CHAPTER 5

DATA ANALYSIS AND INTERPRETATION

5.1 Introduction

The data collected from this research were of the following forms: students’ written responses to geometry tasks, interview transcripts, student concept maps, field and observation notes, and curriculum documents. Qualitative methods were used in analysing this data. Selected cases were analysed in detail to examine the extent to which algebraic thinking (the use of symbols and algebraic relations, the use of different forms of representations and the use of generalisations from patterns) can be a powerful tool for geometric thinking. A method, which is called “data reduction”, was used. With respect to qualitative research, Savenye and Robinson (2004) have pointed out that “the qualitative researcher engages in speculation while looking for meaning in data; this speculation led the researcher to make new observations, conduct new interviews, and looked more deeply for new patterns in this ‘recursive’ process” (p. 1059).

Miles and Huberman (1994) assert that “coding is analysis” (p. 56). In this study I used descriptive and pattern codes. Descriptive codes denote different cognitive processes underlying algebraic and geometric thinking as well as their relationships. Pattern codes denote emergent themes, configurations, or explanations (ibid., p.69). And using to what Glaser and Strauss (1967) (as quoted by Jaworski, 1994) call the “constant comparative method” I identified particular incidents and coded each incident into as many categories as possible, as categories emerged or as data emerged that fitted an existing category.

The structure of the analysis of data to be followed comprises three different phases: 1) the Pilot Study, 2) the main study – Euclidean Geometry course, and 3) the main study -Analytic Geometry course.

5.2 The Pilot Study

5.2.1 Analysis of the Pilot Study Data
I clustered the student’s written responses according to their academic and professional background. The background of the seven selected students is presented in the table below:

Table 5.1: Academic and professional background of seven cases studies (Pilot Study)

<table>
<thead>
<tr>
<th>Number of students/Category</th>
<th>Academic background</th>
<th>Professional background</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 / (I)</td>
<td>Grade 12</td>
<td>None</td>
</tr>
<tr>
<td>1 / (II)</td>
<td>Grade 12</td>
<td>Grade 7+ 2 years lower primary school teacher training course</td>
</tr>
<tr>
<td>1 / (III)</td>
<td>Grade 12</td>
<td>Grade 10+2 years upper primary school teacher training course</td>
</tr>
<tr>
<td>2 / (V)</td>
<td>Grade 10</td>
<td>Grade 10+2 years upper primary school teacher training course</td>
</tr>
<tr>
<td>1 / (VI)</td>
<td>Grade 10</td>
<td>Grade 10+3 years technical college teacher training course</td>
</tr>
</tbody>
</table>

The results showed that even though the students had been attending Analytic Geometry, they did not access that knowledge to solve these tasks. Task 1 could be solved using knowledge from either Euclidean or Analytic Geometry.

The results also showed that three of the students (one student each from category I, II, and VI) accessed algebraic and geometric knowledge but they used it inappropriately for Task 1. Here is an example of the Student I’s (student of category I) solution of Task 1 (Fig 5.1). He correctly sketched the geometric figure and correctly conjectured about the segments DE and BC. He wrote the segments DE and BC are parallel to each other. However, he used this conjecture as a given instead of a conclusion of the theorem. During the process of the demonstration of this theorem he correctly wrote the (algebraic) relations $AB = 2AD$ and $AC = 2AE$, nevertheless he did not use these relations for the solution of the task. He used the congruency of angles to show the conjecture, however, the justification of the congruency was incorrect. For example he wrote $<D \equiv <B$ because they were opposite angles.
Besides, two of them (one student each from category II and VI) after identification of the given (premise) and the conclusion, i.e. the relation $p \Rightarrow q$, they used a property of the conclusion to prove the conclusion. Here we refer to the example of Student II’s solution as shown in Fig 5.2.
Student II correctly conjectured $DE \parallel BC$. He also correctly wrote the (algebraic) relations $DC = \frac{1}{2} AC$ and $AE = \frac{1}{2} AB$. However, he did not use them for proving. Besides, he used the property of parallelism to show the congruency of certain angles that led him nowhere.

The other Student I and one of the students of category V accessed some algebraic and geometric concepts, but they could not use them to solve the tasks. Student V’s solution to Task 1 serves as an example to show this:
Figure 5.3: Student of Category V - Task 1

Although Student V correctly sketched the figure and identified the conjecture, he could not carry on the demonstration of the theorem.

Student III and one of the students of category V accessed all key algebraic and geometric concepts and they used them successfully for Task 1. An example of the solution of Student III is given below:
Student III conjectured $BC = 2DE$. He used an auxiliary theorem that if a line bisects two sides of a triangle, then it is parallel to the third side to show that the triangles ADE and ABC are similar to each other. Then, using the criterion of proportion for similar triangles he came up with the conclusion.

Although all students were aware that they had to find a pattern and then generalize with a formula, they faced difficulties in identifying patterns, making generalisations, and reasoning.
processes in Task 2. A solution of one of the students is presented to illustrate this point as follows:

Figure 5.5: Solution 1 of Task 2

This student faced difficulties in the process of deducing a formula. It seems that he started off with a triangle, some squares, some pentagons, some hexagons to see whether he could get a pattern and finally a formula. It seems, however, that he was unsuccessful, and he tried to draw a 70-sided polygon to get the number of its diagonals from the pictorial representation. Meanwhile, it seems that he abandoned this approach as we can see that he tried to erase the solution. Further on (Fig 5.6) we can see how he continued his solution using a similar approach but with some systematization. However, he also was unsuccessful. It seems that one of the difficulties the student faced was in the stating the correct definitions of a diagonal and of a polygon. He wrote that a 2-sided “polygon” had 2 diagonals; a triangle possessed 3 diagonals and so forth.
All results presented above are summarized in Table 5.2

Table 5.2: Results of the pilot test in accessing and using knowledge of a sample

<table>
<thead>
<tr>
<th>Result Group</th>
<th>Accessed all key concepts and used them appropriately for solving Task 1</th>
<th>Accessed most key concepts and did not use them appropriately for solving Task 1</th>
<th>Accessed some concepts and did not use them for solving Tasks 1 and 2</th>
<th>Accessed some concepts and did not use them for solving Task 2</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Group</th>
<th>None</th>
<th>1</th>
<th>1</th>
<th>2</th>
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<tr>
<td>Group II</td>
<td>None</td>
<td>1</td>
<td>None</td>
<td>1</td>
</tr>
<tr>
<td>Group III</td>
<td>1</td>
<td>None</td>
<td>None</td>
<td>1</td>
</tr>
<tr>
<td>Group V</td>
<td>1</td>
<td>None</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Group VI</td>
<td>None</td>
<td>1</td>
<td>None</td>
<td>1</td>
</tr>
<tr>
<td>Total</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>7</td>
</tr>
</tbody>
</table>

5.2.2 Discussion of the Pilot Study Data

The students came from different academic and professional backgrounds. This fact partially seems to account for different categories of accessing and use of algebraic and geometric knowledge in the tasks amongst the students. Although these students passed Euclidean Geometry and were attending Analytic Geometry, they showed difficulties in accessing and using knowledge in both domains when solving the tasks of the pilot test. These difficulties seem to be caused by lack of development of some indicators of algebraic and geometric thinking, namely, making relations, identifying patterns, making generalisations and reason. Therefore, tasks were designed in collaboration with the lecturers of these disciplines mostly assessing these indicators of algebraic and geometric thinking for the main study.

5.2.3 Emerging Issues of the Pilot Study Data

In this section, I describe some issues emerging from the analysis of the students’ written responses on the pilot test:

1- Students entering Universidade Pedagógica (Pedagogical University) enrolled in the course of mathematics teacher training came from different academic and professional backgrounds. Accordingly, they experienced different geometry teaching and learning contexts.
**Insight 1:** The differing contexts of teaching and learning develop students academically differently.

2- Although UP may create a homogeneous milieu for teaching and learning of geometry, it seems that students still face difficulties in accessing and using algebraic and geometric knowledge at least in the first year.

**Insight 2:** The University mathematics education setting, however suitably packaged, may come too late in the development of students’ capabilities to use algebraic thinking in geometry.

3- It seems that students needed to develop algebraic and geometric thinking during the course through different pedagogical activities (e.g. construct appropriate tasks where the indicators of algebraic and geometric thinking are assessed).

**Insight 3:** In order to develop algebraic and geometric thinking during a course, a range (rather than one form) of powerful activities are needed.

### 5.2.4 Conclusion of the Pilot Study Data

The Pilot Study enabled me to rearrange some aspects concerned with appropriateness of the tasks for exploring the interplay between algebraic and geometric thinking for the next stages (the Main Study – Euclidean Geometry Course and the Main Study – Analytic Geometry Course) as well as to find out the first-year students’ academic and professional background at the beginning of their degree at UP. From the students’ results I concluded that the differing contexts of teaching and learning developed students academically differently before entering UP. Although UP has been making efforts to create a homogeneous milieu for teaching and learning of geometry, it seemed that students still faced difficulties in accessing and using algebraic and geometric knowledge at least in the first year. These results allowed me to be open to other factors, besides the conceptual, such as contextual and structural issues which may influence the students’ understanding and their use of algebraic knowledge and thinking for solving geometric tasks.

### 5.3 The Main Study – Euclidean Geometry Course
The first component of the main study comprehends the Euclidean Geometry course. The data analysis of this component consists of Diagnostic test, Interviewing Phase 1, and Elaboration and Concept Mapping Task data. The overall analysis structure adopted for each data source is as follows: 1- Analysis, 2- Discussion, 3- Emerging Issues, and 4- Conclusion. Moreover there is a section on “Research question: Discussion” where a holistic view is presented for the entire Euclidean Geometry course data analysis and interpretation regarding the research questions of the study.

5.3.1 Analysis of the Diagnostic Test Data

In order to develop the Diagnostic test and later on to analyse the students’ written responses to this test a literature review was accomplished to clarify some essential indicators or elements that portray algebraic and geometric thinking. In this study, three indicators of algebraic thinking were considered on the one hand, namely, use of symbols and algebraic relations (use of variables, use of formulae, and use of other symbols, such as abbreviations and acronyms); use of representations (tables, graphs, formulae, equations, arrays, identities, and functional relations); and use of patterns and generalisations (Charbonneau, 1996, Dindyal, 2003, and Wheeler, 1996). On the other hand, geometric thinking was portrayed by three main elements: Visualization processes which is characterized as “the visual representation of a geometrical statement” (a new way of looking at the situation in order to suggest an inductive generalisation) or “the heuristic exploration of a complex geometrical situation” (proof and verification in one process and an explanation of why the generalisation holds, however, not regarded as final and strict but as provisional and plausible only) (Duval, 1998, Hershkowitz, 1998, and Polya, 1973); construction processes only dependent on connections between mathematical properties and the technical constrains of the tools used; and reasoning in relation to discursive processes for extension of knowledge, for proof, or for explanation (written or verbally) which is exclusively dependent on the corpus of propositions – definitions, axioms, and theorems (Duval, 1998).

The students’ written responses to this Diagnostic test (algebraic and geometric knowledge base, connectedness and strategies) were analysed in light of the conceptual model for algebraic thinking in geometrical understanding drawn from the ideas of Stillwell (1998), Charbonneau (1996) and Duval (1998) (see the description of the model in Chapter 3, Fig 3.1) and the framework on (knowledge) transfer and learning from Prawat (1989) to see whether students accessed and utilized their intellectual resources (algebraic and geometric
knowledge base, strategies and disposition) in problem situations where those resources might be relevant. Disposition category was not to be noticed in the students’ written responses. That is why, it was not considered in the case of the Diagnostic test data analysis.

The Diagnostic test was constructed in collaboration with the lecturer of Euclidean Geometry with two main purposes: for the lecturer it constituted a diagnostic test and for me a pre-test. The lecturer wanted to assess the newcomers’ previous geometric knowledge and I intended to use the same tasks to find out how the students bring their algebraic thinking and knowledge in working with geometric tasks. Collaboration between the researcher and the teacher is highly fostered by the qualitative research approach adopted in this study Woods (1992, p. 389) quoting Pollard (1984); Woods (1985, 1986, 1989); Hustler, Cassidy, and Cuff (1986); and Woods and Pollard (1988).

From 32 test scripts, out of 16 points (maximum score) one student scored 12 points, two scored 9 points, three students scored 8 points, six students scored 7 points, and the remaining twenty scored 6 points and below. This assessment was qualitatively carried out, that is, I went through the tasks and analysed the students’ responses according to the following procedure for each task:

- 4 points: good connections among algebraic and geometric concepts
- 3 points: reasonable connections among algebraic and geometric concepts
- 2 points: weak connections among algebraic and geometric concepts
- 1 point: blank response or wrong connections among algebraic and geometric concepts

These categories allowed me to rank the 32 students as either low achievers (LA) or high achievers (HA) in this test in order to select 8 for my target sample (Appendix 6).

By good connections I meant the student can use most of algebraic concepts correctly to solve the task. For example, the solution below shows a task scored with 4 points (Fig 5.7):

**Task 1: What is the Pythagorean Theorem? Please, demonstrate this theorem. (Use a right triangle)**
Figure 5.7: Student’s solution to Task 1

This student used trigonometry relations $\sin \alpha$, $\cos \alpha$ and $\sin^2 \alpha + \cos^2 \alpha = 1$ to demonstrate Pythagorean Theorem. Although the identity $\sin^2 \alpha + \cos^2 \alpha = 1$ is firstly yielded using Pythagorean Theorem we can notice that the student correctly uses these algebraic relations and transformations to get the Pythagorean formula $b^2 = c^2 + a^2$. For this reason I awarded 4 points for this task.

Another example showing a task where I scored 3 points is presented in Fig 5.8 below.

Task 3: Observe cube [ABCDEFGH] below. Can you fit any additional pyramid(s) congruent to [ABCDH] into the cube? If ‘yes’, draw it (them).

3. Observe o cubo [ABCDEFGH] abaixo.

Caberá no cubo mais alguma pirâmide congruente a [ABCDH]? Se sim, apresente-a(s).
[At most three congruent pyramids fit in the cube with 27 cm$^3$ as the volume of a pyramid is 9 cm$^3$ and it has the same altitude as the altitude of the cube (the translation of the student’s last sentence)].

This solution brings together two parts an algebraic and a geometric part. In the algebraic part we can see some algebraic formulae used for concrete values by the student. The idea was to see whether the students generalized for any cube not for a particular one as this student did. Although the student got a correct conclusion he could not draw the three pyramids in the cube.

Those who made weak connections (2 points) used some of basic algebraic concepts correctly for example an acronym for a pyramid [EFGHC], nevertheless the key algebraic concepts were not accessed to solve the task (Fig 5.9).
1 point was assigned to students who had blank responses or had simply written “I forgot” or “Impossible”. I assigned 1 instead of zero points to this category because I believed that the student had something in mind even though he had put a blank response. I confirmed this fact in Interviewing Phase 1.

A student’s answer to Task 1 was “I forgot it” (Fig 5.10). In Task 2 a student wrote “It is impossible” (Fig 5.11). (It is to notice that the figure in this solution was correctly divided in four congruent parts. This division happened during Interviewing Phase 1).

Task 2: Consider the figure below. Divide the figure in four congruent parts.
It is the aim here to analyse in-depth the eight target students Diagnostic test responses to portray their patterns regarding algebraic and geometric thinking. I labeled them as Student 1, Student 2 and so forth. The students 3, 7 and 8 were the HA and 1, 2, 4, 5, and 6 the LA according to my ranking for the Diagnostic test (see Table 5.3). This order happened during the process of data collection process. I kept this order to facilitate the analysis.

Table 5.3: Target students ranking for the Diagnostic test

<table>
<thead>
<tr>
<th>Student nº/Category/Item</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Total Points</th>
<th>Observation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1(II)</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>7</td>
<td>LA</td>
</tr>
<tr>
<td>2(IV)</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>7</td>
<td>LA</td>
</tr>
<tr>
<td>3(I)</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>8</td>
<td>HA</td>
</tr>
<tr>
<td>4(I)</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>7</td>
<td>LA</td>
</tr>
<tr>
<td>5(IV)</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>7</td>
<td>LA</td>
</tr>
<tr>
<td>6(III)</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>6</td>
<td>LA</td>
</tr>
<tr>
<td>7(I)</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>9</td>
<td>HA</td>
</tr>
<tr>
<td>8(VII)</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>8</td>
<td>HA</td>
</tr>
</tbody>
</table>

5.3.1.1 Analysis of Task 1

Task 1 was as follows:
What is the Pythagorean Theorem? Please, prove this theorem.
(Use the right triangle).

The students might have come across the Pythagorean Theorem in their earlier work in mathematics according to the secondary syllabuses. The aim here was to find out how the students recalled algebraic and geometric knowledge embedded in this important theorem and how they proved it. Student 1 wrote down “I forgot” for this task (Fig 5.10). Student 2 did not give a verbal form of the theorem and presented a wrong algebraic formula of it. Besides, he gave an incomplete pictorial representation (he just drew a right triangle and its altitude). Accordingly he could not prove the theorem (Fig 5.12).

Student 3 gave a deficient verbal form of Pythagorean Theorem (he did not use the concept of length) as well as an incomplete pictorial representation; however, he presented a correct algebraic representation of it. Instead of proving the theorem he used it to solve an example he suggested (Fig 5.13).
[In a right triangle the sum of the squares of the legs is equal to the square of the hypotenuse (the translation of the student’s sentence above)].

Student 4 gave a correct algebraic representation and an incomplete pictorial representation as he did not sketch the congruent triangles in it. Moreover, he wrote down a wrong verbal form of the theorem (he used the concept of proportion). He presented a key idea to prove the theorem, i.e. he was aware that the area of the square of the hypotenuse side is equal to the sum of the areas of the squares of the other two sides which coincide with the legs of a right triangle (Fig 5.14).
Figure 5.14: Student 4’s solution to Task 1

[The square of the hypotenuse is directly proportional to the sum of the squares of the legs (the translation of the student’s last sentence)].

Student 5 wrote down a deficient verbal representation of the theorem as well as an incomplete pictorial representation. He presented a correct algebraic representation of it. He verified the theorem using a Pythagorean triple 3-4-5 (Fig 5.15).

Figure 5.15: Student 5’s solution to Task 1

[In a right triangle the square of the hypotenuse is equal the sum the squares of the legs (the translation of the student’s sentence above)].
Student 6 presented a deficient verbal form, an incomplete pictorial representation as well as an incorrect algebraic representation of the theorem. In Fig 5.16, we can see a square in the formula. She completed it during Interviewing Phase 1 after hinting. She tried to prove the theorem but was unsuccessful.

Figure 5.16: Student 6’s solution to Task 1

[The sum of the square of the hypotenuse is equal the sum of the squares of the legs (the translation of the student’s sentence above)].

Students 7 and 8 succeeded in twofold representation: algebraic and pictorial. Nevertheless, they presented a deficient verbal representation of the theorem. Student 7 used the trigonometry identity $\sin^2 \alpha + \cos^2 \alpha = 1$ to prove the theorem. However, this identity is yielded using Pythagorean Theorem (Fig 5.7). Student 8 presented a proof (Fig 5.17), but he did not justify why the quadrilateral EHGF was a square and why the four right triangles were congruent to each other.
5.3.1.2 Discussion of Task 1

Table 5.4 summarizes the result of the analysis regarding the students’ solution to Task 1.

Table 5.4: The analysis summary of the students’ solution to Task 1

<table>
<thead>
<tr>
<th>Student /Cognitive Processes</th>
<th>VR</th>
<th>AR</th>
<th>PR</th>
<th>Proof</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>None</td>
<td>None</td>
<td>None</td>
<td>None</td>
</tr>
<tr>
<td>2</td>
<td>None</td>
<td>Not correct</td>
<td>Incomplete</td>
<td>Failed</td>
</tr>
<tr>
<td>3</td>
<td>Incomplete</td>
<td>Correct</td>
<td>Incomplete</td>
<td>Failed</td>
</tr>
<tr>
<td>4</td>
<td>Not correct</td>
<td>Correct</td>
<td>Incomplete</td>
<td>Failed</td>
</tr>
<tr>
<td>5</td>
<td>Not correct</td>
<td>Correct</td>
<td>Incomplete</td>
<td>Verification (Pythagorean triple)</td>
</tr>
<tr>
<td>6</td>
<td>Not correct</td>
<td>Not correct</td>
<td>Incomplete</td>
<td>Failed</td>
</tr>
<tr>
<td>7</td>
<td>Incomplete</td>
<td>Correct</td>
<td>Correct</td>
<td>Reasonable</td>
</tr>
<tr>
<td>8</td>
<td>Incomplete</td>
<td>Correct</td>
<td>Correct</td>
<td>Reasonable</td>
</tr>
</tbody>
</table>

The codes and the categories used in this table are as follows:

“VR”: Verbal Representation (the theorem text)
“AR”: Algebraic Representation (algebraic formula or relation)
“PR”: Pictorial Representation (the sketch)
“None”: Blank or “I forgot” response
“Not correct”: wrong answer
“Correct”: All key concepts present
“Incomplete”: Some key concepts missed
“Failed”: Unsuccessful proof
“Verification”: Verification of Pythagorean formula using Pythagorean triple
“Reasonable”: Acceptable proof with some pitfalls
“Proof”: Reasoning

Table 5.4 shows that Student 7 and Student 8 presented correct algebraic and pictorial representation of the theorem and accessed key algebraic and geometric concepts (e.g. trigonometry identities or relations, area formulae of a square and triangles, congruency of triangles, etc) and used them to solve the Task 1 reasonably correctly. Student 4 presented a correct algebraic and an incomplete pictorial representation of the theorem. However, he did not access some other key algebraic and geometric concepts. Consequently, he failed in proving the theorem. It seems that the verbal representation did not influence in the success of the solution of this task as all students missed to give a correct verbal representation of the theorem. It also seems that a correct algebraic representation alone did not trigger these students to access key algebraic and geometric concepts towards a successful solution of this task.

It can be deduced that all students articulated their thoughts about the Pythagorean Theorem through their written responses. Even Student 1 was also aware of this theorem, that is why, he wrote down “I forgot it”. It means that once he encountered this theorem but in that moment he could not recall it. In terms of organization of knowledge, Student 7 and 8 seemed to possess key algebraic and geometric concepts connected to each other regarding this theorem. As result they presented a reasonable proof of it. Student 4 accessed some key algebraic and geometric concepts, although these concepts were not sufficient for conducting a successful proof of the theorem. The remaining students seemed to face difficulties in accessing key algebraic and geometric concepts related to this theorem.

5.3.1.3 Emerging issues of the analysis of Task 1

In this section, I describe major issues from the analysis of Task 1.
1- The verbal representation of the theorem seems to be problematic for all students. They simply skipped it or they were not able to correctly write it down (Anton and Rorres, 1991).

*Insight 1:* It seems the verbal representation is not accessible to students. Without this verbal representation, effective representation of proofs is likely to be hindered.

2- Most students correctly wrote down the algebraic representation of the theorem, although it seems not to be sufficient to succeed in proving it (Anderson, 1990 as quoted by Dindyal, 2003).

*Insight 2:* It seems the algebraic representation is high in activation (the momentary availability of a knowledge component) but low in strength (the durability of the knowledge component over long term).

3- It seems that the students, who produced algebraic and pictorial representations, accessed and used key algebraic and geometric concepts successfully to solve the task (Dreyfus, 1991 as quoted by Dindyal, 2003).

*Insight 3:* Effective learning is a matter of integrating representations and switching flexibly between them. That is, students who access multiple representations of the same mathematical object, make available at each moment, and are able to conveniently choose one or the other show conceptual understanding (effective learning).

### 5.3.1.4 Analysis of Task 2

Task 2 was as follows:

Consider the figure below. Divide the figure in four congruent parts.

![Figure](image-url)
Student 1 wrote down “Impossible” for this task (Fig 5.11). Student 2 and 6 presented a division of the figure into three congruent squares after completing the missing fourth square (Fig 5.18).

Additionally, Student 6 divided the figure into 6 congruent right triangles (Fig 5.18). Student 3 presented the figure divided into four parts with the same area, three of which were congruent (Fig 5.19).

Students 4, 5 and 7 presented the figure divided into several non congruent parts. In Fig 5.20 illustrates an example of this kind of solution.
Student 8 presented a blank response to this task.

5.3.1.5 Discussion of Task 2

The analysis summary of the eight student’s written responses to Task 2 is presented in Table 5.5.

Table 5.5: The analysis summary of the students’ solution to Task 2

<table>
<thead>
<tr>
<th>Student/Approach</th>
<th>Synthetic(Intuition)</th>
<th>Analytic(Reasoning)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>None</td>
<td>None</td>
</tr>
<tr>
<td>2</td>
<td>Incomplete</td>
<td>None</td>
</tr>
<tr>
<td>3</td>
<td>Incomplete</td>
<td>None</td>
</tr>
<tr>
<td>4</td>
<td>Not correct</td>
<td>None</td>
</tr>
<tr>
<td>5</td>
<td>Not correct</td>
<td>None</td>
</tr>
<tr>
<td>6</td>
<td>Incomplete</td>
<td>None</td>
</tr>
<tr>
<td>7</td>
<td>Not correct</td>
<td>None</td>
</tr>
<tr>
<td>8</td>
<td>None</td>
<td>None</td>
</tr>
</tbody>
</table>

Most of the students used a synthetic approach, i.e. they used visualization and construction processes to solve the task. None of them succeeded in this task. Here the categories “None”, “Incomplete”, “Not correct” have the same meaning as in Table 5.4.

5.3.1.6 Emerging issues of the analysis of Task 2

1- Most of the students used a synthetic approach (visualization and construction processes).
2- None of them was able to solve the task correctly though one student (Student 3) almost got the correct answer.

3- It seems that for these kinds of tasks these students were compelled to use their intuition (synthetic approach) rather than reasoning (analytic approach) (Stillwell, 1998).

*Insight: It seems easier for these students to use their intuition rather than their reasoning in the tasks where they are qualitatively presented (no quantities).*

### 5.1.3.7 Analysis of Task 3

Task 3 was as follows:

Observe cube [ABCDEFGH] below. Can you fit any additional pyramid(s) congruent to [ABCDH] into the cube? If ‘yes’, draw it (them).

![Diagram of a cube with added pyramids](image)

Only Student 1 used key algebraic concepts to solve it. He compared the volume formulae of a cube (he wrote $V = A_b \cdot h$, $A_b$ - base area and $h$ - altitude) and of a pyramid (he wrote $V = \frac{A_b \cdot h}{3}$) to show that at most three congruent pyramids fitted into the cube. He showed it with a concrete cube of side 3 cm. He numerically concluded that the volume of a cube was three times as big as the volume of a pyramid with the same base area and altitude. He wrote down pyramids [EFGHB] and [BCGFE]. He only constructed pyramid [EFGHB] that leans on the original pyramid [ABCDH] by the segment line BH (Fig 5.21). However, pyramids [EFGHB] and [BCGFE] intersect each other, though they do not intersect the original pyramid. The three pyramids do not intersect each other and fit into the cube are [ABCDH] (the original), [ABFEH] and [BCGFH].
Figure 5.21: Student 1’s solution to Task 3

[The student wrote: At most three congruent pyramids fit in the cube with 27 cm$^3$ as the volume of a pyramid is 9 cm$^3$ and it has the same altitude as the altitude of the cube.]

The solution of Student 2 is presented as follows:
3. Observe o cubo [ABCDEFGH] abaixo.

![Diagram of cube]

Caberá no cubo mais alguma pirâmide congruente a [ABCDH]? Se sim, apresente-a(s).

Fig 5.22: Student 2’s solution to Task 3

[The student wrote: Yes, there is another (pyramid) which is [EFGHB]. Then, ABCDH=EFGHB.]

Student 2 concluded that only two pyramids [EFGHB] and the original fitted into the cube. He also constructed pyramid [EFGHB].

In turn Student 3 similarly solved as Student 2 writing down that only two pyramids fitted into the cube, namely the original and pyramid [EFGHB]. He also constructed it (Fig 5.23).

![Diagram of cube]

3.

Figure 5.23: Student 3’s solution to Task 3

[The student wrote: Yes, there is another congruent pyramid (to the original) that fits into the cube which is [EFGHB].]
Student 4 only wrote down that more than three congruent pyramids fitted into the cube beside the original that is a total of four pyramids. He only wrote down pyramids \([ABEFC]\), \([ABCDE]\) and \([EFGHB]\) and he did not construct any pyramid. However, it seems that he possessed a mental picture of a pyramid because he gave an algebraic representation of a pyramid. These pyramids intersect each other. His solution is as follows:

\[\text{Figure 5.24: Student 4\textapos;s solution to Task 3}\]

\[\text{[The student wrote: Yes, beside the original pyramid we can get three more pyramids, namely \([ABEFC]\), \([ABCDE]\) and \([EFGHB]\).]}\]

Student 5 wrote down that seven more congruent pyramids to the original fitted into the cube, that is they are eight pyramids in total. He presented the following pyramids: \([EFGHB]\), \([HGCDA]\), \([AEHDC]\), \([ABFEG]\), \([ADEHF]\), \([BCFGD]\) and \([DCGHE]\). These pyramids intersect each other. He justified his response saying that each face of a cube could be considered as a base of a pyramid. He did not construct any pyramid, a response that is similar to what Student 4 did. See his response below:

\[\text{Figure 5.25: Student 5\textapos;s solution to Task 3}\]

\[\text{[The student wrote: Yes, (there are some congruent pyramids) that fit into the cube, as each face of it can be considered as a base of a pyramid.]}\]
Student 6 solved this task as follows:

She constructed three more congruent pyramids to the original, namely [EFGHC], [ABFEG] and [DCGHA]. That is she seemed to conclude that four pyramids including the original fitted into the cube. Pyramid [EFGHC] intersects the others.

Student 7 asserted that some pyramids congruent to the original could fit into the cube. He constructed three pyramids. According to his solution it seems he concluded that at least four pyramids might fit into the cube, as he indicated pyramids [ADHEC], [DCGHA] and [ABFEG]. At least pyramid [ADHEC] intersects pyramid [DCGHA] and with the original pyramid (Fig 5.27).

[The student wrote: Yes, some congruent pyramids (to the original) can fit into the cube, such as:]
Student 8 wrote down three congruent pyramids to the original fitted into the cube, namely [FGHEA], [EFBAG] and [CDHGB]. It seems that he assumed four pyramids fitted into the cube, namely the three pyramids he suggested and the original. At least Pyramid [FGHEA] intersects pyramid [EFBAG]. He did not construct any pyramid. The same happened with Students 4 and 5 (Fig 5.28).

![Figure 5.28: Student 8’s solution to Task 3](image)

[The student wrote: Yes, [FGHEA], [EFBAG] and [CDHGB].]

### 5.3.1.8 Discussion of Task 3

Table 5.6 shows how I categorized the students’ written responses related to Task 3. The codes were produced according to the aim of the study, namely to see whether the students appropriately utilized algebraic thinking and knowledge in understanding and working with geometry.

Table 5.6: The analysis summary of the students’ solution to Task 3

<table>
<thead>
<tr>
<th>Student</th>
<th>Algebraic thinking</th>
<th>Geometric thinking</th>
<th>Strategy</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>S</td>
<td>F</td>
<td>R</td>
</tr>
<tr>
<td>1</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>2</td>
<td>+</td>
<td>_</td>
<td>_</td>
</tr>
<tr>
<td>3</td>
<td>+</td>
<td>_</td>
<td>_</td>
</tr>
<tr>
<td>4</td>
<td>+</td>
<td>_</td>
<td>_</td>
</tr>
<tr>
<td>5</td>
<td>+</td>
<td>_</td>
<td>_</td>
</tr>
<tr>
<td>6</td>
<td>+</td>
<td>_</td>
<td>_</td>
</tr>
<tr>
<td>7</td>
<td>+</td>
<td>_</td>
<td>_</td>
</tr>
<tr>
<td>8</td>
<td>+</td>
<td>_</td>
<td>_</td>
</tr>
</tbody>
</table>

The explanation of the codes is as follows:
We can see in this task that all students accessed algebraic and geometric thinking. However, only Student 1 accessed key algebraic concepts to partially solve this task. He used the volume formula of a cube and a pyramid, and the relationships between these two formulae. He realized that at most three congruent pyramids including the original fitted into the cube, as the volume of the given pyramid is one third of the volume of the cube with the same altitude and base area, though it did not help the student to visualize and construct the three congruent pyramids in the cube. The other students only used some algebraic symbols to represent pyramids, congruency and so on. This algebraic thinking seemed not sufficient to prompt connections with geometric thinking and solve the problem completely. However, it seemed that algebraic thinking partially aided geometric thinking to get more insight into the task. All students were aware of the key geometric concept “pyramid”. We can infer it through the algebraic and pictorial representation of a pyramid they provided.

From the table above we can conclude that all students but one used an empirical approach (as a means of development of intuition) that is according to their responses it seemed that they were guessing how many pyramids could fit into the cube through trial and errors using visualization and construction processes- the inference of a general law from particular instances (The Oxford English Reference Dictionary, 1995). In turn, Student 1 used a deductive approach (a means of discovery) - the inferring of particular instances from a general law- (The Oxford English Reference Dictionary, 1995). He used the volume formulae
and relations to conclude that at most three congruent pyramids fitted into the cube. Both approaches used separately it seems not to help students to succeed in geometric problem solving environment. Schoenfeld (1986) observed that unless students learn to take advantage of both approaches to geometry and learn to profit from the interaction of those two approaches; students will not reap the benefits of their knowledge. In turn, Hershkowitz (1998) corroborates this view explaining that reasoning takes place when by experimentation [e.g. construction by ruler and compass or geometrical software] and inductive generalisation [by visualization processes], one extends her geometrical knowledge about shapes and relations and extends her “vocabulary” of legitimate ways of reasoning. Deductive reasoning [dependent exclusively on the corpus of propositions- definitions, axioms, and theorems] then, becomes a vehicle for understanding and explaining why and inductively discovered conjecture might hold.

5.3.1.9 Emerging issues of the analysis of Task 3

1- The use of the different properties of algebraic thinking (within analytic approach) in connection seems to aid students to partially understand some ideas or concepts in geometry (see Student 1). On the contrary, using them separately seems to hinder understanding in geometry.
2- Using the properties of geometric thinking (within synthetic approach) separately leads to weak understanding of geometric ideas or concepts.
3- It seems that the interaction between deductive and empirical approaches (or between analytic and synthetic approaches) to geometry (unfortunately not to be noticed in these subjects) helps students to succeed in geometric problem solving environment (Schoenfeld, 1986).

Insight: Profiting from the interaction between deductive and empirical approaches to geometry students may reap the benefits of their knowledge.

5.3.1.10 Analysis of Task 4

Task 4 was as follows:
A quadrilateral possesses two diagonals, a pentagon five diagonals, and a hexagon possesses... diagonals. Determine an algebraic expression for the total number of diagonals of a polygon with n sides.

In this task the students have to provide an algebraic generalisation. None of the eight students provided an algebraic expression to determine the total number of diagonals of a polygon. Only Student 2 saw a pattern in a sequence to determine the total number of diagonals of the following polygon by working from the total number of diagonals of previous polygon (Table 5.7).

Table 5.7: Student 2’s strategy to solution of Task 4

<table>
<thead>
<tr>
<th>Number of sides of the polygon</th>
<th>Total number of diagonals</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>2+3=5</td>
</tr>
<tr>
<td>6</td>
<td>5+4=9</td>
</tr>
<tr>
<td>7</td>
<td>9+5=14</td>
</tr>
</tbody>
</table>

This strategy is efficient when we know the total number of diagonals of the previous polygon. However, it does not allow producing a general formula for this sequence of numbers which relates the number of the polygon sides and the number of its diagonals (Fig 5.29). He did not present a pictorial representation.
Student 7 tried to relate the number of the sides of a polygon and the number of its diagonals and he labeled \( n \) and \( a_n \) respectively (Table 5.8).

Table 5.8: Student 7’s strategy to Task 4

<table>
<thead>
<tr>
<th>Number of sides of the polygon (n)</th>
<th>Total number of diagonals ( a_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2n-6</td>
</tr>
<tr>
<td>5</td>
<td>2n-5</td>
</tr>
<tr>
<td>6</td>
<td>2n-3</td>
</tr>
<tr>
<td>7</td>
<td>2n-2</td>
</tr>
</tbody>
</table>

Because of this result, Student 7 concluded that it was impossible to determine a general algebraic expression to determine the total number of diagonals of any polygon as each polygon possessed properties different from the other regarding the number of its diagonals (Fig 5.30).
Student 7’s solution to Task 4

[The student wrote: It is impossible to determine an exact formula for the determination of the number of diagonals of an n-sided polygon... Each polygon possesses its own properties regarding the number of its diagonals.]

Student 3 also concluded it was impossible after determining the total number of diagonals of a hexagon that was 3. He did not present any picture (Fig 5.31).

Student 4 tried to determine the number of diagonals of a hexagon (9 diagonals) and he stopped there. He did not present any pictorial representation (Fig 5.32).
Student 5 inductively tried to find the total number of diagonals of some polygons starting from a quadrilateral up to an octagon. He just presented a picture of a heptagon. He correctly wrote down the total number of diagonals from a quadrilateral to an octagon. He tried to get a generalizing formula of the total number of diagonals of any polygon, but he could not carry on further his solution.

Student 1 after determining the number of diagonals of a hexagon (9 diagonals) he carried on in determining the number of diagonals of a heptagon (10 diagonals). He missed to count the remaining 4 diagonals, as the total diagonal number of a heptagon is 14. He drew an octagon but he did not count its diagonals. Besides, he missed to draw 3 of its diagonals (Fig 5.34).
Student 6 only constructed a hexagon and its diagonals and did not conclude anything about the number of its diagonals (Fig 5.35). Student 8 had a blank response.

5.3.1.11 Discussion of Task 4

The analysis summary of the students’ written responses to Task 4 is presented in Table 5.9.

Table 5.9: The analysis summary of the students’ solution to Task 4

<table>
<thead>
<tr>
<th>Student/Approach</th>
<th>Synthetic(Intuition)</th>
<th>Analytic(Reasoning)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Visualization</td>
<td>Construction</td>
</tr>
<tr>
<td>1</td>
<td>±</td>
<td>±</td>
</tr>
<tr>
<td>2</td>
<td>_</td>
<td>_</td>
</tr>
<tr>
<td>3</td>
<td>_</td>
<td>_</td>
</tr>
</tbody>
</table>
The codes “+”, “±”, and “_” used in this table has the same meaning as the codes used in Table 5.6.

This task allows the interplay between synthetic and analytic approaches, and accordingly between algebraic and (purely) geometric thinking. Table 5.9 shows that, in general, the students faced difficulties in accessing key algebraic and geometric concepts to solve this task. Student 2 made a relation between the numbers of diagonals of two neighbouring polygons and discovered a pattern. Nevertheless this pattern did not lead to a general formula for the number of diagonals of an n-sided polygon. In turn, Student 7 correctly constructed and visualized the hexagon and its diagonals, but it seems that he used a pattern to calculate the number of diagonals of a heptagon which did not work. He related the number of sides and diagonals of a polygon and constructed some formulae which did not lead to a general formula.

5.3.1.12 Emerging issues of the analysis of Task 4

1- The nature of task 4 required students to apply synthetic and analytical approaches (or empirical and deductive approaches). According to the figures of Table 5.9 it seems that only a synthetic approach was not enough for solving this task. It also required an analytical approach.

2- A student used some properties of analytical approach (relation and patterning), even though he was not able to produce a formula as a generalisation of the total number of diagonals of an n-sided polygon.

Insight: Profiting from the interaction between deductive and empirical approaches to geometry students may reap the benefits of their knowledge.

5.3.2 Discussion of the Diagnostic Test Data
The Diagnostic test students’ responses (algebraic and geometric knowledge base, connectedness and strategies) were analysed in light of the conceptual model suggested above and the framework on learning and transfer drawn on Prawat’s work.

In Task 1 the target students had to use algebraic symbols and relations (use of variables/parameters, use of formulae, and use of other symbols, such as abbreviations and acronyms) while they demonstrate Pythagorean Theorem. Additionally, they had to translate verbal language into algebraic symbols/relations and pictures, i.e. into different representations, and they had to generalize the theorem for any right triangle. The algebraic form \( a^2 + b^2 = c^2 \) of the theorem was easily recalled by most of the target students (six students out eight); however, all of them could not verbalize the theorem correctly. This fact is explained by Anton and Rorres (1991) that it is easier to manipulate equations (in this case an algebraic identity) than to write out in words. Only two students out of eight correctly used pictorial representation for solving this task. Again, there seemed to be a conflict between different forms of representations; in this case the verbal and the algebraic or even the pictorial. Most students did not flexibly switch from one form of representation to another. This fact is corroborated by Markowitz et al (1986) in Schwartz and Bruckheimer (1988) that the passage between different representations is difficult. On the contrary, Greeno (1983) indicated that analogies can facilitate the construction of relationships between units of knowledge:

If the domains are represented by entities that have relations that are similar, the analogy may be found easily, but if the representation of either domain lacks these entities, the analogy may be impossible to find. Consequently, an analogy can be used in facilitating the acquisition of representational knowledge in a domain. (p. 288)

Dindyal (2003) also noticed that his subjects could not flexibly switch from one form of representation to another. His subjects could only use one form of representation, algebraic in his case. He continues saying that the algebraic form is high in activation and hence recalling the algebraic form was fairly easy for his subjects.

The results showed the students who made algebraic and pictorial representational links accessed key algebraic and geometric concepts and procedures towards a proof (see Table 5.4). Only Student 7 and 8 succeeded in proving to completion. However, Student 7 used a
trigonometry identity to prove the Pythagorean Theorem (Fig 5.7). This identity is firstly yielded using Pythagorean Theorem. It means that the basic knowledge is Pythagorean Theorem. In spite of this, the student showed ability to access (algebraic and geometric) knowledge and strategy as he used concepts from different mathematical domains (algebra, geometry and trigonometry).

In Task 2 the students might use either a synthetic or an analytic approach. All of them, but two used a synthetic approach. Student 1 just wrote down “Impossible” and Student 8 had a blank response. Only Student 3 presented three congruent parts in the form of L-shape (the correct shape for the four congruent parts). Student 2 and 6 had a key idea towards the solution. They started dividing the figure into three congruent squares but they went astray. The remaining student did not access any key idea towards the solution.

In turn, in Task 3, students might also opt between a synthetic approach and an analytic approach. Only Student 1 used an analytic approach. Although he could not represent the correct pyramids, he realized that at most three pyramids congruent to the original pyramid fitted into the cube.

We can see algebraic knowledge and thinking (volume formulae of a cube and a pyramid and the links between these formulae) contributed to partially answer a geometric task, though it did not help the student to visualize and construct the three congruent pyramids in the cube (Fig 5.21). The remaining students used a synthetic approach and according to their answers seemed that they were guessing how many pyramids could fit into the cube through trial and errors using visualization and construction. According to Schoenfeld (1986), Student 1 used a deductive approach (as a means of discovery). However, he could not use an empirical approach (as a means of development of intuition), while his colleagues only used an empirical approach. Schoenfeld observed that unless students learn to take advantage of both approaches to geometry and learn to profit from the interaction of those two approaches; students will not reap the benefits of their knowledge. In turn, Hershkowitz (1998) corroborates this view explaining that reasoning takes place when, by experimentation [e.g. construction by ruler and compass or geometrical software] and inductive generalisation [by visualization processes], one extends her geometrical knowledge about shapes and relations and extends her “vocabulary” of legitimate ways of reasoning. Deductive reasoning [dependent exclusively on the corpus of propositions- definitions, axioms, and theorems]
then, becomes a vehicle for understanding and explaining why and inductively discovered conjecture might hold.

The nature of Task 4 is different from the others. This task required of the students to interplay between synthetic and analytical approaches. It means that these approaches should be used conjointly to solve this task. The first step the students should do is to construct polygons and the respective diagonals. Thereafter they should visualize the total number of diagonals of each polygon and inductively discover a pattern which it might help to yield a formula of the total number of of diagonal of any polygon under certain conditions. Furthermore, they could deductively explain or proof the formula inductively discovered (not required for this task).

5.3.3 Emerging Issues of the Diagnostic Test Data

From the analysis of the students’ written responses to the Diagnostic test, we observe the following:

1- The differences in teaching and learning experiences and contexts amongst the students led to observe the differences in the approaches they used in the problem solving situations particularly in geometric tasks.

2- The tasks of the Diagnostic test required different algebraic and geometric concepts and allowed different forms of knowledge connectedness and strategies.

3- Cognitive processes underlying algebraic and geometric thinking appeared in all tasks; however, some of them were more preponderant in some tasks than in others, that is why, I eliminated in the analysis of some tasks those cognitive processes which were underrepresented.

4- The target students in general accessed algebraic and geometric knowledge and some of them used it to solve the tasks.

5- It seemed that most of them faced difficulties in flexibly switching from one form of representation to another (e.g. verbal, numerical, algebraic and pictorial representations) (Schwartz and Bruckheimer, 1988; Greeno, 1983; and Dindyal, 2003).

6- It seemed that most of the students who made algebraic and pictorial representational links accessed key algebraic and geometric concepts and procedures towards a proof (Prawat, 1989 and Dreyfus, 1991 as quoted by Dindyal, 2003).
7- For Task 2 and 3 most of the students used a synthetic approach which seemed very difficult to them to encounter a way out to solve the task. The only student who used an analytic approach seemed to trigger him to get more insight for the solution of Task 3 (Stillwell, 1998).

8- It seemed that the interaction of these two approaches (synthetic and analytic or deductive and empirical) may allow students to reap the benefits of their knowledge (Schoenfeld, 1986).

5.3.4 Conclusion of the Diagnostic Test Data

Regarding research question 1 the Diagnostic test results showed that the target students solved some geometric tasks using both algebraic and geometric thinking. At this stage I could not find out about the meanings the students held of different algebraic and geometric concepts due to the nature of the data. I followed up the tasks of the Diagnostic test during Interviewing Phase 1 to attain this response. Research question 2 was partially answered through Task 3. Only Student 1 accessed and used algebraic thinking to get a geometrical insight to solve the problem. Nevertheless he was not able to obtain the solution. The remaining students tried to solve it using geometrical thinking only (visualization and construction processes) but were unsuccessful. This result replicated what Schoenfeld (1986) found that unless students learn to take advantage of both deductive (as a means of discovery) and empirical (as a means of development of intuition) approaches to geometry and learn to profit from the interaction of those two approaches; students will not reap the benefits of their knowledge. The Diagnostic test data did not present enough evidence to answer research question 3.

5.3.5 Analysis of Interviewing Phase 1

In order to generate meaning in these transcripts, qualitative techniques were used (Kvale, 1996). Firstly, I thoroughly read through each transcript to highlight issues/sub issues and made respective comments, resulting in the following matrix (see an example in Table 5.10) for each task of the Diagnostic test and for each student.

Table 5.10: A scheme for analyzing students’ transcripts (Task 1)
S6: O Teorema de Pitágoras diz: A soma do quadrado da hipotenusa é igual a soma do quadrado dos catetos. E a fórmula é esta \[ BC = AB + AC \].

S6 presents an incorrect verbal representation. She writes a wrong Pythagorean Theorem (PT) formula.

Contradiction between verbal and algebraic representation of PT

R: Sim, sim. Aqui vem a palavra “quadrado”, aqui no enunciado. Como aparece aqui na fórmula?
S6: Tem que ser ao quadrado.

S6 corrects herself putting squares in the formula, i.e. \[(BC)^2 = (AB)^2 + (AC)^2\]

h1: square in verbal and in the formula? (h1 stands for hinting 1)

R: Agora como pode demonstrar esta fórmula?
S6: Creio que não. Não me recordo...
S6: Acho que tentaram demonstrar. Já não me lembro.

S6 does not remember the proof.
r1: proof of PT? (r1 stands for recall 1)
inert knowledge

S6: Eu devo tentar igualar a... então tenho que tentar ficar com uma variável. Devo tentar isolar a

S6 seems to confound proving and to solve an equation using PT formula.

Proof = solution of an equation?

R: Mas o que se pretende é demonstrar este teorema. Demonstrar que a hipotenusa ao quadrado é igual a soma dos quadrados dos catetos. Tem alguma ideia ou...
S6: Não me recordo. Foi há muito tempo. Foi há muito tempo. Parei e fui trabalhar e passam 8 anos.

S6 asserts that she was out of school 8 years ago and she does not remember these concepts.

-h2: to prove that \((BC)^2 = (AB)^2 + (AC)^2\).
-inert knowledge

This stage is called in the literature as noting patterns or themes (Miles and Huberman, 1994). In this step software ATLAS.ti 5.5 was used for producing the quotations and the respective codes (issues/sub issues) from the transcripts (see an extract below).

**P 9: Student 6 P.doc - 9:1 [E: (Lê o enunciado). O Teorema..] (8:8) (Super)**

Codes: [verbal vs algebraic representation contradiction]

No memos

S7: (Lê o enunciado). O Teorema de Pitágoras (TP) diz: A soma do quadrado da hipotenusa é igual a soma do quadrado dos catetos. E a fórmula é esta \[ BC = AB + AC \]. (A linguagem falada contradiz a linguagem simbólica)

**P 9: Student 6 P.doc - 9:2 [R: Sim, sim. Aqui vem a palavra..] (9:9) (Super)**

Codes: [recall “square in verbal and in the formula?”]

No memos

R: Sim, sim. Aqui vem a palavra quadrado aqui no enunciado. Como aparece aqui na fórmula?
Thereafter, I dealt with each student’s data on a case-by-case basis, as it was quite clear, upon reading the data that the students responded differently to the questions. This step is about seeing plausibility (ibid.). Accordingly, a description and a possible interpretation of each student’s data were produced to permit a way to achieve more integration among diverse pieces of data within one student’s data and later across the eight students’ data. *Clustering* (ibid.) was another tactic used in my analysis. This tactic allowed me to ask myself “what things are like each other? Which things go together and which not?”

5.3.5.1 **Analysis of Task 1**

Task 1 was as follows:

> What is the Pythagorean Theorem? Please, prove this theorem.

(Use the right triangle).

Student 1 (S1) justified his “I forgot it” response to the Pythagorean Theorem task (see Fig 5.10). He contended that he learnt this theorem long time ago when he was schooling. Since then he had never used it, although he was teaching mathematics to upper primary school (grades 6 and 7), but not this topic. It could be argued that he had the necessary mathematical schemas; however, he was not able to access and use them. This knowledge is considered as “inert” knowledge (Chinnappan quoting Bereiter and Scardamalia, 1998). He tried to retrieve some isolated concepts (triangle, equal sides, and square). I tried to help him to recall his prior knowledge from what I found to be a key concept within the concepts he mentioned (see an extract of his transcript below). According to Chinnappan the key concepts anchor other concepts. See the extracts of the Portuguese transcripts in Appendix 9.

**S1 (Student 1):** (S11:21-23: Student 1, Interviewing Phase 1, and lines 21 to 23) [For example, here we have a triangle... eh... these two sides must be equal, mustn’t they? (Inaudible). Then, the square... this side (inaudible). Eh... I have a vague idea.]

**R (Researcher):** (RI:24: Researcher, Interviewing Phase 1, and line 24) [Yes, what you can remember, as...]

**S1:** (S11:25-32) [(Silence). This is... a leg, this is also a leg, it is a hipothenuse I guess. The square of the legs... (Inaudible). The Pythagorean Theorem ... Then, I really put it “I’ve forgotten ” because in fact now I do not have it like this... I know... I know more or less where The Pythagorean Theorem comes from. Then, I needed to go back and read some books in order to recall it. I’ve learnt this theorem, I’ve learnt this theorem. But I forgot it. I really forgot it. I forgot it. I forgot it. I’ve not been dealing with these concepts for a long time. I’ve learnt these concepts, but it has been long without dealing with these concepts.]
R: (RI:33-35) [But at least the text of the Pythagorean Theorem. Not its proof as it demands more, at least the text. What does that theorem say? Do you have any idea or a formula?]
S1: (S1I:36) [I forgot it; I forgot it.]
R: (RI:37) [You forgot it. I cannot force you.]
S1: (S1I:38) [I forgot it completely.]
R: (RI:39-41) [But... yes, yes, I think I heard an interesting concept. You have mentioned the word “square”, haven’t you? Now, how can you apply the word “square” in the Pythagorean formula?]

........................................................................................................................................................................
S1: (S1I:44, 45) [The hypotenuse is equal to the square of a leg. (Inaudible).]

This time S1 at least related the hypotenuse to a leg even with a wrong connection. I tried to help him to recall the basic idea to get the correct Pythagorean formula and I insisted giving several hints as follows.

R: (RI:62) [But maybe if you got a(n) (algebraic) square (in your formula), it has to do maybe with the area of a square.]
S1: (S1I:63) [Yes, it has to do with the area. Yes, it has.]
R: (RI:64) [Then, what is the relationship amongst those areas?]
S1: (S1I:65) [Ah, well, that is it. The area... (long silence).]
R: (RI:66) [The area...]
S1: (S1I:67, 68) [The area of... of... this part where for ex. the ... because this is a square, it is not a rectangle. It is a square. (Inaudible).]
R: (RI:69) [The area of a square.]
S1: (S1I:70-71) [Let’s consider the area of the square of which side is the hypotenuse. Then, its area is as double as the area of the square of which side is one of the legs.]

At least S1 has provided with a particular case of the Pythagorean formula when a right triangle possesses legs of the same lengths, which is \[ h^2 = 2l^2 \] where \( h \) is the length of the hypotenuse and \( l \) the length of a leg. However, he conceived the meaning of the concept “double” the same as the meaning of the concept “squared”. Moreover, it seemed that he misused the concepts algebraic square, geometric square, area of a square, hypotenuse, and leg. May be this has happened because he got confused between the algebraic representation of the Pythagorean Theorem (formula) and its geometric representation.

R: (RI:72) [Then, how would you write it in form of a formula?]
S1: (S1I:73) [It would be almost like this. The area... of this square is...]
R: (RI:74) [Which square?]
S1: (S1I:75-77) [This square which is the hypotenuse is equal to the area of the square on the leg side squared. I mean, squared because it is as double as the area of the square on the leg side.]
R: (RI:78) [You are saying it is as double as... the area...]
S1: (S1I:79) [(Nodding). The double.]
R: (RI:80) [Then what has the double to do with the exponent?]
S1: (S1:81-83) [It is the double, yah... Not the exponent... It is the double. Then, it would be like this: Equal to as the double as the area of the square on the leg side, the area of the square on the leg side.]

I asked him what would happen with the formula if the lengths of the legs were different. He tried to check the formula he has suggested with different sized legs (which was not the case as he unintentionally checked it with equal sized legs) and it did not work.

R: (RI:84, 85) [Now, if the legs are different sized what happens with the Pythagorean Theorem? But being this angle... what kind of angle is this one?]
S1: (S1:86) [It is a right angle.]
R: (RI:87, 88) [It is a right angle. Now, let’s suppose that the legs are different sized. Do you think whether this theorem work for this case?]
S1: (S1:89-91) [(Long silence reflecting and uttering some inaudible words). Well, I think it will work, because we can see the area, we can see the area. Then, ah, it is fine, it doesn’t work, it doesn’t work.]
R: (RI:92) [Why doesn’t it work?]
S1: (S1:93-98) [It doesn’t work because if I have here, for instance, 3 cm and here 5 cm I cannot say that the area of the square on the hypotenuse side will be as double as the area of the square on one of the leg side.]
R: (RI:99) [Why?]
S1: (S1:100, 101) [Because the area on this leg side, the area of the square on this leg side with 3 cm is different from the area of the square on the hypotenuse side with 5 cm. Then... (Long silence).]
R: (RI:102) [Then...]
S1: (S1:103, 104) [I should have a quick look and skim the Pythagorean Theorem. May be I could remember it. But now I have been guessing in order to... (laughing).]
R: (RI:105, 106) [Never mind... My intention is to see how your dormant knowledge can...]
S1: (S1:107) [Awake.]
R: Despertar, esse é que é o objectivo. (RI:108) [Awake. This is my objective.]
S1: (S1:109) [Ah, ok.]
R: (RI:110) [Yes, but...]
S1: (S1:111-113) [I was saying that this formula can only work if the lengths of these sides are equal in a right triangle. If the lengths of these sides are equal I think I can write (the formula) down in this way. (Silence).]
R: (RI:114) [Continue.]
S1: (S1:115-121) [Then, if the legs are equal I can say that the area of the square on the hypotenuse side is as double as the area of the square of one of the legs side, if the legs are equal. But now you have suggested another idea (inaudible). Let’s suppose the legs are different. If the legs are different, then my little idea does not work. Because the area, the area of... on this side for instance with 3 cm is 9 cm², and here it is 25 cm².]
R: (RI:122) [Continue.]
S1: (S1:123-124) [Then, the theory that says it is as double as the area of one of the squares on the leg side does not work. It does not work.]
I asked him whether there were other formulas for the Pythagorean Theorem which could work for a right triangle with different sized legs. He recalled a key idea of the theorem, namely, the square area on the hypotenuse which is equal to the sum of the areas of the squares on the legs. However, he wrote down a specific formula, that is $A_h = A_i + A_j$, where $A_h$ is the area of the square on the hypotenuse and $A_i$ the area of the squares on the legs. He used the same symbol for different areas. I thought this symbol was a slippery mistake. So I probed this asking him what an area formula of a square was. He wrote down $A_h = h^2$ and $A_i = l^2$. I realized that this was a conceptual mistake. S1 thought that the Pythagorean Theorem worked for a right triangle with equal sized legs.

R: (RI1:125, 126) [Now, another idea, maybe ... ah... (Inaudible). Is it the only relation of the Pythagorean Theorem or there is another one? That says that... the area of this square...]

S1: (S1I:127-129) [I could write for instance the area of this square that (whose side) is the hypotenuse, it is equal to the sum of the area (the square’s) of the first leg and the area (the square’s) of the second leg given that the legs are different, given that the legs are different.]

R: (RI:130, 131) [This is regarding the Pythagorean Theorem... eh... or ... that the area of the square...]

S1: (S1I:132) [Ah, that the area is equal to the sum, yes, I think so.]

R: (RI:133, 134) [Now, what is a formula of the area of... of... of the square of which side is the hypotenuse? Can you write it down?]

S1: (S1I:135) [A formula?]

R: (RI:136) [Yes, a formula.]

S1: (S1I:137, 138) [It is that formula of the area eh... squared which is equal, well, in this case, is the hypotenuse, it is the squared hypotenuse.]

R: (RI:139) [And the area of this...]

S1: (S1I:140, 141) [Also the area is equal eh... (The leg) squared plus... (Silence).]

R: (RI:142, 143) [But here, it is like the legs that are different sized, then (inaudible) maybe you should use another symbol.]

I hinted him to use different letters (symbols) for different sized legs. But he ignored my hint and preferred to sketch to help himself to think and get motivation using his own words (see Fig 5.36 and an extract of his transcript).
S1: (S1I:144) [Well, I... (Silence). Perhaps using a ruler.]
R: (RI:145) [A ruler. Do you want to use a ruler?]
S1: (S1I:146-148) [To be more visible, I want... (Inaudible) I want to see whether what I am really thinking of in fact (is)... Ok, now I feel comfortable.]

Using a sketch S1 also found out that the formula he suggested did not work for different legs, as he has chosen a non Pythagorean triple 3-5-6. And he explained that the formula did not work because of lacking of precision in sketching (see an extract of his transcript).

R: (RI:159) [Why have you chosen 5 for one leg and 3 for the other?]
S1: (S1I:160) [To distinguish the legs in order to see whether (the area) of this square is equal to the sum of these two areas.]
R: (RI:161) [Could you choose any other numbers?]
S1: (S1I:162) [Yes, I could. I have chosen these numbers arbitrarily. (Inaudible).]
R: (RI:163) [With several attempts, may be you will remember, won’t you?]
S1: ((S1I:164-171) [ (Laughing). (Silence). Then, we have here 3 cm, 5 cm, and here 6 cm. (This sketch) was roughly drawn. In this case we have for this area (silence) 36 cm², here 9 cm², and here 25 cm². Then, to check the idea of doubling, I’d rather say, the area A₁ is equal to the sum of the areas A₂ and A₃. Then, 36 cm² has to be equal to the sum of 9 cm² and 25 cm². The result is 34 cm². If I had sketched precisely, I would have had it equality. These are my ideas concerning to the Pythagorean Theorem...]

It seems that S1 relies on good constructions to solve geometric tasks successfully. I asked him to substitute in the Pythagorean formula the area formulae of the squares, he wrote down a formula which appears in Fig 5.36 meaning \( h^2 = l^2 + l^2 \). S1 faced difficulties in recalling his prior knowledge even with some hints. Concerning to the proof of this theorem, I sketched another picture of a classical proof of the Pythagorean Theorem similar to the
picture of Fig. 5.37 to see whether S1 was able to recall it. The points E, F, G, and H have not been midpoints as the picture suggests.

S1 contended that he did not remember anything about that picture. Meanwhile he remembered the following picture from his schooling time, and then he sketched it (Fig 5.38).

He added saying that they had divided the three squares in small squares evenly. And the total sum of the small squares of the two leg squares had been equal to the total number of the small squares of the hypotenuse square. To get this result they should have used a Pythagorean triple for the lengths of the right triangle sides.

Similarly to Student S1, S2 stated that he learnt this theorem long time ago. And he also claimed he taught mathematics to grades 6 and 7 and therefore, he has not used this theorem in his teaching for long time. He affirmed that when knowledge is used, it makes it easier to be accessible later (see extract below).
S2: (S2I:7, 8) [Maybe, to say that (these issues) were dealt with long time ago. This is often difficult to recall what we had learnt due to lack of practice.]
R: (RI:9) [Yes, yes...]
S2: (S2I:10) [The use of knowledge permits to access later.]

S2: (S2I:30-33) [Don’t we use knowledge; we will lose some of it. After finishing a training teacher course, I was teaching grades 6 and 7.]

I asked him about the Pythagorean Theorem text (verbal representation) or “linguistic medium”, a term borrowed from Pallascio et al (1993). S2 gave a verbal representation as we can see in the following extract.

S2: (S2I:52, 53) [The square of the hypotenuse is equal to the sum of the two legs.]

Further he corrected by himself giving some other alternatives for the verbal representation of the theorem, still with some difficulties.

S2: (S2I:55-57) [But my objective was to demonstrate... the squared“c” is equal to... the hypotenuse, this is to say that the square of the hypotenuse is equal to the square of the legs sum, that is the square of “a” plus the square of “b” (“a” and “b” are the lengths of the legs).]

Pallascio et al (1993) found out that transposing a written explanation (text) of a problem into a model (e.g. an algebraic formula) and in this case the opposite direction illustrates the main difficulties that students encounter. According to a verbal explanation of S2 the relations \((a + b)^2\) and \(a^2 + b^2\) were identical. Pallascio et al argued that this might happen due to shortage of geometric vocabulary and in my case shortage of algebraic vocabulary. Besides, he was not able to transfer the algebraic knowledge to geometric knowledge as, for instance, the expression \(a^2\) means in geometry the area of a square of which length side is \(a\). He carried on sketching a picture used to prove the Pythagorean Theorem similar to Fig 5.36. Unfortunately S2’s sketch is not visible.

I asked him whether the quadrilaterals sketched around the right triangle were rectangles or squares. He was not sure about it. I improved his sketch and I asked him whether he had seen that picture. He still seemed not to remember and he contended that his teacher used a picture to demonstrate the Pythagorean Theorem, but that picture was a triangle. Due to time constraint I could not carry on the interview and we moved to Task 2.
S3 gave a “condensed form” of verbal representation of the Pythagorean Theorem. He missed the concept “length”. Regarding the proof of the theorem he asserted that at school they only used formulae to calculate values.

**S3:** (S3:57) *We were only given formulae. Thereafter we used them to calculate the lengths of the sides.*

I sketched a picture to help S3 recall a proof of the theorem similar to the picture of Fig 5.36. S3 seemed to remember that picture and the respective proof. He seemed to think that to prove the Pythagorean Theorem one needed to get the area of the right triangle. But he gave up his explanation saying “I don’t know”.

**R:** (RI:74) *Well, let’s see perhaps... have you ever seen this picture? A right triangle?*

**S3:** (S3:75) *Yes.*

**R:** (RI:76, 77) *Thereafter, you sketch squares of which sides coincide with the sides of the (right) triangle. Have you ever seen this picture?*

**S3:** (S3:78) *Yes, I have.*

**R:** (RI:79, 80) *How did you use this picture to prove the theorem? This is one of the pictures used to prove this theorem.*

**S3:** (S3:81) *Yes, yes.*

**R:** (RI:82) *Do you see how you used it?*

**S3:** (S3:83, 84) *I think to get the area of the triangle I think we have used the formula of the square area, which is the square of the length of its side.*

**R:** (RI:85) *Continue.*

**S3:** (S3:86) *Then, having the length of the side squared... Sorry, I don’t know.*

I hinted further using algebraic and geometric concepts (the formula which relates the areas of the circumscribed squares and congruent triangles) to see whether S3 retrieved his “inert” knowledge. S3 remembered that formula; however, he did not remember how they used it to prove the theorem. Besides, he was not able to explain how congruent triangles were used for that purpose. Finally, he contended that they did not prove the theorem at all.

**R:** (RI:87, 88) *Do you remember this formula (the formula which relates the areas of the circumscribed squares to the right triangle)??*

**S3:** (S3:89) *Yes, I do.*

**R:** (RI:90) *How have you used this formula to prove the theorem?*

**S3:** (S3:91) *(Silence).*

**R:** (RI:92-93) *Have you used congruent triangles to prove this formula?*

.................................................................................................................................................................................................

**S3:** (S3:96) *Congruent triangles?*

**R:** (RI:97-98) *Do you remember these (congruent) triangles sketched like this? (I sketched the congruent triangles).*

**S3:** (S3:99) *No, no.*
R: (RI:100) [Haven’t you used this method?]
S3: (S3I:101) [No, no. This formula was not proved at school.]
R: (RI:102) [Then, how have you used this picture? What was it for?]
S3: (S3I:103) [Silence.]
R: (RI:104) [If you have seen this picture I think you have used it to explain something.]
S3: (S3I:105) [Yes.]
R: (RI:106-109) [That is why I would like to remind you on whether you have used congruent triangles in this picture in order to prove that the sum of the small squares is equal to the area of the large square. Do you understand what I mean?]
S3: (S3I:110) [No, no.]
R: (RI:111) [Then, don’t you know what the picture was used for?]
S3: (S3I:112) [No.]

I sketched another picture similar to Fig 5.37 to see whether S3 could evoke other ideas on the Pythagorean Theorem proof. He remembered that picture in an examination as a task. Meanwhile, he seemed not to remember fully what it was for, as he pointed out unconnected concepts regarding that task (e.g. co-sinus formula). And he asserted that they had not used that picture to demonstrate the Pythagorean Theorem.

R: (RI:113-115) [What about this other picture? We have a square and another square inside and around we have right triangles. They are four. Do you remember this picture as well? Have you used it to prove the theorem?]
S3: (S3I:116) [Yes, yes (silence). This picture... (Silence). This picture... ]
R: (RI:117) [Yes.]
S3: (S3I:118-121) [I remember seeing this picture, I think, in an examination, in an examination. It was for proving... for proving the formulae of... how to determine the triangle. It was for proving eh... how to determine the area of a square through that formula... the co-sinus formula.]
R: (RI:122) [Have you seen it in an examination?]
S3: (S3I:123) [Yes, in an entry examination.]
R: (RI:124) [Haven’t you used it to prove the theorem?]
S3: (S3I:125) [No, no.]

I showed another picture to S3 (Fig 5.39). The triangles ABC, CBD, and ACD are right and similar to each other. The concept of similarity of triangles plays an important role for the proof of the Pythagorean Theorem using this picture. S3 contended he never used that picture to prove the theorem. I asked him what they had done at school. He affirmed that at school they used to apply formulae to solve tasks and they had not proved the Pythagorean Theorem.
R: (RI:126-129) [Have you used another method where ... you use similarity of triangles? You get three similar right triangles and you use the criteria of similarity of triangles to prove the Pythagorean Theorem. Do you know this method? Have you used it in the previous grades?]
S3: (S3I:130) [No.]
R: (RI:131) [Then, what did you do at secondary school (to prove the Pythagorean Theorem)?]
S3: (S3I:132, 133) [At Secondary schools we only applied formulae to solve tasks. I don’t remember whether we demonstrated this theorem or not.]
R: (RI:134) [The Pythagorean Theorem?]
S3: (S3I:135, 136) [Yes. They only explained what the Pythagorean Theorem was for and they provided its formula. They did not provide a proof.]

S4 explained a key idea of proving Pythagorean Theorem using a formula \( A_1 = A_2 + A_3 \) and consequently \( l^2 = m^2 + n^2 \). He also presented a picture, but in the verbal representation, he used the concept “directly proportional” which has nothing to do with the theorem.

S4: (S4I:7-15) [According to my resolution, eh according to the sketch I did, eh here I’ll try to explain better (what I’ve done). According to my opinion this theorem has to do somehow with squares. Then, in this case these areas are equal (appointing the formula \( A_1 = A_2 + A_3 \)). I considered the longest side \( l \). This is a square, this is a square. Then, it is valid \( l^2 = m^2 + n^2 \), where \( A_1 = l^2, A_2 = n^2 \), and \( A_3 = m^2 \). Then, the Pythagorean Theorem says that... the sum... the longest... the hypotenuse is the sum of the square... the square of the hypotenuse is equal the sum of... of the squares of the legs. It means that the hypotenuse... the square of the hypotenuse is directly proportional to the square... to the sum of the squares of the legs.)

S4 used the concept of perimeter of a triangle, i.e. \( P = l + m + n \), to explain how the formula \( l^2 = m^2 + n^2 \) came about. When I asked him how the perimeter comes into play in the demonstration of the theorem, he simply got stuck.

R: (RI:22) [How did you find out that the area of the big square is equal to the sum of the areas of the small squares?]
S4: (S4I:26, 27) [Well, well I... I realized that the perimeter of the triangle is the sum of all lengths of its sides.]
R: (RI:28) [Now, what has the perimeter to do with demonstration?]
S4: (S4I:33) [It is difficult for me to explain.]

When I asked him where those ideas came from, S4 claimed that they were his own ideas. Additionally, he said he had never learnt any proof of the Pythagorean Theorem. I asked him further how the picture he sketched helped him towards the proof. He explained that the “definition” of the theorem helped him to understand how the hypotenuse is related to the legs. Under “definition” it should be understood the text of the theorem. He added saying that the square of the hypotenuse was directly proportional to the sum of the legs squares. I found out that S4 used “directly proportional” to mean “equal to”. In other words, the square of the hypotenuse was equal to the sum of the legs squares.

S4: (S4I:73) [Directly proportional means equal to. This is to say that.]

I asked him why he meant “directly proportional” as “equal to”. He explained using the inverse proportion. For him inverse proportion stood for the arithmetic operation of division.

S4: (S4I:73) [Maybe, maybe... Speaking about the inverse (proportion). Maybe when we speak about the inverse proportion, we are speaking about division. Eh, eh direct proportion stands for equal to, doesn’t it?]

I was interested in his ideas about direct and inverse proportion and I asked him to present an example of two directly proportional magnitudes. He drew on an example from Physics (Fig 5.40).

Figure 5.40: Student 4’s Physics formula example

I found out that his idea on direct proportion was influenced by the way he interpreted the Newtonian second law of motion, $F = ma$ where $F$ is the force, $m$ is the mass, and $a$ is the acceleration. He read this law as follows:

S4: (S4I:83-85) [Now I cannot find (an example). For example, I don’t know whether..., eh the (Newtonian) second law of motion. If I’m not mistaken, this
law reads that, the force is directly proportional to the mass... ya it is directly proportional to the product of the mass and its acceleration."

The way S4 read the formula \( F = ma \), “directly proportional” may be replaced by “equal to”. I asked him using direct proportion as equality how he could interpret the formula \( F^2 = m^2 + n^2 \). Note that the former formula on its right side is a product and the latter formula on its right side is a sum. S4 excused himself saying that he used direct proportion as equality but he was not able to explain it in terms of a relation. I realized that S4 was aware that to have two directly proportional magnitudes, there should be a relation of the type presented in Fig 5.40.

**S4:** (S4I:95-97) *Maybe in terms of relation, I don’t know (how to explain). I use directly proportional to mean equality (in a formula).*

S4 recognized his limitations in proving. And he claimed that during his schooling there was no the culture of proving. They simply used to apply formulae to solve tasks.

**S4:** (S4I:112-116) *Well, about the same task...maybe to say that... it was not easy to prove (this theorem) because the habit of proving is not activated (on me) yet. (In my schooling time) I did not have such capacity of proving.. Normally, (the teacher) would give us some formulae to resolve tasks. That is why you can notice some flat ideas, but they are my opinions.*

S5 similarly presented a “condensed form” of the verbal representation as in the case of S3. He did not use the concept “length”. He verified the Pythagorean Theorem formula with a Pythagorean triple and presented an incomplete picture (Fig 5.41). He ascribed a verification of a formula for a proof.

![Figure 5.41: Student 5’s sketch for Pythagorean Theorem](image)

**S5:** (S5I:5-9) *In a right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs. Then, I sketched a triangle with these lengths.*
attributed 3 to this leg and 4 to the other. To the hypotenuse, the longest side, I attributed 5. Then, the square of the hypotenuse is equal to the sum of the squares of the legs. I demonstrated (this theorem by checking the formula using the lengths suggested).]

I asked him whether we could choose any other triple to check the Pythagorean Theorem formula. He said that one should choose that triple of which square roots are integers. This explanation does not work even with a Pythagorean triple. For example, the triple chosen by S5, namely 3-4-5 their square roots are not integers.

I asked him whether he had ever proved the Pythagorean Theorem. S5 asserted he knew a method and he started to sketch a picture (Fig 5.42). And he presented a proof using similarity of triangles as follows:

![Figure 5.42: Student 5’s proof of the Pythagorean Theorem](image)

S5 correctly explained most of geometric concepts involved in the picture he sketched and he used some algebraic symbols to identify some geometric objects (triangles, segments, lengths, etc). Besides, S5 correctly used algebraic concepts (proportional relation of corresponding lengths in two similar triangles and algebraic transformations) to get to
Pythagorean Theorem formula. However, he was not able to justify the similarity of triangles. S5 asserted that he had learnt this proof on his own in a book for teaching purposes.

R: (RI:83) [Where have you learnt this proof from?]
S5: (S5I:84) [I was teaching... I was teaching at a secondary school. (Learning through teaching.)]
R: (RI:85) [As a teacher?]
S5: (S5I:86) [Yes.]
R: (RI:87) [Where have you learnt this proof from? Have you learnt on your own in a book or...]
S5: (S5I:88, 89) [There is a book where we can find it. It might be in a different form from mine, but there is a such book.]

I asked him whether he learnt another method of proving the Pythagorean Theorem. He referred to the Pythagorean triple mentioned above used in constructing small squares on the sides of the right triangle in order to check the Pythagorean formula geometrically (Fig 5.43).

Figure 5.43: Student 5’s geometrical verification of a Pythagorean triple

S5: Por ex. usando aquelas medidas que eu tenho aqui. Posso considerar que este lado mede 4. (Desenha em silêncio a fig acima). Suponhamos que esses quadradinhos são iguais. Então, os quadrados construídos sobre os catetos, a soma dos quadrados construídos sobre os catetos é igual aos quadrados construídos sobre a hipotenusa. (S5I:95-98) [For instance, using the lengths I suggested above I attribute 4 to this side. (He sketches a picture silently). Let’s suppose that these small squares are equal. Then, the sum of the number of the small squares on the legs sides is equal to the number of the small squares on the hypotenuse side.]

S6 presented an incorrect verbal representation. She wrote a wrong Pythagorean formula.
S6: (S6I:11,12) [The Pythagorean Theorem says: The sum of the square of the hypotenuse is equal to the sum of the squares of the legs. And the formula is $BC = AB + AC$.]

S6 corrected herself putting squares in the formula, i.e. $(BC)^2 = (AB)^2 + (AC)^2$ after I hinted her with the concept square she used in the verbal representation.

R: (RI:14) [Yes, yes. Here (you have mentioned) the word” (you have mentioned) square. How does it reflect in this formula?]
S6: (S6I:16) [It must be squared.]

S6 asserted that she finished her vocational course 8 years ago and since than she has been working. That is why she did not remember those concepts. When I noticed she was unwilling to answer my questions I gave up hinting. This corroborates what Prawat (1989) designated as disposition. She tended to avoid difficult learning task and she was apt to withdraw when obstacles to learning were confronted.

S7 presented a pictorial representation and used algebraic symbolization to name triangles, angles and lengths. Moreover, S7 presented a “condensed” form of verbal representation. He missed the concept “length”. S7 delivered coherent and interconnected algebraic concepts to prove the Pythagorean Theorem. However, he used a trigonometry identity, which is proved using this theorem, that is he used a consequence of the Pythagorean Theorem to prove this theorem.

S7: (S7I:18-29) [To prove the Pythagorean Theorem I departed from the fundamental trigonometry identity $\sin^2 \alpha + \cos^2 \alpha = 1$. Then, I determined sine and co-sine of $\alpha$ from this triangle, namely $\sin \alpha = \frac{c}{b}$ and $\cos \alpha = \frac{a}{b}$. Then I replaced them in the formula. Then, I got $\frac{c^2}{b^2} + \frac{a^2}{b^2} = 1$. It turned into $\frac{c^2 + a^2}{b^2} = 1$. Finally it became $c^2 + a^2 = b^2$.]

S7 asserted that he learnt it at school and he also learnt the inverse proof: from the Pythagorean formula to get the trigonometry identity. He showed it as follows:

S7: (S7I:50-55) [I divide both sides of the formula $c^2 + a^2 = b^2$ by $b$. Sorry, by $b^2$. It turns into $1 = \left(\frac{a}{b}\right)^2 + \left(\frac{c}{b}\right)^2$. Then, I replace $\sin \alpha$ and $\cos \alpha$ in the previous formula and I get $\sin^2 \alpha + \cos^2 \alpha = 1$.]
I asked him what they had learnt first, the Pythagorean Theorem or the trigonometry identity. He said that first they were taught the trigonometry formula without demonstration and they had used it to prove the Pythagorean Theorem. Additionally, he contended that he did not know another method of proving the Pythagorean Theorem.

S8 presented an algebraic representation of the Pythagorean Theorem and a “condensed” verbal representation for not using the concept “lengths”. Moreover, S8 sketched a picture (see Fig 5.44) and explained the concepts involved in it (squares, congruent triangles, areas and area comparison amongst them).

**S8: (S8I:13-21)** [Now, to prove (this theorem) I used this method. I have here a big square and the other small square inscribed in it (silence). We also have congruent right triangles. What was the principle I used? If we subtract the area of (the big) square with the area of (the small square) we get the remaining area. This area entails three… four right triangles.]

![Figure 5.44: Student 8’s proof of the Pythagorean Theorem in the interview](image)

S8 learnt this proof at school in grade 8. He presented area formula for squares and triangles, symbols representing segments and lengths, algebraic relations and transformations. When asked why he affirmed that the quadrilateral EFGH was a square he justified that its sides were equal.
R: Fundamentaram porque este quadrilátero que está dentro é um quadrado? (RI:37) [Have you justified why this inner quadrilateral is a square?]

S8: Essas distâncias que temos aqui (HG=GF=FE=EH) são iguais. (S8I:45) [The sides are equal (HG=GF=FE=EH).]

I hinted to him showing that a rhombus also possessed equal sides. Accordingly, one needed to show that a square possessed four right angles (actually it suffices to show that it possesses two right angles). He realized that his justification was wrong.

R: (RI:55) [How to show, that these angles are right? In order to be a square its angles must be right. The sides are equal. But it can be a square or a rhombus. A rhombus possesses…]

S8: (S8I:60) [Equal sides (silence). Therefore, it is wrong (my justification).]

S8 affirmed that the teacher drew a picture and then he proved the theorem. They had not justified why the inner quadrilateral was a square. And in that moment he did not have any idea how to justify that statement.

5.3.5.2 Discussion of Task 1

I clustered the eight students according to the context within which they encountered the Pythagorean Theorem. From the categories that emerged from the data I looked for elements of algebraic thinking and geometric thinking that the students accessed and used during the interview. These categories emerged after the process involving two different types of tasks: A recall task was mainly used in form of the Diagnostic test tasks. This kind of task permitted to determine aspects of knowledge that is functionally available to students. A hinting task was used during this interview phase to examine the levels of connectedness of knowledge schemas that is “the degree to which new nodes of information is connected with one another to form a single well-defined structure” (Chinnappan et al, 1999, p. 168).

Category 1: Learnt the Pythagorean Theorem but not used it further for so long

S1, S2 and S6 affirmed that they had learnt the Pythagorean Theorem when they were schooling long time ago and they never used it further. This context contributed to having
“inert knowledge”. Accordingly, they were not able to access and use their previous knowledge appropriately. In Prawat’s terms due to their “inert knowledge” the adequacy of their cognitive structures was weak regarding the key concepts necessary for this task. Although they were under the same context category, they responded to the interview differently, due to different geometry learning contexts. Under pictorial representation all of them showed difficulties in sketching a complete picture. At least they all sketched a right triangle. Moreover, S1 and S2, after some hinting also sketched the squares on the sides of the right triangle. S6 responded hesitantly during the hinting process confirming that those were concepts she had learnt long time ago and was not able to recall them. Accordingly, her ability in sketching was limited.

Under verbal and algebraic representation categories, S1 only remembered isolated concepts “(algebraic) square”, “legs”, and “hypotenuse”. Even after some hinting using a sketch of the Pythagorean Theorem, S1 did not get either the correct text of the theorem or the correct Pythagorean formula. At least he got the core idea behind the Pythagorean Theorem, though with some pitfalls. He wrote the area of the square on the hypotenuse is equal to the areas sum of the squares on the legs. He considered lengths of two equal legs. From this idea, he correctly wrote the respective algebraic formula. S2 gave an incorrect text of the theorem and he tried to correct himself using the Pythagorean formula, but he was not successful. S2 seemed to not possess the core idea as in the case of S1, even after he sketched a picture similar to S1’s sketch supplemented with some interviewer’s hinting. S6 gave an incorrect text of the theorem and a wrong Pythagorean formula. After interviewer’s hinting, she corrected herself and wrote the correct formula. Although all the students got an incorrect text of the Pythagorean Theorem, S2 and S6 got the correct formula of the theorem. Besides, they (S2 and S6) had difficulties in transferring the algebraic language into a verbal (geometric) language as the algebraic formula of the theorem did not help them to translate it into the text of the theorem. S1 translated a verbal (geometric) language into an algebraic language when he wrote his core idea of the theorem in an algebraic formula. The reasoning category (proof) showed that all students of this group were aware of a proof learnt at school. However, none of them accessed the necessary concepts to prove the theorem. S1 presented a vague idea of a proof learnt at school, meanwhile it constituted a verification of a Pythagorean triple applied to a sketch. S2 and S6 only asserted that they were taught a proof at school but they could not remember even when given hints. It seemed that “inert
knowledge” hindered the prevalence of the model for algebraic thinking in geometrical understanding regarding this task.

Category 2: Just learnt to use Pythagorean formula to solve tasks

S3 and S4 claimed that at school they only used the Pythagorean formula to solve tasks. I also probed them regarding the three categories I created namely; verbal representation, algebraic representation, pictorial representation, and proof (reasoning). Verbal representation: S3 gave a condensed form of verbal representation. He missed the concept “length”. That is, the square of the hypotenuse (length) is equal to the sum of the squares of the legs (length). S4 presented a verbal representation as he used the concept “directly proportional” in it. He used this concept to mean “equal to”. Algebraic representation: Both correctly wrote the Pythagorean formula, although only S4 seemed to have a deeper understanding of the geometrical meaning of that formula. He related the Pythagorean formula to the areas of the squares on the sides of a right triangle. According to him this additional idea came up through self study. Pictorial representation: They produced sketches of the Pythagorean Theorem. S3 only sketched a right triangle and S4 additionally drew squares around it. S3 used his sketch for calculating the lengths of the sides of a right triangle with the help of the Pythagorean formula, while S4 used his sketch to interpret this formula geometrically. Reasoning: None of them was able to prove the theorem. Their excuse was that they had not proved the Pythagorean Theorem at school. S4 tried to prove it using “own ideas” but he was unsuccessful. I hinted them towards a proof; nevertheless, they missed most of the key concepts. Accordingly, they were not able to prove the theorem. These students were not able to access and utilize their intellectual resources towards a proof because of lack of them. It seemed that S3 was performance oriented, as he was satisfied with the knowledge taught by the teacher. In turn, S4 seemed to be mastery oriented due to his efforts to broaden his knowledge through self study. Although both students faced difficulties in solving this task, at least S4 showed some conceptual understanding.

Category 3: Self study for teaching purposes

Verbal and algebraic representations: S5 presented a condensed form of the verbal representation and he correctly wrote the Pythagorean formula. Pictorial representation and reasoning: He presented a proof of the Pythagorean Theorem using the principle of similar triangles. He correctly used the principle of similar triangles and algebraic transformations in
his sketch. However, he was unable to explain why the triangles were similar to each other. He stated that he had learnt that proof by himself in a book for teaching purposes. For him it seems that verification and proof mean the same as he firstly presented a verification of a Pythagorean triple and he considered it as a proof. S5 is a special example where we can see learning taking place through teaching. He was forced to learn the proof in order to teach. Meantime, he still faced some difficulties in justifying some concepts. This student was forced through teaching to be mastery oriented (see extracts RI:83 to S5I:88, 89 in Section 5.3.5.1 and S5I:245-248; and S5I: 250-254 in Section 5.3.5.4). Accordingly, I noticed some conceptual understanding in him.

Category 4: Proof learnt at school

S7 and S8 asserted that they had learnt the Pythagorean Theorem and its proof at school. Verbal and algebraic representations: Both presented the Pythagorean formula and a condensed form of the verbal representation. Pictorial representation and reasoning: They sketched a picture suited to the type of the proof they had learnt at school. S7 sketched a right triangle and used the fundamental trigonometry identity to get the Pythagorean formula. He correctly used algebraic transformations to get that formula. I asked him what they had first learnt, the trigonometry identity or the Pythagorean formula. He affirmed that they first had learnt the trigonometry identity without proving it. Afterwards, they had used it to prove the Pythagorean formula. It is known that the fundamental trigonometry identity is proved using the Pythagorean formula. S7 mentioned this fact saying that they had also proved the “converse theorem”, that is from the Pythagorean formula to get the fundamental trigonometry identity. He also presented a proof of it.

S8 presented one of the known proofs of the Pythagorean Theorem in the literature. This proof is rich in connections between algebraic and geometric concepts. The key concepts to be taken into account in this proof are for example, area formula of squares and triangles, triangle congruency, angles relation, and the relationship between areas of triangles and squares. S8 showed mastery in algebraic relations and transformations in connections with related geometric concepts. However, he was not able to explain why a certain constructed picture constituted a square. He simply said his teacher had not justified it only he had constructed the picture and asserted that it was a square. These two students showed mastery in solving this task. However, they were not able to explain certain concepts due to lack of teaching of those concepts.
5.3.5.3 Emerging issues of the analysis of Task 1

In this section, I describe key issues that emerged from the analysis of Task 1 interview.

1- Students entering Universidade Pedagógica (Pedagogical University) enrolled in the course of mathematics teacher training came from different academic and professional backgrounds. Accordingly, they experienced different geometry teaching and learning contexts.

*Insight 1:* The differing contexts of teaching and learning develop students academically differently.

2- The students who learnt the Pythagorean Theorem and not used it further showed difficulties in accessing and using it even with hints.

*Insight 2:* Knowledge learnt and not used further may become inert and difficult to access later (Chinnappan quoting Bereiter and Scardamalia, 1998 and Prawat, 1989).

3- Most of students’ ideas were related to their teachers’ use of the same in their classes except for two students (S4 and S5) who tried to expand their knowledge further through self study.

*Insight 3:* Institutional relationship to student’s knowledge was strongly noticeable and it reshaped mostly what students knew.

4- A special case of S5 showed that learning can take place through teaching.

*Insight 4:* Teaching can be a healthy environment for learning.

5- The students who had difficulties in flexibly switching from one form of representation to another (e.g. verbal, algebraic, and pictorial) belong to the categories 1 and 2.

*Insight 5:* Inert knowledge and procedural teaching impair conceptual understanding.

6- The students who flexibly switched from one form of representations to another belong to categories 3 and 4.
Insight 6: Effective teaching and self learning for teaching purposes contribute (partially) to conceptual understanding

7- It seems that the students, who produced algebraic and a complete pictorial representations, accessed and used key algebraic and geometric concepts successfully towards a proof.

Insight 7: Connectedness of key concepts and procedures provide access and use of knowledge (Prawat, 1989 and Dreyfus, 1991 as quoted by Dindyal, 2003).

5.3.5.4 Analysis of Task 2

Under the umbrella of developing a code inductively: a data-driven approach adopted by this study I chose a sub sample of five case studies, one of which represented the contextual categories emerged during the process of the analysis of Task 1 (Section 5.3.5.2 and Boyatzis, 1998), except for Category 2 where I chose two subjects as I found their data interesting for the purpose of the study. The raw data collected from this sub sample was the basis for developing the code which is supported by the data reduction method (Boyatzis, 1998). Moreover qualitative researchers tend to view reliability as a fit between what they record as data and what actually occurs in the setting under study, rather than the literal consistency across different observations. They also view validity as the degree to which a study generates theory, description, or understanding (Bogdan and Biklen, 1982) rather than the generalisation of findings. Accordingly, I chose the interesting cases according to my view within the four categories yielded in the analysis of Task 1, namely S1 from Category 1, S3, and S4 from Category 2, S5 from Category 3, and S7 from Category 4.

An in depth data analysis was carried out for these five case studies. In the analysis of Task 2 transcripts, I used the same procedure as in the analysis of Task 1. Table 5.11 shows the first stage of the analysis of S1’s transcripts to Task 2, which involved elements of the conceptual model developed in Chapter 3.

Table 5.11: A scheme for analyzing students’ transcripts (Task 2)

<table>
<thead>
<tr>
<th>Extract of transcript (Portuguese)</th>
<th>Comments</th>
<th>Issue/ Sub issue</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1: Eh... Eu usei uma régua e tentei tantas estratégias quantas possíveis...</td>
<td>S1 used several specific strategies through construction using a ruler.</td>
<td>-Construction processes -Visualization processes</td>
</tr>
<tr>
<td>S1: Divida a figura em quatro partes congruentes... Então, eu parti do princípio que partes congruentes quer dizer quatro partes, quatro partes iguais. Eu usei uma régua (rindo)... Eu tentei... Divida a figura em quatro partes iguais... Eu compreendo [a pergunta] nesta maneira... nesta maneira. Quando eu usei uma régua eu tentei... tentei... e só consigo três partes iguais.</td>
<td>S1 explained the concept congruency as equality of pictures and tried to construct such pictures using a ruler. He got three congruent parts instead of four as required.</td>
<td>-Construction processes -Specific strategies</td>
</tr>
<tr>
<td>---</td>
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</tr>
<tr>
<td>R: E como você fez para ter as três partes iguais? S1: Eu fiz nesta maneira e nesta maneira (mostrou na figura como construiu três quadradinhos congruentes) e eu consegui três... três... partes iguais. Eu não consigo quatro partes iguais (rir). R: Que é uma outra tentativa que você pode tentar? É com diversas tentativas que nós podemos obter a solução... S1: Para ver se for possível ou impossível. Eu tentei unir estes dois pontos (estava tentando dividir a figura com segmentos que unem os seus vértices)... E eu obtive quatro partes desiguais. O princípio é aquele para dividir a figura que nós devemos usar rectas... segmentos de rectas... para unir os pontos. Eu fiz muitas tentativas (como) esta... R: Isto a terceira tentativa... S1: Sim, é... também não me dá... R: Que é que dá aí? S1: Quatro partes, mas não são iguais... eles não são iguais... Eu tentei sempre unir pontos, como esta figura contem pontos, este ponto, este ponto... Eu tentei unir os pontos com segmentos que dividem a figura a fim de conseguir uma subdivisão... que dê... que dê... 4 partes iguais, mas eu não consegui.Acho que fiz essa tentativa no rascunho... ah não. Eu puz impossível.</td>
<td>Through several specific strategies (construction and visualization processes) he got four non congruent parts and gave up stating that it was impossible.</td>
<td>-r1: Three congruent parts? -r2: Another strategy? -r3: What have you got? -Construction processes -Visualization processes</td>
</tr>
<tr>
<td>R: A minha pergunta é: Esta figura talvez terá aparecido de que figura? É uma parte de uma outra figura. Donde terá aparecido? S1: (Silêncio). Pôde ter vindo de um quadrado. R: Porque? S1: Se eu prolongo a figura, fico com um quadrado (completa a figura com um quadrado). I hinted to S1 that he could exercise visualization. He responded positively to.</td>
<td>I hinted to S1 that he could exercise visualization. He responded positively to.</td>
<td>-h1: How does this picture came about from? -h2: Why from a square? -Visualization processes</td>
</tr>
</tbody>
</table>
R: Tiňha visto antes isso?
S1: Sim, vi... vi... Mas como se tirou uma parte, quando dividido, eu consigo obter somente três partes (mostra como dividiu a figura em três quadrados pequenos congruentes). Quando eu a dividi desta maneira eu apanho apenas três partes... mas se eu puxar um bocadinho a imaginação e vir que a figura é uma parte de um quadrado, e há uma parte que falta desse quadrado, quando eu trago a parte que falta apanho quatro partes iguais (quatro quadrados pequenos congruentes).

S1 accessed the first key idea for solving this task: the picture was a part of a square. Meanwhile he was not able to carry on further to completion. He asserted he could get four congruent parts after completing the missing part of the square.

-h3: Divide further into smaller squares.
-Construction processes
-Visualization processes

R: Mas a tarefa requer-nos conseguir as quatro partes congruentes da figura dada sem adicionar qualquer coisa.
S1: (Quis apagar a peça adicionada).
R: Você pode deixá-lo. Agora... depois que você se dividiu a figura em três quadrados congruentes pode você continuar dividi-lo em quadrados congruentes menores?
S1: Sim, eu posso... Eu posso...
R: Como?
S1: Eu tento dividir nesta maneira... Estimar o centro de cada quadradinho [dos três quadrados congruentes obtidos previamente). (Juntou os pontos médios dos lados de cada quadrado com segmentos).
R: Dessa divisão pode você ter uma ideia de como...?
S1: Sim.
R: Qual seria essa ideia, então?
S1: (Em silêncio contava os quadrados pequenos). Então, tenho 12 quadrados pequenos... (fêz mentalmente a seguinte operação algébrica 12:4=3). Yah, ok... Então, cada três quadrados pequenos (rir) eu fico com eh... a primeira parte, a segunda parte... (construia as partes da figura com três quadrados congruentes pequenos e apanhou quatro partes). Yeh!
R: Quantas partes congruentes você conseguiu?
S1: Bem, consegui três partes congruentes no tamanho e na forma; mas tenho quatro partes congruentes com a mesma área mas com formas diferentes.

S1 explained to him the four congruent parts had to be constructed from the existing picture without adding anything. He tried to erase the part he had added. But I discouraged him to do that as this part would help him to think further.

I hinted to him whether the process of dividing the picture in smaller (congruent) squares was possible. He got the idea and used it. After dividing the picture in small squares he counted them and found out that each part should entail three small squares. He got four parts with the same area and three of which were congruent to each other. This strategy can be generalized to similar tasks involving other pictures.

R: Suas partes não podem satisfazer as exigências da tarefa porque você tem três partes congruentes, mas a quarta parte é diferente daquelas, embora tenha a mesma área. Congruencia tem que ser...
S1: Na forma e na área (rir).
R: E tamanho... Pode você tentar uma outra estratégia? Talvez... Ao menos você descobriu um princípio. Qual foi?

S1 understood the concept congruency in synthetic geometrical language: two objects are congruent to each other when they possess the same shape and size. He was aware that his response was not correct according to this definition.

-Construction processes
-General strategy
-Patterning
-Visualization processes

-R reasoning
A partir do princípio de dividir cada quadrado em quatro quadrados menores. Não, dividi primeiro... Eu dividi a figura [dada] em três partes iguais (quadrados). Depois disso eu dividi cada parte em quatro quadrados pequenos.

R: E então...

S1: E eu vi que eu tive 12 quadrados pequenos e eu dividi... 12 quadrados pequenos por 3. Desculpe, 12 quadrados pequenos por 4, então eu apanho quatro quadrados (meaning quatro partes) com 3 quadrados pequenos cada um. Agora eu necessito procurar a forma dos quadradinhos [das 4 partes requeridas]...

R: Esta é a tarefa que você deve fazer em seguida...

S1: Sim, para procurar a forma dos quadrados (que significam as partes) de modo que...

(Silêncio longo ao refletir e ao tentar as formas das partes no papel).

---

He explained his strategy how he got four parts with the same area three of which were congruent. He was aware at which stage he was and what was missing to solve the task.

---

Task 2 was as follows:

Consider the figure below. Divide the figure in four congruent parts.

S1 explained the concept of congruency as equality of pictures and tried to construct them using a ruler. He got three congruent parts instead of four as required. He tried to understand the task as he read and repeated the text of the task.

S1: (S1I:234-240) [Divide the picture into four congruent parts... Then, I considered congruent parts as four equal parts, four equal parts. I used a ruler]
[laughing]... I tried... Divide the picture into four congruent parts... I understand in this way... in this way. When I used a ruler I tried... I tried... and I just got three equal parts.]

I prompted him to try to use other strategies for solving the task. He got four non congruent parts using other specific strategies (through construction and visualization processes) and gave up the task stating that it was impossible. Unfortunately, S1 used the same picture given in the test to try those strategies and erased them while trying one after another strategy (Fig 5.45).

R: (RI:241) [And how have you got the three congruent parts?]  
S1: (S1I:242-244) [I did this way and this way. (He used his previous picture to show how he got them.) And I got three... three...equal parts. I cannot get four equal parts.]  
R: (RI:245, 246) [Can you try another strategy? We can find a solution by trying several strategies.]  
S1: (S1I:247-250) [To see whether (the task) is solvable or insolvable, I tried to join these two points... (He was trying to divide the picture through joining its vertices). And I got four non congruent parts. The principle of dividing the picture was to construct lines... segment lines... joining the points. I tried many strategies...]  
R: (RI:251) [This is your third attempt...]  
S1: (S1I:252) [Yes, it is... this also does not work...]  
R: (RI:253) [What have you got?]  
S1: (S1I:254-258) [Four parts, but they are not equal... they are not equal... I always tried to join the points as this picture possesses points, this point, this point... I tried to join the points constructing segment lines in order to get four equal parts, but I could not. I think I also used this strategy in a rough paper... oh no. I think it is impossible.]  

I hinted him to exercise visualization and S1 responded to it positively. He accessed the first key idea for solving this task: the picture was a part of a square. Meanwhile he was not able to carry on further to completion. He asserted that he could get four congruent parts after completing the missing part of the square. See the following excerpt of his transcript.

R: (RI:261, 262) [My question is: where might this picture come from?]  
S1: (S1I:263) [(Silence). It might have come from a square.]  
R: (RI:264) [Why?]  
S1: (S1I:265, 266) [If I extend the picture it becomes a square (he adds a small square to the picture).]  

............................................................................................................................  
R: (RI:271) [Have you got this idea before?]  
S1: (S1I:272-277) [Yes, I have... I have... I only got three equal parts because one part was taken out (he shows how he got the three congruent small squares). When
I divided this picture I only got three parts... but if I reason further, I can see that this picture is a part of a square and there is a part missing and if I add it to the picture I get four equal parts (four small congruent parts)].

I explained to him that the four congruent parts had to be constructed from the existing picture without adding anything. He tried to erase the part he had added. But I discouraged him to do that as this part would help him to think further. I hinted him whether the process of dividing the picture further in smaller (congruent) squares was possible. He got the idea and used it. After dividing the picture in small squares he counted them and found out that each part should entail three small squares. He got four parts with the same area and three of which were congruent to each other. During the process of sectioning he used middle point, congruent squares, and patterning notions.

**R:** (RI:278, 279) [But the task requires four congruent parts without adding anything.]

**S1:** (S1:280) [(He intended to erase the part added to the picture.)]

**R:** (RI:281, 282) [You can leave it. After dividing the picture into three congruent squares, can you continue dividing it into smaller congruent squares?]

**S1:** (S1: 283) [Yes, I can... I can...]

**R:** (RI:284) [How?]

**S1:** (S1:285-287) [I try to divide in this way... I guess where the centre of each square is. (He refers to the three congruent squares obtained previously). (He joined the mid points of the sides of each square through segments.)]

**R:** (RI:288) [From that division do you have an idea on how to proceed?]

**S1:** (S1:289) [Yes.]

**R:** (RI:290) [Then, what is your idea?]

**S1:** (S1:291-295) [(Silently, he counted the small squares). Then, I got 12 small squares... Yah, ok... Then, each group of three small squares (laughing)... I have eh... the first part, the second one... (He was constructing parts with three congruent small squares and he got four parts. He might mentally have done the arithmetic calculation 12:4=3.)]

**R:** (RI:296) [How many congruent parts have you got?]

**S1:** (S1: 297, 298) [Well, I got three congruent parts in shape and size, but I have four congruent parts with the same area.]

S1 understood the concept congruency in synthetic geometrical language: two objects are congruent to each other when they possess the same shape and size. He was aware that his response was not correct according to this definition.

**R:** (RI:303-305) [The parts you constructed are not congruent to each other. You have three congruent parts, but the fourth is not, though it possesses the same area as the other parts. Congruency means...]

**S1:** (S1:306) [(The parts should possess the same) shape and area (laughing).]
R: (R1:307, 308) [And size... Can you try another strategy? May be... at least you discovered a key idea. Which was that?]

He explained his strategy on how he got the four parts with the same area. He sectioned the picture into small congruent squares and he counted them all totaling 12. Afterwards he divided 12 by 4 and he concluded that each congruent part should entail three small squares. However, he was stuck when constructing the four congruent parts sought. Because he tried several strategies but he ended up with four parts with the same area three of which with the same L-shape. He was aware at which stage he was and what was missing to solve the task.

S1: (S1I:309-311) [The strategy consisted in dividing each square into four small squares. No, I first divided ... I divided the picture given into three equal parts (squares). Afterwards, I divided each part into four small squares.]

R: (R1:312) [And then...]

S1: (S1I:313-316) [I counted 12 small squares and I divided... 12 small squares by 3. Sorry, 12 small squares by 4, then I got four squares (four parts) with 3 small squares for each. Now, I need to search for the shape of these squares (the 4 parts sought)...]

R: (R1:317) [This is the step you should do next...]

S1: (S1I:318, 319) [Yes, to search for the shape of the squares (meaning the parts sought) in order to... (He was reflecting silently and attempting to get the shapes of the parts.)]

According to S1’s explanation of one of his strategies above it seemed that an analytic approach (numeric framework and algebraic thinking through patterning) helped him to systematise and organise his ideas (reason) and hence to be focused on his search for solution through visualization and construction processes (synthetic approach) as the following extract of his transcript points out:

S1: (S1I:321, 322) [(Long silence). Then, I have... (Silence). Yeah, that is fine. I think I got it. I got one... two, three, and four (parts).]

R: (R1:323) [Are these four congruent parts or not?]

S1: (S1I:324, 325) [There are four congruent parts. There are four. I got this, this, this, and this one. There are four.]

R: (R1:326) [What is your response, then? Is it possible or impossible?]

S1: (S1I:327) [It is possible, it is possible (laughing).]

R: (R1:328) [How did you find it...?]

S1: (S1I:329) [It is interesting, it is very interesting. Yes, it is possible.]

R: (R1:330) [Well, I think...]

S1: (S1I:331, 332) [Actually, these parts are equal, this, this, this, and this one. They are four.]

The final picture S1 came up with is presented below.
S3 used a general strategy of dividing the picture in 12 small congruent squares. He saw that the original picture was a square divided into other four congruent squares one of which was taken out. Similarly, he divided each of the remaining squares into other four squares. And in each divided square he took a small square out. In his words "it remains three fourths of the square". He concluded that those three fourths of the squares were four parts with the same area three of which had the same shape. In other words he used patterning to see such parts with the same shape and area. (See his solution in Fig 5.46).

S3 recognized that he did not get the correct solution. Because the parts he got had the same area but not the same shape. And he concluded that it was not possible under those conditions to get four congruent parts.

**S3:** (S3I:152-164) *I divided this picture into three equal parts. Yes, this picture was divided into three congruent parts. We have three squares, three equal squares. Then to divide into four equal parts, I mean, we have... four parts in a square, ie we can construct four small squares in it. In a square we can draw four other squares. But we want those squares to be... be similar. In each square we have four parts. If we take a part out, a part out, we will have three fourths of this square. Similarly for this square we take a part out and three parts remain, and so forth in order to get similar parts. I don’t know whether...*
R: (RI:165) [Continue, continue.]
S3: (S3I:166-168) [In this part (square) remain three parts, in this part (square) also remain three parts and so forth. And the parts we took out, form another three parts.]
R: (RI:169) [Then, what is your conclusion?]
S3: (S3I:170-175) [My conclusion is that we took a part out in this square. Also we took a part out in this square, and so forth. Then we get four, four pictures with... with the same size, I can say with the same area.]
R: (RI:176, 177) [Then, Is your solution responding the task? What does the task say? Divide...]
S3: (S3I:178) [(Divide) the picture into four congruent parts.]
R: (RI:179) [Did you manage those four congruent parts?]
S3: (S3I:180) [No, I have not.]
R: (RI:181) [What have you got?]
S3: (S3I:182) [I’ve simply got four parts with the same area.]
R: (RI:183) [In order to be congruent what are the conditions?]
S3: (S3I:184, 185) [The four parts should have the... the same geometric configuration, i.e. they should be equal in shape and size.]
R: (RI:186) [Now, is it possible to get the four congruent parts sought?]
S3: (S3I:187) [No, no.]

I mentioned that he almost got the correct result and I encouraged him to carry on thinking.
S3 accepted the challenge but said that it was difficult.

R: (RI:188, 189) [Why do you say it is not possible? You were about to get the solution.]
S3: (S3I:190) [Yes, it might be possible, but it is difficult.]
R: (RI:191-193) [Now, I want you to overcome this obstacle because you succeeded in 50% of the solution or more than 50%. From there you can get four congruent parts. Now, how (can you proceed)?]

I revisited the strategy S3 used in order to find the stage where he got stuck. S3 knew that the congruent parts should entail three congruent squares. Through visualization and construction processes he always found four parts with the same area, three of which were congruent. In those strategies he mainly used axial symmetry, middle point, congruent squares, and patterning concepts. That is why he said “I always end up with a non congruent part”. I: asked him what shape the congruent parts might have. After several trials he was able to get the four congruent parts. Fig 5.47 shows some of his strategies.
S3 said a key idea in order to break through towards the solution was the shape the four congruent parts should possess, the L-shape (pattern of the congruent parts).

R: (RI:202-203) [Perhaps I will pose a question to enlighten a bit. Well, you divided this picture in small squares. How do they look like?]
S3: (S3:204) [All of them are equal.]
R: (RI:205, 206) [Another question, I if want four congruent parts how many small squares should each part entail?]
S3: (S3I:207) [It should entail three.]
R: (RI:208) [Now, how can you get those four parts? Play the parts and see.]
S3: (S3:209) [(He was trying several strategies in silence). Always I end up with a non congruent part.]
R: (RI:210) [Sorry?]
S3: (S3I:211) [Always I get a part which is different to the other.]
R: (RI:212, 213) [Each part entails three small squares. Now, you should play around those parts to get them.]
S3: (S3I:214) [(Trying several strategies in silence).]
R: (RI:215) [You are almost there. You've started up well. Only you need to finish up.]
S3: (S3I:216) [(Trying several strategies in silence).]
R: (RI:217) [In your opinion, what shape should the parts look like?]
S3: (S3I:218, 219) [It works; it works... one, two, three, four... (He got the solution and smiled with joy).]

R: (RI:220) [Explain how you got it.]
S3: (S3II:221, 227) [If we took (a small square) out in each square we got four equal parts. In a square we took one out. How can I explain? A square can be divided into four equal small squares, then we can... we can take a small square out. Another square entail four equal small squares. If we take a small square out it forms a picture of this...this... type. (He referred to the L-shapes). Then...
getting such pictures we can construct (four congruent parts).

I questioned why he did not get the solution previously even with the key idea of L-shape. He had difficulty in explaining it but he asserted that maybe the way he constructed the parts misled him. And on that moment he used another way to construct the parts. It is important to say that the L-shape is a key idea towards the solution. However, it does not suffice. It requires one to correctly visualize in order to construct the L-shapes properly.

**R:** (RI:231-234) *What is the difference between the first strategy and this one? There, you also got the L-shapes. Why did you not succeed in the former strategy and you did in the latter?*

**S3:** (S3I:235) *I think the difference lies in these two here.*

**R:** (RI:236) *In these two.*

**S3:** (S3I:237-239) *Yes. And this too. I don’t know how to explain but... I don’t know how to explain. I think because here I did like this: These three equal pictures I linked to the other three. I avoided using the previous strategy.*

S4 found the task rather unusual and he did not believe it had a solution (i.e. he considered it unsolvable).

**S4:** (S4I:140-143) *(He reads the text of Task 2). (He laughs). Actually I found this task a paradox, because I have never seen it. I have never seen it. When I saw this picture I was astonished, because I have never seen a similar task. Truly, I did not believe in existence of a solution for this task.*

Meanwhile he had tried to solve it in the Diagnostic test. He was not sure about the correctness of his solution. See an extract of his transcript and the division of the picture he had done in the Diagnostic test (Fig 5.48). He got two pairs of congruent parts using axial symmetry and middle point notions. And he verified that the sides of the parts were not equal for every part.

**R:** (RI:144) *But you have already divided...*

**S4:** (S4I:145-149) *I mean this... this was... was a way out I found. It was a possible solution, let’s say so, I found. Even after solving it I was not convinced that I got the solution. I trusted myself but I was not sure. I looked at this side*
and at the other, and I do not believe that they are equal.]
R: (RI:150) [Why don’t you believe?]
S4: (S4I:151) [This side is not equal to that side.]

But he saw that the picture was a square but with a missing part. And he tried another strategy drawing two diagonals, two perpendiculars, and completing the missing part (Fig 5.49).

R: (RI:152) [In terms of...]
S4: (S4I:153-158) [In terms of the distance, in terms of the distance, if I have a square, I can draw a diagonal. I would have this side equal to that one... Having... having this point as... having this segment as a perpendicular passing through the midpoints of these sides... But what happens in this task, in this task according to my point of view a part is missing.]

R: (RI:168) [In your opinion this picture...]
S4: (S4I:169) [It’s a square; it’s a square with a missing part.]
R: (RI:170) [Continue.]
S4: (S4I:171, 172) [Then, being this picture a square I divide this square into two equal parts through this diagonal.]
R: (RI:173) [Continue.]
S4: (S4I:174-179) [Then, one part was taken out, it was taken out, and this (part) remained.... according to my explanation this is a midpoint. This diagonal divides this square into two equal parts. Afterwards, I looked for a midpoint, a midpoint of the remaining part. Then, constructing a parallel line to this side I divided the picture into two parts which I consider to be equal.]

Figure 5.49: S4’s solution 2 to Task 2

S4 gave up this strategy as it yielded more than four parts and tried another strategy where he got two pairs of congruent parts through parallel segments and using axial symmetry (Fig 5.50).
Figure 5.50: S4’s solution 3 to Task 2

**R:** (RI:180) [Why do you consider them equal?]

**S4:** (S4I:181, 182) [For a simple reason, the distance from here to here I believe to be the same as the distance from here to here.]

**R:** (RI:183) [Are you sure?]

**S4:** (S4I:184-189) [More ore less, but I... (Laughing). This segment is longer (than the other). According to my reasoning I divided this into two parts and also I divided that into two parts. May be my concern was to divide and get four parts. According to my reflection when I got home, I saw that this side was equal to this side from the division I made.]

**R:** (RI:190) [Which side?]

**S4:** (S4I:191, 192) [This part is equal to that part. And this part is equal to that part. I concluded that I was not lucky in my solution: to divide the picture into four congruent parts.]

He used a definition of congruency. He recognized that the parts he constructed were not congruent. This time he was referring to the picture of Fig 5.48. After several attempts he stated that it was unsolvable. He showed mastery orientation. He tried several (specific) strategies to solve the task but he was not satisfied with and carried on solving it at home.

**R:** (RI:193) [Why were you not lucky?]

**S4:** (S4I:194) [Because I think that... if the parts are congruent they must be equal in shape and in size.]

**R:** (RI:195) [And what is happening here?]

**S4:** (S4I:196-199) [Here I’ve got pictures with the same shape but with different sizes. That is why I think I was not lucky. The shape is the same. But the sizes are different. Accordingly, the pictures don’t go or match with the criteria of congruency which means to have the same shape and size.]

**R:** (RI:200) [And what is your conclusion?]

**S4:** (S4I:201-203) [My conclusion is that this solution is not correct. As I said at the outset of... of... my analysis about this picture, I think it is not possible to divide it into four congruent parts (laughing). This was my attempt to solve it.]

**R:** (RI:204) [You are of the opinion that this task has no solution.]

**S4:** (S4I:205) [Yes, I call it an insolvable task.]

**R:** (RI:206) [Ok.]

**S4:** (S4I:207) [This was a way, abstraction I had. So to say it was an attempt. After reflecting and trying several strategies, I thought this was the correct one. When I got home I carried on trying to solve the task. If I had noticed that this
segment was longer than the other I would have concluded that the task was insolvable.]

I hinted him towards a general strategy (the same strategy used by S1 and S3). Even with some hints S4 was doubtful on the solvability of the task. However, when he divided the picture into small congruent squares he reasoned by himself and he concluded that each congruent part should entail three congruent small squares. Meanwhile, he was stuck when constructing those four congruent parts. He always found four parts with the same area three of which were congruent in several attempts he did.

**R:** (RI:238, 239) [In one of your strategies you mentioned that this picture came about from a square, didn’t you?]

**S4:** (S4:I:240) [Yes, I did. This division I did, I took into account that this picture was a square.]

**R:** (RI:241, 242) [Now, using the concept square is it possible to continue dividing the picture into small squares?]

**S4:** (S4:I:243) [To continue dividing into small squares?]

**R:** (RI:244) [I mean to construct small squares in this picture considering it as a square.]

**S4:** (S4:I:245, 246) [(He sketches a square). I believed that a part of it was taken out, it is probably in the middle point.]

**R:** (RI:247) [Can you take a part out and construct small squares in the remaining picture?]

**S4:** (S4:I:248) [Let’s see. (He sketches accordingly). We take this part out, don’t we?]

**R:** (RI:249) [Yes, carry on.]

**S4:** (S4:I:250, 251) [No way, they are three small squares. I tried to construct the small squares, but also it is not possible to get four congruent parts.]

**R:** (RI:252) [How many small squares have you got?]

**S4:** (S4:I:253) [Here you are... in terms of grouping they are one, two, three small squares.]

**R:** (RI:254) [And they are asked for...]

**S4:** (S4:I:255) [Four.]

**R:** (RI:256) [Is it possible to divide further into smaller squares?]

**S4:** (S4:I:257) [I think it is possible.]

**R:** (RI:258) [Is it possible?]  

**S4:** (S4:I:259) [I think so.]

**R:** (RI:260) [Then, carry on.]

**S4:** (S4:I:261) [(Silence).]

**R:** (RI:262) [What are you doing?]  

**S4:** (S4:I:263) [I’m dividing into smaller squares.]

**R:** (RI:264) [How are you dividing them?]

**S4:** (S4:I:265, 266) [I’m dividing... I’m trying... trying dividing each small square into four equal smaller parts.]

**R:** (RI:267) [How is it possible to do that division?]

**S4:** (S4:I:268, 269) [In the same way I divided the initial square into four equal parts, I can divide each small square into four equal parts.]
S4: (S4I:271, 273) [I can consider each small square as a unit. (He carries on dividing into smaller squares). Would it be possible to get the solution?]
R: (RI:274) [What do you mean?]
S4: (S4I:275, 276) [I refer to my theory that the task is insolvable. My conclusion is that the picture does not yield four congruent parts.]
R: (RI:277) [Let’s try, ok?]
S4: (S4I:278, 279) [Uhm… This is geometry, geometry, geometry… (He was speaking alone and low while solving the task).]
R: (RI:280) [What are you doing? Are you speaking to yourself?]
S4: (S4I:281, 282) [(Laughing). No, no. My reasoning is as follows: These smaller squares are equal, aren’t they?]
R: (RI:283) [Continue.]
S4: (S4I:284, 285) [I divided each square, which lies in the interior into four equal parts.]
R: (RI:286) [Continue.]
S4: (S4I:287-292) [There are 12 smaller squares in total, aren’t there? There are 12. It is asked for dividing this picture into four equal parts, into four equal parts. Then according to the criterion of congruency two pictures are congruent when they possess the same shape and size. There are 12 smaller squares. It means that each pair (part), each pair, according to my reasoning, each pair should entail three smaller squares.]
R: (RI:293) [But 12 divided by 3 is equal to 4 parts.]
S4: (S4I:294, 295) [Yes, there are four parts. But each part entails three small squares.]
R: (RI:296) [Continue your reasoning. You are looking for…]
S4: (S4I:297-302) [I’m looking for a pencil. I would like to shade (the parts)... to check if it is true or not. Then, here we have three, it is a part. I can shade it. No, no, it cannot be so. (Speaking low and shading the parts). It is difficult to construct the... the geometrical parts. Let me try this way. No way. It is not correct (laughing). I can see the solution but I’m failing in organizing my thought. (He carries on in shading the parts). No, no I cannot find the solution.]
R: (RI:303) [What is your conclusion?]
S4: (S4I:304-305) [I’ve got three... they are three congruent parts, but a part is not, it is not.]

I hinted the L-shape to him and the starting point where to construct the first part, otherwise it would not work. Finally he was able to draw the four congruent parts. It is interesting to notice that the L-shape is also the shape of the original picture. Although S4 used shading strategy to help him to visualise the congruent parts, he used all the hints needed to succeed in Task 2 (Fig 5.51). He recognized lack of organization (of his knowledge) although he understood what is sought. S4 was astonished about this exercise and wanted to explore more strategies.

R: (RI:306) [You have tried several strategies.]
S4: (S4I:307-309) [I’ve been trying several strategies which are this, this, this, and this... any division I made, I end up... with the same result.]
R: (RI:310, 311) [Why don’t you search for another strategy? I see that you always get stuck in the same difficulties.]

S4: (S4I:312-314) [The possibilities I have here... I’m going to use another strategy... three... three... three... (Sketching congruent parts with the three congruent small squares and trying several strategies). No way. I’ll give up (laughing).]

R: (RI:315) [Why do you have to give up? You are almost there.]

S4: (S4I:316-323) [What is needed here is... what matters here is congruency. Always I end up with three congruent parts. I always end up with three congruent parts. I always get it. But I always find a different part. This result is not the solution sought. (He revised the strategies he used and he still ends up with the same result.) I always face the same problem, even if I start from this side or the other.]

R: (RI:324) [If you try the other side besides these two you have already chosen, what would happen?]

S4: (S4I:325) [This side.]

R: (RI:326) [From here.]

S4: (S4I:327-329) [From here? I would have, one, two, and three. This would be a part. One, two, and three, it’s another part. Let’s see one, two, and three (he did not succeed).]

R: (RI:330) [Try using this shape (the L-shape)].

S4: (S4I:331) [Using this shape, one, two, and three...]

R: (RI:332) [Starting from here...]

S4: (S4I:333-341) [One, two, and three, ok... I would also have one, two, and three. These are already shaded, eh... Here it would be four, five, and six, it’s another part, then (laughing), four, five, and six, then I would have seven, eight, and nine... (He got the four congruent parts and he looked happy). This task is terribly difficult! When I discovered that it should entail three small squares for each part, three small squares for each, I knew the maximum should be twelve small squares. No, it should be more than twelve. The issue here is the organization (of knowledge). And the other issue is congruency. I shouldn’t say that the parts are congruent when they are different. I don’t know whether there is another strategy for this task.]

R: (RI:342) [I think you can try it.]

S4: (S4I:343) [That is true.]

R: (RI:344) [Then what is your conclusion after all this?]

S4: (S4I:345) [It is... it is solvable. It is solvable. I don’t know whether anyone got it!]

Figure 5.51: S4’s final solution to Task 2
S5 used specific strategies through construction and visualization processes without success. One of the strategies could have led to a general strategy of dividing the picture into small squares. But he gave up. He mainly used middle point, segment lines, axial symmetry, and congruent squares. Unfortunately I could not present the construction strategies S5 yielded because he used the picture which appeared in the text to solve the task. And he erased the strategies which led him nowhere. Meanwhile one of his pictures looked like the picture of Fig 5.52 which is described by him in the following excerpt of his interview.

![Image of Fig 5.52: One of S5’s solutions to Task 2](image)

I used his latter strategy to explore the student’s ideas. He correctly visualized the picture that it came about from a square. It seemed that he was hindered in pursuing this strategy, because he was not sure whether the three parts he constructed were congruent and constituted squares.

R: (RI:120) [Now, are these 3 parts congruent?]
S5: (S5I:121, 122) [I... I... I thought... I was doubtful... because of my method to show that this point was the midpoint of this segment here.]

R: (RI:123) [Could you prove it?]

S5: (S5I:124-127) [I could not prove it. I... I... I think, I measured here and here to see whether they (the segments) were equal. It was difficult to affirm surely that this was a midpoint. If the length were precise (a whole number) and (hence) divisible I would say that they were equal. I would measure the length of the segment and divide by two. They would be equal.]

R: (RI:128) [Now, what do you think? Where this picture came about from?]

S5: (S5I:129) [I think... from a square.]

R: (RI:130) [Why do you think so?]

S5: (S5I:131) [If I drew this missing part, I would have a square.]

From this idea of a square I tried to help him to carry on with this general strategy of dividing the picture further into smaller congruent parts (in this task, congruent squares). S5 correctly said if it were a square he could divide the square into four congruent parts which were also squares.

R: (RI:132, 133) [You would have a square. Now, how can you use this concept “square” to get ideas for solving this task?]

S5: (S5I:134, 135) [If it were a square, if it were a square I could divide the picture into 4 equal parts.]

R: (RI:136) [Can you sketch it?]

S5: (S5I:137) [He completes the missing part and gets a square and divides it into 4 equal parts.]

R: (RI:138) [This part is not given.]

S5: (S5I:139) [If we take this part out, it will become more complicated than it is now.]

S5 seemed not to believe in the properties of the geometrical objects, as he after dividing the square into four congruent parts he still doubted whether the three parts left after taking one part out were congruent. He measured the lengths to certify. Even after the first hint he was not able to relate the same idea to carry on with the process of dividing the squares into four other smaller congruent squares.

Through some graded hints S5 was responding to them. Assumption: through graded hinting students can organize their knowledge and hence access and use it.

R: (RI:140) [But taking this part out, how many parts would remain?]

S5: (S5I:141) [To sketch the diagonals?]

R: (RI:142) [No. You divided the square into 4 parts. Taking out one part, how many parts will remain?]

S5: (S5I:143) [3 parts will remain.]

R: (RI:144) [You can draw them.]

S5: (S5I:145) [(Sketching the three parts.)

R: (RI:146) [How many squares would there be?]
S5: (S5I:147) [There are 3.]
R: (RI:148) [Are they congruent or non congruent?]
S5: (S5I:149) [I would measure. (He measured the lengths of the sides of the squares).]
R: (RI:150) [Now, we have 3 congruent squares, haven’t we?]
S5: (S5I:151) [Yes, we have.]
R: (RI:152) [Is it possible to construct further smaller congruent squares?]
S5: (S5I:153) [To construct smaller squares?]
R: (RI:154) [Yes, by dividing the picture into smaller squares.]
S5: (S5I:155) [(Silence).]
R: Tendo um quadrado quantos quadradinhos podemos construir? (RI:156) [Having a square, how many small squares can we construct?]
S5: Tendo um quadrado podemos dividir este por dois... Poderia ter 4 quadradinhos. (S5I:157) [Having a square we can divide it into two...We could get 4 small squares.]
R: Cada quadrado teria quantos quadradinhos? (RI:158) [How many small squares can we get for each square?]
S5: (S5I:159) [Four.]
R: (RI:160) [You can draw them.]
S5: (S5I:161) [(Drawing the small squares). 4 here, 4 here, and 4 here.]
R: (RI:162) [What relationship is there amongst these small squares?]
S5: (S5I:163) [I think these small squares have the same size.]
R: (RI:164) [Which sizes?]
S5: (S5I:165) [(Of the sides).]
R: (RI:166) [How do you consider these small squares?]
S5: (S5I:167) [They are equal.]
R: (RI:168) [Then, using this idea, is it possible to construct 4 congruent parts in this picture?]
S5: (S5I:169) [(Silence).]
R: (RI:170) [How many small squares have you got?]
S5: (S5I:171) [There are 12, there are 12.]
R: (RI:172) [Then if you want 4 congruent parts, how many small squares will each part entail?]
S5: (S5I:173) [It will entail... (silence).]
R: (RI:174) [It will entail...]
S5: (S5I:175) [Three.]
R: (RI:176) [What do you mean by three?]
S5: (S5I:177) [Three pairs.]
R: (RI:178) [Three pairs?]
S5: (S5I:179) [When we have 12 small squares, we need 4.]
R: (RI:180) [What do you mean by we need 4?]
S5: (S5I:181) [Four equal parts.]
R: (RI:182) [Then do you mean that each part entail 4 small squares?]
S5: (S5I:183) [(Silence). Each part entails 3.]
R: (RI:184) [There are 4 congruent parts. And how many small squares does each part entail?]
S5: (S5I:185) [Three.]

After getting the idea that each congruent part should entail three small congruent squares S5 tried several strategies to get the four congruent parts but he was unsuccessful. He stated that
"It is very complicated. It seemed easy when we discovered that the congruent parts should entail three (congruent small) squares". I guided him to get them and finally he got them.

R: (RI:190) [If you chose a picture with three small squares, what would it be? And can you shade it?]
S5: (S5I:191) [We could choose for example this one.]
R: (RI:192) [And what would the next one be?]
S5: (S5I:193) [This one.]
R: (RI:194) [Exactly, what is the other part?]
S5: (S5I:195) [It can be this one.]
R: (RI:196) [How many congruent parts have you got?]
S5: (S5I:197) [So far I got 3.]
R: (RI:198) [And what about the fourth part?]
S5: (S5I:199) [I got this part with 3 congruent small squares...]
R: (RI:200) [Is this part congruent to the others?]
S5: (S5I:201) [No.]
R: (RI:202) [What is another strategy?]
S5: (S5I:203) [(Trying to get the 4 congruent parts.]
R: (RI:204) [What is up?]
S5: (S5I:205) [This strategy does not work either.]
R: (RI:206) [Maybe, you can try another strategy. Start up from this side.]
S5: (S5I:207, 208) [It is very complicated. It seemed easy when we discovered that the congruent parts should entail 3 small squares.]
R: (RI:209) [Start up from the other side.]
S5: (S5I:210) [(He tried another strategy). It does not work either.]
R: (RI:211) [Does it still not work? Now, try this side.]
S5: (S5I:212) [This part is congruent to this part, to this part, this part to this part, and this part to this part.]

Asked to revise the successful strategy, he showed that he understood it. See lines 213 to 234 of his transcript.

R: (RI:213) [What do you think? What strategy have you used to get the solution?]
S5: (S5I:214) [Firstly, I divided the picture into three equal parts and I got these squares.]
R: (RI:215) [And then?]
S5: (S5I:216, 217) [In turn, each square was divided into four equal parts. Then we got twelve small squares in total.]
R: (RI:218) [Continue.]
S5: (S5I:219) [As we are looking for four equal parts, then we divide twelve by three, sorry by four.]
R: (RI:220) [And then?]
S5: (S5I:221-224) [We should have three equal parts, then divide... the three squares... I was saying we had twelve small squares and we were looking for four parts. We divided by four. And we understand had to divide the twelve small square ehhh... we had to find three small squares.]
R: (RI:225) [Continue.]
S5: (S5I:226, 227) [Therefore we constructed the squares in L-shape. (Difficulties in explaining his strategy?)]
R: (RI:228) [You mean each congruent part should entail three small squares, isn’t it?]
S5: (S5I:229) [Yes, yes.]
R: (RI:230) [You made several attempts and you ended up with three congruent parts instead of four, isn’t it?]
S5: (S5I:231) [Yes, yes.]
R: (RI:232) [Now, how have you got four congruent parts?]
S5: (S5I:233, 234) [We need to search for three small squares. And then to use that… to use that in order to get four equal parts.]

S5 complained about his geometric knowledge (thinking) reason for the difficulties he was facing during the solution of this task. He explained this fact saying that at school, they had learnt it in the syllabuses algebra and geometry. Meanwhile, it was first taught algebra and geometry afterwards. And most of times geometry was not taught at all. To fill up his gaps in geometry he had carried out a self study.

R: (RI:235) [ Do you want to say anything else about this task?]
S5: (S5I:236-242) [I see that my capital difficulty is my weakness in geometry because…It’s true that I learnt… mathematics, but I had skipped geometry, I had skipped geometry. I found… ahhh… the most difficult part to see the relationship between…the squares. I got the idea of dividing into three squares. But I did not get the idea of carrying on this process of dividing into smaller squares. It means that I lacked… to see that relationship. May be after dividing the picture into three squares, I might divide into smaller squares.]
R: (RI:243, 244) [ I did not catch up what you said. You said you had learnt mathematics but you had skipped geometry.]
S5: (S5I:245-248) [When we learnt mathematics, we were taught algebra and geometry. They were also included in the syllabus. But it would happen that algebra was taught and… and… geometry was left as the latter topic. And often times; geometry was not taught at all.]
R: (RI:249) [What course have you done?]
S5: (S5I: 250-254) [I did a vocational teacher training course. I don’t mean that my case is a general issue. I learnt some geometrical notions, but I did not learn them deeply. The geometrical knowledge I possess is a result of my self study. (Self study for filling up his gaps).]

S5 ended up with the following picture as a solution for Task 2 (Fig 5.53).
S7 divided the picture into two different pairs of congruent triangles (Fig 5.54 and an extract of his transcript). The picture presented entails two different solutions of S7 distinguished by the colour of the segment lines. Being the blue segments the first strategy and the grey ones the final strategy. After trying the first strategy and realized that it did not get the solution sought he affirmed that the task was insolvable.

R: (RI:103) [Are the parts you have divided congruent?]
S7: (S7I:104) [(Silence).]
R: (RI:105) [How have you made the division?]
S7: (S7I:106) [If the tasks required two (congruent) parts, it would be this solution.]
R: (RI:107) [Would three (congruent) parts be possible?]
S7: (S7I:108) [Three (congruent) parts... (Silence). Here you are three (congruent) parts.]
R: (RI:109) [Four (congruent) parts?]
S7: (S7I:110) [Four (congruent) parts? (Laughing). Four (congruent) parts...]
R: (RI:111) [The purpose is to get four congruent parts.]
S7: (S7I:112, 113) [Firstly, I divided into two (congruent) parts. Afterwards, I divided in four parts. But they are not congruent. But I think they are similar.]
R: (RI:114) [Here it is sought four congruent parts. What was your answer?]
S7: (S7I:115) [It is impossible.]

In the first strategy, S7 sectioned the picture into two pairs of congruent triangles. He used
axial symmetry and segment notions. Similarly, S7 through graded hinting accessed and used the knowledge needed for the solution of the task. However, during prompting process he took long time reflecting before answering the questions.

R: (RI:116, 117) [Is it impossible? Perhaps you can try other strategies. Can you tell me how this picture came about from?]

S7: (S7I:118, 119) [From a square. (It seemed that a key idea for solving the task is in mind of S7. But he needed some hints to activate it).]

R: (RI:120) [Why?]

S7: (S7I:121-123) [If I extend this segment and the other one parallel to the other sides I get a square, the issue is that we took a part out. Dividing this square we get four (congruent) parts.]

R: (RI:124, 125) [Dividing this square we can get four (congruent) parts.]

S7: (S7I:126) [We can divide a square into four congruent parts.]

R: (RI:127) [I’m referring to the picture given and not the square.]

S7: (S7I:128) [(Laughing).]

R: (RI:129, 130) [Using the concept square how can you transform this picture in order to get four congruent parts?]

S7: (S7I:131) [(Long silence).]

R: (RI:132, 133) [You got three congruent squares. Is it possible to divide them further into smaller squares?]

S7: (S7I:134) [(Trying to get smaller squares silently).]

R: (RI:135) [How many small squares have you got? And what is the relationship amongst them?]

S7: (S7I:136) [(Silence). (Laughing).]

R: (RI:137) [You are smiling, may be you got something interesting, didn’t you?]

S7: (S7I:138) [(Silence).]

R: (RI:139) [Can you explain me what you are doing?]

S7: (S7I:140) [(Long silence).]

R: (RI:141) [Have you got any result?]

S7: (S7I:142) [(Laughing). I changed my mind.]

R: (RI:143) [why have you changed your mind?]

S7: (S7I:144-148) [I firstly divided into three (congruent) squares. I divided them into smaller squares. I found in each square four small squares. Then, adding them I got twelve small squares, twelve small squares. Now I wanted to divide the picture into four equal parts. Then, I divided 12 small squares by 4, 4 parts. Then, it means each part entails three small squares.]

R: (RI:149) [Continue.]

S7: (S7I:150-151) [Then I divided 4 small squares, sorry 3, 1,2, and 3 it is the first part; 1,2, and 3 the second part; 1,2, and 3 the third part; and 1,2, and 3 the fourth part.]

R: (RI:152) [Then, what is your conclusion?]

S7: (S7I:153) [(Laughing). There is a solution. It is possible, it is possible.]

The final picture sketched by S7 is shown in Fig 5.54. Although S7 took a long time of reflection in solving Task 2, he was able to construct the four congruent parts with fewer hints than the rest of the subjects. I tried to elicit some more strategies in him and he
suggested dividing the picture into small congruent triangles. See an extract of his transcript and the respective sketch in Fig 5.55. However, he concluded that with 12 congruent small triangles did not work because the L-shape could not be constructed.

R: (RI:154) [By the way, can you find another strategy to solve this task?]  
S7: (S7I:155-157) [Uhm... (Long silence). Besides this strategy... besides the strategy of squares, there is a method of triangles. (S7 after solving the task successfully he suggests another strategy).]  
R: (RI:158) [Can you explain to me how you will implement it?]  
S7: (S7I:159) [Can I draw?]  
R: (RI:160) [Make yourself comfortable.]  
S7: (S7I:161, 162) [(Firstly he sketched three squares; secondly he sketched the diagonals of all squares getting thus 6 congruent triangles).]  
R: (RI:163) [What is your conclusion?]  
S7: (S7I:164-170) [Now, I got 1, 2, 3, 4, 5, and 6. I got 6 congruent triangles. The 6 congruent triangles divided by 4 equals... equals to a number... a decimal number. Then, it does not equal to a whole number. We need to divide further. I’m joining these two vertices. Thus, we get 4 triangles in this square. Similarly in this square, I get 4 triangles and the other square too. Then, in total we have 1, 2, 3, 4... We get 12 triangles. Then dividing 12 by 4 parts we get 3, 3 triangles.]  
R: (RI:171) [How would the 4 congruent parts look like?]  
S7: (S7I:172) [Well, the first part (is) 1, 2, 3... (Long silence).]  
R: (RI:173) [How many congruent parts should you construct?]  
S7: (S7I:174) [(Trying several strategies).]  
R: (RI:175) [What is your conclusion?]  
S7: (S7I:176) [1, 2, 3... 1, 2, 3... (Trying several strategies in silence).]  
R: (RI:177) [Then what is your reasoning?]  
S7: (S7I:178, 179) [I got 4 parts, but only some are congruent, this is to say, they possess the same size and shape.]  
R: (RI:180) [Continue.]  
S7: (S7I:181-183) [I got 3 parts with the same shape and one... one different, one different. Then they are not congruent. I think there is a strategy. If there might be other strategies (laughing). Never say the strategies are over.]  

Figure 5.55: S7’s other strategy
5.3.5.5 Discussion of Task 2

All students except S3 used synthetic approaches (visualization and construction processes only) to solve this task. Meanwhile all of them had a key idea towards an analytic approach (number framework and algebraic thinking through patterning aiding visualization and construction processes): the division of the picture into three congruent squares. However, they did not utilize it further except S3 who almost got the correct answer. I describe below a categorization of the subjects according to the strategy used and connectedness of their knowledge.

*Synthetic strategy*

All students but one (S3) used visualization and construction processes driven by their intuition to solve the task. None of them got the four congruent parts sought and they stated that the task was either insolvable or impossible or difficult. The strategies they used led to a certain result, although they did not readily transfer to new, potentially relevant situations (specific strategies) Prawat (1989).

*Hybrid strategy (analytic and synthetic strategies)*

S3 used a numerical approach to see patterns in the picture combined with visualization and construction processes and got four parts with the same area and three of which were congruent. He used numerical terminology while explaining his strategies like “three fourths of” and he mentally considered each square as a unit and arithmetically divided the 12 units of the squares by 4 and obtained each congruent part to entail three units of the squares. This strategy might lead to a new, potentially relevant situation (general strategy) (ibid.). In this approach we noticed an indicator of the presence of algebraic thinking in the solution of Task 2 (patterning). My claim is supported by Vennebush *et al* (2005) saying that “algebraic thinking is used in any activity that combines a mathematical process with one of the big ideas in algebra, such as understanding patterns (…)” (p. 87). Other researchers corroborate the same view (Dindyal, 2003; Vennebush *et al*, 2005 quoting Kriegler, 2004; and Greenes and Findell, 1998).
The idea that each congruent part sought should entail three small congruent squares was a result of analysis which is at the heart of algebraic thinking (algebra) (Charbonneau, 1996). He put it this way:

There are two steps involved in solving a geometrical problem analytically. First, the problem is purported to be solved. In other words, the relations between the different objects involved in the problems are considered to be true. This consideration constitutes a hypothesis. Second, this hypothesis having been formulated, one searches for objects which can be constructed or can be arrived at from the objects which are known in the problem. If one of these objects can be constructed, then, from it, one can construct the object which is sought. (ibid., p. 23)

This was what happened during the process of solving this task through this strategy. The first step we considered, was the hypothesis that the picture given in the task is a part of a square (three fourths of the entire square) which was firstly divided into four congruent squares. The second step from this hypothesis one searched for objects (in this case to divide each of the three squares into four small congruent squares) which could be constructed from the objects, which were known in the problem (the three congruent squares). Through reasoning (counting the total number of small congruent squares and dividing it by the four congruent parts and seeking for patterns), and through intuition (construction and visualization processes), one can construct the object which is sought (in this case the four L-shaped congruent parts).

Connectedness of concepts: reasoning versus intuition

I noticed that during the hinting process all students responded positively regarding a numerical approach and patterning (each congruent part entails three small squares). However, all of them showed difficulties in visualizing and constructing the four L-shapes. Even some of them (S4 and S5) used most of the hints available to get to a solution. This was an indication that their intuition was still weak, although their reasoning seemed to be robust. This again corroborates the Schoenfeld’s (1986) claim that unless students learn to take advantage of both deductive (as a means of discovery) and empirical (as a means of development of intuition) approaches to geometry and learn to profit from the interaction of those two approaches; students will not reap the benefits of their knowledge.
In this task S3 needed fewer hints (a hint only see the table of hinting frequency in Table 5.12) than the rest of the students. S5 showed more difficulties in accessing knowledge (16 hints) and the others were between S3 and S5 on the hinting frequency table namely S1 (4 hints), S7 (3 hints), and S4 (9 hints) respectively.

Table 5.12: Hinting frequency table to Task 2

<table>
<thead>
<tr>
<th>Student</th>
<th>Hints (H) to Task 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>H1: My question is: where might this picture come about from?</td>
</tr>
<tr>
<td></td>
<td>H2: Why from a square?</td>
</tr>
<tr>
<td></td>
<td>H3: After dividing the picture into three congruent squares, can you continue dividing it into smaller congruent squares?</td>
</tr>
<tr>
<td></td>
<td>H4: Can you try another strategy? May be... at least you discovered a key idea. Which was that?</td>
</tr>
<tr>
<td>S3</td>
<td>H1: In your opinion, what shape should the parts look like?</td>
</tr>
<tr>
<td>S4</td>
<td>H1: Now, using the concept square is it possible to continue dividing the picture into small squares?</td>
</tr>
</tbody>
</table>
According to Table 5.12, we can infer that algebraic thinking through patterning aided by a numerical approach facilitated most of the students for getting the L-shaped pictures. However, when they got to the point where they should construct the four congruent L-shapes, they faced difficulties.

*General strategy: division into small congruent parts and patterning*

S7, after using the general method of dividing the picture in congruent small squares, when prompted he suggested and applied the same strategy but using congruent small triangles. He looked for the multiple of 4 in the number of the small triangles and came up with 12 small congruent triangles (Fig 5.55). However, this number did not work for getting four congruent parts as the minimum multiple of 4 compatible with the L-shapes is 24 congruent triangles (Fig 5.56).
5.3.5.6 Emerging issues of the analysis of Task 2

The analysis of students’ responses to Task 2 in the interview yielded some emerging issues, which concern the strategies they used in relation to the connectedness of their knowledge.

1- All students used intuition to solve Task 2 through visualization and construction processes (synthetic strategies). However, they faced difficulties in solving it. Besides these strategies seemed to be specific and applied to a concrete situation.

   **Insight 1:** Specific strategies lead to a certain result, although they do not readily transfer to new, potentially relevant situations. In particular in geometry specific strategies (synthetic strategies) lack a general method (strategy) (Hansen, 1998 and Prawat, 1989).

2- All students through hinting process used both intuition (visualization and construction processes) and reasoning (a numerical approach and patterning) to solve the task successfully.

   **Insight 2:** The interaction of deductive (as a means of discovery) and empirical (as a means of development of intuition) approaches to geometry leads students to reap the benefits of their knowledge (Schoenfeld, 1986).

3- In Euclidean Geometry analytic strategies (general strategies) seem to be more efficient than synthetic strategies (specific strategies), meanwhile they are difficult to access.

   **Insight 3:** In Euclidean Geometry general strategies readily transfer to new, potentially relevant situations. However, general strategies seem to be more difficult to access than specific strategies. (Prawat, 1989).
4- In Task 2 most of the students seemed to possess robust schemas as they needed few hints to solve it. Only S4 and S5 needed more hints showing weak schemas maybe due to lack of confidence and/or knowledge.

*Insight 4: A schema with components that are effectively organized is one for which minimal levels of cueing are required for activation. When a greater level of hinting support is needed for access, the knowledge schema is either less extensive or less well connected.* (Lawson and Chinnappan, 2000, p. 31). *Moreover affective and motivational components (e.g., lack of confidence) can hinder the use of a strategy or skill on a transfer task* (Prawat, 1989).

5.3.5.7 Analysis of Task 3

Similarly I analysed Task 3 using the raw data of the five students mentioned above. Table 5.13 shows the first stage of the analysis of S1’s transcripts to Task 3.

Table 5.13: A scheme for analyzing students’ transcripts (Task 3)

<table>
<thead>
<tr>
<th>Extract of transcript (Portuguese)</th>
<th>Comments</th>
<th>Issue/ Sub issue</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1: (Lê o enunciado do terceiro exercício em voz audível). Bom, aqui eu... eh... eu aqui (tracei) essa pirâmide. Eu pensei da seguinte maneira. Talvez estou a pensar certo ou estou a pensar errado. Eu pensei da seguinte maneira.</td>
<td>S1 learnt a definition and use it to algebraically solve the task relating the volume formulae of a cube and a pyramid.</td>
<td>-Visualization processes</td>
</tr>
<tr>
<td>R: O que se quer é a sua compreensão.</td>
<td></td>
<td>-General strategy</td>
</tr>
<tr>
<td>S1: Eu pensei da seguinte maneira. Nós temos aqui um cubo.</td>
<td></td>
<td>-Reasoning</td>
</tr>
<tr>
<td>R: Continue.</td>
<td></td>
<td>-Specific strategy</td>
</tr>
<tr>
<td>S1: Um cubo. O cubo é como se fosse um... prisma. É um prisma por assim dizer. É um prisma. O cubo é um... Ok, eh...a pirâmide, esta pirâmide que está aqui dentro tem a mesma altura, e tem a mesma área da base com o cubo.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
igual a área da base vezes altura sobre 3. Dividi, deu 9 cm³. Então, no cubo podem caber 3 pirâmides congruentes, visto que o cubo de 27 cm³ de volume, uma pirâmide com a mesma área da base e mesma altura tem 9 cm³ de volume, logo cabem 3 vezes.

R: Para o exercício...
S1: Para o exercício, então... caberá no cubo mais alguma pirâmide congruente a [ABCDH]? Então [ABCDH]. Se sim apresente-a(s). Então, temos aqui [ABCDH] está aqui representado, [EFGHB] bom, eu aqui, peguei na primeira imaginação, eu inverto, vi, peguei isto como base EFGH, é a base, depois tem a altura, EFGH é a base, B... deste lado, portanto, base e o segmento que vai partir por exemplo de H até B. Então, nesse caso tem-se esta parte como altura. Da mesma forma que temos ABCD do B partimos para o H, também podemos ter EFGH, do F partimos para o B e temos a altura de um triângulo. Então observado o cubo deste lado, temos BCGF, do F podemos ir até H. (Silencio). Do F, não, não vamos para H. Do F não vamos para H, vamos até E. Do F vamos até E. Temos um triângulo a base está deste lado onde...

R: B... B... (queria perceber o raciocínio do estudante).
S1: Temos BCGF, então vai ser a base duma outra pirâmide que vai se projectar para o ponto E.

R: Estas pirâmides não se intersectam entre si ou intersectam-se? Porque o importante é como caber, né.

S1: Yah, caber, yah, não se intersectar, vamos ver, não funciona. Bom, o que eu sei, esta aqui que projeta para o H, e este do H que projeta para o B não se intersectam. Acho que não.

R: Enconstam-se apenas assim, vamos ver. Então, tem ABCDH e depois a base...

S1: A base é esta, né? Esta é base.

R: Mas pode desenhar esta pirâmide, esta aqui?

S1: Esta segunda?

R: Yah, para ver como é que fica? Talvez, yah...

S1: Então fica, este. (Silencio). Então, deste ponto... desce para aqui. Então os pontos todos... (Silencio longo).

R: Será que não se intersectam? O que é que acha?

S1: Estas duas pirâmides acho que não se intersectam.

R: Não se intersectam?

S1: Não se intersectam. Enconstam-se apenas no segmento HB. Que vai servir por uma... chamada geratriz, (uma encosta)... de D para H sobe o ponto baixo, o ponto alto. Então este ponto desce até aqui.

R: Este ponto do lado de cá, não é? Esta parte...

S1: Vem assim, deste ponto vai descendo até este ponto, então ali se encostam assim, mas não se intersectam.

R: Encostam-se nesta aresta ou enconstam-se numa face, o que acha?

S1: Eu acho que se encostam apenas... numa aresta. Não, não, encostam-se apenas numa... numa aresta isso mesmo.

R: Numa aresta ou num lado talvez?

S1: Lado não, lado não, lado não. Eu digo lado não, porque... se eu tenho... (algo imperceptível). Então B até H... (Silencio).

R: Ok, mas, ok, e qual seria a outra?
S1: Yah, (silencio). A terceira... (silencio). Não pensei nesse assunto de se encostarem, eu vi isso no volume, e não vi nas pirâmides (rindo-se) como figuras, como sólidos, aliás.
R: Aqui vem apresente-as, não é?
S1: Yah.
R: Além de serem 3 [pirâmides]...
S1: Yah.
R: É preciso saber quais são, não é?
S1: Yah. Duas...duas... duas consigo ver. Duas figuras, uma assim e outra... onde a... a maior inclinação encontra-se no segmento BH, a maior inclinação encontra-se no segmento BH. (Silencio). As outras duas... a outra que falta está separada. (Silencio).
R: Está separada, o que significa está separada.
S1: Eu vejo aqui. (Silencio). Eu vejo aqui. (Silencio).
R: Isto é uma pirâmide, não é?
S1: Yah. É de facto. É esta aqui. Esta HEGF e... e... e B, ok. Então, a outra (nunca eu apanho aqui assim). (Silencio longo).
R: Quer construir?
S1: Estou aqui para construir, depois atracejar para me facilitar a percepção. (Silencio). Aqui tenho o B, o B (querendo dizer o C), o D, o A e o H. (Silencio). Então, falta o terceiro, a terceira pirâmide. A terceira pirâmide que havia projectado ela... Assim já passa a intersectar.
R: Esta... esta...
S1: Yah, yah.
R: Intersecta qual pirâmide ou intersecta as duas?
S1: O que estou a ver era BCGE acho que vai para aqui.
R: Intersecta as outras né?
S1: Sim, intersecta. Mas só que assim como está sobra dois espaços laterais, que esses dois espaços laterais assim juntos formam uma outra pirâmide.
R: Mas como pode fazer isso?
R: Que segmento é?
S1: HB, HB. Neste segmento. Quando juntamos as duas pirâmides encontramos este segmento, então sobram espaços (à volta) vazios dentro do cubo.
R: Mas o que acha, será que haveria uma outra escolha de maneira a encontrar as 3 pirâmides bem nítidas e não se intersectam? É possível encontrar essa possibilidade? Ou acha que este é o caminho para encontrar as pirâmides.
R: Estes dois espaços vazios...
S1: Ah...
R: Podemos construir de tal maneira que possa constituir uma outra pirâmide ou...

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Awareness and organization of knowledge</th>
<th>Construction processes</th>
<th>Reasoning</th>
<th>Visualization processes</th>
</tr>
</thead>
<tbody>
<tr>
<td>h6: Draw the 3rd pyramid</td>
<td>S1 correctly solved the task using an algebraic strategy. However, he could not transfer or translate the algebraic strategy into geometric strategy as he stated he had not thought about the construction or the position of these three congruent pyramids particularly regarding the intersection of the pyramids.</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>h7: Another clearer construction strategy?</td>
<td></td>
<td></td>
<td></td>
<td>Visualization processes</td>
</tr>
<tr>
<td>h8: Can you draw or highlight the two triangular pyramids?</td>
<td></td>
<td></td>
<td></td>
<td>-</td>
</tr>
<tr>
<td>h9: Can you visualize the two triangular pyramids?</td>
<td></td>
<td></td>
<td></td>
<td>-</td>
</tr>
</tbody>
</table>
S1: Eu ainda estava a ver se os espaços vazios ficavam... ficavam duas pirâmides... triangulares parece-me.
R: Pode desenhar?
S1: Estava a tentar fazer isso mesmo. Ficavam duas pirâmides triangulares, que ao juntar... Ficavam duas pirâmides triangulares. Mas, aqui precisam duas pirâmides quadrangulares. Então, sobram duas pirâmides triangulares. Só que estas triangulares quando já juntamos formam uma pirâmide quadrangular.
R: Pode mostrar aqui, como vão ser essas pirâmides triangulares? Pode mencionar?
S1: (Silencio longo).
R: Consegue visualizar essas duas pirâmides triangulares? Disse duas pirâmides?
S1: Yah, consigo imaginar, mas eu (rindo-se) não, não estou a ver como posso verificar né, em forma de... consigo, não sei como posso explicar.
R: Pode esboçar talvez. Como é que... ou... se puder na própria figura ou aqui, pois...
S1: (Silencio). (Algo imperceptível). Tinha que arranjar uma base... Tudo eu nesta pequena imaginação que estou a tentar, partí do principio que o segmento BH, é o sótão onde as duas pirâmides se encontram. A pirâmide de cima para baixo, de baixo para cima. Elas se encontram no segmento que é, de facto eu chamo aqui, uma aresta. Encontram-se numa aresta. Elas encontram-se, deixam um espaço vago deste lado, deixam um espaço vago deste. As duas pirâmides, uma virada para baixo, a outra de baixo para cima. Há um ponto, há um ponto, que é o ponto HB onde elas se encontram, se intersectam. Se intersectam não, se encontram, enconstam-se. E aqui onde se encontram, uh, deixa aqui, se tocam assim por exemplo, deixa aqui um espaço vazio, um espaço vazio... deste... um espaço vazio que este espaço vazio tem este quadrado todo... tendo este quadrado todo já não dá para dizer que no quadrado temos o ponto E (algo imperceptível), já não dá. (Silencio). Encontram-se aqui neste ponto, encontram-se neste ponto. Encostam-se e fica aqui um vácuo, um vácuo. Consigo entender que aqui é um vácuo. E então a ideia que eu tenho, eu digo que são dois... são dois vácuos que juntados formam uma... uma outra pirâmide. Fica um vácuo deste lado e fica um outro vácuo deste lado. Duas pirâmides têm um ponto, um segmento onde se encontram. Fica um vácuo deste, fica um vácuo doutro. Esses dois vácuos, eu não consigo idealizar em forma de figuras dois vácuos. Mas eu penso que esses dois vácuos juntados formam uma pirâmide congruente às outras duas. (Silencio). .................................................................
S1: Eu defendi que existiam mais... mais duas pirâmides. Partindo do princípio do volume da pirâmide partindo do volume do... do volume cubo. Eu sustento que existe.

---

Task 3 was as follows:

3. Observe cube [ABCDEFGH] below. Can you fit any additional pyramid(s) congruent to [ABCDH] into the cube? If ‘yes’, draw it (them).]
S1 learnt a definition and used it to algebraically solve the task using the volume formulae of a cube and a pyramid. He referred to the following definition: The volume of a pyramid inserted in a right prism with the same base and altitude is as one third as the volume of that prism. (Paraphrased by me). Thereafter he numerically related the two volume formulae and concluded that at most three congruent pyramids could be inserted in the cube.

S1: (S1I:335-337) [(He reads Task 3 aloud.) Well, here I... eh... here I sketched this pyramid. I thought in this way. Maybe I’m thinking correctly or wrongly. I thought in this way.]
R: (RI:338) [What matters is your understanding.]
S1: (S1I:339) [I thought in this way. We have a cube here.]
R: (RI:340) [Continue.]
S1: (S1I:341-343) [A cube, a cube is like... a prism. It is a prism anyway. It is a prism. The cube is a... Ok, eh... a pyramid, this pyramid inserted here has the same altitude and base as the cube.]
R: (RI:344) [Continue.]
S1: (S1I:345-357) [I’ll repeat. The pyramid has the same altitude and base. So one of the definitions, I call it definition, to get the volume of... a pyramid, the volume of a pyramid from the volume of a right prism, let’s say, when we have a pyramid inserted in a right prism with the same base and altitude, then the area (volume) of the pyramid can be inserted three times into... the prism with the same altitude and base. I used this theory and (inaudible). Given BC equals to 3 cm, you can choose any edge, can’t you? Then I determined the volume of the cube. The volume of the cube equals the base multiplied by the altitude, and then the base in this case is 9 cm\(^2\) multiplied by 3 we get 27 cm\(^3\). Then the volume of the pyramid, of the pyramid equals to the base multiplied by the altitude divided by 3 and the result is 9 cm\(^3\). At most, we can insert three congruent pyramids in the cube because we have a cube with 27 cm\(^3\) and a pyramid with the same base area and altitude with 9 cm\(^3\) of its volume. Hence, at most three pyramids can be inserted.]

S1 correctly visualized and constructed a second pyramid and he described correctly its position in the cube (Fig 5.57). However, it seems he did not handle properly the geometric objects, for instance he used the concept of altitude of a triangle which was out of the context of the task. Maybe he wanted to refer to the altitude of a pyramid. Also from the excerpt of his transcript below I inferred that he did not understand the conditions of the task regarding
insertion of the pyramids in the cube. He did not understand that the pyramids must not intersect each other (S1I:374-376 and S1I:400, 401).

Figure 5.57: S1’s sketch 1 to Task 3

R: (RI:358) [What about the task itself?]
S1: (S1I:359-368) [The task itself, then... Can you fit any additional pyramid(s) congruent to [ABCDH] into the cube? Then, [ABCDH]... If ‘yes’, draw it (them). Then, we have outlined [ABCDH] here. Well, [EFGHB], here I used a sudden imagination, I inverted, I saw, I chose EFGH as the base. This is the base, thereafter there is the altitude. EFGH is the base, B... this side, hence, the base and the segment which departs from, for instance, H to B. Then, in this case one has this part as the altitude. It was the same procedure used for (the base) ABCD, we departed from B to H. Also we use the same procedure for (the base) EFGH, from F we depart to B and we have the altitude of a triangle. So visualizing the cube from this side, we have (the base) BCGF. We depart from F to H. (Silence). From F we don’t, don’t depart to H. From F we don’t depart to H, but we depart to E. From F we depart to E. We have a triangle of which base we find this side where...

R: (RI:369) [B...B... (I wanted to understand S1’s thinking).]
S1: (S1I:370,371) [We have BCGF which is the base for another pyramid of which vertex is the point E.]

R: (RI:372, 373) [Do these pyramids intersect each other or not? The issue is to insert them without intersecting each other.]

S1: (S1I:374-376) [Yah, to insert without intersecting, let’s see, it doesn’t work. Well, what I see, this one with vertex in H and the other with the vertex in B don’t intersect. I think they don’t.]

R: (RI:377) [They intersect each other. Let’s see. Then, we have ABCDH and the base...]

S1: (S1I:378) [This is the base, isn’t it? This is the base.]

R: (RI:379) [Can you sketch this pyramid, this one?]

S1: (S1I:380) [The 2nd one?]

R: (RI:381) [Yah, to see how it looks like. Maybe, yah...]

S1: (S1I:382, 383) [Then, it comes out this... (Silence). Then from this point...it comes down. Then every point...(Long silence).]

R: (RI:384) [Would they intersect? What do you think?]
S1: (S1I:385) [I think they do not intersect each other.]

R: (RI:386) [Don’t they intersect?]
S1: (S1:387-389) [They don’t. They only intersect at segment HB which serves as an edge. From D to H, we go up from a bottom point to the top point.]
R: (R:390) [Do you refer to this point from this side? This part…]
S1: (S1:391, 392) [It comes from this point down to that point, then there they touch but don’t intersect each other.]
R: (R:393) [Do they touch at this edge or they touch at a face? What do you think?]
S1: (S1:394, 395) [I think they only touch at an edge. No, no, they only touch at... at an edge. That is right.]
R: (R:396) [At an edge or at a segment?]
S1: (S1:397, 398) [Not at a segment, not at a segment, not at a segment. I say not at a segment because… if I have...(inaudible). Then B to H... (Silence).]
R: (R:399) [Ok, ok, what is another pyramid?]

S1 correctly solved the task using an algebraic strategy. However, he could not transfer or translate the algebraic strategy into geometric strategy as he stated he had not thought about the construction or the position of these three congruent pyramids particularly regarding the intersection of the pyramids. After being aware of this fact he showed some understanding. He noticed that the three pyramids he chose intersected each other. He correctly visualized two remaining parts of the cube after constructing the second pyramid EFGHB and he stated that joining those two parts; it would become the third pyramid.

S1: (S1:400, 401) [Yah, (silence). The third... (silence). I did not consider the issue of intersecting each other. I only thought about the issue of volume. I did not see the pyramids as solids in the sketch.]
R: (R:402) [Here you are: If ‘yes’, draw them. Isn’t it?]
S1: (S1:403) [Yes.]
R: (R:404) [Beyond knowing that they are three (pyramids)...]
S1: (S1:405) [Yes.]
R: (R:406) [We need to know which ones, don’t we?]
S1: (S1:407-409) [Yes, two... two I can see them. Two pyramids one lies in this way and the other... where... where segment BH has the greater inclination, where segment BH has the greater inclination (silence). The other two...the other missing (pyramid) is separated (silence).]
R: (R:410) [It is separated. What do you mean “it is separated?”]
S1: (S1:411) [I can see here... (silence). I can see here... (silence).]
R: (R:412) [This is a pyramid, isn’t it?]
S1: (S1:413, 414) [Yes, it is. Here you are. This is HEFG and... and... and B, ok. Then, the other... (long silence).]
R: (R:415) [Do you want to sketch?]
S1: (S1:416, 419) [Yes, I do. And I want to draw to facilitate my understanding. (Silence). Here I have B, the B, (he was indicating C), D, A, and H. (Silence). Then, the third is missing, the third pyramid. The third pyramid I considered previously, it... intersects (another one).]
R: (R:420) [This one...this one...]
S1: (S1:421) [Yes, yes.]
R: (R:422) [Which pyramid does it intersect? Does it intersect the two other
S1 was convinced about the correctness of his algebraic solution (analytic strategy) and the "definition" learnt at school. However, he was aware of lacking a geometrical construction proof (synthetic strategy) to corroborate his algebraic solution. S1 tried to construct the third missing pyramid. He found out that two triangular pyramids would form the third square pyramid sought. But he acknowledged that still it needed a proof of it. It is interesting that he stated that he visualized the two triangular pyramids but he could not construct them or even highlight them.

R: (RI:427) [How can you show it?]
S1: (S11:428-431) [(Silence). I cannot justify that. But I can... but I can see it. Because these two pyramids when eh... they intersect at this segment, segment AB... (silence). From this, yah... it is not this one. (Inaudible). Is this AB? No, it is not at that one. It is at this segment, sorry, at this segment.]
R: Que seamento é? (RI:432) [What segment is that?]
S1: (S11:433, 434) [HB, HB, at this segment. When we join the two pyramids they intersect at this segment. Then, two extra parts remain in the cube.]
R: (RI:435-437) [What do you think? Is there any other clear strategy where we can construct the three pyramids which would not intersect each other? Is it possible to find such strategy? Or is this the only strategy available?]
S1: (S11:438- 441) [I think that... (silence). I just hasted to solve... Well, I set out seeing that the first base was occupied by the pyramid given. This is the first base. Then, I chose the top base to get another pyramid. This base is not occupied, right? I constructed this pyramid. It is upside down. Thereafter I see two extra parts. (Silence).]
R: (RI:442) [These two extra parts...]
S1: (S11:443) [Ah...]
R: (RI:444) [Can we construct the two extra parts in such a way that we get another pyramid or...]
S1: (S11:445, 446) [I was still thinking whether the extra parts were... were two triangular pyramids, but I think they look like so.]
R: (RI:447) [Can you draw them?]
S1: (S11:448-451) [I was trying to do so. I get two triangular pyramids. When I join them together... I get two triangular pyramids. But here it is sought two (more) square pyramids. Then, two triangular pyramids remain. These two triangular pyramids when joined together I get a square pyramid.]
R: (RI:452) [Can you show me how those triangular pyramids look like? Can you name them?]
S1: (S11:453) [(Long silence.)]
R: (RI:454) [Can you visualize those triangular pyramids? Did you say two pyramids?]
S1: (S1I:455, 456) [Yes, I can imagine them, but eh (laughing) I cannot verify it in terms of... I don’t know how to explain it.]

R: (R1:457) [Maybe you can sketch them. How is it... or... if you can do it in the picture already given or here, as...]

S1: (S1I:458-476) [Silence). (Inaudible). I need to get a base... what I have been trying through my little imagination is to start from the point that segment BH is where the two pyramids intersect: the pyramid which lies upside down and the other with the vertex on top. They intersect at a segment which I name an edge. They intersect at an edge. Two extra empty parts of the cube are left around the two pyramids: the pyramid which lies upside down and the other with the vertex on top. There is a point; there is a point which is HB where they intersect, where they intersect. Not where they intersect, but where they touch each other. I cannot get it. (Silence). They intersect in this point. They intersect in this point. They touch each other and one extra part remains on this side and another one on that side. I understand that here we have an extra part. Then, my idea is that there are two... extra parts which, when joined, turn to be another... another pyramid. One extra part remains on this side and another one on that side. I cannot visualize these extra parts as a picture. However I think these two extra parts, when joined, they become another pyramid congruent to the other two.]

S1 relied on algebraic solution and definition to be sure about the existence of at most three congruent pyramids in the cube.

S1 was still convinced that into the cube could be inserted three congruent pyramids, though he was unable to visualize even construct them. He tried to sketch the pyramids but he was not able to get the solution and gave up. See his sketch in Fig 5.58.

It is interesting to see how S1 was convinced about the correctness of the definition learnt at school so as the respective algebraic solution to the problem even with difficulties in constructing and visualizing the solution geometrically. We can see reasoning prevailing against intuition in the case of S1. Algebraic approach aided S1 to be aware of the correct solution of that geometric problem.

R: (R1:477-478) [This is one strategy, but there might be another clearer one. Let’s see. This pyramid, that one...]

S1: (S1I:479, 480) [Silence). Well, we are... are limited. We can see a cube and... (Silence). (He was looking around as he were looking for something).]

R: (R1:481) [Do you want to construct a cube?]

S1: (S1I: 482) [Yes.]

R: (R1:483) [Do you need a pencil?]

S1: (S1I: 484-494) [No, thank you. I have got one. Maybe this way I can get an idea. (Inaudible). Then, this base has a segment which departs from here to this point which is segment BH. There is a segment more or less like this (silence). Well, we get a segment from here to here, another from here to here. Regarding the pyramid upside down eh... we have a segment from here to here, another from here to here, and the base is this one. This point departs from here, more
or less like the way I present it (silence). Then, this base and these points form a pyramid upside down... (silence). I see these points which depart from... (silence) divide... (inaudible). (Silence). Then... (silence) these small triangles... (inaudible).

R: (RI:495) [Continue].
S1: (RI:498) [Yah, I think you used a strategy...]
S1: (S1I: 499) [Yes.]
R: (RI:500) [But you need to prove that such extra parts form a pyramid.]
S1: (S1I: 501) [Laughing.]
R: (RI:502, 503) [But it’s possible that there will be another clearer strategy.
Now the issue is to find it.]
S1: (S1I: 504) [The other strategies... Starting from the point that there is a
triangle... a pyramid.]
R: (RI:505) [Yes, the pyramid which we spoke about.]
S1: (S1I: 506, 507) [Yes, we already had a pyramid. And we need to search for
other pyramids congruent to that one, that one.]
R: (RI:508) [Yes.]
S1: (S1I: 509) [Can you fit any additional pyramid(s) congruent to [ABCDH]
into the cube? If ‘yes’, draw it (them).]
R: (RI:510) [Thus, you firstly...]
S1: (S1I: 511, 512) [I stated that there were two more... two more pyramids. I
relied on the pyramid volume. I relied on the... cube volume. I argue that it
does exist.]
R: (RI:513) [Do you have anything else regarding this task? Or can we move
on further?]
S1: (S1I: 514) [Silence. Well, I can stop here (laughing). We can stop here.]
R: (RI:515) [Ok.]
S1: (S1I: 516) [Let’s stop here. I cannot work it out any more. Let’s stop here.]

Figure 5.58: S1’s sketch 2 to Task 3

S3 used a specific strategy (synthetic) and got two pyramids inserted in the cube. He correctly
visualized that those pyramids intersected on the edge BH only (Fig 5.59 and Fig 5.60).

S3: (S3I:283-290) [It’s possible. For example, we have a cube. A cube has six
faces. There are six faces. So the base is one of the faces. The base of the
pyramid given is one of the faces. And this face here and the base lie on one of these faces and the vertex is a point like a middle point on another face, of the opposite face. It is in one... of the vertices. So, if this one is in one of the vertices, one of the vertices, it's logic, it's reasonable too, that we can get another pyramid, but with an opposite vertex which coincides with... the vertex will be... the vertex of this pyramid will be one of the vertices of... it will coincide with one of the vertices of the base.]

R: (RI:291) [How many pyramids can be inserted in this cube?]
S3: (S3I:292) [Two pyramids can be inserted at most sharing a common edge.]
R: (RI:293, 294) [Why do you say that at most two pyramids can be inserted in the cube? Can't you insert more than two?]
S3: (S3I:295) [At most two pyramids can be inserted because, for instance, we have a cube.]

R: (RI:296) [Carry on.]
S3: (S3I:297-308) [We have a cube. (Silence). The base, the base, for instance lies on this face here. And the vertex (the pyramid vertex) lies in one of the vertices of the opposite face. It's more or less like this. And the question whether we can insert another pyramid, we can insert another pyramid. Because this pyramid occupies, how can I explain, it's not the half (of the cube), but it occupies... there is a part, for instance, if we visualize this picture in lateral perspective in relation to the pyramid, we can see a thing more or less like this of the cube. If we visualize the cube in lateral perspective, we can see the cube this way and the pyramid that way. The pyramid lies like a diagonal. Then, whether it is possible to construct another pyramid which might take half the square of this face, the face of the cube, I think we can get one...one more pyramid which takes the other half of the cube.]

R: (RI:309) [What do you mean? How many additional pyramids can be inserted in the cube?]
S3: (S3I:310) [There are two.]
R: (RI:311) [There are two. Which ones are they?]
S3: (S3I:312-314) [There... there are these two which lie... (silence). If we can insert pyramid ABCDH (the original pyramid), I think the other pyramid will be EFGHB. We can insert one more pyramid.]

Figure 5.59: S3’s sketch from the Diagnostic test
S3 succeeded, after a hint, to use H as a common vertex for two pyramids ABCDH and EABFH. Thereafter he got a third pyramid using the same principle and he realized that the three pyramids did not intersect each other. S3 exercised his intuition to visualize and construct geometric objects.

**R:** (RI:315) [Isn’t there any other pyramid that can be inserted besides these two?]

**S3:** (S3:316) [Do you want me to get another pyramid without intersecting these two?]

**R:** (RI:317) [Yes.]

**S3:** (S3:318, 319) [No, I don’t think so. (Silence). Let me see... let’s construct another, for instance, CDHGB. It intersects one... one of the previous pyramids.]

**R:** (RI:320, 321) [Well, let’s see whether these two pyramids intersect each other. ABCDH is the original pyramid. Thereafter we have EFGHB with vertex in B. Do they intersect? Sorry, they don’t...]

**S3:** (S3:322) [They only share a common edge.]

**R:** (RI:323) [What edge is that?]

**S3:** (S3:324) [It is BH.]

**R:** (RI:325, 326) [But now, do you think whether these two pyramids occupy the whole space in the cube or there is some space left?]

**S3:** (S3:327) [There is a space left. For instance, this part... (silence) AFE.]
R: (RI:328, 329) [So, there is a space left. Putting together those spaces left, can we get another pyramid?]
S3: (S3I:330-337) [It might form a pyramid. But a pyramid is a solid. For instance, there is a part left here. And we can get another part here. But we cannot get... if we put together these parts, they don’t form a solid. It can become a solid, but we know that a solid must be a compact geometric object. For instance, this solid (showing one of the artefacts of geometric solids lying on the table) sectioned apart doesn’t form a solid. I don’t know whether I understood well the matter.]
R: (RI:338, 339) [We can see that these two pyramids don’t occupy the whole cube space. So, what do you think? Can we insert one more pyramid?]
S3: (S3I:340) [(Silence.)]
R: (RI:341) [What are you thinking of?]
S3: (S3I:342, 343) [I was thinking, for instance, of this part. It becomes a picture which stretches up to this point. I don’t know whether it becomes a pyramid.]
R: (RI:344) [It is a space. It is not a superficial domain (related to the cube face).]
S3: (S3I:345) [Yes, it is a space.]
R: (RI:346, 347) [Maybe if you choose another pyramid it might work instead.]
S3: (S3I:348-350) [This part left is not even a part. It is a... a straight line. (S3 visualizes the part left as a D1 object. Meanwhile minutes ago we considered it as a D3 object and not a D2 object).]
R: (RI:351) [I did not understand. What part do you refer to?]
S3: (S3I:352, 353) [This part left. It is a part... eh. What is it? It is a line. Yes, it is a line.]
R: (RI:354) [Maybe if you drew a picture here once more.]
S3: (S3I:355) [Do you want me to draw a cube?]
R: (RI:356) [No, draw what you refer to, because you say that...]
S3: (S3I:357, 358) [(Silence). There is a part... left, this part. (Silence). The problem is this part left, this one. There is a space here. (Silence).]
R: (RI:359-361) [Let’s see. First, maybe let’s consider another face. Instead of considering this pyramid, let’s consider another face. Be for instance this one. Let’s suppose face ABFE. Can you sketch a pyramid with face ABFE?]
S3: (S3I:362) [Do you want me to rule out what I was doing?]
R: (RI:363-365) [Give up the pyramid you have chosen as we cannot clearly visualize the others. Then, let’s consider for instance face EADH. How can we...]
S3: (S3I:366) [Do you mean a pyramid which does not intersect the original pyramid?]
R: (RI:367) [Yes, those pyramids which can be inserted in the cube without intersecting each other. Can you check whether...]
S3: (S3I:368) [(He was silently trying using face ABFE).]
R: (RI:369) [No, use vertex in H instead of D.]
S3: (S3I:370) [I don't understand.]
R: (RI:371) [Choose ABFE or AEFB as a pyramid base and H as a vertex.]
S3: (S3I:372) [AEFB... (He was trying to sketch the pyramid).]
R: (RI:373) [What do you think about that pyramid?]
S3: (S3I:374) [It’s possible.]
R: (RI:375) [What is possible?]
S3: (S3I:376, 377) [It’s possible to get another pyramid which doesn’t intersect the other, which doesn’t intersect the other.]
R: Qual é a posição relativa entre as pirâmides? (RI:378) [What is the position between these pyramids?]
S3: (S3I:379) [They have a common face.]
R: (RI:380) [What face is that?]
S3: (S3I:381) [ABH.]
R: (RI:382) [How many pyramids have you already inserted so far?]
S3: (S3I:383) [Two, two pyramids.]
R: (RI:384) [What space is left?]
S3: (S3I:385) [(Silence.) This part here...we can get another pyramid with base FBCG and vertex in H.]
R: (RI:386) [Do these pyramids intersect each other?]
S3: (S3I:387) [No, no.]
R: (RI:388) [Is all the space of the cube occupied? Or is there any part left?]
S3: (S3I:389) [There is a little space left.]
R: (RI:390) [There is a little space left...]
S3: (S3I:391) [(Reflecting silently). No, nothing is left.]
R: (RI:392) [Nothing is left.]
S3: Nada. (S3I:393) [Nothing.]
R: (RI:394) [Then, how many pyramids can you insert at most?]
S3: (S3I:395) [Three.]
R: (RI:396) [Three pyramids. Are such pyramids congruent?]
S3: (S3I:397) [No.]
R: (RI:398) [Aren’t they congruent?]
S3: (S3I:399) [They are.]
R: (RI:400) [Why are they congruent?]
S3: (S3I:401) [They are, because of the (properties) of a cube.]
R: (RI:402) [Because of the (properties) of a cube?]
S3: (S3I:403, 404) [Because of the (properties) of the cube and the vertex is... the vertex of the pyramids is a common point. (He referred to common vertex H.)]
R: (RI:405, 406) [Now, which pyramids did you find that are congruent to pyramid given and that can well be inserted and occupy the whole space?]
S3: (S3I:407) [EABFH. (Silence).]
R: (RI:408) [And so?]
S3: (S3I:409) [FBCGH.]
R: (RI:410) [What is the rule you found to get the solution?]
S3: (S3I:411) [All pyramids converge to a common point. (Silence).]
R: (RI:412) [And what else?]
S3: (S3I:413-415) [They converge to a common point. (Silence). The edges... there are common edges too. For instance edge BH is common for the three... three pyramids. (Silence).]

S3 gave a wrong pyramid formula; hence he found a contradiction between analytic and synthetic approaches. He tried to solve the contradiction by saying that the sum of the volumes of the two newly constructed pyramids was equal the volume of the pyramid given in the task. It seemed that S3 departmentalized his knowledge to the extent to not see the
relationship between the two approaches even with a hint provided by the interviewer.

R: (RI:416) [Have you ever learnt about the volume formula of a pyramid and that of a cube?]
S3: (S3:417) [The volume formula? Yes, I have.]
R: (RI:418) [What is the volume formula of a cube?]
S3: (S3:419, 420) [The cube volume is the base area multiplied by its altitude. Then, the cube volume is \( a^3 \) where; “\( a \)” is the length of the cube edge.]
R: (RI:421) [And what is the volume formula of the pyramid?]
S3: (S3:422, 423) [It is... the base area multiplied by the altitude divided by two. In this case the altitude is the length of the edge.]
R: (RI:424) [What is the relation between the cube and the pyramid volume?]
S3: (S3:425, 426) [Both volumes have the same form, ie the base area... the base area multiplied by the altitude. Besides for the pyramid volume we must divide by two.]
R: (RI:427) [How many pyramids can be inserted in the cube?]
S3: (S3:428) [Two.]
R: (RI:429) [What was our conclusion using the geometrical approach?]
S3: (S3:430) [At most three pyramids could be inserted.]
R: (RI:431) [Is there any contradiction?]
S3: (S3:432-434) [There is. I think there is a contradiction because the volume of... of... the volume of the original pyramid is equal to the sum of the volumes of the other two.]
R: (RI:435) [Do you mean that the pyramids are not congruent?]
S3: (S3:436) [No, there are not.]
R: (RI:437, 438) [You said they are not congruent. Why aren’t they congruent? Firstly you said the three pyramids were congruent and now you changed your mind.]
S3: (S3:439) [Sorry, they are congruent, they are congruent.]
R: (RI:440) [The three pyramids are congruent. Are their volumes equal?]
S3: (S3:441) [Yes, they are.]
R: (RI:442) [Now, how come there is a contradiction with the formulae?]
S3: (S3:443-446) [I think there is a contradiction with the formulae because I determined them considering the vertex of the pyramid lying in the middle of the opposite face. If we considered... this base area and its vertex in the middle point, I think we would get this formula.]
R: (RI:447) [What formula?]
S3: (S3:448) [The volume is equal the base area multiplied by the altitude divided by two.]
R: (RI:449, 450) [Do you mean that this volume formula of a pyramid only works when its vertex is in the middle of the face and it doesn’t work for all cases?]
S3: (S3:451) [I think so, because if it worked there would not be any contradiction.]
R: (RI:452) [How did you learn the volume formula of a pyramid at school?]
S3: (S3:453) [It was like this, but I learnt it long time ago.]
R: (RI:454) [Instead of two it was not three?]
S3: (S3:455) [(Silence.) I think it was two. I don’t remember, but I think it was two.]
S4 used a synthetic approach to get at most four pyramids inserted into the cube. When he was aware that the pyramids must not intersect each other, he changed saying that at most two pyramids could be inserted into the cube. He floated between the two solutions above and finally he decided to stick to the latter solution after some reflection through visualization and construction processes. He asserted he could not translate his synthetic approach into the language of "letters". It seemed that he could visualize the geometric objects but he could not translate it analytically using algebraic symbols (e.g. acronyms). During interview he sketched a picture (Fig 5.61) to help him visualize better according to his own words. He correctly visualized the position of two pyramids ABFEG and ABFEG he considered. They only intersect in segments HB and AG. Besides, S4 recognized that his algebraic knowledge was far better than his geometrical knowledge. That fact was due to the teaching process he underwent during his schooling time. Algebra was taught well different from geometry.

S4: (S4I:347-358) [I don’t know whether I did Task 3. Then... Task 3 is a nice task but very difficult. If it were designed for assessment purpose I would get a low mark (laughing). It is a geometric task and I believe that eh... I recognize, I’m... not good at geometry. I don’t know why. I have not been exploring enough geometry because maybe I haven’t had teachers who motivated me to enjoy this subject. I think this factor also contributes to...I enjoy other areas of mathematics mainly algebra because my teachers motivated us and taught us this subject well. Differently it happened with geometry. The teachers overlooked geometry topics. That is why I recognize I don’t enjoy geometry. My skills at geometry are poor because I didn’t explore it enough. But as to this task I said that... the task reads: Can you fit any additional pyramid(s)...]  
R: (RI:359) [Can you read the whole text from the beginning to get the global idea?]  
S4: (S4I:360-362) [(He reads the entire text). Well, I presented some possible cubes (meaning pyramids) (acronyms). Congruent pyramids must be of the same shape and size.]  
R: (RI:363) [Can you explain how you solved this task?]  
S4: (S4I:364-367) [Yes, besides the original pyramid we can get three more
pyramids which are ABEF... ABE... AB... Where is it? ABEFC, that is, it's a pyramid with base ABEF and with the center (vertex) in C.

R: (RI:368) [Carry on.]

S4: (S4I:369-370) [And the other is AB... ABCD congruent to this one but with center, with vertex C (meaning E).]

R: (RI:371) [Carry on.]

S4: (S4I:372-376) [Another would be EFGH, but with center B. That is, eh, one pyramid has its vertex on this side and with a base. The other has its vertex on the other side and with a base. I believe... this was my understanding. That is, at most four congruent pyramids can be inserted in the cube.]

R: (RI:377) [Do they intersect each other?]  

S4: (S4I:378-388) [Well, maybe because of their geometric shapes I believe at most... Because when we represent... there is this one. It's clear that there cannot be any perpendicular to this one. Because what it's going on is that I didn’t analyze the task properly. I can see in real situation. Suppose that this cube were a container with water inside, a container with water inside. I insert inside eh... an object with a shape, with a shape of a pyramid, with a shape of a pyramid. Then, it’s an object eh... it will take some space, won’t it? What is happening here is that the position of this pyramid impedes another pyramid to be inserted. That is why; at most there can be only one more pyramid which can be inserted without leaving an empty space in the middle and with the same base. It just touches it and fits in the cube. It would be more or less like this.]

R: (RI:389) [So, how many pyramids can be inserted at most in the cube?]  

S4: (S4I:390) [At most two pyramids.]

R: (RI:391) [What about your previous solution? (I was pointing out his written solution).]

S4: (S4I:392-394) [I withdraw, I withdraw, I withdraw my previous solution because the answer to the question 'can you fit any additional pyramid(s) congruent to ABCDH' is yes. But under the condition, under the condition that the pyramids mustn’t intersect, the answer is one more pyramid could be inserted only.]

R: (RI:395) [It's to occupy the cube space... ]  

S4: (S4I:396) [Entirely.]

R: (RI:397) [Yes, it's to occupy the cube space entirely.]

S4: (S4I:398) [Yes.]

R: (RI:399) [Then, how many pyramids can be inserted at most?]  

S4: (S4I:400, 401) [This pyramid occupied this space in this way... No, I hold my previous idea.]

R: (RI:402) [What idea?]  

S4: (S4I:403) [The idea of the three pyramids.]

R: (RI:404) [Then the total number of the pyramids would be... ]  

S4: (S4I:405) [Including the original one they make four.]

R: (RI:406, 407) [You mean that four pyramids can be inserted in the cube without intersecting each other and take the whole space.]

S4: (S4I:408-417) [This pyramid leaves a space here, another there, and there. Perhaps, I cannot represent it in terms of letters. But this cube leaves... from this edge some space and from this edge another space. And these spaces need... to be occupied. I don’t know how to represent it in terms of letters to show them clearly how they look like. But I assure that they do exist, they do exist. These spaces do exist. These spaces must be occupied so that the space...  


R: (R1:418) [Following your reasoning, how many pyramids can be inserted in that existing space?]
S4: (S4:419) [Three.]
R: (R1:420) [Three.]
S4: (S4:421, 422) [This would be an edge, this part here. Our cube is this one. It leaves a space here from this edge.]
R: (R1:423-425) [How can you represent that space as a pyramid? You stated that three more pyramids could be inserted in the cube without intersecting and occupying the entire cube space. You chose these pyramids. Do you think that these pyramids satisfy those conditions?]
S4: (S4:426-428) [No, no, these pyramids don’t satisfy those conditions. If I chose this base I would choose this center (vertex)... and this base with this center... umh...]
R: (R1:429) [Can you sketch it?]
S4: (S4:430). [Yes, it’s a matter of visualizing better. (He starts sketching while uttering some words low.)]
R: (R1:431) [You sketching a cube, aren’t you?]
S4: (S4:432). [Yes. (He was sketching the cube and the pyramids.)]
R: (R1:433) [You are sketching a pyramid. What is its base and vertex?]
S4: (S4:434-441). [This is the vertex and that one is the base. (He was referring to pyramid ABFEG.) Considering that the other base was occupied by the original pyramid (referring to base ABCD), this pyramid (ABFEG) tangentially touches here. And it is inclined like this (silence). I think they are congruent. Ia, this ends up here where it is the vertex of the other pyramid (vertex G). Thinking carefully, I believe it ends up here.]
R: (R1:442) [Then, how many pyramids can be inserted in the cube?]
S4: (S4:443) [In terms of... (Laughing). One, no more pyramids can be inserted.]
R: (R1:444) [What is the total number of the pyramids?]
S4: (S4:445) [Two, two.]
After a hint S4 worked with the volume formulae of a cube and a pyramid. He forgot the volume formula of a pyramid. Thereafter he was reminded the correct formula of a pyramid. Meanwhile when asked to relate the volumes of the two solids he used his previous solution to support that relationship: the volume of the cube is as double as the volume of a pyramid inserted in it. That is at most two pyramids can be inserted in the cube. And he represented it in an identity. He transformed it and resulted in a wrong relationship $2a^3 = 3a^3$. He handled it as it were an equation and ended up with $a = 0$ (Fig 5.62). He got another hint to relate the two volume formulae through algebraic transformations only. S4 wanted some picture support to do that and asked whether he should consider a pyramid inserted in the cube. He did some algebraic transformations which confused him. He got other hints that the altitude of the pyramid should be replaced by the edge of the cube in an appropriate transformation. Thereafter he replaced it and ended up with the right relation: the volume of the cube is as three times as the volume of the pyramid after some hinting (Fig 5.63). Accordingly he concluded that at most three congruent pyramids can be inserted in the cube. It is to note that S4 misspoke some passages of his written solution during the interviewing.

**R:** (RI:446) [Do you know the volume formulae for a pyramid and for a cube?]

**S4:** (S4:447-451) [I don’t remember clearly, but I think the cube volume is $a^3$ where; “a” is the cube edge. Ahhh, the pyramid volume... the pyramid volume... the pyramid volume... Ahhh, the pyramid volume... I think, I don’t know. I think it is the base area, ahhh, multiplied by altitude... I’m not... divided by two. I think this is the volume formula of the pyramid, but I’m not sure about it... but, I’m sure about the cube volume.]

**R:** (RI:452) [The volume formula of a pyramid...]

**S4:** (S4:453) [I think it is this one, but I’m not sure.]

**R:** (RI:454) [Have you learnt this formula?]

**S4:** (S4:455) [Yes, I have. I have learnt it surely.]

**R:** (RI:456) [Haven’t you learnt it divided by three?]

**S4:** (S4:457-460) [Divided by three, divided by three, divided by three, ...(thoughtful). Yeah, my thought is floating between a formula divided by three and another by two. I’m not... I just know that... there is a formula divided by two. A triangle has a formula divided by two (weak memory). I don’t know, I don’t... know, I don’t know but it’s trustingly.]

**R:** (RI:461) [Why is it trustingly?]

**S4:** (S4:462-468) [The reason is that I saw a deduction of... of... ahhh... the volume formula of a pyramid in a textbook. There I found the formula divided by three. I think that textbook contains such deduction. That is why I said that it’s trustingly. Maybe I’m wrong. I don’t trust that two appears in the pyramid volume formula. I think it appears in the triangle formula, in the triangle formula, not in the pyramid... (A hint helped him to remember a dormant knowledge.) I think the cube, the pyramid, ehhh... cone, and... pyramid volume formulae, I think they are divided by three.]
R: (RI:469) [Now, let’s suppose it is divided by three. How would you relate the two volumes?]
S4: (S4:470-473) [How can I relate the two volumes? Ehhh, considering that they are... well, if I say that the base area, I’m referring to this base. And this base is formed by an edge, isn’t it? Ehhh, if I... if I... if I consider, if I consider that there are two pyramids inside, isn’t it?]
R: (RI:474) [Carry on.]
S4: (S4:475-491) [If I consider two pyramids, then it means that the cube volume is as double as the volume of the pyramid, if I consider two pyramids. Because it would be the sum of the two pyramids volumes to get the cube volume. Then, ehhh... If I reason like that, it means that I can replace here the respective expression and here as well, if I consider two pyramids. Now if I consider one pyramid only...and relate, but I’m not sure whether it’s only one. But I think at most we can get one pyramid. Sorry, at most we can get two pyramids because we already have one pyramid. At most we get two pyramids. If we consider that there are two pyramids at most, then we would have
\[
\frac{2A_s h}{3} = a^3
\]
where; \(A_s\) is the base area, \(h\) is the altitude of the pyramid, and \(a^3\) is the cube volume. There is an important detail to consider. The altitude of the pyramid is equal the cube edge as the altitude is perpendicular to the base. If the altitude is perpendicular to the base, it must be equal to the cube edge. Besides the base area is...is... a square. Then we have... These transformations don’t help me much.]
R: Porque? (RI:492) [Why?]
S4: Ehhh... porque o que vou ter será aresta nula. (S4I:493) [Ehhh... because I get the edge equals to zero.]
R: Nula porque? (RI:494) [Why is the edge equal to zero?]
S4: Por um simples motivo. Se eu disser três a ao cubo menos dois a ao cubo são polinómios do mesmo grau, podem-se subtrair, então seria a ao quadrado igual a zero. (S4I:495, 496) [For a simple reason, I have $3a^3 - 2a^3$. They are two polynomial expressions with the same degree. I can subtract and I get $a^2 = 0$. (It must be $a^3 = 0$.)]
R: E a outra solução? (RI:497) [What is the other solution?]
S4: É que todas são as mesmas. Mais ou menos zero, então isso é zero. E este resultado não me estimula, não me estimula muito (rindo-se). Talvez eu esteja a cometer algum erro. Mas eu tenho quase a certeza de que a altura é igual. (S4I:498-500) [All solutions are the same, zero. This result doesn’t satisfy me, it doesn’t satisfy me much (laughing). May be I’m wrong. But I’m sure that the altitude is equal to the edge.]
R: Vamos partir das fórmulas, o sr partiu da figura e diz que o volume do cubo é duas vezes o volume da pirâmide. (RI:501, 502) [Let’s depart from the formulae. You used the picture and stated that the cube volume is as double as the volume of the pyramid.]
S4: Sim. (S4I:503) [Yes.]
R: Agora a partir destas fórmulas, o volume do cubo e o volume da pirâmide. A partir daqui vamos relacionar as duas fórmulas. (RI:504-505) [Using these formulae, the cube and the pyramid volumes, let’s relate the two formulae.]
S4: Há uma coisa que preciso perceber. Ao relacionar isto eu tenho que considerar exactamente uma pirâmide inscrita aqui dentro? (S4I:506, 507) [There is something I want to understand. Do I need to consider a pyramid inserted in the cube in order to relate the two formulae?]
R: Esta pirâmide. (RI:508) [Consider this pyramid.]
S4: Esta pirâmide. Ok, tá bom. Ehhh... atendendo e considerando pelo menos algo em que estou seguro é que a base é... é... um cubo não é? É um quadrado. Então, vou dizer que a área da base é a aresta ao quadrado. Então, ehhh... sendo aresta, vezes aresta, então teria aqui o volume da pirâmide é a ao quadrado sobre três, não é? Ehhh... então se eu considerar, se eu considerar essas duas expressões para as poder relacionar, ehhh... teria o que é? Teria o volume da pirâmide, o volume da pirâmide seria... podia ter três vezes o volume da pirâmide igual a a ao cubo vezes aresta. A aresta ao cubo seria mais ou menos igual a quê? Tenho que relacionar o volume da pirâmide (VP) com o volume do cubo (VC). Então seria o mesmo que ter o que é? Três vezes o VP ehhh... sobre a altura da pirâmide. Então teria ainda ehhh... posso muito bem dizer que .... Eu posso ainda calcular a raiz, que seria $a = \sqrt[3]{\frac{3VP}{h}}$. Então de acordo com esta expressão posso vir substituir aqui. Então o volume VP é da pirâmide. Então este é do cubo (referindo-se ao VC). VC seria a ao cubo. Então

$$a = \sqrt[3]{\frac{3VP}{h}}$$

isto é $a = \pm \sqrt[3]{\frac{3VP}{h}}$. Então eleva-se esta expressão ao cubo. Esta expressão seria... teria... mais ou menos o que é? (Segue-se as seguintes transformações algébricas):

$$VC = \left(\pm \sqrt[3]{\frac{3VP}{h}}\right)^3 = \left(\frac{3VP}{h}\right)^2 \cdot \sqrt[3]{\frac{3VP}{h}} = \frac{3VP}{h} \cdot \sqrt[3]{\frac{3VP}{h}}.$$ (S4I:509-524) [This pyramid, ok that’s fine. Ehhh... at least what I am sure of is that the base is... is... a cube, isn’t it? It’s a square. Then the base area is the edge squared. Then, ehhh... it’s the edge, (we multiply) by the edge. I would get for the pyramid volume, the pyramid volume would be a squared divided by three. Ehhh... if I consider, if I consider the two expressions to relate, ehhh... what would I get? I would get the pyramid volume... the pyramid volume would be... I would get three times the volume of a pyramid is equal $a^3$ multiplied by the edge. What would the edge squared be equal? I need to relate the pyramid volume to the cube volume. Then, what would I get? Three multiplied by PV ehhh... divided by the pyramid altitude. Then I would get ehhh... I can say that...I can get $a = \sqrt[3]{\frac{3VP}{h}}$. Then, I replace this expression here. The PV is the pyramid volume and CV is the cube volume. CV equals to $a^3$. Then, we get $a = \sqrt[3]{\frac{3VP}{h}}$, that is $a = \pm \sqrt[3]{\frac{3VP}{h}}$. (He replaces this expression in the cube volume formula and he gets the following algebraic transformation:}
$VC = \left( \pm \sqrt[3]{\frac{3VP}{h}} \right)^2 = \pm \sqrt{\left( \frac{3VP}{h} \right)^2} = \sqrt{\left( \frac{3VP}{h} \right)^2}.$

R: Qual é a conclusão que tira? Qual é a relação entre o VP e o VC? (RI:525) [What is your conclusion? What is the relation between PV and CV?]

S4: A relação que existe é três vezes, não é, ao VP, três vezes ao VP. Embora não seja uma relação concisa, uma vez que eu tenho aqui uma raiz, em termos de expressões talvez seja esta. Mais ou menos o cubo, não é?... (Aparentava estar confuso após essas transformações algébricas). (S4I:526-529) [The relation is three times PV, three times PV. Although these algebraic transformations don’t state that clearly as I got here a root. Maybe is this expression. (It seemed that S4 got confused after accomplishing the algebraic transformations in Fig 5.63).]

R: (RI:530, 531) [Where you get the altitude... What do you replace the altitude with?]

S4: (S4I:532) [The altitude?] 

R: (RI:533) [The pyramid altitude.]
S4: (S4:534) [Ahhh... With the edge.]
R: (R:535) [I think in order to get fewer variables you need to replace the altitude \( h \) from here...]
S4: (S4:536) [From where?]
R: (R:537) [From this relation.]
S4: (S4:538) [Three multiplied by \( PV \) equals to \( a^3 \) multiplied by \( a \). Then I get \( PV \) equals to \( a^3 \).]
R: (R:539) [What does \( a^3 \) mean?]
S4: (S4:540) [Three multiplied by \( PV \) equals to the cube volume.]
R: (R:541) [Then, what is the relation?]
S4: (S4:542, 543) [\( CV \) is the sum of \( PV \), \( PV \), and \( PV \), which are the three pyramids. (He writes down \( VC = VP + VP + VP \).)]
R: (R:544) [Then, how many pyramids can be inserted?]
S4: (S4:545) [There is already a pyramid, and then we can additionally insert two more pyramids.]
R: (R:546) [What is your conclusion?]
S4: (S4:547) [We can additionally insert two more pyramids.]

It seemed that an algebraic approach (analytic strategy) was clearer and more convincing than a geometric approach (synthetic strategy) to S4. He considered the algebraic approach clearer, obvious, and scientific. He visualized and constructed pyramid ABFEG which he correctly saw that it did not intersect the original one. It only touched it in its edge. Meanwhile any third pyramid would intersect one of those as there is only one construction solution to Task 3, namely the all three pyramids should have the same vertex at point H. The algebraic solution elicited S4’s awareness about the existence of at most three pyramids inserted in the cube, although he was unable to construct them.

R: (R:548) [Why have you changed your mind? Firstly you said that there were four. Thereafter you said...]
S4: (S4:549) [One pyramid, I have changed my mind because of this systematic deduction which makes sense...]
R: (R:550) [Why does it make sense?]
S4: (S4:551-553) [It makes sense because the altitude is in fact equals to the edge. The deduction also shows another thing. \( CV \) is three times \( PV \). It means that there are three pyramids inserted in the cube. This is an obvious and clear proof.]
R: (R:554, 555) [Now, how can you represent the three pyramids here? You represented one pyramid and we got the original one. Where does the other pyramid lie?]
S4: (S4:556) [(Reflecting silently.) This one is the original and I got this one... there is some space left.]
R: (R:557) [Have you found the third pyramid?]
S4: (S4:558, 559) [I haven’t found the third one yet. But I think the third one... (silence) can just lie here with its centre... in G. Its base would be this one, isn’t it?]
R: (R:560) [Continue.]
S4: (S4:561-562) [Its base would be ABEF with the center in G. It would not be here because it could intersect the other. It’s like that.]

R: (R:563) [Doesn’t it intersect the others, does it?]

S4: (S4:564) [I believe it touches them tangentially, not in terms of intersecting the others. I think it only touches.]

R: (R:565) [Let’s see the first pyramid. What is it? Here we are.]

S4: (S4:566) [The second, the second is here, it’s here. This is its base and the center.]

R: (R:567) [Does it intersect the first pyramid?]

S4: (S4:568) [The second one?]

R: (R:569) [Yes.]

S4: (S4:570) [No! It touches it tangentially.]

R: (R:571) [Where does it touch it?]

S4: (S4:572, 573) [It touches more or less in these edges, in these lines. It doesn’t intersect, it touches. Because the original pyramid lies on the corner of the cube, it leaves a space. And the second pyramid crosses over the original one (paraphrased).]

R: (R:574) [And the third pyramid?]

S4: (S4:575-577) [The third one is this. Here you are. Its centre is this. The first two... two pyramids don’t take the entire space of the cube. They leave some space. This one leaves a space here and the other there. It’s my opinion (laughing).]

R: (R:578, 579) [The intention was to explore certain issues. What did you think about this task?]

S4: (S4:580-588) [Well, this task required a lot of abstraction. It’s not easy; it’s not easy to think without algebraic expressions support. It’s not easy. My view is that it’s that it’s easy to be mislead when you think without calculation support. If we visualize we can see that here we get a space, another space there, and we can naturally realize that in each space we can occupy at least with a unit of a cube. However, when we use calculations, we get at most three pyramids inserted. I mean firstly we needed abstraction abilities to take this diagnostic test. We needed certainly imagination also, to be able to imagine, but based in scientific support. (Algebra aiding geometry).]

R: (R:589) [What do you mean by scientific support?]

S4: (S4:590-595) [It does not suffice just to think. It’s necessary to have a purpose, to have a scientific knowledge or in other words to make relation between things. For example, to get to know about congruency you need to know its properties. If we deal with certain stuff, then we should make relations among the entities of that stuff. We should make relations among them. This is the support that, at that time, I did not possess or I could not imagine that such support existed. (Shortage of key concepts).]

R: (R:596) [But I can see that you brought...]

S4: (S4:597) [Maybe, maybe I didn’t know that I brought some knowledge (laughing).]

R: (R:598) [Basically who was doing the calculations was you and not me.]

S4: (S4:597) [This is the famous theory of some educators which argue that one is born with knowledge. The role of the teacher is only to elicit that one’s knowledge so that s/he can be aware of.]

S5 used visualization processes and concluded 8 pyramids could be inserted in the cube. He
justified that each face of the cube could be a pyramid base. When asked to count the cube faces he got 6 faces. It seemed that he did not understand the task. When I explained to him that the pyramids must not intersect each other, he stated that only one pyramid could be inserted in the cube and any other inserted would intersect the original one.

S5: (S5I:257, 258) [(He reads the task). I saw that this pyramid lies on this base, this base here. I counted... the cube faces.]
R: (RI:259) [Continue.]
S5: (S5I:260, 264) [I saw that a cube could lie on each face. For instance, a cube can lie on this face. This would be its base. Sorry, it’s not a cube, but a pyramid. It could be a pyramid base. The other face could also be a pyramid face. Each face could be a pyramid face. (S5 did not understand the task. Was it due to the wording of the task?)]
R: (RI:265) [How many pyramids are there?]
S5: (S5I:266) [(He counts the faces of the cube and finds out that they are 8). Ah, there are 8 (laughing).]
R: (RI:267) [There are 8. What?]
S5: (S5I:268) [There are 8 faces. Then, excluding the original pyramid we get 7.]
R: (RI:269) [Does a cube have 8 faces?]
S5: (S5I:270) [(Silence).]
R: (RI:271) [Let’s count once again.]
S5: (S5I:272) [Maybe.]
R: (RI:273) [Show me the faces of the cube.]
S5: (S5I:274) [For example, BCGF.]
R: (RI:275) [One face.]
S5: (S5I:276) [BAEF.]
R: (RI:277) [Two faces.]
S5: (S5I:278) [This face up EFGH.]
R: (RI:279) [Third face.]
S5: (S5I:280) [CDHG.]
R: (RI:281) [Another.]
S5: (S5I:282) [I can have ABCD.]
R: (RI:283) [Another.]
S5: (S5I:284) [I don’t know if I counted ADHE.]
R: (RI:285) [Not yet. Then, how many are there?]
S5: (S5I:286) [(Silence). There... There are 6 faces.]
R: (RI:287) [Where does eight come from?]
S5: (S5I:288) [(Silence).]
R: (RI:289) [How many pyramids did you construct on each face?]
S5: (S5I:290) [On each face I constructed only one pyramid. We can construct a pyramid base on each face.]
R: (RI:291) [How many pyramids can you construct on each face?]
S5: (S5I:292) [Only one.]
R: (RI:293) [What comes next?]
S5: (S5I:294, 295) [(He counts once again the number of the faces). I... I mean, I took into account for example this side... ABCD. This one replaces the base of this pyramid.]
R: (RI:296) [Carry on.]
S5: (S5I:297) [It means each of the faces can serve as the base.]
R: (RI:298) [Carry on.]
S5: (S5I:299) [It serves as the base.]
R: (RI:300, 301) [But the issue here is we want to know how many pyramids can be inserted in this cube. It’s not just a matter of constructing the pyramids.]
S5: (S5I:302) [It has to do with uhhhh…]
R: (RI:303) [It’s to occupy the space fully.]
S5: (S5I:304-307) [Yes, yes, It’s not... it’s not to be occupied only by... by this pyramid, but also by another pyramid. There mustn’t be ...let’s suppose, this pyramid of which base is this one. It means that this point would... would... would intersect this pyramid.]
R: (RI:308, 309) [The purpose is to know how many pyramids congruent to this one can be additionally inserted in the cube without intersecting each other.]
S5: (S5I:310) [I see.]
R: (RI:311) [It’s a sort of inserting. How many pyramids can be inserted inside the cube?]
S5: (S5I:312) [Under that condition I cannot see any other possibility.]
R: (RI:313) [What difficulty are you facing?]
S5: (S5I:314, 315) [Because if I... if I considered this... this face as base, thus to get this point here, there would not be any intersection. If I consider this other face it will be the same.]
R: (RI:316) [What is your conclusion? How many pyramids can be inserted in this cube?]
S5: (S5I:317) [Only one pyramid can be inserted.]
R: (RI:318) [Which one?]
S5: (S5I:319) [This one, this one, the original.]
R: (RI:320) [Why?]
S5: (S5I:321, 322) [If you take any other face as a pyramid base, the pyramids will always intersect each other.]
R: (RI:323) [Why have you drawn such conclusion? Have you tried all possible pyramids?]
S5: (S5I:324, 325) [Silence. I cannot grasp that. How is it possible to construct any other pyramid which doesn’t intersect the original? I cannot grasp that.]

After the hint on the volume, S5 got the volume formulae of the cube and of the pyramid with some interviewer’s help and he correctly related the volume of a cube and a pyramid inserted in it. Thereafter, he correctly stated that at most three congruent pyramids could be inserted in the cube. It seemed that algebraic approach helped S5 to envisage the correct solution of the task.

R: (RI:326) [Have you related the pyramid volume to the cube volume?]
S5: (S5I:327) [No.]
R: (RI:328) [Do you know the formulae?]
S5: (S5I:329) [The cube volume is the edge raised by three. The pyramid volume is the base area multiplied by its altitude.]
R: (RI:330, 331) [Is the pyramid volume the base area multiplied by its altitude? Haven’t you missed anything in the formula?]
S5: (S5I:332) [It’s divided by three.]
R: (RI:333) [Why divided by three?]
S5: (S5I:334) [Silence.]
R: (RI:335) [Did you learn the pyramid volume at school?]
S5: (S5I:336) [I have forgotten it. I can go and have a look.]
R: (RI:337) [It’s divided by three, it’s divided by three. How can you proceed based on this?]
S5: (S5I:338, 339) [(Silence.) The cube volume is the edge raised by three. The pyramid volume is the base area multiplied by its altitude divided by three. (Silence.)]
R: (RI:340) [What is the base area of this pyramid?]
S5: (S5I:341) [The base area of this pyramid is a square.]
R: (RI:342) [It’s a square. Then, it follows...]
S5: (S5I:343) [The square area is \(l^2\). Then we get \(VP = \frac{l^2h}{3}\).]
R: (RI:344) [What does \(l\) stand for?]
S5: (S5I:345) [The square area.]
R: (RI:346) [No, I’m referring to the side \(l\).]
S5: (S5I:347) [Do you mean the side of the base area?]
R: (RI:348) [It’s the edge of the cube.]
S5: (S5I:349) [The cube edge.]
R: (RI:350) [Then, instead of using \(l\) you can replace it with...]
S5: (S5I:351) [(Reflecting). The edge... the cube edge, it’s not the cube volume!]
R: (RI:352) [Here you are \(l^2\). Then what comes next?]
S5: (S5I:353) [It’s \(a^3\).]
R: (RI:354) [Then, what stands for the altitude of the pyramid?]
S5: (S5I:355) [The pyramid altitude... is equal the length of the square side.]
R: (RI:356) [Then, instead of \(h\) I can replace it by...]
S5: (S5I:357) [\(a\).]
R: (RI:358) [Then, it follows...]
S5: (S5I:359) \[VP = \frac{a^3}{3}\].
R: (RI:360) [What is the relation you can draw from the volume of the cube and of the pyramid?]
S5: (S5I:361) [It means that the pyramid volume is one third of the cube volume.]
R: (RI:362) [How many pyramids of the original type can be inserted in the cube?]
S5: (S5I:363) [3.]

S5 faced difficulties in constructing the three congruent pyramids. He said that algebraically it was easy to see that at most three congruent pyramids could be inserted because it was a logic relation but by visualization it seemed impossible to construct them without intersecting each other. After the interviewer's hint he correctly constructed the three pyramids.
Meanwhile he could not visualize and explain why they did not intersect each other. It is interesting that S5 used the same principle the interviewer suggested before to complete the construction task correctly (Fig 5.64). To S5 logic deductive reasoning was a key to envisage the correct answer as through intuition was difficult to visualize.

Figure 5.64: S5’s sketch to Task 3

R: (RI:364) [Which ones? How many are those?]
S5: (S5I:365) [3 pyramids.]
R: (RI:366, 367) [Is it possible to represent these pyramids in the cube? We already have one. And what are the others? They mustn’t intersect.]
S5: (S5I:368) [(Silence.)]
R: (RI:369) [Using the volume formulae is it possible to have 3 non-intersecting pyramids?]
S5: (S5I:370-373) [Using the volume formulae it’s possible. Alias, it’s a reasonable relation, isn’t it? (Laughing). Using visualization it seems impossible. But I have, for instance, this... for instance ABFEG. (Silence). (Algebra served as an aid for S5 to clarify a geometric solution which through synthetic approach would be difficult to visualize).]
R: (RI:374-376) [It seems there is a contradiction between the algebraic approach and the geometric approach as you said algebraically there were at most 3 pyramids inserted in the cube, but geometrically it seemed impossible to get them.]
S5: (S5I:377-379) [There might be a contradiction, but I cannot... I cannot grasp, however, the (algebraic) deduction is logic. But I cannot draw them. (He accepts the possibility of existing a contradiction because he cannot visualize the pyramids.) It’s complicated. (He spoke low).]
R: (RI:380) [I didn’t understand your point.]
S5: (S5I:381) [It’s complicated. (He was trying some strategies in silence.).]
R: (RI:382) [What face and vertex are you considering?]
S5: (S5I:383) [I’m using this face...]
R: (RI:384) [What face?]
S5: (S5I:385) [FEAB.]
R: (RI:386) [FEAB and the vertex...]
S5: (S5I:387) [G.]
R: (RI:388) [The pyramids can touch each other but they mustn’t intersect.]
S5: (S5I:389) [Yes. (Trying some strategies, however, he realized that they didn’t work.).]
R: (R1:390) [Instead of choosing vertex G, use vertex H keeping the same face. What do you think?]
S5: (S5I:391) [(Silence).]
R: (R1:392) [Can you draw the pyramid?]
S5: (S5I:393) [(Drawing).]
R: (R1:394) [What do you think? Does that pyramid intersect the original?]
S5: (S5I:395) [No.]
R: (R1:396) [Why don’t they intersect?]
S5: (S5I:395-400) [(Silencio). They don’t intersect because this segment doesn’t coincide... just it was... just it was constructed on the other side. (His explanation was not correct. I took the teacher’s role and the pupil just accepts the solution as correct without understanding. This is a disadvantage of the hinting strategy.).]
R: (R1:401, 402) [The two pyramids are linked to a triangular face ABH. They don’t intersect. Let’s search for the third pyramid.]
S5: (S5I:403) [We can carry on using the same point H.]
R: (R1:404) [Why?]
S5: (S5I:405, 406) [We use point H once again because we saw that pyramid ABFEH and the original don’t intersect, but they fit together.]
R: (R1:407) [Carry on.]
S5: (S5I:408-410) [So, we can construct the third pyramid using the same reasoning. That is using the same approach. (S5 is influenced by the correct answer to reason in the same way.).]
R: (R1:411) [And what comes next?]
S5: (S5I:412) [(Silence).]
R: (R1:413) [What face is missing?]
S5: (S5I:414) [(Silence). It’s the face ABEH. (Difficulties in visualizing.).]
R: (R1:415, 416) [Is it possible to get a pyramid with such a face as the base and with vertex H?]
S5: (S5I:417) [(Silence). No.]
R: (R1:418) [What would be the other possible face?]
S5: (S5I:419) [(Silence).]
R: (R1:420) [What face have you chosen?]
S5: (S5I:421) [BCGF.]
R: (R1:422) [What about that face?]
S5: (S5I:423) [(Silence). I think this one also fits.]
R: (R1:424) [Why does it fit?]
S5: (S5I:425) [BCGFH.]
R: (R1:426) [Why does it fit?]
S5: (S5I:427) [(Silence).]
R: (R1:428) [Maybe you need to show where the faces fit with the others.]
S5: (S5I:429-431) [I got face DCGH and this vertex. Meanwhile there is another pyramid BAEF which has the same base. It’s a regular pyramid. (He didn’t answer the question. He just presented some disconnected ideas.).]
R: (R1:432) [Don’t those 3 pyramids intersect each other?]
S5: (S5I:433, 434) [(He got a correct solution but he was not able to explain it.).]
R: (R1:435) [What did you think of this task?]
S5: (S5I:436- 438) [It’s interesting. It requires a lot of thinking. My difficulty was not to relate the volumes. My reasoning detached from the picture only. That is why I faced a lot of difficulties. It’s insightful to relate the volumes in order to solve it. (Difficulties occurred when using a synthetic approach only).]

Also S7 did not understand the task. He thought each face of the cube could be a base of a pyramid. After interviewer's explanation he said at most two pyramids could be inserted in the cube.

S7: (S7I:191, 192) [Yes, we can additionally insert more pyramids congruent to the original. Yes, we can insert and here you are some of them.]
R: (RI:193) [How many pyramids can one insert at most?]
S7: (S7I:194) [Here we can insert...4 at most. The original is here, so, we can insert 3 pyramids more.]
R: (RI:195) [Why 4?]
S7: (S7I:196, 197) [(Long silence). I think that... in this cube we can insert 3 pyramids, 4 pyramids, 3 pyramids besides the original.]
R: (RI:198) [Why have you drawn such a conclusion?]
S7: (S7I:199) [I drew this conclusion because the cube comprises 4 faces.]
R: (RI:200) [Carry on.]
S7: (S7I:201-203) [It has 4 faces. Therefore, each face constitutes a pyramid base. Each face constitutes a pyramid base. That is why we can also get 4 pyramids, because each face...]
R: (RI:204) [How many faces does a cube have?]
S7: (S7I:205) [4.]
R: (RI:206) [Why?]
S7: (S7I:207) [Why? (Laughing and counting the faces of the cube). 6.]
R: Então quantas pirâmides? (RI:208) [Then, how many pyramids have you got?]
S7: (S7I:209) [6, 5 or 6 with this one.]
R: (RI:210) [How many pyramids can be inserted in the cube without intersecting each other?]
S7: (S7I:211, 212) [Without intersecting each other! (It seems that the student didn’t understand the task).]
R: (RI:213) [What do you mean to insert in? It means to occupy the space entirely.]
S7: (S7I:214, 215) [Without intersecting each other? The pyramids? These pyramids intersect each other because the segments intersect each other and that is why they will intersect somewhere.]
R: (RI:216, 217) [Might be there at least a pyramid which doesn’t intersect the original? They can touch each other but they mustn’t intersect.]
S7: (S7I:218) [Can they touch each other?]
R: (RI:219) [They can touch each other on the faces.]
S7: (S7I:220) [They can touch each other on the faces.]
R: (RI:221) [Yes, it’s like a puzzle, the pieces fit together.]
S7: (S7I:222) [(Long silence uttering some words low).]
R: (RI:223) [Then, your opinion eh...]
S7: (S7I:224) [In order to non-intersecting we can just insert a pyramid.]
I presented to S7 the concept of volume. It seemed that he got confused about it and I helped him with some hints. For instance, he said "the volume of a square". I thought it was a slippery mistake but further on I realized that he forgot some geometric concepts. Because of using the same formula for the cube and the volume of the pyramid, he said both volumes were equal. It is interesting how S7 compartmentalized his knowledge to the extent not to relate the conclusion he had made using the formulae and the picture given. It was clear from the picture that the volume of the pyramid was less than the volume of the cube. I hinted to him in this regard and he was aware that there was a problem there. Meanwhile when I asked him which factor we should multiply to the formula of the cube volume to get the pyramid volume, he answered correctly though insecure. He correctly answered that at most three pyramids could be inserted in the cube after relating the two formulae through algebraic transformations (Fig 5.65).

R: (RI:225- 227) [Let’s talk about the volumes: the cube and the pyramid volumes. Can you relate the two volumes? How big is the cube volume in relation to the pyramid volume?]
S7: (S7I:228) [(Inaudible).]
R: (RI:229) [Are you aware of the formulae?]
S7: (S7I:230) [The square volume…]
R: (RI:231) [Which volumes are you talking about?]
S7: (S7I:232) [Of the square.]
R: (RI:233) [Of the square? The square volume? What is the square volume?]
S7: (S7I:234) [(Silence).]
R: (RI:235) S[Does the square have a volume? Which one?]
7: (S7I:236) [(Laughing).]
R: (RI:237) [Why does the square have a volume?]
S7: (S7I:238, 239) [A pyramid has a volume as it has a base. It possesses a volume. Now, we have to determine the square volume. (It seems he doesn’t grasp the notion of volume).]
R: (RI:240, 241) [We aren’t dealing with the square volume, but with the cube and the pyramid volumes.]
S7: (S7I:242) [The cube volume.]
R: (RI:243) [Yes, the cube volume.]
S7: (S7I:244) [Does the cube possess a volume?]
R: (RI:245) [Yes, it does, because it has a base area and an altitude.]
S7: (S7I:246-248) [A base area and a altitude, then the base area is a square and the altitude is the side of the cube. (It seems he doesn’t master certain geometric concepts).]
R: (RI:249) [What is the cube volume formula?]
S7: (S7I:250) [Is the cube volume the side raised by three? The cube volume is its side raised by three.]
R: (RI:251) [What about the pyramid volume?]
S7: (S7I:252-254) [The pyramid volume is also the base area multiplied by the altitude. (Long silence). The base area is a square. The pyramid volume is the
side squared multiplied by the altitude... Is the same as the cube altitude? It’s the square side.]

R: (RI:255) [Carry on.]

S7: (S7I:256) [It’s the side raised by three. The cube volume is equal the pyramid volume.]

R: (RI:257) [Looking at the picture, are both volumes equal?]

S7: (S7I:258) [(Laughing). The calculations led us to that conclusion.]

R: (RI:259) [Do you think the pyramid volume is equal the cube volume?]

S7: (S7I:260, 261) [There is a problem. It’s interesting S7 didn’t notice the contradiction between the algebraic and geometric approach. He was aware of it after he was told about.]

R: (RI:262, 263) [There is a factor which is multiplied in the formula of the pyramid volume. What factor is that?]

S7: (S7I:264) [Isn’t it divided by three?]

R: (RI:265) [It’s divided by three, it’s divided by three. Do you remember the formula?]

S7: (S7I:266) [(Laughing).]

R: (RI:267) [When did you learn it?]

S7: (S7I:268) [It was in grade 9.]

R: (RI:269) [In grade 9?]

S7: (S7I:270) [I don’t rember it. It was long time ago.]

R: (RI:271) [How many pyramids can be inserted in the cube?]

S7: (S7I:272, 273) [The cube volume... (Silence). The cube volume... (He was doing some algebraic transformations). How many pyramids can be inserted in the cube?]

R: (RI:274) [You can consider it the other way round.]

S7: (S7I:275) [The pyramid can be inserted three times in the cube. It can be inserted three times.]

R: (RI:276) [Then, how many pyramids can be inserted in the cube?]

S7: (S7I:277, 278) [Since we insert the pyramid three times in the cube; we can insert three pyramids in the cube. (Good relation between algebraic and geometric concepts).]
When asked to represent the three pyramids in the cube he was not sure whether it would be possible with the picture given. He asked himself "Is it possible to represent the three pyramids in this cube?" In the meantime S7 correctly visualized the two remaining pyramids and explained why those three pyramids did not intersect each other with some difficulties (Fig 5.66). It seemed that the algebraic approach aided S7 to be focused in the search of the pyramids. Hence, reasoning aided intuition in case of S7.

![Figure 5.66: S7’s sketch to Task 3 during interview](image)

R: (RI:279) [Can you represent them?]
S7: (S7I:280, 281) [(Laughing). Whether I can represent them. (It seems S7 faces difficulties in representing geometrically the pyramids).]
R: (RI:282) [You can try it. Afterwards explain how you got it.]
S7: (S7I:283-286) [(He was trying it silently. There were some inaudible utterances, meanwhile what I understood was he faced some difficulties regarding segments as concrete objects, for instance a pen. A pen occupies space. Hence, if we constructed pyramids of which edges were like pens, then it wouldn’t be possible to construct three pyramids in the cube).]
R: (RI:287) [What does the algebraic formula mean?]
S4: (S7I:288) [Three pyramids can be inserted in the cube.]
R: (RI:289) [They mustn’t intersect, though they can touch each other on the faces.]
S7: (S7I:290-292) [Do the givens allow us to analyse it with or without a sketch? (It seems S7 thinks the givens might not be sufficient for an algebraic and a geometric approach to the task or he asks for another hint).]
R: (RI:293) [What is to be analysed?]
S7: (S7I:294-297) [This task here, I wonder whether this cube has enough space to insert in it three pyramids. (It seems S7 thinks the cube given is not appropriate to represent in it three pyramids according to the conclusion got from the algebraic approach).]
R: (RI:298) [I cannot catch your point.]
S7: (S7I:299, 300) [The three pyramids… Is there enough space in this cube to sketch the three pyramids? Here you are a cube...(Laughing). I think I got a pyramid.]
R: (RI:301) [I didn’t understand!]
S7: (S7I:302) [I think I got a pyramid.]
R: (RI:303) [What kind of pyramid?]
S7: (S7I:304) [I got a pyramid, which leans on the original pyramid.]
R: (RI:305) [What is it?]
S7: (S7I:306) [This pyramid.]
R: (RI:307) [What is its base and vertex?]
S7: (S7I:308-310) [ABEF and vertex... vertex H. (It seems an algebraic approach served as a springboard for a geometric approach as the algebraic approach motivated S7 to search for a geometric solution).]
R: (RI:311) [What is the position of that pyramid?]
S7: (S7I:312, 313) [There is a touch between them. And that touch happens... happens here on segment AB, segment AB and vertex H. This face is common for this pyramid and the original.]
R: (RI:314) [Do you still hold the idea that the givens are not sufficient for the construction task? What do you think?]
S7: (S7I:315-318) [(Laughing). I think... I think the geometric givens the task offers are sufficient. I got a cone (pyramid) and I’m going to search for the missing pyramid. (It seems a good disposition in the problem solving environment helps in reasoning as one works relaxed).]
R: (RI:319) [Here you got two pyramids.]
S7: (S7I:320) [It’s missing a pyramid.]
R: (RI:321) [Yes, it’s missing a pyramid.]
S7: (S7I:322, 323) [Three pyramids can be inserted. (He was searching for the missing pyramid silently). I got base DCGH, then... (Long silence).]
R: (RI:324) [Have you already got the missing pyramid?]
S7: (S7I:325) [(Long silence). Here you are.]
R: (RI:326) [Which one?]
S7: (S7I:327) [Base FGCB and vertex H.]
R: (RI:328) [Why do you say this is the third pyramid? Doesn’t it intersect the others?]
S7: (S7I:329-331) [It’s so, once... it seems segment AB, sorry, AH intersects segment FG. But in fact there is no intersection. Why? Because we have a cube in which the vertices of the pyramid faces are inclined.]
R: (RI:332) [Carry on.]
S7: (S7I:333-340) [The pyramid faces are inclined. And we have a part of the cube, edge FG, it seems there is a touch, but there is not. With the cube base we construct a pyramid, and with this another cube face we can also construct a pyramid with the same vertex, vertex H. This way there is no intersection; there is just a little touch with...with some faces, with some faces. For instance face BC(H) is common for pyramid BCGF(H). They are sharing it. (S7 has a correct reasoning but he cannot explain it clearly and correctly).]
R: (RI:341) [What conclusion do you draw?]
S7: (S7I:342) [The conclusion I draw, surely there are three pyramids which can be inserted in the cube (laughing).]
R: (RI:343) [Can you explain to me why you have drawn such conclusion?]
S7: (S7I:344-347) [I have drawn that conclusion based on the volumes of the solids. I based on the cube and pyramid volumes. I yielded the relation between the cube and the pyramid volume. The cube volume is three times as big as the pyramid volume; it means three pyramids can be inserted in the cube.]
R: (RI:348) [Is there a contradiction between the picture and the algebraic approach?]
5.3.5.8 Discussion of Task 3

All students except S1 used synthetic approaches (visualization and construction processes only) to solve Task 3. None of them got a completely correct solution. S1 used an analytic approach evoking the formulae of the cube and pyramid volumes and respective relations. He at least concluded that at most three pyramids could be inserted in the cube. However, he was not able to represent all three pyramids. During interviewing I noticed that most of them did not understand the wording of the task. This influenced their written responses in the Diagnostic test. Even so, after understanding the wording of the task they faced difficulties in solving it through the same strategies, i.e. synthetic approaches. A categorization of the subjects according to the strategy used and connectedness of their knowledge is presented below.

*Synthetic strategy*

All students except S1 used visualization and construction processes only driven by their intuition to solve the task. None of them got the three congruent pyramids sought. The strategies they used led to a certain result, although they did not readily transfer to new, potentially relevant situations (specific strategies) Prawat (1989). The non-intersecting condition was only grasped by S3. The rest of the students became aware of that condition after the interviewer’s hint.

*Hybrid strategy (analytic and synthetic strategies)*

S1 used an algebraic approach to demonstrate the relation between the cube and the pyramid volume. He used the volume formulae of a cube and a pyramid in light of a “definition” learnt at school: “the volume of a pyramid inserted in a right prism with the same base and altitude is equal to one third of the volume of that prism”. He used numerical values for the lengths and substituted in the formulae to verify that the cube volume is three times that of the pyramid. Accordingly, he correctly asserted that at most three pyramids could be inserted in the cube. This strategy might lead to a new, potentially relevant situation (general strategy) (ibid.) as the volume formulae are generalizable for any cube and any pyramid. In this approach we noticed some indicators of the presence of algebraic thinking in the solution of Task 3 (formulae, symbols, acronyms, and generalisation). It was odd for S1 to not have
understood the wording of the task even after getting a correct algebraic solution and relating the algebraic solution to a correct geometric statement. When I asked him to sketch three non-intersecting pyramids in the cube he asserted: Yah, (silence). The third... (silence). I did not think about the issue of intersecting each other. I only thought about the issue of volume. I did not visualize the pyramids as solids in the sketch (S11:400, 401). S1 thought at most any three pyramids could be inserted in the cube no matter whether they intersected or not. However, after becoming aware of the non-intersecting condition, S1 realized that the three pyramids he chose did not satisfy the condition of non-intersecting pyramids. Moreover, he correctly visualized that two of them only touched at an edge. There are several possible explanations for this result. Firstly, Douady (1998) states that the dual nature of geometry enables the interplay and interconnection between mathematical language (e.g. algebra) and the language of pictures, between the synthetic approach (where at each step what you say has a meaning in relation to a figure) and the analytic approach (using coordinates to facilitate transfer to a numeric or algebraic framework, which allows blind calculation). Douady asserts that the analytic approach may allow blind calculation. Maybe when S1 used this approach, he was only focused on calculations and ignored visualizing the pyramids. Secondly, this again may corroborate the Schoenfeld’s (1986) claim that unless students learn to take advantage of both deductive (as a means of discovery) and empirical (as a means of development of intuition) approaches to geometry and learn to profit from the interaction of those two approaches; students will not reap the benefits of their knowledge. It seemed that S1 was better at deductive than at empirical approaches regarding this task. Another possible explanation is argued by Lewis (1964): “perhaps the most difficult part of this proof (task) is visualizing the three pyramids in the prism” (p. 620). S1 perhaps faced difficulties in visualizing the pyramids in the cube.

**Connectedness of concepts: reasoning versus intuition**

I noticed that during the hinting process all students except S1 faced difficulties in both synthetic and analytic strategies (S3, S4, S5, and S7). S1 failed only in synthetic strategy although he correctly visualised the second pyramid he constructed and he hypothesized that the remaining space which consisted of two separated triangular pyramids would form the third square pyramid. He was convinced about his hypothesis according to his own words: *I stated that there were two more... two more pyramids. I relied on the pyramid volume. I relied*
on the... cube volume. I argue that they do exist (S1I: 511, 512). For S1 algebra formulae and relations caused a positive impact in his geometric knowledge. He was sure of the existence of three non-intersecting pyramids, although he could neither draw nor visualize them in the cube: ...I cannot visualize these extra parts as a picture. However, I think these two extra parts joined they become another pyramid congruent to the other two (S1I:458-476). S4 and S5 also corroborated S1’s position. S4 stated “One pyramid, I have changed my mind because of this systematic deduction which makes sense....” (S4I:549). By “systematic deduction” he referred to the algebraic transformations he accomplished during interviewing. For S4 these transformations changed his mind. When he used a synthetic approach he came up with a certain idea. However, when he was confronted with an analytic approach he was confident about the solution of the task which was different from the synthetic solution. He added saying “It makes sense because the altitude in fact equals to the edge. The deduction also shows another thing. CV is three times PV. It means that there are three pyramids inserted in the cube. This is an obvious and clear proof” (S4I:551-553). He finally summarized his ideas about this task regarding the two approaches. He was of the opinion that in this task algebra enlightened geometry for understanding some geometric concepts which would otherwise be difficult for most of these particular students.

Well, this task required a lot of abstraction. It’s not easy; it’s not easy to think without algebraic expressions support. It’s not easy. I’m of the opinion that it’s easy to be mislead when you think without calculation support. If we visualize we can see that here we get a space, another space there, and we can naturally realize that in each space we can occupy at least with a unit of a cube. However, when we use calculations, we get at most three pyramids inserted. I mean firstly we needed abstraction abilities to take this diagnostic test. We needed certainly imagination also, to be able to imagine, but based in scientific support. (S4I:580-588)  

S5 also stated that “Using the volume formulae it’s possible. Alias, it’s a logic relation, isn’t it? (Laughing). Using visualization it seems impossible. But I have, for instance, this... for instance ABFEG. (Silence)” (S5I:370-373). Further, he added, “It’s interesting. It requires a lot of thinking. My difficulty was not to relate the volumes. I based my thinking on the picture only. That is why I faced a lot of difficulties. It’s insightful to relate the volumes in order to solve it” (S5I:436- 438). In turn S7 explained his strategy in terms of reasoning aiding intuition too: I have drawn that conclusion based on the volumes of the solids. I based on the cube and pyramid volumes. I yielded the relation between the cube and the pyramid volume. The cube volume is three times as big as the pyramid volume; it means three pyramids can be
inserted in the cube (S7I:344-347).

In Table 5.14 we see that S1 used fewer hints (14) than the other students. However, he did not provide a clear synthetic solution. The hints were concerned about the relation between an analytic strategy to a synthetic strategy. In turn, S3 and S5 used most of hints 36 (synthetic – 21; analytic – 5; and relation analytic/synthetic - 10) and 35 hints (synthetic - 11, analytic - 14, and relation analytic/synthetic - 10 hints) respectively. S4 (synthetic – 12; analytic – 13; and relation analytic/synthetic – 3); and S7 (synthetic – 6; analytic – 13; and relation analytic/synthetic – 7) were in between. It is noted that only S7 got the correct solution without being provided with all hints.

Table 5.14: Hinting frequency table to Task 3

<table>
<thead>
<tr>
<th>Student</th>
<th>Hints (H) to Task 2</th>
</tr>
</thead>
</table>
| S1      | H1: Do these pyramids intersect each other? The issue is to insert them without intersecting each other.  
         | H2: Can you sketch this pyramid, this one?  
         | H3: Do they touch at this edge or they touch at a face? What do you think?  
         | H4: Do they intersect at an edge or at a segment?  
         | H5: Ok, ok, what is another pyramid?  
         | H6: Do you want to sketch?  
         | H7: What do you think? Is there any other clearer strategy where we can construct the three pyramids which do not intersect each other? Is it possible to find such strategy? Or is this the only strategy available?  
         | H8: Can you show me how those triangular pyramids look like? Can you name them?  
         | H9: Can you visualize those triangular pyramids? Have you said two pyramids?  
         | H10: Maybe you can sketch them. How is it... or... if you can do it in the picture already given or here, as...  
         | H11: This is a strategy, but there might be another clearer. Let’s see. This pyramid, that...  
         | H12: Only you need to prove that such extra parts form a pyramid.  
         | H13: But it’s possible that there will be another clearer strategy. Now the issue is to find it.  
         | H14: Yes, we already had a pyramid. And we need to search for other pyramids congruent to that one, that one. |
| S3      | H1: Why do you say that at most two pyramids can be inserted in the cube? Can’t you insert more than two?  
         | H2: There are two. What are they?  
         | H3: Isn’t there another pyramid to be inserted besides these two?  
         | H4: But now, do you think whether these two pyramids occupy the whole space in the cube or there is some space left?  
         | H5: Then, there is a space left. Putting together those spaces left, can we get another pyramid?  
         | H6: We can see that these two pyramids don’t occupy the whole cube space. What do you think? Can we insert one more pyramid?  
         | H7: It is a space. It is not a superficial domain (related to the cube face).  
         | H8: Maybe if you choose another pyramid it might work.  
         | H9: Maybe if you drew here once more a picture.  
         | H10: Let’s see. First, maybe let’s consider another face. Instead of considering this pyramid, let’s consider another face. Be for instance this one. Let’s suppose face ABFE. Can you sketch a pyramid with face ABFE?  
         | H11: Give up the pyramid you have chosen as we cannot clearly visualize the others. Then, let’s consider for instance face EADH. How can we... |
H12: No, use H instead of D as vertex.

H13: Choose ABFE or AEFB as a pyramid base and H as a vertex.

H14: What is the position between these pyramids?

H15: How many pyramids have you already inserted so far?

H16: Do these pyramids intersect each other?

H17: Then, how many pyramids can you insert at most?

H18: Three pyramids. Are they congruent?

H19: Why are they congruent?

H20: Now, what are the pyramids you found?

H21: What is the rule you found to get the solution?

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H22: Have you ever learnt the volume formula of a cube?

H23: What is the volume formula of a cube?

H24: And what is the volume formula of the pyramid?

H25: What is the relation between the cube and the pyramid volume?

H26: How many pyramids can be inserted in the cube?

-----------------------------------------------------------------------------------------------

H27: What was our conclusion using the geometrical approach?

H28: Is there any contradiction?

H29: Do you mean that the pyramids are not congruent?

H30: You said they are not congruent. Why aren’t they congruent? Firstly you said the three pyramids were congruent and now you changed your mind.

H31: The three pyramids are congruent. Are their volumes equal?

H32: Now, how come there is a contradiction with the formulae?

H33: Do you mean that this volume formula of a pyramid only works when its vertex is in the middle of the face and it doesn’t work for all cases?

H34: Instead of two it was not three?

H35: If we had here three, I think, there would be...

H36: Can we put three?

-----------------------------------------------------------------------------------------------

S4

H1: Do they intersect each other?

H2: Then, how many pyramids can be inserted at most in the cube?

H3: Yes, it’s to occupy the cube space entirely.

H4: Then, how many pyramids can be inserted at most?

H5: You mean that four pyramids can be inserted in the cube without intersecting each other and occupying the space entirely?

H6: According to your idea how many pyramids can be inserted in that existing space?

H7: How can you represent that space as a pyramid? You stated that three more pyramids could be inserted in the cube without intersecting and occupying the entire cube space. You chose these pyramids. Do you think that these pyramids satisfy those conditions?

H8: Can you sketch it?

H9: You are sketching a pyramid. What is its base and vertex?

H11: Then, how many pyramids can be inserted in the cube?

H12: What is the total number of the pyramids?

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H13: Do you know the volume formula of a pyramid and of a cube?

H14: Haven’t you learnt it divided by three?

H15: Now, let’s suppose it is divided by three. How would you relate the two volumes?

H16: What is the other solution?

H17: Let’s depart from the formulae. You used the picture and stated that the cube volume is as double as the volume of the pyramid.

H18: Consider this pyramid.

H19: What is your conclusion? What is the relation between PV and CV?

H20: Where you get the altitude... What do you replace the altitude with?

H21: The pyramid altitude.

H22: From this relation.

H23: What does mean?

H24: Then, what is the relation?

H25: Then, how many pyramids can be inserted?

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H26: Now, how can you represent the three pyramids here? You represented one pyramid
and we got the original one. Where does the other pyramid lie?

| S5 | H1: Does a cube have 8 faces?  
|    | H2: Let’s count once again  
|    | H3: Show me what the cube faces are.  
|    | H4: Then, how many are there?  
|    | H5: How many pyramids can you construct on each face?  
|    | H6: But the issue here is, we want to know how many pyramids can be inserted in this cube.  
|    | It’s not only a matter of constructing the pyramids.  
|    | H7: It’s to occupy the space fully.  
|    | H8: The purpose is to know how many pyramids congruent to this one can be additionally inserted in the cube without intersecting each other.  
|    | H9: It’s a sort of inserting. How many pyramids can be inserted inside the cube?  
|    | H10: What is your conclusion? How many pyramids can be inserted in this cube?  
|    | H11: Why have you drawn such conclusion? Have you tried all possible pyramids?  

|     | H12: Have you related the pyramid volume to the cube volume?  
|     | H13: Do you know the formulae?  
|     | H14: Is the pyramid volume the base area multiplied by its altitude? Haven’t you missed anything in the formula?  
|     | H15: It’s divided by three, it’s divided by three. How can you proceed based on this?  
|     | H16: What is the base area of this pyramid?  
|     | H17: What does l stand for?  
|     | H18: Then, instead of using l you can replace it with...  
|     | H19: Then, what stands for the altitude of the pyramid?  
|     | H20: Then, instead of h I can replace it by...  
|     | H21: What is the relation you can draw from the volume of the cube and of the pyramid?  
|     | H22: How many pyramids of the original type can be inserted in the cube?  
|     | H23: Is it possible to represent these pyramids in the cube? We already have one. And what are the others? They mustn’t intersect.  
|     | H24: Using the volume formulae is it possible to have 3 non-intersecting pyramids?  
|     | H25: It seems there is a contradiction between the algebraic approach and the geometric approach as you said algebraically there were at most 3 pyramids inserted in the cube, but geometrically it seemed impossible to get them.  

|     | H26: What face and vertex are you considering?  
|     | H27: The pyramids can touch each other but they mustn’t intersect.  
|     | H28: Instead of choosing vertex G, use vertex H keeping the same face. What do you think?  
|     | H29: Can you draw the pyramid?  
|     | H30: What do you think? Does that pyramid intersect the original?  
|     | H31: The two pyramids are linked to a triangular face ABH. They don’t intersect. Let’s search for the third pyramid.  
|     | H32: What face is missing?  
|     | H33: Is it possible to get a pyramid with such face as base and with vertex H?  
|     | H34: What would be another possible face?  
|     | H35: Maybe you need to indicate the faces where they fit.
| S7 | H1: How many faces does a cube have?  
H2: Then, how many pyramids have you got?  
H3: How many pyramids can be inserted in the cube without intersecting each other?  
H4: What do you mean to insert in? It means to occupy the space entirely.  
H5: Might be there at least a pyramid which doesn’t intersect the original? They can touch each other but they mustn’t intersect.  
H6: Yes, it’s like a puzzle, the pieces fit together.  
H7: Let’s speak about the volumes: the cube and the pyramid volumes. Can you relate the two volumes? How big is the cube volume related to the pyramid volume?  
H8: Do you know the formulae?  
H9: Does the square have a volume?  
H10: Why does the square have a volume?  
H11: We aren’t dealing with the square volume, but with the cube and the pyramid volumes.  
H12: Yes, it does, because it has a base area and a altitude.  
H13: What is the cube volume formula?  
H14: What about the pyramid volume?  
H15: According to the picture are both volumes equal?  
H16: Do you think the pyramid volume is equal the cube volume?  
H17: There is a factor which is multiplied in the formula of the pyramid volume. What factor is that?  
H18: It’s over three, it’s over three. Do you remember the formula?  
H19: How many pyramids can be inserted in the cube?  
H20: Can you represent them?  
H21: What does the algebraic formula mean?  
H22: They mustn’t intersect, though they can touch each other on the faces.  
H23: What is its base and vertex?  
H24: What is the position of that pyramid?  
H25: Why do you say this is the third pyramid? Doesn’t it intersect the others?  
H26: Is there a contradiction between the picture and the algebraic approach? |

According to these figures I inferred that reasoning driven by algebraic thinking (volume formulae and relationships) aided to some extent S1’s and S7’s intuition to get the correct solution without providing them with all available hints, although S1 was not able to provide a clear synthetic solution. S3, S4, and S5 faced difficulties in both strategies so as in the relation between the two strategies. They required all hints to get the correct solution.

5.3.5.9 Emerging issues of the analysis of Task 3

1- Both tasks compelled all students (except S3 in Task 2 and S1 in Task 3) to use visualization and construction processes only driven by their intuition that is, they firstly tackled these tasks using synthetic strategies.

*Insight: It seems easier for these students to use their intuition rather than their reasoning in tasks which are qualitatively presented (no quantities).*
2- All students claimed that the analytic strategies used in these tasks (in Task 2 a numerical approach and patterning in Task 3 mainly volume formulae and relations) were insightful towards a geometric solution.

*Insight 2: Algebraic thinking can be seen as a way of layering meanings on each other, connecting between ways of knowing and seeing, rather than as a way of replacing meaning of each other. These meanings become reshaped as learners exploit available tools in algebra to move the focus of their attention onto new objects and relationships in geometry.*

3- All students except S1 faced difficulties in remembering the pyramid volume formula. Meanwhile all of them correctly related the volume formulae of a pyramid to that of a cube algebraically and they correctly transferred it to a geometric statement.

*Insight 3: It seemed that most of these students as opposed to the subjects of Dindyal’s research had good understanding of the algebraic and geometric concepts although they were not able to recall correctly some of them. In other words, the students possessed knowledge with high strength but low activation according to Anderson’s terminology (Anderson, 1990 as quoted in Dindyal, 2003).*

5.3.6 Discussion of Interviewing Phase 1

The Interviewing Phase 1 transcripts also were analysed in terms of algebraic and geometric knowledge base, connectedness and strategies using the conceptual model suggested above and the framework on learning and transfer drawn on Prawat’s work.

In Task 1 the students spontaneously explained their first encounter with Pythagorean Theorem and additionally they explained about their previous geometric experience. I used this information to cluster them according to their school geometry background.

S1 affirmed he had learnt the Pythagorean Theorem when he was schooling a long time ago and he had never used it further. This context contributed to his “inert knowledge”. Accordingly, he was not able to access and use his previous knowledge appropriately. It seemed that “inert knowledge” hindered the prevalence of the model for algebraic thinking in geometrical understanding in him regarding this task.

S3 and S4 claimed that at school they only used Pythagorean formula to solve tasks. These students were not able to access and utilize their intellectual resources towards a proof
because of lack of them. It seemed that S3 was performance oriented as he was satisfied with the knowledge taught by the teacher. In turn S4 seemed to be mastery oriented due to his efforts to broaden his knowledge through self study. Although both students faced difficulties in solving this task, at least S4 showed some conceptual understanding.

S5 was a special example where we can see learning taking place through teaching. He was forced to learn the proof in order to teach. Meantime, he still faced some difficulties in justifying some concepts. This student was forced through teaching to be mastery oriented. Accordingly, I noticed some conceptual understanding in him.

S7 asserted that he had learnt the Pythagorean Theorem so as its proof at school. He showed performance in solving this task. However, he was not able to explain certain concepts due to lack of teaching of those concepts.

All students used synthetic approaches (visualization and construction processes only) to solve Task 2 and Task 3 (except S3 for Task 2 and S1 for Task1) though they did not succeed. S1 and S3 also used analytic strategies (number approach, patterning, formulae, and relations). Therefore, I noticed that during the hinting process all students showed that their intuition was still weak, although their reasoning seemed to be robust. One of the reasons for this phenomenon might be found in a statement of some of these students:

S3: Só dávamos as fórmulas. Daí usávamos para calcular as medidas dos lados. (S3I:57) [We were only given formulae. Thereafter we used them to calculate the lengths of the sides.]

S4: Bem, sobre o mesmo exercício, eh... eh, talvez dizer que não... não foi assim tão fácil provar porque o hábito de provar ainda não está... ainda... ainda não está activado, ainda não existia em mim essa capacidade de provar, normalmente dá-se isto e aquilo e pede-se para calcular. E o porquê ainda não existia na altura. Podem ainda aparecer justificações infundadas mas são o meu parecer. (S4I:112-116) [Well, about this task...maybe to say that... it was not easy to prove (this theorem) because the habit of proving is not activated in me yet. In my schooling time I did not question the reason why of things. Normally, the teacher gave us some formulae to solve tasks. That is why you can notice some flat ideas, but they are my opinions.]

S5: Quando nós fizemos o curso, né, tínhamos álgebra, também tínhamos geometria. O que acontecia é que se dava álgebra e... e deixava-se geometria para o fim. E muitas das vezes parecia que quase não se dava. Dava se cabo da geometria. (Ensino deficiente da geometria). (S5I:245-248) [When we learnt mathematics, we were taught algebra and geometry was in the syllabuses too. But it would happen that algebra was taught and... and... geometry was left as the latter topic. And often times, geometry was not taught at all.]

S7: [We were only given formulae. Thereafter we used them to calculate the lengths of the sides.]
These statements show that these students might have been exposed more to algebra teaching than to geometry teaching during their schooling time. Besides even in the geometric lessons they might have been taught more formulae and algebraic relations than diagrammatic representations for the purpose of *locality* use to direct perceptual inference (Larkin and Simon, 1987 as quoted by Koedinger and Anderson, 1990). That is it seemed that the diagrammatic representations they used before were merely an auxiliary tools for formulae application as they mentioned above, instead of being sources which “allow easy perceptual inferences to replace hard symbolic inferences” (ibid., p. 518).

The students’ school mathematics background (especially algebra and geometry) might (partially) explain why their reasoning driven by algebraic thinking (e.g. patterning, formulae and relationships) aided to some extent intuition of some of them to get the correct solution without providing them with all available hints (S1, S3, and S7 in Task 2 and S1 and S7 in Task 3). S4 and S5 faced difficulties in synthetic and analytic strategies and in the relation between the two strategies in both tasks. They required all hints to get the correct solution. S3 faced difficulties in Task 3 only.

It is important to mention that the hinting process might in some cases bias my findings. However, what I saw was that some students asserted that the analytic strategies shed some light to understand a geometric solution in relation to the geometric concepts involved (Section 5.3.5.8).

### 5.3.7 Emerging Issues of Interviewing Phase 1

Emerging issues concerning the Interviewing Phase 1 transcripts are as follows:

1- Students entering Universidade Pedagógica enrolled in the course of mathematics teacher training came from different academic and professional backgrounds. Accordingly, they experienced different geometry teaching and learning contexts. It seems that differing contexts of teaching and learning develop students academically differently.

2- Students who possessed inert knowledge showed difficulties in accessing it and using it even with hints. Hence, knowledge learnt and not used further may become inert and difficult to access later.
3- Institutional relationship to student’s knowledge was strongly noticeable and it reshaped mostly what students knew although it was also noticed that some of them expanded their knowledge through self study.

4- Students who experienced effective teaching or carried out a self study flexibly switched from one form of representation to another (e.g. verbal, algebraic, and pictorial) opposed to those who possessed inert knowledge or used procedural knowledge. It was also noticed that the former group of students successfully accessed and used key algebraic and geometric concepts towards a correct solution.

5- It seems easier for these students to use their intuition (synthetic strategies) rather than their reasoning (analytic strategies) in the tasks which are qualitatively presented (no quantities). However, they faced difficulties in getting a solution. Besides these strategies seemed to be specific and applied to a concrete situation.

6- All students, through hinting process, used both intuition (visualization and construction processes) and reasoning (a numerical approach, patterning, formulae and relations) to solve the tasks. Accordingly, it showed that the interaction of deductive (as a means of discovery) and empirical (as a means of development of intuition) approaches to geometry led these students to reap the benefits of their knowledge.

7- In geometry, analytic strategies (general strategies) seem to be more efficient than synthetic strategies (specific strategies), although they are difficult to access. This confirms what is argued in the literature, that specific strategies lead to a certain result, although they do not readily transfer to new, potentially relevant situations. In particular in geometry, synthetic strategies lack a general method (strategy).

8- Most of the students claimed that the analytic strategies were insightful towards a geometric solution. This results seems to corroborate my assumption that algebraic thinking can be seen as a way of layering meanings on each other, connecting between ways of knowing and seeing, rather than as a way of replacing meaning of each other. These meanings become reshaped as learners exploit available tools in algebra to move the focus of their attention onto new objects and relationships in geometry.

9- It seemed that most of these students as opposed to the subjects of the Dindyal’s research had good understanding of the algebraic and geometric concepts although they were not able to recall correctly some of them. In other words the students possessed knowledge with high strength but low activation.

10- In some tasks most of the students seemed to possess robust schemas as they needed few hints to solve them. Meanwhile other students needed more hints showing weak schemas
maybe due to lack of confidence and/or knowledge. These facts can be explained using the literature that a schema with components that are effectively organized is one for which minimal levels of cueing are required for activation. When a greater level of hinting support is needed for access, it is argued that the knowledge schema is either less extensive or less well connected. Moreover affective and motivational components (e.g., lack of confidence) can hinder the use of a strategy or skill on a transfer task.

5.3.8 Conclusion of Interviewing Phase 1

The analysis of the transcripts showed that the target students solved geometric tasks using both algebraic and geometric thinking. It seemed easier for these students to use their intuition (synthetic strategies) rather than their reasoning (analytic strategies) in the tasks which are qualitatively presented. However, they faced difficulties in getting a solution. Besides these strategies seemed to be specific and applied to a concrete situation. In such tasks I needed to provide hints to most of the students either using visualization and construction processes or reasoning driven by algebraic thinking. I noticed that their reasoning was stronger than their intuition. When I prompted them to elicit their intuition I was forced to supply them with all key concepts to get the solution. While when I elicited their reasoning through algebraic thinking I noticed that the intuition of some of them was elicited and became robust towards the solution. All students, through hinting process, used both intuition (visualization and construction processes) and reasoning (a numerical approach, patterning, formulae and relations) to solve the tasks. Accordingly, it showed that the interaction of deductive (as a means of discovery) and empirical (as a means of development of intuition) approaches to geometry led these students to reap the benefits of their knowledge.

Most of the students claimed that the analytic strategies were insightful towards a geometric solution. This result seems to corroborate my assumption that algebraic thinking can be seen as a way of layering meanings on each other, connecting between ways of knowing and seeing, rather than as a way of replacing meaning of each other. These meanings became reshaped as learners exploited available tools in algebra to move the focus of their attention onto new objects and relationships in geometry. Although algebraic thinking shed light in geometrical understanding, still the students needed to possess visualization and construction skills to accomplish the geometric tasks successfully.

5.3.9 Analysis of the Elaboration and Concept Mapping Task
During this activity I intended to find out about “knowledge connectedness” of the students from two different perspectives. That is why I constructed Elaboration and Concept Mapping Task (Appendix 4). The Elaboration Task aimed to assess how different knowledge schemas that are relevant to a particular problem can be related one to the other (external connectedness). The Concept Mapping Task was used in the form of yielding true propositions using the concepts presented to examine the levels of connectedness of knowledge schemas (internal connectedness). Due to the nature of the tasks, I only collected the students’ scripts and some field notes. I went through all scripts and field notes for each student. I used the software Atlas.ti for coding instances as they appeared (for example see Fig 5.67 for Elaboration Task and Fig 5.68 for Concept Mapping Task).

He tried to construct another task. He explained the solution verbally. This time he only used a picture, symbols and verbal solution.

Figure 5.67: S1’s solution to Elaboration Task

S3 constructed synthetic statements (geometric properties) although instead of using synthetic concepts like "congruency" and "small" he uses analytic concepts "equals to" and "a half".
5.3.9.1 Analysis of the elaboration task

The Elaboration Task was as follows:

1. Suppose that you have the following concepts and theorems:
   
   a) The sum of the measures of the three (interior) angles of every triangle is 180°.

   b) If two parallel lines are cut by a transversal, then the alternate interior angles are congruent.

   c) If two angles and the side between them in one triangle are congruent to two angles and the side between them in another triangle, then the two triangles are congruent.

   d) A diagonal of a parallelogram divides the parallelogram into two congruent triangles.

1.1 Construct a task where you can use some or all concepts and theorems given.

1.2 Solve this task mentioning the concepts and theorems used.

S1 constructed a task where he used a lot of symbols and acronyms instead of text and a picture. His solution involves use of formulae, equations, symbols, unknowns and algebraic transformations (Fig 5.69). He used the picture to visualize the angles and the respective relations.
Figure 5.69: S1’s solution to Elaboration Task

[Translation: Observe the picture where (straight line) ‘a’ is parallel to (straight line) b and A is the point of intersection between the straight lines s and t. The measures of angles \( \angle \) and \( \angle ABC \) are, respectively 130° and 30°, determine the measure of angle \( \angle CBA \).]

It is interesting that S1 to visualize the statements given for understanding purposes he sketched some pictures (Fig 5.70). S1 translated the statements into pictures. He used some symbols and acronyms to identify some geometric concepts. This phenomenon might be explained that Euclidean geometry course helped to elicit the S1’s intuition and simultaneously to use some symbols, acronyms, and formulae, which facilitated his reasoning.
S3 started off with a picture and its identification using symbols and acronyms before he constructed the task (Fig 5.71). His task could be solved using the definition of a diagonal (a line segment joining the two pairs of opposite vertices). Meanwhile he used formulae, equation, unknown, symbols and algebraic transformations to solve it. In his context algebraic thinking aided him to solve a geometric task although he misapplied a geometric property when he considered BD as an angle bisector. He also used synthetic statements to justify his formulae and algebraic transformations with help of the picture.
[Translation: Knowing that ABCD is a parallelogram and (angle) $\varnothing = 35^\circ$ and (angle) $\zeta = 110^\circ$. Prove that BD is a diagonal of ABCD.

Solution: 1- $\varnothing = \beta$ (Statement b); 2- $\zeta = \gamma$ (Statement b); 3- $\beta + \gamma + \xi = 180^\circ$ (Statement a); 4- $\xi = 35^\circ$ (from 3); 5- $\xi = \epsilon$ (Statement b); 6- $\xi = j$ and $\varepsilon = \beta$ (adjacent angles); 7- BD is a common side of the triangles ABD and BCD; 8- $\triangle ABD \cong \triangle BCD$ (from 6 to 7); and BD is a diagonal of ABCD (Statement d).]
S4 also constructed a task from a picture he sketched beforehand and used algebraic thinking (acronyms, equations, formulae, symbols, and unknown) throughout his solution (Fig 5.72). This task is also a good example where we can see the important role of algebraic thinking in geometric tasks solution. He also used synthetic statements to justify the formulae and algebraic transformations with recourse of the picture.

Figure 5.72: S4’s solution to Elaboration Task
Given the picture aside, determine the measures of angles $\beta$ and $\chi$.

Solution: 1- $\varphi = 75^\circ$ (vertical angles); 2- $\varphi = \chi$ (Statement b); 3- $\chi = \Theta + \gamma$ where $\gamma = 20^\circ$ (vertical angles); 4- $\Theta = 55^\circ$ (from 2 and 3); 5- $\varphi + \alpha = 180^\circ$ (picture); 6- $\alpha = 105^\circ$ (from 1 and 5); 7- $\varphi + \alpha + \beta = 180^\circ$ ($\Delta ABC$); and 8- $\beta = 20^\circ$ (from 1, 6, and 7).

S5’s task consisted of a picture identified by symbols and acronyms. He mainly used text in it. He used algebraic thinking (acronyms, formulae, and symbols) to solve it (Fig 5.73). Besides, we can also notice some elements of geometric thinking (visualization and construction) by using synthetic statements. The second S5’s solution is an attempt to prove the Statement d which according to the task it should be taken as knowledge to be used without proving it. He correctly proved it using a synthetic strategy although he used algebraic symbols and acronyms to identify geometric objects.
[Translation: In the picture the two pairs of straight lines (s and r) and (v and t) are parallel to each other. \([ABCD]\) is a parallelogram. Prove that: a) the sum of the interior angles of a parallelogram is 360° and b) \(\triangle ABC \cong \triangle ACD\) with diagonal \(AC\).

Solution: a) Given: 1- \([ABCD]\) is a parallelogram; 2- \(v \parallel t \land s \parallel r\); 3- Interior angles of \([ABCD]\) are \(\angle A, \angle B, \angle C, \text{and} \angle D\).
Conclusion: $\angle A + \angle B + \angle C + \angle D = 360^\circ$.

Proof: 1- Considering diagonal $AC$; 2- $\triangle ABC$:
$\angle CAB + \angle ABC + \angle BCA = 180^\circ$ (Statement a); 3- $\triangle ADC$:
$\angle CAD + \angle ADC + \angle DCA = 180^\circ$ (Statement a); 4- $\angle CAB + \angle CAD + \angle ABC + \angle BCA + \angle ACD = 360^\circ$; 5- $\angle A + \angle B + \angle C + \angle D = 360^\circ$.

b) Given: 1- $[ABCD]$ is a parallelogram; 2- Diagonal $AC$; 3- $v||t \land s||r$.

Conclusion: $\triangle ABC \cong \triangle ACD$.

Proof: 1- $\alpha = \gamma$ (Statement b); 2- $\theta = \sigma$ (Statement b); 3- $AC$ is a common side for both triangles; 4- $\triangle ABC \cong \triangle ACD$ (Statement c).

S7 also constructed a task with a picture identified by symbols and acronyms accompanied by text. He presented two different kinds of solutions (Fig 5.74). The first solution used geometric properties so as algebraic thinking (symbolization, formulae, equation, and unknown). The second solution used mainly geometric properties (synthetic statements); meanwhile we can see some algebraic thinking (acronyms and symbolization).
Figure 5.74: S7’s solution to Elaboration Task

[Translation: The picture aside represents parallelogram $[ABCD]$ and the two pairs of straight lines $(a, b)$ e $(c, d)$ are parallel to each other. a) Determine the measures of angles $\alpha$, $\beta$ and $\gamma$. b) Prove that triangles $[ABC]$ and $[ACD]$ are congruent.

Solution: a) $a\parallel b$ and $c\parallel d$; $\beta = 30^\circ$ (vertical angles); $\gamma = \beta = 30^\circ$ (Statement b); $\alpha + \gamma + 45^\circ = 180^\circ$; $\alpha = 105^\circ$.}
b) Triangles \([ABC]\) and \([ACD]\) are congruent because: 1- Diagonal \(AC\) of parallelogram \([ABCD]\) part of the straight line \(c\) divides the picture into two equal parts or into equal triangles (Statement d); 2- Both triangles share a same side, side \(AC\); besides \(a\parallel b\) and \(c\parallel d\); then, \(\gamma \cong \angle ADC\) (the alternate interior angles in two parallel lines- Statement c); and the angles divided by diagonal \(AC\) are also congruent; then the triangles are congruent.]

The second S7’s solution is an attempt to prove the Statement d which according to the task it should be taken as knowledge to be used without proving it. He used some synthetic statements which are not correct in geometry or better to get to such statements it is required to prove some other statements. In step 2 for example he concluded that \(\gamma \cong \angle ADC\) because \(AC\) was a common side for triangles \([ABC]\) and \([ACD]\) and \((a\parallel b\) and \(c\parallel d)\). Moreover he mistakenly considered these geometric statements as Statement c, maybe he wanted to refer to Statement b, because in brackets he wrote down “alternate interior angles in two parallel lines”. In the literature (Lewis, 1964 and Serra, 1997) to show that the opposite angles of a parallelogram are congruent it is used an auxiliary theorem that a diagonal of a parallelogram divides the parallelogram into two congruent triangles. Moreover in the end he wrote down an ambiguous geometric property that “the angles divided by diagonal \(AC\) are also congruent”.

5.3.9.2 Discussion of the elaboration task

Table 5.15 below summarizes the result of the analysis regarding the students’ solution to the Elaboration Task.

Table 5.15: The analysis summary of the students’ solution to Elaboration Task

<table>
<thead>
<tr>
<th>Student /Cognitive Processes</th>
<th>T</th>
<th>AR</th>
<th>PR</th>
<th>Reasoning</th>
<th>Intuition</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>Little</td>
<td>Yes</td>
<td>Yes</td>
<td>Correct</td>
<td>Correct</td>
</tr>
<tr>
<td>S3</td>
<td>Reasonable</td>
<td>Yes</td>
<td>Yes</td>
<td>Incomplete</td>
<td>Incomplete</td>
</tr>
</tbody>
</table>
The codes and the categories used in this table are as follows:

“T”: Task text

“AR”: Algebraic Representation (formulae, acronyms, symbols, relation, unknown, equations)

“PR”: Pictorial Representation (visualization and construction processes)

“Little”: Few words

“Reasonable”: Between few and many words

“Much”: Many words

“Reasoning”: Student uses AR to solve the task

“Intuition”: Student uses PR to solve the task

“Correct”: All key concepts present

“Incomplete”: Some key concepts present

“Not correct”: All key concepts missed

Table 5.15 shows that S1, S4, and S5 used key algebraic and geometric concepts to solve the tasks they constructed. Accordingly they used reasoning and intuition in connection. In turn, it seemed that S3 and S7 still faced difficulties in connecting reasoning and intuition as they used incorrect geometric properties though they correctly used algebraic representation.

It needs to be noted that these students had successfully finished the Euclidean Geometry course at Universidade Pedagógica when they did this activity. It is interesting also to note that all students started the Elaboration Task with a picture and later they produced the actual task based on that picture. All of them used algebraic thinking to identify the picture elements (vertices, sides, and angles) so as in the solution of most part of the tasks. At this stage it seemed that these students used diagrammatic representations in this activity with two purposes mentioned in Koedinger and Anderson (1990):

First, one can use locality of objects in the diagram to direct inference, and second perceptual inferences can be made more easily than symbolic inferences (p. 517).
For the first purpose I realized that these students when they constructed the tasks based on the pictures (diagrams) they sketched beforehand, they wanted to gather all required information in the same locality which is often easier to find in a diagram than in a list of statements. For the second purpose that diagrams allow easy perceptual inferences to replace hard symbolic ones, I noticed that these diagrams aided the students to problem search, that is, the search for a problem solution. We can see it in their solutions. They usually justified each step of their solutions pointing to the statements given which were also reflected in the pictures they sketched.

The Elaboration Task allowed students to construct tasks and solve them using predominantly synthetic strategies (for example Fig 5.75).

Prove that the opposite sides of a parallelogram are congruent.

Given: ABCD is a parallelogram.

Concl.: $AD \cong BC$ and $AB \cong DC$.

1- $\triangle ABC \cong \triangle CDA$ (Statement d).

2- $AD \cong BC$ and $AB \cong DC$ (Congruent triangles).

Figure 5.75: An example of a task solved synthetically

Meanwhile only two students S5 and S7 used synthetic strategies only for solving parts of their tasks although they utilized algebraic symbols and acronyms to identify some geometric objects and concepts (congruency, angles, vertices, triangles, etc). The remaining students solved them using hybrid strategies (analytic and synthetic strategies).

5.3.9.3 Emerging issues of the elaboration task

1- All students strongly used algebraic thinking in constructing tasks as well as in solving them.

Insight 1: It seems that the prolonged exposure to algebraic teaching during schooling period obliged these students to apply algebraic thinking in solving geometric tasks even after attending the Euclidean Geometry course at Universidade Pedagógica.

2- Three students out of five used their reasoning and intuition in connection and succeeded
Insight 2: The interaction of deductive (as a means of discovery) and empirical (as a means of development of intuition) approaches to geometry leads students to reap the benefits of their knowledge (Schoenfeld, 1986).

3- The Euclidean Geometry course at Universidade Pedagógica seemed to elicit students’ intuition regarding the use of diagrammatic representations.

Insight 3: It is likely that students of geometry have had more prior experience with geometric images than with formal notations, and since diagrams typically have the locality feature, students are likely to find perceptual inferences in this domain easier than symbolic inferences (Koedinger and Anderson, 1990).

5.3.9.4 Analysis of the concept mapping task

The Concept Mapping Task was as follows:

2. Suppose you have the following concepts: triangles, angles, parallelism, midsegment, similarity, angle bisector, ratio and proportion. Construct true statements connecting the concept of “triangle” with one or more concepts given.

It seemed that for S1 the Pythagorean Theorem formula was the insight he got to make this statement (Fig 5.76). That is the hypotenuse is lengthier than the other legs. This idea I drew from Interviewing Phase 1 when he stated that “Let’s consider the area of the square of which side is the hypotenuse. Then, its area is as double as the area of the square of which side is one of the legs” (S1I:70, 71). Further on he explained "I was saying this formula only works if the lengths of these sides are equal in a right triangle. If the lengths of these sides are equal I think I can write (the formula) in this way. (Silence)" (S1I:111-113). Symbolically we can write down $h^2 = 2l^2$ where $h$ is the hypotenuse length and $l$ is the leg length, that is, the hypotenuse length is bigger than the leg length. However, it is not to rule out the right triangle he sketched during interviewing when he was attempting to prove the Pythagorean Theorem. It might also help his intuition to visualize that the hypotenuse is lengthier than the legs.

Besides his statement suggested a wording of algebraic formulae $h > a$ and $h > b$ where $h$ is the hypotenuse and $a$ and $b$ are legs of a right triangle. Accordingly, S1 seemed to use
algebraic thinking to reason geometrically.

Figure 5.76: S1’s solution to Concept Mapping Task

[Translation:

In a right triangle the opposite side to the right angle is lengthier than the other sides.]

S3 constructed synthetic statements (geometric definitions and properties) although instead of using synthetic concepts like "congruency" and "small" he used analytic concepts "equals to" and "a half" (Fig 5.77).

Figure 5.77: S3’s solution to Concept Mapping Task

[Translation:

1- An equilateral triangle is a triangle having three equal angles. (Angles)

2- The intersection between two pairs of parallel lines results in four points and joining them it results in several triangles. (Parallelism)
3- A midsegment of a triangle is a half of the hypotenuse. (Midsegment)

4- An angle bisector of a triangle divides the triangle angles into two equal angles. (Angle bisector)

5- If two triangles have two equal angles and a common side, then they are similar. (Similarity)]

Statements 1 and 4 are correct and they constitute basic concepts of Euclidean Geometry course. Statement 2 seems to be an obvious result after sketching the picture of Elaboration Task (Fig 5.71). Statement 3 is not correct for all types of triangles and Statement 5 sounds like A.S.A. (Angle, Side, Angle) congruency postulate of triangles. It is evident that congruency is a special case of similarity. Meanwhile, I think S3 was not aware about this relation as in geometry this postulate is known as A.S.A. for congruent triangles.

S4 produced geometric statements using algebraic thinking (formulae, acronyms, symbols, ratios, and proportions). He started off with a picture and thereafter he constructed the statements from the picture (Fig 5.78).
Figure 5.78: S4’s solution to Concept Mapping Task

[Translation:

*DB* is the angle bisector of $\angle B$. The following statements are true:

1.- $\angle A + \angle B + \angle C + \angle D = 360^\circ$. (The sum of the measures of the interior angles of a quadrilateral).

2.- $\Delta ABC$ is a right triangle with the right angle in $A$. It is correct $a^2 = b^2 + c^2$.

3.- It is correct $\Delta BCD \cong \Delta BDE$.

a) A.A. (Angle, Angle) postulate on similarity: $\angle D$ is a common angle
and $\beta = \gamma$.

b) The two triangles are similar, and then it is correct
\[
\frac{BE}{BC} = \frac{DE}{DC} = \frac{DB}{DB} = k = 1.
\]

Statement 1 is out of the context of Concept Mapping Task into consideration. Statement 2 refers to Pythagorean Theorem. Statement 3 seems to be similarity of triangles. However, he used the congruency symbol. He used A.A. postulate on similarity to prove his statement. Further on he used S.S.S. (Side, Side, Side) postulate on similarity to prove the same statement and he ended up with a special case of similarity where constant $k=1$ which is the condition for congruency of triangles. Moreover, he misused the acronym of Statement 3, which should be $\triangle BCD \cong \triangle BDE$ to comply with the ratios.

S5 produced synthetic statements and also used some algebraic thinking elements (ratio, equals to, and proportion) (Fig 5.79).

2. **Proposições Verdadeiras**
   
   a) A bissetriz de um triângulo divide a base em dois segmentos iguais.
   b) Linha média de um triângulo é um segmento que une os pontos médios dos lados desse triângulo.
   c) Uma reta paralela a um dos lados de um triângulo, determina um único triângulo semelhante ao triângulo dado.
   d) A razão de semelhança de dois triângulos semelhantes é igual ao quádruplo das retas entre os lados proporcionais.
   e) Um triângulo é um polígono constituído por três ângulos.

(Figure 5.79: S5’s solution to Concept Mapping Task)

[Translation:]
True statements:

1- An angle bisector of a triangle divides the base into two equal segments.

2- A mid-segment of a triangle is the segment which joins the midpoints of the triangle sides.

3- A parallel line to one of the sides of a triangle determines another triangle similar to the latter triangle.

4- The ratio of two similar triangles is equal to the ratios of the proportional sides.

5- A triangle is a polygon with three angles.

Statement 1 is not correct. It seems that S5 confounded it with the definition of a median of a triangle. Besides he used the concept “base” which is only applicable for isosceles triangle. In Statement 2 instead of saying “the segment which joins the midpoints of the triangle sides”, it should be “the segment which joins the midpoints of two sides of a triangle”. Statement 3 and 4 seem to be attempts of some theorems on similarity of triangles. For Statement 3 might be, “If a line is parallel to one side of a triangle, then the ratios of the measures of corresponding segments of the other two sides will be equal” (Lewis, 1964, p. 334). And for Statement 4, “Two triangles are similar if there exists a correspondence between the vertices in which the ratios of the measures of corresponding sides are equal” (Ibid., p.346). Statement 5 is a triangle definition, although in the literature it is defined in terms of the number of sides.

S7 produced synthetic statements although some of them with help of some algebraic concepts (ratio, proportion, and equals to) (Fig 5.80).
True statements:

1- In similar triangles the ratio of the proportional sides is constant.

2- In a triangle, if the (interior) angles are equal, then the sides are equal likewise.

3- In a triangle, if one of the (interior) angles is right (measured 90°), then the other two angles are named acute angles and their sum is 90°.

4- A triangle has at most only one angle measured 90° (right angle).

Statement 1 is a variant of S.S.S. Theorem on similarity. Statement 2 constitutes a property of the equilateral triangle. Statements 3 and 4 are corollaries of Statement a of Elaboration Task.

5.3.9.5 Discussion of the concept mapping task

Table 5.16 shows the analysis summary of the students’ solution to the Concept Mapping Task. The same coding process was used for this analysis with slight changes in comparison to the previous.

Table 5.16: The analysis summary of the students’ solution to Concept Mapping Task
Because of the nature of the data collected in this activity the coding was slightly changed as follows:

“Correct”: All key concepts present and complex statements
“correct”: All key concepts present and simple statements

By complex statements, it should imply those statements which entail a vast range of concepts and simple statements otherwise.

Table 5.16 shows that all students used algebraic thinking to construct the statements from the concepts suggested. Although in the table is indicated that only S4 used a pictorial representation to yield the statements, I am aware that the other students produced the statements through mental pictures (Hadamard, 1945). That is why I considered that all students also used their intuition (visualization and construction processes) to produce such statements.

In this activity I also noticed that the interplay between reasoning and intuition played a role in accessing and using key algebraic and geometric concepts in connection towards a correct solution (for example S4 and S7).

5.3.9.6 Emerging issues of the concept mapping task

1- All students used algebraic thinking in constructing geometric statements.

*Insight 1: The same as in Section 5.3.9.3.*

2- Three students out of five used their reasoning and intuition in connection and succeeded in the Concept Mapping Task.

*Insight 2: The same as in Section 5.3.9.3.*
3- Most students did not sketch pictures in relation to the geometric statements.

*Insight 3: The use of iconic support in mathematical problem solving is of paramount importance, wherever students meet a certain concept for the first time. However, there is a need to move along the continuum to abstraction so that image schemata – a bridge between abstract logical structures and particular concrete images and experience – may become more flexible and abstract.*

5.3.10 Discussion of the Elaboration and Concept Mapping task

In both activities it is to notice that all students used algebraic thinking to solve the geometric tasks. Besides, the interplay between reasoning and intuition was beneficial for correctly solving them. At this stage it seemed that the students used diagrammatic representation for the purposes Koedinger and Anderson suggested in geometric problem solving. The reason of the students’ change regarding diagrammatic representation use maybe is due to Euclidean Geometry course they attended during the first semester. Meanwhile in Concept Mapping Task most of the students did not sketch pictures to help them think in yielding true geometric statements. Maybe the possible explanation of this phenomenon is that the use of iconic support in mathematical problem solving is of paramount importance, wherever students meet a certain concept for the first time. However, there is a need to move along the continuum to abstraction so that image schemata – a bridge between abstract logical structures and particular concrete images and experience – may become more flexible and abstract.

5.3.11 Emerging Issues of the Elaboration and Concept Mapping Task

The main emerging issues to these activities I list below:

1- It seems that the prolonged exposure to algebraic teaching during schooling period obliged these students to apply algebraic thinking in solving geometric tasks even after attending Euclidean Geometry course at Universidade Pedagógica.

2- The interaction of deductive (as a means of discovery) and empirical (as a means of development of intuition) approaches to geometry led some of these students to reap the benefits of their knowledge (Schoenfeld, 1986).
3- It is likely that these students have had prior experience with geometric images during Euclidean Geometry course than with formal notations, and since diagrams typically have the locality feature, they were likely to find perceptual inferences in this domain easier than symbolic inferences (Koedinger and Anderson, 1990).

4- The use of iconic support in mathematical problem solving is of paramount importance, wherever students meet a certain concept for the first time. However, there is a need to move along the continuum to abstraction so that image schemata – a bridge between abstract logical structures and particular concrete images and experience – may become more flexible and abstract.

5.3.12 Conclusion of the Elaboration and Concept Mapping Task

The findings of this activity showed that these students used algebraic thinking to solve geometric tasks. Even those tasks whose structure allowed synthetic strategies, these students used some algebraic thinking elements (acronym and symbols) to identify geometric objects (angles, polygons, vertexes, lines, and segments). Moreover, they also used pictures (visualization and construction processes) to inform their analytic (algebraic) strategies. This means that algebraic and geometric concepts mutually aided each other towards students’ conceptual geometric understanding.

5.3.13. Research Questions: Discussion

In this study, the aim was to explore how students work with geometry problems, what algebraic knowledge, if any, they bring to the solving of these problems, and how knowledge of algebra and algebraic ways of thinking promotes or hinders success (ie conceptual geometric understanding and performance) in geometry problem-solving processes. Accordingly, three research questions were considered. I discuss two in this first component of the main study – the Euclidean Geometry course due to the nature of data collected.

The first question and sub questions attempts to find out whether first-year university students use algebraic symbols and algebraic relations, different representations, and patterns and generalisations (aspects of algebraic thinking considered in this study) while solving geometry problems. If yes, then my concern was to find out the meanings held by the target students of different algebraic and geometric concepts involved in problem solving situations
on (1) how they make relationships amongst those concepts; (2) how students use them and (3) how students account for the purpose of those concepts. The first type of meaning concerns the issue of context. The second and the third deals with method and validation aspects of meaning (epistemological meaning). To find out these three aspects of meaning, tests, interviews, and concept maps were designed.

The second question focuses on conceptual geometric understanding (visualization, construction and reasoning working in synergy) and performance possibly triggered by algebraic thinking. Conceptual understanding should be understood as such understanding rich in relationships (Even, 1988) and performance is considered as a result of conceptual understanding (Mousley quoting Knapp et al, 1995). Attempting to see these elements of conceptual geometric understanding in my subjects I referred to the same data collection instruments mentioned in first research question.

The third question is concerned with the algebraic and geometric concepts of the first year university geometry course which comprises the Euclidean Geometry (first semester) and Analytic Geometry (second semester). In order to answer this question, the students reflected on their own learning process during the geometries courses. This reflection consisted in responding to an open ended question on the relation between the two courses and how they assessed their geometrical understanding. The syllabi of those courses were also collected as an additional data source although they were not extensively analysed.

Answers to these questions have been provided in the analysis and concluding sections of the chapter most of which appear under the title “emerging issues”.

Below I present the major research findings regarding the first two research questions organized by themes previously set up and others that emerged during the data analysis of the first component of the main study – the Euclidean Geometry course.

*Algebraic and geometric knowledge base and connectedness*

The Diagnostic test results showed that the target students solved some geometric tasks using both algebraic and geometric thinking.

Only Student 1 accessed and used algebraic thinking (reasoning) to obtain a geometrical insight to solve the problem. Nevertheless he was not able to obtain the solution. The
remaining students tried to solve it using geometrical thinking only (intuition) but were unsuccessful. This result replicated what Schoenfeld (1986) found that unless students learn to take advantage of both deductive (as a means of discovery) and empirical (as a means of development of intuition) approaches to geometry and learn to profit from the interaction of those two approaches; students will not reap the benefits of their knowledge.

The analysis of the transcripts regarding Interviewing Phase 1, I showed that the target students solved some geometric tasks using both algebraic and geometric thinking.

It seemed easier for these students to use their intuition (synthetic strategies) rather than their reasoning (analytic strategies) in the tasks where they are qualitatively presented (no quantities). However, they faced difficulties in getting a solution. Besides these strategies seemed to be specific and applied to a concrete situation. In such tasks I needed hint to most of the students either using visualization and construction processes or reasoning driven by algebraic thinking. I noticed that their reasoning was stronger than their intuition. When I prompted them to elicit their intuition I was forced to supply them all key concepts to get the solution. When I elicited their reasoning through algebraic thinking, I noticed that the intuition of some of them was elicited and became robust towards the solution. All students through hinting process used both intuition and reasoning (a numerical approach, patterning, formulae and relations) to solve the tasks. Accordingly it also showed that the interaction of deductive (as a means of discovery) and empirical (as a means of development of intuition) approaches to geometry led these students to reap the benefits of their knowledge.

**Analytic versus synthetic strategies**

Most of the students claimed that the analytic strategies were insightful towards a geometric solution. This fact seems to corroborate my assumption that algebraic thinking can be seen as a way of layering meanings on each other, connecting between ways of knowing and seeing, rather than as a way of replacing meaning of each other. These meanings became reshaped as learners exploited available tools in algebra to move the focus of their attention onto new objects and relationships in geometry. Although algebraic thinking shed light in geometrical understanding, still the students needed to possess visualization and construction skills to accomplish the geometric tasks successfully.

**Students’ academic and professional background**
Due to the differences in teaching and learning experiences and contexts amongst the students before entering UP, I expected and confirmed the differences in the approaches they used, in the problem solving situations in the Diagnostic test. Meanwhile, after attending the Euclidean Geometry course I noticed on one hand a common tenet regarding the diagrammatic representation use: it seemed that these students used diagrammatic representations with the two purposes mentioned by Koedinger and Anderson (1990). On the other hand, most of them showed a strong tendency to “algebraise” geometry as even those tasks, which could be constructed and solved synthetically; they tried to use analytic (algebraic) strategies even after attending the Euclidean Geometry course. I may explain this fact due to a prolonged exposure of algebra teaching against geometry teaching during schooling time.

5.4 The Main Study – Analytic Geometry Course

The second component of the main study was concerned with the Analytic Geometry course. The data analysis of this component consists of Interviewing Phase 2 regarding the lecturer’s test, and an Open Ended Question to the students in the relation to the two geometries: Euclidean Geometry and Analytic Geometry. The overall analysis structure adopted for each data source is as follows: 1- Analysis, 2- Discussion, 3- Emerging Issues, and 4- Conclusion. Moreover there is a section on “Research question: Discussion” where a holistic view is presented for the entire Analytic Geometry course data analysis and interpretation regarding the research questions of the study.

5.4.1 Analysis of Interviewing Phase 2

The same approach used for Interviewing Phase 1 analysis was used for the analysis of Interviewing Phase 2 transcripts (Section 5.3.5). The matrix in Appendix 7 is an example of a scheme for analyzing students’ transcripts related to Interviewing Phase 2 responses.

The lecturer's test on Analytic Geometry consisted of several tasks (Appendix 5). I interviewed the five target students on Task 3, 4 and 5. I found Task 3 more insightful as during interviewing, I noticed that this task showed the importance assigned to algebra as a tool for geometry (Kline, 1972). Kline states that this importance has to do with the
possibility to “recognize the typical problems of geometry and to bring together problems that in geometrical form would not appear to be related at all” (ibid., p. 314). He asserts that “Algebra brings to geometry the most natural principles of classification and the most natural hierarchy of method” and that “Not only can questions of solvability and geometrical constructability be decided elegantly, quickly, and fully from parallel algebra, but without it they cannot be decided at all. Thus, system and structure were transferred from geometry to algebra” (ibid., p. 314).

I read all the five transcripts and similarly used software Atlas.ti to code the different episodes I found important for answering the research question beforehand posed.

I present Task 3 in Fig 5.81.

![Figure 5.81: Task 3 from the lecturer’s test](image)

[Translation: Determine angle $\alpha$ between the segment lines $AB$ and $CD$ with $A(-3;2;4)$, $B(2;5;-2)$, $C(1;-2;2)$, and $D(4;2;3)$.]

S1 explained how he determined the angle between segments AB and CD using the dot product of two vectors. He used segment and vector interchangeably. Later I searched for the reason he did it. I present his solution to Task 3 on the lecturer’s test in Fig 5.82.
The following extract of his interview shows a detailed explanation on how he solved this task. I saw that he reasonably used algebraic thinking aspects (acronyms, formulae, and algebraic transformations); however, he made some mistakes, namely, a calculation mistake ($6^2 = 30$) and a misuse of symbols ($CD = (D - A)$). He explained that in order to determine the measure of angle $\alpha$ he used a calculator. The extracts of the Portuguese transcripts can be found in Appendix 10.

**S1:** (S1II:3-17) [(He reads the task). Then, I solved this task this way. Determine the angle between segments AB and CD. I have here... I have here two segments in tridimensional space. First, I had to determine the segments. Yes, I had to determine the segments. I have the initial and the terminal points. I have point A and point B. The coordinates of point A are -3, 2, and 4; and for point B 2, 5, and -2. First, I need to know the coordinates of this segment. How? We get segment AB subtracting B and A. Thus, we get two minus, minus three equals five. Subtracting minus and minus becomes positive and we get five. Five minus two equals three and minus two minus four equals minus six. Thus, I get segment AB. Likewise, I get segment CD. Hence, co sinus of alpha consists of segments AB and CD. This is the dot...dot product. It is the dot product. I used these values I have just determined, I calculated the dot product, and... the absolute value of segment AB multiplied by the absolute value of segment CD. I get $\cos \alpha = \frac{21}{40.8}$, then $\alpha$ equals 60°. I got it using a calculator.]
S1 blindly determined the angle between the two segments. After hinting he was aware, that the segments did not intersect that is why he said that they did not form an angle. Accordingly, he stated that his solution was not correct because they yielded the dot product formula using two vectors joined at the initial points.

R: (RII:18-20) [I see that you used some formulae. I want to know eh... Here we have segments AB and CD. And we have to determine the angle between them. I noticed these segments don’t intersect at the initial points. They are set apart.]
S1: (S1II:21) [Yes.]
R: (RII:22) [Now, how have you determined the angle between these segments?]
S1: (S1II:23) [Umh... They are indeed set apart. That is correct.]
R: (RII:24) [When you yielded the formula... Are these vectors?]
S1: (S1II:25) [Yes, they are.]
R: (RII:26) [You have yielded it when using vectors...]
S1: (S1II:27) [We used vectors intersected at the initial points.]
R: (RII:28) [Now, I don’t know. Do you think these segments intersect each other at the initial points?]
S1: (S1II:29) [Silence.]
R: (RII:30) [Do you understand my concern?]
S1: (S1II:31) [Yes, I do, I do, I do. (Silence).]
R: (RII:32) [Here you are point A and point B.]
S1: (S1II:33-35) [It’s clear that they do not intersect each other. (Silence). I’ve just used the formula. When I was studying for this test, I saw this formula, which helped to determine the angle. Ya. (laughing).]
R: (RII:36) [Do you know how this formula appeared?]
S1: (S1II:37) [Yes, we demonstrated it.]
R: (RII:38-39) [How did it appear? Tell me what the position between the two vectors was.]
S1: (S1II:40-42) [The position of the vectors? They intersected each other at the initial points. When I was solving the task, I was not aware of it. I was not aware of it.]
R: (RII:43) [Then, what do you think about your solution?]
S1: (S1II:44) [I think it’s not correct.]
R: (RII:45) [Can you repeat?]
S1: (S1II:46) [I think it’s not correct. (Laughing).]

S1 was confused about the angle between two segments in space (3D) whereby they do not intersect each other. He suggested the extension of one of the segments to enable intersection in order to visualize the angle between them. It may not be enough to enable intersection in the 3D extension. S1 reasoned, as the segments were coplanar (2D). However, even if the extension does not enable intersection between the two segments, we can transfer the 3D picture to 2D and visualize the angle between the two segments as if they intersected each other according to this convention: We will define the angle between any two directed skew
lines to be the angle between any two intersecting lines which are respectively parallel to the skew lines and similarly directed (Lehmann, 1942, p. 281).

R: (RII:47) [Why do you think it’s not correct?]
S1: (S1II:48) [(Silence).]
R: (RII:49) [Well, let’s see. We have this segment.]
S1: (S1II:50) [Yes.]
R: (RII:51) [Let’s suppose we have another segment here.]
S1: (S1II:52) [Yes.]
R: (RII:53) [Is there any angle between these segments?]
S1: (S1II:54) [If we extend the segments, we can get intersection.]
R: (RII:55) [Can you extend them?]
S1: (S1II:56) [(Silence).]
R: (RII:57) [How can you determine the angle between these segments?]
S1: (S1II:58) [(Silence).]
R: (RII:59) [Is there any angle between them?]
S1: (S1II:60) [Well, like this there is no angle. But if we extend, we can get the angle.]
R: (RII:61) [Can you extend them?]
S1: (S1II:62) [(He extends the segments).]
R: (RII:63) [Then, do you think this is the angle between these segments?]
S1: (S1II:64) [Yes.]
R: (RII:65) [Why?]
S1: (S1II:66, 67) [(Silence). I stated yes, because there is an intersection between them. That intersection yields this point that is why I consider this part as the angle between them.]

I asked him whether it was possible to make a parallel translation of segment CD in order to intersect point C and point B. He kept silent. I asked him whether he learnt translation and whether the properties of the object would change after this geometric change. He asserted that the properties of the object would be preserved. He made a translation of segment CD and it intersected segment AB at point B. He explained how he made such translation and used parallel lines and he said he would use a compass in order to preserve the length of the segment. I asked him whether the angles obtained through extension (a translation in the same direction of the segment) and the translations were equal. He hesitantly said they were equal as alternate interior angle. Actually if we consider them as coplanar segments, the angles would be equal as corresponding angles.

R: (RII:68, 69) [Is it possible to make a translation of this segment in order to intersect point B and A? That is, is it possible to make a translation?]
S1: (S1II:70) [(Silence).]
R: (RII:71, 72) [You made an extension. Is it possible to make a parallel translation of this segment in order to intersect at this point?]
S1: (S1II:73) [(Silence).]
R: (RII:74) [You have learnt translation, haven’t you?]
S1: (SIII:75) [Yes, I have.]
R: (RII:76) [When you make a translation of an object A, does it change its properties?]
S1: (SIII:77) [No, it does not. Its properties don’t change.]
R: (RII:78) [Is the angle obtained through translation the same as the angle obtained by extension?]
S1: (SIII:79, 80) [If I make a translation of this segment preserving its properties, I think its length …]
R: (RII:81) [Can you do such a translation and see what happens?]
S1: (SIII:82) [(He is doing a translation silently.)]
R: (RII:83) [You are doing a...]
S1: (SIII:84) [I’m sketching a segment.]
R: (RII:85) [What transformation is that one?]
S1: (SIII:86) [Geometry.]
R: (RII:87, 88) [It’s geometry. But here it’s a translation. You moved this segment to this point. Is it possible to do that or not?]
S1: (SIII:89) [It’s possible.]
R: (RII:90) [Then, how have you done it?]
S1: (SIII:91) [(Silence).]
R: (RII:92) [This is a parallel line...]
S1: (SIII:93) [Ah, it’s a parallel line.]
R: (RII:94) [I think you’ve learnt it in Euclidean Geometry.]
S1: (SIII:95) [Of course.]
R: (RII:96) [And then?]
S1: (SIII:97) [I use a compass to sketch a segment with the same length as this one.]
R: (RII:98) [Is it possible to do that?]
S1: (SIII:99) [(He is sketching). It’s possible.]
R: (RII:100, 101) [Now, you have an angle here. Is it the same angle as the one you got through extension?]
S1: (SIII:102) [No, it is not. ( Laughing).]
R: (RII:103) [Is it not?]
S1: (SIII:104) [No, but ... it is... it is possible.]
R: (RII:105) [How do you call such angles?]
S1: (SIII:106-108) [These are alternate interior angles. We get them if two parallel lines are cut by a transversal. Then, these angles are equal. They are angles... (Silence).]

S1 explained why the formula works using his sketch and geometrical explanation (construction, parallelism, invariance of length, congruent angles, and a theorem).

R: (RII:109-111) [Let’s say you explained your solution using geometric thinking. Now, you are using a formula yielded when two vectors intersect each other.]
S1: (SIII:112) [Yes.]
R: (RII:113) [Now, the issue is the following: Do you think this formula satisfies the geometric transformations you have accomplished?]
S1: (SIII:114-119) [I think so. It does. We can move this segment and get...
another which intersects the second segment using parallelism, parallel lines, and with a compass we preserve the length of the segment. And we get... two equal angles. We get two lines cut by a transversal. These angles are equal. There is a theorem which supports this. If this transformation is possible, then I can use this property.]

S1 had difficulties in comparing a vector and the corresponding segment. He stated that what he knew was to determine a vector from a segment using the initial and the terminal points. After the hinting process he realized that the length of a vector was the same as the length of the corresponding segment.

R: Agora a minha pergunta, o sr usou vectores e aqui fala-se de segmentos. (RII:120) [Now my question is: Did you use vectors instead of segments in the formula?]
S1: Segmentos. (S1II:121) [Segments.]
R: Existe alguma relação entre vectores e segmentos? Por ex. aqui temos o segmento AB e o sr achou o vector AB. Então qual é relação entre vectores e segmentos? (RII:122, 123) [Is there any relationship between vectors and segments? For instance, here we have segment AB and you determined vector AB.]
S1: (S1II:124) [Yes. (Silence.).]
R: (RII:125) [I don’t know whether...]
S1: (S1II:126-129) [I do understand, I do understand. From the segments I got the vectors. Afterwards, I used the vectors. I think there is a relationship. From a segment we can get the corresponding vector with the initial and terminal points. Thus, I used the coordinates of the segment; I used the coordinates of the segment.]
R: (RII:130) [Do you think the vector and the corresponding segment have the same nature?]
S1: (S1II:131) [No, well... from the segment I can get the corresponding vector.]
R: (RII:132) [Then what is the relationship between the vector and the corresponding segment?]
S1: (S1II:133) [(Silence).]
R: (RII:134) [Is it the same concept?]
S1: (S1II:135) [Between the segment and the vector?]
R: (RII:136) [And the vector.]
S1: (S1II:137-139) [(Silence). Well, what I know is that from the segment or better from the segment coordinates I can get the corresponding vector. But I’m not able to explain the essence of the relationship between the two concepts. I have no idea.]
R: (RII:140) [What are the properties of a segment and for a vector?]
S1: (S1II:141) [(Silence).]
R: (RII:142) [Does a segment have a direction?]
S1: (S1II:143-148) [A segment has no direction. But the vector has. Ah, well, to determine this segment I needed to give direction to the vector. I gave it a direction; I gave direction to the vector. I used the initial and the terminal
points, I subtracted them and I got the direction of the vector. From that stage on I have... the segment does not have direction. Once I got the vector from the segment, I gave direction to the vector.]

R: (RII:149) [What is the relationship between the vector length and the segment?]

S1: (SIII:150) [Sorry, I didn’t get it.]

R: (RII:151, 152) [A vector has a direction and a length. What is the relationship between the segment and the vector length?]

S1: (SIII:153) [(Silence).]

R: (RII:154) [The length is a measure, isn’t it?]

S1: (SIII:155) [Yes, the length is a measure indeed.]

R: (RII:156) [And the vector length?]

S1: (SIII:157) [(Silence).].

R: (RII:158) [Is it a measure or not?]

S1: (SIII:159) [Are you referring to the vector, to the vector coordinates?]

R: (RII:160) [What is the length of this vector?]

S1: (SIII:161) [(Silence).]

R: (RII:162) [Have you heard about norm?]

S1: (SIII:163) [Yes, I have.]

R: (RII:164) [What is the norm of a vector?]

S1: (SIII:165) [The norm is the measure of a vector.]

R: (RII:166) [It’s the measure of a vector. What is the vector measure?]

S1: (SIII:167) [It’s the norm of a vector.]

R: (RII:168) [You are thinking...]

S1: (SIII:169) [I’m trying to use a formula.]

R: (RII:170) [Of the norm?]

S1: (SIII:171) [Yes.]

R: (RII:172, 173) [Here you used the norm. (I was referring to the formula used by S1 to determine the angle.)]

S1: (SIII:174) [This is exactly the norm. When I use the norm, the product...]

R: (RII:175) [Where is the norm in this formula?]

S1: (SIII:176) [It’s below in the formula. (See the formula used by S1).]

R: (RII:177) [Which one?]

S1: (SIII:178) [Vector AB, the absolute value of vector AB.]

R: (RII:179) [Is the absolute value of vector AB its norm?]

S1: (SIII:180) [No, the absolute value of vector AB is the norm...]

R: (RII:181) [What do these values $\sqrt{64}$ and $\sqrt{26}$ stand for?]

S1: (SIII:182) [(Silence).]

R: (RII:183, 184) [We have the norm of vector AB multiplied by the norm of vector CD. You said the norm was the absolute value of a vector or its length. Then, $\sqrt{64}$ is...]

S1: (SIII:185, 186) [It’s the measure ... Ah, that’s fine. The norm... sorry, $\sqrt{64}$ is the value of the norm of vector AB.]

R: (RII:187) [What is the length of segment AB?]

S1: (SIII:188) [(Silence).]

R: (RII:189) [It’s exactly the norm of vector AB, isn’t it?]

S1: (SIII:190) [(Laughing). The norm of vector AB.]
Before hinting he knew that the translation preserved the properties of an object but he did not link this idea with vector coordinates invariance. S1 stated the coordinates of a vector would change after translation. But after some hinting he realized that coordinates of the initial and the terminal points of a vector would change but the vector coordinates would be preserved. Hence, he correctly explained why the formula also worked for non-intersecting segments, which is the case of Task 3.

R: (RII:191-193) [It's the same. My question is: When we make a translation of this vector, are its coordinates the same or different and why?]
S1: (SIII:194-197) [These are points in space. When we make a translation of the segment, its point C to this point, it’s clear that its coordinates change. (Maybe S1 considered the coordinates of the initial and terminal points of the segment which would change.)]

R: (RII:198) [Will the vector coordinates change?]
S1: (SIII:199) [Yes.]

R: (RII:200) [Then, the vectors are different because their coordinates are different or...]
S1: (SIII:201) [No, they may not even be different. They may not be different.]

R: (RII:202) [Are the vectors different or not when we make a translation?]
S1: (SIII:203) [(Silence). No, they are not.]

R: (RII:204) [And are their coordinates different or are they the same?]
S1: (SIII:205) [The coordinates will change. The coordinates will change.]

R: (RII:206, 207) [Will different coordinates represent the same vector?]
S1: (SIII:208) [Silence. I don’t think so.]

R: (RII:209) [Suppose we want to make a translation of this vector. This is an example.]
S1: (SIII:210) [Alright.]

R: (RII:211-213) [Let's consider a simple example departing from the origin of the coordinate system. For example we have vector OA which has coordinates (2, 1). Let's make a translation of it to any position. Let's suppose to this point...]
S1: (SIII:214) [Point (0, 0) moves to...]
R: (RII:215) [Point(0, 0)moves to point (1, 1). And this one to... let’s preserve the...]
S1: (SIII:216) [The length.]
R: (RII:217, 218) [The length, yes. Then, we move two units horizontally and a unit vertically. What would be the coordinates of this point?]
S1: (SIII:219) [Three in this case.]

R: (RII:220) [And what about the other coordinate?]
S1: (SIII:221) [Here we’d have the two, two.]

R: (RII:222, 223) [My question is: We made a parallel translation of this vector, didn’t we? Suppose we got vector BC. Are the coordinates of vectors BC and OA the same?]
S1: (SIII:224) [No, they are not. No, they are not.]

R: (RII:225) [Can you write down the coordinates of vector OA?]
S1: (SIII:226) [Here we got point O(0, 0) and point A(2, 1).]
R: (RII:227) [What are the coordinates of vector OA?]
S1: (S1II:228) [The coordinates of vector OA are two... and one (calculating 2-0 and 1-0).]
R: (RII:229) [What about vector BC?]
S1: (S1II:230, 231) [For vector BC we have points B(1, 1) and C(3, 2). Then, the coordinates of vector BC are (2, 1). (Laughing).]
R: (RII:232) [What have you noticed? Translation...]
S1: (S1II:233) [The translation doesn’t change the vector coordinates.]
R: (RII:234) [What is your conclusion?]
S1: (S1II:235-244) [The translation changes the initial and terminal points of a segment, however it doesn’t change the segment coordinates. The initial and terminal points change, but the segment coordinates don’t change. For instance, we have here the point (0, 0) and the point (2, 1). Here we have the point (3, 2). I changed the initial and the terminal point coordinates, but the... the vector coordinates don’t change. Through translation of this segment I can change the initial and the terminal points. But I don’t change the vector coordinates. Accordingly, I can use this formula. Because I only changed the initial and the terminal points of this segment, but I don’t change the vector coordinates. This fact allows me to use this formula to determine the angle between two segments.]

S3 used the formula of the dot product to get the measure of the angle between vectors AB and CD (Fig 5.83). He also constructed a picture to visualize the segments involved. For him it was clear that the segments did not intersect. However, he was aware of the existence of an angle between the two non-intersecting segments.

S3: (S3II:7-12) [It was to determine... the angle (between segments AB and CD). First, I determined (segment) AB as it were a vector and also CD as a...

Figure 5.83: S3’s solution to Task 3 on the lecturer’s test
vector. I got these coordinates. To determine the angle I used the dot product which says (vector) \( \mathbf{a} \) multiplied by (vector) \( \mathbf{b} \), this is a product between two vectors, equals the absolute value of 1 multiplied by the absolute value of 2 multiplied by cosines of the angle (between segments \( \mathbf{AB} \) and \( \mathbf{CD} \)). I used this formula to determine the angle.]

S3 sketched the vectors involved and explained that to visualize the angle; a segment should be moved parallel towards the initial point of the other segment.

\[ S3: \text{(S3II:15)} \] [This is segment AB. This one is segment CD.]

\[ R: \text{(RII:16, 17)} \] [I see that these segments are set apart. How can you determine the angle between them?]

\[ S3: \text{(S3II:18-20)} \] [Yes, I considered the following... We have this segment and another one. If I move one of them to this position, in parallel, if I move this one to this position, I always get an angle between them.]

S3 correctly explained the relationship between a vector and the corresponding segment. He said the relationship between the two lay in length.

\[ R: \text{(RII:32, 33)} \] [I see that you used vectors instead of segments. What is the relationship between segments and vectors?]

\[ S3: \text{(S3II:34)} \] [The relationship between segments and vectors?]

\[ R: \text{(RII:35)} \] [Yes, here we deal with segments.]

\[ S3: \text{(S3II:36)} \] [Yes.]

\[ R: \text{(RII:37, 38)} \] [Here we deal with segments. How do you use segments as they were vectors? For instance you said vector \( \mathbf{AB} \) or is it also segment \( AB \)?]

\[ S3: \text{(S3II:39)} \] [The \( \mathbf{AB} \)? I meant the vector of segment \( \mathbf{AB} \).]

\[ R: \text{(RII:40)} \] [You meant the vector of segment \( \mathbf{AB} \). Then, what is the relationship between the two concepts?]

\[ S3: \text{(S3II:41)} \] [There is a relationship.]

\[ R: \text{(RII:42)} \] [What is it?]

\[ S3: \text{(S3II:43, 44)} \] [We can determine a vector because we have the coordinates. I think so. A segment has the initial and the terminal points.]

\[ R: \text{(RII:45)} \] [Then, the relationship between a segment and a vector lies in...]

\[ S3: \text{(S3II:46)} \] [I think it lies in length.]

\[ R: \text{(RII:47)} \] [It lies in length. Anything else Do you want to add anything else?]

\[ S3: \text{(S3II:48)} \] [No.]

S3 had the idea of translation of segments. But he doubted whether the coordinates of the vectors would change after translation or not for determining the measure of the angle. This question also arose amongst some of his classmates.
**R:** (RII:51) [You mentioned a segment translation when you were using
vectors, haven’t you?]

**S3:** (S3II:52) [Yes, I have. But we haven’t learnt about translation yet.]

**R:** (RII:53) [You have not mentioned it now, but you have it previously.]

**S3:** (S3II:54, 55) [It was... was an idea I had. Anyway, we have two lines, and
there exists an angle between these lines.]

**R:** (RII:56) [Of course, there must be an angle. Now, how do you determine
the measure of that angle?]

**S3:** (S3II:57, 58) [My idea was to construct a parallel line to this one and
move to this position. Hence, I get this angle between these two lines.]

**R:** (RII:59-62) [You had an idea about translation. Now how do you relate
this geometric transformation with the calculations you did? Can you explain
me about that?]

**S3:** (S3II:63) [I did not relate the translation with any calculation.]

**R:** (RII:64) [I think the vectors can show that...]

**S3:** (S3II:65-68) [By the way, after taking this test there was a debate, not a
debate... there was... about this task. Some of my classmates and I did not
know exactly, after translation, whether we had to change the coordinates or
not to determine the measure of the angle.]

**R:** (RII:69) [What was your conclusion?]

**S3:** (S3II:70, 71) [We discussed a lot, however, we did not draw any
conclusion. The question remained unsolved.]

During the hinting process, S3 was able to see that the formula worked for determining the
measure of the angle between any two segments due to the invariance of the vector
coordinates after translation.

**R:** (RII:72, 73) [Let’s suppose that we have this vector with the initial point
at the coordinate system origin. It’s a simple vector. What are its
coordinates?]

**S3:** (S3II:74) [Do you want me to write down the coordinates of this vector?]

**R:** (RII:75) [Yes, suppose vector AB. Can you write down its coordinates?]

**S3:** (S3II:76) [(He wrote down AB = (2; 1)].

**R:** (RII:77, 78) [Now, let’s make a translation of this vector to this position.
(See Fig 5.84). What point would be this one?]
S3: (S3II:81) [(Silence).]
R: (RII:82, 83) [What point is this one? I made a translation. Let’s suppose vector DC. What are its coordinates?]
S3: (S3II:84) [They would be the same, two and one.]
R: (RII:85) [What is your conclusion?]
S3: (S3II:86, 87) [There isn’t any... there isn’t any... regarding the vector there isn’t any... regarding the coordinates of the vector there isn’t any difference.]
R: (RII:88, 89) [What do you think when you make a translation of this vector to this position and its coordinates remain unchanged what is your conclusion?]
S3: (S3II:90) [The coordinates of the vector remain the same.]
R: (RII:91) [Then, what does it mean to work with the translated vector?]
S3: (S3II:92) [They are parallel vectors. The coordinates are the same because the vectors are parallel.]
R: (RII:93) [You are drawing a conclusion...]
S3: (S3II:94) [Yes, I am. The coordinates remain the same.]
R: (RII:95) [Now you know why...]
S3: (S3II:96) [Yes.]
R: (RII:97) [What do you know?]
S3: (S3II:98) [I know that if we make a parallel translation, the coordinates of the vector don’t change.]

S4 explained how he used the formula of dot product to get to the solution. He stated he accomplished algebraic transformations. For him the measure of the angle was an unknown, hence he used these transformations like a solution of an equation (Fig 5.85).
**S4**: [S4II:6-30] *Firstly, I have points. We can determine vectors with these points. I give a direction to this segment. The direction is from A to B and it becomes a vector. I determine its coordinates subtracting the coordinates of the initial and terminal points. This is vector AB. Likewise; I determine the coordinates of vector CD. These vectors are in 3D. To determine the angle between these two vectors I learnt a formula. I have here the coordinates of vector AB and CD. The angle between these vectors is exactly the product of two vectors, the dot product, that is, the dot product is equal the absolute values of the two vectors multiplied by the angle searched. I use the vector coordinates to determine the absolute values of the vectors. I also determined the dot product of these vectors using the respective coordinates. Afterwards, I determined the angle searched. I used algebraic procedures and came up with cosines of beta equals 21 over the square root of this value. Then, beta is the arcsines of this value, which is the inverse function. I got an approximate value of the angle.*

S4 considered a vector as a directed line segment. This was his justification of using vectors.
to determine the measure of the angle between segments AB and CD.

R: (RII:31) [You said you used this formula to solve the task.]
S4: (S4II:32) [Yes.]
R: (RII:33, 34) [We have here segments AB and CD. How do you use vectors instead of segments?]
S4: (S4II:35) [Let me understand your question. You ask me how I transfer...]
R: (RII:36) [How do you transfer from segments to vectors?]
S4: (S4II:37-39) [We can determine a vector from a segment. That is, from a segment I can determine a vector. In turn, this vector belongs to a segment.]
R: (RII:40) [What is meant that a vector is part of a segment?]
S4: (S4II:41, 42) [It is simple to explain. Let’s suppose that I have... points A and B.]
R: (RII:43) [Continue.]
S4: (S4II:44-49) [This is a segment. Depending on my purpose if I intend to determine an angle based on vectors, I can transform this segment into a directed segment. (For S4 a vector is a directed segment.) Before transforming the segment, it was only a simple segment. From now on, it is a directed segment, which is a vector.]

S4 used translation to explain how two skew vectors form an angle between them. He sketched the vectors to visualize the angle between them.

R: (RII:50, 51) [In order to visualize an angle between two vectors, these must intersect at the initial point, is it not?]
S4(S4II:52, 53) [It is not compulsory that two vectors must intersect at the initial points. I can determine the angle between two skew vectors like these ones.]
R: (RII:54) [How do you do that to...?]
S4: (S4II:55, 56) [Indeed, if I have these skew vectors I sketch like this. Moreover, I get this angle between them.]
R: (RII:57) [What transformation have you done?]
S4: (S4II:58-65) [I moved this vector to this position. I consider both vectors lying on the same plane. Let us suppose to define an angle only when the vectors intersect at the initial points. We would rule out the other possibilities, which is not true. The fact that the vectors do not intersect at the initial points, it does not mean that there is no angle between them. I think it is clear my position. Unfortunately, I did not sketch it. I did not sketch it. Ah, it is here, it is here.]
R: (RII:66) [Ah, you made a translation of this vector.]
S4: (S4II:67) [Yes.]

S4 stated the dot formula is generalizable for intersecting and non-intersecting vectors. This assertion underpins what the literature states that analytic strategies (algebraic transformations) are generalizable and synthetic strategies (geometric transformations) are concrete for each case (Hansen, 1998). However, he was not able to explain that the translated vectors keep their coordinates unchanged. This property allows using the formula
for determining the measure of an angle between any two vectors.

R: (RII:68, 69) [How do you explain translation in your algebraic transformations?]
S4: (S4II:70) [How can I explain translation in my algebraic transformations?]
R: (RII:71, 72) [How have you yielded the formula of the dot product? Was it with two vectors intersecting at the initial points or with two vectors separated apart?]
S4: (S4II:73) [I think the vectors met at the initial points.]
R: (RII:74) [The vectors met at the initial points. Now...]
S4: (S4II:75) [I think it was something like this. A projection was made of a vector on the other.]
R: (RII:76, 77) [How does the same formula, which was yielded with two intersecting vectors at the initial points, serve in the case of non-intersecting vectors?]
S4: (S4II:78-87) [What... what... we are doing, indeed, apparently I think... the vectors can be set apart, but it is possible to make a transformation in order to get projections. I think that in mathematics we do not have a unique strategy. It is possible to make transformations. If I write down 4, I can arrange another way... if I write down 2, I can arrange another way to write it, for instance 4 over 2. Nobody can refuse that. I mean, what counts is to apply correctly the properties correctly. This was what I did. Indeed, I would not be able to determine the angle between these two vectors, if I did not make such translation. Otherwise, it would be nonsense.]

S4 explained why he believed the formula worked for determining the angle of any two vectors. He stated that two parallel vectors had the same coordinates (equivalent vectors). He used a metaphor of a bucket saying that the properties of a bucket would remain unchanged even after moving it around. Meanwhile, he did not verify it deductively; he used only his intuition to explain the vector coordinates invariance after translation.

R: (RII:88 and 89) [you are saying that the translated vector and this one are...]
S4: (S4II:90) [They are, they are parallel.]
R: (RII:91) [They are parallel. What do their coordinates look like? Are they different or the same?]
S4: (S4II:92) [They are not different, no way. They are the same.]
R: Mesmo depois da translação? (RII:93) [Are they the same even after translation?]
S4: (S4II:94) [Even after translation, the coordinates remain the same. Otherwise, the angle would not be the same.]
R: (RII:95) [Are the coordinates the same after translation?]
S4: (S4II:96-98) [The coordinates must be the same as I said. Otherwise, the angle would not be the same.]
R: (RII:99) [Then, after translation the coordinates don’t change.]
S4: (S4II:100) [No, they don’t.]
R: (RII:101) [What is the explanation of that? If we have a movement.]
S4: (S4II:102-104) [I think the parallel vectors have the same coordinates. The parallel vectors, which lie on the same plane, have the same coordinates due to being parallel.]
R: (R1:105) [Do you make such statement because you have deduced it, you have seen somewhere, or you believe in it? What is the reason?]
S4: (S4II:106-112) [I cannot explain it. I cannot exactly explain it. But I believe in it. Firstly, I applied that principle that the coordinates only were transformed because of translation in some, some units... I believe that those units keep the coordinates unchanged. I cannot support it further. But this was the process I used.]
R: (R1:113) [Have you ever verified it when you make a translation of a vector its coordinates remain the same?]
S4: (S4II:114). [I have never verified it.]
R: (R1:115) [Meanwhile you believe in it.]
S4: (S4II:116). [Yes, I do.]
R: (R1:117, 118) [I think it is worth to verify it. Make a translation of a vector and verify whether its coordinates remain unchanged in any position, parallel translation.]
S4: (S4II:119) [(Silence).]
R: (R1:120) [Do you think it is interesting to do that?]
S4: (S4II:121-123) [Yes, it is. I have never thought about it. My reasoning was not feasible regarding verification whether translation changes or not the coordinates. I have never thought about it.]
R: (R1:124) [Does a translated object change its properties?]
S4: (S4II:125-127) [No, I think, that object was subject to movement and it changed its position, but its size and shape remain unchanged. Let us suppose a bucket. We can put it here. If I put it at 5 meters high, it remains the same bucket.]
R: (R1:128) [Is that explanation enough to explain why the coordinates...]
S4: (S4II:129, 130) [This is the issue. I do not know whether this explanation is... It can be. Now, I do not know whether it is enough or not.]
R: (R1:131) [Do you want to add anything else?]
S4: (S4II:132) [I do not think so. I do not think so.]

Before determining the angle between AB and CD, S5 wanted to know the relationships between the corresponding straight lines. And he concluded that they were parallel, hence the angle between them was zero. He used the notion of matrix to arrive at that conclusion (Fig 5.86). Nevertheless, he confused matrix with determinant. Besides, he applied the notion of determinant in a wrong way. The relationship between two straight lines in space consists of two cases: a) coplanar lines (intersecting or parallel lines) and b) not coplanar lines (skew lines). And to get to know which case is which we compose the scalar triple product of three vectors. If this product is zero the straight lines are coplanar and if it is different to zero they are not coplanar. Afterwards, we compare the direction vectors of the lines. If they are collinear the two straight lines are parallel and if not they are intersecting in the case of coplanar lines. In the case of not coplanar lines if the direction vectors are not collinear, the
lines are skew.

S5: (S5II:4-8) [Task 3 reads, determine angle alpha between segments AB and CD where the coordinates of point A are -3, 2, and 4; B(2, 5, -2); C(1, -2, 2); and D(4, 2 e 3). Then, I have these lines. My idea was to find a vector for each line, a vector for line between points A and B so as for line between points C and D.]

R: (RII:9) [Continue.]

S5: (S5II:10) [Afterwards I used a formula to determine the angle.]

R: (RII:11) [Let us see how you used those ideas.]

S5: (S5II:12) [I determined vector AB. I could called it a. I found these values.]

R: (RII:13) [What comes next?]

S5: (S5II:14-17) [I determined this matrix. Well, I think I got confused. I determine this matrix and got zero. Then I concluded that the angle was zero grades. Accordingly the segments were parallel. It might be so. Maybe if I got another formula. There is another formula. Can I use it?]

R: (RII:18, 19) [You can use it. Meanwhile, before you use it, I want to know why you used matrix and what for.]

S5: (S5II:20-22) [I wanted to know the relationship between the lines. Then, if the lines were not... not... not equal zero, I would conclude that the lines were not parallel. Moreover, I would determine the angle between them.]

R: (RII:23) [Firstly, you wanted to find out the relationship between...]

S5: (S5II:24) [I wanted to find out the relationship between the lines.]

R: (RII:25) [Why does the matrix show the relationship between the lines?]

S5: (S5II:26-28) [As...as...here for instance I got (the matrix) equals zero, it means that the lines are parallel. They would be parallel. The angle between two parallel lines equals zero grades.]

R: (RII:29, 30) [Of course, I just wanted to know how you relate the matrix and the relationship between the lines.]

S5: (S5II:31-33) [We had just learnt a topic on planes. We had been learning coplanar vectors. Vectors are coplanar when... when... eh... when the matrix equals zero.]

R: (RII:34) [What do coplanar vectors mean?]

S5: (S5II:35) [They are... are... they can lie on the same plane or they lie on parallel planes.]

R: (RII:36, 37) [How does the matrix appear to indicate that when it equals zero the lines are parallel to each other?]

S5: (S5II:38) [I use the coordinates of these vectors; I build the matrix; and I determine it.]

R: (RII:39-41) [I am still not grasping what the matrix has to do with the relationship between the lines. Maybe, you have not deduced it. Meanwhile, I believe that the lecturer has explained how the matrix appears.]

S5: (S5II:42) [Yes, he has.]

R: (RII:43) [I think he has explained how he got the matrix.]

S5: (S5II:44-49) [Yes, he has. I do not remember now, but there is... we used to... (See Fig 5.86). We used to build this matrix from these, let us say, coefficients. After determining it, we would get zero. We would conclude that the vectors would be coplanar. They would lie on parallel lines.]
Figure 5.86: S5’s “matrix”

R: (RII:50) [Now, what is the difference between matrix and determinant?]
S5: (S5II:51) [(Silence).]
R: (RII:52) [Have you heard about determinant?]
S5: (S5II:53) [Yes, I have. Yes, I have.]
R: (RII:54) [What is the difference between matrix and determinant?]
S5: (S5II:55, 56) [Well, determinant is... is... for example we can get, let us suppose the values of x, y, and z. I can determine for example the determinant of x. (Cramer formula).]
R: (RII:57) [Can you give an example?]
S5: (S5II:58-60) [For example if I want to determine the determinant of x, I do not... not... not use x. Then, I work with these values standing on this side. (Laplace theorem). Meanwhile, I think in essence both concepts are the same.]
R: (RII:61) [The difference between...]
S5: (S5II:62, 63) [The difference between...between determining determinant... In essence there is no difference. What I have done was... (Inaudible).]
R: (RII:64) [In a determinant, the number of rows and the number of lines are the same, aren’t they?]
S5: (S5II:65) [Yes, yes.]
R: (RII:66-70) [In a matrix, the number of rows and lines can be different. I wonder how you have done the calculations like those. Matrix algebra has different operations to those you have done. That is why I asked you about the difference between matrix and determinant. The calculations you have accomplished are determinant operations. Moreover, the configuration of a matrix uses brackets.]
S5: (S5II:71) [Are they brackets?]
R: (RII:72 and 73) [Yes, now I wonder whether you were talking about determinant or about matrix.]
S5: (S5II:74, 75) [Yes, I... I also understand that beyond functioning (as a determinant), it functions as an absolute value as well.]
R: (RII:76) [Does it function as an absolute value?]
S5: (S5II:77) [It functions as an absolute value as well. If I had a negative number, I would determine its absolute value.]
R: (RII:78) [Is this an absolute value?]
S5: (S5II:79) [Yes, yes.]
R: (RII:80) [Have you used absolute values in determinants?]
S5: (S5II:81-84) [In determinants? In determinants... the problem... we have not learnt determinants in the classroom. It was supposed that we already know this concept. We are obliged to search on our own.]
R: (RII:85) [You have not learnt...]
S5: (S5II:86, 87) [At the university we have not learnt this topic. Maybe we should have learnt in calculus. This topic is covered in grade 12.]
R: (RII:88) [Ah, you have learnt it in grade 12.]
S5: (S5II:89, 90) [I have learnt it in grade 12. However, I passed grade 12 many years ago. When we need to use this topic, we search this topic, which we do not remember it and it was supposed that we have learnt it previously.]

S5 used the dot product to determine the angle between vectors a and b. He correctly used algebraic thinking (formula, a trigonometric relation, acronyms, vectors, coordinates, and vector norm) (Fig 5.87).

R: (RII:91) [Now you can use the other strategy you proposed.]
S5: (S5II:92 and 93) [The other strategy, for example, was a formula to determine the angle. (See Fig 5.87).]

R: (RII:95) [Let us suppose that the result is \( \frac{1}{2} \).]
S5: (S5II:96) [Then, \( \cos \alpha = \frac{1}{2} \). I would search for an angle of which cosines equals to \( \frac{1}{2} \).]
R: (RII:97) [What does this formula determine?]
S5: (S5II:98) [In this case we are determining the angle between vectors a and b.]
S5 was confused about the relationship between a segment and the respective vector. Although he correctly stated that he would turn the segment into a vector and that the vector had a direction. He tried to explain it using the concept of line and normal which confused him more. For S5 to determine an angle between two segments it was not the same as to determine the angle between the corresponding vectors. According to him it was the same as to determine the angle between the corresponding straight lines. It seemed that his confusion lay in non-intersecting segments, which might not form an angle. Meanwhile, the lines might intersect somewhere and hence, they might form an angle.

R: (RII:99, 100) [Now, here we have segments instead of vectors. How do you replace segments with vectors? Is there a relationship between segments and vectors?]
S5: (S5II:101) [There is.]
R: (RII:102) [What is that relationship?]
S5: (S5II:103, 104) [When we have for instance a segment or a line, we can determine the corresponding... the corresponding ...normal vector.]
R: (RII:105) [Sorry, can you explain once again through...?]
S5: (S5II:106) [Through a picture?]
R: (RII:107) [Yes, through a picture. You spoke about normal vector. I cannot understand properly.]
S5: (S5II:108) [Neither can I.]
R: (RII:109) [There is segment AB.]
S5: (S5II:110, 111) [Let us suppose we have this segment, this line for example and another line here. (Silently he sketches a coordinate system.)]
R: (RII:112) [We have segment AB. What is this coordinate system for?]
S5: (S5II:113) [It is for representing the normal segment.]
R: (RII:114) [Do you mean a normal segment or segment AB?]
S5: (S5II:115) [Segment AB.]
R: (RII:116) [What comes next?]
S5: (S5II:117, 118) [We can find... ... I turned segment AB into vector a.]
R: (RII:119, 120) [Then the segment is the same as vector because you said you turned segment AB into vector a.]
S5: (S5II:121) [This is because... the... the... the vector must have a direction.]
R: (RII:122) [What about the segment?]
S5: (S5II:123) [We have segment AB.]
R: (RII:124) [What is the relationship between this vector and the corresponding segment?]
S5: (S5II:125) [(Silence).]
R: (RII:126) [Are different names for the same entity?]
S5: (S5II:127) [No, I think a segment and a vector... (Silence).]
R: (RII:128) [Here we are determining the angle between vectors a and b.]
S5: (S5II:129) [Yes, yes.]
R: (RII:130, 131) [You said, “to determine the angle between segments AB and CE, firstly I turn these segments into vectors. Thereafter I use this formula.”]
S5: (S5II:132) [Yes, yes.]
R: (RII:133, 134) [Now, I want to know whether to determine the angle between segments AB and CD is the same thing as to determine the angle between vectors a and b.]
S5: (S5II:135-140) [Here I am not determining the angle between AB (and CD). On the contrary, I am determining the angle between lines AB and CD. Maybe, here there would be another line like this. (It seems that S5 distinguishes in this case the angle between segments AB and CD from the angle between the lines AB and CD because the segments are set apart. Moreover, he uses line hoping that anywhere they would intersect each other and form an angle. Meanwhile, in space non-parallel lines can be skew lines, which do not intersect each other and apparently do not form an angle.).]
R: (RII:141) [What is that line?]
S5: (S5II:142-144) [It would be CD. Nobody said that these lines lie in this way. It is my suggestion. Here for instance, for instance I have AB... (Inaudible). This angle... (Silence).]

S5 moved in parallel segment CD and merged point C and point A of segment AB in order to visualize the angle between these segments. He explained that the formula worked for any two segments because of the norm of the vectors. Meanwhile, it seemed that he was unclear about this concept. After some hinting he at least realized that a norm of a vector represented its length which was equal to the length of the corresponding segment. However, he did not know how to determine the norm of a vector. Moreover, it seemed that S5 was insecure about the algebraic and geometric concepts involved in this task. He often tried to give an explanation, which was out of context of the task. For example he stated “I consider the norm
of segment $AB$ as the difference between vectors” (S5II:198-199). As it is known, the norm is the magnitude of a vector and the algebra of magnitudes is different from the algebra of vectors although we can find some similarities.

R: (RII:145) [Now, in your sketch point C merges with point A.]
S5: (S5II:146) [Yes, it does.]
R: (RII:147) [It is clear that the coordinates of C have changed. It cannot be the same point.]
S5: (S5II:148) [Here for example we have point B.]
R: (RII:149) [We can at least notice that these segments do not intersect at the...]
S5: (S5II:150) [They are not; they are not.]
R: (RII:151-154) [They are set apart. Now, how do you explain that this formula also works for this case? For example, we have segment AB and segment CD. Do you think there is an angle between these segments?]
S5: (S5II:155) [Between these segments? In the position, they appear...]
R: (RII:156) [They do not intersect at the initial points.]
S5: (S5II:157) [They are not.]
R: (RII:158, 159) [How can we use a formula yielded when the vectors intersect at the initial points?]
S5: Sim, sim. (S5II:160) [Yes, yes.]
R: (RII:161, 162) [How can we generalize this formula for this case as well? Do you understand my question?]
S5: (S5II:163) [Yes, yes. (Silence).]
R: (RII:164) [What do you think? You said it works. Now, why does it work?]
S5: (S5II:165, 166) [I think it works because of their norms. We should determine the norms of each line.]
R: (RII:167) [What is a norm?]
S5: (S5II:168) [A norm is...is a vector, which...is perpendicular to a line.]
R: (RII:169) [A norm?]
S5: (S5II:170) [Yes.]
R: (RII:171) [Now, when we want to determine the norm of vector $a$, what is it?]
S5: (S5II:172) [(Silence).]
R: (RII:173) [Is the norm of a vector a vector or a number?]
S5: (S5II:174) [A number.]
R: (RII:175) [What does that number represent?]
S5: (S5II:176) [The length.]
R: (RII:177) [Of which is the length?]
S5: (S5II:178) [The length of the vector.]
R: (RII:179, 180) [Is the norm the length of this vector? Do you have this idea?]
S5: (S5II:181) [Yes, I think so, but I am not sure.]
R: (RII:182) [Why aren’t you sure?]
S5: (S5II:183) [I am not sure, but the norm is the... the magnitude of a vector.]
R: (RII:184) [You said this formula also worked for this case because of the vector norms.]
S5: (S5II:185) [The norm for each vector.]
R: (RII:186-188) [The norm for each vector. Does the length of segment AB have to do with the norm of vector AB? Are they different things?]
S5: (S5II:189, 190) [No, segment AB is not the same as its norm.]
R: (RII:191) [The measure or the length of segment AB. Do you use length?]
S5: (S5II:192) [Yes, we do.]
R: (RII:193) [What is the difference between the length of segment AB and the norm of vector AB?]
S5: (S5II:194) [In this case, there is no difference.]
R: (RII:195) [Is it the same thing?]
S5: (S5II:196) [Yes, it is.]
R: (RII:197) [Why is it the same thing?]
S5: (S5II:198-199) [I consider the norm of segment AB as the difference between vectors.]

S5 suggested extending the segments into lines in order to visualize the angle as they would intersect each other if the lines were coplanar. But the question still remained unclear because even with the extension, the segments would not intersect at the initial points and the formula was yielded while the two segments (or vectors) intersected at the initial points (Fig 5.88). He suggested a way out saying that he would use trigonometry relations like cosines to determine the angle. For that purpose he suggested to construct a perpendicular to one of the lines (Fig 5.89). I notice that S5 suggested any idea which came in his mind to solve the task although such ideas seemed to be out of context of the task. That is why I kept in questioning him about the dot product formula. I argued about the length of the vectors because after the extension the length became bigger. I asked him whether the formula worked under these conditions. He suggested another idea: to make a translation of one of the segments and to join at the initial point of the other segment. He correctly said that translation preserved length and direction of the segment. However, its coordinates would change according to him. I asked him whether the formula worked for changing coordinates. He stated that he had no idea.
Let us see the following. You said we had to sketch lines like this, didn't you?

I think so.

Extending like this, we can see that...

There is an angle between them.

Extending this segment, it intersects the other segment, considering coplanar segments. Do you think whether the formula works for this case or just in case when the segments intersect at the initial points?

Well, I think this case will fall into the standard case, because we would sketch a perpendicular vector to this line.

What comes next?

I would determine the angle.

How would you determine the angle?
S5: (S5II:218, 219) [For example, if I had the measure of this, this vector I could use this ratio or another.]
R: (RII:220) [What would you use?]
S5: (S5II:221) [For this case, I would use cosines, which is the opposite leg over the hypotenuse.]
R: (RII:222) [What about the other case?]
S5: (S5II:223, 224) [In that case, I need to sketch another segment perpendicular to this line.]
R: (RII:225, 226) [Now, there is an interesting issue. You said the formula worked for both cases because of the invariance of lengths. However, when you extend the segments their lengths increase.]
S5: (S5II:227) [Yes, yes.]
R: (RII:228, 229) [How can you explain the length increase in the formula? Do you understand my query?]
S5: (S5II:230) [Yes, yes.]
R: (RII:231) [The norm of vector CD will increase.]
S5: (S5II:232) [I got your point.]
R: (RII:233-236) [I wanted to know why you use this formula in this case, because this formula was yielded when the vectors met at the initial points. However, when we extend this segment to intersect that condition its length increases. I am getting confused.]
S5: (S5II:237) [I think instead of extending the segments, we can join their initial points.]
R: (RII:238) [How do you join the initial points?]
S5: (S5II:239) [Silence. We can move this segment to this position.]
R: (RII:240, 241) [Are you suggesting moving this segment to this position? How do you call this process of turning the segment to this position?]
S5: (S5II:242) [Moving the segment.]
R: (RII:243) [Moving the segment. There is an appropriate term... I do not know whether you have learnt translation.]
S5: (S5II:244) [Translation.]
R: (RII:245, 246) [When we make a translation of this segment, does it change anything like length, direction...?]
S5: (S5II:247, 248) [No, it does not. Only we need to watch out the angle when we make a translation.]
R: (RII:249-251) [We are making a translation of this segment so that point A merges with point C. And here it lies point D. Now, this is my question. Are we working with coordinates?]
S5: (S5II:252) [Yes, we are.]
R: (RII:253-255) [When I make a translation of this segment, I keep its length and direction. This is a parallel translation. Now, what happens with the coordinates after translation? Do they change or remain the same?]
S5: (S5II:256) [They change.]
R: (RII:257, 258) [They change. Does it mean that this formula would change? The initial coordinates changed.]
S5: (S5II:259) [(Silence).]
R: (RII:260) [Do you understand?]
S5: (S5II:261, 262) [Yes, I do. We have two parallel lines, however, when this point moves to this position changes its coordinates.]
R: (RII:263-266) [Now, do you think this formula would work for this case?}
Or do you suggest another formula?
S5: (S5II:267) [Now I have no idea. (Laughing).]

After hinting S5 understood that translation preserved the vector coordinates, that is why the dot formula worked to determine the angle between any two segments. It seemed that S5 used algebraic thinking blindly. He did not relate the algebraic concepts with the geometric concepts involved in this task. Meanwhile, I noticed that it was not lack of knowledge, but lack of organization of that knowledge in a meaningful way. The hinting process served to organize his knowledge.

R: (RII:268-269) [Let us see. We are working with coordinates. We have point O. Can you tell me its coordinates?]
S5: (S5II:270) [Zero, zero, zero.]
R: (RII:271-274) [Let us consider the plane for simplicity. Afterwards we can generalize in space. Let us consider point O(0; 0) as the initial point of a vector and the terminal point A(2; 1). (Similar to Fig 5.84). What are the coordinates of vector OA?]
S5: (S5II:275) [Two and one.]
R: (RII:276, 277) [They are two and one. Now, let us make a translation keeping its direction and length. We get vector BC. What are the coordinates of vector BC?]
S5: (S5II:278, 279) [When we make a translation of this segment, I think the coordinate system moves as well...]
R: (RII:280, 281) [How do you determine the coordinates of a vector? Can you write down the coordinates of vector OA?]
S5: (S5II:282) [The coordinates of vector OA are two and one.]
R: (RII:283) [What are the coordinates of vector BC?]
S5: (S5II:284) [Vector BC, let us suppose that... in this case it would be two, the y varies and it is three.]
R: (RII:284) [What are the coordinates of point B?]
S5: (S5II:285) [The coordinates of point B are zero and one.]
R: (RII:286) [And the coordinates of point C?]
S5: (S5II:287) [The coordinates of point C are two and three.]
R: (RII:288) [How do you determine the coordinates of vector BC?]
S5: (S5II:289, 290) [We are going to have C-B. Then, two minus zero equals to two. Three minus one equals to two.]
R: (RII:291) [Why do you have three?]
S5: (S5II:292) [Well, I put three.]
R: (RII:293) [Why? Here, it is one.]
S5: (S5II:294) [Here it is one. Well, it becomes two.]
R: (RII:295) [Then, you can put here two.]
S5: (S5II:296) [Then, two minus one equals to one.]
R: (RII:297) [Now compare the coordinates of vector BC and OA.]
S5: (S5II:298) [Vectors BC and OA. The coordinates are the same. The coordinates are the same.]
R: (RII:299) [What does it mean?]
S5: (S5II:300) [It means that the coordinates are preserved.]
R: (RII:301) [Under which conditions?]
S5: (S5II:302) [When we make translation, the coordinates are preserved.]
R: (RII:303) [When we translate what?]
S5: (S5II:304) [When we make translation of vectors, in our case vector OA to vector BC, the coordinates are preserved.]
R: (RII:305) [Then, this formula works for any type...]
S5: (S5II:306) [It works for any type of translation.]
R: (RII:307) [Why?]
S5: (S5II:308) [The coordinates are preserved even if we make translation of vectors from one position to another.]
R: (RII:309) [Why do you state so?]
S5: (S5II:310) [I concluded like this because of this deduction.]

S7 also used the dot product to determine the angle between segments AB and CD. He intuitively used translation of vectors as he saw that the segments did not intersect at the initial points and the formula was yielded under this condition (Fig 5.90).

S7: (S7II:7-17) [Well, we have to determine the angle between these two lines... in space. We have these, these three points. Then we can sketch two segments AB. I have points A, B, and C. They are three points in space. Then we have to determine the angle between AB and CD. Sorry, there are four points A, B, C, and D. Then the segments are AB and CD. I determined the angle between these two segments. I considered segment AB as a vector u. I determined its coordinates by subtracting points A and B or B and A, it does not matter. It is the difference between these two points. Likewise, I did for segment CD and I got vector v. Then, I did A minus B. The coordinates of point A are -3, 2, and 4...]
R: (RII:18) [You have already solved this task in here.]
S7: (S7II:19-23) [Here I did B minus A and D minus C. It can also be A minus B and C minus D. Then, to get the angle between these segments or between these vectors, they should intersect at the initial points. If they do not intersect, there is no angle between them. That is why I considered vectors u and v intersecting at the initial points. I considered these two vectors intersecting at the initial points.]
R: (RII:24) [Carry on.]
S7: (S7II:25-34) [Then, vectors u and v intersecting at the initial points form an angle and I call angle alpha. This is the angle we want to determine. To determine angle alpha, angle alpha we have to use this formula, cosines of alpha equals to the dot... dot product of these two vectors over the product of their norms. The norm of Vector u equals to the square root of the values of x squared and y, the sum squared of x, y, and z. Also I calculated the dot product of these two vectors and I got 21. Then to get angle alpha I used the inverse function, which is arccosines of this value.]
S7 was disturbed because in the task appeared four points, which suggested two non-intersecting segments. For S7 in such case there was no angle between those segments. To solve the problem he intuitively joined the two segments at the initial points before using the formula. Hence, S7 before using a formula he wanted to make sense of it. However, he could not support his solution.

R: Não sei se tem mais alguma coisa a dizer sobre este exercício? (RII:36) [Do have anything else to add to this task?]

S7: Bom, esta foi a minha ideia. Uma vez que nos dão quatro pontos, ficaria mais claro se tivessem dado três pontos. Assim, os dois vectores teriam a mesma origem. Seria fácil tirar o ângulo. Mas neste caso nos dão quatro pontos. E por isso eu na prova fiz uma advertência tomando dois vectores $u$ e $v$ com a mesma origem. Caso contrário seria impossível tirar o ângulo entre dois vectores ou segmentos de rectas se eles não se intersectam ou não têm a mesma origem. (E7 reflectiu antes de aplicação da fórmula. Pois viu que os dois segmentos não se intersectavam e a fórmula foi deduzida com dois vectores que tinham a mesma origem. Intuitivamente criou dois vectores que partiam da mesma origem para poder aplicar a fórmula. Embora esta fórmula sirva para qualquer tipo de segmentos e em qualquer disposição). (S7II:37-46) [Well, this was my idea. It was clearer if they provided us with three points instead of four. Thus, the two vectors would have the same initial point. It would be easier to get the angle. But in this case, we have four points. That is why in the test I warned about this using vectors $u$ and $v$ with the same initial point. Otherwise, it would be impossible to get the angle between these two vectors or segments. (S7 reflected before he used the formula. He saw that the segments did not intersect at the initial points and the formula was yielded under that condition. Intuitively he constructed two vectors with the same initial point in order to use the formula, although this formula is generalizable for determining the angle between any two segments).]

R: E aí aonde queria chegar. Porque aqui nós temos segmentos, que não estão na mesma origem. Agora como consegue traçar o ângulo entre estes dois segmentos que não estão na mesma origem considerando os respectivos vectores aplicados na mesma origem? O que lhe possibilita a fazer isso? Esta
relação? Não sei se percebeu a minha pergunta? (RII:47-50) [Actually, I wanted to question about this issue. Here we have non-intersecting segments. Now you transformed that situation using two vectors with the same initial point. What knowledge have you used to support such transformation? Are you with me?]

S7: Estou a ver. (S7II:51) [Yes, I am.]

R: Como dissesse se fossem três pontos poderíamos traçar os dois vectores e veríamos que estariam na mesma origem. Mas aqui tem dois segmentos e são dois segmentos que não se intersectam num mesmo ponto. O que lhe possibilita a fazer o que fez e ter a certeza que está a achar o ângulo entre estes dois segmentos? (RII:52-55) [As you said, if we had three points, we would construct two vectors with the same initial point. However, here we have non-intersecting segments. What knowledge supports your solution and why are you sure that the angle you determined is the angle between these segments?]

S7: (Silêncio). Bom... (silêncio). (S7II:56) [(Silence). Well... (Silence).]

S7 considered segment, vector and line geometric objects determined by two points. He was aware that segment and vector were different geometric objects. However, he was not able to explain it fully. He said a segment is the length; the distance between two points and a vector was not. Nevertheless, he accepted that the vector had a length which was its norm. He was aware that the segments given were not coplanar. But in order to determine the angle between them it was necessary to consider them coplanar and collinear as only under this condition these segments could intersect and form an angle between them.

R: (RII:57) [Why have you turned segments into vectors?]

S7: (S7II:58) [Why have I turned segments into vectors?]

R: (RII:59) [You have here segments AB and CD and you turned them into vectors.]

S7: (S7II:60-63) [Given two points we can determine a vector. Also given two points we can determine a segment. Moreover, through two points we can define a line.]

R: (RII:64) [What is the relationship between a segment and a vector.]

S7: (S7II:65, 66) [To determine the vector here I subtract the points. The difference between A and B yields a vector.]

R: (RII:67) [What is the difference between a vector and the corresponding segment? Or...]

S7: (S7II:68) [(Silence).]

R: (RII:69) [Is it the same thing?]

S7: (S7II:70) [There is a difference.]

R: (RII:71) [What is the difference between segment AB and vector AB?]

S7: (S7II:72-75) [When we deal with segment AB, we are considering the length or the distance between two points. Vector AB is not. It is... (Silence). It is... (Silence).]

R: (RII:76) [Does a vector have a length?]

S7: (S7II:77) [Yes, it has. It is the norm of a vector. Yes, it has.]

R: (RII:78-80) [Here we have two non-intersecting segments. How do you
support the relationship between the angles between these two segments and between the corresponding vectors intersected at the initial point? Is the same angle?]
S7: (S7II:81-84) [Well, it is the following idea. I got confused. We have the concept of collinear vectors and coplanar vectors. And... (Silence). Well, these two vectors, these two vectors are coplanar, because both lie on... both lie on the same plane.]
R: Os vectores $\vec{AB}$ e $\vec{CD}$ estão no mesmo plano? (RII:85) [Do vector $\vec{AB}$ and $\vec{CD}$ lie on the same plane?]
S7: (S7II:86, 87) [No, I considered these vectors as coplanar vectors. If these vectors are coplanar, then,...]
R: (RII:88) [Continue.]
S7: (S7II:89, 90) [Then, we can consider these vectors... as intersecting at the initial point.]
R: (RII:91) [Continue.]
S7: (S7II:92-100) [Under this condition, we can determine the angle between these vectors. (S7 always sought answers to all questions. Although he showed some limitations in explaining certain concepts due to maybe lack of knowledge. Meanwhile, he reasoned before he used the formula.)]
R: (RII:101) [Anything else?]
S7: (S7II:102) [No.]

5.4.2 Discussion of Interviewing Phase 2

All students used analytic approaches (the dot product formula, coordinates, vector, trigonometry relations, matrix, acronym, equation, and unknown) to solve Task 3. Meanwhile, some of them (S3, S4, and S7) used pictures to inform their analytic approach. During interviewing, I noticed that S1 and S5 used the formulae in a rote fashion while S3, S4, and S5 showed connections between synthetic and analytic approaches with some limitations. Below is a categorization of the subjects according to the strategy used, connectedness of their knowledge, and disposition (mastery and performance orientation).

Analytic strategy

S1 and S5 used purely analytic strategy. S1 applied the dot product formula to solve Task 3. He reasonably used algebraic thinking aspects (acronyms, formulae, and algebraic transformations); however, he made some mistakes, namely, a calculation mistake ($6^2 = 30$) and a misuse of symbols ($\vec{CD} = (D - A)$). He explained that to determine the measure of angle $\alpha$ he used a calculator. S5, before determining the angle between AB and CD, wanted to know the relationships between the corresponding straight lines. He used the notion of matrix to arrive at that conclusion. Nevertheless, he confused
matrix with determinant. Besides, he applied the notion of matrix inappropriately. In addition, he stated that the “matrix” result was zero; hence, the lines were parallel. He said the angle between parallel lines was zero.

Hybrid strategy (analytic and synthetic strategies)

S3, S4, and S7 used analytic strategy informed by some aspects of synthetic strategy. All of them used the dot product formula to solve Task 3. In the meantime, to make sense of the formula they sketched a picture. During interviewing, they explained the meaning of the picture to inform the formula. To visualize the angle between two non-intersecting segments; they moved one of the segments (vectors) towards the initial point of the other segment (vector). They applied an isometric geometric transformation known as translation.

Mastery oriented group

Three students (S3, S4, and S7) applied the dot product formula with understanding. They were aware that the dot product formula worked for determining the measure of the angle between any two segments. Moreover, the synthetic strategies (e.g. translation, extension, intersection, skew lines, parallel lines, and intersecting lines) helped them to enforce their understanding of the generalizability aspect of this formula. Some excerpts of their transcripts show this fact.

(S3II:18-20): Yes, I considered the following... We have this segment and that one. If I move one of them to this position, in parallel, if I move this one to this position, I always get an angle between them.

S3 was sure that the non-intersecting segments form an angle and through translation, he was able to visualize it. He knew that isometric transformations in geometry preserve the properties of the objects. This synthetic knowledge helped him to apply the dot product formula with understanding.

(S4II:58-65): I moved this vector to this position. I considered both vectors lying on the same plane. Let us suppose to define an angle only when the vectors intersect at the initial points. We would rule out the other possibilities, which is not true. The fact that the vectors do not intersect at the initial points, it does not mean that there is no angle between them. I think it is clear my position. Unfortunately, I did not sketch it. I did not sketch it. Ah, it is here, it is here.
(S4II:78-87): What... what... we are doing, indeed, apparently I think... the vectors can be separated apart, but it is possible to make a transformation in order to get projections. I think that in mathematics we do not have a unique strategy. It is possible to transform. If I write down four, I can arrange another way... if I write down two, I can arrange another way to write it, for instance four over two. Nobody can refuse that. I mean, what counts is to apply correctly the properties. This was what I did. Indeed, I would not be able to determine the angle between these two vectors, if I did not make such translation. Otherwise, it would be nonsense.

S4 raised an interesting idea of different forms of representation for the same mathematical entity. He wanted to support the idea of generalizability. That is the mathematical entity (the dot product formula) “represents” different angles between segments (skew lines, intersecting lines, and parallel lines). He also referred to translation of segments in order to visualize the angle between non-intersecting segments.

(S7II:37-46): Well, this was my idea. It was clearer if they provided us with three points instead of four. Thus, the two vectors would have the same initial point. It would be easier to get the angle. In this case, we have four points. That is why in the test I warned about this using vectors $u$ and $v$ with the same initial point. Otherwise, it would be impossible to get the angle between these two vectors or segments.

I included S7 in this group because he reasoned before applying the dot product formula. Although he stated that the non-intersecting segments do not form an angle, he understood that the formula of dot product was yielded while the vectors met at the initial points. Hence, a geometric transformation had to be applied to the segments in order to satisfy the condition of the formula.

Performance oriented group

The group of students (S1, and S5) applied the dot product formula blindly, although the solution was correct. (S5 used it blindly later during interviewing). This happened because the dot product formula is generalizable for determining the measure of the angle between any two segments. Their “blindness” state was uncovered during interviewing. They faced difficulties in explaining why the dot product formula worked for determining the angle between any two segments. Some extracts of their transcripts corroborates this phenomenon.

(S1II:23): Umh... They are indeed separated apart. That is correct.
S1 uncovered his blindness through hinting process. He was taken by surprise when he noticed that the segments did not intersect each other. In addition, he was honest about his performance orientation. He just wanted to use the formula to get the measure of an angle.

S5 tried to conceal his performance orientation answering to almost every question. Meanwhile, some of his responses seemed to be out of context of the question. To reveal it I had to ask a critical question of which answer was “Now I have no idea. (Laughing)” (S5II:267).

Connectedness of concepts: reasoning versus intuition

I noticed that during the hinting process all students, in general faced difficulties in informing their analytic strategy. However, the mastery-oriented group (S3, S4, and S7) showed some understanding when they used their intuition (S3 and S4) and reasoning (S7) to support their analytic strategy. S3 and S4 used the correct synthetic transformation (translation) to support the dot product formula use while S7 used analytic concepts (collinear and coplanar vectors) to inform his analytic solution. S3 and S4 showed limitations in explaining analytically (reasoning) the invariance of the vector coordinates after translation. Below are the extracts of S3’s and S4’s transcripts, which underpin this phenomenon.

(S3II:54, 55): It was... was an idea I had. Anyway, we have two lines, and there exists an angle between these lines.
(RII:56): Of course, there exists an angle. Now, how do you determine the measure of that angle?
(S3II:57, 58): My idea was to construct a parallel line to this one and move to this position. Hence, I get this angle between these two lines.
(RII:59-62): You had an idea about translation. Now how do you relate this geometric transformation with the calculations you did? Can you explain me about that?
(S3II:63): I did not relate the translation with any calculation.
(RII:64): I think the vectors can show that...
(S3II:65-68): By the way, after taking this test there was a debate, not a debate... there was... about this task. Some of my classmates and me did not know exactly, after translation, whether we had to change the coordinates or not to determine the measure of the angle.
(RII:69): What was your conclusion?
(S3II:70, 71): We discussed a lot, however, we did not draw any conclusion. The question remained unsolved.
Likewise, S4 had strong synthetic position with regard the invariance of the vector coordinates after translation. He calls the original vector and the image vector as “parallel vectors” (collinear vectors in the literature). Meanwhile he accepted that he missed to verify it analytically. Moreover, he used a metaphor of translation of a bucket, which preserved its properties after translation.

(RII:88 and 89): You are saying that the translated vector and this one are...
(S4II:90): They are... they are parallel.
(RII:91): They are parallel. What do their coordinates look like? Are they different or the same?
(S4II:92): They are not different, no way. They are the same.
(RII:93): Are they the same even after translation?
(S4II:94): Even after translation, the coordinates remain the same. Otherwise, the angle would not be the same.
(RII:95): Are the coordinates the same after translation?
(S4II:96-98): The coordinates must be the same as I said. Otherwise, the angle would not be the same.
(RII:99): Then, after translation, the coordinates do not change.
(S4II:100): No, they do not.
(RII:101): What is the explanation of that? If we have a movement,...
(S4II:102-104): I think the parallel vectors have the same coordinates. The parallel vectors, which lie on the same plane, have the same coordinates due to being parallel.
(RII:105): Do you make such statement because you have deduced it, you have seen somewhere, or you believe in it? What is the reason?
(S4II:106-112): I cannot explain it. I cannot exactly explain it. However, I believe in it. Firstly, I applied that principle that the coordinates only were transformed because of translation in some, some units... I believe that those units keep unchanged the coordinates. I cannot support it further. However, I used this process.
(RII:113): Have you ever verified it when you make a translation of a vector its coordinates remain the same?
(S4II:114): I have never verified it.
(RII:115): Meanwhile you believe in it.
(S4II:116): Yes, I do.
(RII:117, 118): I think it is worth to verify it. Make a translation of a vector and verify whether its coordinates remain unchanged in any position, parallel translation.
(S4II:119): (Silence).
(RII:120): Do you think it is interesting to do that?
(S4II:121-123): Yes, it is. I have never thought about. My reasoning was not feasible regarding verification whether translation changes or not the coordinates. I have never thought about.
(RII:124): Does a translated object change its properties?
(S4II:125-127): No, I think that the object was subject to movement and it changes its position, but its size and shape remain unchanged. Let us suppose a bucket. We can put it here. If I put it at 5 meters high, it remains the same
bucket.

(RII:128): Is that explanation sufficient to explain why the coordinates...

(S4II:129, 130): This is the issue. I do not know whether this explanation is...

It can be. Now, I do not know whether it is sufficient or not.

In turn, S7 was initially disturbed because of the non-intersecting segments. According to him, it was impossible to get an angle between non-intersecting segments. To solve this problem he used the corresponding collinear vectors with the same initial point in order to apply the dot product formula. This phenomenon might be explained that S7 was not able to use his intuition to see that translation of segments allowed getting the angle between non-intersecting segments.

(S7II:37-46): Well, this was my idea. It was clearer if they provided us with three points instead of four. Thus, the two vectors would have the same initial point. It would be easier to get the angle. In this case, we have four points. That is why in the test I warned about this using vectors u and v with the same initial point. Otherwise, it would be impossible to get the angle between these two vectors or segments.

In the meantime, S7 used his reasoning to explain why the collinear vectors with the same initial point was a correct explanation for determining the angle between the two non-intersecting segments, although he did not explain it fully. This knowledge is possible thanks to the convention: “We will define the angle between any two directed skew lines to be the angle between any two intersecting lines, which are respectively parallel to the skew lines and similarly directed” (Lehman, 1942, p. 281).

(RII:78-80): Here we have two non-intersecting segments. How do you support the relationship between the angles between these two segments and between the corresponding vectors intersected at the initial point? Is the same angle?

(S7II:81-84): Well, it is the following idea. I got confused. We have the concept of collinear vectors and coplanar vectors. And... (Silence). Well, these two vectors, these two vectors are coplanar, because both lie on ... both lie on the same plane.

(RII:85): Do vector AB and CD lie on the same plane?

(S7II:86, 87): No, I considered these vectors as coplanar vectors. If these vectors are coplanar, then,...

(RII:88): Continue.

(S7II:89, 90): Then, we can consider these vectors... as intersecting at the initial point.

(RII:91): Continue.
The performance oriented group (S1 and S5) needed many hints to understand why the dot product formula was generalizable for determining the angle of any two segments (See their transcripts above). They only used reasoning regarding the procedures how to use the dot product formula and to get the solution. There were minor slippery calculation mistakes and symbolization errors. These students showed that they had unstructured and disorganized knowledge (Lawson and Chinnappan, 2000). The hinting process helped them in some extent to organize and structure their knowledge.

5.4.3 Emerging Issues of Interviewing Phase 2

The analysis of the students’ Interviewing Phase 2 transcripts yielded some emerging issues, which concern the strategies they used, the connectedness of their knowledge, and their disposition (mastery and performance orientation).

1- All students used reasoning to solve Task 3 of the lecturer’s test through algebraic thinking aspects such as the dot product formula, coordinates, vector, trigonometry relations, matrix, acronym, equation, and unknown (analytic strategies). Although they got the correct solution, some of them blindly used the formulae. Others attempted to make sense of the formulae using their intuition through visualization and construction processes (synthetic strategies).

Insight 1: Analytic strategies are methods, which usually lead to simpler and more elegant solutions than the synthetic strategies; however, these strategies allow blind calculations (Anton and Rorres, 1991; and Douady, 1998).

2- All students, through hinting process used both intuition (visualization and construction processes) and reasoning (algebraic approaches and generalisation) to solve the task successfully.

Insight 2: The interaction of deductive (as a means of discovery) and empirical (as a means of development of intuition) approaches to geometry leads students to reap the benefits of their knowledge (Schoenfeld, 1986).

3- In Analytic Geometry, analytic strategies (general strategies) seem to be more efficient and easier to access than synthetic strategies (specific strategies), meanwhile synthetic strategies
seem to be more insightful than analytic strategies.

Insight 3: In Analytic Geometry general strategies readily transfer to new, potentially relevant situations and seem to be more accessible than specific strategies. (Prawat, 1989).

4- In Task 3 all students seemed to possess robust schemas regarding analytic strategies, however, they faced difficulties in relating analytic and synthetic schemas as they needed some hints (the mastery oriented group) and a lot of hints (the performance oriented group) for clarification purposes.

Insight 4: A schema with components that are effectively organized is one for which minimal levels of cueing are required for activation. When a greater level of hinting support is needed for access, we argued that the knowledge schema is either less extensive or less well connected. (Lawson and Chinnappan, 2000, p. 31).

5- All students and especially the performance-oriented group seemed to profit from the hinting process for organizing and structuring their knowledge schemas.

Insight 5: The hinting process may be adopted by the pedagogy to enhance students’ organization and structuring of their knowledge schemas so as their mastery and performance dispositions.

5.4.4 Conclusion of Interviewing Phase 2

The analysis of the transcripts showed that three of the target students (S3, S4, and S7) solved Task 3 from the lecturer’s test using both algebraic and geometric thinking and the other two (S1 and S5) used algebraic thinking only. It seemed easier for these students to use their reasoning (analytic strategies) rather than their intuition (synthetic strategies) in the tasks where they are quantitatively presented (numbers). They all rapidly got the solution. However, some students blindly used these strategies maybe due to their generalizability nature. To tackle this “blindness” phenomenon I needed to hint to most of the students either using visualization and construction processes or reasoning driven by algebraic thinking. I noticed that their reasoning was stronger than their intuition. When I prompted them to elicit their intuition, I was forced to supply them most of the key concepts for clarification purposes, while they were able easily to handle the formulae and algebraic transformations. All students
through hinting process used both intuition (visualization and construction processes) and reasoning (algebraic transformations, generalisation, formulae and relations) to solve the task.

Considering the academic background of my target students, I noticed that in Analytic Geometry they needed intuition (synthetic strategies) to inform their reasoning (analytic strategies). In the contrary, in Euclidean Geometry they needed reasoning (analytic strategies) to inform their intuition (synthetic strategies). However, I observed that this balance still needed to take plane in my target students’ mind. In other words, in Analytic Geometry, my students needed geometric thinking to aid their algebraic thinking and in Euclidean Geometry, they needed algebraic thinking to aid their geometric thinking. This mutuality in geometry seems to underpin what Schoenfeld (1986) found that the interaction of deductive (as a means of discovery) and empirical (as a means of development of intuition) approaches to geometry leads students to reap the benefits of their knowledge.

5.4.5 Analysis on the Students’ Opinions on the Relationship between Euclidean Geometry and Analytic Geometry

During Interviewing Phase 2, I posed a general question to the students on relationship between Euclidean Geometry and Analytic Geometry to see whether their geometrical understanding evolved during these two courses and what helped them to get such understanding. The analysis was organized according to the opinion of each student.

The main questions posed to the students are as follows:

1. After attending Euclidean Geometry course and now you have been attending Analytic Geometry course, is there a relationship between these two geometries?
2. If the answer is “yes”, what is the relationship? Moreover, how does this relationship help your geometric understanding?

S1 said that there was a relationship between Euclidean Geometry and Analytic Geometry. Meanwhile he faced difficulties in explaining what relationship was that. After insisting, he stated that the relationship was not obvious like in mathematics. He gave an example of an algebra topic, the set of the whole and the natural numbers. He said it was clear to see the relationship between these two set of numbers because the properties of the set of natural numbers were applicable in the set of the whole numbers. In the meantime, he said that the theorems learnt in Euclidean Geometry course had support in Analytic Geometry course. However, he added that it was difficult at that moment to explain it clearly.
R: Vê alguma relação entre estas duas cadeiras ou são diferentes? [R: Is there any relationship between these two subjects? Are they different?]

S1: Há relação. Acho que há relação. Porque na Geometria Euclidiana se puxa muitos conhecimentos que damos... no ensino secundário. Há muitos conhecimentos que são vistos com mais ênfase na Geometria Euclidiana. Na Geometria Analítica... acho que há uma relação. Só que agora acho difícil dizer qual é. Mas há relação. [S1: There is a relationship, I think there is a relationship. Euclidean Geometry course dealt with school geometry. Of course, we dealt in depth many topics learnt in secondary school. The Analytic Geometry... I think there is a relationship, but now I have difficulties to explain it, but there is a relationship.]

R: Mas qual é o seu sentimento ao nível de conceitos? Como vê, como tem ajudado, talvez uma cadeira ajuda à outra ou são diferentes? Nesses termos... [R: How do you see that relation regarding the concepts? How do you see it? How helpful is one subject to the other? Are they different?]

S1: Não, bom, a relação não é assim muito... não é como na Matemática, agente vê por ex uma pessoa vê numa classe e é suporte para 7ª, 8ª, por ex aprendemos os números relativos e aplicamos as propriedades que aprendemos nos números naturais. Aí é fácil ver a relação. Agora entre a Geometria Analítica e a Geometria Euclidiana há momentos que na Geometria Analítica suportamos teorias que aprendemos na Geometria Euclidiana. É mais ou menos isso. Eu acho que é isso. Agora é me difícil dizer... [S1: Well, the relationship is not so very... it is not like in mathematics. For instance, you learn a topic in a specific grade. That topic is needed in the following grade. For example, we learn integers applying the properties learnt in the set of natural numbers. Like this, it is easy to see the relationship. Now, between Analytic Geometry and Euclidean Geometry, I can say that there are moments in Analytic Geometry where we support the theorems learnt in Euclidean Geometry. It is more or less what I think. Now it is difficult to explain it.]

S1 asserted the relationship between Euclidean and Analytic Geometry helped in his geometrical understanding. He added saying that it was very important to do a lot of reading to ensure that understanding.

R: E como tem ajudado isso no seu pensamento geométrico? A maneira de ver a Geometria Euclidiana e depois a Geometria Analítica? Acha que está a desenvolver ou complica mais? [R: How does that relationship help in your geometric understanding? First, you attended Euclidean Geometry and afterwards Analytic Geometry. Do you think is your understanding evolving or are you getting more confused?]

S1: (Rindo-se). Desenvolve, desenvolve. O que é necessário é fazer leituras. É importante fazer muita leitura. [S1: (Laughing). It is evolving. It is evolving. However, it is needed a lot of reading. Reading is very important.]

R: Tem uma última palavra antes de terminar? [R: Do you have anything else to say?]

S1: (Silencio). [(Silence).]

R: Tem a última palavra? [R: Do you have anything else to say?]
S1: Acho que podemos parar aqui. (Rindo-se). [S1: We can stop here. (Laughing).]

S3 also stated that there was a relationship between the two geometries. According to him, this relationship happened at the level of the pictures. He gave an example on triangles. He said in Euclidean Geometry they learnt similarity on triangles and the respective postulates, while in Analytic Geometry, they learnt to determine lengths and measures of the elements of triangles through vectors.

R: Bem, agora estás a ver a Geometria Analítica e já viste a Geometria Euclidiana. Estás a ver alguma relação entre as duas geometrias? Qual é a tua ideia? [R: You have attended Euclidean Geometry and now you are attending Analytic Geometry. Do you see any relationship between the two geometries?]

S3: Tem alguma relação. [S3: There is a relationship.]

R: Qual é a relação que vês? [R: What is the relationship?]

S3: Nas duas disciplinas tratamos de... tratamos de figuras praticamente. Figuras, planos, a relação entre... como é que vou explicar. Tem a ver mais... na geometria analítica, nós temos planos e rectas. E tivemos as suas projeções. Sim, as projeções. [S3: In both subjects we... we... we deal with pictures actually such as planes. The relationship... How can I explain? In Analytic Geometry, we have planes, lines, and their projections, yes, their projections.]

R: E não viram triângulos na geometria analítica? [R: Have you learnt triangles in Analytic Geometry?]

S3: Mas também através... vimos triângulos através de vectores. Sim, praticamente tem relação. A relação de figuras geométricas. [S3: We have learnt triangles through vectors. Yes, there is a relationship, the relationship between geometric figures.]

R: E o tratamento das figuras geométricas na geometria analítica é diferente do tratamento que se faz na geometria Euclidiana? [R: Is the way you learn geometric figures in Analytic Geometry different from Euclidean Geometry?]

S3: Não. [S3: No.]

R: É o mesmo tipo de tratamento? [R: Do you learn in the same way?]

S3: Não, não é exactamente o mesmo tipo de tratamento. Na geometria analítica nós vemos através... por ex. os triângulos... enquanto que na geometria Euclidiana vimos a semelhança, a semelhança de triângulos, vimos... todos os critérios de semelhanças de triângulos, enquanto que na geometria analítica vimos as medidas, as dimensões dos triângulos através de vectores. [No, it is not exactly in the same way. In Euclidean Geometry, we learnt similarity on triangles. We learnt all the similarity postulates, while in Analytic Geometry we learn lengths and measures of triangles through vectors.]

S3 did not say anything about how that relationship helped in his geometrical understanding. I recognize my methodology mistake because I did not insist after he answered he had nothing else to tell about his geometric knowledge.
S4 acknowledged that the two geometries dealt with the same objects but in different perspectives. He stated that Euclidean Geometry treats those objects in a classical way almost devoid of algebra and calculus. It concerned in seeking solutions within itself using ruler and compass to get the actual measures without support of algebra, while in Analytic Geometry, for example, he constructed a triangle using operations with coordinates. The same triangle in Euclidean Geometry he constructed given the measures of its sides. Besides, he asserted that the two geometries were intrinsically related because one could solve the same task in both geometries using different perspectives and get the same result.

S4: The two geometries are not different, indeed. Both deal with the same topics but in different perspectives. Euclidean Geometry treats the concepts in a classical way almost devoid of algebra and calculus. It concerns in seeking solutions within itself using ruler and compass to get the actual measures without support of algebra, while Analytic Geometry, for example,... a triangle is constructed using operations with coordinates. The same triangle in Euclidean Geometry is constructed given for example the measures of its sides. Through operations with...
coordinates, I can get the same measures obtained in Euclidean Geometry. I cannot separate Analytic Geometry from Euclidean Geometry. It is a matter of perspectives or approaches, but the result is the same.]

S4 asserted that both geometries helped in his geometrical understanding. Meanwhile he remarked that first one had to understand each geometry separately before relating or combining.

R: O que é que acha essas duas abordagens quando estão em combinação ajudam a perceber melhor os conceitos ou confundem mais qual é a sua opinião? [R: What do you think? Do both perspectives combined help to understand better the concepts or cause confusion?]

S4: Bem, de forma alguma. Se os conceitos não se entendem mesmo em separado, não vão ser juntos que se vão compreender. Isto é se a pessoa percebe a geometria Euclidiana como ela é sem relacioná-la à geometria analítica não tem porque não perceber se se fizer uma combinação ou uma análise das duas ou uma junção das duas geometrias. Eu acho indiretamente nós usamos alguns argumentos da geometria Euclidiana na... na geometria analítica. Quando dizem determine a distância pode-se medir com régua e saber que vai ser 5. Em vez de dizer que a distância é 5 dizem (o ponto) A (tem coordenadas) x e y e (o ponto) B (tem coordenadas) x e z. Mas é a mesma distância que vou obter. O alvo da análise é o mesmo só em pontos de vistas diferentes. Mas os resultados obtidos obviamente são os mesmos. [S4: Well, they do help. Meanwhile, if you do not understand the concepts separately, you also cannot understand them combined. In other words, if you understand Euclidean Geometry separately from Analytic Geometry, sure you will understand both geometries combined. I think indirectly in Analytic Geometry we use some concepts of Euclidean Geometry. To the question, determine a distance you can measure directly with a ruler and get 5 or the question reads, determine the distance between \( A(x;y) \) and \( B(u,v) \). The distance will be the same, but in different perspectives.]

S4 was very talkative and I posed an unexpected question, which later I felt to be important for all the target students.

In which geometry do you understand the concepts better? And why?

He understood the concepts better in Analytic Geometry because he had lack of experience in Euclidean Geometry. I wanted to know what he meant with “lack of experience”. He asserted that to handle geometric objects through coordinates was easier and more interesting rather than through synthetic statements like, “the sum of the measures of two sides of a triangle is always bigger than the measure of the third side”. This assertion corroborates the position that through coordinate geometry we can solve the geometric problems in more elegant, quick, and fuller way than in Euclidean geometry (Kline, 1972 and Anton and Rorres, 1991).
R: E o que acha em temos de abstracção onde os conceitos geométricos são melhores entendidos? [R: Where are the concepts better understood?]

S4: Bem, eu acho que os conceitos geométricos... talvez por não ter abordado tantas coisas em geometria Euclidiana, não tenho tanta capacidade para relacionar isto. Mas pude ver que é mais fácil perceber as coisas na geometria analítica que na geometria Euclidiana. É mais fácil. Entende-se melhor o que se faz. Quando se coloca um triângulo no espaço ou no plano percebe-se melhor. Percebe-se melhor. [S4: Well, I think, the geometric concepts... Sorry, due to lack of experience in Euclidean Geometry, I have difficulties in this discipline. Meanwhile I noticed that it is easier to understand better the concepts in Analytic Geometry than in Euclidean Geometry. It is easier. You understand better. When you treat a triangle in plane or in space, you understand better.]

R: Qual é a razão ou o segredo para se perceber melhor? [What is the reason behind what you are saying?]

S4: Quer dizer pelo facto de eu poder determinar as suas coordenadas, por ex. determinar o triângulo através do método de coordenadas tal e tal eu posso facilmente obter os seus segmentos de rectas que... que me foram dados. E eu ao unir aquilo aí de facto é interessante. Unir e ver que aquilo constitui um sólido geométrico conhecido. Em vez de usar a propriedade que diz que a soma de dois lados dum triângulo é maior, sempre maior que o terceiro. Eu acho que (os conceitos na geometria analítica) são mais visíveis talvez pelo apoio da própria álgebra que se usa. Torna-se um bocadinho mais convincente. [S4: To determine the elements of a triangle trough coordinates is easy and interesting instead of using a property, which reads the sum of the measures of two sides of a triangle is always bigger than the measure of the third side. (It seems that for this student the concepts in AG are better grasped than in EG because of the rigor of algebra, which it is convincing).]

S4 went on talking about his weaknesses in geometry. I did not ask him in this regard. He did it voluntarily. He firstly recognized that geometry was a difficult subject and he did not know how to handle it. He called it “lack of skill of imagination and abstraction before solving the task”. Due to this difficulty, he asserted that he ran straightaway to formulae before analysing the task. This happened especially in Analytic Geometry. Meanwhile he was hopeful that this difficulty would be overcome in the future.

R: Não sei se tem uma última palavra para terminarmos a nossa conversa? [R: Is there anything else you would like to add?]

S4: Talvez dizer que... é difícil. (Rindo-se). A geometria em si é difícil. É difícil. E não sei o que se pode fazer, não é? Em relação a geometria para que se possa criar uma grande capacidade de abstracção. Porque eu creio que há um... um grande erro, que nós cometemos. Quando nos deparamos com um problema (geométrico) corremos para escrever algumas expressões sem antes fazer uma abstracção no espaço. O que é exactamente isso. E creio que é isso que nos falta. Falta-nos bastante porque não conseguimos ter a ideia antes de escrever alguma coisa. Quer dizer falta-nos a capacidade de imaginação no
concreto dos problemas da geometria analítica. Eu acho que ainda falta em mim. Eu espero ainda conseguir cultivar essa capacidade de abstração dum problema de geometria analítica antes de correr para uma fórmula. [S4: Maybe to say... (Laughing). Geometry itself is difficult. It is a difficult subject. I do not know how to tackle this problem. I think... there is a... big mistake we make when we solve geometric tasks. We run to formulae before we analyse it. This mistake persists in me. We are short of the skill of imagination before using any formula especially in Analytic Geometry.]

R: Está a falar do esboço geométrico ou que tipo de imaginação? [R: What kind of imagination you are talking about? Geometric figures?]

S4: Não só do esboço geométrico mas também o que estaríamos por detrás do esboço geométrico por ex. [S4: I am referring not only to geometric figures. Also to what is behind that geometric figure.]

R: Por detrás? [R: What do you mean with what is behind?]

S4: O que eu preciso para esboçar? Determine o ângulo formado entre os vectores. O que tenho de fazer? [S4: What do I need to sketch a picture? What should I do?]

R: Mas depois de ter dado as duas geometrias não deu para cultivar essa capacidade? [R: After attending two geometries, have you not cultivated such skill yet?]

S4: Bem, eu creio que é aos poucos. Agente não nasce correndo. Agente nasce gatinhando. E creio aos poucos eu vou lá chegar. Eu vou lá chegar. [I think it is coming slowly. A baby is not born running. It takes time to get there. I will get there.]

S5 said did not see a direct relationship between Euclidean and Analytic Geometry. He thought maybe because they did not cover the syllabuses of Euclidean Geometry. However, he stated that he saw a slight relationship in construction of geometric objects and in proving. He explained that in Analytic Geometry, they departed from a geometric object to prove or to yield an equation or a formula.


S5: Bom, na Geometria Euclidiana eu... até agora a relação não é muito directa. Porque, bom, não sei se é o problema se calhar não cumprimos o programa. Não sei o que aconteceu. Nós terminamos... [S5: Well, in Euclidean Geometry, I... I still do not see a direct relationship, maybe because we did not cover the syllabus. I do not know what happened.]

R: Não cumpriram o programa de que disciplina? [R: Which subject you did not cover the syllabus?]

S5: Estou a supor. Se calhar não demos tudo. Porque na Geometria Euclidiana o que nós vimos foi... mais construção de triângulos. Onde há um pouco de relação é nas construções básicas de segmentos, onde se diz construir uma recta paralela, uma recta perpendicular. Então para se fazer um esboço é
preciso ter esses conceitos, uma recta perpendicular, uma recta paralela. Na Geometria Euclidiana vimos muito isso. (Geometria Euclidiana relaciona-se com a Geometria Analítica nas construções de objectos geométricos?). A construção desses elementos básicos, figuras geométricas como planos, circonferências. Tem alguma relação mas não é aquela directa é em termos de construção. Ter a noção do que é por ex uma recta paralela. O que é uma recta perpendicular. Nós construímos muito na Geometria Euclidiana. O que eu posso dizer se calhar a cultura que se cria na Geometria Euclidiana de demonstração ou da igualdade de duas figuras é o que se pode transferir para se tentar puxar as fórmulas. Era muito exigente nas demonstrações. Também se parte duma figura para demonstrar. Então se parte duma figura para se compor uma equação. (Geometria Euclidiana relaciona-se com a Geometria Analítica no raciocínio lógico e demonstrações?). Embora não esteja a ver aquela relação em termos de tratamento. Eu acho que pode ser isso. (Rindo-se). [S5: I am supposing. Maybe we have not covered the syllabus. In Euclidean Geometry, we learnt triangle construction. We learnt a lot of construction in EG. I see a slight relationship in construction of segments, parallel lines, and perpendicular. (EG is related to AG in construction of geometric objects). I also see relationship in construction of planes and circles, but it is not a direct relationship. I also see relationship in proving and in congruency of geometric objects, which in Analytic Geometry we do it with help of a formula. Also in Analytic Geometry, we depart from a geometric object to prove or to yield an equation. (EG is related to AG in logical reasoning and proof). I cannot see a relationship in terms of treatment of topics. (Laughing).]

S5 explained what was missing in his geometrical understanding even after attending the two geometries: visualizing a picture before rushing to formulae. He clarified why he had not assimilated this skill yet although in Euclidean Geometry, they made many geometric constructions. He excused that there was discontinuity of constructions in Analytic Geometry. Besides, he missed some Euclidean Geometry topics due to registering late in this course. Meanwhile, he decided to overcome this difficulty through continuous practicing of sketching because he would be a mathematics teacher and he needed to have that skill.

R: Com base na geometria que deu e na geometria que está a dar agora deu para aumentar o seu conhecimento geométrico, a entender melhor os conceitos geométricos ou acha que falta mais alguma coisa? Qual é a sua ideia? [R: After attending EG and now you are attending AG, has your understanding improved? Is there anything missing?]

S5: A minha ideia é essa. Eu vejo que a nossa preocupação quando aparece um problema é correr para a fórmula. Corre directamente para a fórmula. Aparece um problema de geometria agente quer ver a fórmula, como é que é a fórmula sem ver o esboço. Eu acho que o principal é mesmo começar com o esboço. É mesmo começar com o esboço. O trabalho que eu tenho que fazer agora é mesmo exercitar fazendo esboços. (Uma boa estratégia começar com o esboço?). [S5: I think that our concern, when we solve a geometric task, is to rush to formulae. Rushing straightaway to formulae without
visualizing the picture is our concern. I think the most important thing is to begin with a sketch. It is to begin with a sketch. My concern now is to practice sketches.

R: Mas essa capacidade de ir primeiro ao esboço e depois à fórmula não adquiriram durante o curso de Geometria Euclidiana? Já que falou que na Geometria Euclidiana faziam muitas construções. . [R: That skill of sketching first and then formulae, have not you cultivated during these two geometries?]

S5: Eu penso que de alguma maneira pode ter influenciado. Porque... (a Geometria Euclidiana) pode ter mostrado o que é preciso. Na Geometria Euclidiana o que me aconteceu foi seguinte. Eu comecei um pouco mal. Quando eu chego estávamos a ver triângulos, a demonstração de teoremas. Eu já tinha... conseguia (perceber) os conceitos básicos, o conceito de triângulo, triângulos semelhantes, era um pouco fácil. Se os triângulos são semelhantes o que posso fazer. Qual é o comportamento dos seus segmentos. Qual é o comportamento dos seus ângulos. Agora aqui (na Geometria Analítica) é que um problema. Não vem portanto o esboço. Então é preciso primeiro ter o esboço. [R: What is your last word concerning what we have been discussing?]

S5: It helped me to see my weaknesses. Last semester, I faced difficulties in proving. I worked on proving until I succeeded. The lecturer used to rate high marks for proving. My problem was not to justify the steps of a proof.

R: Eu vou insistir não, não, não por causa da, da, da nota mas por causa do conhecimento. (Boa atitude mastery orientation rather performance orientation as a mathematics teacher). Porque eu preciso desse conhecimento como professor de matemática. [S5: I am going to insist, not because of marks, but because of knowledge. (Mastery orientation behaviour). I need that knowledge as a mathematics teacher.)

S7 gave an example of how the two geometries related to each other. Moreover, he said that the Euclidean Geometry served as a foundation for Analytic Geometry. That is why the order should be attending first Euclidean Geometry and afterwards Analytic Geometry. See an extract of his interview below.
R: Agora, o sr deu a Geometria Euclidiana e agora está a ver a Geometria Analítica. Está a ver alguma relação entre as duas geometrias? [R: You attended EG and now you are attending AG, do you see any relationship between these two subjects?]

S7: (Silencio). Bom, há uma relação. Por ex há partes, há axiomas, propriedades ou regras que vimos na Geometria Euclidiana e estamos a dar na Geometria Analítica. Por ex, demos rectas, tem um axioma na Geometria Analítica, na Euclidiana desculpe, que diz por dois pontos sempre passa uma recta. E na Geometria Analítica estamos a aplicar este axioma. Através de dois pontos nós conseguimos determinar a expressão analítica dessa recta que passa por esses dois pontos. E muitos outros axiomas que nós utilizamos da geometria Analí... Euclidiana. Há uma relação sim. [S7: (Silence). Well, there is a relationship. For example, there are some topics such as axioms, properties or rules we learnt in EG and we have been applying in AG. For instance, we learnt lines. There is an axiom in EG, which says through two points passes a line. In AG we used this axiom. We determined the equation of a line passing through two points. Moreover, so many other axioms we learnt in EG, we have been using them in AG. There is a relationship.]

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S7: É melhor a Geometria Analítica ser vista depois da Geometria Euclidiana. [S7: It is better to attend AG after attending EG.]

R: Porque? [R: Why?]

S7: É mais interessante. A Geometria Euclidiana parece por ex uma base. Porque há muitos conceitos para aprender na Geometria Euclidiana. Como o conceito de triângulo, a relação dos ângulos... muita coisa. [S7: It is more interesting. EG is like the foundation. There are so many concepts to be learnt in GE such as triangle, angle relationship... so many things.]

S7 asserted that relationship helped not only in his geometrical understanding but also in understanding other subjects.

R: Essa relação tem ajudado de alguma maneira a compreender melhor a geometria ou acha que é confuso? [R: Does this relationship help you in understanding better geometry? Does it confuse you?]

S7: Não. Ajuda, ajuda. Não só a geometria mas também as outras cadeiras. (E7 acha que essa relação ajuda a compreender outras cadeiras, isto é a Geometria Analítica e a Euclidiana em conjunto faz com que os conceitos de geometria sejam aplicados nas outras cadeiras, especialmente dentro da própria Matemática o que o Stillwell constatou também). Por ex para demonstrar num pentágono regular, para demonstrar que as trissectrizes dum ângulo dum pentâgono regular são iguais. Nós vimos este exercício. Isto para demonstrar tavemos que recorrer à Geometria não Euclidiana. (Queria dizer Geometria Euclidiana). [S7: Yes, it does help. It does help not only in geometry but also in other subjects. (Stillwell evoked this multidisciplinary application of geometry). For instance, we have proved a theorem in AG and we appealed to non-EG]

R: A Geometria Euclidiana ou não Euclidiana? [R: Did you say non-EG or EG?]

S7: A Geometria Euclidiana. [EG]
5.4.6 Discussion on the Students’ Opinions on the Relationship between Euclidean Geometry and Analytic Geometry

The students’ opinions on relationship between Euclidean and Analytic Geometry and its influence in their geometrical understanding were diverse. I attempted to see the commonalities and differences in those opinions, which I report as follows.

*Relationship between Euclidean and Analytic Geometry: Synthetic to analytic perspective*

All of them asserted that there was a relationship between the two geometries. Although some of them (S1 and S5) added that that relationship was not obvious for different reasons.

Most of the students (except S1) explicitly or implicitly asserted that both geometries deal with the same objects in different perspectives. The reviewed literature corroborates this idea. These perspectives are called geometric modes: Synthetic versus Analytic Geometry.

Stillwell (1998) describes geometry as a dual subject in its essence:

> First, there is the visual, self-contained, synthetic side, which seems intuitively natural; then the algebraic, analytic side, which takes over when intuition fails and integrates geometry into the larger world of mathematics. (p. 105)

These students grasped these perspectives as on one hand, synthetic perspective- the treatment of geometric objects almost devoid of algebra, on the other hand, analytic perspective- the treatment of geometric objects with aid of algebra. The following extracts of their transcripts underpin this position.

S3 stated that the same object (triangle) was treated in both geometries where in Euclidean Geometry they learnt similarities on triangles and the respective postulates. This treatment is almost devoid of algebraic concepts except for example the use of ratio, acronyms, and abbreviations. In Analytic Geometry, S3 said, they used vectors (vector coordinates) to determine measures of the elements of triangles.
S3: No, it is not exactly in the same way. In Euclidean Geometry, we learnt similarity on triangles. We learnt all the similarity postulates, while in Analytic Geometry we learn lengths and measures of triangles through vectors.

S4 clearly explained the way the two geometric modes worked. In Euclidean Geometry, they used instruments ruler, straightedge, and compass to construct geometric figures and get the measures of the respective elements without support of algebra (synthetic procedures), while in Analytic Geometry they used operations and coordinates to get the same measures (analytic procedures).

S4: Indeed, the two geometries are not different. Both deal with the same topics but in different perspectives. Euclidean Geometry treats the concepts in a classical way almost devoid of algebra and calculus. It concerns in seeking solutions within itself using ruler, straightedge, and compass to get the actual measures without support of algebra, while Analytic Geometry, for example, ... a triangle is constructed using operations with coordinates. The same triangle in Euclidean Geometry is constructed given for example the measures of its sides. Through operations with coordinates, I can get the same measures obtained in Euclidean Geometry. I cannot separate Analytic Geometry from Euclidean Geometry. It is a matter of perspectives or approaches, but the result is the same.

S5 mentioned a very important feature of Analytic Geometry the treatment of geometric objects through equations and formulae. This was the idea discovered by Fermat and Descartes, a method devoted to the application of algebra in geometry and geometry in algebra. The ultimate aim was the geometric solution to problems by algebraic methods.

S5: I also see relationship in proving and in congruency of geometric objects, which in Analytic Geometry we do it with help of formulae. Also in Analytic Geometry, we depart from a geometric object to prove or to yield an equation.

S7 similarly corroborated the S5’s opinion the treatment of geometric objects through equations.

S7: (Silence). Well, there is a relationship. For example, there are some topics such as axioms, properties or rules, we learnt in Euclidean Geometry and we have been applying in Analytic Geometry. For instance, we learnt lines. There is an axiom in Euclidean Geometry, which says through two points it passes a line. In Analytic Geometry, we used this axiom. We determined the equation of a line passing through two points. Moreover, so many other axioms we learnt in Euclidean Geometry, we have been using them in Analytic Geometry. There is a relationship.
In addition, he stated that Euclidean Geometry was a sort of a foundation for Analytic Geometry as it supplied with empirical concepts (intuitive level) to Analytic Geometry. In turn, Analytic Geometry applied these concepts upgrading them to abstract concepts (reasoning level).

**R:** What do you think? Is Analytic Geometry supporting Euclidean Geometry or is it the other way round?

**S7:** I think, Euclidean Geometry aids Analytic Geometry. In Euclidean Geometry, we learnt a lot of axioms and difficult tasks, which have application in Analytic Geometry.

**R:** Now what is the role of Analytic Geometry?

**S7:** (Silence). Well, as I said, the name Analytic Geometry explains everything. Its role is to study geometry analytically, numerically, graphically, and geometrically. We interpret the several aspects of geometry analytically.

**R:** What is role of Euclidean Geometry?

**S7:** It is better to attend Analytic Geometry after attending Euclidean Geometry.

**R:** Why?

**S7:** It is more interesting. Euclidean Geometry is like the foundation. We have so many concepts to learn in Euclidean Geometry such as triangle, angle relationship... so many things.

S1 did not explain in detail what he meant with “I can say that there are moments in Analytic Geometry where we support the theorems learnt in Euclidean Geometry”. Meanwhile, I inferred that he learnt theorems in Euclidean Geometry, which also learnt in Analytic Geometry, meanwhile using other approaches. Another possible interpretation would be they used the theorems of Euclidean Geometry to support the theorems learnt in Analytic Geometry. The term “support” suggests at least these two interpretations.

**Relationship between Euclidean and Analytic Geometry: Aid for geometrical understanding**

This issue brought different opinions amongst the students. This question raised affective responses on part of some students.

I noticed, for example, that S5 was very humble in explaining his geometrical understanding. He only presented his weaknesses in understanding. However, he insightfully explained the relationship between the two geometries. This is an indication that his geometrical understanding evolved during the geometry courses. Besides, I also saw that the fact he was able to identify his weaknesses showed that he was progressing in his geometrical understanding. His weaknesses consisted of “lack of visualization” before starting solving the task. That is why he rushed to formulae. He added that he still needed
sketches in Analytic Geometry to nurture his visualization skills. However, he added saying that in that discipline almost disappeared sketches.

**S5:** I think that our concern, when we solve a geometric task, is to rush to formulae. Rushing straightaway to formulae without visualizing the picture is our concern. I think the most important thing is to begin with a sketch. It is to begin with a sketch. My concern now is to practice sketches. I think somehow I was influenced in sketching. What happened to me in Euclidean Geometry is the following. I started attending Euclidean Geometry a little bit late. I started learning triangles and proof of theorems. I could understand the basic concepts, triangle, similar triangles; it was a little bit easy concerning angles and sides. Now, in Analytic Geometry, it is a big problem. We rarely deal with sketches. Then, we need first sketches.

S1, S4, and S7 promptly said the relationship between the two geometries helped them in their geometrical understanding. S7 added saying it also helped in understanding other subjects. S4 raised important issues regarding geometrical understanding.

**S4:** Well, they do help. Meanwhile, if you do not understand the concepts separately, you also cannot understand them combined. In other words, if you understand Euclidean Geometry separately from Analytic Geometry, sure you will understand both geometries combined. I think indirectly in Analytic Geometry we use some concepts of Euclidean Geometry. To the question, determine a distance you can measure directly with a ruler and get 5 or the question reads, determine the distance between A(x,y) and B(u,v). The distance will be the same, but in different perspectives.

The first issue deals with the question of understanding in mathematics portrayed as cumulative structuring. He states that before combining the two geometries for understanding purposes one firstly needs to understand how Euclidean Geometry is constructed. Afterwards, one should understand both geometries combined.

For the second issue he pointed out the same S5’s weakness in geometrical understanding: lack of imagination or visualization skills before solving a geometric task.

**S4:** Maybe to say... (Laughing). Geometry itself is difficult. It is a difficult subject. I do not know how to tackle this problem. I think… there is a... big mistake we make when we solve geometric tasks. We run to formulae before we analyse it. This mistake persists in me. We are short of the skill of imagination before using any formula especially in Analytic Geometry.

**S4:** What do I need to sketch a picture? What should I do?
Moreover, he explained what constituted imagination or visualization skills to him. He presented it in the form of two questions, of which answers I drew on Hershkowitz’s arguments (1998). To the question, “What do I need to sketch a picture?” I think has to do with visualization processes before sketching any concrete picture. Hershkowitz points out three progressing stages of these processes: 1) A new way of looking at the situation in order to suggest a generalisation; 2) its proof and verification in one process; and 3) an explanation of “why” the generalisation holds. In practice, I think, firstly, before we sketch a picture, we need in our minds to visualize a “general” picture, which entails all the conditions or aspects required by the task or even more. Secondly, we need to relate the elements and the properties of that picture against the conditions of the task to see whether they match, and finally, we need to explain to ourselves why that picture serves for the purpose of the task and related tasks.

The S4’s second question, “What should I do?” seems to do with construction processes. According to Duval (1998) construction processes depend only on connections between mathematical properties and the technical constrains of the tools used.

The third issue is regarding interplay between synthetic approaches and analytic approaches. He stated that he understood better the concepts in Analytic Geometry than in Euclidean Geometry. This is a controversial statement and I think it depends on the individuals. Meanwhile, there are claims, which support S4’s position (Chapter 3) on one hand. On the other hand, Stillwell argues that the synthetic aspect of geometry seems intuitively natural, which is the case of Euclidean Geometry. According to common sense, what is natural suggests something easy to be perceived (S5 is an example).

S4: Well, I think, the geometric concepts... Sorry, due to lack of experience in Euclidean Geometry, I have difficulties in this discipline. Meanwhile I noticed that it is easier to understand better the concepts in Analytic Geometry than in Euclidean Geometry. It is easier. You understand better. When you treat a triangle in plane or in space, you understand better. To determine the elements of a triangle trough coordinates is easy and interesting instead of using a property, which reads the sum of the measures of two sides of a triangle is always bigger than the measure of the third side.

S5: What happened to me in Euclidean Geometry is the following. I started attending Euclidean Geometry a little bit late. I started learning triangles and proof of theorems. I could understand the basic concepts, triangle, similar triangles; it was a little bit easy concerning angles and sides. Now, in Analytic
Geometry, it is a big problem. We rarely deal with sketches. Then, we need first sketches.

S3 did not say anything about his geometrical understanding. However, according to the first question I inferred that his geometrical understanding also evolved during the courses.

### 5.4.7 Emerging Issues on the Students’ Opinion on the Relationship between Euclidean Geometry and Analytic Geometry

The analysis of the students’ opinion in relationship to Euclidean and Analytic Geometry yielded some emerging issues, which concern the geometric modes and their geometrical understanding.

1- Most of the students (except S1) explicitly or implicitly asserted that both geometries deal with the same objects in different perspectives.

**Insight 1:** First, there is the visual, self-contained, synthetic side, which seems intuitively natural; then the algebraic, analytic side, which takes over when intuition fails and integrates geometry into the larger world of mathematics (Stillwell, 1998, p. 105).

2- Euclidean Geometry is a sort of a foundation for Analytic Geometry as it supplied with empirical concepts (intuitive level) to Analytic Geometry. In turn, Analytic Geometry applied these concepts upgrading them to abstract concepts (reasoning level).

**Insight 2:** The use of iconic support in mathematical problem solving is of paramount importance, wherever students meet a certain concept for the first time. However, there is a need to move along the continuum to abstraction so that image schemata – a bridge between abstract logical structures and particular concrete images and experience – may become more flexible and abstract. (Watson et al, 1993; Campbell et al, 1995; and San, 1996).

3- S4 and S5 pointed out that lack of imagination or visualization skills obliged them to rush to formulae for solving geometric tasks, which constituted their weakness in geometrical understanding.

**Insight 3:** Geometrical understanding consists of three cognitive processes, which are closely connected and their synergy is cognitively necessary for proficiency in geometry: Visualization processes, construction processes, and reasoning (Duval, 1998).
4- S4 stated that before combining the two geometries for understanding purposes one firstly needs to understand how Euclidean Geometry is constructed. Afterwards, one should understand both geometries combined.

*Insight 4: The axioms are assumed to be, or at least widely accepted within the mathematical community, obvious truths. In other words, all mathematical facts are constructed from the basic ones by proof. Thus, mathematics can metaphorically be considered as a ‘building’. It is from this perspective that understanding in mathematics can be portrayed as cumulative structuring (Mousley, 2003).*

5- S4 described visualization and imagination skills under the question: What do I need to sketch a picture?

*Insight 5: Hershkowitz (1998) points out three progressing stages of visualization skills: 1) A new way of looking at the situation in order to suggest a generalisation; 2) its proof and verification in one process; and 3) an explanation of “why” the generalisation holds.*

6- S4 described construction skills under the question: What should I do (in order to sketch a picture)? (In bracket added by me).

*Insight 6: According to Duval (1998), construction processes depend only on connections between mathematical properties and the technical constrains of the tools used.*

7- S4’s and S5’s approaches to geometry reveal that we cannot favor a geometric mode against the other. One should lean where one’s mind is inclined.

*Insight 7: Pedagogy should be flexible in accommodating the diversity of forms of thinking.*

5.4.8 Conclusion on the Students’ Opinion on the Relationship between Euclidean Geometry and Analytic Geometry
The target students possessed a vast range of ideas around the issues on relationship between Euclidean Geometry and Analytic Geometry so as on how they assessed their geometric understanding after attending these two geometries.

The common tenet (except for S1) refers to their approaches to these two geometric modes. They explicitly or implicitly asserted that both geometries deal with the same objects in different perspectives. These perspectives are known in the literature as synthetic and analytic aspects of geometry.

These students grasped these perspectives as on one hand a perspective where the treatment of geometric objects was almost devoid of algebra (synthetic aspect of geometry), while on the other hand a perspective where the treatment of geometric objects occurred with aid of algebra (analytic aspect of geometry).

In addition, a student (S7) thought a step further saying that Euclidean Geometry was a sort of a foundation for Analytic Geometry as it supplied with empirical concepts (intuitive level) to Analytic Geometry. In turn, Analytic Geometry applied these concepts upgrading them to abstract concepts (reasoning level). Accordingly, he supported the order of geometries in the curricula: first Euclidean Geometry and afterwards Analytic Geometry. This position is supported by findings of some empirical studies such as Watson et al, 1993; Campbell et al, 1995; and San, 1996.

Two students (S4 and S5) pointed out their weakness in the geometrical understanding namely lack of visualization or imagination skills. Accordingly, they straightaway rushed to formulae when solving geometric tasks. The rest of the students asserted they benefited from the two geometries in their geometrical understanding except S3 who had no idea about. However, according to his explanation on relationship between Euclidean and Analytic Geometry I inferred that he also was profiting from these courses in his geometrical understanding.

A student (S4) raised important issues on visualization skills posing the following question “What do I need to sketch a picture?” and on construction skills posing another question “What should I do (in order to sketch a picture?”).

Finally, I noticed two different forms of thinking namely a student (S5) who favored synthetic aspect of geometry for evolving his geometrical understanding and another (S4) who preferred analytic aspect of geometry for better understanding of the concepts. The other
students did not mention about this issue. Meanwhile for their explanation on relationship of both geometries they seemed to fit the S5’s position.

5.4.9 Research Question: Discussion

Likewise, in the main study component of Analytic Geometry I discuss the findings in light of the three research questions of the study namely:

1. How do first-year university students solve geometry problems? To what extent do they use algebraic knowledge and thinking in solving such problems? What kinds of meanings do students make of different algebraic and geometric concepts involved in problem solving situations?
2. To what extent does algebraic knowledge and thinking aid students’ conceptual understanding and problem solving performance in geometry?
3. To what extent is geometric work linked to algebraic thinking in the first year university geometry course at the Universidade Pedagógica in Maputo?

The rationale of each research question is found in Section 5.3.13.

Answers to these questions have been provided in the analysis and concluding sections of the chapter most of which appear under the title “emerging issues”.

Below I present the major research findings regarding the research questions organized by themes previously set up and others unexpected emerged during the data analysis of the second component of the main study – Analytic Geometry course.

Algebraic and geometric knowledge base and connectedness

All students used reasoning to solve Task 3 of the lecturer’s test through algebraic thinking aspects such as the dot product formula, coordinates, vector, trigonometry relations, matrix, acronym, equation, and unknown (analytic strategies); although they got the correct solution, some of them used the formulae in a rote fashion. Others attempted to make sense of the formulae using their intuition through visualization and construction processes (synthetic strategies).
All students, through hinting process, used both intuition (visualization and construction processes) and reasoning (formula, coordinates, vector, trigonometry relations, acronym, equation, and unknown, and generalisation) to solve the task successfully.

All students and especially the performance-oriented group seemed to profit from the hinting process for organizing and structuring their knowledge schemas. The mastery-oriented group showed reasonable, organized, and structured knowledge schemas. The hinting process only served to refine their knowledge schemas structure and organization.

*Analytic versus synthetic strategies*

In Analytic Geometry, analytic strategies (general strategies) seem to be more efficient and easier to access than synthetic strategies (specific strategies), meanwhile synthetic strategies seem to be more insightful than analytic strategies.

In Task 3 all students seemed to possess robust schemas regarding analytic strategies, however, they faced difficulties in relating analytic and synthetic schemas as they needed some hints (the mastery-oriented group) and a lot of hints (the performance-oriented group) for clarification purposes.

*Geometric modes: Euclidean (synthetic) Geometry and Analytic Geometry*

Most of the students explicitly or implicitly asserted that both geometries deal with the same objects in different perspectives. Euclidean Geometry is a sort of a foundation for Analytic Geometry as it supplied with empirical concepts (intuitive level) to Analytic Geometry. In turn, Analytic Geometry applied these concepts upgrading them to abstract concepts (reasoning level).

*Geometrical Understanding*

I noticed that in Analytic Geometry, the students needed intuition (synthetic strategies) to inform their reasoning (analytic strategies). In the contrary, in Euclidean Geometry they needed reasoning (analytic strategies) to inform their intuition (synthetic strategies). However, I observed that this balance still needed to take place in my target students’ mind. In other words, in Analytic Geometry, my students needed geometric thinking to aid their algebraic thinking and in Euclidean Geometry, they needed algebraic thinking to aid their geometric thinking. This mutuality in geometry seems to underpin what Schoenfeld (1986)
found that the interaction of deductive (as a means of discovery) and empirical (as a means of development of intuition) approaches to geometry leads students to reap the benefits of their knowledge.

This imbalance I observed in my subjects, which hindered their geometric conceptual understanding, might (partially) be explained by some students’ opinions. They pointed out that lack of imagination or visualization skills obliged them to rush to formulae for solving geometric tasks using them in a rote fashion. They acknowledged this phenomenon as their weakness in geometrical understanding. However, they proposed solutions to meet this difficulty saying that before combining the two geometries for understanding purposes one firstly needs to understand how Euclidean Geometry is constructed. Afterwards, one should understand both geometries combined. A student (S4) raised two critical issues regarding attaining imagination and visualization skills through two questions (1) what does one need to sketch a picture? And (2) what should one do (in order to sketch a picture?). In Section 5.4.6, I proposed answers to these two questions with recourse of literature. However, S4’s and S5’s approaches to geometry revealed that we cannot favor a geometric mode against the other. One should lean where one’s mind is reclined. Accordingly, pedagogy should be flexible in accommodating the diversity of forms of thinking.

5.5 Summary

The results in general showed that algebraic and geometric thinking mutually aided each other towards students’ geometric conceptual understanding. This phenomenon is explained by the dual nature of the discipline of geometry (synthetic and analytic aspects). However, in most of my subjects I observed a dichotomous approach to geometry, that is, a clear split of these two aspects when they solved geometric tasks either in the Euclidean and Analytic geometry courses. Accordingly, most of these students did not show a holistic understanding of geometry concepts. The dichotomous approach to geometry (partially) can be explained by Gascón’ notion of “thematic autism” which the next chapter refers to.