CHAPTER 6

CONCLUSIONS AND IMPLICATIONS

1.1 Introduction

This study investigated how first-year university students at Universidade Pedagógica- UP (Pedagogical University) in Maputo-Mozambique bring their knowledge and thinking of algebra in understanding and working with geometry. The study explored how these students connected and used algebraic and geometric concepts and investigated whether this connection promoted students’ conceptual understanding and problem solving performance in geometry.

At UP, in the first academic year geometry, the students attend Euclidean Geometry course (first semester) and Analytic Geometry course (second semester). In the first semester, the geometry content incorporates origin of geometry, axiomatic approach of Euclidean Geometry, geometric constructions, triangles, quadrilaterals, circumference and sphere, elements of plane trigonometry, and spatial geometry. In turn, in the second semester it comprises vectors and vector algebra, straight line in plane, circumference, coordinate transformations, conics, types of coordinate systems, parametric equations, straight line and plane in space, and quadric surfaces. Research studies have generally tended to focus on the teaching and learning of either algebra or geometry but rarely on the connections between the two areas of mathematics (Chapter 3). According to the geometry content at UP, the students use algebra as a tool to solve geometric tasks and to learn new content.

This chapter provides a summary of the study and a discussion of the findings in the context of the current literature. This chapter also addresses limitations of the study, implications for curriculum and instruction, and recommendations for future research.

1.2 Summary of the Study and its Findings
The compartmentalization of mathematics into distinct subjects areas, such as geometry, arithmetic, algebra, calculus, functional analysis, etc, often fails to present mathematics as a coherent whole to students, a great deal of whom find it quite difficult to make links between these areas. In this regard, mathematical syllabi have undergone reform at UP emphasizing multidisciplinary connections in general and a link between different concepts in the curricula in particular (Ismael, 2003). Accordingly, connections between algebra and geometry might help students to appreciate how one complements the other. Connections between algebra and geometry are not new but have their roots several centuries ago. According to Hansen (1998), Descartes achieved this goal of finding a general method in geometry by the introduction of coordinate systems and the creation of analytic geometry (coordinate geometry). An example to illustrate this idea is as follows. If we are to show that a line, which bisects two sides of a triangle, is parallel to the third side, in the Euclidean geometry, then we can use, for example, the following knowledge schemas: We need to use an auxiliary theorem, which states that a line bisecting one side of a triangle and is parallel to a second side, bisects the third side of the triangle. Afterwards, it is recommendable to approach the theorem using an indirect proof, where at each step what we say has a meaning in relation to a figure (Lewis, 1964, pp. 325, 326). While in the coordinate geometry context, we can use a direct proof, approaching it, for example, by setting any coordinates for the vertexes of the triangle and using collinear vectors and formulae of midpoint coordinates. It might be simpler to use vector algebra geometrically. Analyzing the solutions we can conclude that using traditional Euclidean geometry requires one to be acquainted with previously demonstrated theorems, postulates and axioms. Besides, one needs to make complicated connections and be bound to figures at each step. However, using coordinate geometry requires one to know vector algebra equations, which are simpler to deduce and to recall. Afterwards, one applies them directly to solve the problem. In other words, through coordinate geometry we can solve the geometric problems in more elegant, quick, and fuller way than in Euclidean geometry.

A model was adapted from the literature (Charbonneau, 1996; Stillwell, 1998; and Duval, 1998) to explore how algebraic thinking might be an aid to geometrical understanding (Chapter 3, Fig. 3.1). It shows that the cognitive processes (symbolization, relations, and abstraction), which underlie algebraic thinking, are interconnected. These cognitive processes might jointly be used to aid any of the cognitive processes which constitute geometrical understanding (visualization processes, construction processes, and reasoning), either separately or jointly. Prawat’s (1989) framework on transfer and learning appeared to be
relevant to analyse the data collected in the study: (Knowledge connectedness and communication, general and specific strategy, and mastery and performance disposition).

1.2.1 Purpose of the Study

In this study, the aim was to explore how students work with geometry problems, what algebraic knowledge, if any, they bring to the solving of these problems, and how knowledge of algebra and algebraic ways of thinking promotes or hinders success (ie conceptual geometric understanding and performance) in geometry problem-solving processes. Accordingly, three research questions were considered:

1. How do first-year university students solve geometry problems? To what extent do they use algebraic knowledge and thinking in solving such problems? What kinds of meanings do students make of different algebraic and geometric concepts involved in problem solving situations?
2. To what extent does algebraic knowledge and thinking aid students’ conceptual understanding and problem solving performance in geometry?
3. To what extent is geometric work linked to algebraic thinking in the first year university geometry course at the Universidade Pedagógica in Maputo?

According to the ideas of Stillwell (1998) the synthetic side of geometry is concerned with visualization and construction processes driven by intuition, thus corresponding to van Hiele Level 1 and partly Level 2. From Level 2 (partly) to Level 5 it seems to characterize the analytic side of geometry as analysis, abstraction, deduction, and rigor are mainly analytic concepts and especially in my case algebraic stances where reasoning plays a key role. Hence, the concern of this study was not to place the learner’s understanding according to the van Hiele levels discontinuously. On the contrary, its concern was in placing the learner’s understanding in terms of the two aspects of geometry, synthetic and analytic being algebra (algebraic thinking) the bridge between the two aspects, in the Mozambique context particularly at Universidade Pedagógica.

The research in this study is therefore theoretically significant as it explored how algebra may serve as a tool into geometrical understanding, a conceptual relationship that has not received any focused attention in the conceptualization of the van Hiele model of understanding.
1.2.2 Methodology

The main body of the research was done in three phases, which can be analysed separately. The three phases were the Pilot Study, the Main Study – Euclidean Geometry Course, and the Main Study – Analytic Geometry Course.

The Pilot Study was designed to explore students’ algebraic and geometric knowledge and thinking and their ability to access and use algebraic knowledge and thinking in geometry problem solving. I constructed a 50-minute pilot test (Appendix 1) and administered to 26 first-year students at UP (UP mainly aims at training secondary and high school pre-service teachers for different subjects including mathematics). The tasks of the pilot test were developed with adaptations from different sources (Lewis, 1964 and Alvarinho et al., 1992) and validated by the lecturer of the Euclidean and Analytic Geometry courses at UP. This method enabled me to rearrange some aspects concerned with appropriateness of the tasks in terms of interplay between algebraic and geometric thinking for the next stages of the study as well as to find out the first-year students’ academic and professional background at the beginning of their degree at UP.

In the Main Study - Euclidean Geometry Course, a 60-minute Diagnostic test (Appendix 2) was administered at the start of the Euclidean Geometry course to 32 first-year university students at UP. This test assessed the students’ proficiency in school geometry and algebra when they enter UP. Students’ responses to this Diagnostic test were examined and analysed (Section 5.3.1). Afterward, I selected 14 students (8 low and 6 high achievers) for Interviewing Phase 1. These students were individually interviewed using Free Recall and Hinting Task on their Diagnostic test responses (Appendix 3). Thereafter, I selected 8 students (5 low and 3 high achievers) students for in depth follow up study. This was a convenience sample as during interviewing process, I found their conversation interesting for the purpose of this study. After analyzing the transcripts of the 8 students I came up with four contextual background categories of students (Section 5.3.5.2). Accordingly, I selected 5 (2 low and 3 high achievers) students out of 8 as my target sample. All categories were represented in this target sample.

I administered a Elaboration and Concept Mapping Task (Appendix 4) to the target sample. The Elaboration Task required students to construct tasks from a given set of concepts or theorems. The Concept Mapping Task required students to form true propositions using the
given concepts. The lecturer of Euclidean Geometry noted that these concepts were some of the most critical for the course.

In the Main Study – Analytic Geometry Course, I worked with the same target students. They all passed Euclidean Geometry course and they enrolled and attended Analytic Geometry course in Semester II. In the middle of Semester II, Interviewing Phase 2 took place, I interviewed the target sample based on the lecturer’s test (Appendix 5) using a Free Recall and Hinting Task. In this interviewing, I posed an open-ended question on the relationship between Euclidean and Analytic Geometry and its influence on geometric conceptual understanding.

Thus, transcripts from the interviews with the students, and the artefacts collected from the students (Pilot test, Diagnostic test, and written responses to the Elaboration and Concept Mapping Task) constituted the bulk of the data for this study. The data collection was carried out during two semesters.

1.2.3 Data Analysis

Data from the interviews and the artefacts collected from the students were analysed using constant comparative method for an inductive analysis of qualitative data (Sanger, 1994 and Boyatzis, 1998). Data from interview transcripts were first coded and unitized using a qualitative software for data analysis, ATLAS.ti. The artefacts were also analysed and coded using the same software. In this study, I used descriptive and pattern codes. Descriptive codes denote different cognitive processes underlying algebraic and geometric thinking as well as their relationships. Pattern codes denote emergent themes, configurations, or explanations (ibid., p.69). Using what Glaser and Strauss (1967) (as quoted by Jaworski, 1994) call the “constant comparative method”. I identified particular incidents and coded each incident into as many categories as possible, as categories emerged or as data emerged that fitted an existing category. Memos were written about important aspects of the data. This helped me to condense the mass of data collected. Under the umbrella of developing a code inductively, a data-driven approach adopted by this study, I chose a sub sample of five case
studies, one of which represented the contextual categories that emerged during the process of analysis of Task 1 (Section 5.3.5.2 and Boyatzis, 1998). For Category 2, I chose two subjects as I found their data interesting for the purpose of the study. The raw data collected from this sub sample was the basis for developing the codes, which is supported by the data reduction method (Boyatzis, 1998). I describe and discuss the findings of the study in the next section.

1.3 Discussion of the Findings

The structure of the discussion of findings from the analysis of data to be followed comprises three different phases: 1) the Pilot Study, 2) the main study – Euclidean Geometry course, and 3) the main study - Analytic Geometry course.

1.3.1 Pilot Study

The Pilot Study enabled me to rearrange some aspects concerned with appropriateness of the tasks in terms of the interplay between algebraic and geometric thinking for the next stages of the study as well as to find out the first-year students’ academic and professional background at the beginning of their degree at UP. From the analysis of students’ responses, I concluded that the differing contexts of teaching and learning developed students academically differently before entering UP. Although UP has been making efforts to create a homogeneous milieu for teaching and learning of geometry, it seemed that students still faced difficulties in accessing and using algebraic and geometric knowledge at least in the first year. These results allowed me to be open to other factors, besides the conceptual, such as contextual and structural issues, which may influence the students’ understanding and their use of algebraic knowledge and thinking for solving geometric tasks.

1.3.2 Main Study – Euclidean Geometry Course

Below I present the key research findings regarding the first two research questions organized by themes previously set up and others that emerged during the data analysis of the first component of the main study – the Euclidean Geometry course.

*Algebraic and geometric knowledge base and connectedness*
The Diagnostic test results showed that the target students solved some geometric tasks using both algebraic and geometric thinking. Only Student 1 accessed and used algebraic thinking (reasoning) to get a geometrical insight to solve the problem (Stillwell, 1998). Nevertheless he was not able to accomplish the solution. The remaining students tried to solve it using geometrical thinking only (intuition) but were unsuccessful. This result replicated what Schoenfeld (1986) found that unless students learn to take advantage of both deductive (as a means of discovery) and empirical (as a means of development of intuition) approaches to geometry and learn to profit from the interaction of those two approaches; students will not reap the benefits of their knowledge.

The analysis of the transcripts regarding Interviewing Phase 1 showed that the target students solved some geometric tasks using both algebraic and geometric thinking. It seemed easier for these students to use their intuition (synthetic strategies) rather than their reasoning (analytic strategies) in the tasks which are qualitatively presented. However, they faced difficulties in getting a solution. Besides these strategies seemed to be specific and applied to a concrete situation. In such tasks, I needed to provide hints to most of the students either using visualization and construction processes or reasoning driven by algebraic thinking. Their reasoning was stronger than their intuition. When I prompted them to elicit their intuition, I was forced to supply them all key concepts to get the solution. Meanwhile, when I elicited their reasoning through algebraic thinking I noticed that, the intuition of some of them was elicited and became robust towards the solution. All students, through hinting process used both intuition and reasoning (a numerical approach, patterning, formulae and relations) to solve the tasks. Accordingly, it also showed that the interaction of deductive (as a means of discovery) and empirical (as a means of development of intuition) approaches to geometry led these students to reap the benefits of their knowledge.

The findings from the analysis of data from the Elaboration and Concept Mapping Task showed that these students used algebraic thinking to solve geometric tasks. Even those tasks which structure allowed synthetic strategies, these students used some algebraic thinking elements (acronym and symbols) to identify geometric objects (angles, polygons, vertexes, lines, and segments). Moreover, they also used pictures to inform their analytic (algebraic) strategies. This means that algebraic and geometric concepts mutually aided each other towards students’ conceptual geometric understanding.

*Analytic versus synthetic strategies*
Most of the students claimed that the analytic strategies were insightful towards a geometric solution. This finding seems to corroborate my assumption that algebraic thinking can be seen as a way of layering meanings on each other, connecting between ways of knowing and seeing, rather than as a way of replacing meaning of each other. These meanings became reshaped as learners exploited available tools in algebra to move the focus of their attention onto new objects and relationships in geometry. Although algebraic thinking shed light in geometrical understanding, the students still needed to possess visualization and construction skills to accomplish the geometric tasks successfully.

Students’ academic and professional background

The differences in teaching and learning experiences and contexts amongst the students before entering UP was linked to the differences in the approaches they used, in the problem solving situations in the Diagnostic test. Meanwhile, after attending the Euclidean Geometry course I noticed on one hand a common tenet regarding the diagrammatic representation use: it seemed that these students used diagrammatic representations for the two purposes mentioned by Koedinger and Anderson (1990). On the other hand, most of them showed a strong tendency to “algebraise” geometry as even those tasks, which could be constructed and solved synthetically; they tried to use analytic (algebraic) strategies even after attending the Euclidean Geometry course. This was due to a prolonged exposure of algebra teaching against geometry teaching during schooling time.

1.3.3 Main Study – Analytic Geometry Course

I present below the major findings regarding the research questions organized by themes previously set up and others that emerged during the data analysis of the second component of the main study – Analytic Geometry course.

Algebraic and geometric knowledge base and connectedness

All students used reasoning to solve Task 3 of the lecturer’s test through algebraic thinking aspects such as the dot product formula, coordinates, vector, trigonometry relations, matrix, acronym, equation, and unknown (analytic strategies); although they got the correct solution, some of them blindly used the formulae. Others attempted to make sense of the formulae using their intuition through visualization and construction processes (synthetic strategies).
All students through hinting process used both intuition (visualization and construction processes) and reasoning (formula, coordinates, vector, trigonometry relations, acronym, equation, and unknown, and generalisation) to solve the task successfully (Schoenfeld, 1986).

All students and especially the performance-oriented group seemed to profit from the hinting process for organizing and structuring their knowledge schemas. The mastery-oriented group showed reasonable organized and structured knowledge schemas. The hinting process only served to refine their knowledge schemas structure and organization.

*Analytic versus synthetic strategies*

In Analytic Geometry, analytic strategies (general strategies) seem to be more efficient and easier to access than synthetic strategies (specific strategies), meanwhile synthetic strategies seem to be more insightful than analytic strategies.

In Task 3 all students seemed to possess robust schemas regarding analytic strategies, however, they faced difficulties in relating analytic and synthetic schemas as they needed some hints (the mastery-oriented group) and a lot of hints (the performance-oriented group) for clarification purposes.

*Geometric modes: Euclidean (synthetic) Geometry and Analytic Geometry*

Most of the students (except S1) explicitly or implicitly asserted that both geometries deal with the same objects in different perspectives.

A student (S7) observed that Euclidean Geometry is a sort of a foundation for Analytic Geometry as it supplied with empirical concepts (intuitive level) to Analytic Geometry. In turn, Analytic Geometry applied these concepts upgrading them to abstract concepts (reasoning level).

*Geometrical Understanding*

I noticed that In Analytic Geometry the students needed intuition (synthetic strategies) to inform their reasoning (analytic strategies). In the contrary, in Euclidean Geometry they needed reasoning (analytic strategies) to inform their intuition (synthetic strategies). However, I observed that this balance still needed to take plane in my target students’ mind.
In other words, in Analytic Geometry, my students needed geometric thinking to aid their algebraic thinking and in Euclidean Geometry, they needed algebraic thinking to aid their geometric thinking. This mutuality in geometry seems to underpin what Schoenfeld (1986) found that the interaction of deductive (as a means of discovery) and empirical (as a means of development of intuition) approaches to geometry leads students to reap the benefits of their knowledge.

1.4 Limitations of the Study

The limitations of this study include the reliance on the interview tasks and the responses of the target students only, which did not allow an adequate triangulation of data.

Another limitation was that this study took a long time to be concluded which may allow in some extent, outdated literature. Accordingly, the results might not be original even considering that in Mozambique context a similar study was not carried out yet.

A further limitation of the study is that the conceptual model for algebraic thinking in geometrical understanding adapted from the literature may not be suitable for the context, in which the students evolved academically allowing some bias in interpretation of data.

The process of translation from Portuguese into English of most of the data collected might also bias the actual meaning of the students’ responses.

The sample included only five target students from an institution (Universidade Pedagógica). It means that the results of the study are not generalizable for Mozambique context in general or for Universidade Pedagógica in particular.

1.5 Implications for Curriculum and Instruction

The findings of this study have implications for the high school mathematics and for the Universidade Pedagógica curriculum and instructional practice. The implications for the curriculum describe the algebraic knowledge and skills necessary for successful transition to geometry courses. The implications for instruction describe how the algebraic thinking of students can be developed for improving their geometrical understanding.
1.5.1 Implications for curriculum

This study and others (Dindyal, 2003; Gascón, 2004) highlight the close connection between algebra and geometry, in my case and in Gascón’s case linked to the synthetic and analytic aspects of geometry, in high school and the first-year university geometry curriculum (at Universidade Pedagógica). In Mozambique, high school geometry is not only the study of Euclidean Geometry but also Analytic Geometry. At Universidade Pedagógica the same subjects appeared in the first-year curriculum. However, the contents are broadened and deepened. Accordingly, there need to be alternative ways of judging the thinking of the first-year students at UP in particular and of high school students in general, in geometry, which take into account the algebraic thinking required for geometrical understanding.

Similar to the recommendations of Dindyal’s study, it is noted that algebraic thinking is of paramount importance to engaging in the geometry class in the Mozambique context. The students should know not only how to manipulate symbols, equations, formulae, etc but also demonstrate the epistemological and contextual meanings behind those concepts. Most of my subjects showed proficiency in manipulating algebraic concepts (a partial understanding of epistemological meaning—method), although some of them lacked contextual and other aspects of epistemological meaning (validation) in their thinking. To foster students’ thinking in this regard, I recommend a geometry curriculum, which assesses those different aspects of meaning.

Patterning and generalisation aspects of algebraic thinking constituted a difficulty on the students’ side, meanwhile these cognitive processes showed to be very important for geometrical understanding. Most of the students faced difficulties in recognizing number patterns, which would allow them to see a geometric pattern. Besides, they were not aware what formulae were suitable for a general solution to certain typical problems of geometry and brought together problems that in geometrical form would not appear to be related at all (Kline, 1972). Such weaknesses in patterning and generalisation process can be addressed through appropriate curriculum.

1.5.2 Implications for instruction

This study shows that algebraic thinking is extremely important in the study of first-year geometry at Universidade Pedagógica and in high schools. Accordingly, instruction of
geometry should ensure that students possess the required ability to think algebraically in order to facilitate geometrical understanding.

As a possible strategy in geometry instruction, the student should possess algebraic thinking skills prior to attending a geometry class. In turn, the teacher should be able to explain the meanings of the different aspects of algebraic thinking whenever and wherever they appear along the geometry class. Furthermore, the teacher should assess those meanings in tasks, in tests, and in other means of assessment to ensure students’ understanding. In this regard, the recall and hinting processes in the geometry classrooms might be useful according to my experience during interviewing. These processes elicit the students’ knowledge and they might help in organizing and structuring their knowledge schemas (Lawson and Chinnapan, 2000).

The teaching of geometry should carefully focus on ascertaining that the various aspects of algebraic thinking investigated and omitted in this study are present, explained, and assessed whenever and wherever they occur.

The above recommendations deal with algebraic thinking as an aid in geometrical understanding. Meanwhile, this study also dealt with the two aspects of geometry mentioned by Stillwell (1998): synthetic and analytic modes of geometry. Sure algebraic thinking is present in both aspects of geometry no matter whether at a weak or strong form. However, still there is a big concern regarding the rupture between Euclidean Geometry and Analytic Geometry in the curricula worldwide (Gascón, 2004). Certainly, this rupture affects algebraic thinking into geometry as well as ultimately geometrical understanding. This study has underscored this phenomenon. However, a discussion and engagement with this rupture is beyond the scope of this study. Gascón proposed a didactical recommendation to meet this concern.

… las técnicas de la geometría analítica constituyen la respuesta a algunas de las limitaciones que presentan las técnicas sintéticas para resolver problemas genuinamente geométricos planteados sin utilizar coordenadas. Se trata de problemas de construcción o de determinación de figuras geométricas a partir de elementos (puntos, segmentos,...) que mantienen entre sí relaciones que podemos describirse y manipularse más eficazmente con las técnicas analíticas. También puede demostrarse, recíprocamente, que las técnicas
analíticas requieren en muchas ocasiones, de manera casi imprescindible, el uso previo de ciertas técnicas sintéticas que son las que sugieren el diseño de la estrategia que se llevará a cabo posteriormente con las técnicas analíticas. Se cierra así el círculo de la complementariedad entre ambos tipos de técnicas (p. 6). [... the strategies of Analytic Geometry constitute the response to some of the limitations of the synthetic strategies for solving purely geometric tasks designed without use of coordinates. They are tasks of construction and determination of geometric figures through (points, segments...), which possess relationships, which we can describe and handle more efficiently with analytic strategies. Likewise, we can prove, reciprocally, that the analytic strategies require often times, in an almost crucial way, the prior use of certain synthetic strategies, which suggest the design of the strategy to be carried out through analytic means. This is a cycle of complementary between the two strategies.]

Gascón points out a crucial recommendation for instruction of Euclidean and Analytic Geometry as a “cycle of complementary”. Actually, this is a motivational recommendation for geometry teachers as they work with their students. The implication for this recommendation requires necessarily designing appropriate geometric tasks, where on one hand we can show our students of the limitations of the synthetic strategies and the use of analytic strategies to solve them, and vice versa. Dindyal (2003) corroborates this point of view by stating that, “In general, students are more interested in studying what makes sense to them” (p. 203).

This study demonstrated the importance of algebraic thinking in geometry in an institution where algebra and geometry are taught together (Analytic Geometry syllabus, Chapter 2) and separately (Euclidean Geometry syllabus, Chapter 2). However, the results showed that even so students faced difficulties in effectively using algebraic thinking in geometrical contexts with understanding. This phenomenon may be due to “thematic autism” (rupture between Euclidean and Analytic Geometry) argued by Gáscon (2004).

6.6 Comments on the Theoretical Framework

This study used grounded theory to analyse the data and also adapted models from earlier studies such as Charbonneau (1996), Duval (1998) and Stilwell (1994, 1998) that explored
algebraic thinking and geometrical understanding. Furthermore, the study used framework of transfer and learning by Prawat (1989).

6.7 Recommendations for Future Research

This study investigated how first-year university students at Universidade Pedagógica in Maputo-Mozambique bring their knowledge and thinking of algebra in understanding and working with geometry. The conceptual model for algebraic thinking in geometrical understanding (Chapter 3, Fig 3.1) was drawn using the reviewed literature. My interest, therefore, was to find out the extent to which this model was relevant to the context of my study with students from Universidade Pedagógica. Prawat’s framework on transfer and learning appeared to be relevant. Future studies may refine or adopt a different model for algebraic thinking in geometrical understanding in order to suit their contexts.

At dispositional category I looked for the way the teaching and learning process was taking place in students themselves whether the orientation was that of mastery or/and of performance. I further researched whether these two types of disposition energized or hindered the use of strategies or skills on the tasks. Future studies may look for the way the teaching and learning process takes place in the classrooms to see which type(s) of the dispositional orientation(s) occur(s) in students.

This study was carried out with only five target students from one institution. Further research may use varied sample spread over more than one tertiary institution and high schools.

Earlier, I quoted Gascón about the rupture between Euclidean and Analytic Geometry, which certainly affects algebraic thinking into geometry. As a consequence, students’ geometrical understanding may be affected likewise. This study underscores this phenomenon. However, the solution of this rupture is beyond the scope of this study. Gascón proposed a didactical recommendation to meet this concern. There is a need for further research in this direction in order to corroborate Gascón’s recommendation: to reach “the cycle of complementary” of these two geometric modes.

6.8 Reflections on my Thesis Journey
In this section, I reflect on some of the important ideas I have learned in my research journey. I reflect on the dilemmas of doing research, and on what this research meant for my target students.

My research journey “was strewn with various dead-ends and the constant need to rethink basic assumptions” (Silverman, 2000, p. 26). Fortunately, I lectured Euclidean Geometry and Analytic Geometry at my University and found myself beginning in familiar territory. Nevertheless, soon, I realized that I had a long way to go to reach the peak of my research.

Firstly, I struggled to find a settled theoretical orientation. Accordingly, I did literature review and attended several conferences. At PME 28 Conference I met some researchers who recommended me to read Dindyal’s (2003) PhD thesis. This thesis dealt with algebraic thinking in geometry at high school level. This was a breakthrough for my work regarding theoretical orientation. My new-found enthusiasm was supported by my supervisor. In the meantime, the connections among cognitive processes are, in general, very complex to describe (Prawat, 1989). Hence, I was challenged by the need to be more precise in categorization during my data analysis. For instance, I found difficult to identify what fitted into intuition and what fitted as reasoning.

Secondly, other challenges were associated with practical contingencies. For example, the audio recordings were of poor quality due to the noise caused by rebuilding and extension of the infrastructures of Universidade Pedagógica (UP). The transcription process was a hard work as I had to rewind and forward the tape several times to capture the real situation. It is to notice that the only available and secure venue was the UP building to ensure the collaboration of the subjects. Once I changed the venue, consequently, half of the subjects missed the interviewing.

My story was plenty of setbacks. In this section I only mentioned some of them. Fortunately, my supervisor was a source of inspiration, enthusiasm, and dedication throughout my research journey. He instilled in me the zeal and direction needed through the demanding ploys of this thesis.

My advice to other researchers fully complies with Silverman’s twelve-point guide. I experienced most of these points. These points are as follows: 1) Begin in familiar territory; 2) find a settled theoretical orientation; 3) narrow down your topic; 4) do not try to reinvent
the wheel; 5) keep writing; 6) begin data analysis early; 7) use your supervisor; 8) use other resources and opportunities; 9) do not expect a steady learning curve; 10) keep a research diary, 11) earmark blocks of working time; and 12) do not reproach yourself (pp. 28, 29). Personally, I would like to add a point which was a source of hope and confidence: trust in God.

My target students stated that this research was a means to discover new strategies and ideas. Moreover, they asserted that they learnt a lot and it was an interesting experience. Some extracts of their transcripts underscore this position.

**S1:** Não, eu... bom. Fiquei muito satisfeito aqui, eu vi, descobri duas coisas. Primeiro, que aqui de facto isto se pudesse se dividir. Eu pus impossível, de facto, eu peguei dividir, mas esqueci-me que podia fazer subdivisões até o ponto de apanhar isto aqui. Portanto, achei muito interessante. (S1I:845-848)  
*Well, I was very glad to participate in this interviewing. I saw, discovered two things. First, it is possible to divide (the picture). I wrote it is impossible, I forgot that I could divide (the picture) in order to get this solution. Hence, I found very interesting.*

**S3:** ... aprendi, aprendi muita coisa. (S3I:661)  
*[...] I learnt, I learnt a lot of things.*

**S4:** Valeu muito para mim. Aprendi bastante. Eu creio, quem sabe daqui algum tempo o sr poderá ser o meu arguente no meu trabalho de licenciatura (em geometria). Espero que a sua investigação corra tudo bem. (S4I: 779, 778)  
*[It was worth it. I learnt a lot. I believe, who knows, in the future you will become my opponent for my research report (in geometry). I hope your research will succeed.]*

**S5:** Muito obrigado também. O que achei é que agente tem muito que aprender. De facto precisamos de mais noções básicas, mesmo básicas. Então agente precisa de ler conceitos primeiro e tentar interpretá-los. (S5I:508-510)  
*[Thanks a lot too! I found that we have a lot to learn (in geometry). In fact, we need to learn the basic concepts. We first need to read the concepts and then try to interpret these concepts.]*

**S7:** Há coisas que eu descobri. Gostei muito da pergunta 3. Porque a partir de volumes de figuras pude saber quantas vezes uma figura cabe noutra e depois construí-las a partir de volumes. Esta pergunta gostei muito. Também da pergunta 2 gostei porque tinha que dividir a fig duma maneira rápida (eficiente) e a partir de figs pequenas iguais obter as partes iguais procuradas. Foi uma experiencia enorme. (S7I:446-450)  
*[I discovered something. I liked*
Task 3 a lot because from the volumes of the solids I was able to know how many times a solid can be inserted into another solid and then, to draw the solids from the volumes. I liked Task 3 a lot. Also I liked Task 2 because I had to divide the picture rapidly, and from the small equal pictures to get the congruent parts sought. It was a great experience.

These assertions were very encouraging and motivating to me regarding my research. Reflections on their own written responses and some hintings from me were critical procedures to build a sound learning and collaborative environment between my target students and myself throughout the research.

### 6.9 Concluding Statement

This study allowed students to reflect on their geometrical thinking/understanding and learning. Some of them acknowledged and identified their weaknesses in geometrical understanding. Others proposed strategies to meet those weaknesses. Moreover, they attempted to explain the nature of the discipline of geometry regarding the interplay between synthetic and analytic aspects. This is succinctly captured in the following extracts which also appear in Section 5.4.5:

S4: Maybe to say... (Laughing). Geometry itself is difficult. It is a difficult subject. I do not know how to tackle this problem. I think... there is a... big mistake we make when we solve geometric tasks. We run to formulae before we analyse it. This mistake persists in me. We are short of the skill of imagination before using any formula especially in Analytic Geometry.

S5: I think that our concern, when we solve a geometric task, is to rush to formulae. Rushing straightaway to formulae without visualizing the picture is our concern. I think the most important thing is to begin with a sketch. It is to begin with a sketch. My concern now is to practice sketches.

S7: (Silence). Well, there is a relationship. For example, there are some topics such as axioms, properties or rules we learnt in Euclidean Geometry (EG) and we have been applying in Analytic Geometry (AG). For instance, we learnt lines. There is an axiom in EG, which says through two points it passes a line. In AG we used this axiom. We determined the equation of a line passing through two points. Moreover, so many other axioms we learnt in EG, we have been using them in AG. There is a relationship.
These metacognitive skills are very important towards effective learning and Ridley et al (1992) explicate:

Metacognitive skills include taking conscious control of learning, planning and selecting strategies, monitoring the progress of learning, correcting errors, analyzing the effectiveness of learning strategies, and changing learning behaviors and strategies when necessary. (Ridley, D.S. et al, 1992).

Having used metacognitive skills as a vehicle for learning any subject and especially geometry, I believed this could make a difference in students’ confidence and autonomous learning. Autonomy leads to ownership as students realize they can pursue their own intellectual needs and discover a world of information at their fingertips. This made the thesis journey a significant contribution.