Weighted Approximation For Erdős Weights

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ABSTRACT.

We investigate Mean Convergence of Lagrange Interpolation and Rates of Approximation for Erdős Weights on the Real line.

An Erdős Weight is of the form, $W := \exp[-Q]$, where typically Q is even, continuous and is of faster than polynomial growth at infinity.

Concerning Lagrange Interpolation, we obtain necessary and sufficient conditions for convergence in $L_p\,(1 \leq p < \infty)$ and in particular, sharp results for $p > 4$ and $1 \leq p < 4$.

On Rates of Approximation, we first investigate the problem of formulating and proving the correct Jackson Theorems for Erdős Weights. This is accomplished in $L_p\,(0 < p \leq \infty)$ with endpoint effects in $[-a_n, a_n]$, the Mhaskar-Rahmanov-Saff interval.

We next obtain a natural Realisation Functional for our class of weights and prove its fundamental equivalence to our modulus of continuity.

Finally, we prove the correct converse or Bernstein Theorems in $L_p\,(0 < p \leq \infty)$ and deduce a Marchaud Inequality for our modulus.

DECLARATION

I declare that this dissertation is my own unaided work. It is being submitted for the degree of Doctor of Philosophy in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

SBDamelin

S B Damelin

3rd day of November, 1995.

To my wonderful family who made this all possible,
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PREFACE

(1) The results of Part A of this thesis, will appear in the Canadian Journal of Mathematics, in the form of two joint papers with Professor D.S Lubinsky.

(2) The results of Section 5.1, have been submitted to the Journal of Approximation Theory, in the form of a joint paper with Professor D.S Lubinsky.

(3) The results of Section 5.2, have been submitted to the Journal of Approximation in the form of a paper.

(4) The result of Section 5.3, will appear as a paper in the Eighth Texas Symposium on Approximation Theory.

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Chapter 1

A General Introduction

1.1 Bernstein's Approximation Problem

The subject of weighted polynomial approximation on the real line has its origins in the problem of the famous mathematician, S.N Bernstein, who in the 1910's made the following important observation. As polynomials are unbounded on unbounded sets, he realised the need to weight them. What resulted was the following:

Let $W : \mathbb{R} \longrightarrow (0,1]$ be a weight function satisfying

$$W(x) \geq 0 \, , \, \forall x \in \mathbb{R} \, ,$$

with

$$\lim_{|x| \longrightarrow \infty} x^n W(x) \, dx = 0, \quad n = 0, 1, 2 \ldots .$$

If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a given continuous function, is it true that there exist polynomials P making

$$\sup_{x \in \mathbb{R}} |f(x) - P(x)| \, W(x)$$

arbitrary small? Alternatively, under what conditions on W, are the polynomials dense in the weighted space of continuous functions generated by W [28, 29]? Naturally, this question generalises the well known theorem of Weierstrass, which states that each continuous function can be uniformly approximated by polynomials.

Bernstein's problem was solved in the 1950's by Mergelyan, Achieser and Pollard in various forms but we choose to state the following version due to Dzrbasjan and Carleson.

Theorem 1.1.1. (The possibility of approximation by weighted polynomials). *Let* $W(x) := \exp(-Q(x))$, $Q : \mathbb{R} \longrightarrow \mathbb{R}$, *$Q$ even and $Q(e^x)$ convex in $(0, \infty)$. The following are equivalent:*

(i) $\forall$ *continuous* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *with*

$$\lim_{|x| \longrightarrow \infty} fW(x) = 0$$

and $\forall \varepsilon > 0, \exists$ *a polynomial P such that*

$$\|(f - P)\, W\|_{L_\infty(\mathbb{R})} < \varepsilon;$$

(ii)

$$\int_{\mathbb{R}} \frac{Q(x)}{1 + x^2} dx = \infty. \tag{1.1}$$

We remark that (1.1) although quite simple to absorb, has had far reaching consequences on weighted approximation up to this day. This is borne out in the following:

Corollary 1.1.2. *Let $\gamma > 0$ and set*

$$W_\gamma(x) := \exp\left(-\frac{|x|^\gamma}{2}\right). \tag{1.2}$$

Then the polynomials are dense for W_γ iff $\gamma \geq 1$.

This corollary essentially tells us that at least for polynomial approximation on $\mathbb{R}$, our weight must decay at least as fast as $\exp(-|x|)$. Quite naturally, we are lead by the reasons above and others, for example the determancy of the moment problem [28], to two important classes of weights on $\mathbb{R}$.

1.2 The Freud Class

A weight $W(x) := \exp(-Q(x))$ is said to be a Freud weight if Q is of smooth polynomial growth at infinity. They are named after the Hungarian mathematician, Geza Freud, who, while working on problems related to weighted approximation, Orthogonal Fourier Series, and Lagrange interpolation, discovered that there had been a complete lack of results regarding general orthogonal polynomials on infinite intervals[42]. For example, it is well known, that the theory of rates of approximation on finite intervals depends heavily on trigonometric approximation. Here, heavy use is made of the orthogonal trigonometric polynomials,

$$(\cos(n\theta), (\sin(n\theta)))_{n=0}^{\infty}.$$

Freud realised that for many questions of weighted approximation on the real line, one needed a proper understanding of the weighted orthonormal polynomials $(P_j(x))_{j=0}^{\infty}$, satisfying

$$\int_{\mathbb{R}} P_n(x) P_m(x) W^2(x) dx = \delta_{m,n}.$$

A classical example of a Freud Weight is (1.2) of which the Hermite Weight,

$$W_2(x) := \exp\left(\frac{-x^2}{2}\right) \tag{1.3}$$

is a special example.

1.3 The Erdős Class

A Weight $W(x) := \exp(-Q(x))$ is said to be an Erdős Weight if Q is of faster than polynomial growth at infinity. They were named by D.S Lubinsky, after the Hungarian Mathematician, Paul Erdős, who was the first to consider them, obtaining the contracted zero distribution of their orthogonal polynomials, as well as investigating the asymptotic behavior of the largest zeros of their orthogonal polynomials. Some classical examples of Erdős Weights are

$$W_{k,\alpha}(x) := \exp(-\exp_k(|x|^\alpha)) \quad k \geq 1, \alpha > 1 \tag{1.4}$$

where $\exp_k(x) = \exp(\exp(\ldots(\exp(x))))$ denotes the kth iterated exponential, and

$$W_{A,B}(x) := \exp\left(-\exp\left(\log\left(A + x^2\right)\right)^B\right) \; B > 1, A \text{ large enough.} \tag{1.5}$$

We see that to some extent, the Freud and Erdős classes are analogues to entire functions of finite and infinite order.

We mention that there is of course, a third naturally occurring class of weights, the class of Q, where Q is of slower than polynomial growth at infinity. The canonical example is the Stieltjies-Wigert weight or log-normal distribution,

$$W^2(x) := \exp\left(-k(\log x)^2\right), k > 0.$$

We observe, however, that for most questions of weighted approximation on the real line, Theorem 1.1.1 forces us to work with the former two classes of weights, although the latter class has been investigated for other related problems.

It is then not suprising that the theory of orthogonal polynomials for both Freud and Erdős Weights and the theory of weighted approximation on $\mathbb{R}$ have developed in parallel over the last twenty years. The idea, of course, is to obtain a complete understanding of the orthogonal polynomials generated by these two classes of weights. For example, the asymptotics of their zeros, their bounds and so on. Although Freud initiated this study, his results have been surpassed in almost every respect in both sharpness and generality by many including Bauldry, Bonan, Levin, Lubinsky, Magnus, Mate, Mthembu, Mhaskar, Nevai, Rahmanov, Saff, Sheen, Totik and Ulman. See[13, 24, 26, 28, 29, 42] and later chapters.

1.4 Infinite finite Range Inequalities

When dealing with weights on $\mathbb{R}$, one realises immediately that unlike weights on finite intervals, these weights are of unbounded support. It took a Freud and Nevai inspiration [42] to allow us effectively to work on finite intervals when dealing with weighted polynomials $(P_n W)$ on $\mathbb{R}$. They developed the so called Infinite-Finite Range Inequality. The idea was to consider a given expression of the form,

$$g(x) = x^n \exp[-Q(x)]$$

and to determine its maximum at

$$q_n,\ n \geq 1, \tag{1.6}$$

the so called Freud Number given by

$$n = q_n Q'(q_n). \tag{1.7}$$

They effectively showed that most of the time, the quantity (PW) "lives" in an interval like $[-q_n, q_n]$, so that the interval depends on the degree of the polynomial, n and not on the polynomial in question. The sharp form of (1.6) was obtained independently by Rahmanov and then Mhaskar and Saff[37, 38]. We have:

Definition 1.4.1 (Mhaskar-Rahmanov-Saff number). *Let $W := \exp(-Q)$, where $Q : \mathbb{R} \longrightarrow \mathbb{R}$ is even, continuous and $xQ'(x)$ is positive and increasing in $(0,\infty)$ with limits 0 and ∞ at 0 and ∞. For $u > 0$, the Mhaskar-Rahmanov-Saff number a_u is the positive root of the equation*

$$u = \frac{2}{\pi}\int_0^1 a_u t Q'(a_u t)\frac{dt}{\sqrt{1-t^2}}. \tag{1.8}$$

Under the conditions on Q above, which guarantee that $Q(s)$ and $Q'(s)$ increase strictly in $(0,\infty)$, a_u is uniquely defined, increases with u and grows roughly like $Q^{-1}(u)$, where Q^{-1} is the inverse of Q on $(0,\infty)$.

We remark that it is often possible to use something other than a_u that would require less of $xQ'(x)$, namely, that it be quasi-increasing for large x, for example $Q^{-1}(u)$. However, this often complicates formulations and so is omitted. Here, a function

$$f : (a,b) \longrightarrow (0,\infty)$$

is quasi-increasing if $\exists C > 0$ such that

$$a < x < y < b \Longrightarrow f(x) < Cf(y).$$

Mhaskar and Saff then used a_u to prove the infinite-finite inequality [38]

$$\|P_n W\|_{L_\infty(\mathbb{R})} = \|P_n W\|_{L_\infty[-a_n\, a_n]}, \tag{1.9}$$

holding for all polynomials P_n of degree $\leq n$, $n \geq 1$ and where Q is as in Definition 1.4.1, is convex or is of the form $|x|^\alpha$, $\alpha \geq 1$.

These inequalities have been improved and generalised since then for example to $L_p\,(0 < p \leq \infty)$. See [24, 26, 39] and later chapters.

It is instructive to see some concrete representations of a_u. For example, for $W_\gamma(x)$ defined by (1.2), $a_u \sim u^{\frac{1}{\gamma}}$, whereas, for $W_{k,\alpha}(x)$ defined by (1.4), $a_u \sim (\log_k u)^{\frac{1}{\alpha}}$ where $\log(\log(\log\ldots()))$ denotes the kth iterated logarithim. Also, $Q_{k,\alpha}(a_u) \sim u\left\{\prod_{j=1}^{k} \log_j u\right\}^{-\frac{1}{2}}$. See [26] and later chapters.

1.5 Entire Functions

Let $W = \exp(-Q)$ be a Freud or Erdös Weight. By Carleman's Theorem, if Q is continuous, we know that there exist two entire functions G_1 and G_2 such that for a given $\varepsilon > 0$

$$\begin{aligned} 1-\varepsilon &< \frac{W(x)}{G_1(x)} < 1+\varepsilon\,,\ \forall x \in \mathbb{R} \\ 1-\varepsilon &< \frac{W^{-1}(x)}{G_2(x)} < 1+\varepsilon\,,\ \forall x \in \mathbb{R} \end{aligned}$$

It was D.S Lubinsky, who initiated the study of approximating Freud or Erdös Weight type weighted polynomials of the form $P_n(x)\,V'(a_x x)$ by entire functions, whose representation could be explicitly written down [28, 42]. For many of our main results, it will be important to consider theorems on polynomial approximation of W^{-1}. The following theorem of Clunie and Kovari will be frequently used.

Theorem 1.5.1. (Clunie and Kovari). *Let* $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ *be an increasing function and* *suppose that it has the representation*

$$\phi(r) = \phi(1)\exp\left(\int_1^r \frac{\psi(p)}{p}dp\right)\ r \geq 1 \tag{1.10}$$

for some positive increasing function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ *. Assume further that for some* $C > 1$ *and every* $r \geq 1$,

$$\psi(Cr) - \psi(r) \geq 1. \tag{1.11}$$

Then there exists an entire function G *with positive coeficients*

$$G(r) = \sum_{j=0}^{\infty} g_{2j} r^j \tag{1.12}$$

such that

$$C_2 \leq \frac{G(r)}{\phi(r)} \leq C_1 \tag{1.13}$$

where C_1 *and* C_2 *depend only on* C *and not* r.

1.6 Towards Lagrange Interpolation and Rates of Approximation for Erdős Weights

The primary aim of this thesis concerns the approximation of functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ by weighted polynomials of Erdős type. Several problems were considered.

1.6.1 Lagrange Interpolation for Erdős Weights

Following from earlier work of Nevai, Bonan, Lubinsky, Knopmacher, Mthembu and Matjila, we investigated mean convergence of Lagrange Interpolation for Erdös Weights. We obtained Necessary and Sufficient conditions for L_p $(1 \leq p < \infty)$ and in particular, sharp results for $p > 4$ and $1 \leq p < 4$.

1.6.2 Rates of Approximation for Erdős Weights

Jackson Theorems

Following from from earlier work of Ditzian, Lubinsky and Totik, we investigated the problem of formulating and proving the correct Jackson Theorems for Erdős Weights. This was accomplished in L_p $(0 < p \leq \infty)$. An interesting feature here is that the degree of the approximation improves towards the endpoints of the Mhaskar-Rahmanov-Saff interval $[-a_n\ a_n]$. This is in contrast to the Freud case.

K-Functionals

Following from earlier work of Ditzian, Lubinsky and Totik, we investigated the problem of formulating the correct Realisation Functional for our modulus of continuity and prove its equivalence. We deduce classical properties of our modulus, including Marchaud Inequalities.

Converse Theorems

Following from earlier work of Ditzian, Lubinsky and Totik, we investigated the problem of formulating and proving the correct Converse Bernstein type Theorems for Erdős Weights. This was accomplished in L_p $(0 < p \leq \infty)$ with endpoint effects in $[-a_n\ a_n]$.

1.7 General Information

This thesis consists of two parts. Part 1 deals with the quantitative theory of Lagrange Interpolation for Erdős Weights, while Part 2 considers the question of rates of approximation for Erdős Weights. Both parts contain in turn, their own chapters, historical background, definitions and theorems and are thus self contained and can be read independently of each other. To encourage "reader" friendliness, we have in many places, resorted to stating well known results with references.

Throughout, $\mathcal{P}_n$ denotes the class of polynomials of degree $\leq n$, $C, C_1, C_2...$ denote positive constants independent of n, x and $P_n \in \mathcal{P}_n$. The same symbol does not necessary denote the same constant in different occurrences. We write $C \neq C(L)$ to indicate that C is independent of L. Finally we introduce some more notation.

(1) $c_n \sim d_n$ means that $C_1 \leq \frac{c_n}{d_n} \leq C_2$ for some $C_j > 0$, $j = 1, 2$ and the relevant range of n.

(2) $a_n = O(b_n)$ means that $a_n \leq C_3 b_n$ for some $C_3 > 0$.

(3) $a_n = o(b_n)$ means that $\lim_{n \longrightarrow \infty} \left| \frac{a_n}{b_n} \right| = 0$.

Similar notation is used for functions and sequences of functions.

Part I

Lagrange Interpolation for Erdős Weights

Chapter 2

Introduction and Statement of Results

One of the most quantitative and explicit methods of approximating a given function f is that of polynomial interpolation. In this first part, we consider the problem of weighted Lagrange interpolation for Erdős Weights.

2.1 Some Historical Backround

Let us be given a Freud or Erdős Weight, $W : \mathbb{R} \longrightarrow \mathbb{R}$. We can then define, for this weight, a unique set of orthonormal polynomials

$$p_n(x) := p_n\left(W^2, x\right) = \gamma_n x^n + ..., \tag{2.1}$$

with $\gamma_n = \gamma(W^2) > 0$
and satisfying

$$\int_{\mathbb{R}} p_n\left(W^2, x\right) p_m\left(W^2, x\right) W^2(x)\, dx = \delta_{m,n}. \tag{2.2}$$

See[13, 42].

We write W^2 not W as we weight each $p_n = p_n(W^2, x)$ by W. It is well known that p_n has

n real zeros $(x_{j,n})_{j=1}^{n}$ and we order them as follows

$$-\infty < x_{n,n} < x_{n-1,n} < ...x_{2,n} < x_{1,n} < \infty. \tag{2.3}$$

Now for each $1 \leq j \leq n$, let us define the fundamental polynomials of Lagrange Interpolation by

$$l_{j,n}(x) = \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{x - x_{j,n}}{x_{j,n} - x_{k,n}} = \frac{p_n(x)}{p_n'(x_{j,n})(x - x_{k,n})} \in \mathcal{P}_{n-1} \tag{2.4}$$

satisfying

$$l_{j,n}(x_{k,n}) = \delta_{j,k}. \tag{2.5}$$

Then for a given $f : \mathbb{R} \longrightarrow \mathbb{R}$, we define the Lagrange Interpolation Polynomial of degree $\leq n-1$ to f by

$$L_n[f](x) := \sum_{j=1}^{n} f(x_{j,n})\, l_{j,n}(x). \tag{2.6}$$

For large classes of Freud and Erdős Weights, mean convergence of Lagrange Interpolation is an extensively researched and widely studied subject. We survey some of the literature but refer the reader to [33, 35, 41, 44] for more on this subject, and its corresponding analogue on finite intervals.

We begin with the following form of the Erdös-Turan Theorem as extended by Shohat. See[13, chapter 2, pg 97].

Theorem 2.1.1.(Erdős-Turán). *If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is Riemann integrable in each finite interval and there exists an even entire function G with all non-negative Maclaurin series coefficients such that*

$$\lim_{|x|\longrightarrow\infty} \frac{f^2(x)}{G(x)} = 0$$

and

$$\int_{\mathbb{R}} G(x)\, W^2(x)\, dx < \infty.$$

Then

$$\lim_{n\longrightarrow\infty} \|(f - L_n[f])\, W\|_{L_2(\mathbb{R})} = 0. \tag{2.7}$$

Remark

For "nice" weights W like $W_{k,\alpha}$ and $W_{A,\beta}$, Theorem 1.5.1 allows us to choose G with

$$G(x) \sim W^{-2}(x)(1+|x|)^{-1-\kappa},\ \forall x \in \mathbb{R} \text{ and } \kappa > 0.$$

So that we can ensure (2.7) holds if

$$\lim_{|x|\longrightarrow\infty} (fW)(x)(1+|x|)^{\frac{1}{2}+\frac{\kappa}{2}} = 0. \tag{2.8}$$

2.2 Mean Convergence for $p \neq 2$ for Freud Weights

P. Nevai and his Ph.D. student, S. Bonan, essentially completed the study for the Hermite Weight defined in (1.3). Nevai [45] proved:

Theorem 2.2.1. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous with*

$$\lim_{|x|\longrightarrow\infty} f(x)(1+|x|)W_2^2(x) = 0.$$

Then for every $p > 1$, if $L_n[f]$ denotes the Lagrange interpolation polynomial of degree $\leq n-1$ to f at the zeros of $p_n = p_n(W^2, x)$,

$$\lim_{n\longrightarrow\infty} \|(f - L_n[f])W\|_{L_p(\mathbb{R})} = 0. \tag{2.9}$$

Moreover, if (2.9) holds for some weight W and for every continuous f with compact support, then W satisfies

$$\int_{\mathbb{R}} \left[\frac{\frac{W(x)}{W_2^2(x)}}{1+|x|}\right]^p dx < \infty,$$

so that W is quite close to W_2.

S. Bonan in his Ph.D. thesis, obtained precise necessary and sufficient conditions for the Hermite weight, as well as obtaining results for the generalised Hermite weight

$$W_2^\gamma(x) := |x|^\gamma \exp\left(-\frac{x^2}{2}\right)\ x \in \mathbb{R},\ \gamma > 1.$$

Here is one of his results for $\gamma = 0$ [1].

Theorem 2.2.2. *Let $W_2(x) = \exp\left(-\frac{|x|^2}{2}\right)$. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous and let $L_n[f]$ be the Lagrange Interpolation polynomial of degree $\leq n-1$ at the zeros of $p_n = p_n(W_2^0, x)$. Then if*

$$f(x)W(x) = O\left(|x|^{-2}\right), \ |x| \longrightarrow \infty$$

we have for $0 < p < \infty$, and

$$\delta < \begin{cases} 1 - \frac{1}{p} & 0 < p < 4 \\ \frac{2}{3} + \frac{1}{3p} & p > 4 \end{cases}$$

$$\lim_{n \longrightarrow \infty} \left\| (f - L_n[f]) W_0^2 (1 + |x|)^{\delta} \right\|_{L_p(\mathbb{R})} = 0.$$

A. Knopmacher and D.S Lubinsky, on the other hand, deduced sufficient conditions for mean convergence for a large class of Freud weights including $W(x) := \exp\left(\frac{-x^m}{2}\right)$, $m = 2, 4, 6..$ [19].

2.3 Necessary and Sufficient Conditions for Mean Convergence of Lagrange Interpolation for Freud Weights.

The possibility of obtaining identical, necessary and sufficient conditions for mean convergence of Lagrange Interpolation for large classes of Freud and Erdős Weights, arises from the correct bounds for the orthonormal polynomials, together with the asymptotics and distribution of their zeros, obtained recently by E.Levin, D.S Lubinsky and T. Mthembu[24, 26]. D.Matjila and D.S Lubinsky tackled the Freud case[33]. For notational simplicity, we recall their main result for $W_\gamma(x)$, $\gamma > 1$ given by (1.2).

Theorem 2.3.1. *Let $W(x) = W_\gamma(x) = \exp\left(\frac{-|x|^\gamma}{2}\right)$, $\gamma > 1$. Given $f : \mathbb{R} \longrightarrow \mathbb{R}$, let $L_n[f]$ denote the Lagrange Interpolation polynomial to f at the zeros of $p_n(W^2, x)$. Let*

$1 < p < \infty$, $\Delta \in \mathbb{R}$, $\alpha > 0$ *and*

$$\tau := \frac{1}{p} - \min(1, \alpha) + \max\left(0, \frac{\gamma}{6}\left(1 - \frac{4}{p}\right)\right).$$

Then for

$$\lim_{n \longrightarrow \infty} \left\| (f - L_n[f]) W_0^2 (1 + |x|)^{\Delta} \right\|_{L_p(\mathbb{R})} = 0$$

to hold for every continuous $f : \mathbb{R} \longrightarrow \mathbb{R}$ *satisfying*

$$\lim_{|x| \longrightarrow \infty} (fW)(x)(1 + |x|)^{\alpha} = 0,$$

it is necessary and sufficient that

(1) $\Delta > \tau$ if $1 < p \leq 4$

(2) $\Delta > \tau$ if $p > 4$ and $\alpha = 1$

(3) $\Delta \geq \tau$ if $p > 4$ and $\alpha \neq 1$.

2.4 Necessary and Sufficient conditions for Mean Convergence of Lagrange Interpolation for Erdős Weights

In describing analogous results for Erdős Weights, we need a class of weights W^2, for which suitable bounds are available for $p_n(W^2,)$. These were found in [26] and L_p analogues in [31].

2.5 Statement of results

For our purposes, the following subclass of weights from [26] is suitable:

Definition 2.5.1. *Let* $W := \exp[-Q]$, *where* $Q : \mathbb{R} \longrightarrow \mathbb{R}$ *is even, continuous,* $Q^{(2)}$ *exists in* $(0, \infty)$ *and the function,*

$$T^*(x) := 1 + \frac{xQ^{(2)}(x)}{Q^{(1)}(x)} \tag{2.10}$$

is increasing in $(0, \infty)$, *with*

$$\lim_{x \longrightarrow \infty} T^*(x) = \infty, \; T^*(0+) := \lim_{x \longrightarrow 0+} T^*(x) > 1. \tag{2.11}$$

Moreover, we assume for some $C_1, C_2, C_3 > 0$

$$C_1 \leq \frac{T^*(x)}{\frac{xQ^{(1)}(x)}{Q(x)}} \leq C_2,\ x \geq C_3 \tag{2.12}$$

and for every $\varepsilon > 0$

$$T^*(x) = O\left(Q(x)^{\varepsilon}\right),\ x \longrightarrow \infty. \tag{2.13}$$

Then we write $W \in \mathcal{E}_1^*$.

The new restrictions over those in [26] are (2.13) and $Q \geq 0$. The latter is easily achieved by replacing Q by $Q + |Q(0)|$. The former is needed in simplifying the formulation of our theorems. We note that the restriction is a weak one, since one has typically for each $\varepsilon > 0$,

$$T^*(x) = O\left(\log Q'(x)\right)^{1+\varepsilon} \quad x \to \infty.$$

In fact, one can show that for any weight W satisfying our conditions except possibly for (2.13) we have,

$$meas\mathcal{E}_r = meas\{x \geq r : T^*(x) \geq \varepsilon(\log Q'(x))^{1+\varepsilon}\}$$

satisfies

$$\int_{\mathcal{E}_r} \frac{dx}{x} \to 0, \quad r \to \infty \text{ [32]}.$$

Here *meas* denotes Lebegue measure.

The principal example of $W = \exp[-Q] \in \mathcal{E}_1^*$, is $W_{k,\alpha} = \exp(-Q_{k,\alpha})$ given by (1.4) with $\alpha > 1$. For this W,

$$T^*(x) = T^*_{k,\alpha}(x) = \alpha\left[1 + x^{\alpha}\sum_{l=1}^{k}\prod_{j=1}^{l-1}\exp_j(x^{\alpha})\right],\ x \geq 0. \tag{2.14}$$

Here (2.12) holds in the stronger form:

$$\lim_{x \longrightarrow \infty} \frac{T^*(x)}{\frac{xQ^{(1)}(x)}{Q(x)}} = 1 \tag{2.15}$$

and (2.13) holds in the stronger form

$$\lim_{x \longrightarrow \infty} \frac{T^*(x)}{\left[\prod_{j=1}^{k} \log_j Q(x)\right]} = \alpha. \tag{2.16}$$

We remark that here,

$$T^*(a_u) \sim \prod_{j=1}^{k} \log_j(u). \tag{2.17}$$

For $\alpha \leq 1$, the second part of (2.11) fails, but this can be circumvented by considering $W_{k,\frac{\alpha}{2}}(A+x^2)$, with A large enough to guarantee $T^*(0+) > 1$.

Another more slowly decaying example of $W = \exp[-Q] \in \mathcal{E}_1^*$ is given by $W_{A,B}(x)$ for which

$$T^*(x) = \frac{2x^2}{A+x^2}\left[\frac{\beta-1}{\log(A+x^2)} + \beta\left\{\log\left(A+x^2\right)\right\}^{\beta-1}\right] + \frac{2A}{A+x^2}. \tag{2.18}$$

Again (2.12) holds in the stronger form, (2.15), while (2.13) holds in the stronger form

$$\lim_{x \longrightarrow \infty} \frac{T^*(x)\log x}{\log Q(x)} = \beta. \tag{2.19}$$

We begin with our first result for $1 < p < \infty$.

Theorem 2.5.2. *Let $W := \exp[-Q] \in \mathcal{E}_1^*$. Let $L_n[.]$ denote the Lagrange Interpolation to f at the zeros of $p_n(W^2,.)$. Let $1 < p < \infty$, $\Delta \in \mathbb{R}$, $\kappa > 0$. Then for*

$$\lim_{n \longrightarrow \infty} \left\|(f - L_n[f])\, W (1+Q)^{-\Delta}\right\|_{L_p(\mathbb{R})} = 0 \tag{2.20}$$

to hold for every continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying

$$\lim_{|x| \longrightarrow \infty} |fW|(x)\,(\log|x|)^{1+\kappa} = 0, \tag{2.21}$$

it is necessary and sufficient that

$$\Delta > \max\left\{0, \frac{2}{3}\left(\frac{1}{4} - \frac{1}{p}\right)\right\}. \tag{2.22}$$

At first, the choice of the extra weighting factor $(1+Q)$ in (2.20) may seem rather severe. After all, Q grows faster than any polynomial. However, even if f vanishes outside a fixed finite interval, we need such a factor if $p > 4$.

Theorem 2.5.3. *Let W, L_n be as above and $p > 4$. Suppose that measurable $U : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies*

$$\liminf_{x \longrightarrow \infty} U(x)\, x^{-\left(\frac{3}{2}-\frac{1}{p}\right)} Q(x)^{\frac{2}{3}\left(\frac{1}{4}-\frac{1}{p}\right)} > 0. \tag{2.23}$$

Then there exists continuous $f : \mathbb{R} \longrightarrow \mathbb{R}$ vanishing outside $[-2, 2]$ such that

$$\limsup_{n \longrightarrow \infty} \|L_n[f]\, W U\|_{L_p(\mathbb{R})} = \infty. \tag{2.24}$$

So for $p > 4$, no growth restriction on f, however severe, allows us a weighting factor weaker than a power of $1 + Q$. One can formulate versions of Theorem 2.5.2 for $p > 4$ that involve $\Delta = \frac{2}{3}\left(\frac{1}{4} - \frac{1}{p}\right)$, and then one has to introduce extra factors in (2.20), such as negative powers of $1 + |x|$ and negative powers of T^* or $\log(2 + Q)$. Unfortunately, one then needs extra hypotheses on T^* to avoid very complicated formulations. One of the complicating features here, is that T^* may grow faster than any power of $|x|$ (as in (2.14) for $k \geq 2$), like a power of x (as in (2.14) for $k = 1$), or slower than any power of $|x|$ (as in (2.18)). Moreover, one has to compare T^* to $\log Q$. We spare the reader the details.

For $p \leq 4$, the weighting factor $1 + Q$ is unnessesarily strong. Indeed, Theorem 2.5.2 does not extend the classical Erdős-Turán theorem, i.e. Theorem 2.1.1 for $p = 2$. Following is our extension.

Theorem 2.5.4. *Let $W := \exp[-Q] \in \mathcal{E}_1^*$. Let $1 < p < 4$, and $\alpha \in \mathbb{R}$. Let $L_n[f]$ denote the Lagrange interpolating polynomial to f at the zeros of $p_n(W^2, .)$. Then the following are equivalent.*

(a) For every continuous $f : \mathbb{R} \longrightarrow \mathbb{R}$ with

$$\lim_{|x| \longrightarrow \infty} |f(x)|\, W(x)\, (1 + |x|)^{\alpha} = 0, \tag{2.25}$$

we have

$$\lim_{n \longrightarrow \infty} \|(f - L_n[f])\, W\|_{L_p(\mathbb{R})} = 0. \tag{2.26}$$

(b)

$$\alpha > \frac{1}{p}. \tag{2.27}$$

Thus our result extends Theorem 2.1.1 for $\alpha > \frac{1}{p}$.

We next show that Theorem 2.5.4 is sharp in the sense, that we cannot insert any positive power of $1+|x|$ inside the L_p norm in (2.26), at least when $\alpha > \frac{1}{p}$.

Theorem 2.5.5. *Let $W := \exp[-Q] \in \mathcal{E}_1^*$. Let $1 < p < 4$ and $\Delta \in \mathbb{R}$. Then the following are equivalent:*

(a) *For every $\alpha > \frac{1}{p}$ and every continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying* (2.25), *we have*

$$\lim_{n \longrightarrow \infty} \left\| (f - L_n[f])(x) W(x) (1+|x|)^{\Delta} \right\|_{L_p(\mathbb{R})} = 0. \tag{2.28}$$

(b)

$$\Delta \le 0. \tag{2.29}$$

What about a sharp form for $p = 4$? The following points the way.

Theorem 2.5.6. *Let $W := \exp[-Q] \in \mathcal{E}_1^*$. Suppose that a measurable function $U : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies*

$$\lim_{x \longrightarrow \infty} U(x) x^{\frac{-3}{4}} (\log Q(x))^{\frac{1}{4}} = \infty. \tag{2.30}$$

Then there exists a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ vanishing outside $[-2, 2]$, such that

$$\limsup_{n \longrightarrow \infty} \|L_n[f] WU\|_{L_4(\mathbb{R})} = \infty. \tag{2.31}$$

If, for example, $Q(x)$ grows faster than $\exp(x^{3+\varepsilon})$, some $\varepsilon > 0$, then Theorem 2.5.6 shows that we cannot choose $U \equiv 1$ and hope for convergence. So there is no analogue of Theorem 2.5.4 for $p = 4$. However, it seems that a negative power of $\log Q$, rather than the $1+Q$ required for $p > 4$, will allow some analogue of Theorem 2.5.2 for $p = 4$.

We note that with more work, we can replace continuity of f in Theorems 2.5.2, 2.5.4 and 2.5.5 by Riemann integrability and we can replace, in Theorems 2.5.4 and 2.5.5 $(1+|x|)^{\alpha}$, $\alpha > \frac{1}{p}$, by $(1+|x|)^{\frac{1}{p}} (\log(2+|x|))^{\frac{1}{p}+\varepsilon}$, some $\varepsilon > 0$, (and so on).

Furthermore, the methods of proof of Theorem 2.5.2 and 2.5.3 rely heavily on estimates and results of [26, 31], whereas those of Theorems 2.5.4-2.5.6 rely on [20, 21, 26, 31].

2.6 Some more notation

The nth *Christoffel function* for a weight W^2 is

$$\begin{aligned} \lambda_n(x) \quad &: = \lambda_n\left(W^2, x\right) = \inf_{P_{n-1} \in \mathcal{P}_{n-1}} \int_{\mathbb{R}} \frac{\left(P_{n-1}W\right)^2 dt}{P_{n-1}^2(x)} \\ &= \frac{1}{\sum_{j=0}^{n-1} p_j^2(x)}. \end{aligned} \tag{2.32}$$

The *Christoffel* or *Cotes numbers* are

$$\lambda_{jn} = \lambda_n\left(W^2, x_{j,n}\right) \quad 1 \leq j \leq n. \tag{2.33}$$

The fundamental polynomials l_{jn} of (2.4) admit the representation

$$l_{jn}(x) = \lambda_{jn} \frac{\gamma_{n-1}}{\gamma_n} p_{n-1}\left(x_{j,n}\right) \frac{p_n(x)}{x - x_{j,n}}. \tag{2.34}$$

The reproducing kernel for W^2 is

$$\begin{aligned} K_n(x,t) \quad &: = K_n\left(W^2, x, t\right) = \sum_{j=0}^{n-1} p_j(x)\, p_j(t) \\ &= \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x)\, p_{n-1}(t) - p_n(t)\, p_{n-1}(x)}{(x-t)}. \end{aligned} \tag{2.35}$$

(the Christoffel-Darboux formula).

Given measurable $f : \mathbb{R} \longrightarrow \mathbb{R}$ with $f(x)\, x^j W^2 \in L_1(\mathbb{R}) \ \ \forall j \geq 0$, the nth partial sum of its orthonormal expansion with respect to W^2, is denoted by $S_n[f](x)$, and admits the representation

$$S_n[f](x) = \int_{\mathbb{R}} K_n(x,t)\, f(t)\, W^2(t)\, dt. \tag{2.36}$$

We introduce the **Hilbert Transform** of $g \in L_1(\mathbb{R})$ by

$$H[g](x) := \lim_{\varepsilon \longrightarrow 0+} \int_{|x-t| \geq \varepsilon} \frac{g(t)}{x-t} dt, \tag{2.37}$$

which exists a.e. [49].

We may then use the Christoffel-Darboux formula for $K_n(x,t)$ to rewrite (2.36) as

$$S_n[f] = \frac{\gamma_{n-1}}{\gamma_n} \left\{ p_n H\left[f p_{n-1} W^2\right] - p_{n-1} H\left[f p_n W^2\right] \right\}. \tag{2.38}$$

Finally, we define an auxillary quantity

$$\delta_n := (nT^*(a_n))^{-\frac{2}{3}}, \ n \geq 1. \tag{2.39}$$

This quantity is useful in describing the behaviour of $p_n(\exp[-2Q])$ near $x_{1,n}$. For example,

$$\left| \frac{x_{1,n}}{a_n(Q)} - 1 \right| \leq \frac{L\delta_n}{2}. \tag{2.40}$$

Here L is independent of n.

We often use the fact that δ_n is much smaller than any power of $\frac{1}{T^*(a_n)}$ (see Chapter 3) .We also use the function

$$\Psi_n(x) := \max\left\{ \sqrt{1 - \frac{|x|}{a_n} + L\delta_n}, \left[T^*(a_n) \sqrt{1 - \frac{|x|}{a_n} + L\delta_n} \right]^{-1} \right\}, \ |x| \leq a_n \tag{2.41}$$

and set

$$\Psi_n(x) := \Psi_n(a_n), \ |x| > a_n. \tag{2.42}$$

This function is used in describing the spacing of zeros of p_n, the behaviour of Christoffel functions and so on. Finally, we adopt the following conventions: Set

$$x_{0,n} := x_{1,n}(1 + L\delta_n); \ x_{n+1,n} := x_{n,n}(1 + L\delta_n), \tag{2.43}$$

$$I_{j,n} := (x_{j,n}, x_{j-1,n})\,;\, 1 \le j \le n \tag{2.44}$$

and

$$|I_{j,n}| := x_{j-1,n} - x_{j,n},\; 1 \le j \le n. \tag{2.45}$$

Also, in proving our quadrature estimates, we use

$$f_{j,n}(x) := \min\left\{\frac{1}{|I_{j,n}|}, \frac{|I_{j,n}|}{(x - x_{j,n})^2}\right\}\left[\left|1 - \frac{|x|}{a_r}\right| + L\delta_n\right]^{-\frac{1}{4}} \tag{2.46}$$

and the characteristic function of $I_{j,n}$

$$\chi_{j,n}(x) := \chi_{I_{j,n}}(x) := \begin{cases} 1, x \in I_j, \\ 0, x \notin I_{j,n} \end{cases} \tag{2.47}$$

Chapter 3

Technical estimates, Hilbert Transforms and Quadrature

Throughout this chapte., let $W \in \mathcal{E}_1^*$.

3.1 Technical Lemmas

In this section, we gather technical estimates from various sources. We begin by recalling a number of estimates from [26].

Lemma 3.1.1. *(a)* Uniformly for $n \geq 1$ and $|x| \leq a_n$

$$\lambda_n\left(W^2, x\right) \sim \frac{a_n}{n} W^2(x)\, \Psi_n(x). \tag{3.1}$$

(b) For $n \geq 1$

$$\left|\frac{x_{1,n}}{a_n} - 1\right| \leq C\delta_n \tag{3.2}$$

and uniformly for $n \geq 2$ and $1 \leq j \leq n-1$

$$x_{j,n} - x_{j+1,n} \sim \frac{a_n}{n} \Psi_n(x_{j,n}). \tag{3.3}$$

(c) For $n \geq 1$

$$\sup_{x \in \mathbb{R}} |p_n W|(x) \left|1 - \frac{|x|}{a_n}\right|^{\frac{1}{4}} \sim a_n^{-\frac{1}{2}} \tag{3.4}$$

and

$$\sup_{x\in\mathbb{R}} |p_n W|(x) \sim a_n^{-\frac{1}{2}} (nT^*(a_n))^{\frac{1}{6}}. \tag{3.5}$$

(*d*) Let $0 < p \leq \infty$, $K > 0$. There exists $C > 0$ such that for $n \geq n_0$ and $P_n \in \mathcal{P}_n$

$$\|P_n W\|_{L_p(\mathbb{R})} \leq C \|P_n W\|_{L_p[-a_n(1-K\delta_n),\, a_n(1-K\delta_n)]} \tag{3.6}$$

Moreover, given $r > 1$, there exists $C_1 > 0$ such that for $n \geq n_0$ and $P_n \in \mathcal{P}_n$

$$\|P_n W\|_{L_p(|x|\geq a_{rn})} \leq \exp\left[-C_1 \frac{n}{T^*(a_n)^{\frac{1}{2}}}\right] \|P_n W\|_{L_p[-a_n\, a_n]}. \tag{3.7}$$

(*e*) For $n \geq 1$

$$\frac{\gamma_{n-1}}{\gamma_n} \sim a_n. \tag{3.8}$$

(*f*) Uniformly for $n \geq 2$ and $1 \leq j \leq n-1$

$$1 - \frac{|x_{j,n}|}{a_n} + L\delta_n \sim 1 - \frac{|x_{j+1,n}|}{a_n} + L\delta_n \tag{3.9}$$

and

$$\Psi_n(x_{j,n}) \sim \Psi_n(x_{j+1,n}). \tag{3.10}$$

Here, L is chosen so large enough that (2.40) is true.

(*g*) Uniformly for $n \geq 2$ and $2 \leq j \leq n-1$

$$\begin{aligned} &\frac{a_n^{\frac{3}{2}}}{n} \Psi_n(x_{j,n}) \left(1 - \frac{|x_{j,n}|}{a_n} + L\delta_n\right)^{\frac{1}{2}} |p_n' W|(x_{j,n}) \\ \sim\ & a_n^{\frac{1}{2}} |p_{n-1} W|(x_{j,n}) \sim \left(1 - \frac{|x_{j,n}|}{a_n} + L\delta_n\right)^{\frac{1}{4}}. \end{aligned} \tag{3.11}$$

Proof. (*a*) This is part of Theorem 1.2 in [26, p 204].

(*b*) (3.2) is part of Corollary 1.3 in [26, p.205]. We note however that the proof there actually establishes

$$1 - \frac{x_{1,n}}{a_n} \leq C\delta_n$$

which is the more difficult part of (3.2). The easier converse inequality

$$1-\frac{x_{1,n}}{a_n}\geq -C\delta_n$$

is not discussed in [26], but requires only a little extra effort. Next, (3.3) is Corollary 1.3 in [31] (A weaker form of (3.3) appears in Corollary 1.3 in [26]).

(*c*) This is Corollary 1.4(a) in [26].

(*d*) This is Theorem 1.5 in [26, p206]. We note there is a minor oversight in the proof of Theorem 1.5 in [26], for $0<p<\infty$. The proof in [26, pp231 – 236] correctly shows that

$$\|PW\|_{L_p[-a_{4n},\, a_{4n}]}\leq C\,\|PW\|_{L_p[-a_n(1-K\delta_n),\, a_n(1+K\delta_n)]},$$

with C independent of n and P. To estimate $\|PW\|_{\mathbb{R}\setminus[-a_{4n},a_{4n}]}$, an appeal is made to Lemma 2.5 in [26, p.215], and unfortunately that lemma is incorrect. It should actually read as follows: For $r>0$ and $s>1$, $n\geq 1$ and $P\in\mathcal{P}_n$

$$\left\|PW\left|Q'\right|^r\right\|_{L_p(|x|\geq a_{sn})}\leq \exp\left[\frac{-Cn}{T\left(a_n\right)^{\frac{1}{2}}}\right]\|PW\|_{L_p[-a_{sn},a_{sn}]}.$$

The assertion is easily proved using the method of [26, p.231]. The case $r=0$ gives (3.7).

(*e*) This is (10.33) in [26].

(*f*) (3.9) is (9.9) in [26] and (3.10) follows immediately from (3.9).

(*g*) This is Corollary 1.4(b) in [26]. □

We include a full proof of (3.2) and (3.6).

The proof of (3.2)

Note that we already have

$$1-\frac{x_{1,n}}{a_n}\leq C\delta_n.$$

For the converse inequality one needs the following:

If $K>0$ is large enough, then for n large enough and $R\in\mathcal{P}_{2n}$,

$$\int\limits_{|x|\geq a_n(1+K\delta_n)}|R|W^2(x)dx\leq\frac{1}{2}\int\limits_{|x|\leq a_n(1+K\delta_n)}|R|W^2(x)dx.$$

This follows by using (4.18) of [26] and the method of Theorem 1.5 in [26].

Then,

$$\begin{aligned}
&a_n(1+K\delta_n)-x_{1,n}\\
&=\inf_{\substack{P\in\mathcal{P}_{2n-2}\\P\geq 0}}\left(\frac{\int_{\mathbb{R}}(a_n(1+K\delta_n)-x)P(x)W^2(x)dx}{\int_{\mathbb{R}}P(x)W^2(x)dx}\right)\\
&\geq\frac{1}{2}\inf_{\substack{P\in\mathcal{P}_{2n-2}\\P\geq 0}}\left(\frac{\int_{|x|\leq a_n(1+K\delta_n)}(a_n(1+K\delta_n)-x)P(x)W^2(x)dx}{\int_{\mathbb{R}}P(x)W^2(x)dx}\right)\geq 0. \ \blacksquare
\end{aligned}$$

The proof of (3.6)

First note that under more general conditions on W (see part II), we have $\forall\ P\in\mathcal{P}_n$ and $s>1$,

$$\|PW\|_{L_p(|x|\geq a_{sn})}\leq e^{-C_1n/T(a_n)^{1/2}}\|PW\|_{L_p(|x|\leq a_{sn})}.$$

Then as in the proof of Lemma 2.5 in [26], we deduce that

$$\left\|PW|Q'|^r\right\|_{L_p(|x|\geq a_{sn})}\leq\exp\left[-C_1n/T(a_n)^{1/2}\right]\|PW\|_{L_p(|x|\leq a_{sn})}. \ \blacksquare$$

Next, we recall some results from [30, 31], involving mostly the fundamental polynomials of Lagrange Interpolation.

Lemma 3.1.2. (*a*) Let $0 < p < \infty$. Then for $n \geq 2$

$$\|p_nW\|_{L_p(\mathbb{R})} \sim \tag{3.12}$$

$$a_n^{\frac{1}{p}-\frac{1}{2}} \times \begin{bmatrix} 1 & ,p < 4 \\ (\log n)^{\frac{1}{4}} & ,p = 4 \\ (nT^*(a_n))^{\frac{2}{3}\left(\frac{1}{4}-\frac{1}{p}\right)} & ,p > 4 \end{bmatrix}.$$

(*b*) Uniformly for $n \geq 1$, $1 \leq j \leq n$, $x \in \mathbb{R}$

$$|l_{j,n}(x)| \sim \frac{a_n^{\frac{3}{2}}}{n}(\Psi_n W)(x_{j,n})\left(\left(1 - \frac{|x_{j,n}|}{a_n} + L\delta_n\right)^{\frac{1}{4}}\left|\frac{p_n(x)}{x - x_{j,n}}\right|.\right) \tag{3.13}$$

(*c*) Uniformly for $n \geq 1$, $1 \leq j \leq n$, $x \in \mathbb{R}$

$$|l_{j,n}(x)|\, W(x)\, W^{-1}(x_{j,n}) \leq C. \tag{3.14}$$

(*d*) For $n \geq 2$, $1 \leq j \leq n-1$, $x \in [x_{j,n}\ x_{j+1,n}]$

$$l_{j,n}(x)\, W(x)\, W^{-1}(x_{j,n}) + l_{j+1,n}(x)\, W(x)\, W^{-1}(x_{j+1,n}) \geq 1. \tag{3.15}$$

Proof. (*a*) This is Theorem 1.1 in [31].

(*b*) and (*c*). These are Theorem 1.2 in [31].

(*d*) is a special case of the main result in [30].□

Next, some technical estimates on the growth of $a_u, Q(a_u), T^*(a_u)$ ect.

Lemma 3.1.3. (*a*) Given $r > 0$, there exists x_0 such that, for $x \geq x_0$ and $j = 0, 1, 2$, $\frac{Q^{(j)}(x)}{x^r}$ is increasing in (x_o, ∞).

(b) Uniformly for $u \geq C$ and $j = 0, 1, 2,$

$$a_u^j Q^{(j)}(a_u) \sim u T^*(a_u)^{j-\frac{1}{2}}. \tag{3.16}$$

(c) Let $0 < \alpha < \beta$. Then, uniformly for $u \geq C$, $j = 0, 1, 2,$

$$T^*(a_{\alpha u}) \sim T^*(a_{\beta u}); \; Q^{(j)}(a_{\alpha u}) \sim Q^{(j)}(a_{\beta u}). \tag{3.17}$$

(d) Given fixed $r > 1$

$$\frac{a_{ru}}{a_u} \geq 1 + \frac{\log r}{T^*(a_{ru})}, \; u \in (0, \infty). \tag{3.18}$$

Moreover,

$$a_{ru} \sim a_u \quad u \in (1, \infty). \tag{3.19}$$

(e) Uniformly for $t \in (C, \infty)$

$$\frac{a_t'}{a_t} \sim \frac{1}{t T^*(a_t)} \tag{3.20}$$

(f) Uniformly for $u \in (C, \infty)$, and $v \in \left[\frac{u}{2}, 2u\right]$, we have

$$\left|\frac{a_u}{a_v} - 1\right| \sim \left|\frac{u}{v} - 1\right| \frac{1}{T^*(a_u)}. \tag{3.21}$$

Proof. (a) This is lemma 2.1*iii* in [26, *p*207].

(b) $-$ (f) are part of Lemma 2.2 in [26, *p*208 $-$ 209]. □

Our final lemma in this section concerns estimates that specfically follow from (2.13). Recall that δ_n was defined by (2.39).

Lemma 3.1.4. (a) Let $\varepsilon > 0$. Then

$$a_n \leq C n^{\varepsilon}, T^*(a_n) \leq C n^{\varepsilon}, \; n \geq 1. \tag{3.22}$$

(b) Given $A > 0$, we have

$$\delta_n \leq C T^*(a_n)^{-A}, \; n \geq 1. \tag{3.23}$$

(c) Let $0<\eta<1$. Uniformly for $n\geq 1$, $0<|x|\leq a_{\eta n}$, $|x|=a_s$, we have

$$C_1 \leq T^*(x)\left(1-\frac{|x|}{a_n}\right) \leq C_2 \log\frac{n}{s}. \tag{3.24}$$

Proof. (a) From (3.16) for $j=0$, we have

$$Q(a_n) \sim nT^*(a_n)^{-\frac{1}{2}} \leq nT(a_1)^{-\frac{1}{2}}.$$

Since Q grows faster than any power of x (Lemma 3.1.3 (a)) ,we deduce

$$a_n \leq n^{\varepsilon}$$

for n large enough.

Also (2.13) then shows that

$$T^*(a_n) = O\left(Q(a_n)^{\varepsilon}\right) \leq Cn^{\varepsilon}.$$

(b) This follows as

$$\delta_n \leq n^{-\frac{2}{3}} T^*(a_n)^{-\frac{2}{3}},$$

that is δ_n decays faster than a power of n, while $T^*(a_n)$ grows slower than any power of n.

(c) Firstly if $\frac{|x|}{a_n}\leq\frac{1}{2}$, then

$$T^*(x)\left(1-\frac{|x|}{a_n}\right) \geq T^*(0+)\frac{1}{2} > \frac{1}{2}.$$

If $\frac{|x|}{a_n}\geq\frac{1}{2}$, write $|x|=a_s$, so that $s\leq\eta n$. Then

$$T^*(x)\left(1-\frac{|x|}{a_n}\right) \geq T^*(a_s)\left(1-\frac{a_s}{a_{\frac{s}{\eta}}}\right) \geq C_1,$$

by Lemma 3.1.3(d).

So we have the lower bound in (3.24). We proceed to the upper bound. We can assume that $x=a_s$, $s\geq 1$, and $n\geq n_0$. Then using the inequality

$$1-u \leq |\log u|, \; u\in(0,1),$$

we obtain

$$\begin{aligned}\left(1-\frac{|x|}{a_n}\right) &\leq \left|\log\frac{a_s}{a_n}\right| = \int_s^n \frac{a_t'}{a_t}dt \\ &\leq C\int_s^n \frac{dt}{tT^*(a_t)} \leq \frac{C}{T^*(a_s)}\log\frac{n}{s} = \frac{C}{T^*(x)}\log\frac{n}{s}.\square\end{aligned}$$

3.2 The Hilbert Transform

We begin by recalling the definition of the **Hilbert transform** of a $g \in L_1(\mathbb{R})$ given by (2.37).

$$H[g](x) := \lim_{\varepsilon \longrightarrow 0+} \int_{|x-t|\geq\varepsilon} \frac{g(t)}{x-t}dt.$$

It is well known that H is a bounded operator from $L_p(\mathbb{R}) \longrightarrow L_p(\mathbb{R})$, $1 < p < \infty$ [49].

In the 1970's, B.Muckenhoupt, while investigating mean convergence of orthogonal polynomial expansions for the Hermite Weight given by (1.3), initiated the the study of the boundedness of the Hilbert transform between weighted L_p spaces. This lead ultimately to his A_p condition which we will state without proof as it will be important in what follows.

Theorem 3.2.1. (Muckenhoupt's A_p condition)*Let* $U : \mathbb{R} \longrightarrow [0,\infty)$ *be measurable,* $1 < p < \infty$ *and* $q = \frac{p}{p-1}$. *Then*

$$\|H(f)U\|_{L_p(\mathbb{R})} \leq C_1 \|fU\|_{L_p(\mathbb{R})}$$

iff $\exists C_2$ *such that for every interval* $[a,b]$,

$$\frac{1}{b-a}\|U\|_{L_p[a,b]}\left\|U^{-1}\right\|_{L_p[a,b]} \leq C_2. \tag{3.26}$$

Here $C_1, C_2 \neq C_1, C_2(f)$.

We now present two lemmas on bounded operators. The first is adapted from a result in König [21] and the second, essentially appears in 1970 papers of Muckenhoupt [41, $p\,449-451$] and later in König's paper[21], but is of course implied by results on the weighted L_p boundedness of Hilbert Transforms such as Theorem 3.2.1.

Throughout we adopt the notation

$$\|g\|_{L_p(d\mu)} := \left(\int_\Omega |g|^p \, d\mu\right)^{\frac{1}{p}}$$

for μ measurable functions g on a measure space (Ω, μ).

Lemma 3.2.2. *Let $1 < p < \infty$ and $q := \frac{p}{p-1}$. Let (Ω, μ) be a measure space, $k, r : \Omega^2 \longrightarrow \mathbb{R}$ and*

$$S_k[f](u) := \int_\Omega k(u,v) f(v) \, d\mu(v) \tag{3.26}$$

for μ measurable $f . \Omega \longrightarrow \mathbb{R}$.
Assume that

$$\sup_u \int_\Omega |k(u,v)| \, |r(u,v)|^q \, d\mu(v) \leq M. \tag{3.27}$$

$$\sup_v \int_\Omega |k(u,v)| \, |r(u,v)|^{-p} \, d\mu(u) \leq M. \tag{3.28}$$

Then S_k is a bounded operator from $L_p(d\mu)$ to $L_p(d\mu)$.
More precisely,

$$\|S_k\|_{L_P(d\mu) \longrightarrow L_P(d\mu)} \leq M. \tag{3.29}$$

Proof. We sketch this, as no proof is given in [21], though such lemmas are standard. First use the dual expression for the L_p norm of $T_k[f]$, then Fubini's theorem and finally Hölder's inequality to show that

$$\|S_k[f]\|_{L_P(d\mu)} \leq \|f\|_{L_P(d\mu)} \sup_g \left[\int_\Omega \left|\int_\Omega k(u,v) g(u) \, d\mu(u)\right|^q d\mu(v)\right]^{\frac{1}{q}},$$

where the sup is taken over all g with $\|g\|_{L_q(d\mu)} = 1$.
Let us call the sup J. So we must show that J is bounded by M. Using Hölder's inequality on the inner integral in J gives

$$\begin{aligned} &\left|\int_\Omega k(u,v) g(u) \, d\mu(u)\right|^q \\ \leq \; &\left[\int_\Omega |k(u,v)| \, |r(u,v)|^{-p} \, d\mu(u)\right]^{\frac{q}{p}} \int_\Omega |k(u,v)| \, |r(u,v)|^q \, |g(u)|^q \, d\mu(u) \end{aligned}$$

$$\leq \ M^{\frac{q}{p}} \int_{\Omega} |k(u,v)| \, |r(u,v)|^q \, |g(u)|^q \, d\mu(u)$$

Substituting this into J and using Fubini's Theorem gives,

$$J \leq M^{\frac{1}{p}} \sup_g \left[\int_{\Omega} |g(u)|^q \int_{\Omega} |k(u,v)| \, |r(u,v)|^q \, d\mu(v) \, d\mu(u) \right]^{\frac{1}{q}}$$

$$\leq M^{\frac{1}{p}} M^{\frac{1}{q}} = M. \square$$

Lemma 3.2.3. Let $1 < p < 4$. Then

$$\left\| H[g](x) \left| 1 - \frac{|x|}{a_n} \right|^{-\frac{1}{4}} \right\|_{L_p(\mathbb{R})} \leq \left\| g(x) \left| 1 - \frac{|x|}{a_n} \right|^{-\frac{1}{4}} \right\|_{L_p(\mathbb{R})}. \tag{3.30}$$

Here C is independent of n and $g \in L_p(\mathbb{R})$.

Proof. The proof appears with $a_n = \sqrt{2n+2}$ in [21], but we sketch the ideas of the proof here. Consider the operator S_k given by (3.26) with

$$k(u,v) := \frac{\left[\left| \frac{v}{u} \right|^{\frac{1}{4}} - 1 \right]}{(u-v)}$$

Using $r(u,v) := \left| \frac{u}{v} \right|^{\frac{1}{pq}}$, where $q := \frac{p}{p-1}$, Lemma 3.2.2 can be used to show that S_k is bounded from $L_p(\mathbb{R})$ to $L_p(\mathbb{R})$. Comparison of T_k and the bounded operator H show that

$$H_1[g](u) := \lim_{\varepsilon \longrightarrow 0+} \int_{|u-v| \geq \varepsilon} \frac{g(v)}{v-u} \left| \frac{v}{u} \right|^{\frac{1}{4}} dv$$

is bounded from $L_p(\mathbb{R})$ to $L_p(\mathbb{R})$.

Replacing u by $a_n \pm u$, and v by $a_n \pm v$, easily gives the result. $\square$

Our final lemma in this section concerns bounds on the difference between $\frac{1}{(x-x_{j,n})}$ and the Hilbert transform of a weighted characteristic function. Recall the notation (2.45 – 2.47) for $I_{j,n}$, $f_{j,n}$ and $\chi_{j,n}$. In particular, recall that

$$f_{j,n}(x) := \min \left\{ \frac{1}{|I_{j,n}|}, \frac{|I_{j,n}|}{(x-x_{j,n})^2} \right\} \left[1 - \frac{|x|}{a_n} + L\delta_n \right]^{\frac{-1}{4}}.$$

Lemma 3.2.4. Uniformly for $n \geq 1$ and $1 \leq j \leq n$ and $x \in [x_{n,n}, x_{1,n}]$

$$\tau_{j,n}(x) := a_n^{\frac{1}{2}} \left| p_n \left(W^2, x \right) W(x) \right| \left| \frac{1}{x - x_{j,n}} - \frac{1}{|I_{j,n}|} H[\chi_{j,n}](x) \right| \leq C f_{j,n}(x). \tag{3.31}$$

Proof. The idea already appears in [21]. Note first that

$$H[\chi_{j,n}](x) = \log \left| \frac{x - x_{j,n}}{x_{j-1,n} - x} \right| = -\log \left| 1 - \frac{|I_{j,n}|}{x - x_{j,n}} \right|. \tag{3.32}$$

We consider two ranges:

Case 1: $|x - x_{j,n}| \geq 2|I_{j,n}|$. Using the inequality $|t + \log(1-t)| \leq t^2$, $|t| \leq \frac{1}{2}$, we see that

$$\begin{aligned} \left| \frac{1}{x - x_{j,n}} - \frac{1}{|I_{j,n}|} H[\chi_{j,n}](x) \right| &= \frac{1}{|I_{j,n}|} \left| \frac{|I_{j,n}|}{x - x_{j,n}} + \log \left[1 - \frac{|I_{j,n}|}{x - x_{j,n}} \right] \right| \\ &\leq \frac{|I_{j,n}|}{(x - x_{j,n})^2}. \end{aligned}$$

Next, the bounds (3.4), (3.5) show that uniformly in n and x,

$$a_n^{\frac{1}{2}} |p_n W|(x) \leq C \left[\left| 1 - \frac{x}{a_n} \right| + L\delta_n \right]^{\frac{-1}{4}}. \tag{3.33}$$

So, we obtain the result for this range of x.

Case 2: $|x - x_{j,n}| \leq 2|I_{j,n}|$. From the identity

$$a_n^{\frac{1}{2}} (p_n W)(x) = (l_{j,n} W)(x) W^{-1}(x_{j,n}) (x - x_{j,n}) a_n^{\frac{1}{2}} (p_n' W)(x_{j,n}),$$

(for both j and $j-1$)

and from (3.3), (3.9), (3.11), (3.14), we obtain for $|x_{j,n}| \leq 2|I_{j,n}|, 2 \leq j \leq n$

$$a_n^{\frac{1}{2}} |p_n W|(x) \leq C_1 f_{j,n}(x) \min\{|x - x_{j,n}|, |x - x_{j-1,n}|\}. \tag{3.34}$$

For $j = 1$, this holds with the minimum replaced by $|x - x_{j,n}|$. Then for $2 \leq j \leq n$

$$\tau_{j,n}(x) \leq C_2 f_{j,n}(x) \left[1 + \min\{|x - x_{j,n}|, |x - x_{j-1,n}|\} \frac{1}{|I_{j,n}|} \left| \log \left| \frac{x - x_{j,n}}{x_{j-1,n} - x} \right| \right| \right]. \tag{3.35}$$

Since $|I_{j,n}| \geq C_3 \max\{|x - x_{j,n}|, |x - x_{j-1,n}|\}$, we see that with

$$u := \left|\frac{x - x_{j,n}}{x_{j-1,n} - x}\right|$$

we obtain for both signs of the exponent

$$\tau_{j,n}(x) \leq C_4 f_{j,n}(x)\left[1 + 2u^{\pm 1}\left|\log u^{\pm 1}\right|\right].$$

As either u or u^{-1} lies in $[0,1]$ and $t|\log t|$ is bounded for $t \in [0,1]$, we have (3.31). It remains to handle the case $j = 1$. Note that for

$$x \in [x_{n,n}, x_{1,n}]$$

(it is only here that we need this restriction) with $|x - x_{1,n}| \leq 2|I_{1,n}|$, we have

$$|x - x_{0,n}| \sim a_n \delta_n.$$

(See (3.2), (3.3), (2.44), (2.45)).

Then instead of (3.35), we obtain

$$\tau_{1,n}(x) \leq C f_{1,n}(x)\left[1 + C_1 \frac{|x - x_{1,n}|}{a_n \delta_n}\left|\log \sigma \frac{|x - x_{1,n}|}{a_n \delta_n}\right|\right]$$

where $\sigma \sim 1$ independently of x, j and n. As $|x - x_{1,n}| \leq C_2 a_n \delta_n$, the boundedness of $u|\log u|$ in any finite interval in $(0, \infty)$ again gives our result.□

3.3 Some Quadrature Estimates

Estimating certain sums by means of integrals is very crucial in the study of mean convergence of Lagrange Interpolation. These sums involve the product of the Christoffel functions, pth powers of polynomials, $(1 < p < \infty)$ and certain functions taken at the zeros of the orthonormal polynomials. The simplest form of a quadrature sum is the well known Gauss-Jacobi quadrature formula,

$$\sum_{j=1}^{n} P_{2n-1}(x_{j,n})\lambda_{j,n} = \int_{\mathbb{R}} P_{2n-1}(t)\,W^2(t)\,dt\,,\quad \forall P_{2n-1}\in\mathcal{P}_{2n-1}. \tag{3.36}$$

Of course, the measure $d\alpha = W^2(t)\,dt$ can be replaced by more general measures [13].

For applications in the study of mean convergence of Lagrange Interpolation, the most useful quadrature sum estimates are those of the form

$$\sum_{j=1}^{n} |P_m(x_{j,n})|^p \lambda_{j,n} \le C\int_{\mathbb{R}} |P_m(t)|^p W^2(t)\,dt$$

$0 < p < \infty$ and $P_m \in \mathcal{P}_m$, where $C = C(\alpha, p)$ often depends on some function of m and n. These types of inequalities have been investigated by many including: Nevai, Lubinsky, Mate for the generalised Jacobi weights, Shi, for weights on finite intervals and Lubinsky and Matjila, for Freud weights.

We present two quadrature sum estimates, the first of which is really part of a Lebesgue function type estimate. The second involves quadrature sums for polynomials.

Lemma 3.3.1. Let $\beta \in \left(0, \frac{1}{4}\right)$ and

$$\Sigma_n(x) := \sum_{|x_{k,n}|\ge a_{\beta n}} |l_{k,n}(x)|\,W^{-1}(x_{k,n}). \tag{3.37}$$

We have for $|x| \le a_{\frac{\beta n}{2}}$ and $|x| \ge a_{2n}$

$$(\Sigma_n W)(x) \le C. \tag{3.38}$$

Moreover, for $a_{\frac{\beta n}{2}} \le |x| \le a_{2n}$

$$(\Sigma_n W)(x) \le C\left(\log n + a_n^{\frac{1}{2}}\,|p_n W|(x)\,T^*(a_n)^{\frac{-1}{4}}\right). \tag{3.39}$$

Proof. Let $\Sigma_n^*(x)$ denote the sum $\Sigma_n(x)$ omitting those terms $x_{k,n}$ for which $x \in [x_{k+2,n}, x_{k-2,n}]$, (if there are any such k). Here and the sequel, we set for $l \ge 1$

$$x_{1-l,n} := x_{1,n} + l\delta_n;\ x_{n+l,n} := x_{n,n} - l\delta_n. \tag{3.40}$$

Of course the sum $\Sigma_n - \Sigma_n^*$ consists of at most 4 terms. Each of these 4 terms admits the bound in Lemma 3.1.2(c). So

$$|(\Sigma_n - \Sigma_n^*) W(x)| \le C_1. \tag{3.41}$$

Next, by (3.3) and (3.13)

$$(\Sigma_n^* W)(x) \sim a_n^{\frac{1}{2}} |p_n W|(x) \Sigma^*_{|x_{k,n}| \ge a_{\beta n}} \frac{(x_{k,n} - x_{k+1,n})}{|x - x_{k,n}|} \left(1 - \frac{|x_{k,n}|}{a_n} + L\delta_n\right)^{\frac{1}{4}}. \tag{3.42}$$

Now (cf. (3.9))

$$1 - \frac{|t|}{a_n} + L\delta_n \sim 1 - \frac{|x_{k,n}|}{a_n} + L\delta_n,\ t \in [x_{k+1,n}\ x_{k,n}], \tag{3.43}$$

uniformly in k and n. Next, if $x \notin [x_{k+2,n}\ x_{k-2,n}]$, and $t \in [x_{k+1,n}\ x_{k,n}]$

$$\left|\frac{x-t}{x - x_{k,n}} - 1\right| = \left|\frac{t - x_{k,n}}{x - x_{k,n}}\right| \le \frac{x_{k,n} - x_{k+1,n}}{|x_{k\pm 2,n} - x_{k,n}|} \le C.$$

Similarly we bound $\frac{(x - x_{k,n})}{x-t}$. So

$$|x - t| \sim |x - x_{k,n}|,\ t \in [x_{k+1,n}\ x_{k,n}],\ x \notin [x_{k+2,n}\ x_{k-2,n}]. \tag{3.44}$$

In view of the spacing of the zeros (Lemma 3.1.1(b)), we deduce that

$$\begin{aligned}(\Sigma_n^* W)(x) \ &\sim\ a_n^{\frac{1}{2}} |p_n W|(x) \int_{\substack{a_{\beta n} \le |t| \le a_n \\ |t-x| \ge C \frac{a_n}{n} \Psi_n(x)}} \frac{\left(1 - \frac{|t|}{a_n} + L\delta_n\right)^{\frac{1}{4}}}{|t-x|} dt \\ &=\ a_n^{\frac{1}{2}} |p_n W|(x) \int_{\substack{\frac{a_{\beta n}}{a_n} \le |s| \le 1 \\ \left|s - \frac{x}{a_n}\right| \ge C \frac{1}{n} \Psi_n(x)}} \frac{(1 - |s| + L\delta_n)^{\frac{1}{4}}}{\left|s - \frac{x}{a_n}\right|} ds.\end{aligned} \tag{3.45}$$

Note that since δ_n is much smaller than $\frac{1}{T^*(a_n)}$,

$$1 - s + L\delta_n \le C_2 \left(1 - \frac{a_{\beta n}}{a_n}\right) \le C_3 \frac{1}{T^*(a_n)}.$$

(See Lemma 3.[illegible]).

Then we obtain the bound

$$(\Sigma_n^* W)(x) \leq C a_n^{\frac{1}{2}} |p_n W|(x) T^*(a_n)^{-\frac{1}{4}} \int_{\substack{\frac{a_{\beta n}}{a_n} \leq |s| \leq 1 \\ \left|s - \frac{x}{a_n}\right| \geq C \frac{1}{n} \Psi_n(x)}} \frac{ds}{\left|s - \frac{x}{a_n}\right|}$$

Now if $0 \leq x \leq a_{\frac{\beta n}{2}}$ or $x \geq a_{2n}$, then for $n \geq n_o$, we can bound the integral by

$$\begin{aligned} & \int_{\frac{a_{\beta n}}{a_n}}^{1} \frac{ds}{\left|s - \frac{x}{a_n}\right|} \\ \leq \; & \left(1 - \frac{a_{\beta n}}{a_n}\right) \max\left(\left|1 - \frac{a_{2n}}{a_n}\right|^{-1}, \left|\frac{a_{\beta n}}{a_n} - \frac{a_{\frac{\beta n}{2}}}{a_n}\right|^{-1}\right) \leq C_4, \end{aligned}$$

by Lemma 3.1.3(f). In this case the bound (3.4) gives

$$(\Sigma_n^* W)(x) \leq C_5 \left(1 + \left|1 - \frac{|x|}{a_n}\right|^{\frac{-1}{4}} T^*(a_n)^{-\frac{1}{4}}\right) \leq C_6.$$

So we have (3.38). Now let us turn to to the more difficult case where $a_{\frac{\beta n}{2}} \leq x \leq a_{2n}$. We bound the integral in (3.45) as follows:

$$\begin{aligned} & \int_{\substack{\frac{a_{\beta n}}{a_n} \leq |s| \leq 1 \\ \left|s - \frac{x}{a_n}\right| \geq C \frac{1}{n} \Psi_n(x)}} \frac{(1 - |s| + L\delta_n)^{\frac{1}{4}}}{\left|s - \frac{x}{a_n}\right|} ds \\ \leq \; & C_7 \left[\int_{\substack{\frac{a_{\beta n}}{a_n} \leq |s| \leq 1 \\ \left|s - \frac{x}{a_n}\right| \geq C \frac{1}{n} \Psi_n(x)}} \frac{(1-s)^{\frac{1}{4}}}{\left|s - \frac{x}{a_n}\right|} ds + \int_{\substack{\frac{a_{\beta n}}{a_n} \leq |s| \leq 1 \\ \left|s - \frac{x}{a_n}\right| \geq C \frac{1}{n} \Psi_n(x)}} \frac{\delta_n^{\frac{1}{4}}}{\left|s - \frac{x}{a_n}\right|} ds\right] \\ = \; & : C_7[I_1 + I_2]. \end{aligned}$$

Now since $\frac{1}{n}\Psi_n(x)$ is bounded below by a power of n, we see that

$$I_2 \leq C_8 \delta_n^{\frac{1}{4}} \log n.$$

If $x \geq a_n$, we estimate

$$I_1 \leq \int_{\frac{a\beta_n}{a_n}}^{1} \frac{(1-s)^{\frac{1}{4}}}{|s-1|} ds \leq C_9 T^* (a_n)^{-\frac{1}{4}}.$$

If $x < a_n$, we make the substitution $1 - s = \left(1 - \frac{x}{a_n}\right) v$ to get

$$\begin{aligned} I_1 &= \left(1 - \frac{x}{a_n}\right)^{\frac{1}{4}} \int_{\substack{v \in \left[0, \frac{\left(1-\frac{a\beta_n}{a_n}\right)}{\left(1-\frac{x}{a_n}\right)}\right] \\ |v-1| \geq C \frac{\varphi_n(x)}{n\left(1-\frac{x}{a_n}\right)}}} \frac{v^{\frac{1}{4}}}{|v-1|} dv \\ &\leq C_{10} \left(1 - \frac{x}{a_n}\right)^{\frac{1}{4}} \{ \int_{\substack{v \in [0,2] \\ |v-1| \geq C \frac{\varphi_n(x)}{n\left(1-\frac{x}{a_n}\right)}}} \frac{1}{|v-1|} dv + \\ &\int_2^{\left(1-\frac{a\beta_n}{a_n}\right)/\left(1-\frac{x}{a_n}\right)} v^{-\frac{3}{4}} dv \} \\ &\leq C_{11} \{ \left(1 - \frac{x}{a_n}\right)^{\frac{1}{4}} \log n + T^* (a_n)^{-\frac{1}{4}} \} \end{aligned}$$

Combining our estimates for I_1, I_2 and using the bound,

$$a_n^{\frac{1}{2}} |p_n W(x)| \delta_n^{\frac{1}{4}} \leq C,$$

which follows from (3.5), we deduce (3.39) from (3.45).□

In our second quadrature sum estimate, we need the kernel function for the Chebyshev weight

$$v(t) := \left(1 - t^2\right)^{-\frac{1}{2}}, \; t \in (-1, 1) \tag{3.46}$$

If $p_j(v, x) = \sqrt{\frac{2}{\pi}} T_j(x)$ is the jth orthonormal polynomial for v (at least for $j \geq 1$), then

$$K_n(v, x, t) := \sum_{j=0}^{n-1} p_j(v, x) p_j(v, t), \tag{3.47}$$

admits the following estimates[[46], p.36], [[44], p.108].

$$K_n(v, x, x) \sim n, \;\; |x| \leq 1 \tag{3.48}$$

and

$$|K_n(v,x,t)| \le C \min\left\{n, \frac{\sqrt{1-x^2}+\sqrt{1-t^2}}{|x-t|}\right\}, \quad x,t \in [-1,1]. \tag{3.49}$$

Lemma 3.3.2. Let $0<\eta<1$. Let $\phi:\mathbb{R} \longrightarrow (0,\infty)$ be a continuous function with the following property: For $n\ge 1$, there exist polynomials R_n of degree $\le n$ such that

$$C_1 \le \frac{\phi(t)}{R_n(t)} \le C_2, \ |t| \le a_{4n}. \tag{3.50}$$

Then for $n \ge n_0$ and $P_n \in \mathcal{P}_n$,

$$\sum_{|x_{j,n}|\le a_{\eta n}} \lambda_{j,n} \left|P_n W^{-1}\right|(x_{j,n})\,\phi(x_{j,n}) \le C\int_{-a_{4n}}^{a_{4n}} |P_n W|\,\phi. \tag{3.51}$$

Proof Essentially the proof is the same as in [35], and the ideas appeared much earlier [44], [45] but we include the details.

Step 1: An L_1 Christoffel function type estimate.

We first note that for $P_{4n} \in \mathcal{P}_{4n-1}$

$$\begin{aligned}(P_{4n}W)^2(x) &\le \lambda_{4n}^{-1}\left(W^2,x\right)W^2(x)\int_{\mathbb{R}}(P_{4n}W)^2(t)\,dt\\ &\le C_1\frac{n}{a_n}\left(\Psi_{4n}(x)\right)^{-1}\int_{-a_{4n}}^{a_{4n}}(P_{4n}W)^2(t)\,dt,\end{aligned}$$

by Lemma 3.1.1(a), (d).

We deduce that

$$\left\|P_{4n}W\Psi_{4n}^{\frac{1}{2}}\right\|^2_{L_\infty[-a_{4n}\ a_{4n}]} \le C_1\frac{n}{a_n}\int_{-a_{4n}}^{a_{4n}}\left|P_{4n}\Psi_{4n}^{-\frac{1}{2}}W(t)\,dt\right|\left\|P_{4n}W\Psi_{4n}^{\frac{1}{2}}\right\|_{L_\infty[-a_{4n}\ a_{4n}]}$$

and hence that for $|x|\le a_{4n}$,

$$\left|P_{4n}\Psi_{4n}^{\frac{1}{2}}W\right|(x) \le C_1\frac{n}{a_n}\int_{-a_{4n}}^{a_{4n}}\left|P_{4n}\Psi_{4n}^{-\frac{1}{2}}W\right|(t)\,dt.$$

Now we apply this for fixed $|x| \leq a_{4n}$ to

$$P_{4n}(t) := P_{2n}(t) K_n^2\left(v, \frac{x}{a_{4n}}, \frac{t}{a_{4n}}\right)$$

where $P_{2n} \in \mathcal{P}_{2n}$.
We obtain, using (3.48) that

$$\left|P_{2n}\Psi_{4n}^{\frac{1}{2}}W\right|(x) \leq C_2\frac{1}{na_n}\int_{-a_{4n}}^{a_{4n}}\left|P_{2n}\Psi_{4n}^{-\frac{1}{2}}W(t)\right|K_n^2\left(v, \frac{x}{a_{4n}}, \frac{t}{a_{4n}}\right)dt.$$

In particular, applying this to $P_{2n} := P_n R_n$, where $P_n \in \mathcal{P}_n$, and using (3.50), we obtain

$$\left|P_n\Psi_{4n}^{\frac{1}{2}}W\phi\right|(x) \leq C_3\frac{1}{na_n}\int_{-a_{4n}}^{a_{4n}}\left|P_n\phi\Psi_{4n}^{-\frac{1}{2}}W\right|(t)K_n^2\left(v, \frac{x}{a_{4n}}, \frac{t}{a_{4n}}\right)dt. \tag{3.52}$$

Step 2: The general quadrature sum bounded in terms of a special quadrature sum.

We take (3.52) for $x = x_{j,n}$, multiply by $\lambda_{j,n}W^{-2}(x_{j,n})\Psi_{4n}^{-\frac{1}{2}}(x_{j,n})$, and sum over all $|x_{j,n}| \leq a_{\eta n}$. Using our estimate for Christoffel function $\lambda_n(W^2, .)$ in Lemma 3.1.1(a), we obtain

$$\begin{aligned}&\sum_{|x_{j,n}|\leq a_{\eta n}}\lambda_{j,n}\left|P_nW^{-1}\right|(x_{j,n})\Phi(x_{j,n}) \\ \leq\ & C_4\int_{-a_{4n}}^{a_{4n}}|P_nW\Phi|(t)\Sigma_n(t)\,dt,\end{aligned} \tag{3.53}$$

where

$$\Sigma_n(t) := n^{-2}\sum_{|x_{j,n}|\leq a_{\eta n}}\Psi_n(x_{j,n})\Psi_{4n}^{-\frac{1}{2}}(x_{j,n})K_n^2\left(v, \frac{x_{j,n}}{a_{4n}}, \frac{t}{a_{4n}}\right)\Psi_{4n}^{-\frac{1}{2}}(t). \tag{3.54}$$

Then the result will follow if we can show

$$\Sigma_n(t) \leq C_5, \quad |t| \leq a_{4n}. \tag{3.55}$$

Step 3: Estimation of (3.55).

First note that for $|x| \leq a_{\eta n}$

$$\Psi_n(x) \sim \Psi_{4n}(x) \sim \left(1 - \frac{|x|}{a_n}\right)^{\frac{1}{2}}.$$

This follows easily from the fact that $1 - \frac{|x|}{a_{4n}} \geq 1 - \frac{|x|}{a_n} \geq C_6/T^*(a_n)$ for this range. Moreover,

$$\Psi_{4n}(t) \geq \left(1 - \frac{|t|}{a_{4n}} + L\delta_n\right)^{\frac{1}{2}}$$

for $|t| \leq a_{4n}$.

Let us set

$$y_{j,n} := \frac{x_{j,n}}{a_{4n}};\ T := \frac{t}{a_{4n}}.$$

Then we have, using also (3.49) and the spacing in Lemma 3.1.1(b), that

$$\begin{aligned}
\Sigma_n(t)\left(1 - \frac{|t|}{a_n} + L\delta_n\right)^{\frac{1}{4}} &\leq \\
&\leq \frac{C_7}{na_n} \sum_{|x_{j,n}| \leq a_{\eta n}} (x_{j,n} - x_{j+1,n}) \left(1 - \frac{|x_{j,n}|}{a_{4n}}\right)^{-\frac{1}{4}} K_n^2\left(v, \frac{x_{j,n}}{a_{4n}}, \frac{t}{a_{4n}}\right) \\
&\leq C_8 n^{-1} \sum_{|y_{j,n}| \leq a_{\eta n}/a_{4n}} (y_{j,n} - y_{j+1,n})(1 - |y_{j,n}|)^{-\frac{1}{4}} \\
&\quad \times \min\left\{n, \frac{\sqrt{1 - y_{j,n}{}^2} + \sqrt{1 - T^2}}{|y_{j,n} - T|}\right\}^2
\end{aligned}$$

$$\leq C_9 n^{-1} \int_{-1}^{1} (1 - |y|)^{-\frac{1}{4}} \min\left\{n, \frac{\sqrt{1 - y^2} + \sqrt{1 - T^2}}{|y - T|}\right\}^2 dy. \qquad (3.56)$$

In bounding the sum in terms of the integral, we have used (3.9). Let us assume that $1 - n^{-2} \geq T \geq 0$. Then we can continue the above as

$$\begin{aligned}
\Sigma_n(t)(1 - T)^{\frac{1}{4}} &\leq C_{10} n^{-1} \begin{array}{l} \left\{ n^2 \int_{y \in [0,1]: |y - T| \leq \frac{1}{n}(1-T)^{\frac{1}{2}}} (1 - y)^{-\frac{1}{4}} dy \right. \\ \left. + \int_{y \in [0,1]: |y - T| \geq \frac{1}{n}(1-T)^{\frac{1}{2}}} (1 - y)^{-\frac{1}{4}} \frac{1 - y + 1 - T}{|y - T|^2} dy \right\} \end{array} \\
&= C_{10} n^{-1} \begin{array}{l} \left\{ n^2 (1 - T)^{\frac{3}{4}} \int_{w: |1 - w| \leq \frac{1}{n}(1-T)^{-\frac{1}{2}}} w^{-\frac{1}{4}} dw \right. \\ \left. + (1 - T)^{-\frac{1}{4}} \int_{w: |1 - w| \geq \frac{1}{n}(1-T)^{-\frac{1}{2}}} w^{-\frac{1}{4}} \frac{|1 + w|}{|1 - w|^2} dw \right\} \end{array}
\end{aligned}$$

$$
\begin{aligned}
& \text{(substitution } 1-y=(1-T)\,w) \\
\leq \;& C_{11}\,(1-T)^{\frac{1}{4}}.
\end{aligned}
$$

Here we have used the fact that

$$\frac{1}{n}(1-T)^{-\frac{1}{2}} \leq 1.$$

So in this case, we have (3.55). In the remaining case where $1-n^{-2} \leq T < 1$, we continue (3.56) as

$$
\begin{aligned}
\Sigma_n(t)\,(L\delta_n)^{\frac{1}{4}} \;&\leq\; C_{12}n^{-1}\left\{n^2 \int_{y\in[0,1]:|y-T|\leq 4n^{-2}} (1-y)^{-\frac{1}{4}}\,dy \right.\\
&\qquad \left. + \int_0^{1-2n^{-2}} (1-y)^{-\frac{1}{4}}\,\frac{1-y+n^{-2}}{|y-T|^2}\,dy\right\} \\
&\leq\; C_{13}n^{-\frac{1}{2}}.
\end{aligned}
$$

Since $\delta_n^{\frac{1}{4}}$ decays scarcely faster than $n^{-\frac{1}{8}}$ we again have (3.55). □

3.4 A Converse Quadrature Sum Estimate

In this section, we prove a converse Quadrature type estimate that will be needed in proving Theorems 2.5.4-2.5.6. The proof follows that of H.König in [21]. We prove:

Theorem 3.11. *Let $1 < p < 4$. There exists $C > 0$ such that for $n \geq 1$ and $P_n \in \mathcal{P}_{n-1}$*

$$\|P_nW\|_{L_p(\mathbb{R})} \leq C\left\{\sum_{j=1}^{n} \lambda_{j,n}W^{-2}(x_{j,n})\,|P_nW|^p(x_{j,n})\right\}^{\frac{1}{p}} \tag{3.57}$$

Our proof of Theorem 3.4.1 follows that of H.König. We shall divide the proof into several steps: In the sequel, we shall use the abbreviation

$$\mu_{j,n} := \lambda_{j,n}W^{-2}(x_{j,n}) \sim |I_{j,n}| = x_{j-1,n} - x_{j,n}. \tag{3.58}$$

(See (3.1) and (3.3)).

Step 1: Express P_nW as a sum of two terms.

Let $P_n \in \mathcal{P}_{n-1}$. We write (recall (2.47))

$$\begin{aligned}
(P_n W)(x) &= (L_n[P_n] W)(x) = \sum_{j=1}^{n} P_n(x_{j,n})(l_{j,n}W)(x) && (3.59)\\
&= a_n^{\frac{1}{2}}(p_n W)(x) \sum_{j=1}^{n} y_{j,n}\left\{\frac{1}{x - x_{j,n}} - \frac{1}{|I_{j,n}|} H[\chi_{j,n}](x)\right\}\\
&\quad + a_n^{\frac{1}{2}}(p_n W)(x)\, H\left[\sum_{j=1}^{n} y_{j,n} \frac{\chi_{j,n}}{|I_{j,n}|}\right](x)\\
&=: J_1(x) + J_2(x).
\end{aligned}$$

Here

$$y_{j,n} := a_n^{\frac{-1}{2}} \frac{(P_n W)(x_{j,n})}{(p_n' W)(x_{j,n})}. \qquad (3.60)$$

Note that in view of the behaviour of the smallest and largest zeros (see (3.2)) and in view of the infinite-finite range inequality (3.6), it suffices to estimate $\|P_n W\|_{L_p[x_{n,n},\, x_{1,n}]}$ in terms of the right -hand side of (3.57).

Step 2: Estimate $\|J_2\|$

We begin with J_2 as it is easier to handle. Using our bound (3.4) for p_n, and then the weighted boundedness of the Hilbert transform in Lemma 3.2.3 gives:

$$\begin{aligned}
\|J_2\|_{L_p[x_{n,n},\, x_{1,n}]} &\leq C \left\| \sum_{j=1}^{n} y_{j,n} \frac{\chi_{j,n}(x)}{|I_{j,n}|} \left|1 - \frac{|x|}{a_n}\right|^{\frac{-1}{4}} \right\|_{L_p(\mathbb{R})}\\
&= C_1 \left[\sum_{j=1}^{n} \left\{\frac{|y_{j,n}|}{|I_{j,n}|}\right\}^p \int_{I_{j,n}} \left|1 - \frac{|x|}{a_n}\right|^{\frac{-p}{4}} dx\right]^{\frac{1}{p}}.
\end{aligned}$$

Using the spacing (3.3) and also (3.9), one deduces that

$$\int_{I_{j,n}} \left|1 - \frac{|x|}{a_n}\right|^{\frac{-p}{4}} dx \sim |I_{j,n}| \left|1 - \frac{|x_{j,n}|}{a_n} + \delta_n\right|^{\frac{-p}{4}}.$$

Next, from (3.60) and (3.11), we see that

$$|y_{j,n}| \sim |P_n W|(x_{j,n})\, |I_{j,n}| \left|1 - \frac{|x_{j,n}|}{a_n} + \delta_n\right|^{\frac{\pm 1}{4}}. \qquad (3.61)$$

Hence,

$$\begin{aligned}
\|J_2\|_{L_p[x_{n,n},\, x_{1,n}]} &\leq C_2 \left[\sum_{j=1}^{n} |I_{j,n}|\, |P_n W|^p\, (x_{j,n})\right]^{\frac{1}{p}} \\
&\leq C_3 \left[\sum_{j=1}^{n} \lambda_{j,n} W^{-2}\, (x_{j,n})\, |P_n W|^p\, (x_{j,n})\right]^{\frac{1}{p}}.
\end{aligned}$$

by (3.58).

Step 3: Estimate J_1.

By Lemma 3.2.4,

$$|J_1\,(x)| \leq C_4 \sum_{j=1}^{n} |y_{j,n}|\, f_{j,n}\,(x),\ x \in [x_{n,n},\ x_{1,n}].$$

Then

$$\|J_1\|_{L_p[x_{n,n},\, x_{1,n}]} \leq C \left\{\sum_{k=1}^{n} \int_{I_{k,n}} \left[\sum_{j=1}^{n} |y_{j,n}|\, f_{j,n}\,(x)\right]^p dx\right\}^{\frac{1}{p}}.$$

Now using the spacing (3.3), (3.9) and the definition (2.46) of $f_{j,n}$, we see that

$$f_{j,n}\,(x) \sim \frac{|I_{j,n}|}{(x_{k,n} - x_{j,n})^2} \left[\left|1 - \frac{|x_{k,n}|}{a_n}\right| + \delta_n\right]^{\frac{-1}{4}},\ x \in I_{k,n},$$

uniformly in n and $j \neq k$.

We deduce that

$$\|J_1\|_{L_p[x_{n,n},\, x_{1,n}]} \leq C_5\,(S_1 + S_2) \tag{3.62}$$

where

$$S_1 := \left\{\sum_{k=2}^{n} |I_{k,n}| \left[\sum_{\substack{j=1 \\ j \neq k}}^{n} |y_{j,n}| \frac{|I_{j,n}|}{(x_{k,n} - x_{j,n})^2} \left[\left|1 - \frac{|x_{k,n}|}{a_n}\right| + \delta_n\right]^{\frac{-1}{4}}\right]^p\right\}^{\frac{1}{p}} \tag{3.63}$$

and by (2.46)

$$S_2 := \left\{\sum_{k=2}^{n} |y_{k,n}|^p\, |I_{k,n}|^{1-p} \left[\left|1 - \frac{|x_{k,n}|}{a_n}\right| + \delta_n\right]^{\frac{-1}{4}p} dx\right\}^{\frac{1}{p}}.$$

Exactly as in the last part of Step 2, we see that (3.61) gives

$$S_2 \leq C_6 \left[\sum_{j=1}^{n} \lambda_{j,n} W^{-2}(x_{j,n}) \left| P_n W \right|^p (x_{j,n}) \right]^{\frac{1}{p}}.$$

To deal with S_1, we use Lemma 3.2.2 with a discrete measure space. Using (3.61) and (3.58), we see that

$$S_1 \leq C_7 \left\{ \sum_{k=1}^{n} \left[\sum_{j=1}^{n} b_{k,j} \left\{ \mu_{j,n}^{\frac{1}{p}} PW(x_{j,n}) \right\} \right]^p \right\}^{\frac{1}{p}},$$

where

$$b_{k,k} := 0 = b_{1,k} \; \forall k \text{ and for } j \neq k$$

$$b_{k,j} := |I_{j,n}|^{2-\frac{1}{p}} |I_{k,n}|^{\frac{1}{p}} (x_{j,n} - x_{k,n})^{-2} \left[\left| 1 - \frac{|x_{j,n}|}{a_n} \right| + \delta_n \right]^{\frac{1}{4}} \left[\left| 1 - \frac{|x_{k,n}|}{a_n} \right| + \delta_n \right]^{\frac{-1}{4}}.$$

Note the order: $b_{k,j}$ rather than $b_{j,k}$. Defining $B := (b_{k,j})_{k,j=1}^{n}$, we see that if l_p^n denotes the usual (little) l_p space on $\mathbb{R}^n$, then

$$S_1 \leq C_8 \|B\|_{l_p^n \longrightarrow l_p^n} \left[\sum_{j=1}^{n} \mu_{j,n} \left| P_n W \right|^p (x_{j,n}) \right]^{\frac{1}{p}}.$$

So the result follows if we can show that independently of n,

$$\|B\|_{l_p^n \longrightarrow l_p^n} \leq C_9. \tag{3.64}$$

Step 4: We prove (3.64).

This is far more complicated than the analogous proof for the Hermite weight [21], because of the more complicated behaviour of the spacing of the zeros of the orthogonal polynomials. We apply Lemma 3.2.2 with the discrete measure space $\Omega := \{1, 2, ..n\}$ and $\mu(\{j\}) = 1$, $j = 1, 2, ..n$. Moreover, we set there

$$k(k, j) := b_{k,j};\; r_{k,j} := \left(\frac{|I_{j,n}|}{|I_{k,n}|} \right)^{\frac{1}{pq}}.$$

Note that because of the way we order the variables ($b_{k,j}$ rather than $b_{j,k}$), the variable u in

(3.27) – (3.28) is k and the variable v in (3.27) – (3.28) is j. So (3.27 – 3.28) become

$$\sup_k \sum_{\substack{j=1 \\ j\neq k}}^{n} |I_{j,n}|^2 (x_{j,n} - x_{k,n})^{-2} \left[\left|1 - \frac{|x_{j,n}|}{a_n}\right| + \delta_n\right]^{\frac{1}{4}} \left[\left|1 - \frac{|x_{k,n}|}{a_n}\right| + \delta_n\right]^{\frac{-1}{4}} \leq M \tag{3.65}$$

and

$$\sup_j \sum_{\substack{k=1 \\ k\neq j}}^{n} |I_{j,n}| |I_{k,n}| (x_{j,n} - x_{k,n})^{-2} \left[\left|1 - \frac{|x_{j,n}|}{a_n}\right| + \delta_n\right]^{\frac{1}{4}} \left[\left|1 - \frac{|x_{k,n}|}{a_n}\right| + \delta_n\right]^{\frac{-1}{4}} \leq M. \tag{3.66}$$

Recall that given fixed $\beta \in (0,1)$, we have uniformly in l and n

$$|I_{l,n}| \sim \frac{a_n}{n}\left(1 - \frac{|x_{l,n}|}{a_n}\right)^{\frac{1}{2}}, |x_{l,n}| \leq a_{\beta n} \tag{3.67}$$

and

$$|I_{l,n}| \sim \frac{a_n}{n} T^*(a_n)^{-1}\left(1 - \frac{|x_{l,n}|}{a_n} + \delta_n\right)^{-\frac{1}{2}}, |x_{l,n}| \geq a_{\beta n}. \tag{3.68}$$

(See (3.3) and 2.41).

To take account of this dual behaviour of $|I_{l,n}|$, we consider three ranges of $x_{j,n}, x_{k,n}$. It is not difficult to see that we may consider only $x_{j,n}, x_{k,n} \geq 0$.

Range 1: $0 < x_{j,n}, x_{k,n} < a_{\frac{3n}{4}}$.

Using (3.67), we see that if we restrict summation in the sum in (3.65) to $j : |x_{j,n}| \leq a_{\frac{3n}{4}}$, then, the resulting sum is bounded by a constant times

$$I_{11} := \frac{a_n}{n}\left(1 - \frac{x_{k,n}}{a_n}\right)^{\frac{-1}{4}} \int_{\substack{0 \leq t \leq a_{\frac{3n}{8}} \\ |t - x_{k,n}| \geq C_{10}|I_{k,n}|}} \frac{\left|1 - \frac{t}{a_n}\right|^{\frac{3}{4}}}{(t - x_{k,n})^2} dt.$$

We make the substitution

$$1 - \frac{t}{a_n} = \left(\left(1 - \frac{x_{k,n}}{a_n}\right) u\right)$$

in this integral, and use (3.67) again to give

$$\begin{aligned} I_{11} &\leq \frac{1}{n}\left(1-\frac{x_{k,n}}{a_n}\right)^{\frac{-1}{2}} \int_{\substack{0\leq u\leq\left(1-\frac{x_{k,n}}{a_n}\right)^{-1} \\ |1-u|\geq n^{-1}C_{11}\left(1-\frac{x_{k,n}}{a_n}\right)^{\frac{-1}{2}}}} \frac{|u|^{\frac{3}{4}}}{(1-u)^2}du \\ &\leq C_{12}\frac{1}{n}\left(1-\frac{x_{k,n}}{a_n}\right)^{\frac{-1}{2}}\left[n\left(1-\frac{x_{k,n}}{a_n}\right)^{\frac{1}{2}}+1\right] \\ &\leq C_{13}\left[1+\frac{1}{n}T(a_n)^{\frac{1}{2}}\right] \leq C_{14} \end{aligned}$$

by (3.21) and (3.22).

Next, if we restrict summation in (3.66) to $k : |x_{k,n}| \leq a_{\frac{3n}{4}}$, and we use (3.67), we see that the resulting sum is bounded above by a constant times

$$I_{12} := \frac{a_n}{n}\left(1-\frac{x_{j,n}}{a_n}\right)^{\frac{3}{4}} \int_{\substack{0\leq t\leq a_{\frac{4n}{5}} \\ |t-x_{j,n}|\geq C_{15}|I_{j,n}|}} \frac{\left|1-\frac{t}{a_n}\right|^{\frac{3}{4}}}{(t-x_{j,n})^2}dt.$$

The same substitution as before shows that $I_{1,2}$ has a similar upper bound t. hat for $I_{1,1}$ and hence, is bounded independently of j, n.

Range 2: $x_{j,n}, x_{k,n} \geq a_{\frac{n}{2}}$.

Using (3.68), we see that after restricting summation in the sum in (3.65) to $j : |x_{j,n}| \geq a_{\frac{n}{2}}$, then the resulting sum is bounded by a constant times,

$$\begin{aligned} &\sum_{\substack{j:|x_{j,n}|\geq a_{\frac{n}{4}} \\ j\neq k}} \frac{|I_{j,n}|^{\frac{3}{2}}|I_{k,n}|^{\frac{1}{2}}}{(x_{j,n}-x_{k,n})^2} \\ \leq\ & C_{16}|I_{k,n}|^{\frac{1}{2}} \sum_{\substack{j:|x_{j,n}|\geq a_{\frac{n}{4}} \\ j\neq k}} \frac{|I_{j,n}|}{|x_{j,n}-x_{k,n}|^{\frac{3}{2}}} \\ \leq\ & C_{17}|I_{k,n}|^{\frac{1}{2}} \int_{t:|t-x_{k,n}|\geq C_{18}|I_{k,n}|} \frac{dt}{|t-x_{k,n}|^{\frac{3}{2}}} \leq C_{18}. \end{aligned}$$

Similarly, after restricting summation in the sum in (3.66) to $k : |x_{k,n}| \geq a_{\frac{n}{2}}$, then the resulting sum is bounded by a constant times,

$$\sum_{\substack{j:|x_{k,n}|\geq a_{\frac{n}{4}} \\ k\neq j}} \frac{|I_{k,n}|^{\frac{3}{2}}|I_{j,n}|^{\frac{1}{2}}}{(x_{j,n}-x_{k,n})^2}.$$

After swopping the indices, j and k, we see that this is the same as the sum just estimated.

Range 3: $x_{j,n} < a_{\frac{n}{2}}$ **and** $x_{k,n} > a_{\frac{3n}{4}}$; **or** $x_{j,n} > a_{\frac{3n}{4}}$ **and** $x_{k,n} < a_{\frac{n}{2}}$.

Here,

$$|x_{j,n} - x_{k,n}| \geq a_{\frac{3n}{4}} - a_{\frac{n}{2}} \geq C_{19}\frac{a_n}{T^*(a_n)}$$

(See (3.21)). Also, given fixed small $\varepsilon > 0$, we see that

$$|I_{l,n}| \leq C_{20}n^{-\frac{2}{3}+\varepsilon}, \text{uniformly in } l \text{ and } n$$

(See (3.67), (3.68), (3.22), (2.39)). Finally,

$$\left[\left|1-\frac{|x_{k,n}|}{a_n}\right|+\delta_n\right]^{-\frac{1}{4}} \leq C_{21}n^{\frac{1}{6}+\varepsilon}.$$

Then we see after suitably restricting the range of summation in (3.65), we obtain a sum bounded by

$$C_{22}n^{\frac{-1}{2}+2\varepsilon}a_n^{-2}T^*(a_n)^2\sum_j |I_{j,n}| \leq C_{23}n^{\frac{-1}{2}+2\varepsilon}T^*(a_n)^2 a_n^{-1} = o(1).$$

Similarly the sum arising from (3.66) is $o(1)$. So we have completed the proof of (3.64). $\square$

Chapter 4

Necessary and Sufficient Conditions for $1 < p < \infty$

4.1 Sufficiency for Theorem 2.5.2

In proving the sufficiency conditions, we split our functions into pieces that vanish inside or outside $\left[-a_{\frac{n}{8}}, a_{\frac{n}{8}}\right]$. Throughout, we let χ_S denote the characteristic function of a set S. Also, we set for some fixed $\kappa > 0$,

$$\phi(x) := \left(\log\left(2 + x^2\right)\right)^{-1-\kappa}. \tag{4.1}$$

Throughout, we assume that $W = \exp[-Q] \in \mathcal{E}_1^*$, that $1 < p < \infty$ and

$$\Delta > \max\left\{0, \frac{2}{3}\left(\frac{1}{4} - \frac{1}{p}\right)\right\}. \tag{4.2}$$

Lemma 4.1.1. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions from $\mathbb{R} \longrightarrow \mathbb{R}$ such that for $n \geq 1$,

$$f_n(x) = 0, \quad |x| < a_{\frac{n}{8}}; \tag{4.3}$$

$$|f_n W|(x) \leq \phi(x), \; x \in \mathbb{R}. \tag{4.4}$$

Then

$$\lim_{n\to\infty}\left\|L_n[f_n]W(1+Q)^{-\Delta}\right\|_{L_p(\mathbb{R})}=0. \tag{4.5}$$

Proof. Firstly for $|x|\le a_{\frac{n}{18}}$ or $|x|\ge a_{2n}$, Lemma 3.3.1(with $\beta=\frac{1}{9}$) and (4.3), (4.4) show that

$$\begin{aligned}|L_n[f_n]W|(x) &\le \phi\left(a_{\frac{n}{9}}\right)\sum_{|x_{k,n}|\ge a_{\frac{n}{9}}}|l_{k,n}(x)|W^{-1}(x_{k,n})W(x)\\ &\le C_1\phi\left(a_{\frac{n}{9}}\right).\end{aligned}$$

So

$$\begin{aligned}&\left\|L_n[f_n]W(1+Q)^{-\Delta}\right\|_{L_p\left((|x|\le a_{\frac{n}{18}})\cup(|x|\ge a_{2n})\right)}\\ \le\ & C_1\phi\left(a_{\frac{n}{9}}\right)\left\|(1+Q)^{-\Delta}\right\|_{L_p(\mathbb{R})}\le C_2\phi\left(a_{\frac{n}{9}}\right)\end{aligned}$$

Here we have used the fact that Q grows faster than any power of x (Lemma 3.1.3 (a)). Next, for $a_{\frac{n}{18}}\le|x|\le a_{2n}$, Lemma 3.3.1 gives

$$|L_n[f_n]W|(x)\le C_3\phi\left(a_{\frac{n}{9}}\right)\left\{\log n+a_n^{\frac{1}{2}}|P_nW|(x)\,T^*(a_n)^{-\frac{1}{4}}\right\}.$$

Also for this range of x,

$$Q(x)\sim Q(a_n)\sim nT^*(a_n)^{-\frac{1}{2}}.$$

So

$$\begin{aligned}&\left\|L_n[f_n]W(1+Q)^{-\Delta}\right\|_{L_p\left[a_{\frac{n}{18}}\le|x|\le a_{2n}\right]}\\ \le\ & C_4\phi\left(a_{\frac{n}{9}}\right)\left(nT^*(a_n)^{-\frac{1}{2}}\right)^{-\Delta}\left\{\log n\left(a_{2n}-a_{\frac{n}{18}}\right)^{\frac{1}{p}}+a_n^{\frac{1}{2}}T^*(a_n)^{-\frac{1}{4}}\|p_nW\|_{L_p(\mathbb{R})}\right\}\\ \le\ & C_5\phi\left(a_{\frac{n}{9}}\right)\left(nT^*(a_n)^{-\frac{1}{2}}\right)^{-\Delta}(\log n)\left(\frac{a_n}{T^*(a_n)}\right)^{\frac{1}{p}}\end{aligned}$$

$$+C_5\phi\left(a_{\frac{n}{8}}\right)\left(nT^*(a_n)^{-\frac{1}{2}}\right)^{-\Delta}T^*(a_n)^{-\frac{1}{4}}a_n^{\frac{1}{p}}\left\{\begin{array}{ll}1 & ,p<4\\ (\log n)^{\frac{1}{4}} & ,p=4\\ (nT^*(a_n))^{\frac{2}{3}\left(\frac{1}{4}-\frac{1}{p}\right)} & ,p>4\end{array}\right\}$$

by Lemma 3.1.2(a) and Lemma 3.1.3(f).

Since $T^*(a_n)$ and a_n grow slower than any por'tive power of n (Lemma 3.1.4(a)), we see that the right hand side is $o\left(\phi\left(a_{\frac{n}{8}}\right)\right)=o(1)$, because of (4.2). □

Next, we deal with functions that v... h outside $\left[-a_{\frac{n}{8}},a_{\frac{n}{8}}\right]$. We separately estimate the weighted L_p norms of their Lagrange interpolants over $\left[-a_{\frac{n}{8}},a_{\frac{n}{8}}\right]$ and $\mathbb{R}\setminus\left[-a_{\frac{n}{8}},a_{\frac{n}{8}}\right]$.

Lemma 4.1.2. Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of measurable functions from $\mathbb{R}\longrightarrow\mathbb{R}$ such that for $n\geq 1$

$$g_n(x)=0,\ |x|\geq a_{\frac{n}{8}}; \tag{4.6}$$

$$|g_nW| \quad \phi(x),\ x\in\mathbb{R}. \tag{4.7}$$

Then

$$\lim_{n\longrightarrow\infty}\left\|L_n[g_n]W(1+Q)^{-\Delta}\right\|_{L_p\left[|x|\geq a_{\frac{n}{8}}\right]}=0. \tag{4.8}$$

Proof. For $x\geq a_{\frac{n}{8}}$,

$$\begin{aligned}|L_n[g_n](x)| &\leq \sum_{|x_{k,n}|\leq a_{\frac{n}{8}}}|l_{k,n}(x)|W^{-1}(x_{k,n})\phi(x_{k,n})\\ &\leq C_1a_n^{\frac{1}{2}}|p_n(x)|\sum_{|x_{k,n}|\leq a_{\frac{n}{8}}}(x_{k,n}-x_{k+1,n})\frac{\left(1-\frac{|x_{k,n}|}{a_n}+L\delta_n\right)^{\frac{1}{4}}}{|x-x_{k,n}|}\phi(x_{k,n})\\ &\qquad \text{(by Lemma 3.1.2}(b)\text{ and (3.3))}\\ &\leq C_2a_n^{\frac{1}{2}}|p_n(x)|\int_{-a_{\frac{n}{8}}}^{a_{\frac{n}{8}}}\frac{\left(1-\frac{|t|}{a_n}+L\delta_n\right)^{\frac{1}{4}}}{|x-t|}\phi(t)\,dt.\end{aligned}$$

Here we have used the monotonicity of ϕ and (3.44). Next, for $t \in \left[0, a_{\frac{n}{9}}\right]$ and $x \geq a_{\frac{n}{8}}$

$$0 \leq \frac{a_n - t}{x - t} = 1 + \frac{\frac{a_n}{x} - 1}{1 - \frac{t}{x}} \leq 1 + \frac{\frac{a_n}{a_{\frac{n}{8}}} - 1}{1 - \frac{a_{\frac{n}{9}}}{a_{\frac{n}{8}}}} \leq C_3$$

by Lemma 3.1.3(f).

Moreover,

$$1 - \frac{|t|}{a_n} \geq C_4 \frac{1}{T^*(a_n)} >> \delta_n$$

So

$$\begin{aligned} |L_n[g_n](x)| &\leq C_5 a_n^{\frac{1}{4}} |p_n(x)| \int_0^{a_{\frac{n}{9}}} \frac{(a_n - t)^{\frac{1}{4}}}{x - t} \phi(t)\, dt \\ &\leq C_6 a_n^{\frac{1}{4}} |p_n(x)| \int_0^{a_{\frac{n}{9}}} (x - t)^{-\frac{3}{4}} \phi(t)\, dt. \end{aligned}$$

Here if $t = a_s$, $\frac{n}{9} \geq s \geq 1$, we have for $x \geq a_{\frac{n}{8}}$

$$x - t = x\left(1 - \frac{t}{x}\right) \geq a_{\frac{n}{8}}\left(1 - \frac{a_s}{a_{\frac{9s}{8}}}\right) \geq C_7 \frac{a_n}{T^*(a_s)}.$$

So

$$|L_n[g_n](x)| \leq C_5 a_n^{-\frac{1}{2}} |p_n(x)| \int_0^{a_{\frac{n}{9}}} T^*(t)^{\frac{3}{4}} \phi(t)\, dt.$$

Thus

$$\begin{aligned} &\left\| L_n[g_n] W (1+Q)^{-\Delta} \right\|_{L_P\left[|x| \geq a_{\frac{n}{8}}\right]} \\ \leq\ & C_9 a_n^{-\frac{1}{2}} \left[\int_0^{a_{\frac{n}{9}}} T(t)^{\frac{3}{4}} \phi(t)\, dt\right] Q\left(a_{\frac{n}{8}}\right)^{-\Delta} \|p_n W\|_{L_P(\mathbb{R})}. \end{aligned}$$

It is easy to see that the integral involving ϕ in the last right hand side grows slower than any power of n. Then using (4.2) and the estimate on $\|p_n W\|_{L_P(\mathbb{R})}$ provided by Lemma 3.1.2(a), we obtain (4.8).□

We now turn to the most difficult part of the sufficiency proof, namely the estimation of $\left\| L_n[g_n] W (1+Q)^{-\Delta} \right\|_{L_P\left[|x| \leq a_{\frac{n}{8}}\right]}$.

We present the most technical part of this as a separate lemma. Recall the notation $(2.35-2.38)$ f the partial sums $S_n[.]$ of the orthonormal expansions with respect to W^2.

Lemma 4.1.3. Let $\sigma : \mathbb{R} \longrightarrow \mathbb{R}$ be a bounded measurable function. Then, for $n \geq 1$

$$\left\| S_n\left[\sigma\phi W^{-1}\right] W(1+Q)^{-\Delta}\right\|_{L_p\left[|x|\leq a_{\frac{n}{8}}\right]} \leq C\,\|\sigma\|_{L_\infty(\mathbb{R})}. \tag{4.9}$$

Here C is independent of σ and n.

Proof. We split this into several steps. Part of the difficulty lies in that we cannot simply estimate Hilbert Transforms in L_p with the weight $(1+Q)^{-\Delta}$, as it does not satisfy Muckenhoupt's A_p condition(see Theorem 3.2.1). We may assume that $\|\sigma\|_{L_\infty(\mathbb{R})}=1$.

Step 1: Split $S_n[.](x)$ into several terms depending on the location of x.

First note that by (2.38) and by our estimates for $\frac{\gamma_{n-1}}{\gamma_n}$ and p_n (see Lemma 3.1.1 (c), (e)),

$$\left|S_n\left[\sigma\phi W^{-1}\right]\right| W(x) \leq C_1 a_n^{\frac{1}{2}}\left(1-\frac{|x|}{a_n}\right)^{-\frac{1}{4}} \sum_{j=n-1}^{n} |H[\sigma\phi p_j W]|(x). \tag{4.10}$$

Now let us choose $l := l(n)$ such that

$$2^l \leq \frac{n}{8} \leq 2^{l+1} \tag{4.11}$$

Note that our choise of $l = l(n)$ guarantees that

$$2^{l+3} \leq n. \tag{4.12}$$

Define

$$\Im_k := [a_{2^k},\ a_{2^{k+1}}],\ k \geq 1. \tag{4.13}$$

The reason for this choice of intervals is that

$$Q(x) \sim Q(a_{2^k}) \sim 2^k T^*(a_{2^k})^{\frac{-1}{2}},\ \ x \in \Im_k, \tag{4.14}$$

uniformly in k. For $j = n-1,\ n$ and $x \in \Im_k$, we split

$$
\begin{aligned}
H[\sigma\phi p_j W](x) &= \left[\int_{-\infty}^{0} + \int_{0}^{a_{2k-1}} + P.V. \int_{a_{2k-1}}^{a_{2k+2}} + \int_{a_{2k+2}}^{\infty}\right] \frac{\sigma\phi p_j W(t)}{x-t}\,dt \qquad (4.15)\\
&=: I_1(x) + I_2(x) + I_3(x) + I_4(x).
\end{aligned}
$$

Here $P.V$ stands for principle value.

Step 2: Estimation of I_1 and I_2 for $x \in \Im_k$.

We see that (recall $x \geq a_2$)

$$
\begin{aligned}
|I_1(x)| &\leq \int_0^\infty \frac{|p_j W\phi|(-t)}{t+x}\,dt \\
&\leq C_2 a_n^{-\frac{1}{2}} \int_0^{\frac{a_n}{2}} \frac{\phi(t)}{t+a_2}\,dt + C_2 a_n^{-1} \int_{\frac{a_n}{2}}^{\infty} |p_j W|(t)\,dt \\
&\leq C_3 a_n^{-\frac{1}{2}}.
\end{aligned}
$$

Here we have used the bound (3.4), the bound for $\|p_n W\|_{L_1(\mathbb{R})}$ in Lemma 3.1.2(a) and also the form of ϕ (recall (4.1)), which guarantees that

$$\int_0^\infty \frac{\phi(t)}{1+t}\,dt < \infty. \qquad (4.16)$$

Next the bound (3.4) gives

$$
\begin{aligned}
|I_2(x)| &\leq \int_0^{a_{2k-1}} \frac{|p_j W\phi|(t)}{x-t}\,dt \\
&\leq C_4 a_n^{-\frac{1}{2}} \left(1-\frac{x}{a_n}\right)^{-\frac{1}{4}} \int_0^{a_{2k-1}} \frac{dt}{x-t} \\
&= C_4 a_n^{-\frac{1}{2}} \left(1-\frac{x}{a_n}\right)^{-\frac{1}{4}} \log\left(1-\frac{a_{2k-1}}{x}\right)^{-1}.
\end{aligned}
$$

Now

$$1 - \frac{a_{2k-1}}{x} \geq 1 - \frac{a_{2k-1}}{a_{2k}} \geq C_5 \frac{1}{T^*(a_{2k})} \geq C_6 \frac{1}{T^*(x)}.$$

Thus

$$|I_2(x)| \leq C_7 a_n^{-\frac{1}{2}} \left(1-\frac{x}{a_n}\right)^{-\frac{1}{4}} \log\left(C_8 T^*(x)\right).$$

Step 3: Estimation of I_4 for $x \in \Im_k$.

Now using our bound (3.4) again gives

$$
\begin{aligned}
|I_4(x)| &\le \int_{a_{2k+2}}^{\infty} \frac{|p_j W\phi|(t)}{t-x}dt \\
&\le C_9[a_n^{-\frac{1}{2}} \int_{a_{2k+2}}^{2a_{2k+2}} \left|1-\frac{t}{a_n}\right|^{-\frac{1}{4}} \frac{dt}{t-x} + a_n^{-\frac{1}{2}} \int_{2a_{2k+2}}^{\max\{2a_{2k+2},\frac{a_n}{2}\}} \frac{\phi(t)}{t}dt \\
&\quad + \int_{\frac{a_n}{2}}^{\infty} \frac{|p_j W|(t)}{t}dt] \\
&\le C_{10} a_n^{-\frac{1}{2}} [1+J],
\end{aligned}
$$

where

$$J := \int_{a_{2k+2}}^{2a_{2k+2}} \left|1-\frac{t}{a_n}\right|^{-\frac{1}{4}} \frac{dt}{t-x}.$$

(We have used (4.16) and the bound on the L_1 norm of $p_n W$).

Here if $\left|1-\frac{t}{a_n}\right| \le \frac{1}{2}\left(1-\frac{x}{a_n}\right)$, then

$$|t-x| = a_n \left|\left(1-\frac{x}{a_n}\right) - \left(1-\frac{t}{a_n}\right)\right| \ge \frac{1}{2}a_n\left(1-\frac{x}{a_n}\right).$$

Thus

$$
\begin{aligned}
J &\le C_{11}\left[\left(1-\frac{x}{a_n}\right)^{-\frac{1}{4}} \int_{\substack{|1-\frac{t}{a_n}|\ge\frac{1}{2}(1-\frac{x}{a_n}) \\ t\in[a_{2k+2},\,2a_{2k+2}]}} \frac{dt}{t-x}\right. \\
&\quad \left. + a_n^{-1}\left(1-\frac{x}{a_n}\right)^{-1} \int_{\substack{|1-\frac{t}{a_n}|\le\frac{1}{2}(1-\frac{x}{a_n}) \\ t\in[a_{2k+2},\,2a_{2k+2}]}} \left|1-\frac{t}{a_n}\right|^{-\frac{1}{4}} dt\right] \\
&\le C_{12}\left[\left(1-\frac{x}{a_n}\right)^{-\frac{1}{4}} \log\left(1+\frac{a_{2k+2}}{a_{2k+2}-x}\right)\right. \\
&\quad \left. + \left(1-\frac{x}{a_n}\right)^{-1} \int_{|1-s|\le\frac{1}{2}(1-\frac{x}{a_n})} |1-s|^{-\frac{1}{4}} ds\right] \\
&\le C_{13}\left(1-\frac{x}{a_n}\right)^{-\frac{1}{4}} \log\left(C_{14} T^*(x)\right).
\end{aligned}
$$

Step 4: Estimation of $\|S_n[.]\|_{L_p[\Im_k]}$

Combining our estimates for I_j $j = 1, 2, 4$ gives,

$$|I_1 + I_2 + I_4|(x) \leq C_{14} a_n^{-\frac{1}{2}} \left(1 - \frac{x}{a_n}\right)^{-\frac{1}{4}} \log\left(C_{15} T^*(x)\right).$$

Together with (4.10), (4.14) and (4.15), this gives

$$\begin{aligned}
&\left\| S_n\left[\sigma\phi W^{-1}\right] W (1+Q)^{-\Delta} \right\|_{L_p[\Im_k]} \\
\leq\ & Q\left(a_{2^k}\right)^{-\Delta} \left(1 - \frac{a_{2^{k+1}}}{a_n}\right)^{-\frac{1}{4}} \\
& \times \left\{ \begin{array}{l} \left(1 - \frac{a_{2^{k+1}}}{a_n}\right)^{-\frac{1}{4}} \log\left(C_{15} T^*\left(a_{2^{k+1}}\right)\right)\left(a_{2^{k+1}} - a_{2^k}\right)^{\frac{1}{p}} \\ + a_n^{\frac{1}{2}} \sum_{j=n-1}^{n} \left\| P.V. \int_{a_{2^{k-1}}}^{a_{2^{k+2}}} \frac{\sigma\phi p_j W(t)}{x-t} dt \right\|_{L_p[\Im_k]} \end{array} \right\}
\end{aligned}$$

We use M.Riesz's theorem on the boundedness of the Hilbert transform from $L_p(\mathbb{R})$ to $L_p(\mathbb{R})$, (see section 3.2) to deduce that

$$\begin{aligned}
&\left\| P.V. \int_{a_{2^{k-1}}}^{a_{2^{k+2}}} \frac{\sigma\phi p_j W(t)}{x-t} dt \right\|_{L_p[\Im_k]}^p \\
\leq\ & C_{17} \int_{a_{2^{k-1}}}^{a_{2^{k+2}}} |\sigma\phi p_j W|^p(t)\, dt \\
\leq\ & C_{17} a_n^{-\frac{p}{2}} \left(1 - \frac{a_{2^{k+2}}}{a_n}\right)^{-\frac{p}{4}} \left(a_{2^{k+2}} - a_{2^{k-1}}\right).
\end{aligned}$$

Next, note that, in view of (4.12), $n \geq 2^{k+3}$ for $k \leq l$, so

$$\left(1 - \frac{a_{2^{k+1}}}{a_n}\right) \geq \left(1 - \frac{a_{2^{k+2}}}{a_n}\right) \geq \left(1 - \frac{a_{2^{k+2}}}{a_{2^{k+3}}}\right) \geq C_{18} \frac{1}{T^*\left(a_{2^k}\right)}.$$

Moreover,

$$a_{2^{k+1}} - a_{2^k} \leq a_{2^{k+2}} - a_{2^{k-1}} \leq C_{19} \frac{a_{2^k}}{T^*\left(a_{2^k}\right)}$$

Hence,

$$\begin{aligned}
&\left\| S_n\left[\sigma\phi W^{-1}\right] W (1+Q)^{-\Delta} \right\|_{L_p[\Im_k]} \qquad (4.17) \\
\leq\ & C_{20} Q\left(a_{2^k}\right)^{-\Delta} T^*\left(a_{2^k}\right)^{\frac{1}{2}} \log\left(C_{15} T^*\left(a_{2^{k+1}}\right)\right) \left(\frac{a_{2^k}}{T^*\left(a_{2^k}\right)}\right)^{\frac{1}{p}}.
\end{aligned}$$

Step 5: Completion of the proof

The estimation of $S_n[.](x)$ for $x \in -\Im_k = [-a_{2k+1} - a_{2k}]$ is exactly the same as for $x \in \Im_k$. Since we have (4.14), and since a_{2k}, $T^*(a_{2k})$ grow much slower than $Q(a_{2k})$(Lemma 3.1.4 (a)),we obtain

$$\begin{aligned} &\left\|S_n\left[\sigma\phi W^{-1}\right]W(1+Q)^{-\Delta}\right\|^p_{L_p\left[a_2\le|x|\le a_{\frac{n}{8}}\right]} \\ \le\ &\sum_{k=1}^{l}\left\|S_n\left[\sigma\phi W^{-1}\right]W(1+Q)^{-\Delta}\right\|^p_{L_p[\Im_k]} \\ \le\ &C_{21}\sum_{k=1}^{l}2^{\frac{-kp\Delta}{2}}\le C_{22}. \end{aligned}$$

The estimation of $\left\|S_n\left[\sigma\phi W^{-1}\right]W(1+Q)^{-\Delta}\right\|_{L_p[|x|\le a_2]}$ is similar but easier.We split

$$H[\sigma\phi p_jW](x)=[\int_{-\infty}^{-2a_2}+P.V.\int_{-2a_2}^{2a_2}+\int_{2a_2}^{\infty}]\frac{\sigma\phi p_jW(t)}{x-t}dt.$$

The first and third integrals m y be estimated as we did before, and the second is estimated as we did I_3.□

Armed with this lemma, we can complete the estimation of $L_n[g_n]$ over $[-a_{\beta n}, a_{\beta n}]$.

Lemma 4.1.4. Let $\varepsilon \in (0,1)$. Let $\{g_n\}$ be as in Lemma 4.1.2, except that rather than (4.7) ,we assume that

$$|g_nW|(x)\le\varepsilon\phi(x),\ x\in\mathbb{R}, n\ge 1. \tag{4.18}$$

Then,

$$\lim_{n\longrightarrow\infty}\sup\left\|L_n[g_n]W(1+Q)^{-\Delta}\right\|_{L_p\left[|x|\le a_{\frac{n}{8}}\right]}\le C\varepsilon. \tag{4.19}$$

where C is independent of n, $\{g_n\}$ and ε.

Proof. Let

$$\chi_n:=\chi_{\left[-a_{\frac{n}{8}},\,a_{\frac{n}{8}}\right]},$$

$$h_n:=sign(L_n[g_n])\,|L_n[g_n]|^{p-1}\chi_nW^{p-2}(1+Q)^{-\Delta p}$$

and

$$\sigma_n := sign\, S_n[h_n].$$

We shall show that

$$\left\| L_n[g_n] W (1+Q)^{-\Delta} \right\|_{L_p\left[|x|\le a_{\frac{n}{8}}\right]} \le C\varepsilon \left\| S_n\left[\sigma\phi W^{-1}\right] W (1+Q)^{-\Delta} \right\|_{L_p\left[|x|\le a_{\frac{n}{8}}\right]}. \qquad (4.20)$$

Then Lemma 4.1.3 gives the result.

Now using the orthogonality of $f - S_n[f]$ to $\mathcal{P}_{n-1}$, and the Gauss quadrature formula (3.36), we see that

$$\begin{aligned}
\left\| L_n[g_n] W (1+Q)^{-\Delta} \right\|^p_{L_p\left[|x|\le a_{\frac{n}{8}}\right]} &= \int_{\mathbb{R}} L_n[g_n] h_n W^2 \\
&= \int_{\mathbb{R}} L_n[g_n] S_n[h_n] W^2 = \sum_{j=1}^{n} \lambda_{j,n} g_n(x_{j,n}) S_n[h_n](x_{j,n}) \\
&= \sum_{|x_{k,n}|<a_{\frac{n}{8}}} \lambda_{j,n} g_n(x_{j,n}) S_n[h_n](x_{j,n}) \\
&\le \varepsilon \sum_{|x_{k,n}|<a_{\frac{n}{8}}} \lambda_{j,n}\phi(x_{j,n}) W^{-1}(x_{j,n}) \left|S_n[h_n](x_{j,n})\right| \\
&\le C\varepsilon \int_{\mathbb{R}} \phi W \left|S_n[h_n]\right|
\end{aligned}$$

by Lemma 3.3.2.

Note that it is easy to verify the approximation property in Lemma 3.3.2 for ϕ (in fact Jackson's Theorem gives polynomials of degree $o(n)$ satisfying (3.50)).

We can continue this as

$$\begin{aligned}
&= C\varepsilon \int_{\mathbb{R}} \phi\sigma_n W^{-1} W^2 S_n[h_n] \\
&= C\varepsilon \int_{\mathbb{R}} h_n S_n\left[\phi\sigma_n W^{-1}\right] W^2 \\
&= C\varepsilon \int_{-a_{\frac{n}{8}}}^{a_{\frac{n}{8}}} h_n S_n\left[\phi\sigma_n W^{-1}\right] W^2
\end{aligned}$$

for h_n has its support inside $\left[-a_{\frac{n}{8}}, a_{\frac{n}{8}}\right]$.

Using Hölder's Inequality with $q = \frac{p}{p-1}$, we continue this as

$$\begin{aligned}
&\leq C\varepsilon \left(\int_{-a_{\frac{n}{8}}}^{a_{\frac{n}{8}}} \left|h_n W(1+Q)^{\Delta}\right|^q\right)^{\frac{1}{q}} \left(\int_{-a_{\frac{n}{8}}}^{a_{\frac{n}{8}}} \left|S_n\left[\phi\sigma_n W^{-1}\right] W(1+Q)^{-\Delta}\right|^p\right)^{\frac{1}{p}} \\
&= C\varepsilon \left\|L_n[g_n] W(1+Q)^{-\Delta}\right\|_{L_p\left[|x|\leq a_{\frac{n}{8}}\right]}^{p-1} \left\|S_n\left[\phi\sigma_n W^{-1}\right] W(1+Q)^{-\Delta}\right\|_{L_p\left[|x|\leq a_{\frac{n}{8}}\right]}.
\end{aligned}$$

Cancelling the $(p-1)$th power of $\|L_n...\|$ gives (4.20).□

We can now turn to:

The Proof of the Sufficiency Part of Theorem 2.5.2. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous and satisfy (2.21). Let $\varepsilon > 0$. By Corollary 1.1.2, we can choose a polynomial P such that,

$$\left\|(f-P) W\phi^{-1}\right\|_{L_\infty(\mathbb{R})} \leq \varepsilon.$$

Then, for n large enough,

$$\begin{aligned}
&\left\|(f - L_n[f]) W(1+Q)^{-\Delta}\right\|_{L_p(\mathbb{R})} &(4.21)\\
\leq\ &\left\|(f-P) W(1+Q)^{-\Delta}\right\|_{L_p(\mathbb{R})} + \left\|(P - L_n[f]) W(1+Q)^{-\Delta}\right\|_{L_p(\mathbb{R})} \\
\leq\ &\varepsilon \left\|\phi(1+Q)^{-\Delta}\right\|_{L_p(\mathbb{R})} + \left\|(L_n[P-f]) W(1+Q)^{-\Delta}\right\|_{L_p(\mathbb{R})}.
\end{aligned}$$

The first norm in (4.21) is finite as $\Delta > 0$ and as Q grows faster than any power of x. Next, let

$$\chi_n := \chi\left[-a_{\frac{n}{8}}, a_{\frac{n}{8}}\right]$$

and write

$$P - f = (P-f)\chi_n + (P-f)(1-\chi_n) =: g_n + f_n.$$

By Lemma 4.1.1,

$$\lim_{n\longrightarrow\infty} \left\|L_n[f_n] W(1+Q)^{-\Delta}\right\|_{L_p(\mathbb{R})} = 0.$$

Also Lemmas 4.1.2 and 4.1.4 together give

$$\limsup_{n\longrightarrow\infty} \left\|L_n[g_n] W(1+Q)^{-\Delta}\right\|_{L_p(\mathbb{R})} \leq C\varepsilon,$$

with C independent of ε.

Substituting the estimates for $L_n[f_n]$, and $L_n[g_n]$ into (4.21) and then letting $\varepsilon \longrightarrow 0$, gives (2.20). □

4.2 Sufficiency for Theorem 2.5.4

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous and satisfy (2.25) with $\alpha > \frac{1}{p}$. We must show (2.26). Let $\varepsilon \in (0,1)$. We can choose a polynomial P such that

$$\|(f-P)(x)W(x)(1+|x|)^{\alpha}\|_{L_\infty(\mathbb{R})} \leq \varepsilon.$$

(See Corollary 1.1.2). Then for n large enough

$$\begin{aligned} & \|(f-L_n[f])W\|_{L_p(\mathbb{R})} \\ \leq\ & \|(f-P)W\|_{L_p(\mathbb{R})} + \|L_n[P-f]W\|_{L_p(\mathbb{R})} \\ \leq\ & \varepsilon\left\|(1+|x|)^{-\alpha}\right\|_{L_p(\mathbb{R})} + \|L_n[P-f]W\|_{L_p(\mathbb{R})}. \end{aligned} \tag{4.22}$$

The first norm in the right-hand side of (4.22) is, of course, finite as $\alpha p > 1$. Next, Theorem 3.4.1 shows that for large enough n,

$$\begin{aligned} \|L_n[P-f]W\|_{L_p(\mathbb{R})} &\leq C_1\left\{\sum_{j=1}^{n}\lambda_{j,n}W^{-2}(x_{j,n})\,|(P-f)W|^p(x_{j,n})\right\}^{\frac{1}{p}} \\ &\leq C_2\varepsilon\left\{\sum_{j=1}^{n}|I_{j,n}|\,(1+|x_{j,n}|)^{-\alpha p}\right\}^{\frac{1}{p}} \\ &\leq C_3\varepsilon\left\|(1+|x|)^{-\alpha}\right\|_{L_p(\mathbb{R})}. \end{aligned}$$

Substituting into (4.22) and noting that the various constants are independent of ε, gives the result. □

4.3 Sufficiency for Theorem 2.5.5

As $(1+|x|)^{\Delta} \leq 1$ if $\Delta \leq 0$, the limit (2.28) follows from (2.26). □

4.4 Necessity for Theorem 2.5.2

Proof. Fix $1 < p < \infty$, $\Delta \in \mathbb{R}$, $\kappa > 0$, $\delta > 1+\kappa$ and assume the conclusion of Theorem 2.5.2 is true; i.e. (2.20) holds for every continuous function satisfying (2.21). Let X be the space of all continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ with

$$\|f\|_X := \sup_{x \in \mathbb{R}} |fW|(x) (\log(2+|x|))^\delta < \infty.$$

Moreover, let Y be the space of all measurable functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ with

$$\|f\|_Y := \left\| fW(1+Q)^{-\Delta} \right\|_{L_p(\mathbb{R})} < \infty.$$

Each $f \in X$ satisfies (2.20), so the conclusion of Theorem 2.5.2 ensures that

$$\lim_{n \longrightarrow \infty} \|f - L_n[f]\|_Y = 0.$$

Since X is a Banach space, the uniform boundedness principle gives

$$\|f - L_n[f]\|_Y \leq C \|f\|_X \tag{4.23}$$

with C independent of n and f. In particular as $L_1[f] = f(0)$(recall that $p_1(x) = \gamma_1(x)$), we deduce that for $f \in X$ with $f(0) = 0$,

$$\|f\|_Y \leq C \|f\|_X.$$

So for such f,

$$\|L_n[f]\|_Y \leq 2C \|f\|_X. \tag{4.24}$$

Choose g_n continuous in $\mathbb{R}$, with $g_n = 0$ in $[0, \infty) \cup (-\infty, \frac{-1}{2} a_n]$, with

$$\|g_n\|_X = \sup_{x \in \mathbb{R}} |g_n W|(x) (\log(2+|x|))^\delta = 1,$$

and for $x_{j,n} \in \left(-\frac{1}{2}a_n, 0\right)$,

$$g_n W(x_{j,n})(\log(2+|x_{j,n}|))^{\delta} \, sign p_n'(x_{j,n}) = 1.$$

For example, $\left(g_n W(x)(\log(2+|x|))^{\delta}\right)$ can be chosen to be piecewise linear. Then for $x \in [1, a_n]$,

$$\begin{aligned}
|L_n[g_n](x)| &= \left| \sum_{x_{j,n} \in (-\frac{1}{2}a_n, 0)} g_n(x_{j,n}) \frac{p_n(x)}{p_n'(x_{j,n})(x - x_{j,n})} \right| \\
&= |p_n(x)| \sum_{x_{j,n} \in (-\frac{1}{2}a_n, 0)} \frac{(\log(2+|x_{j,n}|))^{-\delta}}{|p_n' W|(x_{j,n})(x + |x_{j,n}|)} \\
&\geq C_1 a_n^{\frac{1}{2}} |p_n(x)| (\log a_n)^{-\delta} a_n^{-1} \sum_{x_{j,n} \in [-\frac{1}{2}a_n, 0)} (x_{j,n} - x_{j+1,n}) \\
&\quad \text{(by Lemma 3.1.1}(g)\text{ and }(b)) \\
&\geq C_2 a_n^{\frac{1}{2}} |p_n(x)| (\log a_n)^{-\delta}.
\end{aligned}$$

Then by (4.24),

$$\begin{aligned}
2C &= 2C \|g_n\|_X \geq \|L_n[g_n]\|_Y \\
&\geq C_3 a_n^{\frac{1}{2}} (\log a_n)^{-\delta} \left\| p_n W (1+Q)^{-\Delta} \right\|_{L_p[1,a_n]} \\
&\geq C_4 a_n^{\frac{1}{p}} (\log a_n)^{-\delta} Q(a_n)^{-\max\{\Delta,0\}} \begin{cases} 1 & , p < 4 \\ (\log n)^{\frac{1}{4}} & , p = 4 \\ (nT^*(a_n))^{\frac{2}{3}(\frac{1}{4}-\frac{1}{p})} & , p > 4 \end{cases}
\end{aligned}$$

Here we used the monotonicity of Q, Lemma 3.1.2(a) and Lemma 3.1.1(d). Note that $[-1, 1]$ does not give a big contribution to the L_p norm of $p_n W$. We obtain a contradiction if $\Delta \leq 0$, for all p. So, $\Delta > 0$. Also, for $p > 4$, we obtain from Lemma 3.1.3(b),

$$2C \geq C_5 a_n^{\frac{1}{p}} (\log a_n)^{-\delta} T^*(a_n)^{\frac{\Delta}{2}+\frac{2}{3}(\frac{1}{4}-\frac{1}{p})} n^{-\Delta+\frac{2}{3}(\frac{1}{4}-\frac{1}{p})}.$$

Since the terms invoving a_n and $T^*(a_n)$ grow to ∞ with n, we see that necessarily

$$\Delta > \frac{2}{3}\left(\frac{1}{4}-\frac{1}{p}\right).\square$$

4.5 Proof of Theorem 2.5.3

This is similar to the previous proof. We let X be the Banach space of continuous functions $f:\mathbb{R}\longrightarrow\mathbb{R}$ vanishing outside $[-2,2]$, with norm

$$\|f\|_X := \|f\|_{[-2,2]}.$$

We let Y be the space of all measurable $f:\mathbb{R}\longrightarrow\mathbb{R}$ with,

$$\|f\|_Y := \|fWU\|_{L_p} \quad < \infty.$$

Assume that we cannot find f satisfying (2.24). Then the uniform boundedness principle gives (4.23) for all $f\in X$. Again, when $f(0)=0$, we obtain (4.24). We now choose $g_n\in X$, with

$$\|g_n\|_X = 1,$$

$$(g_nW)(x_{j,n})\,sign\,(p_n'(x_{j,n})) = 1\ ,\ x_{j,n}\in\ \left[-1,-\frac{1}{2}\right],$$

$$g_n = 0 \text{ in } (-\infty,-2]\cup[0,\infty)$$

and

$$g_nW(x_{j,n})\,sign\,(p_n'(x_{j,n}))\geq 0,\ \ x_{j,n}\in[-2,2].$$

Much as before, we deduce that for $x\geq 1$,

$$|L_n[g_n](x)|\geq Ca_n^{\frac{1}{2}}\frac{|p_n(x)|}{x}.$$

Also by hypothesis, there exists C_1 and C_2 such that,

$$U(x)\geq C_1x^{\frac{3}{2}-\frac{1}{p}}Q(x)^{-\frac{2}{3}\left(\frac{1}{4}-\frac{1}{p}\right)},x\geq C_2.$$

Hence by (4.24)

$$\begin{aligned}
2C &= 2C\|g_n\|_X \geq \|L_n[g_n]\|_Y \\
&\geq C_1 \left\| L_n[g_n](x)\, W(x)\, x^{\frac{3}{2}-\frac{1}{p}} Q(x)^{-\frac{2}{3}\left(\frac{1}{4}-\frac{1}{p}\right)} \right\|_{L_p[C_2,a_n]} \\
&\geq C_2 a_n^{\frac{1}{2}-\frac{1}{p}} Q(a_n)^{-\frac{2}{3}\left(\frac{1}{4}-\frac{1}{p}\right)} \|p_n W\|_{L_p\left[a_{\frac{n}{2}},\, a_n\right]} \\
&\geq C_3 T^*(a_n)^{\frac{1}{4}-\frac{1}{p}},
\end{aligned}$$

much as before, by Lemma 3.1.2(a) and (3.4). Of course this is impossible for large n and we have a contradiction.□

4.6 Proof of Necessity of Theorems 2.5.4 and 2.5.5

We begin with,

Lemma 4.6.1. Let $0 < p < \infty$. Let $0 < A < B < \infty$ and $\xi : \mathbb{R} \longrightarrow (0, \infty)$ be a continuous function such that for $1 \leq s, t < \infty$ with $\frac{1}{2} \leq \frac{s}{t} \leq 2$, we have,

$$A \leq \frac{\xi(a_s)}{\xi(a_t)} \leq B. \tag{4.25}$$

For $n \geq 1$, let $\Im_n \subset [-a_n\ a_n]$ be an interval containing at least two zeros of $p_n(W^2, .)$. Then for $n \geq 1$,

$$\|p_n W \xi\|_{L_p[\Im_n]} \geq C_1 a_n^{\frac{-1}{2}} \left\| \xi(t) \left(\left| 1 - \frac{|t|}{a_n} \right| + \delta_n \right)^{\frac{-1}{4}} \right\|_{L_p[\Im_n]}. \tag{4.26}$$

Here C_1 depends only on A, B (and not on ξ or n or $\Im_n$).

Proof. From (3.15), for $x \in [x_{j+1}, x_{j,n}]$,

$$\max\left\{ l_{j,n}(x)\, W^{-1}(x_{j,n})\, W(x),\, l_{j+1,n}(x)\, W^{-1}(x_{j+1,n})\, W(x) \right\} \geq \frac{1}{2}$$

and hence for such x,

$$|p_n W|(x) \geq \frac{1}{2} \min\left\{ |x - x_{j,n}|\, |p_n' W|(x_{j,n}),\, |x - x_{j+1,n}|\, |p_n' W|(x_{j+1,n}) \right\}$$

$$\geq \; C_2 \frac{n}{a_n^{\frac{3}{2}}} \Psi_n^{-1}(x_{j,n}) \left(\left| 1 - \frac{|x_{j,n}|}{a_n} \right| + \delta_n \right)^{\frac{-1}{4}} \min \{ |x - x_{j,n}|, |x - x_{j+1,n}| \}$$

by (3.11), (3.10) and (3.9).

Let

$$\Im_{j,n} := \left[x_{j+1,n} + \frac{1}{4}(x_{j,n} - x_{j+1,n}), x_{j,n} + \frac{1}{4}(x_{j,n} - x_{j+1,n}) \right],$$

so that $\Im_{j,n}$ has length $\frac{1}{2}(x_{j,n} - x_{j+1,n})$. By (3.3),

$$|p_n W|(x) \geq C_3 a_n^{\frac{-1}{2}} \left(\left| 1 - \frac{|x_{j,n}|}{a_n} \right| + \delta_n \right)^{\frac{-1}{4}}, \; x \in \Im_{j,n}.$$

Then using also (3.9),

$$\begin{aligned} &\int_{x_{j+1,n}}^{x_{j,n}} |p_n W|^p(t) \xi^p(t)\, dt \\ \geq \;& C_4 a_n^{-\frac{p}{2}} \left(\left| 1 - \frac{|x_{j,n}|}{a_n} \right| + \delta_n \right)^{\frac{-p}{4}} \int_{\Im_{j,n}} \xi^p(t) \; dt. \end{aligned}$$

The result follows if we can show that

$$\int_{\Im_{j,n}} \xi^p(t) \; dt \geq C_5 \int_{x_{j+1,n}}^{x_{j,n}} \xi^p(t)\, dt.$$

(The L_p norm of $\xi(t) \left(\left| 1 - \frac{|t|}{a_n} \right| + \delta_n \right)^{\frac{-1}{4}}$ over that part of $\Im_{j,n}$ near the endpoints of this interval, is easily estimated in terms of the rest).

To do this if suffices to show that

$$\xi(t) \sim \xi(x_{j,n}), t \in [x_{j+1,n}\; x_{j,n}].$$

Now in view of (4.25), it suffices to show that if $x_{j+1,n} = a_s$ and $x_{j,n} = a_t$, where $s \geq s_o > 0$ (Here we use the continuity of the map $u :\longrightarrow a_u$) then,

$$1 \leq \frac{s}{t} \leq 2. \tag{4.27}$$

But if $t \geq 2s$, then (3.17) and (3.18) give

$$\frac{x_{j,n}}{x_{j+1,n}} - 1 \geq \frac{a_{2s}}{a_s} - 1 \geq C_6 \frac{1}{T^*(a_s)} \geq C_7 \frac{1}{T^*(a_n)},$$

while our spacing (3.3) gives

$$\frac{x_{j,n}}{x_{j+1,n}} - 1 \leq C_8 \frac{a_n}{n} \frac{\Psi_n(x_{j,n})}{x_{j+1,n}} \leq C_9 \frac{a_n}{n} \Psi_n(a_n) \leq C_{10} a_n \left(nT^*(a_n)\right)^{-\frac{2}{3}}.$$

Our hypothesis shows that $T^*(a_n)^{-1}$ is much larger than any negative power of n, for n large, and we have a contradicton. So (4.27) and the result follow. □

We can now proceed with:

The Proof of the necessity parts of Theorem 2.5.4 and 2.5.5. Fix α, $\Delta \in \mathbb{R}$ and $1 < p < 4$. Assume moreover that we have the convergence (2.28) for every continuous f satisfying (2.25). Let $\eta : \mathbb{R} \longrightarrow (0, \infty)$ be a positive even continuous function, decreasing in $(0, \infty)$, with limit 0 at ∞. We shall assume it decays very slowly later on. Let

$$X := \left\{ f : \mathbb{R} \longrightarrow \mathbb{R} \text{ continuous with } \|f\|_X := \sup_{x \in \mathbb{R}} |fW|(x)(1 + |x|)^{\alpha} \eta(x)^{-1} < \infty \right\}.$$

Moreover, let Y be the space of all measurable functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ with

$$\|f\|_Y := \left\|(fW)(x)(1+|x|)^{\Delta}\right\|_{L_p(\mathbb{R})} < \infty.$$

Each $f \in X$ satisfies (2.25), so the conclusion of Thm 2.5.5 ensures that

$$\lim_{n \longrightarrow \infty} \|(f - L_n[f])\|_Y = 0.$$

Since X is a Banach space, the uniform boundedness principle gives

$$\|(f - L_n[f])\|_Y \leq C \|f\|_X \tag{4.28}$$

with C independent of n and f. In particular as $L_1[f] = f(0)$ (Recall that $p_1(x) = \gamma_1(x)$,) we deduce that for $f \in X$ with $f(0) = 0$,

$$\|f\|_Y \leq C \|f\|_X.$$

So for such f,

$$\|L_n[f]\|_Y \leq 2C \|f\|_X. \tag{4.29}$$

Choose g_n continuous in $\mathbb{R}$, with $g_n = 0$ in $[0, \infty) \cup (-\infty, \frac{-1}{2}a_n]$, with

$$\|g_n\|_X = \sup_{x \in \mathbb{R}} |g_n W|(x)(1 + |x|)^{\alpha} \eta(x)^{-1} = 1,$$

and for $x_{j,n} \in \left(-\frac{1}{2}a_n, 0\right)$,

$$(g_n W)(x_{j,n})(1 + |x_{j,n}|)^{\alpha} \eta(x_{j,n})^{-1} \, sign\left(p_n'(x_{j,n})\right) = 1.$$

For example, $\left(g_n W(x)(1 + |x|)^{\alpha} \eta(x)^{-1}\right)$ can be chosen to be piecewise linear. Then for $x \in [1, \frac{a_n}{4}]$,

$$\begin{aligned}
|L_n[g_n](x)| &= \left| \sum_{x_{j,n} \in [-\frac{1}{2}a_n, 0)} g_n(x_{j,n}) \frac{p_n(x)}{p_n'(x_{j,n})(x - x_{j,n})} \right| \\
&= |p_n(x)| \sum_{x_{j,n} \in [-\frac{1}{2}a_n, 0)} \frac{(1 + |x_{j,n}|)^{-\alpha} \eta(x_{j,n})}{|p_n' W|(x_{j,n})(x + |x_{j,n}|)} \\
&\geq C_1 a_n^{\frac{1}{2}} |p_n(x)| \eta(a_n) \sum_{x_{j,n} \in [-2x, -x)} |I_{j,n}| \frac{(1 + |x_{j,n}|)^{-\alpha}}{(x + |x_{j,n}|)} \\
&\quad \text{(by (3.11))} \\
&\geq C_2 a_n^{\frac{1}{2}} |p_n(x)| \eta(a_n) \int_x^{2x} t^{-1-\alpha} dt \\
&\quad \text{(by (3.3))} \\
&\geq C_3 a_n^{\frac{1}{2}} |p_n(x)| \eta(a_n) x^{-\alpha}
\end{aligned}$$

Then by (4.29),

$$2C = 2C \|g_n\|_X \geq \|L_n[g_n]\|_Y$$

$$\begin{aligned} &\geq C_4 a_n^{\frac{1}{2}} \eta(a_n) \left\| p_n W(x) x^{\Delta-\alpha} \right\|_{L_p[1, \frac{a_n}{4}]} \\ &\geq C_5 \eta(a_n) \left\| x^{\Delta-\alpha} \right\|_{L_p[1, \frac{a_n}{4}]} \end{aligned}$$

by Lemma 4.6.1.

We may assume that η decays so slowly to 0 that,

$$\eta(a_n) \geq (\log\log a_n)^{-1}.$$

(Note that we could have imposed this condition on η at the start but, delayed this for clarity).

Suppose now that $\Delta - \alpha \geq \frac{-1}{p}$. Then we obtain,

$$2C \geq C_6 (\log\log(a_n))^{-1} \log a_n.$$

Then for large n, we obtain a contradiction. So we deduce $\Delta - \alpha < \frac{-1}{p}$ is necessary. Consequently if for a given $\Delta \in \mathbb{R}$, we have the convergence (2.28) for every continuous f satisfying (2.25) and for every $\alpha > \frac{1}{p}$ then, we must have $\Delta \leq 0$. The necessity part of Theorem 2.5.5 is proved.

Finally, for the necessity part of Theorem 2.5.4, we take $\Delta = 0$ in the above and deduce that $\alpha > \frac{1}{p}$.□

4.7 Proof of Theorem 2.5.6

This is similar to the previous proof. We let X be the Banach space of continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ vanishing outside $[-2, 2]$, with norm

$$\|f\|_X := \|f\|_{L_\infty[-2,2]}.$$

We let Y be the space of all measurable $f : \mathbb{R} \longrightarrow \mathbb{R}$ with

$$\|f\|_Y := \|fWU\|_{L_p[\mathbb{R}]} < \infty.$$

Assume that we cannot find f satisfying (2.31). Then the uniform boundedness principle gives (4.28) for all $f \in X$. Again, when $f(0) = 0$, we obtain (4.29). We now choose $g_n \in$

X, with

$$\|g_n\|_X = 1$$

$$(g_n W)(x_{j,n})\, sign\,(p_n'(x_{j,n})) = 1 \;\; x \in \left[-1, -\frac{1}{2}\right]$$

$g_n = 0$ in $(-\infty, -2] \cup [0, \infty)$ and

$$g_n W(x_{j,n})\, sign\,(p_n'(x_{j,n})) \geq 0 \;\; x_{j,n} \in [-2, 2]$$

Much as before, we deduce that for $x \geq 1$,

$$|L_n[g_n](x)| \geq C a_n^{\frac{1}{2}} \frac{|p_n(x)|}{x}$$

Also by hypothesis, given $A > 0$, there exists C_2 such that

$$U(x) \geq A x^{\frac{3}{4}} [\log Q(x)]^{-\frac{1}{4}}, x \geq C_2.$$

Hence by (4.29),

$$\begin{aligned} 2C &= 2C\|g_n\|_X \geq \|L_n[g_n]\|_Y \\ &\geq C_1 A a_n^{\frac{1}{2}} \left\| p_n(x) W(x) x^{\frac{-1}{4}} [\log Q(x)]^{\frac{-1}{4}} \right\|_{L_4[C_2,\, a_n]} \\ &\geq C_3 A a_n^{\frac{1}{4}} [\log n]^{\frac{-1}{4}} \|p_n W\|_{L_4\left[a_{\frac{n}{2}},\, a_n\right]} \end{aligned} \tag{4.30}$$

by (3.16) and (3.22).

Now by Lemma 4.6.1,

$$\begin{aligned} \|p_n W\|_{L_4\left[a_{\frac{n}{2}},\, a_n\right]} &\geq C_4 a_n^{\frac{-1}{2}} \left\| \left(1 - \frac{t}{a_n} + \delta_n\right)^{\frac{-1}{4}} \right\|_{L_4\left[a_{\frac{n}{2}},\, a_n\right]} \\ &= C_4 a_n^{\frac{-1}{4}} \left[\int_{0 \leq s \leq \left(1 - \frac{a_{\frac{n}{2}}}{a_n}\right)/\delta_n} (1+s)^{-1}\, ds \right]^{\frac{1}{4}} \\ &\geq C_5 a_n^{\frac{-1}{4}} \left[\log\left\{1 + C_6 \delta_n^{-1} \left(T^*(a_n)^{-1}\right)\right\}\right]^{\frac{1}{4}} \end{aligned}$$

$$\geq C_6 a_n^{\frac{-1}{4}} (\log n)^{\frac{1}{4}}.$$

Here we make the substitution $1 - \frac{t}{a_n} = \delta_n s$, and also used (3.21) and (3.22) .Finally, using (4.30) ,we obtain

$$2C \geq C_7 A.$$

It is clear that C_7 is independent of A. Of course, this is impossible for large A. So there must exist continuous f vanishing outside $[-2, 2]$ satisfying (2.31) .□

Part II

Rates of Approximation for Erdős Weights

Chapter 5

Introduction and Statement of Results

In chapter 1, it was pointed out that each continuous function f, could be uniformly approximated by weighted polynomials of Erdős type. In this second part, we consider the question of how fast we can approximate our given f in our weighted sense, i.e. we are interested in the degree of our approximation, or more precisely, we estimate how fast

$$E_n[f]_{W,p} := \inf_{P \in \mathcal{P}_n} \|(f-P)W\|_{L_p(\mathbb{R})} \longrightarrow 0 \text{ as } n \longrightarrow \infty\,. \tag{5.1}$$

Here $0 < p \leq \infty$.

Direct and converse results in this field are commonly known as Jackson and Bernstein Theorems and are closely related to the smoothness properties of the approximated function.

5.1 Moduli of Continuity and Jackson Theorems

One of the classical tools used in describing the degree of smoothness of a function, is the modulus of continuity defined by:

$$\hat{w_r}(f,t) := \sup_{0<h\leq t} \|\Delta_h^r(f,x,\mathbb{R})\|_{L_\infty(\mathbb{R})} \quad t > 0, \tag{5.2}$$

where for an interval J , $r \geq 1$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$

$$\Delta_h^r(f, x, J) := \begin{cases} \sum_{i=0}^{r} \binom{r}{i} (-1)^i f\left(x + \frac{rh}{2} - ih\right) & , x \pm \frac{rh}{2} \in J \\ 0 & , \text{otherwise} \end{cases} \tag{5.3}$$

is the nth order symmetric difference of f. If J is not specified, it can be taken as $\mathbb{R}$.

Essentially, $w_{r,p}^{*}(f, .)$ measures how "continuous a function is". For weights on $\mathbb{R}$, analogues of Jackson -Bernstein theorems were initiated by Dzrbasjan but, more intensively studied by Freud in the $1960's - 1970's$ [42]. Freud's principle tools for proving Jackson type theorems was orthogonal polynomials and de la Vallee Poussin sums. Recently, Ditzian and Lubinsky have formulated and proved Jackson Theorems for Freud weights by a different method. Their technique does not use orthogonal polynomials but relies on an approach which goes back to Freud/Brudnyi and more recently, to DeVore, Leviatan and Yu [9, 23]. The approach involves approximating f by a spline (or piecewise polynomial), representing the piecewise polynomial in terms of certain characteristic functions and then, approximating the characteristic functions (in a suitable sense) by polynomials. Their increasing modulus is:

$$w_{r,p}^{*}(f, W, t) := \sup_{0<h\leq t} \|W(\Delta_h^r f)\|_{L_p(|x|\leq\sigma(h))} \tag{5.4}$$

$$+ \inf_{R \text{ of } \deg \leq r-1} \|(f - R)W\|_{L_p(|x|\geq\sigma(t))}.$$

Here,

$$\sigma(t) := \inf\{a_u : \frac{a_u}{u} \leq t\},\ t > 0. \tag{5.5}$$

We remark that their modulus is different, but equivalent to others used in the monograph of Ditzian and Totik[12]. Ditzian and Lubinsky then proved [11] :

Theorem 5.1.1. Let $0 < p \leq \infty$, $r \geq 1$. Let $W := \exp(-Q)$ where $Q : \mathbb{R} \longrightarrow \mathbb{R}$ is even, Q' exists in $(0, \infty)$, $xQ'(x)$ is positive and increasing there and for some $A,\ B,\ \lambda > 1$,

$$A \leq \frac{Q'(\lambda x)}{Q'(x)} \leq B,\ x \geq C. \tag{5.6}$$

Then

$$E_n[f]_{W,p} \leq C_1 \bar{w}_{r,p}\left(f, W, C_2 \frac{a_n}{n}\right).$$

Note here that (5.6) holds in particular for W_γ given by (1.2).

For the corresponding Erdös weight problem, we adopted the method of Ditzian and Lubinsky [11, 12, 27]. This method had the advantage of involving only hypotheses on Q', in contrast with the more complicated approach via orthogonal polynomials, that typically involved hypotheses on Q'' [12, 18, 36, 42]. In the Erdős weight context, some new features arise: The degree of approximation improves toward the endpoints of the Mhaskar-Saff interval, and to reflect this Nikolski-Timan-Brudnyi effect, we need a more complicated modulus of continuity and the proofs become more involved.

We need a suitable class of weights.

Definition 5.1.2. *Let $W := e^{-Q}$, where*

(a) $Q : \mathbb{R} \to \mathbb{R}$ is even, continuous, and Q' is positive in $(0, \infty)$.

(b) $xQ'(x)$ is strictly increasing in $(0, \infty)$ with right limit 0 at 0.

(c) The function

$$T(x) := \frac{xQ'(x)}{Q(x)} \tag{5.7}$$

is quasi-increasing in (C, ∞) for some $C > 0$, and

$$\lim_{x \to \infty} T(x) = \infty. \tag{5.8}$$

(d) $\exists\ C_1, C_2, C_3 > 0$ *such that*

$$\frac{yQ'(y)}{xQ'(x)} \leq C_1 \left(\frac{Q(y)}{Q(x)}\right)^{C_2}, \ y \geq x \geq C_3. \tag{5.9}$$

Then we write $W = e^{-Q} \in \mathcal{E}_1$.

Some Remarks

(a) We notice that $\mathcal{E}_1$ is a much larger class of weights than $\mathcal{E}_1^*$ defined in Definition 2.5.1. The main reason for this, is that here, we are not dependent on the correct bounds of the orthogonal polynomials, as we were in Part A.

(*b*) Much as in Part A, we need (*b*) to ensure the existence of the Mhaskar-Rahmanov-Saff number, a_u, defined by (1.8).

(*c*) The function $T(x)$ plays much the same role as $T^*(x)$ in Part A, i.e, it serves as a measure of the regularity of growth of $Q(x)$. For example for "nice" weights like $W_{k,\alpha}$ given by (1.4)

$$T(x) = \alpha x^\alpha \left[\prod_{j=1}^{k-1} \exp_j(x^\alpha)\right]$$

so that $T(x) \sim T^*(x)$ in this case.

(*d*) As in Part A, (5.9) is a weak regularity condition on T. See (2.13).

We next proceed to define our weighted modulus of continuity/smoothness.

Define for $t > 0$, $\sigma(t)$ given by (5.5). Recall that it has the form

$$\sigma(t) = \inf\{a_u : \frac{a_u}{u} \leq t\}.$$

Further, to reflect endpoint effects, we need our our increment, h, in (5.3) to depend on x, in particular on the function

$$\Phi_t(x) := \left|1 - \frac{|x|}{\sigma(t)}\right|^{\frac{1}{2}} + T(\sigma(t))^{-\frac{1}{2}}, \ x \in \mathbb{R}. \tag{5.10}$$

The function $\Phi_t(x)$ describes the improvement in the degree of approximation near $\pm a_{\frac{n}{2}}$, in much the same way that $\sqrt{1-x^2}$ does for weights on $[-1,1]$.

Set, for $t > 0$, $0 < p \leq \infty$ and $r \geq 1$

$$w_{r,p}(f, W, t) := \sup_{0<h\leq t} \| W \left(\Delta^r_{h\Phi_t(x)}(f)\right) \|_{L_p(|x|\leq\sigma(2t))} \tag{5.11(a)}$$

$$+ \inf_{R \text{ of deg} \leq r-1} \|(f-R)W\|_{L_p(|x|\geq\sigma(4t))}.$$

Further, we define its averaged cousin,

$$\overline{w}_{r,p}(f, W, t) := \left(\frac{1}{t}\int_0^t \| W \left(\Delta^r_{h\Phi_t(x)}(f)\right) \|^p_{L_p(|x|\leq\sigma(2t))} \, dh\right)^{\frac{1}{p}} \tag{5.11(b)}$$

$$+\inf_{R \text{ of deg} \leq r-1} \|(f-R)W\|_{L_p(|x| \geq \sigma(4t))}.$$

If $p=\infty$, we set: $w_{r,p}=\overline{w}_{r,p}$. Clearly,

$$\overline{w}_{r,p}(f,W,t) \leq w_{r,p}(f,W,t).$$

Our modulus consists of a main part and a tail. The main part involves rth symmetric differences over a suitable interval whilst the tail involves an error of weighted polynomial approximation over the remainder of the real line. One can think of the 'main' part of the modulus being controlled by the decreasing function, σ, which is essentially the inverse function of the function

$$u \longrightarrow \frac{a_u}{u}$$

w.. decays to 0 as $u \longrightarrow \infty$. A good way to view the function $\sigma(t)$, is that for purposes of approximation by polynomials of degree at most n, essentially $t = \frac{a_n}{n}$, the main part of the modulus is taken over the range $\left[-a_{\frac{n}{2}}, a_{\frac{n}{2}}\right]$ and the tail over $\mathbb{R}\backslash\left[-a_{\frac{n}{2}}, a_{\frac{n}{2}}\right]$. The tail is necessary because of the inability of $(P_n W)$, $P_n \in \mathcal{P}_n$ to approximate beyond $[-a_n\ a_n]$. The inf is also taken over polynomials of degree $\leq r-1$ to ensure that at least for $f \in \mathcal{P}_{r-1}$, $w_{r,p}(f,W,t) \equiv 0$ [28, 29]. It is possible to replace $\sigma(2t)$ by a somewhat larger term $\sigma(t) - At$ and $\sigma(4t)$ by a somewhat smaller term $\sigma(t) - Bt$, for suitable A, B in our modulus, under additional conditions on Q'. However, it hardly seems worth the effort, as the resulting modulus is almost certainly equivalent to the above one. As evidence of this, see Theorem 5.2.1.

We are ready to state our Jackson Theorems.

Theorem 5.1.3. *Let $W := e^{-Q} \in \mathcal{E}_1$. Let $r \geq 1$ and $0 < p \leq \infty$. Then for $f : \mathbb{R} \to \mathbb{R}$ for which $fW \in L_p(\mathbb{R})$, (and for $p = \infty$, we require f to be continuous, and fW to vanish at $\pm\infty$), we have for $n \geq C_3$,*

$$E_n[f]_{W,p} \leq C_1 \overline{w}_{r,p}(f,W,C_2\frac{a_n}{n}) \leq C_1 w_{r,p}\left(f,W,C_2\frac{a_n}{n}\right) \tag{5.12}$$

where C_j, $j=1,2,3$ do not depend on f or n.

Further, we need for later use the following:

Theorem 5.1.4. *For $n \geq 1$, let $\lambda(n) \in \left[\frac{4}{5}, 1\right]$. Then for $n \geq C_3$*

$$E_n[f]_{W,p} \leq C_1 \overline{w}_{r,p}\left(f, W, C_2\lambda(n)\frac{a_n}{n}\right) \tag{5.13}$$

where C_1, C_2 do not depend on n or f or $\{\lambda(n)\}$.

Moreover,

$$E_n[f]_{W,p} \leq C_1 \inf_{\lambda \in [\frac{4}{5},1]} \overline{w}_{r,p}\left(f, W, C_2\lambda\frac{a_n}{n}\right). \tag{5.14}$$

As our moduli are not monotone increasing in t, we also present a result involving the increasing modulus:

$$\begin{aligned} w^*_{r,p}(f, W, t) &:= \sup_{\substack{0<h\leq t \\ 0<\tau\leq L}} \| W\Delta^r_{\tau h\Phi_h(x)}(f, x, \mathbb{R}) \|_{L_p(|x|\leq\sigma(2h))} \\ &+ \inf_{P\in\mathcal{P}_{r-1}} \| (f-P)W \|_{L_p(|x|\geq\sigma(4t))}. \end{aligned} \tag{5.15}$$

Here L is a fixed (large enough) number independent of f, t.

Theorem 5.1.5. *Under the hypotheses of Theorem 5.1.3,*

$$E_n[f]_{W,p} \leq C_3 w^*_{r,p}\left(f, W, C_4\frac{a_n}{n}\right) \tag{5.16}$$

where C_j, $j = 3, 4$ do not depend on f or n.

It seems likely that one should only really need $\tau = L$ in the definition of $w^*_{r,p}$ but, we have only been able to prove this under additional conditions.

Set:

$$\begin{aligned} w^{\#}_{r,p}(f, W, t) &:= \sup_{0<h\leq t} \left\| W\Delta^r_{Lh\Phi_h(x)}(f, x, \mathbb{R}) \right\|_{L_p(|x|\leq\sigma(2h))} \\ &+ \inf_{P\in\mathcal{P}_{r-1}} \|(f-P) W\|_{L_p(|x|\geq\sigma(4t))}. \end{aligned} \tag{5.17}$$

Then we have:

Theorem 5.1.6. *Assume the hypotheses of Theorem 5.1.3 and further assume that Q''*

exists and is non-negative in $(0,\infty)$ *,and*

$$\frac{Q''(x)}{Q'(x)} \sim \frac{Q'(x)}{Q(x)},\ x \in (0,\infty). \tag{5.18}$$

Moreover, we assume that

$$T'(x) \leq C_1 \frac{T^2(x)}{x},\ x \geq C_1. \tag{5.19}$$

Then

$$E_n[f]_{W,p} \leq C_6 w^{\#}_{r,p}\left(f, W, C_5 \frac{a_n}{n}\right). \tag{5.20}$$

where C_j, $j = 5,6$ *do not depend on* f *or* n.

We note that the additional conditions (5.18) and (5.19) are certainly satisfied for $W_{k,\alpha}$ and $W_{A,\beta}$.

5.2 K-Functionals and Converse Theorems

While K-functionals were introduced in the context of interpolation of spaces, one of their most important applications has been in the analysis of moduli of continuity, and in converse theorems in approximation theory. J.Peetre first made the connection between his K-functional and the modulus of continuity in 1968.

The Ditzian-Totik rth order K-functional has the form

$$K^{*}_{r,p}(f, W, t^r) := \inf_{\substack{g \\ g^{(r-1)} \text{ locally absolutely} \\ \text{continuous}}} \left\{ \|(f-g)W\|_{L_p(\mathbb{R})} + t^r \left\|g^{(r)}W\right\|_{L_p(\mathbb{R})} \right\}. \tag{5.21}$$

Here, $t > 0$, $r \geq 1$ and $p \geq 1$.

We may think of the second term measuring the smooth part of f and the first part measuring the distance of f to that smooth part[12]. The idea is to prove inequalities of the form,

$$\underline{w}_{r,p}(f, W, \alpha t) \leq C_2 K^{*}_{r,p}(f, W, t^r) \leq C_3 \underline{w}_{r,p}(f, W, t), \tag{5.22}$$

for a suitable modulus, $\underline{w}_{r,p}$. Here, $\alpha > 0$ is fixed in advance , $C_1, C_2 > 0$, and t is small enough.

Under mild conditions on W, Ditzian and Totik established the fundamental equivalence of their modulus of continuity and the K-functional[12]. All they assumed was that Q is even, continuous, Q' is continuous and increasing in $(0,\infty)$ and

$$\frac{Q'(x+1)}{Q'(x)} \leq C_4, \quad x > 0.$$

In particular, this holds for $W_\gamma(x)$, $\gamma > 1$ and $W_{1,\alpha}(x)$.

Unfortunately, $K^{\hat{}} \equiv 0$ in L_p $(0 < p < 1)$ [10], so many others have introduced the concept of realisation for $0 < p < 1$ [17]. Set:

$$\overline{K}_{r,p}(f, W, t^r) := \inf_{P\in\mathcal{P}_n}\left\{\|(f-P)W\|_{L_p(\mathbb{R})} + t^r \left\|P^{(r)}W\right\|_{L_p(\mathbb{R})}\right\} \tag{5.23}$$

where the degree n is determined in terms of t by

$$n := \inf\{k\colon \frac{a_k}{k} \leq t\}.$$

Note that here, (compare [5.22]), the inf is taken over polynomials of suitable degree. Z.Ditzian and D.S Lubinsky then proved [11] that if W satisfies the hypotheses of Theorem 5.1.1(which are of course weaker than those of Ditzian/Totik) and omits a Markov-Bernstein Inequality, then (5.22) holds for $p \geq 1$ with $\underline{w}$ replaced by $w^{\hat{}}$ and further for $0 < p < 1$, (5.23) holds with $K^{\hat{}}$ replaced by $\overline{K}$ and with $\underline{w}$ replaced by $w^{\hat{}}$. This yielded converse theorems.

For our purposes, the formulations become more complicated.

We define a suitably modified realisation functional by

$$K_{r,p}(f, W, t^r) := \inf_{P\in\mathcal{P}_n}\left\{\|(f-P)W\|_{L_p(\mathbb{R})} + t^r \left\|P^{(r)}\Phi_t^r W\right\|_{L_p(\mathbb{R})}\right\}, \tag{5.24}$$

where $t > 0$, $0 < p \leq \infty$, and $r \geq 1$ are chosen in advance and

$$n = n(t) := \inf\left\{k : \frac{a_k}{k} \leq t\right\}. \tag{5.25}$$

Further we define the ordinary K-functional by

$$K_{r,p}^{*}\left(f,W,t^{r}\right):=\inf_{\substack{g\\ g^{(r-1)}\text{ locally absolutely}\\ \text{continuous}}}\left\{\left\|(f-g)\,W\right\|_{L_{p}(\mathbb{R})}+t^{r}\left\|g^{(r)}\Phi_{t}^{r}W\right\|_{L_{p}(\mathbb{R})}\right\}. \tag{5.26}$$

We begin with our equivalence result:

Theorem 5.2.1. *Let $W\in\mathcal{E}_1$, $L,\alpha>0$, $r\geq 1$ and $0<p\leq\infty$. Assume that there is a Markov-Bernstein inequality of the form*

$$\left\|R_{n}'\Phi_{\frac{a_n}{n}}W\right\|_{L_{p}(\mathbb{R})}\leq C\frac{n}{a_n}\left\|R_{n}W\right\|_{L_{p}(\mathbb{R})}\quad 0<p\leq\infty,\ R_{n}\in\mathcal{P}_{n}, \tag{5.27}$$

where $C\neq C\left(n,R_{n}\right)$. Then $\exists\ C_{1},C_{2},C_{3}>0$ independent of f and t such that for $t\in(0,t_{0})$,

$$(a)\ \ w_{r,p}\left(f,W,Lt\right)\leq C_{1}K_{r,p}\left(f,W,t^{r}\right)\leq C_{2}w_{r,p}\left(f,W,C_{3}t\right). \tag{5.28}$$

Moreover,

$$(b)\ \ w_{r,p}\left(f,W,t\right)\sim\overline{w}_{r,p}\left(f,W,t\right)\sim K_{r,p}\left(f,W,t^{r}\right) \tag{5.29}$$

uniformly in t and f.

$$(c)\ \ w_{r,p}\left(f,W,\alpha t\right)\leq C_{4}w_{r,p}\left(f,W,t\right). \tag{5.30}$$

Here, C_4 depends on α but not on f and s.

Remark.

The Markov inequality (5.27) was proved for $p=\infty$ in [32], and for $0<p<\infty$ in [17] for $W\in\mathcal{E}_1^{*}$ (see Definition 2.5.1).

Theorem 5.2.1 allows us to deduce a simpler Jackson theorem to Theorem 5.1.3:

Corollary 5.2.2. *Assume the hypotheses of Theorem 5.2.1. Then we have for $n\geq C_1$,*

$$E_{n}\left[f\right]_{W,p}\leq C_{2}\overline{w}_{r,p}\left(f,W,\frac{a_{n}}{n}\right)\leq C_{2}w_{r,p}\left(f,W,\frac{a_{n}}{n}\right). \tag{5.31}$$

Here, C_2 is independent of f and n.

We note that the point of this Corollary is that we have removed the constant from inside the moduli in (5.12).

We have the following converse theorems:

Theorem 5.2.3. *Assume the hypotheses of Theorem 5.2.1. Let $q = \min\{1, p\}$. For $0 < t < C$, determine $n = n(t)$ by (5.25) and let $l = [\log_2 n]$ =the largest integer $\leq \log_2 n$. Then we have,*

$$w_{r,p}(f, W, t)$$

$$\leq C_1 t^r \left[\sum_{k=-1}^{l} (l-k+1)^{\frac{rq}{2}} \left(\frac{2^k}{a_{2^k}} \right)^{rq} E_{2^k}[f]_{W,p}^q \right]^{\frac{1}{q}}. \tag{5.32}$$

where $C_1 \neq C_1(f, t)$ and where we set $E_{2^{-1}} = E_{2^0}$.

We deduce

Corollary 5.2.4. *Assume the hypotheses of Theorem 5.2.1. Then for every $0 < \alpha < r$ the following are equivalent:*

(*a*)

$$w_{r,p}(f, W, t) = O\left(t^{\alpha}\right),\ t \longrightarrow 0. \tag{5.33}$$

(*b*)

$$E_n[f]_{W,p} = O\left(\frac{a_n}{n}\right)^{\alpha},\ n \longrightarrow \infty. \tag{5.34}$$

Finally, we obtain estimates of our modulus in terms of $f^{(r)}$ and deduce the equivalence of the K-functional with the realisation functional for $p \geq 1$.

We need first:

Corollary 5.2.5. *Let $W \in \mathcal{E}_1$, $r \geq 1$, $0 < p \leq \infty$ and assume (5.27). Then $\forall$ nlarge enough and $\forall P_n \in \mathcal{P}_n$ satisfying*

$$\|(f - P_n) W\|_{L_p(\mathbb{R})} \leq L E_n[f]_{W,p} \tag{5.35}$$

for some $L \geq 1$, we have

$$\|(f - P_n) W\|_{L_p(\mathbb{R})} + \left(\frac{a_n}{n}\right)^r \left\| P_n \Phi^r_{\frac{a_n}{n}} W \right\|_{L_p(\mathbb{R})} \sim K_{r,p}\left(f, W, \left(\frac{a_n}{n}\right)^r\right). \tag{5.36}$$

Here, the constants in the $\sim$ relation depend on L but, are independent of n and f. We remark that in particular (5.36) holds for P_n^* the best approximation to f.

We deduce:

Corollary 5.2.6. *Let* $1 < p \leq \infty$ *and assume the hypotheses of Theorem 5.2.1.*

(*a*) *If* $f^{(r)}W \in L_p(\mathbb{R})$, *we have for* $t \in (0, C_2)$,

$$w_{r,p}(f, W, t) \leq C_1 t^r \left\| f^{(r)} \Phi_t^r W \right\|_{L_p(\mathbb{R})}, \tag{5.37}$$

Here $C_j \neq C_j(f, t)$, $j = 1, 2$.

(*b*) *We have for* $t \in (0, C_3)$,

$$1 \leq K_{r,p}^*(f, W, t) / K_{r,p}(f, W, t) \leq C_4. \tag{5.38}$$

Here $C_j \neq C_j(f, t)$, $j = 3, 4$.

5.3 A Marchaud Inequality

Finally, we present a classical property of our modulus, namely a Marchaud Inequality.

Theorem 5.2.7 (Marchaud Inequality).

Let $W \in \mathcal{E}_1$, $q = \min\{1, p\}$, $0 < p \leq \infty$, $r \geq 1$ *and assume* (5.27). *Then* $\forall t > 0$ *small enough*

$$w_{r,p}(f, W, t) \leq C_1 t^r \left[\int_t^{C_2} \frac{w_{r+1,p}(f, W, u)^q \left(\log_2\left(\frac{1}{t}\right)\right)^{\frac{rq}{2}}}{u^{rq}} du + \left(\log_2\left(\frac{1}{tr}\right)\right)^{\frac{rq}{2}} \|fW\|_{L_p(\mathbb{R})}^q \right]^{\frac{1}{q}}. \tag{5.39}$$

Here the C_j, $j = 1, 2$ are independent of f and t.

Chapter 6

Technical Estimates and some Inequalities

6.1 Technical Estimates

We present a series of technical estimates which we will need for later chapters.

Lemma 6.1.1. Let $W \in \mathcal{E}_1$. (a) For some C_j, $j = 1, 2, 3$, and $s \geq r \geq C_3$

$$(\frac{s}{r})^{C_2 T(r)} \leq \frac{Q(s)}{Q(r)} \leq (\frac{s}{r})^{C_1 T(s)}. \tag{6.1}$$

Moreover,

$$(\frac{s}{r})^{C_2 T(r)} \frac{T(s)}{T(r)} \leq \frac{sQ'(s)}{rQ'(r)} \leq \frac{T(s)}{T(r)} (\frac{s}{r})^{C_1 T(s)}. \tag{6.2}$$

(b) Give, $\delta > 0$, there exists C such that

$$T(y) \sim T(y(1 - \frac{\delta}{T(y)})),\ y \geq C. \tag{6.3}$$

(c) Given $A > 0$, the functions $Q'(u)u^{-A}$ and $Q(u)u^{-A}$ are quasi- increasing and increasing respectively for large enough u.

Proof. (a) Firstly, (6.1) follows from the identity

$$\log \frac{Q(s)}{Q(r)} = \int_r^s \frac{T(t)}{t} dt$$

and the fact that T is quasi-increasing. Then, the definition (5.7) of T gives (6.2).

(b) We can reformulate (5.9) as

$$\frac{T(y)}{T(x)} \leq C_1 (\frac{Q(y)}{Q(x)})^{C_2 - 1}.$$

Hence, for $x = y(1 - \frac{s}{T(y)})$, the quasi-increasing nature of T gives

$$C_4 \leq \frac{T(y)}{T(x)} \leq C_1 \exp((C_2 - 1) \int_x^y \frac{T(t)}{t} dt)$$

$$\leq C_1 \exp(C_5 T(y) \log \frac{y}{x}) \leq C_6.$$

Recall here that $T(y)$ is large for large y.

(c) From (6.2) if $s \geq r \geq C$,

$$\frac{Q'(s)s^{-A}}{Q'(r)r^{-A}} \geq \frac{T(s)}{T(r)} (\frac{s}{r})^{C_2 T(r) - 1 - A} \geq C_7.$$

Here we have used the quasi-monotonicity of T, and also that if C is large enough, then $C_2 T(r) - 1 - A \geq 0$. Similarly for $Q(s)s^{-A}$.□

Next some properties of a_u:

Lemma 6.1.2. Let $W \in \mathcal{E}_1$. (a) a_u is uniquely defined and continuous for $u \in (0, \infty)$, and is a strictly increasing function of u.

(b) For $u \geq C$,

$$(i) \quad a_u Q'(a_u) \sim u T(a_u)^{1/2} \tag{6.4}$$

$$(ii) \quad Q(a_u) \sim u T(a_u)^{-1/2} \tag{6.5}$$

(c) Given fixed $\beta > 0$, we have for large u,

$$\begin{aligned} &(i) \quad T(a_{\beta u}) \sim T(a_u) \\ &(ii) \quad Q(a_{\beta u}) \sim Q(a_u) \\ &(iii) \quad Q'(a_{\beta u}) \sim Q'(a_u) \end{aligned} \tag{6.6}$$

(d) Given fixed $\alpha > 1$,

$$\left|\frac{a_{\alpha u}}{a_u} - 1\right| \sim \frac{1}{T(a_u)} \tag{6.7}$$

from which it follows that

$$\frac{a_{\beta u}}{a_u} \longrightarrow 1,\ u \longrightarrow \infty\ \forall \beta > 0. \tag{6.8}$$

(e) If C_2 is as in (5.9), then for some $\delta > 0$,

$$T(a_u) \leq C_1 u^{2\left(\frac{C_2-1}{C_2+1}\right)} = C_1 u^{2(1-\delta)}. \tag{6.9}$$

Moreover, $\forall \varepsilon > 0$

$$a_u = o\left(u^{\varepsilon}\right),\ u \longrightarrow \infty. \tag{6.10}$$

(f) If $\alpha > 1$, then for large enough u,

$$\frac{Q(a_{\alpha u})}{Q(a_u)} \geq C_1 > 1. \tag{6.11}$$

(g) For some C_2, C_3, C_4, C_5, C_6, $u \geq C_3$ and $L \geq 1$,

$$\exp\left(C_4 \frac{\log(C_2 L)}{T(a_u)}\right) \geq \frac{a_{Lu}}{a_u} \geq 1 + C_5 \frac{\log(LC_6)}{T(a_{Lu})}. \tag{6.12}$$

(h) $\exists\, C_j,\ j = 6, 7, 8$ such that for $v \geq u \geq C_6$

$$\left(\frac{a_v}{a_u}\right) \leq C_6 \left(\frac{v}{u}\right)^{\frac{C_7}{T(a_u)}}, \tag{6.13}$$

and

$$\left(\frac{a_v}{v}\right) / \left(\frac{a_u}{u}\right) \leq C_6 \left(\frac{v}{u}\right)^{\frac{C_7}{T(a_u)}-1}. \tag{6.14}$$

In particular, given $\varepsilon > 0$, we have for $v \geq u \geq C_6$

$$\left(\frac{a_v}{a_u}\right) \leq C_6 \left(\frac{v}{u}\right)^{\varepsilon}, \tag{6.15}$$

$$\left(\frac{a_v}{v}\right) / \left(\frac{a_u}{u}\right) \leq C_6 \left(\frac{v}{u}\right)^{\varepsilon-1}. \tag{6.16}$$

Proof. (a) The function $u \to a_u$ is the inverse of the strictly increasing continuous function

$$a \to \frac{2}{\pi} \int_0^1 atQ'(at) \frac{dt}{\sqrt{1-t^2}},\ a \in (0, \infty),$$

which has right limit 0 at 0 and limit ∞ at ∞. (Note that this function is continuous even if Q' is not). So the assertion follows.

(b) For u so large that $T(a_u) > 2$, we have

$$\frac{u}{a_u Q'(a_u)} = \frac{2}{\pi}\left[\int_0^{1-1/T(a_u)} + \int_{1-1/T(a_u)}^1\right] \frac{a_u t Q'(a_u t)}{a_u Q'(a_u)} \frac{dt}{\sqrt{1-t^2}}$$

$$\leq \frac{2}{\pi} T(a_u)^{1/2} \int_0^{1-1/T(a_u)} \frac{a_u Q'(a_u t)}{a_u Q'(a_u)} dt + \frac{2}{\pi} \int_{1-1/T(a_u)}^1 \frac{dt}{\sqrt{1-t^2}}$$

$$\leq \frac{2}{\pi} T(a_u)^{1/2} \frac{Q(a_u) - Q(0)}{a_u Q'(a_u)} + \frac{4}{\pi} T(a_u)^{-1/2}$$

$$\leq \frac{4}{\pi} T(a_u)^{1/2} \frac{Q(a_u)}{a_u Q'(a_u)} + \frac{4}{\pi} T(a_u)^{-1/2} = \frac{8}{\pi} T(a_u)^{-1/2}.$$

Here we also need u so large that $Q(a_u) \geq | Q(0) |$. So we have

$$a_u Q'(a_u) \geq \frac{\pi}{8} u T(a_u)^{1/2}.$$

In the other direction, (6.2) gives for large u,

$$\frac{u}{a_u Q'(a_u)} = \frac{2}{\pi} \int_0^1 \frac{a_u t Q'(a_u t)}{a_u Q'(a_u)} \frac{dt}{\sqrt{1-t^2}}$$

$$\geq C_1 \int_{1/2}^{1} \frac{T(a_u t)}{T(a_u)} t^{C_1 T(a_u)} \frac{dt}{\sqrt{1-t^2}}$$

$$\geq C_2 \frac{T(a_u(1-\frac{1}{T(a_u)}))}{T(a_u)} (1-\frac{1}{T(a_u)})^{C_1 T(a_u)} \int_{1-1/T(a_u)}^{1} \frac{dt}{\sqrt{1-t^2}}$$

$$\geq C_3 T(a_u)^{-1/2}.$$

Here we have used (6.3) and the quasi-monotonicity of T. So we have (6.4)(i). Then (6.5) follows from the definition of T.

(c) We can assume $\beta > 1$. Then by (6.5), and quasi-monotonicity of T,

$$C_1 \leq \frac{T(a_{\beta u})}{T(a_u)} \sim \left[\frac{\beta u}{Q(a_{\beta u})}\right]^2 / \left[\frac{u}{Q(a_u)}\right]^2 \leq \beta^2.$$

The rest of (6.6) follows from (6.4) and (6.5).

(d) Now

$$\alpha u = \frac{2}{\pi} \int_0^1 a_{\alpha u} t Q'(a_{\alpha u} t) \frac{dt}{\sqrt{1-t^2}}$$

$$\geq \frac{2}{\pi} \int_{a_u/a_{\alpha u}}^{1} a_u Q'(a_u) \frac{dt}{\sqrt{1-t^2}}$$

$$\geq C_2 u T(a_u)^{1/2} (1 - \frac{a_u}{a_{\alpha u}})^{1/2}$$

by (6.4). Hence,

$$1 - \frac{a_u}{a_{\alpha u}} \leq C_3 / T(a_u).$$

In the other direction,

$$\alpha u = \frac{2}{\pi} \left[\int_0^{a_u/a_{\alpha u}} + \int_{a_u/a_{\alpha u}}^{1}\right] a_{\alpha u} t Q'(a_{\alpha u} t) \frac{dt}{\sqrt{1-t^2}}$$

$$\leq \frac{2}{\pi} \int_0^{a_u/a_{\alpha u}} a_{\alpha u} t Q'(a_{\alpha u} t) \frac{dt}{\sqrt{1-(\frac{a_{\alpha u} t}{a_u})^2}} + \frac{2}{\pi} a_{\alpha u} Q'(a_{\alpha u}) \int_{a_u/a_{\alpha u}}^{1} \frac{dt}{\sqrt{1-t}}$$

$$\leq \frac{a_u}{a_{\alpha u}} \left[\frac{2}{\pi} \int_0^1 a_u s Q'(a_u s) \frac{ds}{\sqrt{1-s^2}}\right] + \frac{4}{\pi} a_{\alpha u} Q'(a_{\alpha u}) (1 - \frac{a_u}{a_{\alpha u}})^{1/2}$$

$$\leq u + C u T(a_u)^{1/2} (1 - \frac{a_u}{a_{\alpha u}})^{1/2}$$

by (6.4) and (6.6)(i). Then

$$1-\frac{a_u}{a_{\alpha u}}\geq(\frac{\alpha-1}{C})^2\frac{1}{T(a_u)}.$$

(e) We apply (5.9) with $y=a_u$ and $x=C_3$, so that

$$a_uQ'(a_u)\leq C_4Q(a_u)^{C_2}$$

$$\Rightarrow uT(a_u)^{1/2}\leq C_5(uT(a_u)^{-1/2})^{C_2}.$$

Rearranging this gives (6.9). Finally, using (6.4) gives for any $A>0$,

$$Ca_u^A\leq Q\left(a_u\right)\sim uT\left(a_u\right)^{-\frac{1}{2}}\Longrightarrow\frac{\left(a_u\right)^A}{u}\longrightarrow 0,\ u\longrightarrow\infty.$$

So (6.10) follows.

(f) For large enough u,

$$\frac{Q(a_{\alpha u})}{Q(a_u)}=\exp(\int_{a_u}^{a_{\alpha u}}\frac{T(t)}{t}dt)$$

$$\geq\exp(C_6T(a_u)\log(\frac{a_{\alpha u}}{a_u}))\geq\exp(C_7)>1,$$

by (d) of this lemma.

(g) From (5.9) with $y=a_{Lu}$ and $x=a_u$,

$$\frac{T(a_{Lu})}{T(a_u)}\leq C(\frac{Q(a_{Lu})}{Q(a_u)})^{C_2-1}.$$

This forces $C_2>1$, as the left-hand side $\to\infty$ as $L\to\infty$. Then, with the constants in $\sim$ independent of L, (6.5) gives

$$\frac{Q(a_{Lu})}{Q(a_u)}\sim\frac{LuT(a_{Lu})^{-1/2}}{uT(a_u)^{-1/2}}$$

$$\geq CL(\frac{Q(a_{Lu})}{Q(a_u)})^{-(C_2-1)/2}$$

$$\Rightarrow\frac{Q(a_{Lu})}{Q(a_u)}\geq CL^{\frac{2}{1+C_2}}.$$

Then using (6.1),

$$(\frac{a_{Lu}}{a_u})^{C_1T(a_{Lu})}\geq CL^{\frac{2}{1+C_2}}$$

and the right inequality in (6.12) follows. In the other direction, (6.1) and then (6.5) give

$$\frac{a_{Lu}}{a_u} \le \left(\frac{Q(a_{Lu})}{Q(a_u)}\right)^{\frac{1}{C_2 T(a_u)}}$$

$$\le \left(C_1 \frac{LuT(a_{Lu})^{\frac{-1}{2}}}{uT(a_u)^{\frac{-1}{2}}}\right)^{\frac{1}{C_2 T(a_u)}} \le (C_3 L)^{\frac{1}{C_2 T(a_u)}}$$

Here the constants are independent of L and u. Then the left inequality in (6.12) follows. It remains to show (h). Now by (6.5) and then (6.1)

$$C_1 \frac{v}{u} \ge \frac{vT(a_v)^{\frac{-1}{2}}}{uT(a_u)^{\frac{-1}{2}}} \sim \frac{Q(a_v)}{Q(a_u)} \ge \left(\frac{a_v}{a_u}\right)^{C_2 T(a_u)},$$

which implies

$$\left(\frac{a_v}{a_u}\right) \le C_3 \left(\frac{v}{u}\right)^{\frac{C_4}{T(a_u)}}.$$

So we have (6.13) and then (6.14 – 6.16) also follow. □

Lemma 6.1.3 (Infinite-Finite-Range inequality). Let $W \in \mathcal{E}_1, 0 < p \le \infty$ and $s > 1$. Then for some L, C_1, $C_2 > 0$, $n \ge 1$, and $P \in \mathcal{P}_n$,

$$\| PW \|_{L_p(\mathbb{R})} \le C_1 \| PW \|_{L_p(-a_{sn},\, a_{sn})}. \tag{6.17}$$

Moreover,

$$\| PW \|_{L_p(|x| \ge a_{sn})} \le C_1 e^{-C_2 nT(a_n)^{-1/2}} \| PW \|_{L_p(-a_{sn},\, a_{sn})}. \tag{6.18}$$

Remark: Note that (6.9) shows that for some $C_3 > 0$, and large enough n,

$$nT(a_n)^{-1/2} \ge n^{C_3}.$$

We provide a proof as those in the literature [26], [37], [39], ...don't quite match our needs/ hypotheses.

Proof.

We may change Q in a finite interval without affecting (6.17), (6.18) apart from increasing the constants. Note too, that the affect on a_u is marginal, and is absorbed into the fact that $s > 1$. Thus, we may assume that Q' is continuous in $[-1,1]$. This, and the strict monotonicity of $tQ'(t)$ in $(0,\infty)$, allow us to apply existing sup-norm inequalities to deduce that for $P \in \mathcal{P}_n$,

$$\| PW \|_{L_\infty(\mathbb{R})} = \| PW \|_{L_\infty[-a_{sn},\, a_{sn}]} .$$

For a precise reference, see [48] and [16, Theorem 4.5]. Moreover, the proof of Lemma 5.1 in [26, pp.231-232] gives without change

$$| PW |^p (a_{ns}x) \leq \frac{1}{\pi}\frac{2x}{x-1}\int_{-1}^{1} | PW |^p (a_{ns}t)dt, \; x > 1. \tag{6.19}$$

Let $< x >$ denote the greatest integer $\leq x$. Let δ be small and positive, let $l := < \delta n >$ and let $T_l(x)$ denote the Chebyshev polynomial of degree l. Using the identity

$$T_l(x) = \frac{1}{2}[(x+\sqrt{x^2-1})^l + (x-\sqrt{x^2-1})^l], \; x > 1, \tag{6.20}$$

it is not difficult to see that

$$T_l(x) \geq \begin{Bmatrix} \frac{1}{2}\exp(\frac{l}{\sqrt{2}}\sqrt{x-1}), x \in (1, \frac{9}{8}) \\ \frac{1}{2}x^l, x \geq 1 \end{Bmatrix}. \tag{6.21}$$

We now let $m := n + l = n + < \delta n >$, $m' := n + 2l = n + 2 < \delta n >$ and apply (6.19) to $P(x)\, T_l\left(\frac{x}{a_m}\right) \in \mathcal{P}_m$. We obtain for $x > 1$,

$$| PW |^p (a_m x) \leq T_l(x)^{-p}\frac{1}{\pi}\frac{2x}{x-1}\int_{-1}^{1} | PW |^p (a_m t)dt.$$

Replacing $a_m x$ by y, and integrating from $a_{m'}$ to ∞ gives

$$\int_{a_{m'}}^{\infty} | PW |^p (y)dy \leq \left(\int_{-a_m}^{a_m} | PW |^p (s)ds\right)\left(\frac{2}{\pi}\int_{a_{m'}}^{\infty}\frac{y}{y-a_m}T_l(\frac{y}{a_m})^{-p}\frac{dy}{a_m}\right).$$

Here using (6.21),

$$\int_{a_{m'}}^{\infty}\frac{y}{y-a_m}T_l(\frac{y}{a_m})^{-p}\frac{dy}{a_m} = \int_{a_{m'}/a_m}^{\infty}\frac{x}{x-1}T_l(x)^{-p}dx$$

$$\leq C\left(\int_{a_{m'}/a_m}^{9/8}\frac{1}{x-1}\exp(-\frac{lp}{\sqrt{2}}\sqrt{x-1})dx+\int_{9/8}^{\infty}x^{-lp}dx\right)$$

$$\leq C_1(\log(\frac{8}{\frac{a_{m'}}{a_m}-1})\exp(-C_2lp(\frac{a_{m'}}{a_m}-1)^{1/2})+(\frac{9}{8})^{-lp})$$

$$\leq C_3\exp(-C_4nT(a_n)^{-1/2}).$$

Here we have used (6.7) and our choice of l. Now if δ is small enough, $m'\leq sn$. Then (6.18) follows easily, and in turn yields (6.17).□

Lemma 6.1.4. Let $W\in\mathcal{E}_1$, $t>0$ be small enough and $\beta>0$. Put for u large enough

$$t=\frac{\beta a_u}{u}.$$

Set

$$n:=n\,(t)=\inf\{k:\frac{a_k}{k}\leq\frac{\beta a_u}{u}\}. \tag{6.22}$$

Then

(a)

$$\frac{a_n}{n}\leq\frac{\beta a_u}{u}<\frac{a_{n-1}}{n-1}. \tag{6.23}$$

(b)

$$\frac{a_n}{n}\leq\frac{\beta a_u}{u}<2\frac{a_n}{n}. \tag{6.24}$$

(c)

$$u\sim n. \tag{6.25}$$

Proof. (6.23) follows from the definition of n. (6.24) follows from (6.23) as

$$a_{n-1}<a_n.$$

To show (6.25), we first show that $\exists\,\alpha>0$ such that

$$u\leq\alpha n. \tag{6.26}$$

Suppose first that $u \geq n$. Using (6.23) and Lemma 6.1.2 (h), there exists $C > 0$ such that

$$\frac{1}{\beta} \leq \frac{a_u}{u} / \frac{a_n}{n} \leq C \left(\frac{u}{n}\right)^{\frac{-1}{2}}$$

which implies (6.26). Suppose $u \leq n$. Then (6.26) follows with $\alpha = 1$. So it suffices to show that $\exists\, C_1 > 0$ such that

$$u \geq C_1 n.$$

Well, if $n - 1 \geq u$ by (6.23) and Lemma 6.1.2 (h), there exists $C_2 > 0$ such that

$$\beta \leq \frac{a_{n-1}}{n-1} / \frac{a_u}{u} \leq C_2 \left(\frac{n-1}{u}\right)^{-\frac{1}{2}}$$

which implies

$$u \geq C_3 n$$

for some $C_3 > 0$. Further, if $u \geq n - 1$ were done.$\square$

We next present various estimates involving the functions σ, Φ_t, and differences. Throughout, we assume that $W = e^{-Q} \in \mathcal{E}_1$.

Recall that:

$$\sigma(t) := \inf\{a_u : \frac{a_u}{u} \leq t\}, \quad t > 0$$

and

$$\Phi_t(x) := \sqrt{\left|1 - \frac{|x|}{\sigma(t)}\right| + T(\sigma(t))^{-\frac{1}{2}}}, x > 0.$$

Lemma 6.1.5. (a) There exists s_0, v_0 such that for $s \in (0, s_0)$ and $v \geq v_0$, we can write $s = \frac{a_v}{v}$, where $v \geq v_0$. Moreover, we can write

$$\sigma(s) = \sigma\left(\frac{a_v}{v}\right) = a_{\beta(v)} \tag{6.27}$$

where

$$1 \geq \sigma\left(\frac{a_v}{v}\right) / a_v = a_{\beta(v)}/a_v \geq 1 - C/T(a_v). \tag{6.28}$$

In particular,

$$\lim_{v\to\infty}\frac{\beta(v)}{v}=1. \tag{6.29}$$

(b) There exist $C_1, C_2>0$ such that for $\frac{s}{2}\le t\le s$, and $s\le C_1$,

$$1\le\frac{\sigma(t)}{\sigma(s)}\le 1+\frac{C_2}{T(\sigma(t))}. \tag{6.30}$$

Further, for t small enough, we have for some $\varepsilon>0$,

$$T(\sigma(t))=O\left(\frac{\sigma(t)}{t}\right)^{2-\varepsilon}. \tag{6.31}$$

(c) There exist C_3, C_4 independent of s, t, x, such that for $0<t<s\le C_3$,

$$\Phi_s(x)\le C_4\Phi_t(x),\ |x|\le\sigma(s). \tag{6.32}$$

(d) There exists C_5, such that for $0<s\le C_5$, and $\frac{s}{2}\le t\le s$,

$$\Phi_s(x)\sim\Phi_t(x),\ x\in\mathbb{R}. \tag{6.33}$$

(e) Uniformly for $x\in\mathbb{R}$ and $n\ge 1$,

$$\Phi_{\frac{a_n}{n}}(x)\sim\sqrt{\left|1-\frac{|x|}{a_n}\right|+T(a_n)^{-1/2}}. \tag{6.34}$$

Further given $\beta>0$ and $t>0$,we have for some $C_6, C_7>0$ and for all $x\in\mathbb{R}$,

$$\Phi^{\beta}_{\frac{a_n}{n}}(x)\ge C_6T(a_n)^{-\frac{\beta}{2}} \tag{6.35}$$

and

$$\Phi^{\beta}_t(x)\ge C_7T(\sigma(t))^{-\frac{\beta}{2}}. \tag{6.36}$$

Proof. (a) The existence of v for the given s, follows from the fact that $u\to a_u$ is continuous

and

$$\frac{a_u}{u} \to 0, \ u \to \infty.$$

See (6.10).

The continuity of a_u allows us to write $\sigma(s) = a_{\beta(v)}$, some $\beta(v)$. Since

$$\sigma(\frac{a_v}{v}) \le a_v,$$

the left inequality in (6.28) follows. For the other direction, we note that by definition of $\sigma\left(\frac{a_v}{v}\right)$ and $\beta(v)$, we have $\beta(v) \le v$ and

$$\frac{a_{\beta(v)}}{\beta(v)} \le \frac{a_v}{v}$$

so,

$$1 \le \frac{v}{\beta(v)} \le \frac{a_v}{a_{\beta(v)}} \le \left(\frac{Q(a_v)}{Q\left(a_{\beta(v)}\right)}\right)^{\frac{1}{2}}$$

for large enough v, by (6.1). Using (6.5), we obtain

$$1 \le \frac{v}{\beta(v)} \le C\left(\frac{vT(a_v)^{\frac{-1}{2}}}{\beta(v)T\left(a_{\beta(v)}\right)^{\frac{-1}{2}}}\right)^{\frac{1}{2}} \le C_1\left(\frac{v}{\beta(v)}\right)^{\frac{1}{2}}.$$

It follows that $v \le C_2\beta(v)$ and so $v \sim \beta(v)$. Then

$$1 \le \frac{v}{\beta(v)} \le \frac{a_v}{a_{\beta(v)}} \longrightarrow 1, \ v \longrightarrow \infty$$

by (6.7), so we have (6.29). Then (6.7) also gives the right inequality in (6.28).

(b) Write $s = \frac{a_u}{u}$ and $t = \frac{a_v}{v}$. Then as σ is decreasing,

$$1 \ge \frac{\sigma(s)}{\sigma(t)} = \frac{a_{\beta(u)}}{a_{\beta(v)}}$$

If we can show that

$$u \sim v \tag{6.37}$$

then (6.7) gives

$$1 \geq \frac{\sigma(s)}{\sigma(t)} \geq 1 - \frac{C_3}{T(a_v)}$$

which together with (6.6)(i) gives (6.30). We proceed to establish (6.37). Suppose that it is not true, say, for example, we can have

$$\frac{u}{v} \to \infty.$$

For the corresponding s, t, our hypothesis is

$$\frac{1}{2} \leq \frac{t}{s} = \frac{a_v}{a_u}\frac{u}{v} \leq 1.$$

Then

$$\frac{a_v}{a_u} \to 0 \tag{6.38}$$

and (6.1) gives

$$\frac{Q(a_u)}{Q(a_v)} \geq (\frac{a_u}{a_v})^{C_4 T(a_v)} \geq (\frac{a_u}{a_v})^2,$$

for large u, v. But from (6.5),

$$(\frac{a_u}{a_v})^2 \leq \frac{Q(a_u)}{Q(a_v)} \sim \frac{uT(a_u)^{-1/2}}{vT(a_v)^{-1/2}}$$

$$\leq C_5 \frac{u}{v} \leq C_6 \frac{a_u}{a_v},$$

again by our hypotheses on s, t. This contradicts (6.38). So we have (6.37) and hence (6.30).

Finally (6.9), (6.27) and (6.28) gives for some $\varepsilon > 0$,

$$T(\sigma(t)) \leq T(a_u) = O\left(u^{2-\varepsilon}\right) = O\left(\frac{\sigma(t)}{t}\right)^{2-\varepsilon}$$

so that we have (6.31).

(c) Let $\delta > 0$ be fixed. Firstly for $1 - |x|/\sigma(s) \geq \delta/T(\sigma(s))$,

$$\Phi_s(x) \sim \sqrt{1 - \frac{|x|}{\sigma(s)}} \leq \sqrt{1 - \frac{|x|}{\sigma(t)}} \leq \Phi_t(x).$$

Next, for $|1-|x|/\sigma(s)|\leq \delta/T(\sigma(s))$,

$$\Phi_s(x)\sim T(\sigma(s))^{-1/2}.$$

This is bounded by $C_1\Phi_t(x)$ if $|1-|x|/\sigma(t)|\geq \delta/T(\sigma(s))$, for a fixed $\delta>0$. Otherwise, we have $|1-|x|/\sigma(s)|\leq \delta/T(\sigma(s))$ and $|1-|x|/\sigma(t)|\leq \delta/T(\sigma(s))$, so

$$|1-\frac{\sigma(t)}{\sigma(s)}|=|(1-\frac{|x|}{\sigma(s)})-\frac{|x|}{\sigma(s)}(\frac{\sigma(t)}{|x|}-1)|$$

$$\leq C_2\delta/T(\sigma(s)).$$

If δ is small enough, we deduce from (6.7) and (6.9) that

$$T(\sigma(t))\sim T(\sigma(s))$$

and again (6.32) follows.

(d) Write $s=\frac{a_u}{u}$ and $t=\frac{a_v}{v}$. Then we have (6.37), so

$$\left|1-\frac{|x|}{\sigma(t)}\right|=\left|1-\frac{|x|}{\sigma(s)}+\left[\frac{|x|}{\sigma(s)}-1+1\right](1-\frac{\sigma(s)}{\sigma(t)})\right|$$

$$\leq\left|1-\frac{|x|}{\sigma(s)}\right|\left[1+O(\frac{1}{T(\sigma(s))}\right]+O(\frac{1}{T(\sigma(s))}).$$

Then we obtain for $x\in\mathbb{R}$,

$$\left|1-\frac{|x|}{\sigma(t)}\right|^{\frac{1}{2}}\leq C\Phi_s(x).$$

Also $T(\sigma(t))\sim T(\sigma(s))$, so

$$\Phi_t(x)\leq C_1\Phi_s(x).$$

The converse inequality follows similarly.

(e) By (a) of this lemma, we can write

$$\sigma(\frac{a_n}{n})=a_{\beta(n)}=a_{n(1+o(1))}.$$

Recall that

$$\Phi_{\frac{a_n}{n}}(x) = \sqrt{\left| 1 - \frac{|x|}{\sigma(\frac{a_n}{n})} \right| + T(\sigma(\frac{a_n}{n}))^{-1/2}}.$$

Here by (6.6)(i) and (a) of this lemma,

$$T(\sigma(\frac{a_n}{n})) \sim T(a_n)$$

and much as in (d),

$$1 - \frac{|x|}{\sigma(\frac{a_n}{n})} \sim 1 - \frac{|x|}{a_n}$$

for large n and $|x| \leq a_{n/2}$ or $|x| \geq a_{\frac{n}{2}}$. In the range $a_{n/2} \leq |x| \leq a_n$, both the left and right-hand side of (6.34) are $\sim T(a_n)^{-1/2}$.

Finally, note that (6.35) and (6.36) follow from the definition of Φ_t and (6.34). $\square$

Lemma 6.1.6. (a) For $0 < s < t \leq C$,

$$T(\sigma(t))\left(1 - \frac{\sigma(t)}{\sigma(s)}\right) \leq C_1 \log\left(2 + \frac{t}{s}\right). \tag{6.39}$$

(b) For $0 < s < t \leq C$,

$$\sup_{x\in\mathbb{R}} \frac{\Phi_s(x)}{\Phi_t(x)} \leq C_2 \sqrt{\log\left(2 + \frac{t}{s}\right)}. \tag{6.40}$$

Hence, given $\gamma > 0$,

$$\sup_{x\in\mathbb{R}} \left(\frac{s}{t}\right)^{\gamma} \frac{\Phi_s(x)}{\Phi_t(x)} \leq C_3. \tag{6.41}$$

Further if $m \leq n$ and $n, m \geq C_4$, then

$$\sup_{x\in\mathbb{R}} \frac{\Phi_{\frac{a_n}{n}}(x)}{\Phi_{\frac{a_m}{m}}(x)} \leq C_5 \sqrt{\log\left(2 + \frac{n}{m}\right)}. \tag{6.42}$$

Proof

(a) We write $s = \frac{a_u}{u}$ and $t = \frac{a_v}{v}$. Note (with the notation of Lemma 6.1.5) that

$$a_{\beta(u)} = \sigma(s) \geq \sigma(t) = a_{\beta(v)},$$

so $\beta(u) \geq \beta(v)$. Using the inequality

$$1 - u \leq \log \frac{1}{u},\ u \in (0,1]$$

we obtain

$$\begin{aligned} 1 - \frac{\sigma(t)}{\sigma(s)} &\leq \log \frac{\sigma(s)}{\sigma(t)} = \log \frac{a_{\beta(u)}}{a_{\beta(v)}} \\ &\leq C_1 \frac{\log\left(C\frac{\beta(u)}{\beta(v)}\right)}{T\left(a_{\beta(v)}\right)} = C_1 \frac{\log\left(C\frac{\beta(u)}{\beta(v)}\right)}{T(\sigma(t))} \end{aligned} \tag{6.43}$$

by (6.12). Next, $\beta(u) = u(1 + o(1))$ and similarly for $\beta(v)$, so it suffices to show that

$$\log \frac{u}{v} \leq C_2 \log\left(2 + \frac{t}{s}\right). \tag{6.44}$$

But from (6.1) for $s < t$ and small t and then from (6.5),

$$\frac{u}{v} / \frac{t}{s} = \frac{a_u}{a_v} \leq \left(\frac{Q(a_u)}{Q(a_v)}\right)^{\frac{1}{2}}$$

$$\leq C_1 \left(\frac{uT(a_u)^{\frac{-1}{2}}}{vT(a_v)^{\frac{-1}{2}}}\right)^{\frac{1}{2}} \leq C_2 \left(\frac{uT\left(a_{\beta(u)}\right)^{\frac{-1}{2}}}{vT\left(a_{\beta(v)}\right)^{\frac{-1}{2}}}\right)^{\frac{1}{2}} \leq C_3 \left(\frac{u}{v}\right)^{\frac{1}{2}}$$

as $\beta(u) \geq \beta(v)$. So

$$\left(\frac{u}{v}\right)^{\frac{1}{2}} \leq C_4 \frac{t}{s}$$

and we have (6.44).

(b) Now if $x \geq 0$,

$$\left|1 - \frac{x}{\sigma(s)}\right| \leq \left|1 - \frac{x}{\sigma(t)}\right| + \frac{x}{\sigma(t)} \left|1 - \frac{\sigma(t)}{\sigma(s)}\right|$$

$$\le \left|1-\frac{x}{\sigma(t)}\right| + \left(\left|1-\frac{x}{\sigma(t)}\right|+1\right)\left|1-\frac{\sigma(t)}{\sigma(s)}\right|.$$

Using part (a), we obtain

$$\left|1-\frac{x}{\sigma(s)}\right|^{\frac{1}{2}} \le C_{12}\Phi_t(x)\sqrt{\log\left(2+\frac{t}{s}\right)}.$$

Since $\sigma(s) \ge \sigma(t)$ also

$$T(\sigma(s))^{\frac{-1}{2}} \le T(\sigma(t))^{\frac{-1}{2}}.$$

So (6.40) holds. Then (6.41) and (6.42) follow.□

Lemma 6.1.7. (a) Let $L, > 0$. Uniformly for $n \ge 1$, and $|x|, |y| \le a_u$, such that

$$|x-y| \le L\frac{a_u}{u}\sqrt{\left|1-\frac{|y|}{a_u}\right|}, \tag{6.45}$$

we have

$$W(x) \sim W(y) \tag{6.46}$$

and

$$1-\frac{|x|}{a_{2u}} \sim 1-\frac{|y|}{a_{2u}}. \tag{6.47}$$

(b) Furthermore, if $s > 0$ then uniformly for $u \ge 1$ and $|x|, |y| \le a_{us}$ such that

$$|x-y| \le L\frac{a_u}{u}T(a_u)^{\frac{-1}{2}}$$

we have

$$W(x) \sim W(y).$$

(c) Let $L, M > 0$. For $t \in (0, t_0)$, $|x|, |y| \le \sigma(Mt)$ such that

$$|x-y| \le Lt\Phi_t(x), \tag{6.48}$$

we have (6.46) and

$$\Phi_t(x) \sim \Phi_t(y). \tag{6.49}$$

(d) Recall the difference operator Δ_h^r defined by (5.3). Then we have $\forall x \in \mathbb{R}$, $\forall P \in \mathcal{P}_{r-1}$, $r \geq 1$, $\beta \in \mathbb{R}$ and $t > 0$

$$\begin{aligned} &(i)\ \Delta^r_{h\Phi_t^\beta(x)} P(x) \equiv 0. \\ &(ii)\ r!\left(h\Phi_t^\beta(x)\right)^r = \Delta^r_{h\Phi_t^\beta(x)} x^r. \end{aligned} \tag{6.50}$$

Proof

(a) It suffices to prove (6.46), (6.47) for large u. Moreover, (6.46) and (6.47) are immediate for $|x| \leq C$, and large u. Let us suppose that $C \leq x \leq y \leq x + L\frac{a_u}{u}\sqrt{|1 - \frac{|x|}{a_u}|}$. Then as $Q'(s)$ is quasi-increasing for large s,

$$0 \leq Q(y) - Q(x) \leq C_1 Q'(y)(y - x).$$

We have then (6.46) for

$$y - x = O(\frac{1}{Q'(y)}). \tag{6.51}$$

We shall show that

$$a_u Q'(y)\sqrt{|1 - \frac{y}{a_u}|} \leq C_2 u, \tag{6.52}$$

so that (6.45) implies (6.51) and hence (6.46). If firstly, $0 < y \leq \frac{a_u}{2}$, then

$$a_u Q'(y)\sqrt{|1 - \frac{y}{a_u}|} \leq C_3 a_u Q'(y) \int_{1/2}^1 \frac{dt}{\sqrt{1 - t^2}}$$

$$\leq C_4 \int_{1/2}^1 a_u t Q'(a_u t) \frac{dt}{\sqrt{1 - t^2}} \leq C_5 u.$$

If on the other hand, $\frac{a_u}{2} \leq y \leq a_u$,

$$a_u Q'(y)\sqrt{|1 - \frac{y}{a_u}|} \leq C_6 \int_{y/a_u}^1 a_u t Q'(a_u t) \frac{dt}{\sqrt{1 - t^2}} \leq C_7 u.$$

So we have (6.52) in all cases. Next from (6.45) and as $y \leq a_u$,

$$1 \leq \frac{1 - \frac{x}{a_{2u}}}{1 - \frac{y}{a_{2u}}} = 1 + \frac{y - x}{a_{2u}(1 - \frac{y}{a_{2u}})} = 1 + O(\frac{1}{u\sqrt{1 - \frac{y}{a_{2u}}}})$$

$$= 1 + O(\frac{1}{u\sqrt{1-\frac{a_u}{a_{2u}}}}) = 1 + O(\frac{T(a_u)^{1/2}}{u}) = 1 + o(1);$$

by (6.7) and (6.9).

(b) This follows much as in (6.46) using Lemma 6.1.2(b), (c) and (6.8).

(c) Write $Mt = \frac{a_u}{u}$, so that $|x|, |y| \leq \sigma(Mt) \leq a_u$, and we can recast (6.48) as

$$|x - y| \leq C_1 \frac{a_u}{u}[\sqrt{1-\frac{|x|}{a_u}} + T(a_u)^{-1/2}] \leq C_2 \frac{a_{2u}}{2u}\sqrt{1-\frac{|x|}{a_{2u}}}$$

by (6.7), (6.33) and (6.34). Then (a) gives (6.46), and (6.49) follows easily from (6.47).

(d) This follows from the definition of Δ_h^r.□

Lemma 6.1.8. Let $W \in \mathcal{E}_1$, $0 < \delta < 1$; L, $M > 0$ and $0 < p \leq \infty$.

(a) Let $s \in (0,1)$ and $[a,b]$ be contained in one of the ranges

$$|x| \leq \sigma(t)\left[1 - \left(\frac{s}{2\delta\sigma(t)}\right)^2\right] \tag{6.53}$$

or

$$|x| \geq \sigma(t)\left[1 + \left(\frac{s}{2\delta\sigma(t)}\right)^2\right]. \tag{6.54}$$

Then

$$\int_a^b |f(x \pm s\Phi_t(x))|\,dx \leq \frac{2}{1-\delta}\int_{\overline{a}}^{\overline{b}} |f(x)|\,dx \tag{6.55}$$

where

$$\left\{\begin{array}{c}\overline{a}\\ \overline{b}\end{array}\right\} := \left\{\begin{array}{c}\inf\\ \sup\end{array}\right\}\{x \pm s\Phi_t(x) : x \in [a,b]\}. \tag{6.56}$$

(b) Let $r \geq 1$, $t \in \left(0, \frac{1}{M}\right)$, $h \in (0, Mt)$ and $[a,b]$ be as above with $s = Mrt$. Define $\overline{a}$ and $\overline{b}$ by (6.56) with $s = Mrt$. Assume moreover that

$$[a,b] \subseteq [-\sigma(Lt), \sigma(Lt)]. \tag{6.57}$$

Then for some $C \neq C(a,b,t,g)$

$$\left\|\Delta^r_{h\Phi_t(x)}(g,x,\mathbb{R})\,W(x)\right\|_{L_P[a,b]} \le C \inf_{P\in\mathcal{P}_{r-1}} \|W(g-P)\|_{L_P[\bar{a},\bar{b}]} \le C\,\|Wg\|_{L_P[\bar{a},\bar{b}]}. \tag{6.58}$$

Proof. (*a*) Define $\kappa = \pm 1$ and $u(x) := x + \kappa s \Phi_t(x)$.

We shall assume that $[a,b]$ is contained in the range (6.53) and also $a \ge 0$. The case where $a < 0$ is similar, as is the case when $[a,b]$ is contained in the range (6.54). Then for $x \in [a,b]$,

$$u'(x) = 1 + \frac{\kappa s}{2\sqrt{1-\frac{x}{\sigma(t)}}}\left(-\frac{1}{\sigma(t)}\right) \ge 1-\delta,$$

by (6.53). Hence u is increasing in $[a,b]$ and writing $v := u(x)$ gives

$$\int_a^b |f(x \pm s\Phi_t(x))|\,dx = \int_a^b |f(u(x))|\,dx$$

$$= \int_{u(a)}^{u(b)} |f(v)|\,\frac{dx}{du}dv, \quad v = u(x)$$

$$\le \frac{1}{1-\delta}\int_{u(a)}^{u(b)} |f(v)|\,dv$$

$$= \frac{1}{1-\delta}\int_{\bar{a}}^{\bar{b}} |f(x)|\,dx$$

in this case. The extra 2 in (6.55) takes care of having to split $[a,b]$ into two intervals if $a < 0 < b$.

(*b*) Now recall that we have

$$W(x)\,\Delta^r_{h\Phi_t(x)}(g(x))$$

$$= \sum_{i=0}^{r}\binom{r}{i}(-1)^i\,W(x)\,g\left(x + \left(\frac{r}{2}-i\right)h\Phi_t(x)\right).$$

Also (6.46) gives

$$W(x) \sim W\left(x + \left(\frac{r}{2}-i\right)h\Phi_t(x)\right)$$

uniformly in i and for $|x| \leq \sigma(Lt)$ and $h \leq Mt$. Thus we obtain from part (a)

$$\left\| W(x)\,\Delta^r_{h\Phi_t(x)}(g(x)) \right\|_{L_p[a,b]}$$

$$\leq C \sup_{0\leq i\leq r} \int_a^b |gW|^p \left(x + \left(\frac{r}{2} - i \right) h\Phi_t(x) \right) dx$$

$$\leq \frac{2C}{1-\delta} \int_{\overline{a}}^{\overline{b}} |gW|^p (x)\, dx.$$

Note that for $0 \leq i \leq r$, (6.53) with $s = Mrt$ gives

$$|x| \leq \sigma(t) \left(1 - \left[\frac{Mrt}{2\delta\sigma(t)} \right]^2 \right)$$

$$\leq \sigma(t) \left(1 - \left[\frac{ih}{4\delta\sigma(t)} \right]^2 \right)$$

so the range restrictions of (a) are satisfied.

Finally recall that by (6.50) for $P \in \mathcal{P}_{r-1}$,

$$\Delta^r_{h\Phi_t(x)}(P, x, \mathbb{R}) \equiv 0.$$

Hence

$$\left\| \Delta^r_{h\Phi_t(x)}(g, x, \mathbb{R})\, W(x) \right\|_{L_P[a,\, b]}$$

$$= \left\| \Delta^r_{h\Phi_t(x)}(g - P, x, \mathbb{R})\, W(x) \right\|_{L_P[a,\, b]}$$

$$\leq C \left\| (g - P)\, W \right\|_{L_P[\overline{a},\, \overline{b}]}.$$

It remains to take inf's over P.□

6.2 Some Inequalities

In this section, we prove an extension of the Markov-Bernstein inequality (5.27).

Theorem 6.2.1. *Let $W \in \mathcal{E}_1$ and assume (5.27). Let $0 < p \leq \infty$ and define for $n \geq 1$,*

$$\Psi_n(x) := \left(1 - \left(\frac{x}{a_n}\right)^2\right)^2 + T(a_n)^{-2}, \ x \in \mathbb{R}. \tag{6.59}$$

Then for $n \geq C_1, 0 \leq l \leq n$ and $\forall P \in \mathcal{P}_n$ we have,

$$\left\|P^{(l+1)}\Psi_n^{(l+1)/4}W\right\|_{L_p(\mathbb{R})} \leq C_2\left\{\frac{n}{a_n} + \frac{l}{a_n}T(a_n)^{\frac{1}{2}}\right\}\left\|P^{(l)}\Psi_n^{l/4}W\right\|_{L_p(\mathbb{R})} \tag{6.60}$$

$$\leq C_3\frac{n}{a_n}[l+1]\left\|P^{(l)}\Psi_n^{l/4}W\right\|_{L_p(\mathbb{R})}. \tag{6.61}$$

Here $C_j \neq C_j(n, l, P)$ $j = 2, 3$.

We remark that (6.60) and (6.61) will hold with constants depending on l if we replace $\Psi_n^{1/4}$ by $\Phi_{\frac{a_n}{n}}$.

More precisely,

$$\left\|P^{(l+1)}\Phi_{\frac{a_n}{n}}^{l+1}W\right\|_{L_p(\mathbb{R})} \leq C_4^l\left\{\frac{n}{a_n} + \frac{l}{a_n}T(a_n)^{\frac{1}{2}}\right\}\left\|P^{(l)}\Phi_{\frac{a_n}{n}}^{l}W\right\|_{L_p(\mathbb{R})} \tag{6.62}$$

$$\leq C_5^l\frac{n}{a_n}[l+1]\left\|P^{(l)}\Phi_{\frac{a_n}{n}}^{l}W\right\|_{L_p(\mathbb{R})} \tag{6.63}$$

where $C_i \neq C_j(n, P)$ $j = 4, 5$.

We need several lemmas.

Lemma 6.2.2. Let $s > 1$ and $n \geq C_1$. Then there exist polynomials R of degree $o(n)$ such that uniformly for $|x| \leq a_{sn}$

$$R(x) \sim \Phi_{\frac{a_n}{n}}(x) \sim \Psi_n^{\frac{1}{4}}(x) \tag{6.64}$$

and

$$|R'(x)/R(x)| \leq \frac{C_1}{a_n}\Psi_n^{\frac{-1}{2}}(x). \tag{6.65}$$

Proof. Let

$$u(x) := \left(1 - x^2\right)^{-\frac{3}{4}}, \ x \in [-1, 1]$$

be the ultraspherical weight on $(-1, 1)$ and let $\lambda_n(u, x)$ be the Christoffel function corresponding to u satisfying

$$\lambda_n^{-1}(u,x) \in \mathcal{P}_{2n-2}.$$

Then it is known [46, p.36], that given $A > 0$ we have uniformly in n and $|x| \leq 1 - \frac{A}{n^2}$

$$\lambda_n(u,x) \sim \frac{1}{n}\left(1 - x^2\right)^{-\frac{1}{4}} \tag{6.66}$$

and

$$\left|\lambda_n'(u,x)\right| \leq \frac{C_1}{n}\left(1 - x^2\right)^{-\frac{5}{4}}. \tag{6.67}$$

Now choose $m := m(n) =$ the largest integer$\leq T(a_n)^{-\frac{1}{2}}$ and put

$$R(x) := \frac{1}{m^2}\lambda_m^{-2}\left(u, \frac{x}{a_{2sn}}\right), x \in [-a_{sn}, a_{sn}].$$

Then by (6.9), R has degree $o(n)$ and by (6.7), (6.9), (6.34), (6.59) and (6.66) we have uniformly for $|x| \leq a_{sn}$,

$$R(x) \sim \Phi_{\frac{a_n}{n}}(x) \sim \Psi_n^{\frac{1}{4}}(x).$$

To prove (6.65), we observe much as in [40, p.228] that

$$\left|\lambda_n^{-1}\left(u, \frac{x}{a_{2sn}}\right)'\right|$$

$$= \frac{\left|\lambda_n'\left(u, \frac{x}{a_{2sn}}\right)\right|}{a_{2sn}\lambda_n^2\left(u, \frac{x}{a_{2sn}}\right)}, \tag{6.68}$$

so that by (6.66), (6.67) and the definition of R we have uniformly for $|x| \leq a_{sn}$,

$$\left|R'(x)/R(x)\right| \leq \frac{C_2}{a_n}\left(1 - \left(\frac{x}{a_{2sn}}\right)^2\right)^{-1}$$

$$\leq \frac{C_3}{a_n}\Psi_n(x)^{\frac{-1}{2}}. \square$$

Our next lemma is an infinite-finite range inequality:

Lemma 6.2.3. Let $W \in \mathcal{E}_1$. Let $0 < p \leq \infty, s > 1$ and Ψ_n be as in (6.59). Then for

$n \geq C_1, \forall P \in \mathcal{P}_n$ and $0 \leq l \leq n$ we have,

$$\left\| PW\Psi_n^{l/4} \right\|_{L_p(\mathbb{R})} \leq C_1 \left\| PW\Psi_n^{l/4} \right\|_{L_p(|x| \leq a_{3sn})}. \tag{6.69}$$

Moreover,

$$\left\| PW\Psi_n^{l/4} \right\|_{L_p(|x| \geq a_{3sn})} \leq C_2 \exp\left[-C_3 n^{C_4}\right] \left\| PW\Psi_n^{l/4} \right\|_{L_p(|x| \leq a_{3sn})}. \tag{6.70}$$

Here, $C_j \neq C_j(n, P, l)$, $j = 1, 2$.

We remark that (6.9) shows that for large n,

$$nT(a_n)^{\frac{-1}{2}} \geq n^{C_3}. \tag{6.71}$$

Proof. First note that by (6.35) and the definition of Ψ_n, given $\beta > 0$ we have,

$$\Psi_n^{\frac{\beta}{4}}(x) \geq T(a_n)^{-\frac{\beta}{2}}, \quad x \in \mathbb{R}. \tag{6.72}$$

Now write $l = 4j + k$, $0 \leq k < 3$. Then for some $0 < \alpha \leq 3$ and C_1 depending on k we have,

$$\begin{aligned} \left\| PW\Psi_n^{l/4} \right\|_{L_p(|x| \geq a_{3sn})} &= \left\| PW\Psi_n^j \Psi_n^{k/4} \right\|_{L_p(|x| \geq a_{3sn})} \\ &\leq C_1 \left\| PW\Psi_n^j x^\alpha \right\|_{L_p(|x| \geq a_{3sn})}. \end{aligned} \tag{6.73}$$

Now $Px^\alpha \Psi_n^j$ is a polynomial of degree $\leq n + l + 3 \leq 3n$ so by (1.18), we may continue (6.73) as

$$\leq C_2 \exp\left[-C_3 nT(a_n)^{-\frac{1}{2}}\right] \left\| PWx^\alpha \Psi_n^j \right\|_{L_p(|x| \leq a_{3sn})}$$

$$\leq C_4 \exp\left[-C_3 nT(a_n)^{-\frac{1}{2}}\right] a_n^\alpha T(a_n)^{\frac{k}{2}} \left\| PW\Psi_n^{j+k/4} \right\|_{L_p(|x| \leq a_{3sn})} \quad (\text{by } (6.72))$$

$$\leq C_5 \exp\left[-C_6 nT(a_n)^{\frac{-1}{2}}\right] \left\| PW\Psi_n^{l/4} \right\|_{L_p(|x| \leq a_{3sn})}$$

by (6.9) and (6.71). □

We can now give the

Proof of Theorem 6.2.1. We prove (6.60). Then (6.61) will follow by (6.9). (6.62) and

(6.63) will follow as

$$\Psi_n^{1/4}(x) \sim \Phi_{\frac{a_n}{n}}(x), \quad x \in \mathbb{R}.$$

Put $s > 1$ and write $l = 4j + k$, $0 \le k \le 3$. Put $Q := P^{(l)}$. Then

$$J := \left\| P^{(l+1)} W \Psi_n^{(l+1)/4} \right\|_{L_P(|x| \le a_{3sn})} = \left\| Q' W \Psi_n^{(l+1)/4} \right\|_{L_P(|x| \le a_{3sn})}$$

$$= \left\| Q' W \Psi_n^{j + \frac{k+1}{4}} \right\|_{L_P(|x| \le a_{3sn})}.$$

Choose by Lemma 6.2.2, R of degree $o(n)$ such that

$$R(x) \sim \Psi_n^{\frac{1}{4}}(x)$$

and

$$|R'(x)/R(x)| \le \frac{C_1}{a_n} \Psi_n^{\frac{-1}{2}}(x)$$

uniformly for $|x| \le a_{3sn}$.

Then continue the above estimate for J as

$$J \le C_2 \left\| Q' W \Psi_n^j R^k \Psi_n^{\frac{1}{4}} \right\|_{L_P(|x| \le a_{3sn})}$$

where C_2 depends only on k. This is in turn can be can continued as

$$\le C_2 \left\| \left(Q \Psi_n^j R^k \right)' \Psi_n^{\frac{1}{4}} W \right\|_{L_P(|x| \le a_{3sn})}$$

$$+ C_2 \left\| \left(\Psi_n^j \right)' R^k Q \Psi_n^{\frac{1}{4}} W \right\|_{L_P(|x| \le a_{3sn})}$$

$$+ C_2 \left\| \Psi_n^j \left(R^k \right)' Q \Psi_n^{\frac{1}{4}} W \right\|_{L_P(|x| \le a_{3sn})}$$

$= T_1 + T_2 + T_3$.

We begin with the estimation of T_1 :

Note that $Q\Psi_n^j R^k$ is a polynomial of degree$\leq n+l+o(n)\leq 3n$. Thus, we can write

$$T_1 \leq C_3 \frac{n}{a_n} \left\| Q\Psi_n^j R^k W \right\|_{L_P(\mathbb{R})}$$

(by (5.27))

$$\leq C_4 \frac{n}{a_n} \left\| Q\Psi_n^j R^k W \right\|_{L_P(|x|\leq a_{3sn})}$$

(by (6.17))

$$\leq C_5 \frac{n}{a_n} \left\| Q\Psi_n^{j+\frac{k}{4}} W \right\|_{L_P(|x|\leq a_{3sn})}$$

$$\leq C_5 \frac{n}{a_n} \left\| P^{(l)} \Psi_n^{\frac{l}{4}} W \right\|_{L_P(\mathbb{R})}. \tag{6.74}$$

Next we estimate T_2 :

Note that for $|x|\leq a_{sn}$ and by straightforward differentiation, (6.7) gives

$$\left|\left(\Psi_n^j\right)'\right|(x) \leq C_6 \Psi_n(x)^{j-\frac{1}{2}} \frac{j}{a_n}.$$

Thus

$$T_2 \leq C_7 \frac{j}{a_n} \left\| P_n^{(l)} \Psi_n^{j-\frac{1}{2}} \Psi_n^{\frac{k}{4}+\frac{1}{4}} W \right\|_{L_P(|x|\leq a_{3sn})}$$

$$\leq C_7 \frac{j}{a_n} \left\| P_n^{(l)} \Psi_n^{\frac{l}{4}-\frac{1}{4}} W \right\|_{L_P(|x|\leq a_{3sn})}$$

$$\leq C_8 \frac{lT(a_n)^{\frac{1}{2}}}{a_n} \left\| P_n^{(l)} \Psi_n^{\frac{l}{4}} W \right\|_{L_P(\mathbb{R})} \tag{6.75}$$

by (6.72).

It remains to estimate T_3 :

Write

$$T_3 \leq C_9 k \left\| \Psi_n^j R^{k-1} R' Q \Psi_n^{\frac{1}{4}} W \right\|_{L_P(|x|\leq a_{3sn})}$$

$$\leq \frac{C_{10} k}{a_n} \left\| \Psi_n^j \Psi_n^{\frac{k-1}{4}} Q W \right\|_{L_P(|x|\leq a_{3sn})}$$

(by (6.65))

$$\leq C_{10} \frac{lT(a_n)^{\frac{1}{2}}}{a_n} \left\| P_n^{(l)} \Psi_n^{\frac{l}{4}} W \right\|_{L_P(\mathbb{R})} \tag{6.76}$$

as in the estimation of T_2.

Combining (6.74), (6.75) and (6.76) gives

$$J \leq C_{11}\left\{\frac{n}{a_n}+\frac{l}{a_n}T(a_n)^{\frac{1}{2}}\right\}\left\|P^{(l)}W\Psi_n^{\frac{l}{4}}\right\|_{L_p(\mathbb{R})}, \tag{6.77}$$

where $C_{11} \neq C_{11}(n, P, l)$.

Finally by (6.69), (6.77) becomes

$$\left\|P^{(l+1)}W\Psi_n^{\frac{l+1}{4}}\right\|_{L_p(\mathbb{R})} \leq C_{12}\left\{\frac{n}{a_n}+\frac{l}{a_n}T(a_n)^{\frac{1}{2}}\right\}\left\|P^{(l)}W\Psi_n^{\frac{l}{4}}\right\|_{L_p(\mathbb{R})}$$

as required where $C_{12} \neq C_{12}(n, P, l)$. $\square$

Chapter 7

Jackson Theorems

In this chapter we prove Theorems 5.1.3-5.1.6.

7.1 Polynomial Approximation of W^{-1}

The result of this section is:

Theorem 7.1.1. *For $n \geq 1$, there exist polynomials G_n of degree at most Cn, such that*

$$0 \leq G_n(x) \leq W^{-1}(x), \; x \in \mathbb{R}; \tag{7.1}$$

and

$$G_n(x) \sim W^{-1}(x), \; | x | \leq a_n. \tag{7.2}$$

We remark that this does not follow from existing results in the literature on approximation by weighted polynomials of the form $P_n(x)W(a_nx)$ [28], [51] as our weights do not satisfy their hypotheses. The methods of Totik [51] can be applied to give sharper results but we base our proof on:

Lemma 7.1.2 There exists an even entire function,

$$G(x) = \sum_{j=0}^{\infty} g_j x^{2j}, g_j \geq 0 \quad \forall j, \tag{7.3}$$

such that

$$G(x) \sim W^{-1}(x),\ x \in \mathbb{R}. \tag{7.4}$$

Proof. Set

$$Q_1(r) := Q(\sqrt{r});$$

$$\psi(r) := rQ_1'(r) = \frac{1}{2}\sqrt{r}Q'(\sqrt{r}).$$

Then ψ is increasing in $(0, \infty)$, and if $\lambda > 1$, $r \geq r_0$, the quasi-increasing nature of Q' gives for some $C \neq C(\lambda)$,

$$\psi(\lambda r) - \psi(r) \geq \frac{1}{2}\sqrt{r}Q'(\sqrt{r})(\sqrt{\lambda}C - 1) \geq 1$$

if λ is large enough. Moreover, $\phi(r) := e^{Q_1(r)}$ admits the representation

$$\phi(r) = \phi(1)\exp(\int_1^r \frac{\psi(s)}{s} ds),\ r \geq 1.$$

By Theorem 1.5.1 , there exists entire

$$G_1(r) = \sum_{j=0}^{\infty} g_j r^j, g_j \geq 0 \ \ \forall j$$

such that

$$G_1(r) \sim \phi(r) := \exp(Q(\sqrt{r})), r \geq r_0.$$

Then, assuming $g_0 > 0$ as we can, we see that

$$G(r) := G_1(r^2)$$

satisfies (7.4).□

In the analogous construction for Freud weights, D.S Lubinsky and Z. Ditzian used as the polynomials G_n, the partial sums of G. However, in the Erdös case, for partial sums of degree $O(n)$, we only have

$$G_n(x) \sim W^{-1}(x)$$

for $|x| \leq q_n$, where q_n was given by (1.7).

Although, $a_n/q_n \to 1, n \to \infty$ for Erdös weights, in effect, q_n is significantly smaller than a_n. So, we use a more sophisticated interpolant:

Proof of Theorem 7.1.1. Let J be a positive even integer (to be chosen large enough later) and let $T_n(x)$ denote the classical Chebyshev polynomial on $[-1,1]$. Let G_n denote the Lagrange interpolant to G at the zeros of $T_n(x/a_n)^J$ so that G_n has degree at most $Jn-1$, and admits the error representation

$$(G - G_n)(x) = \frac{1}{2\pi i}\int_\Gamma \frac{G(t)}{t-x}\left(\frac{T_n(x/a_n)}{T_n(t/a_n)}\right)^J dt$$

for x inside Γ. We shall choose Γ to be the ellipse with foci at $\pm a_n$, intersecting the real and imaginary axes at $\frac{a_n}{2}(\rho+\rho^{-1})$ and $\frac{a_n}{2}(\rho-\rho^{-1})$ respectively. Here we shall choose for some fixed small $\varepsilon > 0$,

$$\rho := 1 + \left(\frac{\varepsilon}{T(a_n)}\right)^{1/2}.$$

Since G has non-negative Maclaurin series coefficients, and satisfies (7.4), we deduce that

$$\delta_n := \| G_n/G - 1 \|_{L_\infty[-a_n,a_n]} \leq C_1 \frac{W^{-1}(\frac{a_n}{2}(\rho+\rho^{-1}))}{(\rho-1)^2} \frac{1}{\min_{t\in\Gamma} | T_n(t/a_n) |^J}.$$

Now for $t \in \Gamma$, we can write $t = \frac{a_n}{2}(z+z^{-1})$ where $|z| = \rho$, so that

$$| T_n(t/a_n) | = | T_n(\tfrac{1}{2}(z+z^{-1})) | = | \tfrac{1}{2}(z^n + z^{-n}) |$$

$$\geq \frac{1}{2}(\rho^n - \rho^{-n}) \geq \exp(C_2 n T(a_n)^{-1/2}).$$

(Recall that $nT(a_n)^{-1/2} \to \infty$ as $n \to \infty$ and in fact grows faster than a power of n). It is important here that C_2 is independent of J. Next

$$\frac{a_n}{2}(\rho+\rho^{-1}) \leq a_n(1+C_3\frac{\varepsilon}{T(a_n)}) \leq a_{2n}$$

if ε is small enough, and n is large enough, by (6.7). Then,

$$W^{-1}(\frac{a_n}{2}(\rho+\rho^{-1})) \leq \exp(C_4 nT(a_n)^{-1/2}),$$

where again it is important that C_4 is independent of J. Since $(\rho-1)^{-2} \sim T(a_n)$ grows no faster than a power of n, we see that choosing J large enough, gives

$$\delta_n \to 0, \ n \to \infty.$$

Then (7.4) gives (7.2).

We now turn to proving (7.1). It suffices to prove

$$0 \leq G_n \leq CW^{-1}$$

for then (7.1) follows on multiplying G_n by a suitable constant. Firstly, we can assume n is even (for odd n, we can use G_{n+1}) so that $H_n(x) := G_n(\sqrt{x})$ is a polynomial of degree at most $\frac{Jn}{2}-1$ (recall T_n and J are even) that interpolates to the entire function $H(x) := G(\sqrt{x})$ at the $\frac{Jn}{2}$ zeros of $T_n(\frac{\sqrt{t}}{a_n})^J$ that lie in $(0, a_n^2)$. Thus, $H_n(x)$ is determined entirely by interpolation conditions. Let γ_n denote the leading coefficient of $T_n(x/\sqrt{a_n})$. Then, the usual derivative-error formula for Hermite interpolation gives for $x \in (0,\infty)$ and some $\xi \in (0,\infty)$,

$$(H-H_n)(x) = \gamma_n^{-J}T_n(\frac{\sqrt{x}}{a_n})^J\frac{H^{(\frac{Jn}{2})}(\xi)}{(\frac{Jn}{2})!} \geq 0.$$

(Recall that H is entire and has non-negative Maclaurin series coefficients). So in $\mathbb{R}$

$$G_n \leq G \leq CW^{-1}.$$

To show that $G_n \geq 0$ in $\mathbb{R}$, we note that it is true in $[-a_n, a_n]$ and we must establish it elsewhere. We use an idea employed in proving the Posse-Markov-Stieltjes inequalities [13,p.30,Lemma5.3] (There the proof is for $(-\infty,\infty)$, but the proof goes through for $(0,\infty)$ with trivial changes). Now H is absolutely monotone in $(0,\infty)$ and $H-H_n$ has $\frac{Jn}{2}$ zeros in $(0,a_n^2]$. If m is the number of zeros of $H_n(x)$ in $[a_n^2,\infty)$, Lemma 5.3 in [13,p.30] gives

$$\frac{Jn}{2}+m\leq \deg(H_n)+1\leq \frac{Jn}{2}.$$

So $m=0$, that is H_n has no zeros in (a_n^2,∞). Thus $H_n\geq 0$ there, so $G_n\geq 0$ in $\mathbb{R}$.□

7.2 Polynomials approximating characteristic functions

Our Jackson theorem is based on polynomial approximations to the characteristic function $\chi_{[a,b]}$ of an interval $[a,b]$. We believe the following result is of independent interest:

Theorem 7.1.3. *Let l be a positive integer. There exist C_1, C_2, n_0 such that for $n\geq n_0$ and $\tau\in[-a_n,a_n]$, there exist polynomials $R_{n,\tau}$ of degree at most $C_1 n$ such that for $x\in\mathbb{R}$,*

$$|\chi_{[\tau,a_n]}-R_{n,\tau}|(x)W(x)/W(\tau)\leq C_1\Big(1+\frac{n\,|x-\tau|}{a_n\sqrt{1-\frac{|\tau|}{a_{2n}}}}\Big)^{-l}. \tag{7.5}$$

We emphasise that the constants are independent of n,τ,x. Our proof will use polynomials from [24] built on the Chebyshev polynomials:

Lemma 7.1.4. *There exist C_1, B, n_1 such that for $n\geq n_1$ and $|\zeta|\leq\cos\frac{\pi}{2n}$, there exists a polynomial $V_{n,\zeta}$ of degree at most $n-1$ with*

$$\|V_{n,\zeta}\|_{L_\infty[-1,1]}=V_{n,\zeta}(\zeta)=1; \tag{7.6}$$

$$|V_{n,\zeta}(t)|\leq\frac{B\sqrt{1-|\zeta|}}{n\,|t-\zeta|},\ t\in(-1,1)\backslash\{\zeta\}. \tag{7.7}$$

Moreover,

$$V_{n,\zeta}(t) \geq \frac{1}{2}, \mid t-\zeta \mid \leq C_1 \frac{\sqrt{1-\mid \zeta \mid}}{n}. \tag{7.8}$$

The constants are independent of n, ζ, t.

Proof: The assertions (7.6), (7.7) are Proposition 13.1 in [24]. The estimate (7.8) follows from the classical Bernstein inequality.□

The polynomials $R_{n,\tau}$, are determined as follows: Let us suppose that, say,

$$a_1 \leq \tau \leq a_n.$$

Later on, we shall suppose that τ exceeds a fixed positive constant. We define

$$\zeta := \frac{\tau}{a_{2lJn}} \tag{7.9}$$

and if G_n are the polynomials of Theorem 7.1.1,

$$R_{n,\tau}(x) := \frac{\int_0^x G_n(s) V_{n,\zeta}(\frac{s}{a_{2lJn}})^{lJ} ds}{\int_0^{\tau^*} G_n(s) V_{n,\zeta}(\frac{s}{a_{2lJn}})^{lJ} ds}. \tag{7.10}$$

The parameters $\tau^* > \tau$ and J are defined as follows: Let $A \in (0,1]$ denote the constant in the quasi-monotonicity of Q', so that

$$Q'(y) \geq AQ'(x),\ y \geq x. \tag{7.11}$$

Let M denote a positive constant such that for say, $u \geq u_0$,

$$Q'(x) \leq MQ'(a_u),\ 1 \leq x \leq a_{2u}. \tag{7.12}$$

The existence of such an M follows from (6.4) and (6.6)(i)

We set

$$H := H(n, \tau, l) := \frac{2l\ n}{Aa_n Q'(\tau)\sqrt{1-\zeta}} \tag{7.13}$$

and if $\tau = a_r$,

$$\tau^* := \tau^*(n,\tau) := \min\{a_{2r}, a_n, \tau + 2\frac{a_n}{n}\sqrt{1-\zeta}H\log H\}. \tag{7.14}$$

The reason for this (complicated!) choice will become clearer later. We assume that $J \geq 4$ is so large that G_n has degree at most $Jn-1$, and also

$$J \geq 16M/A \tag{7.15}$$

where A, M are as above. Note that then $R_{n,\tau}$ has degree at most $Jn + lJn$. We first record some estimates of the terms in (7.10).

Lemma 7.1.5. (a) For $n \geq n_1$, and $C_1 \leq \tau \leq a_n$, we have

$$W(\tau)\int_0^{\tau^*} G_n(s)V_{n,\zeta}(\frac{s}{a_{2lJn}})^{lJ}ds \geq C_2\frac{a_n}{n}\sqrt{1-\zeta}, \tag{7.16}$$

where $C_2 \neq C_2(n,\tau)$.

(b) For $x \in (\tau, a_{2lJn})$,

$$\int_x^{a_{2lJn}} V_{n,\zeta}(\frac{s}{a_{2lJn}})^{\frac{lJ}{2}}ds \leq C_1\frac{a_n}{n}\sqrt{1-\zeta}(1+\frac{n\mid x-\tau\mid}{a_n\sqrt{1-\zeta}})^{-l} \tag{7.17}$$

and for $x \in (-a_{2lJn}, \tau)$,

$$\int_{-a_{2lJn}}^{x} V_{n,\zeta}(\frac{s}{a_{2lJn}})^{\frac{lJ}{2}}ds \leq C_1\frac{a_n}{n}\sqrt{1-\zeta}(1+\frac{n\mid x-\tau\mid}{a_n\sqrt{1-\zeta}})^{-l}. \tag{7.18}$$

Here $C_1 \neq C_1(n,\tau)$.

Proof. (a) Let us denote the left-hand side of (7.16) by Γ. By (7.2) and (7.8),

$$\Gamma \geq C_2W(\tau)\int_{\tau-C_3\frac{a_n}{n}\sqrt{1-\zeta}}^{\tau} W^{-1}(s)ds \geq C_4\frac{a_n}{n}\sqrt{1-\zeta},$$

where we have used (6.46).

(b) These follow in a straightforward fashion from the estimates (7.6), (7.7) and the fact that $J \geq 4$ so $\frac{lJ}{2} > l+1$.□

Now we begin the proof of Theorem 7.1.3. We first show that it suffices to consider τ in the range $[S, a_n]$, for some fixed S.

Proof of Theorem 7.1.3 for $|\tau| \le S$, where S is fixed. Note first, that since for such τ,

$$W(x)/W(\tau) \le W(0)/W(S),\ x \in \mathbb{R},$$

we must only prove there exists $R_{n,\tau}$ of degree at most n such that

$$|\chi_{[\tau,a_n]} - R_{n,\tau}|(x) \le C_1\left(1 + \frac{n|x-\tau|}{a_n\sqrt{1-\frac{|\tau|}{a_{2n}}}}\right)^{-l},$$

for $|x| \le a_{2n}$, and then our infinite-finite range inequality Lemma 6.1.3 gives the rest. Setting here $\xi := \tau/a_n$, $s := x/a_n$, and $U_{n,\xi}(s) := R_{n,\tau}(x) = R_{n,\tau}(a_n s)$, we see that it suffices to show

$$|\chi_{[\xi,1]}(s) - U_{n,\xi}(s)| \le C_2(1 + n|s-\xi|)^{-l}, s \in [-2,2].$$

We have used here that $|\xi| \le \frac{1}{2}$, for large n. The existence of such polynomials is classical. See for example [9]. One could also base them on the $V_{n,\zeta}$ above.□

It suffices to consider $\tau \in [S, a_n]$, where S is fixed

For once this is done, we have the result for all $\tau \in [0, a_n]$. With the result for $\tau \ge 0$, we set

$$R_{n,-\tau}(x) := 1 - R_{n,\tau}(-x), x \in \mathbb{R}.$$

It is not difficult to check the result for $-\tau$ from the corresponding result for τ, using the identity

$$\chi_{[-\tau,a_n]}(x) = 1 - \chi_{(\tau,a_n]}(-x).$$

□

In the sequel, we define $R_{n,\tau}$ by (7.10)-(7.14).

It suffices to prove (7.5) for $\tau \in [S, a_n]$ and $|x| \le a_{2lJn}$

For then (7.5) for this restricted range implies

$$\| (1+[\frac{n(x-\tau)}{a_n\sqrt{1-\frac{\tau}{a_{2n}}}}]^2)^l R_{n,\tau}(x)W(x) \|_{L_\infty[-a_{2lJn},\, a_{2lJn}]} \leq C_3 n^{C_4}$$

where $C_4 \neq C_4(n,\tau)$. Since the polynomial in the left-hand side has degree at most $2l+Jn+lJn \leq \eta 2lJn$, some fixed $\eta<1$, if $l \geq 2$ and n is large enough (as we can assume), then the infinite-finite range inequality Lemma 6.1.3 gives

$$\| (1+[\frac{n(x-\tau)}{a_n\sqrt{1-\frac{\tau}{a_{2n}}}}]^2)^l R_{n,\tau}(x)W(x) \|_{L_\infty(|x|\geq a_{2lJn})} \leq C_5 \exp(-n^{C_6}).$$

Then (7.5) follows for $|x| \geq a_{2lJn}$.□

We can now begin the proof of (7.5) proper. We consider 5 different ranges of x: $[0,\tau)$, $[\tau,\tau^*]$, $(\tau^*,a_n]$, $(a_n,a_{2lJn}]$, $[-a_{2lJn},0]$. Moreover, we set

$$\Delta(x) := | \chi_{[\tau,a_n]} - R_{n,\tau} | (x)W(x)/W(\tau).$$

Proof of (7.5) for $x \in [0,\tau)$. Here using (7.1), and then (7.16),

$$\Delta(x) = \frac{W(x)\int_0^x G_n(s)V_{n,\zeta}(\frac{s}{a_{2lJn}})^{lJ}ds}{W(\tau)\int_0^{\tau^*} G_n(s)V_{n,\zeta}(\frac{s}{a_{2lJn}})^{lJ}ds}$$

$$\leq C\frac{W(x)\int_0^x W^{-1}(s)V_{n,\zeta}(\frac{s}{a_{2lJn}})^{lJ}ds}{\frac{a_n}{n}\sqrt{1-\zeta}}$$

$$\leq C\frac{\int_0^x V_{n,\zeta}(\frac{s}{a_{2lJn}})^{lJ}ds}{\frac{a_n}{n}\sqrt{1-\zeta}}$$

by the monotonicity of W. Then (7.18) gives the result.□

Proof of (7.5) for $x \in [\tau,\tau^*)$. Here

$$\Delta(x) = \frac{W(x)\int_x^{\tau^*} G_n(s)V_{n,\zeta}(\frac{s}{a_{2lJn}})^{lJ}ds}{W(\tau)\int_0^{\tau^*} G_n(s)V_{n,\zeta}(\frac{s}{a_{2lJn}})^{lJ}ds}$$

$$\leq C \frac{\int_x^{\tau^*} \exp(Q(s) - Q(x)) V_{n,\zeta}(\frac{s}{a_{2lJn}})^{lJ} ds}{\frac{a_n}{n}\sqrt{1-\zeta}}$$

by (7.1) and (7.16). Now for $s \in (x, \tau^*)$, the property (7.12) of Q' gives (recall $\tau^* \leq a_{2r}$)

$$Q(s) - Q(x) \leq MQ'(a_r)(s-x) \leq MQ'(\tau)(s-\tau).$$

Then using our bounds on $V_{n,\zeta}$ in (7.6) and (7.7), we have

$$\Delta(x) \leq C_1 \frac{\int_x^{\tau^*} \exp(MQ'(\tau)(s-\tau)) \min\{1, \frac{Ba_{2lJn}\sqrt{1-\zeta}}{n(s-\tau)}\}^{lJ} ds}{\frac{a_{2lJn}}{n}\sqrt{1-\zeta}}$$

$$= C_1 B \int_{\frac{n(x-\tau)}{Ba_{2lJn}\sqrt{1-\zeta}}}^{\frac{n(\tau^*-\tau)}{Ba_{2lJn}\sqrt{1-\zeta}}} \exp(\frac{a_{2lJn}}{a_n}\frac{2lMBu}{AH}) \min\{1, \frac{1}{u}\}^{lJ} du$$

$$\leq C_2 \int_{\frac{n(x-\tau)}{Ba_{2lJn}\sqrt{1-\zeta}}}^{\frac{2}{B}H\log H} g(u) \min\{1, \frac{1}{u}\}^{lJ/2} du$$

for say $n \geq n_1 = n_1(J,l)$ by (7.14), and where

$$g(u) := \exp(\frac{4lMBu}{AH}) \min\{1, \frac{1}{u}\}^{lJ/2}.$$

We claim that if J is large enough,

$$g(u) \leq C_3, u \in [0, \frac{2}{B}H\log H],$$

with C_3 independent of τ, n. Firstly we claim that if l is large enough,

$$H \geq e; H \geq e^{B/2} \tag{7.19}$$

uniformly for $\tau \in [S, a_n]$ and $n \geq n_0\,(J,l)$.

Firstly recall that B, M, J, A are independent of l (see (7.7), (7.11), (7.12), (7.15)). Then also from (6.52) for $\tau \in [S, a_n]$

$$a_n Q'(\tau)\sqrt{1 - \frac{\tau}{a_{2n}}} \leq Cn,$$

with $C \neq C(n, \tau, l)$. Then from (7.13),

$$H \geq \frac{2l}{AC}\left(\frac{1-\frac{\tau}{a_{2n}}}{1-\frac{\tau}{a_{2lJn}}}\right)^{\frac{l}{2}}.$$

Here for $n \geq n_0(J, l)$, using $1-u \leq \log\frac{1}{u}$, $u \in (0,1]$, we obtain

$$\frac{1-\frac{\tau}{a_{2lJn}}}{1-\frac{\tau}{a_{2n}}} = 1+\frac{\tau}{a_{2n}}\frac{1-\frac{a_{2n}}{a_{2lJn}}}{1-\frac{\tau}{a_{2n}}}$$

$$\leq 1+\frac{\log\frac{a_{2lJn}}{a_{2n}}}{1-\frac{a_n}{a_{2n}}} \leq 1+C_1\log(lJ)$$

by (6.7) and the left inequality in (6.12). Thus for $n \geq n_0(J, l)$, uniformly for $\tau \in [S, a_n]$,

$$H \geq \frac{C_2 l}{\sqrt{\log lJ}}.$$

So (7.19) follo if we choose l enough. Then

$$g(u) \leq \exp(\frac{4lMB}{Ae}), u \in (0,1].$$

Next, by elementary calculus, g has at most one local extremum in $[1,\infty)$, and this is a minimum. Thus in any subinterval of $[1,\infty)$, g attains its maximum at the endpoints of that interval. In particular, we must only check that $g(\frac{2}{B}H\log H)$ is bounded. Note that by (7.19),

$$\frac{2}{B}H\log H \geq e > 1.$$

So

$$g(\frac{2}{B}H\log H) = \exp(l\log H\{\frac{8M}{A}-\frac{J}{2}\}-\frac{Jl}{2}\log[\frac{2}{B}\log H]) \leq 1$$

as $J \geq 16M/A$ and $H \geq e^{B/2}$ (See (7.15)). So we have

$$\Delta(x) \leq C_4\int_{\frac{n(x-\tau)}{Ba_{2lJn}\sqrt{1-\zeta}}}^{\infty}\min\{1,\frac{1}{u}\}^{lJ/2}du$$

and then (7.5) follows as $J \geq 4$.□

Proof of (7.5) for $x \in (\tau^*, a_n]$

Here

$$\Delta(x) = \frac{W(x)\int_{\tau^*}^x G_n(s)V_{n,\zeta}(\frac{s}{a_{2lJn}})^{lJ}ds}{W(\tau)\int_0^{\tau^*} G_n(s)V_{n,\zeta}(\frac{s}{a_{2lJn}})^{lJ}ds}$$

$$\leq C_1 \frac{\int_{\tau^*}^x \exp(Q(s)-Q(x))V_{n,\zeta}(\frac{s}{a_{2lJn}})^{lJ}ds}{\frac{a_n}{n}\sqrt{1-\zeta}}$$

$$\leq C_2 \frac{n}{a_n\sqrt{1-\zeta}}\left(e^{Q(\frac{\tau+x}{2})-Q(x)}\int_{\tau^*}^{\frac{\tau+x}{2}} V_{n,\zeta}(\frac{s}{a_{2lJn}})^{lJ}ds + \int_{\frac{\tau+x}{2}}^x V_{n,\zeta}(\frac{s}{a_{2lJn}})^{lJ}ds\right)$$

$$\leq C_3\left\{e^{Q(\frac{\tau+x}{2})-Q(x)}[1+\frac{n(\tau^*-\tau)}{a_n\sqrt{1-\zeta}}]^{-l} + [1+\frac{n(x-\tau)}{a_n\sqrt{1-\zeta}}]^{-l}\right\} \tag{7.20}$$

by (7.7) and (7.17). Here if $\tau^* > \frac{\tau+x}{2}$, the first term in the last two lines can be dropped and we already have the desired estimate. In the contrary case, we must estimate the first term. We note that we can assume that $\tau^* < a_n$, for otherwise the current range of x is empty. We consider two subcases (recall the definition (7.14) of τ^*):

(I) $\tau^* = \tau + 2\frac{a_n}{n}\sqrt{1-\zeta}H\log H$

We shall show that

$$\Gamma := \frac{Q(x)-Q(\frac{\tau+x}{2})}{l\log(1+\frac{n(x-\tau)}{a_n\sqrt{1-\zeta}})} \geq 1. \tag{7.21}$$

Then, the first part of the first term in the right-hand side of (7.20) already gives the desired estimate; the second part of that first term can be bounded by 1. By quasi-monotonicity (7.11) of Q',

$$Q(x) - Q(\frac{\tau+x}{2}) \geq AQ'(\tau)(\frac{x-\tau}{2}).$$

Setting

$$u := \frac{n(x-\tau)}{a_n\sqrt{1-\zeta}},$$

we have

$$\Gamma \geq \frac{AQ'(\tau)\frac{a_n}{n}\sqrt{1-\zeta}u}{2l\log(1+u)} = \frac{u}{H\log(1+u)}.$$

But

$$u \geq \frac{n(\tau^*-\tau)}{a_n\sqrt{1-\zeta}} = 2H\log H.$$

Recall from (7.19) that $H \geq e$. Then since the function $\frac{u}{\log(1+u)}$ is increasing for $u \geq 2H \log H \geq e$, we obtain

$$\Gamma \geq \frac{2H \log H}{H \log(1+2H \log H)}.$$

Using the inequality $1+2t\log t \leq t^2, t \geq 2$, we have

$$\Gamma \geq \frac{2 \log H}{\log H^2} = 1.$$

So we have (7.21) and the result.

(II) $\tau^* = a_{2r}$

In this case, from (6.7),

$$\tau^* - \tau = a_{2r} - a_r \sim \frac{a_r}{T(a_r)} = \frac{\tau}{T(\tau)}.$$

Now if $\tau^* \leq x \leq \tau(1 + \frac{1}{T(\tau)})$, then

$$x - \tau \sim \tau^* - \tau$$

and the second part of the first term in the right-hand side of (7.20) already gives the desired estimate (the first part of the first term can be estimated by 1). If $x > \tau(1 + \frac{1}{T(\tau)})$, then

$$\frac{x}{(\frac{x+\tau}{2})} \geq 1 + \frac{1}{2T(\tau)+1} \geq 1 + \frac{1}{3T(\tau)}$$

for large τ, so from (6.1),

$$\frac{Q(x)}{Q(\frac{x+\tau}{2})} \geq (1 + \frac{1}{3T(\tau)})^{C_2 T(\frac{x+\tau}{2})} \geq C_3 > 1.$$

(Recall that $\frac{x+\tau}{2} > \tau$). Then

$$e^{Q(\frac{\tau+x}{2})-Q(x)}[1 + \frac{n(\tau^* - \tau)}{a_n\sqrt{1-\zeta}}]^{-l} \leq e^{-C_4 Q(x)}[1 + \frac{C_5 n\tau}{a_n T(\tau)\sqrt{1-\zeta}}]^{-l}.$$

This will admit the desired estimate, namely

$$C_6[1 + \frac{n(x-\tau)}{a_n\sqrt{1-\zeta}}]^{-l}$$

provided

$$e^{C_4 Q(x)/(l)} \frac{\tau}{T(\tau)} \geq C_7(x-\tau).$$

But,

$$e^{C_4 Q(x)/(l)} \frac{\tau}{T(\tau)} \geq C_8 \frac{e^{C_4 Q(x)/(l)}}{T(x)} \geq C_9 Q(x) \geq C_{10} x > C_{10}(x-\tau)$$

by (6.5), (6.9) and the faster than polynomial growth of Q, so we have the desired estimate.□

Proof of (7.5) for $x \in (a_n, a_{2lJn}]$

Here, much as in the previous range,

$$\Delta(x) = \frac{W(x)\int_0^x G_n(s) V_{n,\zeta}(\frac{s}{a_{2lJn}})^{lJ} ds}{W(\tau)\int_0^{\tau} G_n(s) V_{n,\zeta}(\frac{s}{a_{2lJn}})^{lJ} ds}$$

$$\leq C_2 \frac{n}{a_n\sqrt{1-\zeta}} \left(e^{Q(\frac{\tau+x}{2})-Q(x)} \int_0^{\frac{\tau+x}{2}} V_{n,\zeta}(\frac{s}{a_{2lJn}})^{lJ} ds + \int_{\frac{\tau+x}{2}}^{x} V_{n,\zeta}(\frac{s}{a_{2lJn}})^{lJ} ds \right)$$

$$\leq C_3\{e^{Q(\frac{\tau+x}{2})-Q(x)} + [1+\frac{n(x-\tau)}{a_n\sqrt{1-\zeta}}]^{-l}\}.$$

We must show that the first term on the last right-hand side admits a bound that is a constant multiple of the second term on the last right-hand side. Let us write $x = a_v$ (so $v \geq n$) and $\frac{\tau+x}{2} = a_u$ (so that $u < v$). If firstly $u \geq \frac{n}{2}$, then

$$Q(x) - Q(\frac{\tau+x}{2}) \geq C_4 Q'(a_{n/2})(\tau - x)$$

$$\geq C_5 \frac{n}{a_n} T(a_n)^{1/2}(\tau - x) \geq C_6 \frac{n(\tau-x)}{a_n\sqrt{1-\zeta}}$$

by (6.4) and (6.7).(Recall that $\xi = \frac{\tau}{a_{2lJn}}$.) In this case the result follows. If $u < \frac{n}{2}$,

$$Q(x) - Q(\frac{\tau+x}{2}) \geq Q(a_n) - Q(a_{n/2})$$

$$\geq C_7 Q(a_n) \geq C_8 n T(a_n)^{-1/2} \geq C_9 n^{C_{10}}$$

by (6.5) and (6.9). Since

$$[1+\frac{n(x-\tau)}{a_n\sqrt{1-\zeta}}]^{-l} \geq n^{-C_{11}}$$

the result again follows.□

Proof of (7.5) for $x \in [-a_{2lJn}, 0]$

Here using the evenness of W and (7.1), (7.16) as before gives

$$\Delta(x) = \frac{W(x)\int_x^0 G_n(s)V_{n,\zeta}(\frac{s}{a_{2lJn}})^{lJ}ds}{W(\tau)\int_0^{\tau^*} G_n(s)V_{n,\zeta}(\frac{s}{a_{2lJn}})^{lJ}ds}$$

$$\leq C_2\frac{n}{a_n\sqrt{1-\zeta}}\left(\int_x^{\frac{x}{2}} V_{n,\zeta}(\frac{s}{a_{2lJn}})^{lJ}ds + e^{Q(\frac{x}{2})-Q(x)}\int_{\frac{x}{2}}^0 V_{n,\zeta}(\frac{s}{a_{2lJn}})^{lJ}ds\right)$$

$$\leq C_3\left\{[1+\frac{n\mid \frac{x}{2}-\tau\mid}{a_n\sqrt{1-\zeta}}]^{-l} + e^{Q(\frac{x}{2})-Q(x)}[1+\frac{n\tau}{a_n\sqrt{1-\zeta}}]^{-l}\right\}.$$

Here $\mid \frac{x}{2}-\tau\mid = \frac{|x|}{2}+\tau \sim \mid x-\tau\mid$. Also, if $\mid x\mid \leq \tau$, then $\tau \sim \tau + \mid x\mid = \mid x-\tau\mid$. Otherwise (recall $\tau \geq S$), we have

$$e^{Q(\frac{x}{2})-Q(x)} \leq e^{-C_4 Q(x)} \leq e^{-C_5|x|} \leq (C_6\mid x\mid)^{-l}.$$

Again as $\mid x\mid \tau \geq C_8(\tau+\mid x\mid) = C_8\mid x-\tau\mid$, the result follows.□

7.3 The Proofs of Theorems 5.1.3 and 5.1.4

In this section, we prove Theorems 5.1.3 and 5.1.4. Recall that our moduli of continuity are

$$\omega_{r,p}(f,W,t) := \sup_{0<h\leq t} \| W\Delta^r_{h\Phi_t(x)}(f,x,\mathbb{R})\|_{L_p(|x|\leq\sigma(2t))}$$

$$+ \inf_{P\in\mathcal{P}_{r-1}} \|(f-P)W\|_{L(|x|\geq\sigma(4t))}$$

and

$$\varpi_{r,p}(f,W,t) := \left(\frac{1}{t}\int_0^t \| W\Delta^r_{h\Phi_t(x)}(f,x,\mathbb{R})\|^p_{L_p(|x|\leq\sigma(2t))}\,dh\right)^{\frac{1}{p}}$$

$$+ \inf_{P\in\mathcal{P}_{r-1}} \|(f-P)W\|_{L(|x|\geq\sigma(4t))}.$$

Here,

$$\sigma(t) = \inf\{a_u : \frac{a_u}{u} \le t\}.$$

We need further moduli of continuity. If I is an interval, and $f : I \to \mathbb{R}$, we define for $t > 0$,

$$\Lambda_{r,p}(f,t,I) := \sup_{0<h\le t} \left(\int_I \mid \Delta_h^r(f,x,I) \mid^p dx\right)^{1/p} \tag{7.22}$$

and its averaged cousin

$$\Omega_{r,p}(f,t,I) := \left(\frac{1}{t}\int_0^t \int_I \mid \Delta_s^r(f,x,I) \mid^p dx\ ds\right)^{1/p}. \tag{7.23}$$

Note that for some C_1 , C_2 depending only on r and p, (not on f, I, t) [8], [47, p.191],

$$C_1 \le \Lambda_{r,p}(f,t,I)/\Omega_{r,p}(f,t,I) \le C_2. \tag{7.24}$$

For large enough n, we choose a partition

$$-a_n = \tau_{0n} < \tau_{1n} < \ldots < \tau_{nn} = a_n \tag{7.25}$$

such that if

$$I_{kn} := [\tau_{kn}, \tau_{k+1,n}], 0 \le k \le n-1, \tag{7.26}$$

then uniformly in k and n,

$$\mid I_{kn} \mid \sim \frac{a_n}{n}\sqrt{1 - \frac{\mid \tau_{kn} \mid}{a_{2n}}}. \tag{7.27}$$

($\mid I \mid$ denotes the length of the interval I). We also set $I_{nn} := \emptyset$. There are many ways to do this. For example, one can start with the classical Chebyshev points scaled to $[-a_n, a_n]$, and then drop an appropriate number near $\pm a_n$. Let us set

$$I_n := [-a_n, a_n] = \cup_{k=0}^{n-1} I_{kn} \tag{7.28}$$

and

$$\theta_{kn}(x) := \chi_{[\tau_{kn}, a_n]}(x) = \chi_{\cup_{i=k}^{n-1} I_{in}}(x). \tag{7.29}$$

We set

$$I_{kn}^* := I_{kn} \cup I_{k+1,n}, 0 \le k \le n-1. \tag{7.30}$$

By Whitney's theorem [47,p195], we can find a polynomial p_k of degree at most r, such that

$$\|f - p_k\|_{L_p(I_{kn}^*)} \le C_2 \Lambda_{r,p}(f, | I_{kn}^* |, I_{kn}^*) \tag{7.31}$$

with $C_2 \neq C_2(f, n, k, I_{kn}^*)$.

Now define an approximating piecewise polynomial/ spline by

$$L_n[f](x) := p_0(x)\theta_{0n}(x) + \sum_{k=1}^{n-1}(p_k - p_{k-1})(x)\theta_{kn}(x). \tag{7.32}$$

We first show that $L_n[f]$ is a good approximation to f.

Lemma 7.1.6. Let $\Psi_n : [-a_n, a_n] \to \mathbb{R}$ be such that uniformly in n, and $x \in [-a_n, a_n]$,

$$\Psi_n(x) \sim \sqrt{1 - \frac{|x|}{a_{2n}}}, \tag{7.33}$$

Then for $0 < p < \infty$,

$$\|(f - L_n[f])W\|_{L_p(\mathbb{R})}$$

$$\le C_1 \left\{ \left[\frac{n}{a_n} \int_0^{C_2 \frac{a_n}{n}} \| W \Delta_{h\Psi_n(x)}^r (f, x, \mathbb{R}) \|_{L_p[-a_n, a_n]}^p \, dh \right]^{\frac{1}{p}} + \|fW\|_{L_p(|x| \ge a_n} \right\} \tag{7.34}$$

and for $p = \infty$, we replace the pth root and integral by $\sup_{0 < h \le C_2 \frac{a_n}{n}}$.

Here, $C_j \neq C_j(f, n)$, $j = 1, 2$. Moreover, the constants are independent of $\{\Psi_n\}$, depending only on the constants in $\sim$ in (7.33). For $p = \infty$, (7.34) holds if we remove the exponents p.

Proof. We first deal with $p < \infty$. Now

$$\| (f - L_n[f])W \|_{L_p(\mathbb{R})}^p = \sum_{j=0}^{n-1} \Delta_{jn} + \| fW \|_{L_p(|x| \ge a_n)}^p, \tag{7.35}$$

where

$$\Delta_{jn} := \int_{I_{jn}} |\, f - L_n[f] \,|^p \, W^p. \tag{7.36}$$

Note that in $(\tau_{jn}, \tau_{j+1,n})$, $L_n[f] = p_j$, so that

$$\begin{aligned}
\Delta_{jn} &= \int_{I_{jn}} |\, f - p_j \,|^p \, W^p \\
&\leq \| W \|^p_{L_\infty(I_{jn})} C_2^p \Lambda^p_{r,p}(f, |\, I^*_{jn} \,|, I^*_{jn}) \ \text{(by (7.31))} \\
&\leq \| W \|^p_{L_\infty(I^*_{jn})} \| W^{-1} \|^p_{L_\infty(I^*_{jn})} \frac{C_3}{|\, I^*_{jn} \,|} \int_0^{|I^*_{jn}|} \int_{I^*_{jn}} |\, W \Delta^r_s(f, x, I^*_{jn}) \,|^p \, dx \, ds,
\end{aligned} \tag{7.37}$$

by (7.23), (7.24). Now from (6.46),

$$\| W \|^p_{L_\infty(I^*_{jn})} \| W^{-1} \|^p_{L_\infty(I^*_{jn})} \sim 1 \tag{7.38}$$

uniformly in j and n. Moreover, uniformly in j, n, and $x \in I^*_{jn}$,

$$|\, I^*_{jn} \,| \sim \frac{a_n}{n} \sqrt{1 - \frac{|\, x \,|}{a_{2n}}} \sim \frac{a_n}{n} \Psi_n(x).$$

Then we can continue (7.37) as

$$\begin{aligned}
\Delta_{jn} &\leq \frac{C_4}{|\, I^*_{jn} \,|} \int_{I^*_{jn}} \int_0^{|I^*_{jn}|} |\, W \Delta^r_s(f, x, I^*_{jn}) \,|^p \, ds \, dx \\
&= \frac{C_4}{|\, I^*_{jn} \,|} \int_{I^*_{jn}} \Psi_n(x) \int_0^{|I^*_{jn}|/\Psi_n(x)} |\, W \Delta^r_{t\Psi_n(x)}(f, x, I^*_{jn}) \,|^p \, dt \, dx \\
&\leq C_5 \frac{n}{a_n} \int_0^{C_6 \frac{a_n}{n}} \int_{I^*_{jn}} |\, W \Delta^r_{t\Psi_n(x)}(f, x, I^*_{jn}) \,|^p \, dx \, dt.
\end{aligned} \tag{7.39}$$

Adding over j gives

$$\sum_{j=0}^{n-1} \Delta_{jn} \leq C_5 \frac{n}{a_n} \int_0^{C_6 \frac{a_n}{n}} \int_{I_n} |\, W \Delta^r_{t\Psi_n(x)}(f, x, \mathbb{R}) \,|^p \, dx \, dt.$$

This and (7.35) give the result. Note that we have also effectively shown that

$$\sum_{j=0}^{n-1} \Omega_{r,p}^p \left(f, \left|I_{jn}^*\right|, I_{jn}^*\right) W^p\left(\tau_{jn}\right) \leq C_5 \frac{n}{a_n} \int_0^{C_6 \frac{a_n}{n}} \int_{I_n} \left|W \Delta_{t\Psi_n(x)}^r (f, x, \mathbb{R})\right|^p dx dt. \tag{7.40}$$

For $p = \infty$, the proof is similar, but easier: We see that

$$\| (f - L_n[f])W \|_{L_\infty(\mathbb{R})}^p \leq \max\{\max_{0 \leq j \leq n-1} \| (f - p_j)W \|_{L_\infty(I_{jn})}, \| fW \|_{L_\infty(|x| \geq a_n)}\}.$$

The rest of the proof is as before.□

Now we can define our polynomial approximation to f:

$$P_n[f] := p_0(x) R_{n,\tau_{on}}(x) + \sum_{k=1}^{n-1} (p_k - p_{k-1})(x) R_{n,\tau_{kn}}(x). \tag{7.41}$$

Note, that this has been formed from $L_n[f]$ of (7.32) by replacing the characteristic function $\theta_{kn}(x) = \chi_{[\tau_{kn}, a_n]}(x)$ by its polynomial approximation $R_{n,\tau_{kn}}(x)$ formed in the previous section.

Lemma 7.1.7. Let $\{\Psi_n\}_n$ be as in the previous lemma. Then

$$\| (L_n[f] - P_n[f])W \|_{L_p(\mathbb{R})}$$

$$\leq C_1 \left\{ \left[\frac{n}{a_n} \int_0^{C_2 \frac{a_n}{n}} \| W \Delta_{h\Psi_n(x)}^r (f, x, \mathbb{R}) \|_{L_p[-a_n, a_n]}^p dh \right]^{\frac{1}{p}} + \|fW\|_{L_p(I_{o,n}^*)} \right\} \tag{7.42}$$

and for $p = \infty$, we replace the pth root and integ:al by $\sup_{0 < h \leq C_2 \frac{a_n}{n}}$.

Proof. We see that if we define $p_{-1}(x) \equiv 0$,

$$(L_n[f] - P_n[f])(x)$$

$$= \sum_{k=0}^{n-1} (p_k - p_{k-1})(x)(\theta_{kn}(x) - R_{n,\tau_{kn}}(x)). \tag{7.43}$$

We shall make substantial use of the following inequality: Let S be a polynomial of degree at most r, and $[a,b]$ be a real interval. Then for all $x \in \mathbb{R}$,

$$| S(x) | \leq C(b-a)^{-1/p}(1 + \frac{\min\{| x-a |, | x-b |\}}{b-a})^r \| S \|_{L_p[a,b]}. \tag{7.44}$$

Here $C \neq C(a,b,x,S)$ but $C = C(p,r)$.

This follows from standard Nikolskii inequalities and the Bernstein-Walsh inequality. See for example [47,p 193].

Hence for $x \in \mathbb{R}$, and $1 \leq k \leq n-1$,

$$| p_k - p_{k-1} | (x) \leq C \mid I_{kn} \mid^{-1/p} (1 + \frac{| x - \tau_{kn} |}{| I_{kn} |})^r \| p_k - p_{k-1} \|_{L_p(I_{kn})}.$$

This is still true for $k=0$ if we recall that $p_{-1} \equiv 0$. Now for $1 \leq k \leq n-1$, (7.31) gives

$$\| p_k - p_{k-1} \|_{L_p(I_{kn})} \leq C_1 \sum_{i=k-1}^{k} \Lambda_{r,p}(f, | I_{in}^* |, I_{in}^*)$$

where $C_1 \neq C_1(f,k,n)$.

This remains true for $k=0$ if we set

$$|I_{-1,n}| := |I_{0,n}|;\ \left|I_{-1,n}^*\right| := \left|I_{0,n}^*\right|;\ \tau_{-1,n} := \tau_{0,n}$$

and

$$\Lambda_{r,p}(f, | I_{-1,n}^* |, I_{-1,n}^*) := \| f \|_{L_p(I_{0n}^*)} = \Omega_{r,p}\left(f, | I_{-1,n}^* |, I_{-1,n}^*\right).$$

Since (see (6.33), (6.34), (7.27)) uniformly in k, n, and $x \in \mathbb{R}$,

$$1 + \frac{| x - \tau_{kn} |}{| I_{kn} |} \sim 1 + \frac{| x - \tau_{k-1,n} |}{| I_{k-1,n} |}$$

we obtain from (7.44) and Theorem 7.1.3, uniformly for $0 \le k \le n-1$ and $x \in \mathbb{R}$,

$$| (p_k - p_{k-1})(x)(\theta_{kn}(x) - R_{n,\tau_{kn}}(x)) | \frac{W(x)}{W(\tau_{kn})}$$

$$\le C_2 \sum_{i=k-1}^{k} | I_{in} |^{-1/p} (1 + \frac{| x - \tau_{in} |}{| I_{in} |})^{r-l} \Omega_{r,p}(f, | I_{in}^* |, I_{in}^*). \qquad (7.45)$$

We consider three different ranges of p:

(I) $0 < p < 1$

Here from (7.43) and then (7.45),

$$\int_{\mathbb{R}} (| L_n[f] - P_n[f] | W)^p$$

$$\le \sum_{k=0}^{n-1} \int_{\mathbb{R}} (| p_k - p_{k-1} || \theta_{kn} - R_{n,\tau_{kn}} | W)^p$$

$$\le \sum_{k=-1}^{n-1} | I_{kn} |^{-1} \Omega_{r,p}^p(f, | I_{kn}^* |, I_{kn}^*) W^p(\tau_{kn}) \int_{\mathbb{R}} (1 + \frac{| x - \tau_{kn} |}{| I_{kn} |})^{(r-l)p} dx. \qquad (7.46)$$

Here if $(r-l)p < -1$,

$$| I_{kn} |^{-1} \int_{\mathbb{R}} (1 + \frac{| x - \tau_{kn} |}{| I_{kn} |})^{(r-l)p} dx = \int_{\mathbb{R}} (1 + | u |)^{(r-l)p} du =: C_3 < \infty.$$

So

$$\int_{\mathbb{R}} (| L_n[f] - P_n[f] | W)^p$$

$$\le C_4 \sum_{k=-1}^{n-1} \Omega_{r,p}^p(f, | I_{kn}^* |, I_{kn}^*) W^p(\tau_{kn}).$$

This is the same as our sum in (7.40) except for the term for $k = -1$. So the est' ate (7.40) gives the estimate (7.42), keeping in mind our choice of $\Omega_{r,p}\left(f, | I_{-1,n}^* |, I_{-1,n}^*\right)$.

(II) $1 \le p < \infty$

From (7.43), (7.45) and then Hölder's inequality,

$$\{| L_n[f] - P_n[f] | (x) W(x)\}^p$$

$$\le C\{\sum_{k=-1}^{n-1} \mid I_{kn} \mid^{-1/p} (1+\frac{\mid x-\tau_{kn}\mid}{\mid I_{kn}\mid})^{r-l}\Omega_{r,p}(f,\mid I_{kn}^* \mid, I_{kn}^*)W(\tau_{kn})\}^p$$

$$\le C\sum_{k=-1}^{n} \mid I_{kn} \mid^{-1} (1+\frac{\mid x-\tau_{kn}\mid}{\mid I_{kn}\mid})^{(r-l)p/2}\Omega_{r,p}^p(f,\mid I_{kn}^* \mid, I_{kn}^*)W^p(\tau_{kn})\cdot S_n(x)^{p/q} \tag{7.47}$$

where $q := p/(p-1)$ and

$$S_n(x) := \sum_{k=1}^{n}(1+\frac{\mid x-\tau_{kn}\mid}{\mid I_{kn}\mid})^{(r-l)q/2}.$$

We shall show that if $(r-l)q/2 < -1$, then

$$\sup_{n\ge 1}\sup_{x\in\mathbb{R}} S_n(x) \le C_1 < \infty. \tag{7.48}$$

Note that $S_n(x)$ is a decreasing function of x for $x \ge a_n = \tau_{nn}$, so it suffices to consider $x \in [0, a_n]$. Recall that

$$\mid I_{kn} \mid \sim \mid I_{k+1,n} \mid \sim \frac{a_n}{n}\sqrt{1-\frac{\mid \tau_{kn}\mid}{a_{2n}}}.$$

It is then not difficult to see that

$$S_n(x) \le C_2\frac{n}{a_n}\int_{-a_n}^{a_n}(1+\frac{n}{a_n}\frac{\mid x-u\mid}{\sqrt{1-\frac{|u|}{a_{2n}}}})^{(r-l)q/2}\frac{du}{\sqrt{1-\frac{|u|}{a_{2n}}}}$$

$$\le C_3 n\int_{-1}^{1}(1+n\frac{\mid \overline{x}-s\mid}{\sqrt{1-s}})^{(r-l)q/2}\frac{ds}{\sqrt{1-s}}$$

where $\overline{x} := x/a_{2n}$, so that

$$1-\overline{x} \ge 1-a_n/a_{2n} \ge C_4 T(a_n)^{-1} \ge C_5 n^{-2}.$$

We make the substitution $(1-s) = (1-\overline{x})w$ to obtain

$$S_n(x) \le C_3 n\sqrt{1-\overline{x}}\int_{0}^{\frac{2}{1-\overline{x}}}(1+n\sqrt{1-\overline{x}}\frac{\mid w-1\mid}{\sqrt{w}})^{(r-l)q/2}\frac{dw}{\sqrt{w}}$$

$$\le C_4 n\sqrt{1-\overline{x}}\{\int_{0}^{1/2}[1+\frac{n\sqrt{1-\overline{x}}}{\sqrt{w}}]^{(r-l)q/2}\frac{dw}{\sqrt{w}}$$

$$+\int_{1/2}^{3/2}[1+n\sqrt{1-\overline{x}}\mid w-1\mid]^{(r-l)q/2}dw$$

$$+\int_{3/2}^{2/(1-\overline{x})}[1+n\sqrt{(1-\overline{x})w}]^{(r-l)q/2}\frac{dw}{\sqrt{w}}\}.$$

(We can omit the third integral if $2/(1-\overline{x}) \leq 3/2$.)

We now make the substitutions $w = n^2(1-\overline{x})v$ in the first integral, $v = n\sqrt{1-\overline{x}}(w-1)$ in the second integral, and $v = n^2(1-\overline{x})w$ in the third integral. It is then not difficult to see that the resulting terms are bounded independent of n and x if l is large enough. So we have (7.48). Then, integrating (7.47) and using (7.40) gives our result.

(III) $p = \infty$

Now

$$|L_n[f]-P_n[f]|(x) \leq C\sum_{k=0}^{n-1}|p_k-p_{k-1}|(x)\,|\theta_{kn}-R_{n,\tau_{kn}}|(x)W(x)$$

$$\leq C\max_{-1\leq k\leq n-1}\Omega_{r,p}(f,|I_{kn}^*|,I_{kn}^*)W(\tau_{kn})\cdot\sum_{k=0}^{n-1}(1+\frac{|x-\tau_{kn}|}{|I_{kn}|})^{(r-l)}.$$

As before, the sum is bounded if l is large enough. Then we can continue this as

$$\leq C_1\{\sup_{0\leq k\leq n-1}\sup_{0<h\leq|I_{kn}^*|}\|\Delta_h^r(f,x,I_{kn}^*)W\|_{L_\infty(I_{kn}^*)}+\|fW\|_{L_\infty(I_{0n}^*)}\}$$

$$\leq C_2\{\sup_{0\leq k\leq n-1}\sup_{0<h\leq Ca_n/n}\|\Delta_{h\Psi_n(x)}^r(f,x,I_{kn}^*)W\|_{L_\infty(I_{kn}^*)}+\|fW\|_{L_\infty(I_{0n}^*)}\}$$

$$\leq C_3\{\sup_{0<h\leq Ca_n/n}\|\Delta_{h\Psi_n(x)}^r(f,x,\mathbb{R})W\|_{L_\infty(-a_n,a_n)}+\|fW\|_{L_\infty(I_{0n}^*)}\}.$$

□

We can now turn to:

The Proof of Theorem 5.1.3. Now recall that $R_{n,\tau}$ has degree at most $2lJn$, where J is as in the proof of Theorem 7.1.3. So $P_n[f]$ has degree at most $2lJn+r$. So, if $M > 3lJ$, we have for large n,

$$E_{Mn}[f]_{W,p} \leq \|(f-P_n[f])W\|_{L_p(\mathbb{R})}$$

$$\leq \{\| (f - L_n[f])W \|_{L_p(\mathbb{R})} + \| (L_n[f] - P_n[f])W \|_{L_p(\mathbb{R})}\}$$

$$\leq C_1 \left\{ \left[\frac{n}{a_n} \int_0^{C_2 \frac{a_n}{n}} \| W \Delta^r_{h\Psi_n(x)}(f, x, \mathbb{R}) \|^p_{L_p[-a_n, a_n]} \, dh \right]^{\frac{1}{p}} \right.$$

$$\left. + \| fW \|_{L_p(|x| \geq a_n(1 - C_2[nT(a_n)^{1/2}]^{-1}))} \right\}. \tag{7.49}$$

Here we have used Lemmas 7.1.6 and 7.1.7, and also (7.27), which implies that

$$| I^*_{0n} | \sim \frac{a_n}{n} \sqrt{1 - \frac{a_n}{a_{2n}}} \sim \frac{a_n}{n} T(a_n)^{-1/2}.$$

Next for

$$Mn \leq j \leq M(n+1) \tag{7.50}$$

we write

$$n = \kappa j,$$

where $\kappa = \kappa(j, n)$. Note that

$$\kappa = \frac{n}{j} \to \frac{1}{M}, \; j \to \infty. \tag{7.51}$$

We set

$$t := t(j) := \frac{M a_j}{3j}.$$

Note that then

$$\frac{t}{a_n/n} = \frac{1}{3} \, \frac{Mn}{j} \, \frac{a_j}{a_n} = \frac{1}{3}(1 + o(1)), n \to \infty. \tag{7.52}$$

Let $\beta > 3$. We claim that for large enough n,

$$a_n(1 - C_2[nT(a_n)^{1/2}]^{-1}) \geq \sigma(\beta t). \tag{7.53}$$

To see this, note from (6.9) that

$$[nT(a_n)^{1/2}]^{-1} = o(T(a_n)^{-1})$$

so that by (6.7), if $1 > \alpha > 3/\beta$,

$$a_n(1 - C_2[nT(a_n)^{1/2}]^{-1}) \geq a_n(1 - o(\frac{1}{T(a_n)})) \geq a_{\alpha n}$$

$$\geq \sigma(\frac{a_{\alpha n}}{\alpha n}) = \sigma(\frac{3t}{\alpha}[1 + o(1)]) \geq \sigma(\beta t),$$

for large enough j, by first (6.28) and then (7.52). Next, we claim that if $0 < \gamma < 3$, then for n large enough,

$$a_n \leq \sigma(\gamma t). \tag{7.54}$$

To see this, note that by (7.52) if $1 < \delta < 3/\gamma$

$$\sigma(\gamma t) = \sigma(\frac{\gamma a_n}{3n}[1 + o(1)]) \geq \sigma(\frac{a_{\delta n}}{\delta n}) = a_{\delta n(1+o(1))} \geq a_n.$$

Here we also used the fact that σ is decreasing, and also (6.28), (6.29) with n large enough. Since also $2t \leq a_n/n \leq 4t$ for large enough n, (see (7.52)) we can recast (7.49) as

$$E_j[f]_{W,p} \leq E_{Mn}[f]_{W,p}$$

$$\leq C_1 \left\{ \left[\frac{1}{2t} \int_0^{C4t} \| W\Delta^r_{h\Psi_n(x)}(f, x, \mathbb{R}) \|^p_{L_p[|x| \leq \sigma(2t)]} \, dh \right]^{\frac{1}{p}} \right.$$

$$\left. + \| fW \|_{L_p(|x| \geq \sigma(4t))} \right\}. \tag{7.55}$$

Now we choose $h_1 := h/(4C)$ and $\Psi_n := \Phi_t/(4C)$ so that $h\Psi_n = h_1\Phi_t$. We must show that (7.33) holds with constants independent of j and n, that is

$$(4C)^{-1}\Phi_t(x) \sim \sqrt{1 - \frac{|x|}{a_{2n}}}, |x| \leq a_n.$$

But for this range of x,

$$\sqrt{1 - \frac{|x|}{a_{2n}}} \sim \sqrt{1 - \frac{|x|}{a_{2n}}} + T(a_{2n})^{-1/2} \sim \Phi_{\frac{a_{2n}}{2n}}(x) \sim \Phi_t(x)$$

by (6.33) and (6.34). Then, with a suitable choice of $P_0 \in \mathcal{P}_{r-1}$, we have using

$$\Delta_{h_1\Phi_t(x)}(P, x, \mathbb{R}) \equiv 0$$

that

$$\begin{aligned}
E_j[f]_{W,p} &= E_j[f - P_0]_{W,p} \\
&\leq C_3 \left\{ \left[\frac{1}{t} \int_0^t \| W\Delta^r_{h_1\Phi_t(x)}(f - P_0, x, \mathbb{R}) \|^p_{L_p[|x| \leq \sigma(2t)]} \, dh_1 \right]^{\frac{1}{p}} \right. \\
&\qquad \left. + \| (f - P_0)W \|_{L_p(|x| \geq \sigma(4t))} \right\} \\
&\leq 2C_3 \left\{ \left[\frac{1}{t} \int_0^t \| W\Delta^r_{h_1\Phi_t(x)}(f, x, \mathbb{R}) \|^p_{L_p[|x| \leq \sigma(2t)]} \, dh_1 \right]^{\frac{1}{p}} \right. \\
&\qquad \left. + \inf_{P \in \mathcal{P}_{r-1}} \| (f - P)W \|_{L_p(|x| \geq \sigma(4t))} \right\} \\
&= 2C_3 \overline{\omega}_{r,p}(f, W, t) = 2C_3 \overline{\omega}_{r,p}(f, W, \frac{Ma_j}{3j}). \square
\end{aligned}$$

The Proof of Theorem 5.1.4.

Obviously (5.14) implies (5.13). The only difference to the above proof is that we choose

$$t_1 := \rho t := \rho \frac{Ma_j}{3j}$$

to replace t above. Then from (7.52),

$$\frac{t_1}{a_n/n} = \frac{\rho}{3}(1 + o(1))$$

and here $\frac{\rho}{3} \in [\frac{4}{15}, \frac{1}{3}]$. Then as $4\rho > 3$, (7.53) above shows that

$$a_n(1 - C_2[nT(a_n)^{1/2}]^{-1}) \geq \sigma(4\rho t) = \sigma(4t_1)$$

and as $\rho \leq 1$, (7.54) above shows that

$$a_n \leq \sigma(2\rho t) = \sigma(2t_1).$$

Moreover, $a_n/n \leq 3t \leq 4t_1$ and $\frac{a_n}{n} \geq 2t = \frac{2t_1}{\rho} \geq 2t_1$. Choosing $h_1 := h/(4C)$ and $\Psi_n(x) := \Phi_{t_1}(x)/(4C)$, we note that (7.33) holds uniformly in ρ. We proceed as before to obtain

$$E_j[f]_{W,p} \leq C_1 \overline{\omega}_{r,p}(f, W, C_2 \frac{\rho a_j}{j})$$

with constants independent of ρ, f, j.□

7.4 The Proof of Theorem 5.1.5

We turn to the proof of Theorem 5.1.5. We provide full proofs only where the details are significantly different, and otherwise refer back. We begin with an analogue of Lemma 7.1.6 for $L_n[f]$ of (7.32).

Lemma 7.1.8

$$\| (f - L_n[f])W \|_{L_p(\mathbb{R})} \leq$$

$$\leq C_1 [\sup_{\substack{0<h\leq a_{3n}/(3n) \\ 0<\tau \leq L}} \| W \Delta^r_{\tau h \Phi_h(x)}(f, x, \mathbb{R}) \|_{L_p[-a_n, a_n]} + \| fW \|_{L_p(|x| \geq a_n)}]. \qquad (7.56)$$

Here L is independent of f, n.

Proof

We do this for $p < \infty$. Recall that the crux of Lemma 7.1.6 is estimation of

$$\Delta_{jn} := \int_{I_{jn}} | f - p_j |^p W^p \leq C_1 \Omega_{r,p}(f, | I^*_{jn} |, I^*_{jn})^p W^p(\tau_{jn})$$

$$\leq \frac{C_2}{|I_{jn}^*|}\int_{I_{jn}^*}\int_0^{|I_{jn}^*|} |W\Delta_s^r(f,x,I_{jn}^*)|^p\, ds\, dx. \tag{7.57}$$

We now choose $L>0$ such that

$$\sup_{x\in\mathbb{R}} \frac{\frac{h}{L}\Phi_{\frac{h}{L}}(x)}{h\Phi_h(x)} \leq \frac{1}{2}. \tag{7.58}$$

This is possible by (6.40). Now we choose

$$\delta_{n,k}(x) := L^{1-k}\frac{a_{3n}}{3n}\Phi_{L^{1-k}\frac{a_{3n}}{3n}}(x),\, k\geq 1.$$

Note that by (7.58),

$$\sup_{x\in\mathbb{R}} \frac{\delta_{n,k+1}(x)}{\delta_{n,k}(x)} \leq \frac{1}{2}. \tag{7.59}$$

In view of (7.27), (6.33) and (6.34), we may assume that L is so large that uniformly in n, j, $x\in I_{jn}^*$,

$$|I_{jn}^*| \leq L\frac{a_{3n}}{3n}\Phi_{\frac{a_{3n}}{3n}}(x) = L\delta_{n,1}(x);\, |I_{jn}^*| \sim \delta_{n,1}(x).$$

Then from (7.57),

$$\Delta_{jn} \leq C_4\int_{I_{jn}^*}\int_0^{L\delta_{n,1}(x)} \frac{1}{\delta_{n,1}(x)} |W\Delta_s^r(f,x,I_{jn}^*)|^p\, ds\, dx$$

$$= C_4\int_{I_{jn}^*}\sum_{k=1}^{\infty}\int_{L\delta_{n,k+1}(x)}^{L\delta_{n,k}(x)} \frac{1}{\delta_{n,1}(x)} |W\Delta_s^r(f,x,I_{jn}^*)|^p\, ds\, dx$$

$$= C_4\int_{I_{jn}^*}\sum_{k=1}^{\infty}\int_{L\delta_{n,k+1}(x)/\delta_{n,k}(x)}^{L} \frac{\delta_{n,k}(x)}{\delta_{n,1}(x)} |W\Delta_{\tau\delta_{n,k}(x)}^r(f,x,I_{jn}^*)|^p\, d\tau\, dx$$

$$\leq C_4\int_{I_{jn}^*}\sum_{k=1}^{\infty}(\frac{1}{2})^{k-1}\int_0^{L} |W\Delta_{\tau\delta_{n,k}(x)}^r(f,x,I_{jn}^*)|^p\, d\tau\, dx.$$

Then

$$\sum_{j=0}^{n-1}\Delta_{jn} \leq C_4\int_{-a_n}^{a_n}\sum_{k=1}^{\infty}(\frac{1}{2})^{k-1}\int_0^{L} |W\Delta_{\tau\delta_{n,k}(x)}^r(f,x,\mathbb{R})|^p\, d\tau\, dx$$

$$\leq 2C_4 \sup_{\substack{0<h\leq a_{3n}/(3n)\\ 0<\tau\leq L}}\int_{-a_n}^{a_n} |W\Delta_{\tau h\Phi_h(x)}^r(f,x,\mathbb{R})|^p\, dx.$$

The rest of the proof is as before.□

The analogue of Lemma 7.1.7 is

Lemma 7.1.9

$$\| (L_n[f] - P_n[f])W \|_{L_p(\mathbb{R})}$$

$$\leq C_1 \{ \sup_{\substack{0<h\leq a_{3n}/(3n) \\ 0<r\leq L}} \| W \Delta^r_{rh\Phi_h(x)}(f, x, \mathbb{R}) \|_{L_p[-a_n, a_n]} + \| fW \|_{L_p(|x|\geq a_n)} \}.$$

Proof

This is exactly the same as the proof of Lemma 7.1.7, except that we substitute for (7.40) the estimate of Lemma 7.1.8.□

Proof of Theorem 5.1.5.

This follows from Lemma 7.1.8 and 7.1.9 exactly as Theorem 5.1.3 followed from Lemma 7.1.6 and 7.1.7.□

7.5 The Proof of Theorem 5.1.6

Using (5.18) and the methods of proof of Lemma 2.2 in [26,p.209], we obtain

$$\frac{a_u'}{a_u} \sim \frac{1}{uT(a_u)}, u \geq C_2 \tag{7.60}$$

and hence

$$\frac{d}{du}(\frac{a_u}{u}) \sim -\frac{a_u}{u^2}, u \geq C_2. \tag{7.61}$$

Since $u \to \frac{a_u}{u}$ is then strictly decreasing for large u, we obtain the identity

$$\sigma(\frac{a_u}{u}) = a_u, u \geq C_3. \tag{7.62}$$

Differentiating this, and using (7.60), (7.61) leads to

$$\frac{\sigma'(t)}{\sigma(t)} \sim -\frac{1}{tT(\sigma(t))}, 0 < t \leq C_4 \tag{7.63}$$

and then using (5.19), we obtain

$$| t\frac{d}{dt}T(\sigma(t)) | \leq C_5 T(\sigma(t)), 0 < t \leq C_4. \tag{7.64}$$

These last two bounds easily give

$$| \frac{d}{dt}[t\Phi_t(x)] | \leq C_5 \Phi_t(x) \tag{7.65}$$

for

$$0 < t \leq C_5; | 1 - \frac{| x |}{\sigma(t)} | \geq \frac{\varepsilon}{T(\sigma(t))}. \tag{7.66}$$

Here ε is any fixed positive number. We now estimate Δ_{jn} a little differently from the way we proceeded after (7.57). Let us make the substitution $s = Lt\Phi_t(x)$ in the right-hand side of (7.57) and keep our choice of $L, \delta_{n,1}(x)$ to deduce that

$$\Delta_{jn} \leq C_6 \int_{I_{jn}^*} \int_0^{a_{3n}/(3n)} \frac{1}{\delta_{n,1}(x)} | W\Delta_{Lt\Phi_t(x)}^r(f, x, I_{jn}^*) |^p | \frac{d}{dt}[t\Phi_t(x)] | \, dt \, dx$$

$$\leq \frac{C_7 3n}{a_{3n}} \int_{I_{jn}^*} \int_0^{a_{3n}/(3n)} \sqrt{\log(2 + \frac{a_{3n}}{3nt})} \, | W\Delta_{Lt\Phi_t(x)}^r(f, x, I_{jn}^*) |^p \, dt \, dx$$

by (7.65) and (6.40). In applying (7.65), we must ensure that the range conditions in (7.66) must hold for $x \in I_{jn}^*$ and $t \leq a_{3n}/(3n)$. In fact if $| x | \leq a_n$, then

$$1 - \frac{| x |}{\sigma(t)} \geq 1 - \frac{a_n}{\sigma(a_{3n}/(3n))} \geq 1 - \frac{a_n}{a_{3n(1+o(1))}}$$

$$\geq C_8 T(a_n)^{-1} \geq C_9 T(\sigma(t))^{-1}$$

by (6.28), (6.29), then (6.7) and then (6.6)(i). Thus,

$$\sum_{j=0}^{n-1} \Delta_{jn}$$

$$\leq \frac{C_8 3n}{a_{3n}} \int_{-a_n}^{a_n} \int_0^{a_{3n}/(3n)} \sqrt{\log(2+\frac{a_{3n}}{3nt})} \mid W\Delta^r_{Lt\Phi_t(x)}(f,x,\mathbb{R}) \mid^p dt\, dx$$

$$\leq C_8 \sup_{0<t\leq a_{3n}/(3n)} \int_{-a_n}^{a_n} \mid W\Delta^r_{Lt\Phi_t(x)}(f,x,\mathbb{R}) \mid^p dx \int_0^1 \sqrt{\log(2+\frac{1}{s})} ds.$$

So under the additional conditions on Q we obtain

$$E_n[f]_{W,p} \leq C_9 \omega^{\#}_{r,p}(f, W, C_{10}\frac{a_n}{n}).\square \tag{7.67}$$

Chapter 8

The Equivalence Theorem

8.1 A Crucial Inequality

In this section, we obtain a crucial inequality introduced in a similar context in [11], in order to obtain an upper bound for our modulus in terms of our realisation-functional. The main idea is to approximate polynomials of degree $\leq n$ by polynomials of degree $\leq r-1$. Here $n \geq n_0$ and $r \geq 1$.

We prove:

Theorem 8.1.1. *Let $W \in \mathcal{E}_1$ and assume (5.27). Let $r \geq 1$, $L > 0$, $0 < p \leq \infty$, $P_n \in \mathcal{P}_n$ and $n \geq C$. Set*

$$P(x) := P_n(x) - \int_{a_{Ln}}^{x} \int_{a_{Ln}}^{u_{r-1}} \dots \int_{a_{Ln}}^{u_1} P_n^{(r)}(u_o)\, du_o \dots du_{r-1} \in \mathcal{P}_{r-1}. \tag{8.1}$$

Then, $\exists\, C_1 > 0$, $C_1 \neq C_1(n, P_n, P)$ such that

$$\|W(P_n - P)\|_{L_p[a_{Ln},\infty)} \leq C_1 \left(\frac{a_n}{n}\right)^r \left\| W P_n^{(r)} \Phi^r_{\frac{a_n}{n}} \right\|_{L_p(\mathbb{R})}. \tag{8.2}$$

We break the proof down into several steps. We begin with:

Lemma 8.1.2. Let $W \in \mathcal{E}_1$, $1 \leq p \leq \infty$. Then for $n \geq C$ and $\forall g \in L_p[a_{Ln}, \infty)$, $\exists\, C_1 >$

0, $C_1 \neq C_1(g,n)$ such that

$$\left\| W(x) \int_{a_{Ln}}^{x} g(u)\, du \right\|_{L_p[a_{Ln},\infty)} \leq \frac{a_n}{nT(a_n)^{\frac{1}{2}}} \|gW\|_{L_p[a_{Ln},\infty)} \tag{8.3}$$

Proof. We notice that

$$\begin{aligned} & W(x)^{\frac{1}{2}} \int_{t}^{x} W(u)^{-\frac{1}{2}} Q'(u)\, du \\ = \; & 2\left[1 - \left[\frac{W(x)}{W(t)}\right]^{\frac{1}{2}}\right] \leq 2 \end{aligned} \tag{8.4}$$

as $t \leq x$.

Next, notice that for $u \geq a_{Ln}$, and n large enough, we have by Lemma 6.1.2

$$Q'(u) \geq CQ'(a_{Ln}) \sim \frac{nT(a_n)^{\frac{1}{2}}}{a_n}, \tag{8.5}$$

so that for $x \geq a_{Ln}$

$$\begin{aligned} & \frac{a_n}{nT(a_n)^{\frac{1}{2}}} W(x)^{\frac{1}{2}} \int_{a_{Ln}}^{x} |gW(u)|\, Q'(u)\, W^{-\frac{1}{2}}(u)\, du \\ \geq \; & C_1 W(x)^{\frac{1}{2}} \int_{a_{Ln}}^{x} \left|gW(u)^{\frac{1}{2}}\right| du \geq W(x) \left|\int_{a_{Ln}}^{x} g(u)\, du\right|. \end{aligned} \tag{8.6}$$

Now recalling Jensen's Inequality for integrals

$$\left|\int f d\mu\right|^{p} \leq \left(\int |f|^{p} d\mu\right)\left(\int d\mu\right)^{p-1}$$

valid for μ measurable functions f and non negative measures μ, gives:

<u>Case 1. $p = \infty$.</u> Here (8.6) gives for $x \geq a_{Ln}$

$$\begin{aligned} W(x)\left|\int_{a_{Ln}}^{x} g(u)\, du\right| & \leq \frac{a_n}{nT(a_n)^{\frac{1}{2}}} W(x)^{\frac{1}{2}} \|gW\|_{L_\infty[a_{Ln},\infty)} \int_{a_{Ln}}^{x} Q'(u)\, W^{-\frac{1}{2}}(u)\, du \\ & \leq C_2 \frac{a_n}{nT(a_n)^{\frac{1}{2}}} \|gW\|_{L_\infty[a_{Ln},\infty)} \text{ (by (8.4))}. \end{aligned}$$

Case2. $1 \leq p < \infty$. Here

$$\begin{aligned}
& \left\| W(x) \int_{a_{Ln}}^{x} g(u)\, du \right\|_{L_p[a_{Ln},\infty)} \\
\leq\ & \frac{a_n}{nT(a_n)^{\frac{1}{2}}} \left[\int_{a_{Ln}}^{\infty} \left[W(x)^{\frac{1}{2}} \int_{a_{Ln}}^{x} |gW(u)|\, Q'(u)\, W^{-\frac{1}{2}}(u)\, du \right]^p dx \right]^{\frac{1}{p}} \\
\leq\ & C_3 \frac{a_n}{nT(a_n)^{\frac{1}{2}}} \left[\int_{a_{Ln}}^{\infty} 2^{p-1} W(x)^{\frac{1}{2}} \int_{a_{Ln}}^{x} |gW(u)|^p\, Q'(u)\, W^{-\frac{1}{2}}(u)\, du dx \right]^{\frac{1}{p}}
\end{aligned}$$

by Jensens Inequality, with $d\mu = W(x)^{\frac{1}{2}} Q'(u)\ \ (u)^{-\frac{1}{2}}$ on $[a_{Ln}, x]$ and $\int d\mu \leq 2$ (see (8.4)).

Then

$$\begin{aligned}
& \int_{a_{Ln}}^{\infty} W(x)^{\frac{1}{2}} \int_{a_{Ln}}^{x} |gW(u)|^p\, Q'(u)\, W^{-\frac{1}{2}}(u)\, du dx \\
=\ & \int_{a_{Ln}}^{\infty} |gW(u)|^p \left[\int_{u}^{\infty} W(x)^{\frac{1}{2}} Q'(u)\, dx \right] W^{-\frac{1}{2}}(u)\, du \\
\leq\ & C_4 \int_{a_{Ln}}^{\infty} |gW(u)|^p \left[\int_{u}^{\infty} W(x)^{\frac{1}{2}} Q'(x)\, dx \right] W^{-\frac{1}{2}}(u)\, du \ \ (\text{as } x > u) \\
\leq\ & C_5 \|gW\|^p_{L_p[a_{Ln},\infty)} . \square
\end{aligned}$$

We are now in the position to give

The Proof of Theorem 8.1.1 for $1 \leq p \leq \infty$.

We will repeatedly make use of (6.35) :

$$\Phi_{\frac{a_n}{n}}(x) \geq CT(a_n)^{-\frac{1}{2}}, \ \ \forall x \in \mathbb{R} . \tag{8.7}$$

Firstly, if $r = 1$, Lemma 8.1.2 with $g = P_n'$ gives

$$\begin{aligned}
\left\| W(x) \int_{a_{Ln}}^{x} P_n'(u_o)\, du_o \right\|_{L_p[a_{Ln},\infty)} & \leq C_1 \frac{a_n}{nT(a_n)^{\frac{1}{2}}} \left\| P_n' W \right\|_{L_p(\mathbb{R})} \\
& \leq C_2 \frac{a_n}{n} \left\| P_n' \Phi_{\frac{a_n}{n}}(x)\, W \right\|_{L_p(\mathbb{R})} \ (\text{by } (8.7)).
\end{aligned}$$

Now apply (8.1). If $r = 2$, we apply Lemma 8.1.2 with

$$g(u_1) = \int_{a_{nL}}^{u_1} P^{(2)}(u_o)\, du_o$$

to give

$$\begin{aligned}
&\left\| W(x) \int_{a_{Ln}}^{x} \int_{a_{Ln}}^{u_1} P_n^{(2)}(u_o)\, du_o du_1 \right\|_{L_p[a_{Ln},\infty)} \\
= &\left\| W(x) \int_{a_{Ln}}^{x} g(u_1)\, du_1 \right\|_{L_p[a_{Ln},\infty)} \leq C_3 \frac{a_n}{nT(a_n)^{\frac{1}{2}}} \|gW\|_{L_p[a_{Ln},\infty)} \\
= &\; C_3 \frac{a_n}{nT(a_n)^{\frac{1}{2}}} \left\| W \int_{a_{nL}}^{u_1} P_n^{(2)}(u_o)\, du_o \right\|_{L_p[a_{Ln},\infty)} \\
\leq &\; C_4 \left(\frac{a_n}{nT(a_n)^{\frac{1}{2}}} \right)^2 \left\| P_n^{(2)} W \right\|_{L_p(\mathbb{R})} \leq C_5 \left(\frac{a_n}{n} \right)^2 \left\| P_n^{(2)} \Phi_{\frac{a_n}{n}}^2(x) W \right\|_{L_p(\mathbb{R})}.
\end{aligned}$$

Applying now (8.1), and an induction argument on r gives the result.□

We now tackle the more complicated case, $0 < p < 1$. For this case we need two lemmas.

Lemma 8.1.3. Let $W \in \mathcal{E}_1$ and assume (5.27). Let $0 < p < 1$, $r \geq 1$, $R_n \in \mathcal{P}_n$, $R \in \mathcal{P}_{r-1}$ and $n \geq C$. Set for $x \in \mathbb{R}$, and $L > 0$

$$g_n(x) := (R_n - R)(x)$$

and

$$J_n(x) := \left\| \left| g_n' W(u) \right|^{1-p} \left(\frac{W(x)}{W(u)} \right)^{\frac{1}{2}} \right\|_{L_\infty[a_{Ln},x]}^{\frac{p}{1-p}}. \tag{8.8}$$

Then

$$\begin{aligned}
\int_{a_{Ln}}^{\infty} J_n(x)\, dx \;\; \leq \;\; C_1 & \left[\sum_{j=1}^{r-1} \left(\frac{a_n}{nT(a_n)^{\frac{1}{2}}} \right)^{(j-1)p} \left\| W\left(R_n^{(j)} - R^{(j)} \right) \right\|_{L_\infty[a_{Ln},\infty)}^p \right. \\
& \left. + \left(\frac{a_n}{nT(a_n)^{\frac{1}{2}}} \right)^{(r-1)p} \left\| W R_n^{(r)} \right\|_{L_\infty(\mathbb{R})}^p \right].
\end{aligned} \tag{8.9}$$

Here $C_1 \neq C_1(n, R_n, R)$.

Proof. Write

$$J_n(x) = \left\| \left| g_n' W(u) \right|^p \left(\frac{W(x)}{W(u)} \right)^{\frac{p}{2(1-p)}} \right\|_{L_\infty[a_{Ln},x]}$$

and set

$$\tau := \frac{\delta a_n}{nT(a_n)^{\frac{1}{2}}}$$

where $\delta > 0$ is chosen small enough so that for $n \geq 1$ and $\forall S \in \mathcal{P}_n$,

$$\left\|S'W\right\|_{L_p(\mathbb{R})} \leq \left(2\delta^{-1}\right) \frac{nT(a_n)^{\frac{1}{2}}}{a_n} \|SW\|_{L_p(\mathbb{R})} . \tag{8.10}$$

(See (5.27) and (6.35)).

Now given $x \geq a_{Ln}$, we set

$$k_o := k_o(x, \tau) = \max\{k_o : x - (k+1)\tau \geq a_{Ln}\}$$

and write

$$J_n(x) \leq I_1 + I_2,$$

where

$$I_1 := \max_{0 \leq k \leq k_o} \left\| |g_n' W|^p (u) \left(\frac{W(x)}{W(u)}\right)^{\frac{p}{2(1-p)}} \right\|_{L_\infty[x-(k+1)\tau,\, x-k\tau]} \tag{8.11}$$

and

$$I_2 := \left\| |g_n' W(u)|^p \left(\frac{W(x)}{W(u)}\right)^{\frac{p}{2(1-p)}} \right\|_{L_\infty[a_{Ln},\, x-(k_o+1)\tau]} . \tag{8.12}$$

First we observe that for $u \in [x-(k+1)\tau,\ x-k\tau]$

$$\frac{W(x)}{W(u)} \leq \exp\left(Q(x-k\tau) - Q(x)\right).$$

Further, as $x - k\tau \geq a_{Ln} > 0$

$$\begin{aligned} Q(x) - Q(x-k\tau) &\geq C_1 k\tau Q'(x-k\tau) \geq C_2 k\tau Q'(a_{Ln}) \geq C_3 \frac{nT(a_n)^{\frac{1}{2}}}{a_n} \frac{\delta a_n k}{nT(a_n)^{\frac{1}{2}}} \\ &= C_3 k\delta \end{aligned}$$

by (6.4). So

$$\left(\frac{W(x)}{W(u)}\right)^{\frac{p}{2(1-p)}} \leq \alpha^k,\ u \in [x-(k+1)\tau, x-k\tau]$$

where $\alpha \in (0,1)$ is independent of x, u, k. Thus we may write

$$\begin{aligned} I_1 + I_2 &\leq \max_{0\leq k\leq k_o} \alpha^k \|g_n' W\|^p_{L_\infty[x-(k+1)\tau,\, x-k\tau]} \\ &\quad + \alpha^{k_o} \|g_n' W\|^p_{L_\infty[a_{Ln},\, x-(k_o+1)\tau]} \\ &\leq \sum_{k=0}^{k_o(x)} \alpha^k \|g_n' W\|^p_{L_\infty[x-(k+1)\tau,\, x-k\tau]} \\ &\quad + \alpha^{k_o} \|g_n' W\|^p_{L_\infty[a_{Ln},\, x-(k_o+1)\tau]}. \end{aligned}$$

Then

$$\begin{aligned} \int_{a_{Ln}}^{\infty} J_n(x)\,dx &= \sum_{m=0}^{\infty} \int_{a_{Ln}+m\tau}^{a_{Ln}+(m+1)\tau} J_n(x)\,dx \\ &\leq \sum_{m=0}^{\infty} \int_{a_{Ln}+m\tau}^{a_{Ln}+(m+1)\tau} \left[\sum_{k=0}^{k_o(x)} \alpha^k \|g_n' W\|^p_{L_\infty[x-(k+1)\tau,\, x-k\tau]} \right. \\ &\quad \left. + \alpha^{k_o} \|g_n' W\|^p_{L_\infty[a_{Ln},\, x-(k_o+1)\tau]}\,dx\right]. \end{aligned}$$

We observe that

$$\int_{a_{Ln}+m\tau}^{a_{Ln}+(m+1)\tau} \|g_n' W\|^p_{L_\infty[x-(k+1)\tau,\, x-k\tau]}\,dx = \int_{a_{Ln}+(m-k-1)\tau}^{a_{Ln}+(m-k)\tau} \|g_n' W\|^p_{L_\infty[x,\, x+\tau]}\,dx$$

and since

$$x \in [a_{Ln}+(m-k-1)\tau, a_{Ln}+(m-k)\tau] \Longrightarrow m \geq k_o \geq m-1,$$

we have

$$\begin{aligned} \int_{a_{Ln}}^{\infty} J_n(x)\,dx &\leq \\ & \sum_{m=0}^{\infty} \left[\sum_{k=0}^{m-1} \int_{a_{Ln}+(m-k-1)\tau}^{a_{Ln}+(m-k)\tau} \alpha^k \|g_n' W\|^p_{L_\infty[x,\, x+\tau]}\,dx \right. \\ & \left. + 2\alpha^{m-1} \int_{a_{Ln}}^{a_{Ln}+\tau} \|g_n' W\|^p_{L_\infty[a_{Ln},\, x]}\,dx\right] \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{s=0}^{\infty}\left[\int_{a_{Ln}+s\tau}^{a_{Ln}+(s+1)\tau}\|g_n'W\|_{L_\infty[x,\,x+\tau]}^p\,dx\left(\sum_{\substack{(m,k)\\ s=m-k-1}}\alpha^k\right)\right]\\
&\quad+2\int_{a_{Ln}}^{a_{Ln}+\tau}\|g_n'W\|_{L_\infty[a_{Ln},\,x]}^p\frac{1}{\alpha(1-\alpha)}dx\\
&\leq C_4[I_3+I_4].
\end{aligned}
$$

Here

$$I_3:=\sum_{s=0}^{\infty}\int_{a_{Ln}+s\tau}^{a_{Ln}+(s+1)\tau}\|g_n'W\|_{L_\infty[x,\,x+\tau]}^p\,dx \tag{8.13}$$

and

$$I_4:=\int_{a_{Ln}}^{a_{Ln}+\tau}\|g_n'W\|_{L_\infty[a_{Ln},\,x]}^p\,dx. \tag{8.14}$$

We begin by estimating I_3. Observe that g_n' is a polynomial of degree $\leq n-1$ in $u\in[x,x+\tau]$, so expanding it in a Tayor series about x gives

$$
\begin{aligned}
|g_n'(u)|^p &= \left|\sum_{j=1}^{n}\frac{g_n^{(j)}(x)(u-x)^{j-1}}{(j-1)!}\right|^p\\
&\leq \sum_{j=1}^{n}\left|g_n^{(j)}(x)\right|^p\tau^{(j-1)p}\\
&\quad\text{(by the inequality, } (a+b)^\alpha\leq a^\alpha+b^\alpha,\ 0<\alpha<1,\ a,b\in\mathbb{R})\\
&\leq \sum_{j=1}^{r-1}\left|R_n^{(j)}(x)-R^{(j)}(x)\right|^p\tau^{(j-1)p}+\sum_{j=r}^{n}\left|R_n^{(j)}(x)\right|^p\tau^{(j-1)p}.
\end{aligned}
$$

Thus using

$$W(u)\leq W(x),\ u\in[x,x+\tau], \tag{8.15}$$

the definition of τ and (8.10) gives

$$
\begin{aligned}
I_3\leq C_5&\left[\textstyle\sum_{j=1}^{r-1}\left\|\left(R_n^{(j)}-R^{(j)}\right)W\right\|_{L_p[a_{Ln},\,\infty)}^p\tau^{(j-1)p}\right.\\
&\left.+\ \tau^{(r-1)p}\textstyle\sum_{j=r}^{n}\left\|R_n^{(j)}W\right\|_{L_p[a_{Ln},\,\infty)}^p\tau^{(j-r)p}\right]
\end{aligned}
$$

$$\leq C_6\left[\sum_{j=1}^{r-1}\left(\frac{a_n}{nT(a_n)^{\frac{1}{2}}}\right)^{(j-1)p}\left\|\left(R_n^{(j)}-R^{(j)}\right)W\right\|_{L_p[a_{Ln},\infty)}^p\right.$$
$$\left.+\tau^{(r-1)p}\sum_{j=r}^{n}\left(\frac{\tau nT(a_n)^{\frac{1}{2}}}{2\delta a_n}\right)^{(j-r)p}\left\|R_n^{(r)}W\right\|_{L_p(\mathbb{R})}^p\right]$$

$$\leq C_7\left[\sum_{j=1}^{r-1}\left(\frac{a_n}{nT(a_n)^{\frac{1}{2}}}\right)^{(j-1)p}\left\|\left(R_n^{(j)}-R^{(j)}\right)W\right\|_{L_p[a_{Ln},\infty)}^p\right.$$
$$\left.+\left(\frac{a_n}{nT(a_n)^{\frac{1}{2}}}\right)^{(r-1)p}\left\|R_n^{(r)}W\right\|_{L_p(\mathbb{R})}^p\right]. \tag{8.16}$$

To estimate I_4 we proceed in a similar way to that of I_3, except that we use Lemma 6.1.7(*b*) instead of (8.15), which we may in view of the definition of τ, (6.7) and (6.9). Combining our estimates for I_3 and I_4 give the lemma.□

Lemma 8.1.4. Let $W \in \mathcal{E}_1$ and assume (5.27). Let $0 < p < 1$, $r \geq 1$, $L > 0$, $R_n \in \mathcal{P}_n$, $R \in \mathcal{P}_{r-1}$ satisfying,

$$(R_n - R)(a_{Ln}) = 0.$$

Then for $n \geq C$ there exists $C_1 \neq C_1(n, R_n, R)$ such that

$$\|W(R_n - R)\|_{L_p[a_{Ln},\infty)}$$
$$\leq C_1\left[\left[\left(\frac{a_n}{nT(a_n)^{\frac{1}{2}}}\right)\|W(R_n' - R')\|_{L_p[a_{Ln},\infty)}^p\right]\right.$$
$$\times\left[\sum_{j=1}^{r-1}\left(\frac{a_n}{nT(a_n)^{\frac{1}{2}}}\right)^{(j-1)(1-p)}\left\|\left(R_n^{(j)}-R^{(j)}\right)W\right\|_{L_p[a_{Ln},\infty)}^{1-p}\right.$$
$$\left.\left.+\left(\frac{a_n}{nT(a_n)^{\frac{1}{2}}}\right)^{(r-1)(1-p)}\left\|R_n^{(r)}W\right\|_{L_p(\mathbb{R})}^{1-p}\right]\right]. \tag{8.17}$$

Proof. Set

$$g_n(x) := (R_n - R)(x)$$

satisfying $g_n(a_{Ln}) = 0$ and write

$$g_n(x) = \int_{a_{Ln}}^{x} g_n'(u)\,du.$$

Then

$$\begin{aligned}\Delta &= \|W(R_n - R)\|_{L_p[a_{Ln},\infty)} = \|W g_n\|_{L_p[a_{Ln},\infty)} \\ &= \left[\int_{a_{Ln}}^{\infty}\left|\int_{a_{Ln}}^{x} g_n' W(u)\frac{W(x)}{W(u)}du\right|^p dx\right]^{\frac{1}{p}}\end{aligned} \tag{8.18}$$

$$\begin{aligned}\leq &\left[\int_{a_{Ln}}^{\infty}\left\||g_n' W(u)|^{1-p}\left(\frac{W(x)}{W(u)}\right)^{\frac{1}{2}}\right\|_{L_\infty[a_{Ln},\infty)}^p \times\right. \\ &\left.\left(\int_{a_{Ln}}^{x}|g_n' W(u)|^p\left(\frac{W(x)}{W(u)}\right)^{\frac{1}{2}}du\right)^p dx\right]^{\frac{1}{p}}.\end{aligned}$$

Now apply Hölders Inequality with $r = \frac{1}{1-p}$, $\sigma = \frac{1}{p}$ satisfying $r^{-1} + \sigma^{-1} = 1$ to give

$$\Delta \leq I_1 I_2$$

where

$$I_1 := \left(\int_{a_{Ln}}^{\infty}\left\||g_n' W(u)|^{1-p}\left(\frac{W(x)}{W(u)}\right)^{\frac{1}{2}}dx\right\|_{L_\infty[a_{Ln},\infty)}^{\frac{p}{1-p}}\right)^{\frac{(1-p)}{p}} \tag{8.19}$$

and

$$I_2 := \left(\int_{a_{Ln}}^{\infty}\int_{a_{Ln}}^{x}|g_n' W(u)|^p\left(\frac{W(x)}{W(u)}\right)^{\frac{1}{2}}dudx\right). \tag{8.20}$$

Now by (8.8) we may write

$$I_1 = \left(\int_{a_{Ln}}^{\infty}J_n(x)\,dx\right)^{\frac{1-p}{p}} \leq C\left[\sum_{j=1}^{r-1}\left(\frac{a_n}{nT(a_n)^{\frac{1}{2}}}\right)^{(j-1)(1-p)}\right.$$

$$\times\left\|W\left(R_n^{(j)} - R^{(j)}\right)\right\|_{L_p[a_{Ln},\infty)}^{1-p}$$

$$+\left(\frac{a_n}{nT(a_n)^{\frac{1}{2}}}\right)^{(r-1)(1-p)}\left\|WR_n^{(r)}\right\|_{L_p(\mathbb{R})}^{1-p}\Bigg] \tag{8.21}$$

(by Lemma 8.1.3).

Also

$$I_2=\int_{a_{Ln}}^{\infty}|g_n'W(u)|^p\int_u^{\infty}\left(\frac{W(x)}{W(u)}\right)^{\frac{1}{2}}dxdu.$$

Now if $x\geq u\geq a_{Ln}$, Lemma 6.2 gives

$$Q'(x)\geq C_1Q'(a_{Ln})\geq C_2\frac{nT(a_n)^{\frac{1}{2}}}{a_n}$$

so that

$$\begin{aligned} I_2 &\leq C_3\frac{a_n}{nT(a_n)^{\frac{1}{2}}}\int_{a_{Ln}}^{\infty}|g_n'W(u)|^p\left[W(u)^{-\frac{1}{2}}\int_u^{\infty}W(x)^{\frac{1}{2}}Q'(x)\,dx\right]du \\ &\leq C_4\frac{a_n}{nT(a_n)^{\frac{1}{2}}}\int_{a_{Ln}}^{\infty}|g_n'W(u)|^p\,du\,. \end{aligned}$$

This gives

$$I_2\leq C_4\frac{a_n}{nT(a_n)^{\frac{1}{2}}}\|(R_n'-R')W\|_{L_p[a_{Ln},\infty)}^p. \tag{8.22}$$

Combining our estimates for I_1 and I_2 give the result.□

We are now in the position to give the

Proof of Theorem 8.1.1 for $0<p<1$. Let $P_n\in\mathcal{P}_n$ and $P\in\mathcal{P}_{r-1}$ be given by (8.1). We first note that if $0\leq l<r$,

$$\left(P_n^{(l)}-P^{(l)}\right)(a_{Ln})=0.$$

Thus applying (8.17) to $P_n^{(l)}$ with r in (8.17) replaced by $r-l$ gives

$$\begin{aligned} &\left\|W\left(P_n^{(l)}-P^{(l)}\right)\right\|_{L_p[a_{Ln},\infty)} \\ \leq\;& C_1\left[\left[\frac{a_n}{nT(a_n)^{\frac{1}{2}}}\left\|W\left(P_n^{(l+1)}-P^{(l+1)}\right)\right\|_{L_p[a_{Ln},\infty)}^p\right]\right. \end{aligned} \tag{8.23}$$

$$\times\left[\sum_{j=l+1}^{r-1}\left(\frac{a_n}{nT(a_n)^{\frac{1}{2}}}\right)^{(j-l-1)(1-p)}\left\|W\left(P_n^{(j)}-P^{(j)}\right)\right\|_{L_p[a_{Ln},\infty)}^{1-p}\right]$$
$$+\left(\frac{a_n}{nT(a_n)^{\frac{1}{2}}}\right)^{(r-l-1)(1-p)}\left\|W\left(P_n^{(r)}\right)\right\|_{L_p(\mathbb{R})}^{1-p}\Bigg].$$

We show that for $k=r-1,r-2,...,0$

$$\left\|W\left(P_n^{(k)}-P^{(k)}\right)\right\|_{L_p[a_{Ln},\ \infty)}\leq C_3\left(\frac{a_n}{nT(a_n)^{\frac{1}{2}}}\right)^{r-k}\left\|WP_n^{(r)}\right\|_{L_p(\mathbb{R})}. \tag{8.24}$$

Firstly, if $k=r-1$, (8.23) with $l=r-1$ gives

$$\left\|W\left(P_n^{(r-1)}-P^{(r-1)}\right)\right\|_{L_p[a_{Ln},\ \infty)}\leq C_4\left(\frac{a_n}{nT(a_n)^{\frac{1}{2}}}\right)\left\|WP_n^{(r)}\right\|_{L_p(\mathbb{R})}.$$

Assume now that (8.24) holds for $r-1,...,k+1$. We prove (8.24) for k.
Substituting (8.24) with $r-1,..k+1$ into (8.23) with $l=k$ gives

$$\left\|W\left(P_n^{(k)}-P^{(k)}\right)\right\|_{L_p[a_{Ln},\ \infty)}\leq C_5\left[\frac{a_n}{nT(a_n)^{\frac{1}{2}}}\left(\frac{a_n}{nT(a_n)^{\frac{1}{2}}}\right)^{(r-k-1)p}\left\|WP_n^{(r)}\right\|_{L_p(\mathbb{R})}^{p}\right.$$
$$\times\left[\sum_{j=k+1}^{r-1}\left(\frac{a_n}{nT(a_n)^{\frac{1}{2}}}\right)^{(j-k-1)(1-p)}\left(\frac{a_n}{nT(a_n)^{\frac{1}{2}}}\right)^{(r-j)(1-p)}\right]\times\left\|WP_n^{(r)}\right\|_{L_p(\mathbb{R})}^{1-p}$$
$$\left.+\left(\frac{a_n}{nT(a_n)^{\frac{1}{2}}}\right)^{(r-k-1)(1-p)}\left\|WP_n^{(r)}\right\|_{L_p(\mathbb{R})}^{1-p}\right]$$
$$\leq C_6\left(\frac{a_n}{nT(a_n)^{\frac{1}{2}}}\right)^{r-k}\left\|WP_n^{(r)}\right\|_{L_p(\mathbb{R})}^{1-p}.$$

Thus (8.24) holds for all k. In particular, we have

$$\|W(P_n-P)\|_{L_p[a_{Ln},\ \infty)}\leq C_7\left(\frac{a_n}{nT(a_n)^{\frac{1}{2}}}\right)^{r}\left\|WP_n^{(r)}\right\|_{L_p(\mathbb{R})}$$
$$\leq C_8\left(\frac{a_n}{n}\right)^{r}\left\|WP_n^{(r)}\Phi_{\frac{a_n}{n}}^{r}(x)\right\|_{L_p(\mathbb{R})}.\square$$

8.2 Equivalence of Modulus and Realisation Functional

In this section we prove Theorem 5.2.1 which establishes the fundamental equivalence of our modulus of continuity and its corresponding realisation-functional. We also deduce Corollary 5.2.2. Throughout for $0 < p \leq \infty$ we set

$$q := \min\{1, p\}.$$

We begin by quickly recalling the definitions of our moduli and realisation functional. See (5.11 (a)), (5.11 (b)) and (5.24). Let $r \geq 1$, $0 < t \leq C$ and let $n = n(t)$ be determined by (5.25). Then we have

$$(1)\ \ w_{r,p}(f, W, t) := \sup_{0<h\leq t} \left\| W\left(\Delta^r_{h\Phi_t(x)}(f)\right)\right\|_{L_p(|x|\leq\sigma(2t))} + \inf_{R \text{ of } \deg\leq r-1} \|(f-R)W\|_{L_p(|x|\geq\sigma(4t))} \tag{8.25}$$

$$(2)\ \ \overline{w}_{r,p}(f, W, t) := \left[\frac{1}{t}\int_0^t \left\| W\left(\Delta^r_{h\Phi_t(x)}(f)\right)\right\|^p_{L_p(|x|\leq\sigma(2t))} dh\right]^{\frac{1}{p}} + \inf_{R \text{ of } \deg\leq r-1} \|(f-R)W\|_{L_p(|x|\geq\sigma(4t))} \tag{8.26}$$

where we set $\overline{w} = w$ for $p = \infty$ and

$$(3)\ \ K_{r,p}(f, W, t^r) := \inf_{P\in\mathcal{P}_n} \left\{\|(f-P)W\|_{L_p(\mathbb{R})} + t^r \left\|P^{(r)}\Phi^r_t(x)W\right\|_{L_p(\mathbb{R})}\right\}. \tag{8.27}$$

We begin with our lower bound.

Lemma 8.2.1. Let $W \in \mathcal{E}_1$, assume (5.27) and let $L > 0$ be fixed. Let $r \geq 1$, $0 < p \leq \infty$ and $0 < t < C$. Then there exists $C_1 \neq C_1(f, t)$ such that

$$w_{r,p}(f, W, Lt) \leq C_1 K_{r,p}(f, W, t^r). \tag{8.28}$$

Proof Let $q = \min\{1, p\}$. Then by Lemma 6.1.5(a), there exists u such that $4Lt =$

$\frac{a_u}{u}$. Now let $n = n(t)$ be determined by (5.25) and recall it has the form

$$n = \inf\left\{k : \frac{a_k}{k} \le t\right\}.$$

Thus by (6.24) and (6.33) we have

$$\frac{a_n}{2n} \le \frac{t}{2} < \frac{a_n}{n} \tag{8.29}$$

and

$$\Phi_t(x) \sim \Phi_{\frac{a_n}{n}}(x) \sim \Phi_{Lt}(x) \quad \forall x \in \mathbb{R}, \tag{8.30}$$

where the constants in the $\sim$ relation are independent of t and x. Also by (6.27) and (6.25), $\exists$ $\beta > 0$ such that

$$\sigma(4Lt) = \sigma\left(\frac{a_u}{u}\right) \ge a_{\frac{u}{2}} \ge a_{\beta n}. \tag{8.31}$$

Choose $P \in \mathcal{P}_n$ such that

$$\|(f-P)W\|_{L_p(\mathbb{R})} + t^r \left\|P^{(r)}\Phi_t^r W\right\|_{L_p(\mathbb{R})}$$

$$\le 2K_{r,p}(f, W, t^r). \tag{8.32}$$

We show that

$$\sup_{0<h\le tL} \| W\left(\Delta^r_{h\Phi_{tL}(x)}(f)\right) \|_{L_p(|x|\le\sigma(2Lt))} \le C_6 K_{r,p}(f, W, t^r) \tag{8.33}$$

and

$$\inf_{R \text{ of deg}\le r-1} \|(f-R)W\|_{L_p(|x|\ge\sigma(4Lt))} \le C_2 K_{r,p}(f, W, t^r). \tag{8.34}$$

This then gives (8.28) using the definition (8.25). We begin with (8.34).

We appeal to Theorem 8.1.1 and choose for our given $P, S \in \mathcal{P}_{r-1}$ as in (8.1) so that (8.2) holds. Using Lemma 3.1 in [11], we may assume that $x \ge 0$. Then

$$\inf_{R \text{ of deg}\le r-1} \|(f-R)W\|^q_{L_p(x\ge\sigma(4Lt))} \le \|(f-S)W\|^q_{L_p(x\ge\sigma(4Lt))}$$

$$\le \|(f-P)W\|^q_{L_p(x\ge\sigma(4Lt))} + \|(P-S)W\|^q_{L_p(x\ge\sigma(4Lt))}$$

$$\leq C_3 \left(K_{r,p}\left(f, W, t^r\right)\right)^q + \|(P - S)\, W\|^q_{L_p(x \geq a_{\beta n})}$$

(by (8.31) and (8.32))

$$\leq C_3 \left(K_{r,p}\left(f, W, t^r\right)\right)^q + C_4 t^r \left\|P^{(r)} \Phi_t^r W\right\|^q_{L_p(\mathbb{R})}$$

(by (8.2) , (8.29) and (8.30))

$$\leq C_5 \left(K_{r,p}\left(f, W, t^r\right)\right)^q .$$

Hence (8.34).

Next we proceed with (8.33).

Let $0 < h \leq Lt$ and write

$$\| W \left(\Delta^r_{h\Phi_{tL}(x)}(f)\right) \|^q_{L_p(|x| \leq \sigma(2Lt))}$$

$$\begin{aligned} &\leq \left\|W \left(\Delta^r_{h\Phi_{tL}(x)}(f - P)\right)\right\|^q_{L_p(|x| \leq \sigma(2Lt))} + \| W \left(\Delta^r_{h\Phi_{Lt}(x)}(P)\right) \|^q_{L_p(|x| \leq \sigma(2Lt))} \\ &= \; : I_1 + I_2. \end{aligned}$$

We first deal with the estimation of I_1. Note that given $A > 0$,

$$|x| \leq \sigma(2Lt)$$

$$\Longrightarrow 1 - \frac{|x|}{\sigma(tL)} \geq 1 - \frac{\sigma(2Lt)}{\sigma(tL)}$$

$$\geq \frac{C_7}{T(\sigma(Lt))} \geq \left(\frac{At}{\sigma(tL)}\right)^2$$

by (6.30) and (6.31), provided t is small enough. Thus (6.53) and (6.57) are satisfied so that by (6.58),

$$I_1 \leq C_6 \|W(f - P)\|^q_{L_p(\mathbb{R})} \leq C_7 K_{r,p}\left(f, W, t^r\right)^q \tag{8.35}$$

by (8.32).

To deal with the estimation of I_2 we observe first much as in [11] that for

$$S(w) := \sum_{l=0}^{r-1} \frac{P^{(l)}(x)(w-x)^l}{l!} \in \mathcal{P}_{r-1}$$

we have by (6.50) that $\Delta^r_{h\Phi_{Lt}(x)} S \equiv 0$.

Thus expanding $P\left(x + \left(\frac{r}{2} - k\right) h\Phi_t(x)\right),\ 0 \le k \le r,$ in a power series about x gives

$$\begin{aligned}
\Delta^r_{h\Phi_{tL}(x)} P(x) &= \sum_{k=0}^{r} \binom{r}{k} (-1)^k P\left(x + \left(\frac{r}{2} - k\right) h\Phi_{tL}(x)\right) \\
&= \sum_{k=0}^{r} \binom{r}{k} (-1)^k \left[\sum_{l=0}^{r-1} + \sum_{l=r}^{n}\right] \frac{\left[\left(\frac{r}{2} - k\right) h\Phi_{tL}(x)\right]^l P^{(l)}(x)}{l!} \\
&= \sum_{k=0}^{r} \binom{r}{k} (-1)^k \sum_{l=r}^{n} \frac{\left[\left(\frac{r}{2} - k\right) h\Phi_{tL}(x)\right]^l P^{(l)}(x)}{l!},
\end{aligned}$$

so that

$$I_2 \le C_8 \sum_{k=0}^{r} \binom{r}{k}^q \sum_{l=r}^{n} \left[\frac{\left(\frac{r}{2}h\right)^{lq}}{l!^q}\right] \left\| P^{(l)} \Phi^l_{tL} W \right\|^q_{L_p(|x| \le \sigma(2Lt))}$$

$$\le C_9 2^{rq} h^{rq} \sum_{l=r}^{n} \left[\frac{\left(\frac{r}{2}h\right)^{(l-r)q}}{l!^q}\right] \left\| P^{(l)} \Phi^l_{tL} W \right\|^q_{L_p(|x| \le \sigma(2Lt))}. \tag{8.36}$$

Now by repeated applications of Theorem 6.2.1, we have by using (8.29) and (8.30),

$$\left\| P^{()} \Phi^l_{tL} W \right\|_{L_p(\mathbb{R})}$$

$$\le C^r_{10} \left\| P^{(r)} \Phi^r_{tL} W \right\|_{L_p(\mathbb{R})} C^{l-r}_{11} \prod_{j=r}^{l-1} \left(\frac{n}{a_n} + \frac{j}{a_n} T(a_n)^{\frac{1}{2}}\right) \tag{8.37}$$

where $C_j,\ j = 10, 11$ are independent of n, x, l, L and h. Now we observe using (6.9) that given $\varepsilon > 0$, we have for n large enough and $r \le l \le n$

$$\prod_{j=r}^{l-1} \left(\frac{n}{a_n} + \frac{j}{a_n} T(a_n)^{\frac{1}{2}}\right)$$

$$\le C_{12} \varepsilon^{l-r} \left(\frac{n}{a_n}\right)^{l-r} l! \tag{8.38}$$

Here it is important that C_{12} does not depend on l, n, h or L and that C_{10} and C_{11} above are independent of ε.

We may now substitute (8.38) into (8.37) so that (8.36) becomes

$$I_2 \le C_{13} h^{rq} \left\| P^{(r)} \Phi_t^r W \right\|_{L_p(\mathbb{R})}^q \sum_{l=r}^{n} \left[\frac{\left(\frac{r}{2} h C_{10} C_{11} \varepsilon \frac{n}{a_n} \right)^{(l-r)q} l!^q}{l!^q} \right]$$

$$\le C_{14} t^{rq} \left\| P^{(r)} \Phi_t^r W \right\|^q \sum_{k=0}^{\infty} \left[\frac{1}{2} \right]^k$$

(if ε is small enough),

$$\le C_{15} t^{rq} \left\| P^{(r)} \Phi_t^r W \right\|^q \le C_{16} K_{r,p} \left(f, W, t^r \right)^q. \tag{8.39}$$

Thus combining (8.35) and (8.39) and taking sup s over $0 \le h \le Lt$ gives (8.33). □

We proceed with the upper bound. This is more difficult than the lower bound and does not follow as easily using for example the methods of [11]. The crux is establishing the following quasi monotonicity type property of $\overline{w}$.

Lemma 8.2.2. There exists C_j, $j = 1, 2$ and $0 < \varepsilon_0 < 1$ such that if $0 < \lambda < \varepsilon_0$ and $0 < s, t < C_1$ with

$$\lambda \le \frac{s}{t} \le \varepsilon_0 \tag{8.40}$$

we have

$$\overline{w}_{r,p} (f, W, s) \le C_2 \overline{w}_{r,p} (f, W, t). \tag{8.41}$$

Remark We remark that the above property is by no means obvious as recall our modulus is not necessarily monotone increasing . We prove it for $p < \infty$ as the case $p = \infty$ is much easier.

Proof. Let us write

$$\overline{w}_{r,p} (f, W, s)$$

$$= \left[\frac{1}{s} \int_0^s \left\| W \left(\Delta_{h\Phi_s(x)}^r (f) \right) \right\|_{L_P(|x| \le \sigma(3t))}^p dh$$

$$+\frac{1}{s}\int_0^s \left\| W\left(\Delta^r_{h\Phi_s(x)}(f)\right)\right\|^p_{L_p(\sigma(3t)\le|x|\le\sigma(2s))} dh\Bigg]^{\frac{1}{p}}.$$

$$+\inf_{R\ \text{of}\ \deg\le r-1}\|(f-R)W\|_{L_p(|x|\ge\sigma(4s))}$$

$$=I_1+I_2. \tag{8.42}$$

Firstly, by choice of s and t, $\frac{s}{t}\le 1$ so that

$$\sigma(4s)\ge\sigma(4t)$$

(recall σ is decreasing). Thus

$$I_2\le\inf_{R\ \text{of}\ \deg\le r-1}\|(f-R)W\|_{L_p(|x|\ge\sigma(4t))}$$

$$\le\overline{w}_{r,p}(f,W,t). \tag{8.43}$$

Next we estimate I_1 :

Write $(I_1)^p\le I_3+I_4$, where

$$I_3:=\frac{1}{s}\int_0^s\left\|W\left(\Delta^r_{h\Phi_s(x)}(f)\right)\right\|^p_{L_p(|x|\le\sigma(3t))}dh$$

and

$$I_4:=\frac{1}{s}\int_0^s\left\|W\left(\Delta^r_{h\Phi_s(x)}(f)\right)\right\|^p_{L_p(\sigma(3t)\le|x|\le\sigma(2s))}dh.$$

We begin with the estimation of I_4. To this end we make use of Lemma 6.1.8. Much as in the proof of Lemma 8.2.1, we have

$$I_4\le C_1\inf_{R\ \text{of}\ \deg\le r-1}\|(f-R)W\|^p_{L_p(|x|\ge\sigma(4t))}\le C_1\overline{w}_{r,p}(f,W,t)^p. \tag{8.44}$$

Here we used that

$$\inf\{x-Mrs\Phi_s(x):\sigma(3t)\le x\le\sigma(2s)\}$$

$$\ge\sigma(3t)-CtT(\sigma(t))^{\frac{-1}{2}}$$

$$\ge\sigma(3t)+o(1/T(\sigma(t)))\ge\sigma(4t)$$

for small t, see (6.7), (6.9) and (6.30).

It remains to estimate I_3 :

As s and t are small enough, we can use Lemma 6.1.5(a) to obtain a large enough positive integer n such that $\frac{a_n}{n} \sim s$ and then much as in chapter 7, construct a partition of $J :=$ $[-\sigma(3t), \sigma(3t)]$

$$-\sigma(3t) = \tau_0 < \tau_1 ... < \tau_n = \sigma(3t)$$

with the following properties: If $J_k = [\tau_k, \tau_{k+1}]$ and $|J_k|$ denotes the lengh of J_k then,

$$\begin{aligned}
&(1)\ \ |J_k| \leq s\Phi_s(x), \quad x \in J_k \\
&(2)\ \ \Phi_s(x) \sim \Phi_s(y) \quad x, y \in J_k \\
&(3)\ \ W(x) \sim W(y), \quad x, y \in J_k
\end{aligned} \tag{8.45}$$

Here the constants in the $\sim$ relation are independent of x, y, s, k.

Then

$$\begin{aligned}
I_3 &= \frac{1}{s}\int_0^s \left\| W\left(\Delta^r_{h\Phi_s(x)}(f)\right)\right\|^p_{L_p(|x|\leq\sigma(3t))} dh \\
&\leq C_2 \sum_k W^p(\tau_k) \int_{J_k} \frac{1}{s}\int_0^s \left|\Delta^r_{h\Phi_s(x)}(f)\right|^p dh dx \\
&= C_2 \sum_k W^p(\tau_k) \int_{J_k} \frac{1}{s}\int_0^{\frac{s\Phi_s(x)}{\Phi_t(x)}} \left|\Delta^r_{u\Phi_t(x)}(f)\right|^p \frac{\Phi_t(x)}{\Phi_s(x)} du dx.
\end{aligned}$$

Now by (6.40) for some $C \neq C(s,t)$

$$\begin{aligned}
\sup_{x\in\mathbb{R}} \frac{s\Phi_s(x)}{t\Phi_t(x)} &\leq C\frac{s}{t}\sqrt{\log\left(2+\frac{t}{s}\right)} \\
&\leq 1,
\end{aligned}$$

if $\frac{s}{t} \leq \varepsilon_0$, where ε_0 is independent of s, t. Then if $\lambda < \varepsilon_0$, we have for

$$\lambda \leq \frac{s}{t} \leq \varepsilon_0,$$

$$C_3 \leq \frac{\Phi_s(x)}{\Phi_t(x)} \leq C_4.$$

Then

$$\begin{aligned} I_3 &\leq C_5 \sum_k W^p(\tau_k) \int_{J_k} \frac{1}{s} \int_0^t \left| \Delta^r_{u\Phi_t(x)}(f) \right|^p du dx \\ &\leq C_6 \frac{1}{t} \int_0^t \left\| W\left(\Delta^r_{h\Phi_t(x)}(f) \right) \right\|^p_{L_p(|x| \leq \sigma(2t))} dh \\ &\leq C_6 \overline{w}_{r,p}(f, W, t)^p. \end{aligned} \tag{8.46}$$

Combining our estimates (8.43), (8.44) and (8.46) give the lemma.□

Lemma 8.2.3. Let $W \in \mathcal{E}_1$ and assume (5.27). Let $r \geq 1$ and $0 < p \leq \infty$. Then for $0 < t < C_1$, there exists $C_2, C_3 \neq C_2, C_3(f,t)$ such that

$$K_{r,p}(f, W, t^r) \leq C_2 \overline{w}_{r,p}(f, W, C_3 t). \tag{8.47}$$

Proof . Put $\frac{t}{2} = \frac{a_u}{u}$ for some $u \geq u_0$ and let $n = n(t)$ be determined by (5.25), so that

$$n = \inf\left\{ k : \frac{a_k}{k} \leq \frac{2a_u}{u} \right\}$$

and

$$\frac{1}{2}\frac{a_n}{n} \leq \frac{a_u}{u} < \frac{a_n}{n}. \tag{8.48}$$

Now it is easy to see that for large enough u and the given n,

$$t = 2\frac{a_u}{u} = \frac{a_n}{n}\lambda(n) C$$

for some $\lambda(n) \in \left[\frac{4}{5}, 1\right]$ and $C > 0$ independent of n. We then apply (5.13),and choose $P \in \mathcal{P}_n$ such that

$$\|(f-P)W\|_{L_p(\mathbb{R})} \leq C_1 \overline{w}_{r,p}(f, W, C_2 t) \tag{8.49}$$

for some $C_1, C_2 \neq C_1, C_2(f,t)$.

We show that for some $C_3 \neq C_3(f,t)$,

$$t^r \left\| P^{(r)} \Phi_t^r W \right\|_{L_P(\mathbb{R})} \leq C_3 \overline{w}_{r,p}(f, W, C_2 t) \tag{8.50}$$

for then by (8.49),

$$\begin{aligned} K_{r,p}(f, W, t^r) &= \inf_{R \in \mathcal{P}_n} \left\{ \|(f-R)W\|_{L_P(\mathbb{R})} + t^r \left\| R^{(r)} \Phi_t^r W \right\|_{L_P} \right. \\ &\leq \|(f-P)W\|_{L_P(\mathbb{R})} + t^r \left\| P^{(r)} \Phi_t^r W \right\|_{L_P(\mathbb{R})} \\ &\leq (C_1 + C_3) \overline{w}_{r,p}(f, W, C_2 t). \end{aligned}$$

Thus we show (8.50).

Now let $\delta > 0$ be a small enough positive number and put $s := \delta t$. It is sufficient at this point of the proof to choose δ small enough so that by Lemma 8.2.2.

$$\overline{w}_{r,p}(f, W, s) \leq C_4 \overline{w}_{r,p}(f, W, C_2 t). \tag{8.51}$$

Later, we will need to choose δ smaller still.

Let us recall much as in Lemma 8.2.1 that we have for $0 < h \leq s$

$$\Delta_{h\Phi_s(x)}^r P(x) = \sum_{k=0}^{r} \binom{r}{k} (-1)^k \sum_{l=r}^{n} \frac{\left[\left(\frac{r}{2} - k\right) h\Phi_s(x)\right]^l P^{(l)}(x)}{l!}. \tag{8.52}$$

Applying (8.52) to $x^r \in \mathcal{P}_r$ and using (6.50) gives

$$(r!)^{-1} \Delta_{h\Phi_s(x)}^r x^r = (h\Phi_s(x))^r = \sum_{k=0}^{r} \binom{r}{k} (-1)^k \frac{\left[\left(\frac{r}{2} - k\right) h\Phi_s(x)\right]^r}{r!}. \tag{8.53}$$

We now combine (8.52) and (8.53) together with (6.63) to give much as in (8.39),

$$\left\| W \Delta_{h\Phi_s(x)}^r P(x) - W (h\Phi_s(x))^r P^{(r)}(x) \right\|_{L_p(|x| \leq \sigma(2s))}^q$$

$$\leq C_6 h^{rq} \left\| W P^{(r)} \Phi_s^r (x) \right\|_{L_P(\mathbb{R})}^q \sum_{l=r+1}^{n} \frac{\left(C_6 \frac{n}{a_n} h\right)^{(l-r)q} l!^q}{l!^q} \tag{8.54}$$

where C_6 is independent of t, n, h, P_n and l.

Now by $(6.8), (6.25)$ and (8.48) we can choose $\alpha > 3$ independent of t, n, h, P_n, l and C_2 such that $a_u < a_{\alpha n}$. Further (if necessary) we make δ in the definition of s smaller still so that

$$\delta < \min\left(\frac{1}{8\alpha}, \frac{1}{2}\right) \tag{8.55}$$

so that

$$2s \leq \frac{t}{4\alpha} \leq \frac{a_{\alpha n}}{\alpha n}.$$

This gives

$$\sigma(2s) \geq \sigma\left(\frac{t}{4\alpha}\right) \geq \sigma\left(\frac{a_{\alpha n}}{\alpha n}\right) \geq a_{\xi n} \tag{8.56}$$

for some fixed $3 < \xi < \alpha$.

It follows that we obtain using $(8.56), (6.33)$ and (6.70),

$$\left\| W \Delta^r_{h\Phi_s(x)} P(x) - W\left(h\Phi_s(x)\right)^r P^{(r)}(x) \right\|^q_{L_p(|x| \leq \sigma(2s))}$$

$$\leq \frac{1}{2} h^{rq} \left\| W P^{(r)} \Phi_s^r (x) \right\|^q_{L_P(|x| \leq \sigma(2s))} \tag{8.57}$$

provided $\frac{n}{a_n} h \leq \Delta$, where Δ is a fixed positive small number independent of t, h, n, P_n and l.

Now by (8.55) and (8.48), it is easy to see that $\Delta s \leq \Delta \frac{a_n}{n}$ so that $\forall 0 < h \leq \Delta s$ we have

$$\left\| W \Delta^r_{h\Phi_s(x)} P(x) \right\|^q_{L_p(|x| \leq \sigma(2s))}$$

$$\geq h^{rq} \left\| W\left(\Phi_s(x)\right)^r P^{(r)}(x) \right\|^q_{L_p(|x| \leq \sigma(2s))}$$

$$- \left\| W \Delta^r_{h\Phi_s(x)} P(x) - W\left(h\Phi_s(x)\right)^r P^{(r)}(x) \right\|^q_{L_p(|x| \leq \sigma(2s))}$$

$$\geq \frac{1}{2} h^{rq} \left\| W P^{(r)} \Phi_s^r (x) \right\|^q_{L_P(|x| \leq \sigma(2s))}$$

(by (8.57))

$$\geq C_7 h^{rq} \left\| WP^{(r)}\Phi_s^r(x) \right\|_{L_p(\mathbb{R})}^q \tag{8.58}$$

by (6.70). Now raising (8.58) to the p/q th powers, integrating for h from 0 to Δs using the fact that $\Phi_s(x) \sim \Phi_t(x)$, $x \in \mathbb{R}$ (see (6.33)) and assuming that $\Delta < 1$ as we can, gives

$$\begin{aligned}
t^{rp} \left\| WP^{(r)}\Phi_t^r(x) \right\|_{L_p(\mathbb{R})}^p &\leq \frac{C_8}{s} \int_0^{\Delta s} \left\| W\Delta_{h\Phi_s(x)}^r P(x) \right\|_{L_p(|x| \leq \sigma(2s))}^p dh \\
&\leq \frac{C_8}{s} \int_0^{s} \left\| W\Delta_{h\Phi_s(x)}^r P(x) \right\|_{L_p(|x| \leq \sigma(2s))}^p dh \\
&\leq \frac{C_8}{s} \int_0^{s} \left\| W\Delta_{h\Phi_s(x)}^r (P-f)(x) \right\|_{L_p(|x| \leq \sigma(2s))}^p dh \\
&\quad + \frac{C_8}{s} \int_0^{s} \left\| W\Delta_{h\Phi_s(x)}^r f(x) \right\|_{L_p(|x| \leq \sigma(2s))}^p dh \\
&\leq C_9 \left\{ \| W(P-f) \|_{L_p(\mathbb{R})}^p + \overline{w}_{r,p}(f, W, s) \right\}
\end{aligned}$$

(by (6.58))

$$\leq C_{10} \overline{w}_{r,p}(f, W, C_2 t)$$

by (8.51) and (8.49). Thus we have (8.50) and the lemma.□

We now combine Lemmas 8.2.1 and 8.2.3 to give

The proofs of Theorem 5.2.1 and Corollary 5.2.2. We have for any $L > 0$ and $0 < t < t_0$,

$$\begin{aligned}
\overline{w}_{r,p}(f, W, Lt) &\leq w_{r,p}(f, W, Lt) \leq C_1 K_{r,p}(f, W, t^r) \\
&\leq C_2 \overline{w}_{r,p}(f, W, C_3 t) \leq C_2 w_{r,p}(f, W, C_3 t)
\end{aligned} \tag{8.59}$$

where C_3 is independent of L, f and t while C_1 and C_2 are independent of f and t but depend on L.

Fix $M > 0$ and choose $L = MC_3$ and $s = C_3 t$ to deduce that

$$w_{r,p}(f, W, Ms)$$

$$\leq C_2 w_{r,p}(f, W, s) \tag{8.60}$$

and similarly

$$\overline{w}_{r,p}(f,W,Ms)$$

$$\leq C_2\overline{w}_{r,p}(f,W,s). \tag{8.61}$$

Then (5.30) holds and (5.31) follows from (5.12), (8.60) and (8.61).

Finally (8.59) then gives

$$w_{r,p}(f,W,s) \sim \overline{w}_{r,p}(f,W,s) \sim K_{r,p}(f,W,s^r)$$

with constants independent of f and s. □

Chapter 9

Applications of Theorem 5.2.1

9.1 Converse Theorems

In this section, we present the proofs for our converse results of polynomial approximation.

We begin with

The proof of Theorem 5.2.3. For each $n \geq 0$, choose P_n^* to be the best approximant to f satisfying

$$\|(f - P_n^*)\, W\|_{L_p(\mathbb{R})} = E_n[f]_{W,p}.$$

Here, we set $P_{2^{-1}}^* = P_0^*$. Now let $t > 0$ be small enough and define n by (5.25). Put $l = [\log_2 n]$ =the largest integer$\leq \log_2 n$ so that $2^l \leq n < 2^{l+1}$.

Then by Theorem 5.2.1 and Corollary 5.2.2

$$\begin{aligned}
& w_{r,p}\left(f, W, \frac{a_n}{n}\right)^q \\
\leq\ & C_1 K_{r,p}\left(f, W, \left(\frac{a_n}{n}\right)^r\right)^q \\
\leq\ & C_2\left[\|(f - P_{2^l}^*)\, W\|_{L_p(\mathbb{R})}^q + \left(\frac{a_n}{n}\right)^{rq} \left\|P_{2^l}^{*(r)} \Phi_{\frac{a_n}{n}}^r W\right\|_{L_p(\mathbb{R})}^q\right] \\
\leq\ & C_3\left[E_{2^l}[f]_{W,p}^q + \left(\frac{a_n}{n}\right)^{rq} \sum_{k=-1}^{l-1} \left\|[P_{2^{k+1}}^* - P_{2^k}^*]^{(r)}\, \Phi_{\frac{a_n}{n}}^r W\right\|_{L_p(\mathbb{R})}^q\right] \\
\leq\ & C_4\left[E_{2^l}[f]_{W,p}^q + \left(\frac{a_n}{n}\right)^{rq} \sum_{k=-1}^{l-1} \left\|[P_{2^{k+1}}^* - P_{2^k}^*]^{(r)}\, \Phi_{\frac{a_{2^{k+1}}}{2^{k+1}}}^r \left(\log\left(2^{l-k}\right)\right)^{\frac{r}{2}} W\right\|_{L_p(\mathbb{R})}^q\right]
\end{aligned}$$

as $r \geq 1$ and by (6.42). This can be continued as

$$\leq C_5 \left[E_{2^l}[f]^q_{W,p} \right.$$

$$\left. + \left(\frac{a_n}{n}\right)^{rq} \sum_{k=-1}^{l-1} (l-k+1)^{\frac{rq}{2}} \left(\frac{2^k}{a_{2^k}}\right)^{rq} \left\| \left[P^*_{2^{k+1}} - P^*_{2^k}\right] W \right\|^q_{L_p(\mathbb{R})} \right]$$

by (5.27) .

We can continue this as

$$\leq C_6 \left[E_{2^l}[f]^q_{W,p} + \left(\frac{a_n}{n}\right)^{rq} \sum_{k=-1}^{l-1} (l-k+1)^{\frac{rq}{2}} \left(\frac{2^k}{a_{2^k}}\right)^{rq} E_{2^k}[f]^q_{W,p} \right]$$

$$\leq C_7 \left(\frac{a_n}{n}\right)^{rq} \left[\sum_{k=-1}^{l} (l-k+1)^{\frac{rq}{2}} \left(\frac{2^k}{a_{2^k}}\right)^{rq} E_{2^k}[f]^q_{W,p} \right]. \tag{9.1}$$

Now by (6.24) we have that $t \sim \frac{a_n}{n}$. Also by (6.33),

$$\Phi_t(x) \sim \Phi_{\frac{a_n}{n}}(x), x \in \mathbb{R}$$

so that

$$K_{r,p}(f, W, t^r) \sim K_{r,p}\left(f, W, \left(\frac{a_n}{n}\right)^r\right)$$

so that by Theorem 5.2.1

$$w_{r,p}(f, W, t) \sim w_{r,p}\left(f, W, \frac{a_n}{n}\right). \tag{9.2}$$

Thus (9.2) becomes

$$w_{r,p}(f, W, t)^q$$

$$\leq C_8 t^{rq} \left[\sum_{k=-1}^{l} (l-k+1)^{\frac{rq}{2}} \left(\frac{2^k}{a_{2^k}}\right)^{rq} E_{2^k}[f]^q_{W,p} \right]$$

where $C_8 \neq C_8(f, t)$.□

We deduce

The Proof of Corollary 5.2.4. Suppose first that

$$w_{r,p}(f, W, t) = O(t^{\alpha}).$$

Then in particular

$$w_{r,p}\left(f, W, \frac{a_n}{n}\right) = O\left(\left(\frac{a_n}{n}\right)^{\alpha}\right), n \longrightarrow \infty,$$

so that by Corollary 5.2.2

$$E_n[f]_{W,p} = O\left(\left(\frac{a_n}{n}\right)^{\alpha}\right).$$

Next suppose $E_n[f]_{W,p} = O\left(\left(\frac{a_n}{n}\right)^{\alpha}\right)$. Let $0 < \varepsilon < 1$. Then, by (5.32)

$$w_{r,p}\left(f, W, \frac{a_n}{n}\right) \le C_1 \left(\frac{a_n}{n}\right)^{r} \left[\sum_{k=-1}^{l} (l-k+1)^{\frac{rq}{2}} \left(\frac{2^k}{a_{2^k}}\right)^{(r-\alpha)q}\right]^{\frac{1}{q}}$$

$$\begin{aligned}
&\le C_1 \left(\frac{a_n}{n}\right)^{\alpha} \left[\sum_{k=-1}^{l} (l-k+1)^{\frac{rq}{2}} \left(\frac{a_n/n}{a_{2^k}/2^k}\right)^{(r-\alpha)q}\right]^{\frac{1}{q}} \\
&\le C_2 \left(\frac{a_n}{n}\right)^{\alpha} \left[\sum_{k=-1}^{l} (l-k+1)^{\frac{rq}{2}} \left(\frac{2^{l+1}}{2^k}\right)^{(r-\alpha)q(-1+\varepsilon)}\right]^{\frac{1}{q}} \text{ (by (6.16))}
\end{aligned}$$

$$\le C_3 \left(\frac{a_n}{n}\right)^{\alpha} \left[\sum_{j=0}^{\infty} j^{\frac{r}{2}q} a^{jq}\right]^{\frac{1}{q}} \text{ (for some } 0 < a < 1)$$

$$\le C_4 \left(\frac{a_n}{n}\right)^{\alpha}. \tag{9.3}$$

Now for $t > 0$ small enough, we may determine n by (5.25) and using (6.24) and (9.2) deduce the result for t.□

9.2 The proofs of Corollaries 5.2.5 and 5.2.6

We begin with

The proof of Corollary 5.2.5. Let $P_n^{\#}$ satisfy the required hypotheses. Then by the definition of $K_{r,p}\left(f, W, \left(\frac{a_n}{n}\right)^r\right)$, we have

$$\left\{\left\|\left(f-P_n^{\#}\right)W\right\|_{L_p(\mathbb{R})}+\left(\frac{a_n}{n}\right)^r\left\|P_n^{\#(r)}\Phi_{\frac{a_n}{n}}^r W\right\|_{L_p(\mathbb{R})}\right\} \tag{9.4}$$

$$\geq K_{r,p}\left(f, W, \left(\frac{a_n}{n}\right)^r\right).$$

Next choose P_n such that

$$\left\{\|(f-P_n)W\|_{L_p(\mathbb{R})}+\left(\frac{a_n}{n}\right)^r\left\|P_n^{(r)}\Phi_{\frac{a_n}{n}}^r W\right\|_{L_p(\mathbb{R})}\right\} \tag{9.5}$$

$$\leq 2K_{r,p}\left(f, W, \left(\frac{a_n}{n}\right)^r\right).$$

Then

$$\begin{aligned}\left\|\left(P_n-P_n^{\#}\right)W\right\|_{L_p(\mathbb{R})}^q &\leq \|(P_n-f)W\|_{L_p(\mathbb{R})}^q+\left\|\left(f-P_n^{\#}\right)W\right\|_{L_p(\mathbb{R})}^q \\ &\leq C_1 K_{r,p}\left(f, W, \left(\frac{a_n}{n}\right)^r\right)^q\end{aligned} \tag{9.6}$$

(by (9.5)).

Further using (5.27), we can write using (9.6)

$$\begin{aligned}\left\|\left(P_n-P_n^{\#}\right)^{(r)}\Phi_{\frac{a_n}{n}}^r W\right\|_{L_p(\mathbb{R})}^q &\leq C_2\left(\frac{n}{a_n}\right)^{rq}\left\|\left(P_n-P_n^{\#}\right)W\right\|_{L_p(\mathbb{R})}^q \\ &\leq C_3\left(\frac{n}{a_n}\right)^{rq}K_{r,p}\left(f, W, \left(\frac{a_n}{n}\right)^r\right)^q.\end{aligned} \tag{9.7}$$

Thus by (9.5) and (9.7)

$$\begin{aligned}&\left(\frac{a_n}{n}\right)^{rq}\left\|P_n^{\#(r)}\Phi_{\frac{a_n}{n}}^r W\right\|_{L_p(\mathbb{R})}^q \\ \leq\ & C_4\left[\left(\frac{a_n}{n}\right)^{rq}\left\|P_n^{(r)}\Phi_{\frac{a_n}{n}}^r W\right\|_{L_p(\mathbb{R})}^q+\left(\frac{a_n}{n}\right)^{rq}\left\|\left(P_n-P_n^{\#}\right)^{(r)}\Phi_{\frac{a_n}{n}}^r W\right\|_{L_p(\mathbb{R})}^q\right]\end{aligned}$$

$$\leq C_5 K_{r,p}\left(f, W, \left(\frac{a_n}{n}\right)^r\right)^q. \tag{9.8}$$

so that (9.4) and (9.8) give the result.□

We can now give

The proof of Corollary 5.2.6(a). We shall show that

$$\left\|W\Delta^r_{h\Phi_t(x)}(f,x,\mathbb{R})\right\|_{L_p[|x|\leq\sigma(2t)]}\leq C_1t^r\left\|f^{(r)}\Phi_t^rW\right\|_{L_p(\mathbb{R})} \tag{9.9}$$

and

$$\inf_{P\in\mathcal{P}_{r-1}}\|W(f-P)\|_{L_p[|x|\geq\sigma(4t)]}\leq C_2t^r\left\|f^{(r)}\Phi_t^rW\right\|_{L_p(\mathbb{R})}. \tag{9.10}$$

We begin with

The Proof of (9.9). We begin with an observation.

If $h>0$ we may write

$$\begin{aligned}|\Delta_h^r(f,x,\mathbb{R})| &= \left|\int_{\frac{-h}{2}}^{\frac{h}{2}}\int_{\frac{-h}{2}}^{\frac{h}{2}}\dots\int_{\frac{-h}{2}}^{\frac{h}{2}}f^{(r)}(x+t_1+\dots+t_r)\,dt_1dt_2..dt_r\right| \\ &\leq h^{r-1}\int_{\frac{-hr}{2}}^{\frac{hr}{2}}\left|f^{(r)}(x+s)\right|ds.\end{aligned} \tag{9.11}$$

Now note that for $s\in\left[-\frac{rh\Phi_t(x)}{2},\frac{rh\Phi_t(x)}{2}\right]$ and $x\in[-\sigma(2t),\sigma(2t)]$ we have by (6.49)

$$\Phi_t(x)\sim\Phi_t(x+s).$$

Thus we may deduce from (9.11) that for $|x|\leq\sigma(2t)$ as

$$\left|W\Delta^r_{h\Phi_t(x)}(f,x,\mathbb{R})\right|\leq C_3h^r\frac{1}{\frac{rh\Phi_t(x)}{2}}\int_{\frac{-rh\Phi_t(x)}{2}}^{\frac{rh\Phi_t(x)}{2}}\left|Wf^{(r)}\Phi_t^r(x+s)\right|ds. \tag{9.12}$$

Case 1. $p>1$. We recall the definition of the maximal function operator

$$M[g](x):=\sup_{u>0}\frac{1}{2u}\int_{-u}^{u}|g(x+s)|\,ds$$

which is bounded from L_p to L_p, $1<p<\infty$. It follows that (9.12) can be rewritten as

$$\left\|W\Delta^r_{h\Phi_t(x)}(f,x,\mathbb{R})\right\|_{L_p[|x|\leq\sigma(2t)]}\leq C_4h^r\left\|M\left[W\Phi_t^rf^{(r)}\right]\right\|_{L_p(\mathbb{R})}$$

$$\leq C_5 t^r \left\| f^{(r)} \Phi_t^r W \right\|_{L_p(\mathbb{R})}.$$

Case 2. $p = 1$. Integrating (9.12), and noting that if $u = x + s$, then for the range of x and s above,

$$\Phi_t(x) \sim \Phi_t(x+s)$$

so we obtain

$$\int_{|x| \leq \sigma(2t)} \left| W \Delta^r_{h\Phi_t(x)}(f, x, \mathbb{R}) \right| dx$$

$$\leq C_6 h^{r-1} \int_{|x| \leq \sigma(2t)} \frac{1}{\Phi_t(x)} \int_{|s| \leq \frac{rh}{2}\Phi_t(x)} \left| W f^{(r)} \Phi_t^r \right| (x+s)\, ds dx$$

$$\leq C_7 h^{r-1} \int_{\substack{u = x+s: |x| \leq \sigma(2t) \\ |s| \leq \frac{r}{2} h \Phi_t(x)}} \frac{1}{\Phi_t(u)} \left| W f^{(r)} \Phi_t^r \right| (u) \int_{|s| \leq \frac{rh}{2}\Phi_t(u)} ds du$$

$$\leq C_8 h^r \int_{\mathbb{R}} \left| f^{(r)} W \Phi_t^r \right| (u)\, du.$$

Next we give

The Proof of (9.10). We mimic the proof of (8.2) for $p > 1$. For the given $t > 0$, write $4t = \frac{a_u}{u}$. Determine $n = n(t)$ by (5.25) and recall $u \sim n$ (see (6.25)) so that

$$\begin{aligned} (a)\ \sigma(4t) &\leq a_u \leq a_{\alpha n} \\ (b)\ \sigma(4t) &\geq a_{\frac{u}{2}} \geq a_{\beta n} \end{aligned} \tag{9.13}$$

for some $\alpha > 1$ and $\beta > 0$.

As in Lemma 3.1 in [11], we may without loss of generality suppose that $x > 0$. Suppose first that $r = 1$. We have

$$\inf_{P \in \mathcal{P}_{r-1}} \|W(f-P)\|_{L_p[x \geq \sigma(4t)]}$$

$$\leq \|W(f - f(a_{\beta n}))\|_{L_p[x \geq a_{\beta n}]} = \left\| W(x) \int_{a_{\beta n}}^x f'(u)\, du \right\|_{L_p[x \geq a_{\beta n}]}$$

$$\leq C_4 \frac{a_n}{nT(a_n)^{\frac{1}{2}}} \|Wf'\|_{L_p[x \geq a_{\beta n}]} \leq C_5 \frac{a_n}{T(a_{\alpha n})^{\frac{1}{2}} n} \|Wf'\|_{L_p[x \geq a_{\beta n}]}$$

$$\leq C_8 \frac{a_n}{T(\sigma(t))^{\frac{1}{2}} n} \|Wf'\|_{L_p[x \geq a_{\beta n}]} \leq C_7 \frac{a_n}{n} \|Wf'\Phi_t\|_{L_p[x \geq a_{\beta n}]} \tag{9.14}$$

by Lemma 8.1.2, (6.6) and (6.36).

Assume (9.14) holds for $1, 2, , , r-1$. Choose $S \in \mathcal{P}_{r-2}$ such that

$$\|W(f' - S)\|_{L_p[|x| \geq \sigma(t)]} \leq C_6 \left(\frac{a_n}{n}\right)^{r-1} \left\| f^{(r)} \Phi_t^{r-1} W \right\|_{L_p(\mathbb{R})}.$$

Set

$$P(x) := f(a_{\beta n}) + \int_{a_{\beta n}}^{x} S(u)\, du$$

Then we can bound the left hand side of (9.10) by

$$\begin{aligned} & \|W(f-P)\|_{L_p[x \geq a_{\beta n}]} \\ \leq\ & C_7 \left\| W(x) \int_{a_{\beta n}}^{x} (f' - S)(u)\, du \right\|_{L_p[x \geq a_{\beta n}]} \\ \leq\ & C_8 \frac{a_n}{nT(a_n)^{\frac{1}{2}}} \left\| f^{(r)} W \Phi_t^{r-1} \right\|_{L_p[x \geq a_{\beta n}]} \leq C_9 t^r \left\| f^{(r)} \Phi_t^r W \right\|_{L_p(\mathbb{R})} \end{aligned} \tag{9.15}$$

and we have our result.□

We deduce

The proof of Corollary 5.2.6(b). Write $t = \frac{a_u}{u}$ and let $n = n(t)$ be determined by (5.25). Firstly

$$\begin{aligned} K_{r,p}(f, W, t^r) &= \inf_{P \in \mathcal{P}_n} \left\{ \|(f-P)W\|_{L_p(\mathbb{R})} + t^r \left\| WP_n^{(r)} \Phi_t^r \right\|_{L_p(\mathbb{R})} \right\} \\ &\geq \inf_g \left\{ \|(f-g)W\|_{L_p(\mathbb{R})} + t^r \left\| Wg^{(r)} \Phi_t^r \right\|_{L_p(\mathbb{R})} \right\} \\ &= K_{r,p}^*(f, W, t^r). \end{aligned} \tag{9.16}$$

Next, we may choose g such that

$$\|(f-g)W\|_{L_p(\mathbb{R})} + t^r \left\| Wg^{(r)} \Phi_t^r \right\|_{L_p(\mathbb{R})} \leq 2K_{r,p}^*(f, W, t^r) \tag{9.17}$$

Also by Corollary 5.2.5, Theorem 5.2.1 and Corollary 5.2.2 we may choose P_n such that

$$\|(P_n - g) W\|_{L_p(\mathbb{R})} \leq C_2 w_{r,p}\left(g, W, \frac{a_n}{n}\right) \tag{9.18}$$

and

$$\left(\frac{a_n}{n}\right)^r \left\|W P_n^{(r)} \Phi_t^r\right\|_{L_p(\mathbb{R})} \leq C_3 w_{r,p}\left(g, W, \frac{a_n}{n}\right). \tag{9.19}$$

Thus by $(9.17 - 9.19)$ we have

$$\begin{aligned}
& K_{r,p}(f, W, t^r) \\
\leq \; & \|(f - P_n) W\|_{L_p(\mathbb{R})} + t^r \left\|W P_n^{(r)} \Phi_t^r\right\|_{L_p(\mathbb{R})} \\
\leq \; & C_4 \left[\|(f - g) W\|_{L_p(\mathbb{R})} + \|(g - P_n) W\|_{L_p(\mathbb{R})} + t^r \left\|W P_n^{(r)} \Phi_t^r\right\|_{L_p(\mathbb{R})}\right] \\
\leq \; & C_5 \left[\|(f - g) W\|_{L_p(\mathbb{R})} + w_{r,p}\left(g, W, \frac{a_n}{n}\right)\right] \\
\leq \; & C_6 \left[\|(f - g) W\|_{L_p(\mathbb{R})} + w_{r,p}(g, W, t)\right] \text{ (by (9.2))} \\
\leq \; & C_7 \left[\|(f - g) W\|_{L_p(\mathbb{R})} + t^r \left\|g^{(r)} \Phi_t^r W\right\|_{L_p(\mathbb{R})}\right] \text{ (by Corollary 5.2.6 (a))}
\end{aligned}$$

$$\leq C_8 K^*_{r,p}(f, W, t^r). \tag{9.20}$$

Then (9.16) and (9.20) give the result.□

9.3 A Marchaud Inequality

In this section we give:

The proof of Theorem 5.2.7.

Proof. First let n be large enough and let P_n^* be the best approximant to f which exists and satisfies,

$$E_n[f]_{W,p} := \|(f - P) W\|_{L_p(\mathbb{R})} \tag{9.21}$$

By Theorem 5.2.1 and Corollary 5.2.2, we may thus write using (9.21),

$$w_{r,p}\left(f, W, \frac{a_n}{n}\right)^q \tag{9.22}$$

$$
\begin{aligned}
&\leq C_9\left[\left\|(f-P_n^*)W\right\|_{L_p(\mathbb{R})}+\left(\frac{a_n}{n}\right)^{rq}\left\|P_n^{*(r)}\Phi^r_{\frac{a_n}{n}}W\right\|_{L_p(\mathbb{R})}\right]\\
&\leq C_{10}w_{r+1,p}\left(f,W,\frac{a_n}{n}\right)^q+\left(\frac{a_n}{n}\right)^{rq}\left\|P_n^{*(r)}\Phi^r_{\frac{a_n}{n}}W\right\|_{L_p(\mathbb{R})}
\end{aligned}
$$

for some $C_9, C_{10} > 0$. Here we use the inequality $(a+b)^\alpha \leq a^\alpha + b^\alpha \;\; a, b > 0,\; 0 < \alpha < 1$. Now choose $l = l(n)$ such that,

$$
r2^{l+2} \geq n \geq r2^{l+1} \tag{9.23}
$$

where $n \geq 2r$, and write,

$$
P_n^*(x) = \sum_{k=0}^{l-1}\left(P^*_{\left[\frac{n}{2^k}\right]}(x) - P^*_{\left[\frac{n}{2^{k+1}}\right]}(x)\right) + P^*_{\left[\frac{n}{2^{l+1}}\right]}(x) \tag{9.24}
$$

where $[x]$ =the largest integer $\leq x$.

Using Corollary 5.2.2 and (9.21) gives for $0 \leq k \leq l$,

$$
\begin{aligned}
&\left\|\left(P^*_{\left[\frac{n}{2^k}\right]}(x)-P^*_{\left[\frac{n}{2^{k+1}}\right]}(x)\right)W\right\|^q_{L_p(\mathbb{R})} && (9.25)\\
&\leq \left\|\left(f-P^*_{\left[\frac{n}{2^{k+1}}\right]}(x)\right)W\right\|^q_{L_p(\mathbb{R})}+\left\|\left(P^*_{\left[\frac{n}{2^k}\right]}(x)-f\right)W\right\|^q_{L_p(\mathbb{R})}\\
&\leq C_{11}w_{r+1,p}\left(f,W,\frac{a_{\left[\frac{n}{2^{k+1}}\right]}}{\left[\frac{n}{2^{k+1}}\right]}\right)^q
\end{aligned}
$$

for some $C_{11} > 0$. Keeping in mind (9.22), we can now combine $(5.27), (6.42), (9.24)$ and (9.25) to give,

$$
\begin{aligned}
\left\|P_n^{*(r)}\Phi^r_{\frac{a_n}{n}}W\right\|^q_{L_p(\mathbb{R})} &\leq C_{12}\left\|\sum_{k=0}^{l-1}\left(P^{*(r)}_{\left[\frac{n}{2^k}\right]}(x)-P^{*(r)}_{\left[\frac{n}{2^{k+1}}\right]}(x)\right)\Phi^r_{\frac{a_n}{n}}W\right\|^q_{L_p(\mathbb{R})} && (9.26)\\
&\quad+\left\|P^{*(r)}_{\frac{n}{2^l}}(x)\,\Phi^r_{\frac{a_n}{n}}W\right\|_{L_p(\mathbb{R})}
\end{aligned}
$$

$$
\leq C_{13}\left\|\sum_{k=0}^{l-1}(k+2)^{\frac{rq}{2}}\left(P^{*(r)}_{\left[\frac{n}{2^k}\right]}(x)-P^{*(r)}_{\left[\frac{n}{2^{k+1}}\right]}(x)\right)\Phi^r_{\left[\frac{a_{n/2^k}}{n/2^k}\right]}W\right\|^q_{L_p(\mathbb{R})} \tag{9.27}
$$

$$+\left\| l^{\frac{rq}{2}} P^{*(r)}_{\left[\frac{n}{2^l}\right]}(x)\, \Phi^r_{\left[\frac{a_{n/2^l}}{n/2^l}\right]} W \right\|^q_{L_p(\mathbb{R})}$$

$$\begin{aligned} \leq\ & C_{14} \sum_{k=0}^{l-1} \left(\frac{q_{\left[\frac{n}{2^{k+1}}\right]}}{\left[\frac{n}{2^{k+1}}\right]} \right)^{-rq} (k+2)^{rq} \left\| \left(P^*_{\left[\frac{n}{2^k}\right]}(x) - P^*_{\left[\frac{n}{2^{k+1}}\right]}(x) \right) \right\|^q_{L_p(\mathbb{R})} \\ & + \left(\frac{a_{\left[\frac{n}{2^l}\right]}}{\left[\frac{n}{2^l}\right]} \right)^{-rq} l^{\frac{rq}{2}} \|fW\|^q_{L_p(\mathbb{R})} \end{aligned} \tag{9.28}$$

some C_{12}, C_{13} and $C_{14} > 0$.

We can now combine (9.32) with (9.29) and (9.26) and express this as an integral as,

$$\begin{aligned} w_{r,p}\left(f, W, \frac{a_n}{n}\right)^q \ \leq\ & C_{15} \left(\frac{a_n}{n}\right)^{rq} \left[\int_{\frac{a_n}{n}}^{C_{16}} \frac{w_{r+1,p}(f, W, u)^q \left(\log_2(nu)\right)^{\frac{rq}{2}}}{u^{rq}} du \right. \\ & \left. + \left(\log_2\left(\frac{n}{r}\right)\right)^{\frac{rq}{2}} \|fW\|^q_{L_p(\mathbb{R})} \right] \end{aligned} \tag{9.29}$$

whereby following the proof carefully, it can be easily seen that C_{15} and C_{16} are independent of f and t.

Now let $t > 0$, small enough and determine n by (5.25). First, observe that using Lemma 6.5(a), (6.24) and (6.10), we obtain constants C_{17} and $C_{18} > 0$ independent of t and n such that,

$$C_{18} \leq \frac{\log n}{\log\left(\frac{1}{t}\right)} \leq C_{17} \tag{9.30}$$

so that using (9.2) and (9.30), (9.29) becomes,

$$\begin{aligned} w_{r,p}(f, W, t)^q \ \leq\ & C_{19}(t)^{rq} \left[\int_t^{C_{20}} \frac{w_{r+1,p}(f, W, u)^q \left(\log_2\left(\frac{1}{t}\right)\right)^{\frac{rq}{2}}}{u^{rq}} du \right. \\ & \left. + \left(\log_2\left(\frac{1}{tr}\right)\right)^{\frac{rq}{2}} \|fW\|^q_{L_p(\mathbb{R})} \right]. \end{aligned}$$

Taking $\frac{1}{q}$th roots gives the result.□

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