (5.125)

For turbulent flow \hspace{1cm} \text{Re}_B = \frac{V_m D P_{fs}/\mu_p}{(\Gamma_0 + \mu_p \cdot W_f/d_f)} \hspace{1cm} (5.126)

and the grain size related Reynolds no

$$\text{Re}_{SB} = \frac{w_{SB}^2 P_{fs}}{(\Gamma_0 + \mu_p \cdot W_f/d_f)}$$  \hspace{1cm} (5.127)

where the settling velocity of a single particle in a Bingham fluid is given by:

$$w_{SB} = \sqrt{2(2/3)(P_s-P_{fs})(d_f g - \pi \Gamma_0)/(C_0 \cdot P_{fs})}$$  \hspace{1cm} (5.128)
Wilson's model known as the "bed slip" model or layered force balance model is related to the solids wall friction coefficient \( f_s \), and the solids-solids interface between the particle rich lower zone and the relatively particle free upper zone.

For each zone the force balance equates the driving and resisting forces per unit length. The upper zone has a driving force that is the product of area and pressure gradient, and a combined resisting force developed by the fluid shear stress at the pipewall and that interface.

For the lower layer, the force produced by the interfacial shear is of opposite sign, contributing to the driving force. The resisting force at the lower boundary has two components. The fluid component is the fluid shear stress at the lower boundary while the remaining component of resisting force arises from granular contact at the lower boundary.

The total friction gradient is given by:

\[
\dot{\iota}_m = \dot{\iota}_{ms} + (1 - R)\dot{\iota}_{mh}
\]

(5.129)

where

\[
\dot{\iota}_{mh} = \dot{\iota}_f [1 + fn((s_e - 1)\text{Cvd})]
\]

(5.130)

and

\[
\dot{\iota}_{ms} = \dot{\iota}_f + fn [(s_e - 1)\text{Cvd}]
\]

(5.131)
5.5.6 Lazarus and Nelson\(^{(66)}\)

This is an empirical derivation based on a large number of experimental results. The contribution of the various slurry parameters are included in a single expression, \( \Psi \) which is then used to determine the overall mixture friction factor based on the base fluid friction factor. \( \Psi \) is given by:

\[
\Psi = \frac{\nu^2}{gD} \left( \frac{\bar{u}}{VD} \right) \cdot \frac{1}{M} \cdot \frac{1}{S} \left[ 1000 \left( \frac{d}{D} \right)^{(0.44) \log(D/d) + 1.38} \tan h(1.38) \right]
\]

(5.132)

where \( M^* = S \cdot \phi_v/(1-\phi_v) \)

(5.133)

The friction factor of the mixture is then obtained from:

\[
f_{m/fb} = 1.2 \, \Psi^{-1.5} \quad \text{for } \Psi < 0.8
\]

(5.134)

\[
f_{m/fb} = 1.45 \, \Psi^{-0.4} \quad \text{for } 0.8 < \Psi < 2.53
\]

(5.135)

\[
f_{m/fb} = 1 \quad \text{for } \Psi > 2.53
\]

(5.136)

The critical values of \( \Psi \) can be utilised for determining the transition from saltation to heterogeneous flow and the transition from heterogeneous to pseudohomogeneous flow.

5.6 Shear-viscosity of Medium and Dense Phase Slurries at Varying Shear Rates

Paramount in the analysis of any rheological dominant model is the correct determination of the viscosity. This can be achieved by the use of analytical methods, viscometry or pipeline pressure gradient results.
Prior to expressing the various influences on viscosity and the methods of obtaining the value it is important to define the various viscosities referred to.

1) **Shear viscosity** = relative viscosity
   = apparent viscosity = Newtonian viscosity
   \[ \mu = \frac{\gamma}{\dot{e}} \]  \hspace{1cm} (5.137)
   = shear stress/shear rate
   \( \eta \) = Bingham coefficient of rigidity
   \( \mu_a \) = viscosity at high rates of shear
   which may be equal to \( \eta \)
   \( \mu_e \) = \( \Gamma^w/(8V/D) \) and can be related to the pseudoplastic and Bingham viscosity terms by
   \[ \mu_e = \eta(1+(\Gamma_y D)/(6\eta V)) \] Bingham (5.138)
   \[ \mu_e = K(8V/D)^{n-1} \] Pseudoplastic (5.139)

   \( K \) = consistency index for pseudoplastic flow

Three different regions are generally observed on stress vs strain curves of a highly concentrated suspension.

a) A low shear rate region in which interparticle forces are dominant.

b) An intermediate region in which the rheological behaviour is a complex function of both solid and liquid properties.

c) A high shear rate region in which the rheological behaviour tends to be largely Newtonian exhibiting well aligned particle movement with very small effects of interparticle forces compared to hydrodynamic forces.
At high shear rates \( u = u_a \) indicating Newtonian behaviour while \( u_0 = \) lowest possible shear rate.

Between these limits \( u_0 > u > u_a \) the shear viscosity varies in a manner specific to the physical and chemical features of the suspension.

The extension of Einstein's \(^{(87)}\) dilute suspension viscosity equation to highly concentrated suspensions resulted in:

\[
\eta = 1 + \sum_{n=1}^{a} k_n \phi^n
\]  

(5.140)

where \( k_1 = 2.5 \) and each of the other coefficients \( k_n \) refer to a specific effect of the complex interactions between the fluid and solid particles for highly concentrated suspensions. In these models (Table 5.1) the solids concentration is used as the only physical parameter and the other effects are accounted for by the empirical coefficients \( k_n \).

The tabulated equations all indicate that there is an exponential relationship between the viscosity index and concentration. They do not however indicate how the size of the particles influence the viscosity.

A comparison of some of the functions and a Non-Newtonian fluid (polymer melt) and short glass fibres is illustrated in figure 5.19.
Table 5.1*(88) Relationships for relative viscosity \( \pi_{rel} = a_0 + a_1 \varphi + a_2 \varphi^2 + a_3 \varphi^3 + a_4 \varphi^4 \ldots \)

<table>
<thead>
<tr>
<th>Researcher</th>
<th>Equation</th>
<th>Coefficients of expansion</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( a_0 )</td>
</tr>
<tr>
<td>Einstein</td>
<td>( \pi_{rel} = 1 + 2,5\varphi )</td>
<td>1</td>
</tr>
<tr>
<td>Gold</td>
<td>( \pi_{rel} = 1 + 2,5 \left( 1 + \frac{5}{16} \varphi \right) )</td>
<td>1</td>
</tr>
<tr>
<td>Burgers and Saito</td>
<td>( \pi_{rel} = 1 - 2,5\varphi + 12,6\varphi^2 \ldots )</td>
<td>1</td>
</tr>
<tr>
<td>De Bruijn</td>
<td>( \frac{1}{\pi_{rel}} = 1 - 2,5\varphi + 1,55\varphi )</td>
<td>1</td>
</tr>
<tr>
<td>De Bruijn</td>
<td>( \frac{1}{\pi_{rel}} = \frac{(1 - \varphi)}{(1 + 1,5\varphi)} )</td>
<td>1</td>
</tr>
<tr>
<td>Vand</td>
<td>( \pi_{rel} = \exp(2,5\varphi) )</td>
<td>1</td>
</tr>
<tr>
<td>Vand</td>
<td>( \pi_{rel} = \exp \left[ \frac{2,5\varphi}{39} \right] \left[ 1 - \frac{\varphi}{164} \right] )</td>
<td>1</td>
</tr>
<tr>
<td>Researcher</td>
<td>Equation</td>
<td>Coefficients of $\phi$ expansion</td>
</tr>
<tr>
<td>----------------------------</td>
<td>---------------------------------------------------------------------------</td>
<td>----------------------------------</td>
</tr>
<tr>
<td></td>
<td>$\pi_{rel} = 1 = 2.5\phi + 7.35\phi^2 + \ldots$</td>
<td>$a_0$</td>
</tr>
<tr>
<td>Vand</td>
<td>$\pi_{rel} = (1 - \phi)^{-2.5}$</td>
<td>1</td>
</tr>
<tr>
<td>Vand, Roscoe, and Brinkman</td>
<td>$\pi_{rel} = (1 - \phi)^{-2.5}$</td>
<td>1</td>
</tr>
<tr>
<td>Roscoe</td>
<td>$\pi_{rel} = (1 - 1.35\phi)^{-2.5}$</td>
<td>1</td>
</tr>
<tr>
<td>Mooney</td>
<td>$\pi_{rel} = \exp \left( \frac{2.5\phi}{1 - K\phi} \right)$ for $K = 1.35$</td>
<td>1</td>
</tr>
<tr>
<td>Mooney</td>
<td>$\pi_{rel} = \exp \left( \frac{2.5\phi}{1 - K\phi} \right)$ for $K = 1.91$</td>
<td>1</td>
</tr>
<tr>
<td>Shaheen</td>
<td>$\pi_{rel} = \exp \left( \frac{2.5\phi}{1 - a\phi n} \right)$ for $n = 0.5$</td>
<td>1+2.5\phi+1.935a^2\phi^2+(3.905a+ 24.435a^3+11,166a^4)\phi^3+\ldots</td>
</tr>
<tr>
<td>Frankel and Arcivos</td>
<td>$\pi_{rel} = -\frac{9}{8} \frac{(\phi/\phi_m)^{1/3}}{1 - (\phi/\phi_m)^{1/3}}$ for $\phi_m = 0.74$</td>
<td>1,156\phi^{15}+222\phi^{25}+1,35\phi+ 1,49\phi^{45}+\ldots</td>
</tr>
</tbody>
</table>
Figure 5.19 The Relative Viscosity as a function of $C_v^{(89)}$

An important parameter is the maximum packing concentration which refers to the highest possible value of concentrations that can be achieved with the solids of specified characteristics in a specific fluid.

The calculation of $\varphi_m$ for a suspension with uniform and non-uniform size distribution is dealt with in section 5.8, suffice it to say that the value is dependent on the shape and size of the individual solid particles, the size distribution and physical and chemical interactions between the solid particles themselves and the carrier fluid.
To overcome these difficulties, a parameter, the intrinsic viscosity has been introduced to represent the surface conditions.

\[ \eta = \lim_{\phi \to 0} \left[ \frac{\eta - 1}{\phi} \right] \]

(5.141)

\( \eta \) is intended to indicate the effective shape of each solid particle as a result of its surface interaction with the carrier fluid. The wet shape of the particle can be different to the dry shape as a result of the S-potential and absorbed layer.

A popular correlation for the low shear-rate region has been obtained by Chong et al(90):

\[ \eta_n = \left[ 1 + \frac{0.75(\phi/\phi_m)}{1-\phi/\phi_m} \right]^2 \]

(5.142)

It has been suggested that the above equation can be used to predict the shear viscosity of bimodal suspensions at low shear rates provided that \( \phi_m \) is known.

An empirical model proposed by Eilers(91) has given good results in the high shear-rate region where the behaviour tends to be Newtonian. If theoretical values of \( \eta = 2.5 \) and \( m = 0.74 \) for spheres are substituted in the original version of Eilers equation then the following can be used to determine the shear viscosity at high shear rates:

\[ \eta_n = \left[ 1 + \frac{0.5[\eta \phi]}{1-(\phi/\phi_m)} \right]^n \]

(5.143)

Further inspection of the literature by Dabak et al(92) suggests that the shear viscosity of suspensions with rigid spheres is represented by:
\[ \eta = \left[ 1 + \frac{2.5 \ f_{ct}(k \phi)}{\eta} \right]^n \]  
(5.144)

where \( k = \text{crowding factor} \)
\( n = \text{suspension dependent parameter} \)

2.5 is a characteristic of uncharged rigid particles. For real systems 2.5 is replaced by \([\eta]\) to account for the effects of non sphericity and other physical and non physical surface interactions of the solid particles with various fluid media.

\( n \) is considered to reflect the level of particle interactions at varying shear rates, assuming that the intrinsic viscosity represents the effective shape of the particles in a specified fluid and remains invariant.

The variation of \( n \) with shear rate is referred to as the particle interaction parameter:

\[ f_{ct} (k \phi) = \phi/(1-k \phi) \]  
(5.145)

The crowding factor has been defined as \( k = 1/\phi^{(93,94)} \) which implies that as \( \phi \to \phi_m \) the suspension will get thicker and will tend to infinity.

The mobility parameter \((\phi/(\phi_m-\phi))\) affects the viscosity and is representative of the relative degree of freedom the particles have to move in the mixture\(^{(95)}\) \((\phi_m-\phi)\) represents the effective space available for the particles to disperse.

The general form of a viscosity equation to calculate the viscosity of suspensions at varying shear rates is:
\[ \mu_{s,a} = \mu(G) = \mu \left[ \frac{1 + [n] \phi \varphi_m}{n(G)(\varphi_m - \phi)} \right]^{n(G)} \] (5.146)

where \( \mu(G) \) and \( n(G) \) are the viscosity and particle interaction parameter as a function of \( G \) respectively.

At low shear rates, viscosity tends to \( \mu_0 \) or \( [\mu(G \rightarrow 0)] = \mu_0 \) and \( \mu(G \rightarrow \alpha) = \mu_a \). At high shear rates \( n = 2 \) and

\[ \mu_a = \left[ 1 + \frac{[\eta] \phi \varphi_m}{2(\varphi_m - \phi)} \right]^2 \] (5.147)

This is a modified form of Eilers model\(^{[87]}\) which predicts Newtonian viscosity at high shear rates for suspensions with spherical or irregular particles better than the other existing correlations.

\([\eta] = 2.5\) for uncharged particles with smooth spherical surfaces whereas \([\eta]\) is uncertain for irregular shapes. Thus if \( \mu_a \) at high shear rates is known it is possible to calculate \([\eta]\) which is assumed to be invariant for each specific material.

The maximum packing density can be obtained experimentally or analytically (section 5.8) by sedimentation or centrifuging. Analytical methods are in general based on ideal packing conditions and do not include the effects of surface chemistry and particle shape.

The difference between the experimental and calculated packing densities has been correlated by the following postulated relationship\(^{[92]}\).

\[ \varphi_{se} = f_{ct} \left[ [n] \frac{n_0}{n_a} \varphi_{mc} \right] \] (5.148)