

Lattice Boltzmann methods for shallow water flow applications.

Tumelo R.A Uoane

School of Computational and Applied Mathematics,
University of the Witwatersrand,
Johannesburg, South Africa.

A Dissertation submitted for the degree of
Masters of Science in
the field of Computational Fluid Dynamics.



June 3, 2011

Declaration

I declare that this dissertation is my own, unaided work. It is being submitted as partial fulfillment for the Degree of Masters of Science to the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

June 3, 2011

Abstract

The lattice Boltzmann (LB) method has become an effective numerical technique for computational fluid dynamics (CFD). The method is based on gas kinetic theory and is used as an alternative numerical method to simulate shallow water equations (SWEs). In this work, an overview of the Shallow Water LB method will be presented and the stability of this approach will be investigated. The stability structure will be used as a guideline to determine the stability of LB equations applied to SWEs. For instant some of the parameters in the LB equations are adjusted based on the stability structure. With this method, stable LB models are prescribed. To verify the stability theory, computational results for a few examples used in the literature are presented.

Acknowledgements

I would like to express my sincere appreciation to my supervisor, Prof M.K Banda for his tireless assistance and encouragement in doing this research. I gratefully appreciate Dr M Maserumula for his financial support on behalf of CSIR. I would also like to thank my parents, daughter Kelebogile, family and my fellow students for their support and encouragement.

Contents

1	Introduction	1
1.1	Shallow Water Equations	1
1.2	Lattice Boltzmann Method	2
1.3	Content of the dissertation	6
2	Shallow Water Flows	8
2.1	Introduction	8
2.2	The Navier-Stokes Equations	8
2.3	The Shallow Water Equations	10
2.4	Numerical Methods	17
3	Lattice Boltzmann Method	19
3.1	Introduction	19
3.2	Continuum Kinetic Theory	19
3.3	Lattice Pattern	21
3.4	Lattice Boltzmann Equation and BGK Approximation	23
3.5	Local Equilibrium Distribution Function	25
3.6	Macroscopic Properties	29
3.7	Recovery of the Shallow Water Equations	31
3.8	Force term	33
3.9	Stability Conditions	36
3.10	Initial and Boundary Conditions	37
3.10.1	Boundary Conditions	37

3.10.2	Initial Conditions	38
3.11	The Basic Solution	39
4	Stability Structure	40
4.1	Introduction	40
4.2	Stability Notion	41
4.2.1	Examples	43
4.3	Determination of Parameters	49
4.3.1	Examples	50
5	Numerical Results	59
5.1	Introduction	59
5.2	Test Examples	59
5.2.1	Example 1: Steady flow over a hump	59
5.2.2	Example 2: Tidal wave flow	65
5.2.3	Example 3: Flow over a sudden-expansion channel	68
6	Conclusion and Future Research	71
A	Computing Equilibrium Distribution Function	72
B	Projection Matrix	74
C	Symmetric Positive Definite Matrix	76
D	Computing Rank	77
E	Using a parameter outside the required range	79

List of Figures

1.1	Top-up versus Bottom-up approach.	6
2.1	Cartesian coordinate system.	9
2.2	Shallow water flow sketch.	11
3.1	The five particle speeds ξ_α in the 2D square lattice.	22
3.2	The seven particle speeds ξ_α in the 2D square lattice.	23
3.3	The nine particle speeds ξ_α in the 2D square lattice.	24
3.4	Inflow and Outflow lattice notes boundaries.	38
4.1	The nine particle speeds ξ_i in the 2D square lattice.	50
5.1	Steady subcritical flow over a hump: Illustration of the free water surface and the bottom profiles.	60
5.2	Steady subcritical flow over a hump: Free water surface using different lattices.	63
5.3	Steady subcritical flow over a hump: Comparison of the free water surface.	64
5.4	Steady subcritical flow over a hump: Comparison of discharge.	65
5.5	Numerical and analytical free surface for the tidal wave flow at time $t =$ 9117.5 s.	66
5.6	Numerical and analytical free surface for the tidal wave flow at time $t =$ 9117.5 s.	67
5.7	Sudden-expansion channel: velocity field, where $g = 0.15$	69

List of Tables

1.1	Evolution from Boltzmann Equation to the Lattice Boltzmann Equation. . .	4
1.2	The summary of drawbacks, cause and treatment of the LGCA	5
5.1	The summary of gravity for different lattices, – shows that there was no convergence.	61
5.2	The summary of the value of λ for different lattices using $g = \frac{1}{3e}$	62
5.3	Comparison of numerical and analytical solutions using L^2 - error norm. . .	68
5.4	The summary of gravity values for $\lambda = 1$ on a 120×60 grid.	69
5.5	The summary of λ values using $g = 0.1667$ ($g = \frac{1}{3e}$).	70

Chapter 1

Introduction

Recently, before the arrival of fast computing machines, solving a dynamical system required sophisticated mathematical techniques and could only be accomplished for a small class of dynamical systems. Structural stability was adopted as the main study for dynamical systems because most systems studied were only known approximately and the parameters of the systems were not known precisely.

Shallow Water Equations (SWEs) are part of computational fluid dynamics where instability often arises. For example, they have been applied to coastal regions, rivers, open channel flow, with numerical methods that are sometimes partially stable. Most of the schemes, for example, finite difference method (FDM) when applied on SWEs, suffer from numerical instability. They produce non-physical oscillations mainly because discretization of the flux and source terms are not well balanced in their reconstructions. Lattice Boltzmann (LB) method was developed to model fluids under such flow regimes hence, adopted in this dissertation.

1.1 Shallow Water Equations

The present dissertation investigates the stability of the LB method applied to SWEs in the domain of Computational Fluid Dynamics (CFD). SWEs are a set of hyperbolic partial differential equations (PDEs) widely known as the Saint Venant equations. These equations describe the flow of a fluid below a pressure surface. Hence, the equations have

a variety of applications which include a wide spectrum of phenomena including water waves. SWEs can be applied to environmental and hydraulic engineering, for example, these include flows in coastal regions, rivers, reservoirs and open channels.

SWEs are derived from depth-integrating the Navier-Stokes (N-S) equations, in the case where the horizontal length scale is much greater than the vertical length scale. Under such conditions the vertical velocity of the fluid is minute and vertical pressure gradients in the momentum equations are nearly hydrostatic. Horizontal pressure gradients are due to the movement of the pressure surface, causing a constant horizontal velocity field throughout the depth of the fluid. Therefore, the vertical velocity is removed by vertically integrating the equation, thus resulting in the SWEs.

Solving SWEs is numerically challenging. So much effort is invested in developing numerical schemes for the resolution of these equations, rather than adding complexity to physical terms. Computational approaches such as FDM, finite volumes (FVM) and finite element (FEM) methods have been applied to simulate SWEs [3, 39, 42, 52]. The treatment of bed slopes and friction forces for some of the methods mentioned above, cause numerical difficulties in obtaining accurate solutions [3, 44]. Solving the SWEs using the above schemes can be viewed as a top-down approach of fluid modelling, starting with a simple conservation law and adding new equations on demand based on theoretical and phenomenological considerations.

The LB method was introduced as an alternative method to solve SWEs due to its appealing features. These include simplicity in programming (parallel implementation) and straight forward incorporation of complex geometry and irregular topology. For example, the LB method was used to simulate SWEs with wind-driven ocean circulation [34, 53]. It also models three-dimensional (3D) planetary geostrophic equations [35].

1.2 Lattice Boltzmann Method

The numerical approach known as LB method is the one used in the dissertation as mentioned in the previous section. The method originates from the basic idea of Ludwig Boltzmann's work, where a gas is viewed as composed interaction of particles as described

in classical mechanics. The statistical measures have to be used because of the number of particles [19]. The LB model reduces the number of possible particles' spatial positions and microscopic momenta of the continuum (Boltzmann equation) significantly. There are a number of evolution stages from the Boltzmann equation to the LB equation, refer to Table 1.1. In this table; ξ is the velocity; ξ_α is the particle velocity in α direction; f is the distribution function; f_α is the particle distribution function in α direction; f^{eq} is the equilibrium function; f_α^{eq} is the local equilibrium function in α direction; Ω is the collision integral; $\tilde{\tau}$ is a relaxation time; τ is a dimensionless single relaxation time and t is time.

Alternatively, the LB method evolves from the Lattice Gas Cellular Automata (LGCA). LGCA is a particular class of Cellular Automata (CA), developed fully from discrete microscopic model of a fluid. Historically, the first model named HPP was proposed by Hardy, de Pazzis and Pomeau in 1973 and developed as part of LGCA [18]. This model was used to simulate the N-S equations but failed due to an insufficient degree of rotational symmetry of the lattice. After a decade in 1987, the FHP model was developed and it was the first successful LGCA to be used to derive macroscopic equations which led to the N-S equations [15]. It was found that LGCA for the N-S equations are plagued by several drawbacks, such as the non-isotropic advection term and numerical noise [45]. The summary of drawbacks, cause and treatment of LGCA can be viewed in Table 1.2. As a result, LB method was developed to treat such drawbacks.

Boltzmann Equation:

$$\frac{\partial f}{\partial t} + \boldsymbol{\xi} \cdot \nabla f = \Omega, \quad \Downarrow$$

Boltzmann Equation using the BGK approximation:

$$\frac{\partial f}{\partial t} + \boldsymbol{\xi} \cdot \nabla f = -\frac{1}{\tilde{\tau}} (f - f^{eq}), \quad \Downarrow$$

Discrete Boltzmann Equation:

$$\frac{\partial f_\alpha}{\partial t} + \boldsymbol{\xi}_\alpha \cdot \nabla f_\alpha = -\frac{1}{\tilde{\tau}} (f_\alpha - f_\alpha^{eq}), \quad \Downarrow$$

Lattice Boltzmann Equation:

$$f_\alpha(\mathbf{x} + \boldsymbol{\xi}_\alpha \Delta t, t + \Delta t) - f_\alpha(\mathbf{x}, t) = -\frac{1}{\tau} (f_\alpha - f_\alpha^{eq})$$

Table 1.1: Evolution from Boltzmann Equation to the Lattice Boltzmann Equation.

The LB method is relatively new and compared to other schemes in computational fluid dynamics it uses a bottom-up approach to fluid modelling. To achieve this, a fluid is described at a molecular level in which case the molecules are transported and collide. The collisions are resolved by different models, for example, the Bhatnagar-Gross-Krook (BGK) model [2]. This differs with other traditional methods in which the top-down approach is used to model fluids, i.e the full continuum-level physics of the fluid is implicit, refer to Figure 1.1. Therefore, the method always has a model which consists of various physical ingredients to be identified one by one and properly allows segregation between relevant and negligible properties, for example, using the collision term. Based on this, the collision term is modelled in a simplified way for numerical treatment. Unlike other methods, the LB method adapts well in simulating complex fluids, for example, in the two-phase flow

Drawbacks	Cause	Treatment
1. Advection term is non-isotropic	The lattice tensor is of rank 4	Higher symmetric lattice are used
2. The Galilean invariance are violated	Fermi-Dirac distribution	Rescaling is used
3. Too much numerical noise	Boolean variables are used	Averaging is used
4. Spurious invariants	Regular lattice	Increasing collision

Table 1.2: The summary of drawbacks, cause and treatment of the LGCA

the method can represent the two phases in one fluid model. Therefore, the LB method became a powerful numerical tool to simulate fluid flow problems, mostly in engineering.

The main advantages of the LB method can be summarized as follows:

1. it is easier to programme since it consists of simple arithmetic calculations;
2. there is only one unknown variable that needs to be determined, the microscopic distribution function;
3. convenient for parallel programming since the current value of the distribution function depends only on the previous condition 2;
4. simulation of flow in complex flow is easily achieved, for example, multiphase flows and flows with variations of boundary conditions and
5. flows in complex geometry are easily simulated, for example, flows through porous media since boundary conditions are easily implemented.

It is well known among LB researchers that instability problems arise frequently, when the LB method is viewed as a finite-difference method for solving the continuum discrete-velocity Boltzmann equations. It becomes clear that numerical accuracy and stability issues should be addressed. The stability structure is used to investigate the stability

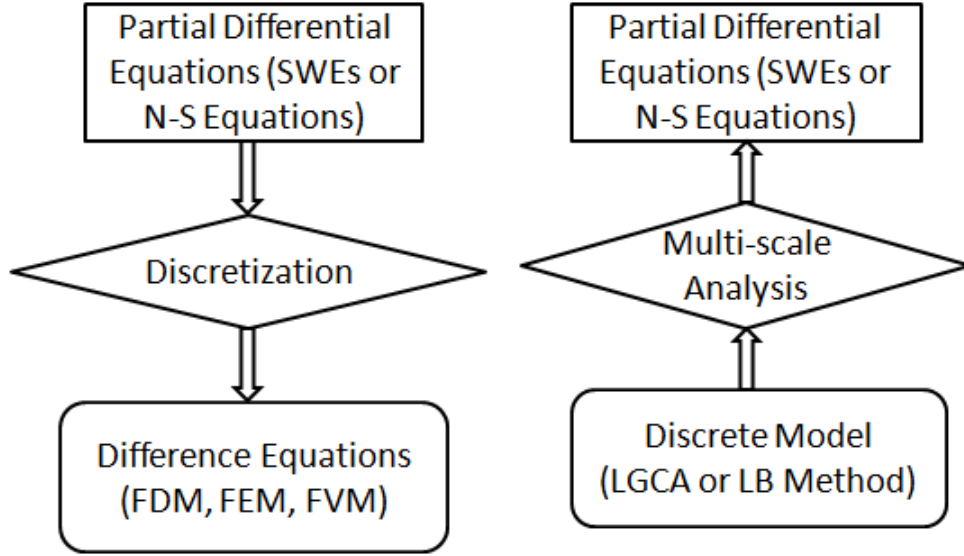


Figure 1.1: Top-up versus Bottom-up approach.

of the LB equations which are currently being applied to simulate SWEs. Stable LB equations will result in stable simulation using LB method.

1.3 Content of the dissertation

The dissertation consists of six chapters. In the first chapter, we briefly introduce SWEs, LB methods and the background of the dissertation. The second chapter presents theory of hydrodynamics and derivations of SWEs. The depth-averaged property is used to achieve the suitable mathematical model for SWEs. Chapter 3 studies the LB theory to solve the SWEs. The most useful technique to the recovery of the SWEs from the LB equations is described in detail, namely the Chapman-Enskog expansion. A brief discussion on stability conditions, boundary conditions and the force term is also covered in the chapter. The relation of the continuum Boltzmann equation and the LB equation will also be discussed. The main core of the dissertation is formed by Chapter 4, where we introduce the stability structure. The stability structure is used to investigate the stability of the LB equations which are currently being applied to simulate SWEs. In Chapter 5, we implement the LB

method and test the results obtained from the stability structure on selected Benchmark problems. In the last chapter, the conclusions and recommendations for future work are presented.

Chapter 2

Shallow Water Flows

2.1 Introduction

Fluid flows obey the conservation of mass and momentum. A set of differential equations representing fluid flow can be derived based on these conservation laws, namely the N-S equations. These equations form a mathematical model for general fluid flow and are used in modelling shallow water flows. The SWEs which are a depth-averaged form of N-S equations are used to describe the horizontal structure of fluid flows. In this chapter, we introduce SWEs and look at their derivations which were obtained from Zhou [51].

2.2 The Navier-Stokes Equations

The governing equations for the incompressible flows are the three-dimensional incompressible N-S equation which can be written in the form

$$\begin{aligned}\nabla \cdot \mathbf{u} &= 0, \\ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} &= -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u} + \mathbf{f},\end{aligned}\tag{2.1}$$

where $\mathbf{u} = (u, v, w)$ is the fluid velocity, ρ the fluid density, t the time; ν the kinematic viscosity, p the pressure and $\mathbf{f}$ represents the body forces (per unit volume) acting on the

fluid and ∇ is the del operator defined as

$$\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k$$

where $\{i, j, k\}$ are unit vectors in their respective directions. The x , y and z are the Cartesian coordinates, refer to Figure 2.1, with u , v and w being their corresponding velocity components. The Laplace operator ∇^2 is a scalar defined as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

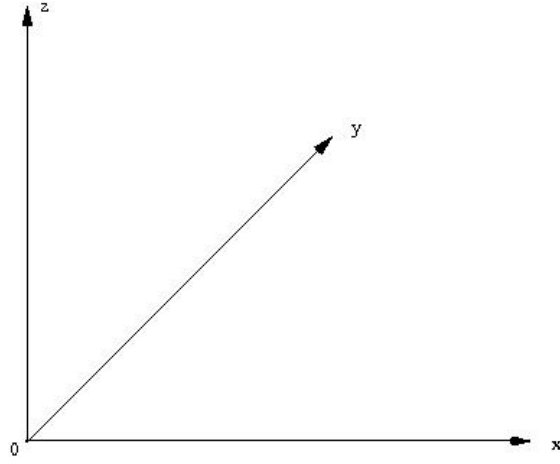


Figure 2.1: Cartesian coordinate system.

Equation (2.1) can be written in tensor form as

$$\frac{\partial u_j}{\partial x_j} = 0 \tag{2.2}$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} + f_i \tag{2.3}$$

where the subscripts i and j are the space direction indices and the Einstein summation convention is used.

The terms in the N-S equations have physical interpretations. The left side of Equation (2.3) describes the inertia terms and consist of two main contributions: the local acceleration and the convective acceleration. These terms can also be viewed as time-rate of change of momentum per unit mass. The right side of the equation is the summation of a

pressure term, the viscous term and body forces. The N-S equations are generally known to have few analytical solutions. As a result various numerical methods were developed to solve the flow problems, taking advantage of computer technology. These made numerical methods a powerful tool for solving flow problems.

2.3 The Shallow Water Equations

The flow of water has always caused amazement in various fields of interest. Usually the depth for flows in rivers, channels, coastal areas, estuaries and harbours is much smaller than the horizontal scale. Therefore, such flows are characterized by horizontal motion, refer to Figure 2.2. These flows can generally be described using two-dimensional and three-dimensional SWEs model. (Note that the assumption of the hydrostatic pressure is often used in the mathematical model to replace the momentum equation in the vertical direction, so the vertical acceleration is ignored [51]).

When a two-dimensional model is used, the depth-averaged quantity provides the horizontal flow structures without vertical velocity. While in three-dimensional model the continuity equation is used to calculate vertical velocity. Both models predict the vertical separation inaccurately [39], suggesting that both models have the same disadvantages. As a result two-dimensional SWEs are widely used as a mathematical model for shallow water flow and are applied in this dissertation.

There are two body forces which act on water flow on earth, namely: the gravitational force which act vertical due to gravity and Coriolis acceleration due to the earth's rotation [13]. In three dimensional space the forces can be written as follows

$$f_x = f_c v, \quad f_y = -f_c u, \quad f_z = -g, \quad (2.4)$$

where u and v are velocity components in x and y directions; $g = 9.81 m/s^2$ is the gravitational acceleration and $f_c = 2\omega \sin\phi$ is the Coriolis parameter in which $\omega \approx 7.3 \times 10^{-5} rad/s$ and ϕ is the earth's latitude.

The two-dimensional SWEs can be derived from depth-integration of the N-S equations, where horizontal length scale is much greater than the vertical scale. The derivations are shown below.

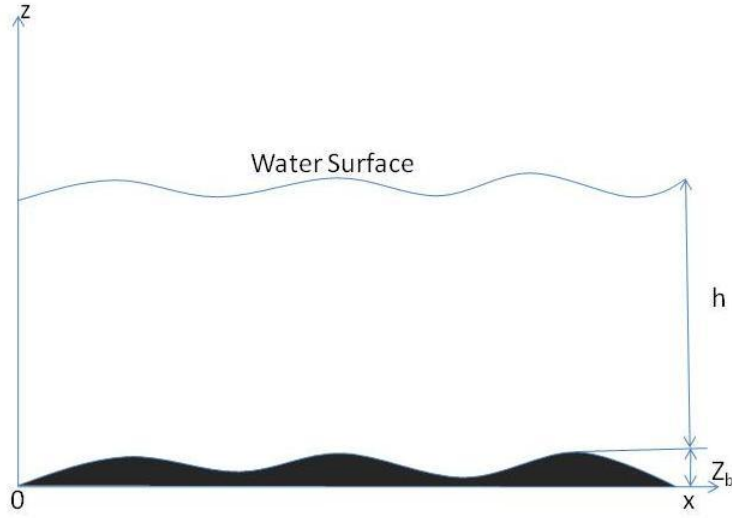


Figure 2.2: Shallow water flow sketch.

If Equation (2.2) is integrated over depth, the continuity equation with respect to depth-averaged quantities is obtained as

$$\int_{z_b}^{h+z_b} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) dz = 0, \quad (2.5)$$

leading to

$$\int_{z_b}^{h+z_b} \frac{\partial u}{\partial x} dz + \int_{z_b}^{h+z_b} \frac{\partial v}{\partial y} dz + w_s - w_b = 0, \quad (2.6)$$

where w_b and w_s are the vertical velocities at the channel bed and the free surface, respectively, h is the water depth and z_b is the bed elevation above the datum, refer to Figure 2.2.

Using the Leibnitz rule [37]:

$$\int_a^b \frac{\partial f(x, y)}{\partial y} dx = \frac{\partial}{\partial y} \int_a^b f(x, y) dx - f(b, y) \frac{\partial b}{\partial y} + f(a, y) \frac{\partial a}{\partial y} = 0, \quad (2.7)$$

the first term on the left side of Equation (2.6) can be written as

$$\int_{z_b}^{h+z_b} \frac{\partial u}{\partial x} dz = \frac{\partial}{\partial x} \int_{z_b}^{h+z_b} u dz - u_s \frac{\partial(h+z_b)}{\partial x} + u_b \frac{\partial z_b}{\partial x}, \quad (2.8)$$

and the second term as

$$\int_{z_b}^{h+z_b} \frac{\partial v}{\partial y} dz = \frac{\partial}{\partial y} \int_{z_b}^{h+z_b} v dz - v_s \frac{\partial(h+z_b)}{\partial y} + v_b \frac{\partial z_b}{\partial y}. \quad (2.9)$$

Substituting Equations (2.8) and (2.9) into (2.6) gives

$$\begin{aligned} \frac{\partial}{\partial x} \int_{z_b}^{h+z_b} u \, dz + \frac{\partial}{\partial y} \int_{z_b}^{h+z_b} v \, dz + \left[w_s - u_s \frac{\partial(h+z_b)}{\partial x} - v_s \frac{\partial(h+z_b)}{\partial y} \right] \\ - \left(w_b - u_b \frac{\partial z_b}{\partial x} - v_b \frac{\partial z_b}{\partial y} \right) = 0, \end{aligned} \quad (2.10)$$

The kinematic conditions are given by

$$w_s = \frac{\partial(h+z_b)}{\partial t} + u_s \frac{\partial(h+z_b)}{\partial x} + v_s \frac{\partial(h+z_b)}{\partial y}, \quad (2.11)$$

and

$$w_b = \frac{\partial z_b}{\partial t} + u_b \frac{\partial z_b}{\partial x} + v_b \frac{\partial z_b}{\partial y}, \quad (2.12)$$

at the water surface and channel bed, respectively. By substituting the above kinematic conditions (2.11) and (2.12) into Equation (2.10), we obtain the continuity equation for shallow water flows:

$$\frac{\partial h}{\partial t} + \frac{\partial(h\bar{u})}{\partial x} + \frac{\partial(h\bar{v})}{\partial y} = 0, \quad (2.13)$$

where $\bar{u}$ and $\bar{v}$ are the depth-averaged velocities, defined as

$$\bar{u} = \frac{1}{h} \int_{z_b}^{h+z_b} u \, dz, \quad \bar{v} = \frac{1}{h} \int_{z_b}^{h+z_b} v \, dz. \quad (2.14)$$

Integrating the x -component of the momentum Equation (2.3) gives

$$\begin{aligned} \int_{z_b}^{h+z_b} \left[\frac{\partial u}{\partial t} + \frac{\partial(u^2)}{\partial x} + \frac{\partial(vu)}{\partial y} + \frac{\partial(wu)}{\partial z} \right] dz = \\ \int_{z_b}^{h+z_b} \left[-\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \right] dz + \int_{z_b}^{h+z_b} f_c v \, dz. \end{aligned} \quad (2.15)$$

Using the Leibnitz rule (2.7), on the first three terms of the left side of Equation (2.15) gives

$$\int_{z_b}^{h+z_b} \frac{\partial u}{\partial t} dz = \frac{\partial}{\partial t} \int_{z_b}^{h+z_b} u \, dz - u_s \frac{\partial(h+z_b)}{\partial t} + u_b \frac{\partial z_b}{\partial t}, \quad (2.16)$$

$$\int_{z_b}^{h+z_b} \frac{\partial u^2}{\partial x} dz = \frac{\partial}{\partial x} \int_{z_b}^{h+z_b} u^2 \, dz - u_s^2 \frac{\partial(h+z_b)}{\partial x} + u_b^2 \frac{\partial z_b}{\partial x}, \quad (2.17)$$

$$\int_{z_b}^{h+z_b} \frac{\partial(vu)}{\partial y} dz = \frac{\partial}{\partial y} \int_{z_b}^{h+z_b} vu \, dz - v_s u_s \frac{\partial(h+z_b)}{\partial y} + v_b u_b \frac{\partial z_b}{\partial y}. \quad (2.18)$$

The last term on the left side of Equation (2.15) can be integrated directly as

$$\int_{z_b}^{h+z_b} \frac{\partial(wu)}{\partial z} dz = w_s u_s - w_b u_b. \quad (2.19)$$

Then Equations (2.16), (2.17), (2.18) and (2.19) are put together and rearranged to obtain

$$\begin{aligned} & \int_{z_b}^{h+z_b} \left[\frac{\partial u}{\partial t} + \frac{\partial(u^2)}{\partial x} + \frac{\partial(vu)}{\partial y} + \frac{\partial(wu)}{\partial z} \right] dz = \\ & \frac{\partial}{\partial t} \int_{z_b}^{h+z_b} u \, dz + \frac{\partial}{\partial x} \int_{z_b}^{h+z_b} u^2 \, dz + \frac{\partial}{\partial y} \int_{z_b}^{h+z_b} vu \, dz \\ & + u_s \left[w_s - \frac{\partial(h+z_b)}{\partial t} - u_s \frac{\partial(h+z_b)}{\partial x} - v_s \frac{\partial(h+z_b)}{\partial y} \right] \\ & - u_b \left(w_b - \frac{\partial z_b}{\partial t} - u_b \frac{\partial z_b}{\partial x} - v_b \frac{\partial z_b}{\partial y} \right). \end{aligned} \quad (2.20)$$

Using the kinematic conditions Equation (2.11) and (2.12) together with (2.14), then Equation (2.20) can be simplified as

$$\begin{aligned} & \int_{z_b}^{h+z_b} \left[\frac{\partial u}{\partial t} + \frac{\partial(u^2)}{\partial x} + \frac{\partial(vu)}{\partial y} + \frac{\partial(wu)}{\partial z} \right] dz = \\ & \frac{\partial(h\bar{u})}{\partial t} + \frac{\partial}{\partial x} \int_{z_b}^{h+z_b} u^2 \, dz + \frac{\partial}{\partial y} \int_{z_b}^{h+z_b} vu \, dz. \end{aligned} \quad (2.21)$$

Applying the second mean value theorem for integrals [38] gives

$$\int_a^b f(x)g(x)dx = f(\zeta) \int_a^b g(x)dx, \quad (2.22)$$

where $f(x)$ and $g(x)$ are continuous on $[a, b]$ with $g(x) \geq 0$ for any $x \in [a, b]$ and $\zeta \in (a, b)$.

The following terms can be expressed as

$$\int_{z_b}^{h+z_b} u^2 = \hat{u}_1 \int_{z_b}^{h+z_b} u \, dz = \hat{u}_1 h \bar{u}, \quad (2.23)$$

and

$$\int_{z_b}^{h+z_b} vu = \hat{u}_2 \int_{z_b}^{h+z_b} v \, dz = \hat{u}_2 h \bar{v}, \quad (2.24)$$

where $\hat{u}_1 = \theta_1 \bar{u}$ and $\hat{u}_2 = \theta_2 \bar{u}$, θ_1 and θ_2 are momentum correction factors which can be determined based on Equations (2.23) and (2.24).

$$\theta_1 = \frac{1}{h\bar{u}^2} \int_{z_b}^{h+z_b} u^2 dz, \quad \theta_2 = \frac{1}{h\bar{v}\bar{u}} \int_{z_b}^{h+z_b} vu \, dz. \quad (2.25)$$

It is observed that using the second mean value Theorem (2.22) implies no change of directions for $u(x, y, z, t)$ and $v(x, y, z, t)$ over the water depth at time t . The velocities are assumed to satisfy $u \geq 0$ or $u < 0$ from channel bed to free surface at the horizontal location (x, y) , and v is treated similarly. This gives the reason why a model based on

two-dimensional SWEs can not be applied to flow separations in the vertical direction [51].

Inserting Equation (2.23) and (2.24) into (2.21) leads to

$$\int_{z_b}^{h+z_b} \left[\frac{\partial u}{\partial t} + \frac{\partial(u^2)}{\partial x} + \frac{\partial(vu)}{\partial y} + \frac{\partial(wu)}{\partial z} \right] dz = \frac{\partial(h\bar{u})}{\partial t} + \frac{\partial(h\theta_1\bar{u}^2)}{\partial x} + \frac{\partial(h\theta_2\bar{u}\bar{v})}{\partial y}. \quad (2.26)$$

In a similar way, when using the y -component of the momentum Equation (2.3), the following equation is obtained,

$$\int_{z_b}^{h+z_b} \left[\frac{\partial v}{\partial t} + \frac{\partial(uv)}{\partial x} + \frac{\partial(v^2)}{\partial y} + \frac{\partial(wv)}{\partial z} \right] dz = \frac{\partial(\bar{v}h)}{\partial t} + \frac{\partial(\theta_2\bar{u}\bar{v}h)}{\partial x} + \frac{\partial(\theta_3\bar{v}^2h)}{\partial y}, \quad (2.27)$$

with θ_3 defined by

$$\theta_3 = \frac{1}{h\bar{v}^2} \int_{z_b}^{h+z_b} v^2 dz. \quad (2.28)$$

Integrating the last term on the right side of Equation (2.15) gives

$$\int_{z_b}^{h+z_b} f_c v dz = f_c h \bar{v}. \quad (2.29)$$

Since the vertical acceleration is insignificant compared to the horizontal effect, the z -component of the momentum Equation (2.3), is reduced with $w = 0$ i.e when $i = 3$ to

$$\frac{\partial p}{\partial z} = -\rho g. \quad (2.30)$$

Integrating, gives

$$p = -\rho g z + C_0, \quad (2.31)$$

where C_0 is the integration constant. Knowing that the pressure at the free surface is the atmospheric pressure p_a , then using boundary conditions $p = p_a$ when $z = h + z_b$ yields

$$C_0 = \rho g(h + z_b) + p_a. \quad (2.32)$$

Substituting Equation (2.32) into (2.31) gives

$$p = \rho g(h + z_b - z) + p_a. \quad (2.33)$$

In practice, p_a is almost constant in the modeling domain. The difference in the atmospheric pressure at the water surface is often insignificant, especially in hydraulic engineering, therefore, we set $p_a = 0$ and Equation (2.33) becomes

$$p = \rho g(h + z_b - z). \quad (2.34)$$

The above Equation (2.34) is usually referred to as "*the hydrostatic pressure approximation*" in shallow water flow. Differentiating Equation (2.34) with respect to x leads to

$$\frac{\partial p}{\partial x} = \rho g \frac{\partial(h + z_b)}{\partial x}. \quad (2.35)$$

Since h and z_b are independent of z then, they should be dependent on x and y . This suggests

$$\int_{z_b}^{h+z_b} \frac{1}{\rho} \frac{\partial p}{\partial x} dz = \frac{h}{\rho} \frac{\partial p}{\partial x}. \quad (2.36)$$

Substituting Equation (2.35) into (2.36) gives

$$\int_{z_b}^{h+z_b} \frac{1}{\rho} \frac{\partial p}{\partial x} dz = \rho g \frac{\partial(h + z_b)}{\partial x}. \quad (2.37)$$

The following approximations are introduced for the second and third terms on the right side of Equation (2.15), respectively, since the acceleration in the z direction is small and $\bar{u}$ (the average of component u along the z direction) is taken.

$$\int_{z_b}^{h+z_b} \nu \frac{\partial^2 u}{\partial x^2} dz \approx \nu \frac{\partial^2(h\bar{u})}{\partial x^2} \quad (2.38)$$

$$\int_{z_b}^{h+z_b} \nu \frac{\partial^2 u}{\partial y^2} dz \approx \nu \frac{\partial^2(h\bar{u})}{\partial y^2}. \quad (2.39)$$

The forth term on the right side of Equation (2.15) may be calculated as

$$\int_{z_b}^{h+z_b} \nu \frac{\partial^2 u}{\partial z^2} dz = \left(\nu \frac{\partial u}{\partial z} \right)_s - \left(\nu \frac{\partial u}{\partial z} \right)_b. \quad (2.40)$$

The right side terms of Equation (2.40) can be approximated with the wind shear stress and the bed shear stress, respectively, in the x direction, giving

$$\left(\nu \frac{\partial u}{\partial z} \right)_s = \frac{\tau_{wx}}{\rho}, \quad \left(\nu \frac{\partial u}{\partial z} \right)_b = \frac{\tau_{bx}}{\rho}. \quad (2.41)$$

By substituting Equation (2.41) into (2.40) leads to

$$\int_{z_b}^{h+z_b} \nu \frac{\partial^2 u}{\partial z^2} dz = \frac{\tau_{wx}}{\rho} - \frac{\tau_{bx}}{\rho}. \quad (2.42)$$

Substitution of Equations (2.26), (2.29), (2.38), (2.39), (2.40) and (2.42) into Equation (2.15) leads to the x momentum equation in the x -direction for the shallow water flows,

$$\begin{aligned} \frac{\partial(h\bar{u})}{\partial t} + \frac{\partial(h\theta_1\bar{u}^2)}{\partial x} + \frac{\partial(h\theta_2\bar{u}\bar{v})}{\partial y} = & -g\frac{\partial}{\partial x} \left(\frac{h^2}{2} \right) + \nu \frac{\partial^2(h\bar{u})}{\partial x^2} + \nu \frac{\partial^2(h\bar{u})}{\partial y^2} \\ & -gh\frac{\partial z_b}{\partial x} - f_ch\bar{v} + \frac{\tau_{wx}}{\rho} - \frac{\tau_{bx}}{\rho}. \end{aligned} \quad (2.43)$$

The momentum equation in the y -direction can be derived in a similar way:

$$\begin{aligned} \frac{\partial(h\bar{v})}{\partial t} + \frac{\partial(h\theta_2\bar{u}\bar{v})}{\partial x} + \frac{\partial(h\theta_3\bar{v}^2)}{\partial y} = & -g\frac{\partial}{\partial y} \left(\frac{h^2}{2} \right) + \nu \frac{\partial^2(h\bar{v})}{\partial x^2} + \nu \frac{\partial^2(h\bar{v})}{\partial y^2} \\ & -gh\frac{\partial z_b}{\partial y} - f_ch\bar{u} + \frac{\tau_{wy}}{\rho} - \frac{\tau_{by}}{\rho}. \end{aligned} \quad (2.44)$$

If velocity profiles for u and v are assumed or already known, the momentum factors θ_1 , θ_2 and θ_3 can be calculated by Equations (2.25) and (2.28) theoretically. In most situations, however there are no valid universal velocity profiles. It is difficult to estimate the momentum correlation forces θ_1 , θ_2 and θ_3 in circulation or separation flow, or in channels with complex geometry. Therefore $\theta_1 = 1$, $\theta_2 = 1$ and $\theta_3 = 1$ are adopted for shallow water flow, which give a good approximation in most situations [4, 24, 27].

Substituting $\theta_1 = 1$, $\theta_2 = 1$ and $\theta_3 = 1$ in Equations (2.43) and (2.44) leads to

$$\begin{aligned} \frac{\partial(h\bar{u})}{\partial t} + \frac{\partial(h\bar{u}^2)}{\partial x} + \frac{\partial(h\bar{u}\bar{v})}{\partial y} = & -g\frac{\partial}{\partial x} \left(\frac{h^2}{2} \right) + \nu \frac{\partial^2(h\bar{u})}{\partial x^2} + \nu \frac{\partial^2(h\bar{u})}{\partial y^2} \\ & -gh\frac{\partial z_b}{\partial x} - f_ch\bar{v} + \frac{\tau_{wx}}{\rho} - \frac{\tau_{bx}}{\rho}, \end{aligned} \quad (2.45)$$

and

$$\begin{aligned} \frac{\partial(h\bar{v})}{\partial t} + \frac{\partial(h\bar{u}\bar{v})}{\partial x} + \frac{\partial(h\bar{v}^2)}{\partial y} = & -g\frac{\partial}{\partial y} \left(\frac{h^2}{2} \right) + \nu \frac{\partial^2(h\bar{v})}{\partial x^2} + \nu \frac{\partial^2(h\bar{v})}{\partial y^2} \\ & -gh\frac{\partial z_b}{\partial y} - f_ch\bar{u} + \frac{\tau_{wy}}{\rho} - \frac{\tau_{by}}{\rho}. \end{aligned} \quad (2.46)$$

The overbars in the Equations (2.45) and (2.46) are dropped for convenience. The continuity Equation (2.13) and the above momentum equations can be concisely written in a tensor form as

$$\frac{\partial h}{\partial t} + \frac{\partial(hu_j)}{\partial x_j} = 0, \quad (2.47)$$

$$\frac{\partial(hu_i)}{\partial t} + \frac{\partial(hu_i u_j)}{\partial x_j} = -\frac{g}{2} \frac{\partial h^2}{\partial x_i} + \nu \frac{\partial^2(hu_i)}{\partial x_j^2} + F_i, \quad (2.48)$$

where F_i is defined as

$$F_i = \frac{\tau_{wi}}{\rho} - \frac{\tau_{bi}}{\rho} - gh \frac{\partial z_b}{\partial x_i} + \Omega_i, \quad (2.49)$$

with Ω_i denoting the Coriolis term given by

$$\Omega_i = \begin{cases} f_c h v, & i = x \\ -f_c h u, & i = y. \end{cases} \quad (2.50)$$

The bed shear stress τ_{bi} in the i direction is given by the depth-averaged velocities,

$$\tau_{bi} = \rho C_b u_i \sqrt{u_j u_j} \quad (2.51)$$

where $C_b = g/C_z^2$ is the bed friction coefficient [51]. The Chezy coefficient is given by $C_z = h^{1/6}/n_b$ where n_b denotes the Manning's coefficient or the Colebrook-White equation

$$C_z = -\sqrt{32g} \log \left(\frac{K_s}{14.8h} + \frac{1.255 \nu C_z}{4 \sqrt{2g} h u} \right), \quad (2.52)$$

K_s is the Nikuradse equivalent sand roughness [14]. The wind shear stress is defined by

$$\tau_{wi} = \rho_a C_w u_{wi} \sqrt{u_{wj} u_{wj}} \quad (2.53)$$

where ρ_a is the density of the air, C_w the resistance coefficient and u_{wi} the component of the wind velocity in the i -th direction [51].

2.4 Numerical Methods

Solving hydrodynamic systems require a sophisticated mathematical technique and can only be accomplished for a small class of systems. N-S equations are part of the hydrodynamic system where a limited number of analytical solutions are found. Based on literature, SWEs are part of hydrodynamic system where analytical solutions are not found. These challenges make numerical methods a popular tool for solving flow equations.

A Semi-Implicit Method for Pressure-Linked Equations (SIMPLE) which is based on finite volume discretization to solve velocity -depth coupling in the SWEs is one of the

methods proposed in the literature to find the solution of SWEs [27, 52]. Other methods like Godunov-type [43] and Semi-implicit methods [6, 8, 31] are also used. The former is based on solving hyperbolic partial differential equations which requires knowledge of an approximate Riemann solver because all properties such as shock and rarefaction waves appear as characteristics in the solution. And the latter is based on a high order finite difference method where certain schemes are embedded for data reconstruction to guarantee the non-oscillatory behavior and to improve the resolution.

All the methods described above are formulated based on a direct solution of the SWEs. Proper initial and boundary conditions were used in schemes like finite element, finite volume and finite difference method to obtain numerical solution of the discretized equations. These methods have common disadvantages, their convective terms, or numerical flux, or source terms require a careful treatment in their numerical procedure.

Lattice Boltzmann (LB) method was developed to model fluids under a variety of flow regimes. The method is based on statistical physics and describes the microscopic picture of particle movements in an extremely simplified way, but on the macroscopic level it gives a correct average description of fluid flow. It is a very promising computational method for simulating fluid flows and in this dissertation, the method will be adopted to simulate SWEs.

Chapter 3

Lattice Boltzmann Method

3.1 Introduction

The LB method was briefly introduced in the first chapter as a numerical technique to solve problems in CFD. The method generally comprises of three components. Firstly, the LB equation, which can be obtained from the discretization of Boltzmann equation, this equation describes the transport of particles. Secondly, the lattice pattern, which is represented by the grid nodes, this determines particles direction. Lastly, the local distribution function, the flow equations are recovered by this LB model, for example, the N-S equations and SWEs. These three components for the LB method are described fully in this chapter together with some related topics. It is important to introduce continuum kinetic theory for the understanding of the Boltzmann equation.

3.2 Continuum Kinetic Theory

In kinetic theory, the single-particle distribution function $f = f(\mathbf{x}, \boldsymbol{\xi}, t)$ represents the probability density of particles at position $\mathbf{x}$, moving with velocity $\boldsymbol{\xi}$ at time t in a phase space. A simple Continuum Boltzmann equation with the BGK equation is given by [2]

$$\frac{\partial f}{\partial t} + \boldsymbol{\xi} \cdot \nabla f = -\frac{1}{\tilde{\tau}} (f - f^{eq}), \quad (3.1)$$

where ∇ is a gradient operator, $\tilde{\tau}$ is the relaxation time and f^{eq} is the Maxwell-Boltzmann equilibrium distribution function in D spatial dimensions given by

$$f^{eq} = \frac{\rho}{(2\pi/3)^{D/2}} \exp \left[-\frac{3}{2}(\boldsymbol{\xi} - \mathbf{V})^2 \right]. \quad (3.2)$$

The fluid density ρ and the velocity $\mathbf{V}$ are macroscopic variables recovered from the moments of distribution function f , with respect to the microscopic particle velocity $\boldsymbol{\xi}$ using

$$\rho = \int f d\boldsymbol{\xi}, \quad \rho \mathbf{V} = \int \boldsymbol{\xi} f d\boldsymbol{\xi}. \quad (3.3)$$

The two particle velocities $\boldsymbol{\xi}$ and $\mathbf{V}$ can be normalized by $\sqrt{3RT}$, where R is the ideal gas constant and T is the temperature, giving $U_s = 1/\sqrt{3}$ which is the speed of sound [9]. The equilibrium distribution function in Equation (3.2) can be expressed as

$$f^{eq} = \frac{\rho}{(2\pi/3)^{D/2}} \exp \left(-\frac{3}{2}\boldsymbol{\xi}^2 \right) \left[1 + 3(\boldsymbol{\xi} \cdot \mathbf{V}) + \frac{9}{2}(\boldsymbol{\xi} \cdot \mathbf{V})^2 - \frac{3}{2}\mathbf{V} \cdot \mathbf{V} \right] \quad (3.4)$$

to second-order accuracy if, the fluid velocity $\mathbf{V}$ is smaller compared with the speed of sound [23]. The continuum Boltzmann (BGK) equation as given in Equation (3.1) can be written in a discrete form

$$\frac{\partial f_\alpha}{\partial t} + \boldsymbol{\xi}_\alpha \cdot \nabla f_\alpha = -\frac{1}{\tilde{\tau}} (f_\alpha - f_\alpha^{eq}), \quad (3.5)$$

where $\boldsymbol{\xi}_\alpha$ is a discrete particle velocity with $\alpha = 1, 2, \dots, M$, (M is the number of directions in the lattice node) with the distribution functions $f_\alpha(\mathbf{x}, t) = f(\mathbf{x}, \boldsymbol{\xi}_\alpha, t)$ and $f_\alpha^{eq}(\mathbf{x}, t) = f^{eq}(\mathbf{x}, \boldsymbol{\xi}_\alpha, t)$. By integrating Equation (3.5) along the characteristics, we obtain

$$f_\alpha(\mathbf{x} + \boldsymbol{\xi}_\alpha \Delta t, t + \Delta t) = f_\alpha(\mathbf{x}, t) - \int_0^{\Delta t} \frac{1}{\tilde{\tau}} [f_\alpha - f_\alpha^{eq}](\mathbf{x} + \boldsymbol{\xi}_\alpha \sigma, t + \sigma) d\sigma, \quad (3.6)$$

and approximating the integral by a simple rectangle rule ($\sigma = 0$) yields

$$f_\alpha(\mathbf{x} + \boldsymbol{\xi}_\alpha \Delta t, t + \Delta t) - f_\alpha(\mathbf{x}, t) = -\frac{1}{\tilde{\tau}} (f_\alpha - f_\alpha^{eq}). \quad (3.7)$$

This is the standard LB equation used today with a single relaxation time denoted as τ and given by $\tau = \Delta t/\tilde{\tau}$. The left hand side of Equation (3.7) represents the streaming of particles and the right hand side denotes collision of particles.

3.3 Lattice Pattern

In LB method, lattice pattern can be used for two main reasons, namely to set up grid points and determining particles motion. This is done for models at microscopic level.

Two-dimensional models will be considered and there are various lattice pattern selections for which the LB method can be written down in simple form [23, 34]. The popular square and hexagon lattices are used in this dissertation, refer to Figure 3.1, Figure 3.2 and Figure 3.3. They are referred to by D2Q5, D2Q7 and D2Q9, respectively, i.e the two-dimensional five, seven and nine velocities according to Qian's notation [32]. The models we have chosen to work with have lattice patterns which have sufficient symmetry, which is a dominant requirement for the recovery of the SWE, excluding that of D2Q5. There are other lattice patterns which are used in LB method which will not be used in this dissertation, for example, D3Q15 and D3Q19.

The theoretical analysis and numerical studies have shown that both D2Q7 and D2Q9 lattices have such a property and satisfactory performance in numerical simulations, but it was proven that D2Q9 lattice gives more accurate results than those of D2Q7 lattice model [36]. Square lattices have advantages when implementing boundary conditions [50] (Note that the D2Q5 model provides us with computational advantage, since we have fewer directions for the recovery of SWEs [34]).

For the lattice square with nine velocities (D2Q9), each particle moves one lattice unit at its velocity along the link as indicated with numbers 1 - 8 and 0 represents the rest particle (particle with zero velocity). The velocity vector of the particles is defined as follows:

$$\xi_{\alpha} = \begin{cases} (0, 0), & \alpha = 0 \\ e \left[\cos \frac{(\alpha-1)\pi}{4}, \sin \frac{(\alpha-1)\pi}{4} \right], & \alpha = 1, 3, 5, 7, \\ \sqrt{2}e \left[\cos \frac{(\alpha-1)\pi}{4}, \sin \frac{(\alpha-1)\pi}{4} \right], & \alpha = 2, 4, 6, 8. \end{cases} \quad (3.8)$$

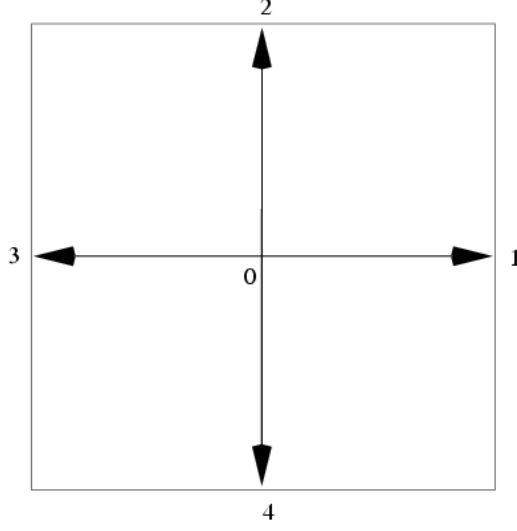


Figure 3.1: The five particle speeds ξ_α in the 2D square lattice.

It can be shown that the D2Q9 lattice has the following basic features

$$\sum_{\alpha} \xi_{\alpha i} = \sum_{\alpha} \xi_{\alpha i} \xi_{\alpha j} \xi_{\alpha k} = 0, \quad (3.9)$$

$$\sum_{\alpha} \xi_{\alpha i} \xi_{\alpha j} = 6e^2 \delta_{ij}, \quad (3.10)$$

$$\sum_{\alpha} \xi_{\alpha i} \xi_{\alpha j} \xi_{\alpha k} \xi_{\alpha l} = 4e^4 (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) - 6e^4 \Delta_{ijkl} \quad (3.11)$$

where

$$\Delta_{ijkl} = \begin{cases} 1, & i = j = k = l, \\ 0, & \text{otherwise.} \end{cases} \quad (3.12)$$

Now if Equation (3.8) is substituted in Equation (3.16) it leads to

$$N_{\alpha} = \frac{1}{e^2} \sum_{\alpha} \xi_{\alpha x} \xi_{\alpha x} = \frac{1}{e^2} \sum_{\alpha} \xi_{\alpha y} \xi_{\alpha y} = 6, \quad (3.13)$$

then substituting (3.13) into (3.24) results into

$$f_{\alpha}(\mathbf{x} + \xi_{\alpha} \Delta t, t + \Delta t) - f_{\alpha}(\mathbf{x}, t) = -\frac{1}{\tau} (f_{\alpha} - f_{\alpha}^{eq}) + \frac{\Delta t}{6e^2} \xi_{\alpha i} F_i(\mathbf{x}, t). \quad (3.14)$$

The above Equation (3.14) is the most common LB equation used to simulate SWEs.

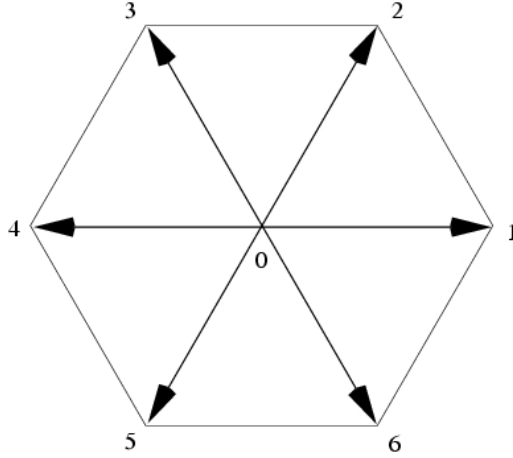


Figure 3.2: The seven particle speeds ξ_α in the 2D square lattice.

3.4 Lattice Boltzmann Equation and BGK Approximation

In the LB method streaming and collision steps can be considered as two different steps. The former involves the movement of particles from one node to the neighboring lattice node in the direction of their velocities, which is given by

$$f_\alpha(\mathbf{x} + \xi_\alpha \Delta t, t + \Delta t) = f'_\alpha(\mathbf{x}, t) + \frac{\Delta t}{N_\alpha e^2} \xi_{\alpha i} F_i(\mathbf{x}, t). \quad (3.15)$$

where f_α is a particle distribution function; f'_α is the value of f_α before streaming; $e = \Delta x / \Delta t$ (Δx is the lattice size and Δt is the time step); F_i is the force term in the direction of i and N_α is the lattice constant given by the lattice pattern as follows

$$N_\alpha = \frac{1}{e^2} \sum_{\alpha} \xi_{\alpha i} \xi_{\alpha i}. \quad (3.16)$$

The collision step, involves interaction of particles where their velocities change direction according to the scattering rule which can be written in the form

$$f'_\alpha(\mathbf{x}, t) = f_\alpha(\mathbf{x}, t) + \Omega_\alpha [f(\mathbf{x}, t)], \quad (3.17)$$

where Ω_α is the collision operator (the speed of f_α is controlled after collision).

The collision operator Ω_α , is generally a matrix decided by the microscopic dynamics. This matrix was found to be very complex to solve, hence the idea of linearizing it around

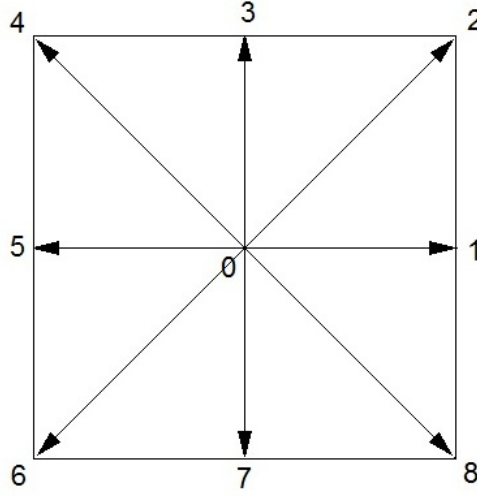


Figure 3.3: The nine particle speeds ξ_α in the 2D square lattice.

its local equilibrium state was introduced [20], i.e Ω_α was expanded about its equilibrium value [30]

$$\Omega_\alpha(f) = \Omega_\alpha(f^{eq}) + \frac{\partial \Omega_\alpha(f^{eq})}{\partial f_\beta} (f_\beta - f_\beta^{eq}) + O[(f_\beta - f_\beta^{eq})^2], \quad (3.18)$$

where f^{eq} is the local equilibrium distribution function.

The LB equation's solution process is characterized by $f_\beta \rightarrow f_\beta^{eq}$, which implies that $\Omega_\alpha(f^{eq}) \approx 0$. Neglecting high order terms leads to

$$\Omega_\alpha(f) \approx \frac{\partial \Omega_\alpha(f^{eq})}{\partial f_\beta} (f_\beta - f_\beta^{eq}), \quad (3.19)$$

which is a linearized collision operator. If the assumption that the local particle distribution relaxes to an equilibrium state at a single rate τ [32] is made, then

$$\frac{\partial \Omega_\alpha(f^{eq})}{\partial f_\beta} = -\frac{1}{\tau} \delta_{\alpha\beta}, \quad (3.20)$$

where $\delta_{\alpha\beta}$ is the Kronecker delta function, defined as

$$\delta_{\alpha\beta} = \begin{cases} 0, & \alpha \neq \beta \\ 1, & \alpha = \beta. \end{cases} \quad (3.21)$$

Substituting Equation (3.20) into (3.19) gives

$$\Omega_\alpha(f) = -\frac{1}{\tau}\delta_{\alpha\beta}(f_\beta - f_\beta^{eq}). \quad (3.22)$$

The above equation results in the BGK collision operator [2]

$$\Omega_\alpha(f) = -\frac{1}{\tau}(f_\alpha - f_\alpha^{eq}), \quad (3.23)$$

where τ is a single relaxation time. The efficiency and simplicity of the LB equation is defined by τ , hence used widely in LB models. Substituting Equations (3.17) and (3.23) into (3.15) results in the most popularly used LB equation given by

$$f_\alpha(\mathbf{x} + \boldsymbol{\xi}_\alpha \Delta t, t + \Delta t) - f_\alpha(\mathbf{x}, t) = -\frac{1}{\tau}(f_\alpha - f_\alpha^{eq}) + \frac{\Delta t}{N_\alpha e^2} \xi_{\alpha i} F_i(\mathbf{x}, t). \quad (3.24)$$

3.5 Local Equilibrium Distribution Function

The most important aspect of the LB method is to determine a suitable local equilibrium function. The local equilibrium function plays an important role because it decides what flow equations are solved by the LB Equation (3.14). A suitable local equilibrium distribution function f_α^{eq} must be derived in order for the solution of LB Equation (3.14) to recover the two-dimensional SWEs (2.47) and (2.48).

The theory of LGCA says that an equilibrium function is the Fermi-Dirac distribution, which is often expanded as a Taylor's series to second order on macroscopic velocity level [15]. It was noticed that the Maxwell-Boltzmann distribution function in the LB Equation (3.14) can only recover the Navier-Stokes equations. Therefore, an Ansatz method was used to find a proper expression for the local equilibrium distribution function for the SWEs. The method uses the assumption on the equilibrium function. It states that equilibrium functions can be expressed as power series in macroscopic velocity [33]:

$$f_\alpha^{eq} = A_\alpha + B_\alpha \xi_{\alpha i} u_i + C_\alpha \xi_{\alpha i} \xi_{\alpha j} u_i u_j + D_\alpha u_i u_i. \quad (3.25)$$

This approach proved to be useful in other flow problems [9]. Its accuracy and flexibility made it famous, hence used for finding suitable equilibrium functions for SWEs. The

equilibrium function has same symmetry hence we can write

$$A_1 = A_3 = A_5 = A_7 = \bar{A}, \quad A_2 = A_4 = A_6 = A_8 = \tilde{A}. \quad (3.26)$$

Similarly this can be applied to B_α , C_α and D_α . For convenience Equation (3.25) can be written in the following form

$$f_\alpha^{eq} = \begin{cases} A_0 + D_0 u_i u_i, & \alpha = 0 \\ \bar{A} + \bar{B} \xi_{\alpha i} u_i + \bar{C} \xi_{\alpha i} \xi_{\alpha j} u_i u_j + \bar{D} u_i u_i, & \alpha = 1, 3, 5, 7, \\ \tilde{A} + \tilde{B} \xi_{\alpha i} u_i + \tilde{C} \xi_{\alpha i} \xi_{\alpha j} u_i u_j + \tilde{D} u_i u_i, & \alpha = 2, 4, 6, 8. \end{cases} \quad (3.27)$$

In order to determine the coefficients in Equation (3.27), conservation of mass and momentum are used as the constraints. The following three conditions must also be satisfied on local equilibrium function for SWEs:

$$\sum_{\alpha} f_{\alpha}^{eq} = h, \quad (3.28)$$

$$\sum_{\alpha} \xi_{\alpha i} f_{\alpha}^{eq} = h u_i(\mathbf{x}, t), \quad (3.29)$$

$$\sum_{\alpha} \xi_{\alpha i} \xi_{\alpha j} f_{\alpha}^{eq} = \frac{1}{2} g h^2 \delta_{ij} + h u_i(\mathbf{x}, t) u_j(\mathbf{x}, t). \quad (3.30)$$

After finding the above equilibrium distribution function, Equation (3.25), then the above constraints (3.28), (3.29) and (3.30) are used together with Equation (3.14) to recover the solution of the two-dimensional SWEs (2.47) and (2.48). The Chapman-Enskog analysis is used at microscopic level for the recovery of SWEs. This will be shown in Section (3.7). By substituting Equation (3.27) into (3.28), results in

$$\begin{aligned} & A_0 - D_0 u_i u_i \\ & + 4\bar{A} + \sum_{\alpha=1,3,5,7} \bar{B} \xi_{\alpha i} u_i + \sum_{\alpha=1,3,5,7} \bar{C} \xi_{\alpha i} \xi_{\alpha j} u_i u_j + 4\bar{D} u_i u_i \\ & + 4\tilde{A} + \sum_{\alpha=2,4,6,8} \tilde{B} \xi_{\alpha i} u_i + \sum_{\alpha=2,4,6,8} \tilde{C} \xi_{\alpha i} \xi_{\alpha j} u_i u_j + 4\tilde{D} u_i u_i = h. \end{aligned} \quad (3.31)$$

Substituting (3.8) into (3.31), and separating the coefficients of h and $u_i u_i$, respectively, leads to

$$A_0 + 4\bar{A} + 4\tilde{A} = h, \quad (3.32)$$

and

$$D_0 + 2e^2\bar{C} + 4e^2\tilde{C} + 4\bar{D} + 4\tilde{D} = 0. \quad (3.33)$$

Substituting Equation (3.27) into (3.29), we obtain

$$\begin{aligned} & A_0\xi_{\alpha i} + D_0\xi_{\alpha i}u_ju_j \\ & + \sum_{\alpha=1,3,5,7} (\bar{A}\xi_{\alpha i} + \bar{B}\xi_{\alpha i}\xi_{\alpha j}u_j + \bar{C}\xi_{\alpha i}\xi_{\alpha j}\xi_{\alpha k}u_ju_k + \bar{D}\xi_{\alpha i}u_ju_j) \\ & + \sum_{\alpha=2,4,6,8} (\tilde{A}\xi_{\alpha i} + \tilde{B}\xi_{\alpha i}\xi_{\alpha j}u_j + \tilde{C}\xi_{\alpha i}\xi_{\alpha j}\xi_{\alpha k}u_ju_k + \tilde{D}\xi_{\alpha i}u_ju_j) = hu_i. \end{aligned} \quad (3.34)$$

Separating the coefficients and using Equation (3.9) leads to

$$2e^2\bar{B} + 4e^2\tilde{B} = h. \quad (3.35)$$

Similarly, inserting Equation (3.27) into Equation (3.30) leads to

$$\begin{aligned} & \sum_{\alpha=1,3,5,7} (\bar{A}\xi_{\alpha i}\xi_{\alpha j} + \bar{B}\xi_{\alpha i}\xi_{\alpha j}\xi_{\alpha k}u_k + \bar{C}\xi_{\alpha i}\xi_{\alpha j}\xi_{\alpha k}\xi_{\alpha l}u_ku_l + \bar{D}\xi_{\alpha i}\xi_{\alpha j}u_ku_k) \\ & + \sum_{\alpha=2,4,6,8} (\tilde{A}\xi_{\alpha i}\xi_{\alpha j} + \tilde{B}\xi_{\alpha i}\xi_{\alpha j}\xi_{\alpha k}u_k + \tilde{C}\xi_{\alpha i}\xi_{\alpha j}\xi_{\alpha k}\xi_{\alpha l}u_ku_l + \tilde{D}\xi_{\alpha i}\xi_{\alpha j}u_ku_k) \\ & = \frac{1}{2}h^2\delta_{ij} + hu_iu_j. \end{aligned} \quad (3.36)$$

Substituting Equation (3.8) into Equation (3.36) results into

$$\begin{aligned} & 2\bar{A}e^2\delta_{ij} + 2\bar{C}e^4u_iu_j + 2\bar{D}e^2u_iu_i + 4\tilde{A}e^2\delta_{ij} \\ & + 8\tilde{C}e^2u_iu_j + 4\tilde{C}e^4u_iu_i + 4\tilde{C}e^2u_iu_i = \frac{1}{2}gh^2\delta_{ij} + hu_iu_j. \end{aligned} \quad (3.37)$$

Separating the coefficients of the above equation leads to the following four relations,

$$2e^2\bar{A} + 4e^2\tilde{A} = \frac{1}{2}gh^2, \quad (3.38)$$

$$8e^2\tilde{C} = h, \quad (3.39)$$

$$2e^2\bar{C} = h \quad (3.40)$$

and

$$2e^2\bar{D} + 4e^2\tilde{D} + 4e^4\tilde{C} = 0. \quad (3.41)$$

From above, Equations (3.39) and (3.40) can be combined together resulting into

$$4\tilde{C} = \bar{C}. \quad (3.42)$$

Based on symmetry of lattices and using the above Equation (3.42), it is feasible to assume the following three relations,

$$4\tilde{A} = \bar{A}, \quad 4\tilde{B} = \bar{B} \quad \text{and} \quad 4\tilde{D} = \bar{D}. \quad (3.43)$$

Now Equations (3.32), (3.33), (3.35), (3.38), (3.39), (3.40) and (3.41) are solved simultaneously, resulting into the following

$$A_0 = h - \frac{5gh^2}{6e^2}, \quad D_0 = -\frac{2h}{3e^2}, \quad (3.44)$$

$$\bar{A} = \frac{gh^2}{6e^2}, \quad \bar{B} = \frac{h}{3e^2}, \quad \bar{C} = \frac{h}{3e^4}, \quad \bar{D} = -\frac{h}{6e^2}, \quad (3.45)$$

$$\tilde{A} = \frac{gh^2}{24e^2}, \quad \tilde{B} = \frac{h}{12e^2}, \quad \tilde{C} = \frac{h}{8e^4}, \quad \tilde{D} = -\frac{h}{24e^2}. \quad (3.46)$$

Therefore, substituting the above Equations (3.44), (3.45) and (3.46) into (3.27), results into

$$f_\alpha^{eq} = \begin{cases} h - \frac{5gh^2}{6e^2} - \frac{2h}{3e^2}u_iu_i, & \alpha = 0, \\ \frac{gh^2}{6e^2} + \frac{h}{3e^2}\xi_{\alpha i}u_i + \frac{h}{3e^4}\xi_{\alpha i}\xi_{\alpha j}u_iu_j - \frac{h}{6e^2}u_iu_i, & \alpha = 1, 3, 5, 7, \\ \frac{gh^2}{24e^2} + \frac{h}{12e^2}\xi_{\alpha i}u_i + \frac{h}{8e^4}\xi_{\alpha i}\xi_{\alpha j}u_iu_j - \frac{h}{24e^2}u_iu_i, & \alpha = 2, 4, 6, 8, \end{cases} \quad (3.47)$$

which is the equilibrium function for solving SWEs. The equilibrium function (3.47) is used together with the LB Equation (3.14) for the solution of SWEs (2.47) and (2.48).

Equation (3.47) can also be written in the form

$$f_\alpha^{eq} = \begin{cases} h + W_\alpha h \left(-\frac{15gh}{8e^2} - \frac{3}{2e^2} u_i u_i \right), \\ W_\alpha h \left(\frac{3gh}{2e^2} + \frac{3h}{e^2} \xi_{\alpha i} u_i + \frac{9}{2e^4} \xi_{\alpha i} \xi_{\alpha j} u_i u_j - \frac{3}{2e^2} u_i u_i \right), & \alpha \neq 0, \end{cases} \quad (3.48)$$

where

$$W_\alpha = \begin{cases} \frac{4}{9}, & \alpha = 0, \\ \frac{1}{9}, & \alpha = 1, 3, 5, 7, \\ \frac{1}{36}, & \alpha = 2, 4, 6, 8. \end{cases} \quad (3.49)$$

3.6 Macroscopic Properties

The purpose of this section is to examine the macroscopic properties of the LB Equation (3.14). Then the remaining task is to determine the macroscopic variables h and u_i , which will be done in the next section. The examination of the macroscopic property of the LB Equation (3.14), is done by taking the zeroth discrete moment of the distribution function in the LB Equation (3.14) over the lattice velocities, leading to

$$\begin{aligned} \sum_\alpha [f_\alpha(\mathbf{x} + \boldsymbol{\xi}_\alpha \Delta t, t + \Delta t) - f_\alpha(\mathbf{x}, t)] &= -\frac{1}{\tau} \sum_\alpha (f_\alpha - f_\alpha^{eq}) \\ &\quad + \frac{\Delta t}{6e^2} \sum_\alpha \xi_{\alpha i} F_i. \end{aligned} \quad (3.50)$$

Note that $\sum_\alpha \xi_{\alpha i} F_i = 0$, which implies that $\frac{\Delta t}{6e^2} \sum_\alpha \xi_{\alpha i} F_i = 0$. Then Equation (3.50) collapses to

$$\sum_\alpha [f_\alpha(\mathbf{x} + \boldsymbol{\xi}_\alpha \Delta t, t + \Delta t) - f_\alpha(\mathbf{x}, t)] = -\frac{1}{\tau} \sum_\alpha (f_\alpha - f_\alpha^{eq}). \quad (3.51)$$

Note that, the cumulative mass and momentum which correspond to the sum of the microdynamic mass and momentum are conserved. This is an explicit constraint to preserve conservative properties in the LB method. Therefore, the mass conservation requires the following identity

$$\sum_\alpha f_\alpha(\mathbf{x} + \boldsymbol{\xi}_\alpha \Delta t, t + \Delta t) \equiv \sum_\alpha f_\alpha(\mathbf{x}, t). \quad (3.52)$$

Substituting Equation (3.52) into (3.51) leads to

$$\sum_{\alpha} f_{\alpha}(\mathbf{x}, t) = \sum_{\alpha} f_{\alpha}^{eq}(\mathbf{x}, t). \quad (3.53)$$

With reference to Equation (3.28), the above Equation (3.53) gives the macroscopic water depth h as

$$h(\mathbf{x}, t) = \sum_{\alpha} f_{\alpha}(\mathbf{x}, t). \quad (3.54)$$

The macroscopic velocity u_i can be found in a similar manner by taking first discrete moment of the distribution function f_{α} in the LB Equation (3.14) over lattice velocities, giving

$$\begin{aligned} \sum_{\alpha} \xi_{\alpha i} [f_{\alpha}(\mathbf{x} + \xi_{\alpha} \Delta t, t + \Delta t) - f_{\alpha}(\mathbf{x}, t)] &= -\frac{1}{\tau} \sum_{\alpha} \xi_{\alpha i} (f_{\alpha} - f_{\alpha}^{eq}) \\ &\quad + \frac{\Delta t}{6e^2} \sum_{\alpha} \xi_{\alpha i} \xi_{\alpha j} F_j. \end{aligned} \quad (3.55)$$

Since $\sum_{\alpha} \xi_{\alpha i} \xi_{\alpha i} = 6e^2 \delta_{ij}$, then Equation (3.55) can be reduced and rearranged to

$$\begin{aligned} \sum_{\alpha} \xi_{\alpha i} [f_{\alpha}(\mathbf{x} + \xi_{\alpha} \Delta t, t + \Delta t) - f_{\alpha}(\mathbf{x}, t)] \\ = -\frac{1}{\tau} \sum_{\alpha} \xi_{\alpha i} (f_{\alpha} - f_{\alpha}^{eq}) + \Delta t F_i. \end{aligned} \quad (3.56)$$

Note again that, the momentum conservation in micro-dynamic variables requires the following identity

$$\sum_{\alpha} \xi_{\alpha i} [f_{\alpha}(\mathbf{x} + \xi_{\alpha} \Delta t, t + \Delta t) - f_{\alpha}(\mathbf{x}, t)] \equiv \Delta t F_i. \quad (3.57)$$

Substituting Equation (3.57) into Equation (3.56), we obtain

$$\sum_{\alpha} \xi_{\alpha i} f_{\alpha}(\mathbf{x}, t) = \sum_{\alpha} \xi_{\alpha i} f_{\alpha}^{eq}(\mathbf{x}, t). \quad (3.58)$$

With reference to Equation (3.29), the above Equation (3.58) gives the macroscopic velocity u_i as

$$u_i(\mathbf{x}, t) = \frac{1}{h(\mathbf{x}, t)} \sum_{\alpha} \xi_{\alpha i} f_{\alpha}^{eq}(\mathbf{x}, t). \quad (3.59)$$

Since the distribution function f_{α} relaxes to its local equilibrium function f_{α}^{eq} via the LB Equation (3.14), then Equations (3.28) and (3.29) are satisfied, i.e mass and momentum conservation is guaranteed on Equations (3.54) and (3.59).

3.7 Recovery of the Shallow Water Equations

In this section the Chapman-Enskog analysis is used to prove that the solution of the LB Equation (3.14) leads to the recovery of macroscopic two-dimensional SWEs (2.47) and (2.48). The method basically allows the restoration of hydrodynamic equations from the Boltzmann equation in the low Knudsen number limit. The procedure starts from the continuous Boltzmann equation and then, it is applied to the discretized LB equation. The present section only deals with the latter. The assumption that Δt is a small time step and equal to ϵ is made, i.e

$$\Delta t = \epsilon. \quad (3.60)$$

Then the LB Equation (3.14) can be expressed as follows

$$f_\alpha(\mathbf{x} + \xi_\alpha \epsilon, t + \epsilon) - f_\alpha(\mathbf{x}, t) = -\frac{1}{\tau} (f_\alpha - f_\alpha^{eq}) + \frac{\epsilon}{6e^2} \xi_{\alpha j} F_j(\mathbf{x}, t). \quad (3.61)$$

Applying the Taylor's expansion to the first term on the left side of Equation (3.61) in time and space around the point (x, t) results in

$$\begin{aligned} & \epsilon \left(\frac{\partial}{\partial t} + \xi_{\alpha j} \frac{\partial}{\partial x_j} \right) f_\alpha + \frac{1}{2} \epsilon^2 \left(\frac{\partial}{\partial t} + \xi_{\alpha j} \frac{\partial}{\partial x_j} \right)^2 f_\alpha + O(\epsilon^3) \\ &= -\frac{1}{\tau} (f_\alpha - f_\alpha^{(0)}) + \frac{\epsilon}{6e^2} \xi_{\alpha j} F_j(\mathbf{x}, t), \end{aligned} \quad (3.62)$$

where $f_\alpha^{(0)} = f_\alpha^{eq}$, and by expanding f_α around the $f_\alpha^{(0)}$ leads to

$$f_\alpha = f_\alpha^{(0)} + \epsilon f_\alpha^{(1)} + \epsilon^2 f_\alpha^{(2)} + O(\epsilon^3). \quad (3.63)$$

Substituting Equation (3.63) into (3.62) results into

$$\begin{aligned} & \epsilon \left(\frac{\partial}{\partial t} + \xi_{\alpha j} \frac{\partial}{\partial x_j} \right) \left(f_\alpha^{(0)} + \epsilon f_\alpha^{(1)} + \epsilon^2 f_\alpha^{(2)} + O(\epsilon^3) \right) \\ &+ \frac{1}{2} \epsilon^2 \left(\frac{\partial}{\partial t} + \xi_{\alpha j} \frac{\partial}{\partial x_j} \right)^2 \left(f_\alpha^{(0)} + \epsilon f_\alpha^{(1)} + \epsilon^2 f_\alpha^{(2)} + O(\epsilon^3) \right) + O(\epsilon^3) \\ &= -\frac{1}{\tau} \left(\epsilon f_\alpha^{(1)} + \epsilon^2 f_\alpha^{(2)} + O(\epsilon^3) \right) + \frac{\epsilon}{6e^2} \xi_{\alpha j} F_j. \end{aligned} \quad (3.64)$$

From the above Equation (3.64) terms with higher order ϵ^3 are ignored. Separating Equation (3.64) by ϵ and ϵ^2 leads to the following two equations

$$\left(\frac{\partial}{\partial t} + \xi_{\alpha j} \frac{\partial}{\partial x_j} \right) f_\alpha^{(0)} = -\frac{1}{\tau} f_\alpha^{(1)} + \frac{1}{6e^2} \xi_{\alpha j} F_j, \quad (3.65)$$

and

$$\left(\frac{\partial}{\partial t} + \xi_{\alpha j} \frac{\partial}{\partial x_j}\right) f_{\alpha}^{(1)} + \frac{1}{2} \left(\frac{\partial}{\partial t} + \xi_{\alpha j} \frac{\partial}{\partial x_j}\right)^2 f_{\alpha}^{(0)} = -\frac{1}{\tau} f_{\alpha}^{(2)}. \quad (3.66)$$

By substituting Equation (3.65) into (3.66) and rearranging, results into

$$\left(1 - \frac{1}{2\tau}\right) \left(\frac{\partial}{\partial t} + \xi_{\alpha j} \frac{\partial}{\partial x_j}\right) f_{\alpha}^{(1)} = -\frac{1}{\tau} f_{\alpha}^{(2)} - \frac{1}{2} \left(\frac{\partial}{\partial t} + \xi_{\alpha j} \frac{\partial}{\partial x_j}\right) \frac{1}{6e^2} \xi_{\alpha k} F_k. \quad (3.67)$$

Taking the summation of Equations (3.65) and the product of ϵ and Equation (3.67) leads to

$$\frac{\partial}{\partial t} \left(\sum_{\alpha} f_{\alpha}^{(0)}\right) + \frac{\partial}{\partial x_j} \left(\sum_{\alpha} \xi_{\alpha j} f_{\alpha}^{(0)}\right) = -\frac{\epsilon}{12e^2} \frac{\partial}{\partial x_j} \left(\sum_{\alpha} \xi_{\alpha j} \xi_{\alpha k} f_k^{(0)}\right). \quad (3.68)$$

By applying the first order accuracy of the force term and substituting Equations (3.14) and (3.47) into (3.68), leads to

$$\frac{\partial h}{\partial t} + \frac{\partial(hu_j)}{\partial x_j} = 0, \quad (3.69)$$

which is the continuity Equation (2.47) for SWEs. Similarly, by taking the summation of the product of $\xi_{\alpha j}$ and Equation (3.65) together with the product of ϵ and Equation (3.67) leads to

$$\begin{aligned} \frac{\partial}{\partial t} \left(\sum_{\alpha} \xi_{\alpha i} f_{\alpha}^{(0)}\right) + \frac{\partial}{\partial x_j} \left(\sum_{\alpha} \xi_{\alpha i} \xi_{\alpha j} f_{\alpha}^{(0)}\right) + \epsilon \left(1 - \frac{1}{2\tau}\right) \frac{\partial}{\partial x_j} \left(\sum_{\alpha} \xi_{\alpha i} \xi_{\alpha j} f_{\alpha}^{(1)}\right) \\ = F_j \delta_{ij} - \frac{\epsilon}{2} \sum_{\alpha} \xi_{\alpha i} \left(\frac{\partial}{\partial t} + \xi_{\alpha j} \frac{\partial}{\partial x_j}\right) \frac{1}{6e^2} \xi_{\alpha j} F_j. \end{aligned} \quad (3.70)$$

Again, by applying the first order accuracy of the force term to the above Equation (3.70) with reference to Equation (3.14) and (3.47) results in

$$\frac{\partial(hu_i)}{\partial t} + \frac{\partial(hu_i u_j)}{\partial x_j} = -\frac{1}{2} g \frac{\partial h^2}{\partial x_i} - \frac{\partial \Lambda_{ij}}{\partial x_j} + F_i, \quad (3.71)$$

where

$$\Lambda_{ij} = \frac{\epsilon}{2\tau} (2\tau - 1) \sum_{\alpha} \xi_{\alpha i} \xi_{\alpha j} f_{\alpha}^{(1)}. \quad (3.72)$$

By considering Equation (3.65) and using Equations (3.14) and (3.47) after some manipulation the following results are obtained

$$\Lambda_{ij} \approx -\nu \left[\frac{\partial(hu_i)}{\partial x_j} + \frac{\partial(hu_j)}{\partial x_i} \right]. \quad (3.73)$$

Substituting Equation (3.73) into (3.71), results into the following equation

$$\frac{\partial hu_i}{\partial t} + \frac{\partial(hu_i u_j)}{\partial x_j} = -\frac{1}{2}g \frac{\partial h^2}{\partial x_i} + \nu \frac{\partial^2(hu_i)}{\partial x_j \partial x_j} + F_i, \quad (3.74)$$

where the kinematic viscosity ν is defined as follows

$$\nu = \frac{1}{6}e^2 \Delta t (2\tau - 1) \quad (3.75)$$

and the force term F_i is given by Equation (2.49). Note that Equation (3.74) is known as the momentum equation for the SWEs, Equation (2.48).

The above proof is of first order accurate for the recovery of SWEs due to force terms. Therefore, the centred scheme can be used to treat the force terms in the LB equation to second-order accuracy of the macroscopic continuity and momentum equations, refer to [51].

3.8 Force term

In fluid mechanics, for fluids to flow there must be external or internal forces that drive the fluids. Therefore, the study of force terms in LB method is important for accurate predictions involving external forces.

Zhou [49, 50] has shown that direct incorporation of force terms into the LB method produces accurate solutions to many flows. This was done by incorporating wind shear stress and the bed slope into the streaming step. This makes the force term to be evaluated in a straight way, allowing any additional natural force terms to be taken into account. After numerical tests, it was realized that for some straight evaluation of force terms, the LB equation gave inaccurate results for other flows. Hence a centred scheme was developed [51] and it is described as follows.

The force term is evaluated at the mid-point between the lattice points and its neighboring point as

$$F_i = F_i(\mathbf{x} + \frac{1}{2}\boldsymbol{\xi}_\alpha \Delta t, t + \frac{1}{2}\Delta t). \quad (3.76)$$

The force term can also be written in the semi-implicit form as

$$F_i = F_i(\mathbf{x} + \frac{1}{2}\boldsymbol{\xi}_\alpha \Delta t, t). \quad (3.77)$$

Substituting the above Equation (3.76) into the LB Equation (3.14) and applying Chapman Enskog expansion gives

$$\begin{aligned} f_\alpha(\mathbf{x} + \boldsymbol{\xi}_\alpha \epsilon, t + \epsilon) - f_\alpha(\mathbf{x}, t) &= -\frac{1}{\tau} (f_\alpha - f_\alpha^{eq}) \\ &+ \frac{\epsilon}{6e^2} \xi_{\alpha i} F_i(\mathbf{x} + \frac{1}{2} \boldsymbol{\xi}_\alpha \epsilon, t + \frac{1}{2} \epsilon), \end{aligned} \quad (3.78)$$

where $\Delta t = \epsilon$ is assumed to be small. Applying the Taylor's expansion to the first term on the left side of Equation (3.78) in time and space around the point (x, t) results in

$$\begin{aligned} f_\alpha(\mathbf{x} + \boldsymbol{\xi}_\alpha \epsilon, t + \epsilon) &= f_\alpha(\mathbf{x}, t) + \epsilon \left(\frac{\partial}{\partial t} + \xi_{\alpha j} \frac{\partial}{\partial x_j} \right) f_\alpha \\ &+ \frac{1}{2} \epsilon^2 \left(\frac{\partial}{\partial t} + \xi_{\alpha j} \frac{\partial}{\partial x_j} \right)^2 f_\alpha + O(\epsilon^3). \end{aligned} \quad (3.79)$$

Similarly,

$$F_i(\mathbf{x} + \boldsymbol{\xi}_\alpha \epsilon, t + \epsilon) = F_i(\mathbf{x}, t) + \frac{\epsilon}{2} \left(\frac{\partial}{\partial t} + \xi_{\alpha j} \frac{\partial}{\partial x_j} \right) F_i + O(\epsilon^2), \quad (3.80)$$

which results from taking Taylor's expansion to the force term on the right side of Equation (3.78) in time and space around the point (x, t) . Substituting Equations (3.79) and (3.80) into (3.78) yields

$$\begin{aligned} \epsilon \left(\frac{\partial}{\partial t} + \xi_{\alpha j} \frac{\partial}{\partial x_j} \right) f_\alpha + \frac{1}{2} \epsilon^2 \left(\frac{\partial}{\partial t} + \xi_{\alpha j} \frac{\partial}{\partial x_j} \right)^2 f_\alpha &= -\frac{1}{\tau} (f_\alpha - f_\alpha^{eq}) \\ &+ \frac{\epsilon}{6e^2} \xi_{\alpha i} F_i + \frac{\epsilon^2}{12e^2} \left(\frac{\partial}{\partial t} + \xi_{\alpha j} \frac{\partial}{\partial x_j} \right) F_i + O(\epsilon^3). \end{aligned} \quad (3.81)$$

Now when expanding f_α around $f_\alpha^{(0)}$ leads to

$$f_\alpha = f_\alpha^{(0)} + \epsilon f_\alpha^{(1)} + \epsilon^2 f_\alpha^{(2)} + O(\epsilon^3), \quad (3.82)$$

where $f_\alpha^{(0)} = f_\alpha^{eq}$. Substituting Equation (3.82) into Equation (3.81) and separating $O(\epsilon)$ and $O(\epsilon^2)$ terms lead to the following two equations

$$\left(\frac{\partial}{\partial t} + \xi_{\alpha j} \frac{\partial}{\partial x_j} \right) f_\alpha^{(0)} = -\frac{1}{\tau} f_\alpha^{(1)} + \frac{1}{6e^2} \xi_{\alpha j} F_j \quad (3.83)$$

and

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \xi_{\alpha j} \frac{\partial}{\partial x_j} \right) f_\alpha^{(1)} + \frac{1}{2} \left(\frac{\partial}{\partial t} + \xi_{\alpha j} \frac{\partial}{\partial x_j} \right)^2 f_\alpha^{(0)} &= -\frac{1}{\tau} f_\alpha^{(2)} \\ &+ \frac{1}{12e^2} \left(\frac{\partial}{\partial t} + \xi_{\alpha j} \frac{\partial}{\partial x_j} \right) F_i. \end{aligned} \quad (3.84)$$

Substituting Equation (3.83) into (3.84) results in the following equation

$$(1 - \frac{1}{2\tau}) \left(\frac{\partial}{\partial t} + \xi_{\alpha j} \frac{\partial}{\partial x_j} \right) f_{\alpha}^{(1)} = -\frac{1}{\tau} f_{\alpha}^{(2)}. \quad (3.85)$$

Taking the summation of Equation (3.83) and the product of ϵ with Equation (3.85) leads to

$$\frac{\partial}{\partial t} \left(\sum_{\alpha} f_{\alpha}^{(0)} \right) + \frac{\partial}{\partial x_j} \left(\sum_{\alpha} \xi_{\alpha j} f_{\alpha}^{(0)} \right) = 0. \quad (3.86)$$

Similarly, by taking the summation of the product of $e_{\alpha j}$ and Equation (3.83) together with the product of ϵ and Equation (3.85), i.e. $(\sum \xi_{\alpha j} [(3.83) + \epsilon (3.85)])$, leads to

$$\begin{aligned} & \frac{\partial}{\partial t} \left(\sum_{\alpha} \xi_{\alpha i} f_{\alpha}^{(0)} \right) + \frac{\partial}{\partial x_j} \left(\sum_{\alpha} \xi_{\alpha i} \xi_{\alpha j} f_{\alpha}^{(0)} \right) \\ & + \epsilon (1 - \frac{1}{2\tau}) \frac{\partial}{\partial x_j} \left(\sum_{\alpha} \xi_{\alpha i} \xi_{\alpha j} f_{\alpha}^{(1)} \right) = F_i. \end{aligned} \quad (3.87)$$

When comparing the two Equations (3.86) and (3.87), it was observed that terms regarding external forces disappear leading to

$$\frac{\partial h u_i}{\partial t} + \frac{\partial h u_i u_j}{\partial x_j} = -\frac{1}{2} g \frac{\partial h^2}{\partial x_i} - \frac{\partial \Lambda_{ij}}{\partial x_j} + F_i, \quad (3.88)$$

where

$$\Lambda_{ij} = \frac{\epsilon}{2\tau} (2\tau - 1) \sum_{\alpha} \xi_{\alpha i} \xi_{\alpha j} f_{\alpha}^{(1)}. \quad (3.89)$$

By considering Equation (3.83) and using Equations (3.14) and (3.47) after some manipulation the following results were obtained [51]

$$\Lambda_{ij} \approx -\nu \left[\frac{\partial h u_i}{\partial x_j} + \frac{\partial h u_j}{\partial x_i} \right]. \quad (3.90)$$

Substituting Equation (3.90) into (3.88), results into the following equation

$$\frac{\partial h u_i}{\partial t} + \frac{\partial h u_i u_j}{\partial x_j} = -\frac{1}{2} g \frac{\partial h^2}{\partial x_i} + \nu \frac{\partial^2 (h u_i)}{\partial x_j \partial x_j} + F_i, \quad (3.91)$$

where the kinematic viscosity ν is defined as follows

$$\nu = \frac{1}{6} e^2 \epsilon (2\tau - 1). \quad (3.92)$$

Equation (3.91) is known as the momentum equation for the SWEs and its accuracy is of second-order for the recovery of SWEs due to the force terms. Other schemes can be used but the centred scheme used above is more accurate in comparison (e.g the basic scheme and the second-order scheme) [51].

3.9 Stability Conditions

It is well known among LB researchers that instability problems arise frequently, when the LB method is viewed as a finite-difference method for solving the continuum discrete-velocity Boltzmann equations. It becomes clear that numerical accuracy and stability issues should be addressed. For example, the work on stability theory was done in [40, 25] based on an explicit difference scheme. Most of the works were based on the von-Neumann stability analysis for the difference scheme and the resulting growth matrix was not treated analytically. In both [40] and [25], approximation methods based on linear algebra were used to compute the eigenvalues of the growth matrix, while perturbation technique was also employed in [25].

Computationally, the LB method was often found stable when certain conditions were met. Firstly, when fluid flows then diffusion phenomena is present. As a result, kinematic viscosity should be positive [40]. Based on Equation (3.92), we have

$$\nu = \frac{1}{6}e^2\Delta t(2\tau - 1) > 0. \quad (3.93)$$

Therefore, from the above equation, it is deduced that

$$\tau > \frac{1}{2}.$$

Secondly,

$$C_r = \sqrt{u_j u_j} \frac{\Delta t}{\Delta x} < 1, \quad (3.94)$$

which can be written in the form

$$\sqrt{u_j u_j} < \frac{\Delta x}{\Delta t} = e,$$

i.e the maximum speed e that the lattice can support is much bigger than the wave speed.

In shallow water flow, the wave speed is $\sqrt{gh}$, therefore

$$\sqrt{gh} < \frac{\Delta x}{\Delta t} = e. \quad (3.95)$$

Since LB method is limited to low speed flows, this implies that the flow is suitable for sub-critical shallow water flows. Therefore, the wave speed should be greater than the

resultant velocity, i.e

$$\sqrt{gh} > \sqrt{u_j u_j}. \quad (3.96)$$

The above Equation (3.96) can also be written in the form

$$F_r = \frac{\sqrt{u_i u_j}}{\sqrt{gh}} < 1, \quad (3.97)$$

which is known as the Froude number in hydraulics. This number can be used to decide the flow state, i.e $F_r > 1$ implies supercritical flow, $F_r = 1$ implies critical flow and for $F_r < 1$ implies sub-critical. From the above Equation (3.97), we can limit our study to shallow water flows in the sub-critical regime. Generally, the LB method becomes stable if the conditions described above are satisfied.

3.10 Initial and Boundary Conditions

3.10.1 Boundary Conditions

Boundary conditions play an important role in the LB method since they can influence the accuracy and stability of the LB method, refer to [55] for more details. The correct implementation of SWE boundary conditions, in the framework of LB schemes, is made complicated by non-availability of analytical methods to assess the consistency of such discretizations. The following boundary conditions will be used and were proposed in [49]:

1. For solid boundary conditions, no-slip or slip boundary conditions are applied. For no-slip condition, the bounce-back scheme is used and for slip conditions, a zero gradient of the distribution function normal to the solid wall is used.
2. For inflow and out flow boundary conditions, if given the velocity and depth, then the distribution function f_i at the boundary can be calculated as follows: After streaming, the unknown f_1 , f_2 and f_8 , refer to Figure 3.4, can be decided as [49]

$$f_1 = f_5 + \frac{2h\mathbf{u}}{3e},$$

$$f_2 = \frac{h\mathbf{u}}{6e} + f_6 + \frac{f_7 - f_3}{2},$$

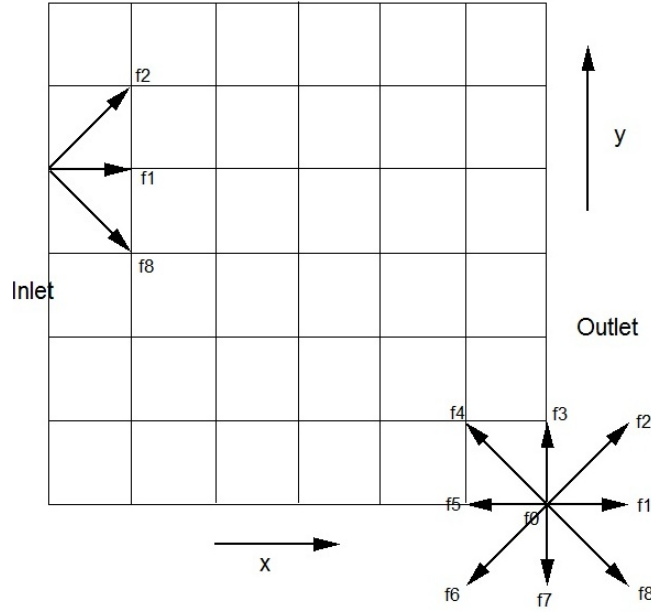


Figure 3.4: Inflow and Outflow lattice notes boundaries.

$$f_8 = \frac{h\mathbf{u}}{6e} + f_4 + \frac{f_3 - f_7}{2},$$

where $e = \Delta x / \Delta t$, Δx is the lattice size and Δt is the time step.

If a zero gradient for velocity or depth at either inflow or outflow boundary is required, the gradient of the distribution function normal to the boundary is set to zero.

3.10.2 Initial Conditions

Initial conditions can generally be specified in two ways, when LB method is used. The first way is to assign a random value between zero and unity for the distribution functions and the second way is to use the local equilibrium distribution function f_α^{eq} with respect to the macroscopic variables [51]. The latter is more convenient for implementing on SWEs because it speeds up the numerical computation.

The consistent initial condition designed in [28] is the other way to set initial conditions, but the approach is not popular in practice. Therefore, in this dissertation the local distribution functions f_α^{eq} are calculated using initial macroscopic variables u , v and h .

3.11 The Basic Solution

The LB algorithm basically consists of two steps the streaming-step and the collision-step as described in Section 3.4. These are usually combined with boundary conditions and initial conditions. When using the LB method, the particle is limited to a number of directions. For example, in two-dimension, the D2Q9 model is usually used and is being adopted in this dissertation.

When finding the solutions for LB method for the SWEs, the following solution procedure can be used [49]:

1. when given initial water depth (h) and velocity (u_i),
2. the $f_i^{(eq)}$ is calculated from Equation (3.47) or (3.48) together with Equation (3.49) and in the next step,
3. the f_i are computed via the LB Equation (3.24), when this step is implemented, proper relaxation time τ is used.
4. the water depth and velocity are updated accordingly using Equations (3.28) and (3.29),
5. steps (2) - (4) are repeated until the solution is obtained.

Chapter 4

Stability Structure

4.1 Introduction

The LB equation is used as an alternative numerical method to simulate SWEs. The method is viewed as a particular discretization of discrete-velocity Boltzmann equation [40], which are hyperbolic equations with stiff source terms. Certain stability criteria were used to ensure a well behaved relaxation limit, namely, the structural stability condition [46], the sub-characteristic condition [21] and dissipative entropy principles [5]. As a result, the LB equations were constructed to satisfy some physical requirements like Galilean invariance and isotropy, to possess a velocity-independent pressure and no compressible effects [23, 54].

Note that the continuum Boltzmann equation satisfies a dissipative entropy condition (Boltzmann's H-theorem) [7]. Since the discrete-velocity LB equation is viewed as a discretized version of the continuum Boltzmann equation, one might expect that the dissipative entropy conditions are satisfied in both equations. But it was proven in [47] that such entropy conditions do not exist for many used LB equations. Hence, conditions like structural stability in [46] were used in such cases.

In the following work, stable LB models will be defined by using stability conditions in [46, 1]. In most of the models used, it is not yet proven that the diffusive limit of the discrete-velocity Boltzmann equation are SWEs at least in the regime of smooth flow. But we can remark that, incompressible fluids are modelled using either SWEs or the

N-S equations. The latter satisfies the diffusive limit of the discrete-velocity Boltzmann equation, see [22], when certain models are used. These models are similar to the ones used in this dissertation. Therefore, it is reasonable to consider the stability condition as a new requirement in constructing LB equations for the SWEs. Note that the present theory is valid only for isothermal models.

To explain how the stability requirement guides the construction of the LB equations, we will show that the LB models for the SWEs are stable using Definition (1), which will be stated in the next section. We will do so by testing the stability structure on some examples which will be shown in the sections below. In other models, we will also investigate the parameter range for which the models are stable. Computational experiments were taken on examples which are used commonly in literature, to confirm the usefulness of the stability structure.

4.2 Stability Notion

We consider the LB equation derived from a particular discretization of the following d -dimensional, N -velocity Boltzmann equation.

$$\frac{\partial f}{\partial t} + \boldsymbol{\xi}_i \cdot \nabla f_i = J_i(f) \quad (i = 1, 2, \dots, N), \quad (4.1)$$

where $f_i(x, t) = f(x, \boldsymbol{\xi}_i, t)$ is a distribution function representing the probability density of the particles at position $\mathbf{x}$ moving with speed $\boldsymbol{\xi}_i$ at time t , i.e $\boldsymbol{\xi}_i$ represents velocity contained in the set of discrete velocities, $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots, \boldsymbol{\xi}_N\}$. The left hand side of Equation (4.1) represents the transport of fluid particles. The right hand side represents the collision term, which describes the interaction of the particles.

Definition 1. *Stability Structure:*

Let f_* be a constant state satisfying $J(f_*) = 0$. The Lattice Boltzmann model (4.1) is called stable at $f = f_*$ if there is an invertible matrix $P \in \mathbb{R}^{N \times N}$ such that $P^T P$ is diagonal and $P J_f(f_*) = -\text{diag}(\lambda_1, \lambda_2, \dots, \lambda_N) P$ with $\lambda_i = 0$ for $i \leq d + 1$ and $\lambda_i > 0$ for $i > d + 1$. Here $J_f(f) \in \mathbb{R}^{N \times N}$ is the Jacobian of $J(f) = (J_1(f), J_2(f), \dots, J_N(f))^T$ [1].

In the above definition, d represents the space dimension, $J(f_*)$ is the Jacobian matrix and λ is an eigenvalue.

Remark 1. *This definition is based on the stability conditions [46] for hyperbolic systems with source terms. It was noted that the usual entropy conditions cannot be used for many used Lattice Boltzmann equations, see [47].*

Remark 2. *In general, a consistent and stable lattice Boltzmann model implies convergence, which can be proven using the approach in [1].*

Definition 2. *We will say that a square matrix A is **diagonalizable** if A is similar to a diagonal matrix. This means that there is an invertible matrix P and diagonal matrix $D = \text{diag}\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ such that $P^{-1}AP$ or $AP = PD$ [10].*

Definition 3. *Two $n \times n$ matrices A and B are said to be similar if a non-singular matrix P exists with $A = P^{-1}BP$ [48].*

Definition 4. *A real symmetric matrix A is positive definite if and only if there exists a real non-singular matrix P such that*

$$A = P^T P,$$

where P^T is a transpose [48].

Theorem 1. *Suppose A and B are similar $n \times n$ matrices and λ is an eigenvalue of A with associated eigenvector $\underline{x}$. Then λ is also an eigenvalue of B and if $A = P^{-1}BP$ then $P\underline{x}$ is an eigenvector associated with λ and the matrix B [48].*

Theorem 2. *If A is an $n \times n$ symmetric matrix and D is a diagonal matrix whose diagonal elements are the eigenvalues of A , then there exists an orthogonal matrix P such that*

$$D = P^T A P$$

[48].

The examples below will be using definitions and theorems stated above.

4.2.1 Examples

To introduce the stability structure, we will consider the following examples below. The models to be used are taken from [51, 34].

Example 1

Consider the following D2Q5-velocity model, with

$$\begin{aligned}\boldsymbol{\xi}_0 &= (0, 0), \\ \{\boldsymbol{\xi}_i : i \neq 0\} &= \{(\pm e, 0)^T, (0, \pm e)^T\}\end{aligned}$$

where e is a lattice constant and usually scaled to be 1. i.e $e = 1$, refer to Figure (3.1) and

$$J_i(f) = \frac{f_i^{eq}(h, u) - f_i}{\tau}.$$

Here τ is a positive constant (relaxation time) and

$$f_i^{eq} = \begin{cases} h - \frac{gh^2}{e^2} \\ \frac{gh^2}{4e^2} + \frac{h\boldsymbol{\xi}_i \mathbf{u}}{2e^2}, & i \neq 0, \end{cases} \quad (4.2)$$

with

$$h = \sum_{i=0}^N f_i,$$

and

$$h\mathbf{u} = \sum_{i=0}^N \boldsymbol{\xi}_i f_i.$$

The model was taken from [34]. Unlike other LB models, this model has no momentum advection term. Therefore, there are two advantages when this model is used, the physical basis and computation. But both advantages are more significant in three dimensions than in two dimensions [34].

To show its stability, it is noted that the following can be verified directly

$$\sum_{i=0}^N f_i^{eq} = h = \sum_{i=0}^N f_i, \quad (4.3)$$

$$\sum_{i=0}^N \xi_i f_i^{eq} = h\mathbf{u} = \sum_{i=0}^N \xi_i f_i.$$

Computing Equation (4.2) gives

$$\frac{\partial f_i^{eq}(h, \mathbf{u})}{\partial f_j} = \begin{cases} 1 - \frac{2gh}{e^2} \\ \frac{gh}{2e^2} + \frac{\xi_j \xi_i}{2e^2}, & i \neq 0, \end{cases} \quad (4.4)$$

see Appendix A.

Definition 5. A projection matrix P is an $n \times n$ square matrix that gives a vector space projection from $\mathbb{R}^n$ to a subspace W . The columns of P are the projections of the standard basis vectors, and W is the image of P . A square matrix P is a projection matrix iff [29]

$$P^2 = P.$$

Remark 3. Eigenvalues of a projection matrix are either 0 or 1 [41].

By using Equation (4.3), we deduce from Equation (4.4) that

$$[f_{f_i}^{eq}(h, \mathbf{u})]^2 = [f_{f_i}^{eq}(h, \mathbf{u})],$$

that is, the Jacobian $[f_{f_i}^{eq}(h, \mathbf{u})]$ is a projection matrix, see Appendix B. Thus, the eigenvalues of

$$J_f(h, \mathbf{u}) = ([f_{f_i}^{eq}(h, \mathbf{u})] - I_5)/\tau,$$

are 0 and $-\frac{1}{\tau}$.

Remark 4. If all the eigenvalues of a matrix A have real parts that are zero or negative with at least one eigenvalue having zero real part, then there is insufficient information available to conclude on the stability of the equilibrium solution. As a result, more analysis must be done [12].

Now take $f_* = f^{eq}(1, \mathbf{u})$ and $e = 1$ then

$$\frac{\partial f_i^{eq}(1, \mathbf{u})}{\partial f_j} = \begin{cases} 1 - 2g \\ \frac{g}{2} + \frac{\xi_j \xi_i}{2}, \quad i \neq 0. \end{cases} \quad (4.5)$$

We want to find A_0 such that

$$A_0 \left[\frac{\partial f_i^{eq}(1, \mathbf{u})}{\partial f_j} \right] = X, \quad (4.6)$$

where X is a symmetric matrix. The square matrix A_0 must be symmetric and positive definite. Choose

$$A_0 = \text{diag} \left[\frac{1}{1 - 2g}, \frac{2}{g} \mathbf{I}_4 \right]. \quad (4.7)$$

Remark 5. Note the following properties of a positive definite matrix (PDM):

- A transpose of a PDM is non-singular.
- Transpose of a PDM is PDM.
- Eigenvalues of a PDM are positive.
- A product of any matrix with its transpose is PDM.

From the choice of A_0 in Equation (4.7), we need to choose g such that A_0 remains positive definite. Therefore, we need to set

$$g < \frac{1}{2} \quad \text{and} \quad g \neq 0,$$

i.e $g \in (0, \frac{1}{2})$, see Appendix C. Substituting Equations (4.5) and (4.7) into (4.6) gives

$$X = \gamma \xi^T \xi + A_0 \Psi_{ij}, \quad (4.8)$$

where γ is a constant and Ψ_{ij} is a square matrix ($N \times N$). Take $g = \frac{1}{3}$, for example, then

$$\frac{\partial f_i^{eq}(1, \mathbf{u})}{\partial f_j} = \begin{cases} \frac{1}{3} \\ \frac{1}{6} + \frac{\xi_j \xi_i}{2}, \quad i \neq 0 \end{cases} \quad (4.9)$$

and

$$A_0 = 3 \operatorname{diag} [1, 2\mathbf{I}_4]. \quad (4.10)$$

From Equations (4.9) and (4.10), we see that

$$A_0 \left[\frac{\partial f_i^{eq}(1, \mathbf{u})}{\partial f_j} \right] = 3\xi^T \xi + A_0 \Psi_{ij} \quad (4.11)$$

is symmetric, where

$$\Psi_{ij} = \begin{cases} \frac{1}{3} \\ \frac{1}{6}, & i \neq 0. \end{cases}$$

We deduce from Equation (4.9) that the rank of $[f_{f_i}^{eq}(1, \mathbf{u})]$ is 3, see Appendix D. Since $[f_{f_i}^{eq}(\rho, \mathbf{u})]$ is a projection matrix and

$$\tau J_f(f_*) = [f_{f_i}^{eq}(1, \mathbf{u})] - I_5$$

then, the rank of $J_f(f_*)$ is 2.

On the other hand, since $A_0[f_{f_i}^{eq}(1, \mathbf{u})]$ is symmetric and A_0 is symmetric positive definite, it is known that there is an invertible matrix P such that

$$A_0 = P^T P \quad \text{and} \quad A_0 \tau J_f(f_*) = P^T \Lambda P$$

with Λ a diagonal matrix. Note that Λ is similar to $\tau J_f(f_*)$. We may as well assume that

$$\Lambda = -\operatorname{diag}(0, 0, 0, 1, 1).$$

Thus we have proven,

Proposition 1. *The 2-dimensional 5-velocity model admits the stability structure with $\lambda_i = 1/\tau$ for $i \geq 2$.*

Remark 6. *The lattice Boltzmann models that are stable can be used as a guideline to construct consistent models which will automatically converge to the shallow water equations in the diffusive limit.*

Remark 7. *We remark that the lattice Boltzmann model (4.2) used above models shallow water equation. Its consistency is not yet proven and it is not within the scope of this work.*

Note that the D2Q5 model cannot be used for the physics containing momentum advection, because its lower degree of isotropy leads to a completely incorrect representation of the momentum flux tensor. It was proven that a sufficient symmetry lattice is needed for correct representation of physics [15], (this is the dominant requirement for the recovery of the correct flow equation).

Example 2

Next we consider D2Q7-velocity model, with

$$\begin{aligned}\xi_0 &= (0, 0), \\ \{\xi_i : i \neq 0\} &= \left\{ e \left[\cos\left(\frac{(i-1)\pi}{3}\right), \sin\left(\frac{(i-1)\pi}{3}\right) \right] \right\}\end{aligned}$$

and

$$f_i^{eq} = \begin{cases} h - \frac{gh^2}{e^2} + \frac{h\mathbf{u}^2}{e^2} \\ \frac{gh^2}{6e^2} + \frac{h\xi_i\mathbf{u}}{3e^2} + \frac{2h(\xi_i\cdot\mathbf{u})^2}{3e^4} - \frac{h\mathbf{u}^2}{2e^2}, & i \neq 0. \end{cases} \quad (4.12)$$

Refer to Figure (3.2), with

$$h = \sum_{i=0}^N f_i,$$

and

$$h\mathbf{u} = \sum_{i=0}^N \xi_i f_i.$$

Similar to Example 1, the following can be verified directly

$$\sum_{i=0}^N f_i^{eq} = h = \sum_{i=0}^N f_i, \quad (4.13)$$

$$\sum_{i=0}^N \xi_i f_i^{eq} = h\mathbf{u} = \sum_{i=0}^N \xi_i f_i.$$

Computing Equation (4.12), gives

$$\frac{\partial f_i^{eq}(h, \mathbf{u})}{\partial f_j} = \begin{cases} 1 - \frac{2gh}{e^2} + \frac{2\xi_j\mathbf{u}}{e^2} - \frac{\mathbf{u}^2}{e^2} \\ \frac{gh}{3e^2} + \frac{\xi_j\xi_i}{3e^2} + \frac{4\xi_j\xi_i\cdot\mathbf{u}}{3e^4} - \frac{2(\xi_i\cdot\mathbf{u})^2}{3e^4} - \frac{\xi_j\mathbf{u}}{e^2} + \frac{\mathbf{u}^2}{2e^2}, & i \neq 0. \end{cases} \quad (4.14)$$

By using Equation (4.13), we deduce from Equation (4.14) that

$$[f_{f_i}^{eq}(h, \mathbf{u})]^2 = [f_{f_i}^{eq}(h, \mathbf{u})],$$

that is, the Jacobian $[f_{f_i}^{eq}(h, \mathbf{u})]$ is a projection matrix. Thus, the eigenvalues of

$$J_f(h, \mathbf{u}) = ([f_{f_i}^{eq}(h, \mathbf{u})] - I_7)/\tau$$

are 0 and $-\frac{1}{\tau}$. Take $f_* = f^{eq}(1, 0)$ and $e = 1$ then

$$\frac{\partial f_i^{eq}(1, 0)}{\partial f_j} = \begin{cases} 1 - 2g \\ \frac{g}{3} + \frac{\xi_j \xi_i}{3}, & i \neq 0. \end{cases} \quad (4.15)$$

We want to find B_0 such that

$$B_0 \left[\frac{\partial f_i^{eq}(1, 0)}{\partial f_j} \right] = Y, \quad (4.16)$$

where Y is a symmetric matrix. The square matrix B_0 must be symmetric and positive definite. Choose

$$B_0 = \text{diag} \left[\frac{1}{1 - 2g}, \frac{3}{g} \mathbf{I}_4 \right]. \quad (4.17)$$

From the above choice of B_0 , we need to choose g such that, B_0 remains positive definite.

Therefore, we set

$$g < \frac{1}{2} \quad \text{and} \quad g \neq 0.$$

Substituting Equations (4.15) and (4.17) into (4.16) gives

$$Y = \gamma \xi^T \xi + B_0 \Psi_{ij}, \quad (4.18)$$

where γ is a constant and Ψ_{ij} is an $(N \times N)$ matrix. Take $g = \frac{1}{4}$, for example, then

$$\frac{\partial f_i^{eq}(1, 0)}{\partial f_j} = \begin{cases} \frac{1}{2} \\ \frac{1}{8} + \frac{\xi_j \xi_i}{2}, & i \neq 0 \end{cases} \quad (4.19)$$

and

$$B_0 = 2 \text{diag} [1, 4\mathbf{I}_4]. \quad (4.20)$$

From Equations (4.19) and (4.20) we see that

$$B_0 \left[\frac{\partial f_i^{eq}(1,0)}{\partial f_j} \right] = 4\xi^T \xi + B_0 \Psi_{ij} \quad (4.21)$$

is symmetric, where

$$\Psi_{ij} = \begin{cases} \frac{1}{2} \\ \frac{1}{8}, & i \neq 0. \end{cases}$$

We deduce from Equation (4.19) that the rank of $[f_{f_i}^{eq}(1,0)]$ is 3. Since $[f_{f_i}^{eq}(h, \mathbf{u})]$ is a projection matrix and

$$\tau J_f(f_*) = [f_{f_i}^{eq}(1,0)] - I_7$$

then, the rank of $J_f(f_*)$ is 4.

On the other hand, since $B_0[f_{f_i}^{eq}(1,0)]$ is symmetric and B_0 is symmetric positive definite, it is well known that there is an invertible matrix P such that

$$B_0 = P^T P \quad \text{and} \quad B_0 \tau J_f(f_*) = P^T \Lambda P$$

with Λ a diagonal matrix. We may as well assume that

$$\Lambda = -\text{diag}(0, 0, 0, 1, 1, 1, 1).$$

Thus we have proven,

Proposition 2. *The 2-dimensional 7-velocity model is stable.*

Remark 8. *The lattice Boltzmann model used above was developed by Zhou [51] using the 7-speed hexagonal lattice shown in Figure (3.2). The model was developed in the same manner to that on the 9-speed square lattice.*

4.3 Determination of Parameters

In the following section, some parameters will be fixed for several LB models. By doing so we will make the assumptions that the LB models are stable for those fixed parameters. The models to be used are taken from [11, 34].

4.3.1 Examples

The following examples are used:

Example 3

Now consider D2Q9-velocity model, with

$$\boldsymbol{\xi}_0 = (0, 0),$$

$$\{\boldsymbol{\xi}_i : i = 1, 2, 3, 4\} = \{(\pm 1, 0)^T, (0, \pm 1)^T\},$$

$$\{\boldsymbol{\xi}_i : i = 5, 6, 7, 8\} = \{(\pm 1, \pm 1)^T\}$$

and

$$\begin{aligned} f_i^{(eq)} = & \omega_i \left(h + \frac{1}{\theta} (h\mathbf{u}) \cdot \boldsymbol{\xi}_i + \frac{1}{2\theta^2} [(P(h) - \theta h)\mathbf{I} + h\mathbf{u}\mathbf{u}] : (\boldsymbol{\xi}_i\boldsymbol{\xi}_i - \theta\mathbf{I}) \right) \\ & + \omega_i g_i \left(\frac{1}{4}h - \frac{3}{8}gh^2 \right). \end{aligned} \quad (4.22)$$

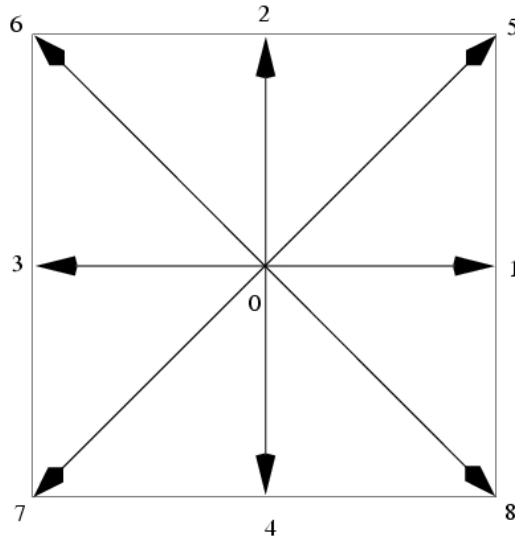


Figure 4.1: The nine particle speeds $\boldsymbol{\xi}_i$ in the 2D square lattice.

Refer to Figure (4.1), where

- $\theta = \frac{1}{3}$ is a constant reference temperature,

- g is gravity,
- $P = \frac{1}{2}gh^2$ is pressure,
- $\mathbf{I}$ is the identity matrix,
- $g_i = (1, -2, -2, -2, -2, 4, 4, 4, 4)^T$, are parameters referred to as a "ghost vector" by Dellar [11]
- and ω_i are the weight functions given in the form

$$\omega_i = \begin{cases} \alpha, & i = 0 \\ \beta, & i = 1, 2, 3, 4 \\ \frac{1 - \alpha - 4\beta}{4}, & i = 5, 6, 7, 8 \end{cases} \quad (4.23)$$

where α and β are parameters,

with

$$h = \sum_{i=0}^N f_i,$$

$$h\mathbf{u} = \sum_{i=0}^N \xi_i f_i$$

and

$$\Pi = \sum_{i=0}^N \xi_i \xi_i f_i.$$

The model was taken from Dellar [11]. To show its stability, it is noted that, the following can be verified directly

$$\sum_{i=0}^N f_i^{eq} = h = \sum_{i=0}^N f_i, \quad (4.24)$$

$$\sum_{i=0}^N \xi_i f_i^{eq} = h\mathbf{u} = \sum_{i=0}^N \xi_i f_i.$$

Note that

$$\begin{aligned} \Pi^{(eq)} &= \sum_{i=0}^N \xi_i \xi_i f_i^{eq} \\ &= P(h)\mathbf{I} + h\mathbf{u}\mathbf{u}, \end{aligned}$$

see [11]. Equation (4.22) can be written as

$$f_i^{eq} = \begin{cases} h + \omega_0 h \left(-\frac{15}{8}gh - \frac{3}{2}\mathbf{u}^2 \right) \\ \omega_i h \left(\frac{3}{2}gh + 3\xi_i \cdot \mathbf{u} + \frac{9}{2}(\xi_i \cdot \mathbf{u})^2 - \frac{3}{2}\mathbf{u}^2 \right), & i \neq 0. \end{cases} \quad (4.25)$$

The above model (4.25) is known as Salmon's equilibria [34]. By substituting the parameters $(\alpha, \beta) = (\frac{4}{9}, \frac{1}{9})$ in Equation (4.23) above leads to

$$\omega_i = \begin{cases} \frac{4}{9}, & i = 0 \\ \frac{1}{9}, & i = 1, 2, 3, 4 \\ \frac{1}{36}, & i = 5, 6, 7, 8, \end{cases} \quad (4.26)$$

these values are usually used in most D2Q9 models. Using Equation (4.26), then Equation (4.25) can be written as

$$f_i^{eq}(h, \mathbf{u}) = \begin{cases} h - \frac{5}{6}gh^2 - \frac{2}{3}h\mathbf{u}^2 & i = 0 \\ \frac{1}{6}gh^2 + \frac{1}{3}h\xi_i \cdot \mathbf{u} + \frac{1}{2}h(\xi_i \cdot \mathbf{u})^2 - \frac{1}{6}h\mathbf{u}^2, & 1 \leq i \leq 4 \\ \frac{1}{24}gh^2 + \frac{1}{12}h\xi_i \cdot \mathbf{u} + \frac{1}{8}h(\xi_i \cdot \mathbf{u})^2 - \frac{1}{24}h\mathbf{u}^2, & 5 \leq i \leq 8. \end{cases} \quad (4.27)$$

Computing

$$\frac{\partial f_i^{eq}(h, \mathbf{u})}{\partial f_j} = \begin{cases} 1 - \frac{5}{3}gh - \frac{4}{3}\xi_j \cdot \mathbf{u} + \frac{2}{3}\mathbf{u}^2 & i = 0 \\ \frac{1}{3}gh + \frac{1}{3}\xi_j \xi_i + \xi_j \xi_i \cdot \mathbf{u} - \frac{1}{2}(\xi_i \cdot \mathbf{u})^2 - \frac{1}{3}\xi_j \cdot \mathbf{u} + \frac{1}{6}\mathbf{u}^2, & 1 \leq i \leq 4 \\ \frac{1}{12}gh + \frac{1}{12}\xi_j \xi_i + \frac{1}{4}\xi_j \xi_i \cdot \mathbf{u} - \frac{1}{8}(\xi_i \cdot \mathbf{u})^2 - \frac{1}{12}\xi_j \cdot \mathbf{u} + \frac{1}{24}\mathbf{u}^2, & 5 \leq i \leq 8. \end{cases} \quad (4.28)$$

By using Equation (4.24), we deduce from Equation (4.28) that

$$[f_{f_i}^{eq}(h, \mathbf{u})]^2 = [f_{f_i}^{eq}(h, \mathbf{u})],$$

that is, the Jacobian $[f_{f_i}^{eq}(h, \mathbf{u})]$ is a projection matrix. Thus, the eigenvalues of

$$J_f(h, \mathbf{u}) = ([f_{f_i}^{eq}(h, \mathbf{u})] - I_9)/\tau$$

are 0 and $-\frac{1}{\tau}$. Take $f_* = f^{eq}(1, 0)$, then

$$\frac{\partial f_i^{eq}(1, 0)}{\partial f_j} = \begin{cases} 1 - \frac{5}{3}g & i = 0 \\ \frac{1}{3}g + \frac{1}{3}\xi_j \xi_i, & 1 \leq i \leq 4 \\ \frac{1}{12}g + \frac{1}{12}\xi_j \xi_i, & 5 \leq i \leq 8. \end{cases} \quad (4.29)$$

We want to find C_0 such that

$$C_0 \left[\frac{\partial f_i^{eq}(1, 0)}{\partial f_j} \right] = Z, \quad (4.30)$$

where Z is a symmetric matrix. The square matrix C_0 must be symmetric and positive definite. For this to hold using Equation (4.29), we see that

$$1 - \frac{5}{3}g > 0 \quad \text{and} \quad g \neq 0$$

i.e, $g \in (0, \frac{3}{5})$. Take $g = \frac{1}{3}$ and substitute into Equation (4.29), then

$$\frac{\partial f_i^{eq}(1, 0)}{\partial f_j} = \begin{cases} \frac{4}{9} & i = 0 \\ \frac{1}{9} + \frac{1}{3}\xi_j \xi_i, & 1 \leq i \leq 4 \\ \frac{1}{36} + \frac{1}{12}\xi_j \xi_i, & 5 \leq i \leq 8. \end{cases} \quad (4.31)$$

By setting

$$C_0 = 9 \operatorname{diag} [1/4, \mathbf{I}_4, 4\mathbf{I}_4], \quad (4.32)$$

from Equations (4.31) and (4.32), we see that

$$C_0 \left[\frac{\partial f_i^{eq}(1, 0)}{\partial f_j} \right] = 3\xi^T \xi + C_0 \Psi_{ij} \quad (4.33)$$

is symmetric, where

$$\Psi_{ij} = \begin{cases} \frac{4}{9}, & \text{for } i = 0 \\ \frac{1}{9}, & \text{for } i = \{1, \dots, 4\} \\ \frac{1}{36}, & \text{for } i = \{5, \dots, 8\}. \end{cases}$$

The rank of $[f_{f_i}^{eq}(1, \mathbf{u})]|_{u=0}$ is 3. Since $[f_{f_i}^{eq}(h, \mathbf{u})]$ is a projection matrix and

$$\tau J_f(f_*) = [f_{f_i}^{eq}(1, \mathbf{u})]|_{u=0} - I_9$$

then, the rank of $J_f(f_*)$ is 6.

On the other hand, since $C_0[f_{f_i}^{eq}(1, \mathbf{u})]|_{u=0}$ is symmetric and C_0 is symmetric positive definite, it is well known that there is an invertible matrix P such that

$$C_0 = P^T P \quad \text{and} \quad C_0 \tau J_f(f_*) = P^T \Lambda P$$

with Λ a diagonal matrix. We may as well assume that

$$\Lambda = -\text{diag}(0, 0, 0, 1, 1, 1, 1, 1, 1).$$

Thus we have proven,

Proposition 3. *The 2-dimensional 9-velocity model is stable when $g \in (0, \frac{3}{5})$.*

Remark 9. *We remark that the stability structure in definition (1) agrees with Salmon's stable equilibria.*

Remark 10. *The D2Q9 model above is stable if $(\alpha, \beta) = (4/9, 1/9)$ in Equation (4.23) and $g \in (0, \frac{3}{5})$. It was shown in [36] that $(\alpha, \beta) = (2/7, 1/7)$ can also be used when modeling N-S equations.*

Example 4

We consider another D2Q9-velocity model, with

$$f_i^{(eq)} = \omega_i \left(h + \frac{1}{\theta} (h\mathbf{u}) \cdot \boldsymbol{\xi}_i + \frac{1}{2\theta^2} [(P(h) - \theta h)\mathbf{I} + h\mathbf{u}\mathbf{u}] : (\boldsymbol{\xi}_i \boldsymbol{\xi}_i - \theta \mathbf{I}) \right) + \omega_i g_i \lambda \left(\frac{1}{4}h - \frac{3}{8}gh^2 \right), \quad (4.34)$$

with constants and parameters defined the same as in the D2Q9 model in (4.22). Substituting for the values of θ , $\mathbf{I}$ and $P(h)$ into Equation (4.34) gives

$$f_i^{(eq)} = \omega_i \left[2h + 3h\boldsymbol{\xi}_i \cdot \mathbf{u} + \frac{9}{4}gh^2\boldsymbol{\xi}_i \boldsymbol{\xi}_i - \frac{3}{2}h\boldsymbol{\xi}_i \boldsymbol{\xi}_i + \frac{9}{2}h(\boldsymbol{\xi}_i \cdot \mathbf{u})^2 - \frac{3}{2}gh^2 - \frac{3}{2}\mathbf{u}^2 \right] + \omega_i g_i \lambda \left(\frac{1}{4}h - \frac{3}{8}gh^2 \right). \quad (4.35)$$

Using the parameters in Equation (4.26), then Equation (4.35) can be written as

$$f_i^{eq}(h, \mathbf{u}) = \begin{cases} \frac{(8+\lambda)}{9}h - \frac{(4+\lambda)}{6}gh^2 - \frac{2}{3}h\mathbf{u}^2 & i = 0 \\ \frac{(1-\lambda)}{18}h + \frac{(1+\lambda)}{12}gh^2 + \frac{1}{3}h\boldsymbol{\xi}_i \cdot \mathbf{u} + \frac{1}{2}h(\boldsymbol{\xi}_i \cdot \mathbf{u})^2 - \frac{1}{6}h\mathbf{u}^2, & 1 \leq i \leq 4 \\ \frac{(\lambda-1)}{36}h + \frac{(2-\lambda)}{24}gh^2 + \frac{1}{12}h\boldsymbol{\xi}_i \cdot \mathbf{u} + \frac{1}{8}h(\boldsymbol{\xi}_i \cdot \mathbf{u})^2 - \frac{1}{24}h\mathbf{u}^2, & 5 \leq i \leq 8. \end{cases} \quad (4.36)$$

Computing

$$\frac{\partial f_i^{eq}(h, \mathbf{u})}{\partial f_j} = \begin{cases} \frac{(8+\lambda)}{9} - \frac{(4+\lambda)}{3}gh - \frac{4}{3}\boldsymbol{\xi}_j \cdot \mathbf{u} + \frac{2}{3}\mathbf{u}^2 & i = 0 \\ \frac{(1-\lambda)}{18} + \frac{(1+\lambda)}{6}gh + \frac{1}{3}\boldsymbol{\xi}_j \boldsymbol{\xi}_i + \boldsymbol{\xi}_j \boldsymbol{\xi}_i \cdot \mathbf{u} - \frac{1}{2}(\boldsymbol{\xi}_i \cdot \mathbf{u})^2 - \frac{1}{3}\boldsymbol{\xi}_j \cdot \mathbf{u} + \frac{1}{6}\mathbf{u}^2, & 1 \leq i \leq 4 \\ \frac{(\lambda-1)}{36} + \frac{(2-\lambda)}{12}gh + \frac{1}{12}\boldsymbol{\xi}_j \boldsymbol{\xi}_i + \frac{1}{4}\boldsymbol{\xi}_j \boldsymbol{\xi}_i \cdot \mathbf{u} - \frac{1}{8}(\boldsymbol{\xi}_i \cdot \mathbf{u})^2 - \frac{1}{12}\boldsymbol{\xi}_j \cdot \mathbf{u} + \frac{1}{24}\mathbf{u}^2, & 5 \leq i \leq 8. \end{cases} \quad (4.37)$$

By using (4.24), we deduce from (4.37) that

$$[f_{f_i}^{eq}(h, \mathbf{u})]^2 = [f_{f_i}^{eq}(h, \mathbf{u})]$$

that is, the Jacobian $[f_{f_i}^{eq}(h, \mathbf{u})]$ is a projection matrix. Thus, the eigenvalues of $J_f(h, \mathbf{u}) = ([f_{f_i}^{eq}(h, \mathbf{u})] - I_9)/\tau$ are 0 and $-\frac{1}{\tau}$. Take $f_* = f^{eq}(1, 0)$, then

$$\frac{\partial f_i^{eq}(1, 0)}{\partial f_j} = \begin{cases} \frac{(8 + \lambda)}{9} - \frac{(4 + \lambda)}{3}g, & i = 0 \\ \frac{(1 - \lambda)}{18} + \frac{(1 + \lambda)}{6}g + \frac{1}{3}\xi_j\xi_i, & 1 \leq i \leq 4 \\ \frac{(\lambda - 1)}{36} + \frac{(2 - \lambda)}{12}g + \frac{1}{12}\xi_j\xi_i, & 5 \leq i \leq 8. \end{cases} \quad (4.38)$$

We want to find D_0 such that

$$D_0 \left[\frac{\partial f_i^{eq}(1, 0)}{\partial f_j} \right] = T \quad (4.39)$$

where T is a symmetric matrix. The square matrix D_0 must be symmetric and positive definite. Choose

$$D_0 = \text{diag} \left[\left(\frac{9}{8 + \lambda - 12g - 3\lambda g} \right), \left(\frac{18}{1 - \lambda + 3g + 3\lambda g} \right) \mathbf{I}_4, \left(\frac{36}{\lambda - 1 + 6g - 3\lambda g} \right) \mathbf{I}_4 \right]. \quad (4.40)$$

From the above choice of D_0 , we need to choose the parameters g and λ such that D_0 is positive definite. Substituting Equation (4.40) into (4.39) gives

$$T = \gamma \xi^T \xi + D_0 \Psi_{ij}, \quad (4.41)$$

where γ is a constant and Ψ_{ij} is an $(N \times N)$ matrix defined as follows

$$\Psi_{ij} = \begin{cases} \frac{(8 + \lambda - 12g - 3\lambda g)}{9}, & \text{for } i = 0 \\ \frac{(1 - \lambda + 3g + 3\lambda g)}{18}, & \text{for } i = \{1, \dots, 4\} \\ \frac{(\lambda - 1 + 6g - 3\lambda g)}{36}, & \text{for } i = \{5, \dots, 8\}. \end{cases}$$

For T to be symmetric, the following relationship is found using the second term in the above Equation (4.41)

$$\frac{6}{1 + 3g - \lambda + 3\lambda g} = \frac{3}{-1 + 6g + \lambda - 3\lambda g}, \quad (4.42)$$

giving the following

$$\lambda = 1.$$

We notice that the model in (4.35) behaves the same as the model in (4.22) when $\lambda = 1$. Then, for the stability structure [1] to hold, $g \in (0, \frac{3}{5})$.

Now take $g = \frac{1}{4}$ and $\lambda = 1$, for example, and substitute these parameters into (4.38) giving

$$\frac{\partial f_i^{eq}(1, 0)}{\partial f_j} = \begin{cases} \frac{7}{12}, & i = 0 \\ \frac{1}{12} + \frac{1}{3}\xi_j\xi_i, & 1 \leq i \leq 4 \\ \frac{1}{48} + \frac{1}{12}\xi_j\xi_i, & 5 \leq i \leq 8, \end{cases} \quad (4.43)$$

and the above Equation (4.40) becomes

$$D_0 = 12 \operatorname{diag} \left[\frac{1}{7}, \mathbf{I}_4, 4\mathbf{I}_4 \right]. \quad (4.44)$$

From Equations (4.44) and (4.43) we see that

$$D_0 \left[\frac{\partial f_i^{eq}(1, 0)}{\partial f_j} \right] = 4\xi^T \xi + D_0 \Psi_{ij} \quad (4.45)$$

is symmetric, where

$$\Psi_{ij} = \begin{cases} \frac{7}{12}, & \text{for } i = 0 \\ \frac{1}{12}, & \text{for } i = \{1, \dots, 4\} \\ \frac{1}{48}, & \text{for } i = \{5, \dots, 8\}. \end{cases}$$

The rank of $[f_{f_i}^{eq}(1, \mathbf{u})]|_{u=0}$ is 3. Since $[f_{f_i}^{eq}(h, \mathbf{u})]$ is a projection matrix and

$$\tau J_f(f_*) = [f_{f_i}^{eq}(1, \mathbf{u})]|_{u=0} - I_9$$

then, the rank of $J_f(f_*)$ is 6.

On the other hand, since $D_0[f_{f_i}^{eq}(1, \mathbf{u})]|_{u=0}$ is symmetric and D_0 is symmetric positive definite, it is well known that there is an invertible matrix P such that

$$D_0 = P^T P \quad \text{and} \quad D_0 \tau J_f(f_*) = P^T \Lambda P$$

with Λ a diagonal matrix. We may as well assume that

$$\Lambda = -\text{diag}(0, 0, 0, 1, 1, 1, 1, 1, 1).$$

We also observe that when $g = \frac{1}{3}$ in Equation (4.38) the parameter λ is arbitrary. Then the D2Q9 model in Equation (4.35) behavior does not change, i.e Equation (4.38) gives

$$\frac{\partial f_i^{eq}(1, 0)}{\partial f_j} = \begin{cases} \frac{4}{9} & i = 0 \\ \frac{1}{9} + \frac{1}{3}\xi_j\xi_i, & 1 \leq i \leq 4 \\ \frac{1}{36} + \frac{1}{12}\xi_j\xi_i, & 5 \leq i \leq 8, \end{cases} \quad (4.46)$$

which is similar to Equation (4.31).

Thus we have proven,

Proposition 4. *If $\lambda = 1$ with $g \in (0, \frac{3}{5})$ or when λ is arbitrary with $g = \frac{1}{3}$, then the D2Q9-velocity model is stable.*

Remark 11. *In [11], it was claimed that the parameter λ was adjustable to give a positive equilibria in the state at rest, i.e, when $\mathbf{u} = 0$. The stability structure (1) above gave only two relationship, when $\lambda = 1$ and arbitrary, proving Dellar's [11] claim.*

From the above examples, it was shown that the stability requirement leads to a good choice of the parameters. When choice of parameters were outside the range required by the stability condition (1), unstable results were obtained, for example, refer to Appendix E. Numerical tests will be done in the next chapter to verify our results.

Chapter 5

Numerical Results

5.1 Introduction

In this chapter we present and verify a number of stability criteria presented in Chapter 4 by Propositions 3 and 4. Different models will be used to test the results obtained from the stability structure on selected Benchmark problems. The main goal is to show that when reasonable ranges for the corresponding parameters are chosen, stable results are obtained. Alternatively, when choices of parameters are outside the suggested range in the propositions, then unstable results are found. The accuracy in our method (stability structure [1]), is demonstrated by comparing the numerical predictions with analytical solutions. The codes to simulate the numerical results of the present chapter were produced using MATLAB 7.5.0 (R2007b). Standard LB Equation (3.7) will be used in the following examples.

5.2 Test Examples

5.2.1 Example 1: Steady flow over a hump

In this example, we show the convergence in time towards the steady flow over the hump. This example is widely used to test numerical schemes for shallow water equations for transcritical and subcritical flows. For example, it was considered by the working group on dam break modelling [17] and used by Vázquez-Cendón [44] to test their scheme with

an upwind discretization for the bed slope source term.

A one-dimensional (1D) steady flow in a 25m long and 1m wide channel with a hump is defined by

$$z_b(x) = \begin{cases} 0.2 - 0.05(x - 10)^2, & \text{if } 8 \leq x \leq 12; \\ 0, & \text{otherwise.} \end{cases} \quad (5.1)$$

The initial conditions are given by

$$h(x, 0) = 2 \text{ m} - z_b(x) \quad \text{and} \quad u(x, 0) = 0 \text{ m/s}$$

as illustrated in Figure 5.1.

When steady subcritical flow passes over the hump on a bed slope, there is surface drop over the hump. The analytical solution is given by Coutal and Maurel [17]. We use this example as our first test problem to verify that Propositions 3 and 4 hold, starting with the former.

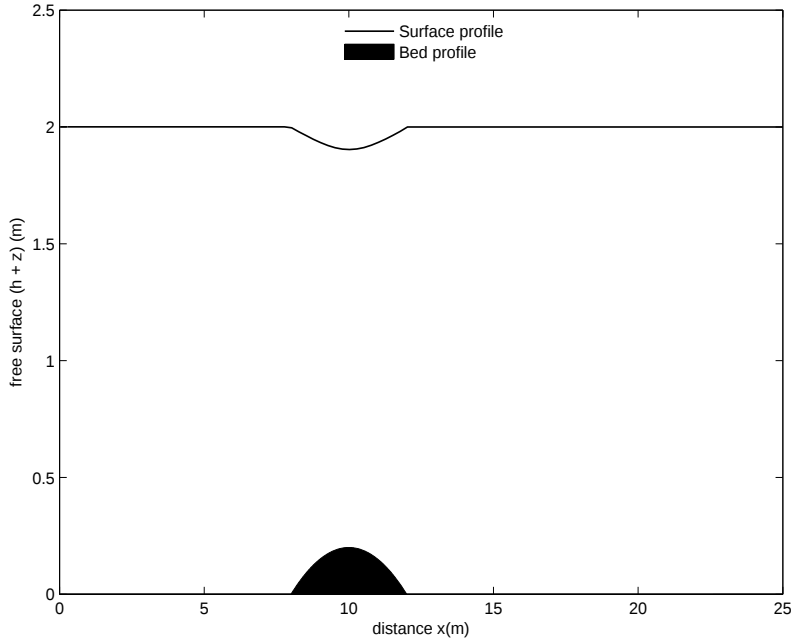


Figure 5.1: Steady subcritical flow over a hump: Illustration of the free water surface and the bottom profiles.

Gravity (g)	Lattice sizes	Number of iterations
0.09	125×50	—
	250×50	—
	500×50	—
0.07	125×50	19513
	250×50	—
	500×50	—
0.03	125×50	19873
	250×50	39170
	500×50	—
0.009	125×50	21333
	250×50	40034
	500×50	59700
0.006	125×50	24165
	250×50	40319
	500×50	60048

Table 5.1: The summary of gravity for different lattices, — shows that there was no convergence.

The following conditions will be imposed on the channel boundaries, the water level $h = 2$ m is used at the outflow boundary condition and the discharge $q = 4.42$ m^2/s is imposed at the inflow boundary condition. The slip or non-slip boundary conditions are used at the solid walls. For the no-slip condition, the bounce-back scheme is used and for slip conditions, a zero gradient of the distribution function normal to the solid wall is used. The lattice speed $e = 15$ m/s and $\tau = 1.5$ are also used.

We define the global relative error R by

$$R = \sqrt{\sum_i \left(\frac{h_i^n - h_i^{n-1}}{h_i^n} \right)^2}, \quad (5.2)$$

as defined in [51]. The h_i^n and h_i^{n-1} represents the local water depth at the current and

Parameter λ	Lattice sizes	Number of iterations
-6	125×50	21333
	250×50	40034
	500×50	59700
0	125×50	21333
	250×50	40034
	500×50	59700
3	125×50	21333
	250×50	40034
	500×50	59700
6.7	125×50	21333
	250×50	40034
	500×50	—
14	125×50	—
	250×50	—
	500×50	—

Table 5.2: The summary of the value of λ for different lattices using $g = \frac{1}{3e}$.

previous time levels, respectively. For the scheme to converge to a steady solution, the convergence criterion is taken as $R < 5 \times 10^{-6}$.

The three lattice sizes, 125×50 , 250×50 and 500×50 which correspond to $\Delta x = 0.2$ m, $\Delta x = 0.1$ m and $\Delta x = 0.05$ m are used in the initial computations to test their effects on lattice solutions. For numerical computation the gravitational acceleration g ranges from 0.006 and 0.09, i.e $g \in (0.006, 0.09)$. Our choice of g for numerical computation was motivated by the fact that $g \in (0, \frac{3}{5})$ and computed on the lattice speed $e = 15$ m/s, giving $g \in (0, \frac{3}{5e})$ (For dimensional purposes it was found useful to divide gravity with the lattice speed). Steady state solutions were obtained from different values of g used in the computation, refer to Table 5.1. When values of g outside the required range were used, the method was unstable. For example, when $g = 0.07$ steady state solution is reached

only at 125×50 lattice point and the solution does not convergence when the grid point is refined. On the other hand when gravity (g) is reduced, better results are obtained (when $g = 0.03$ and 0.006)

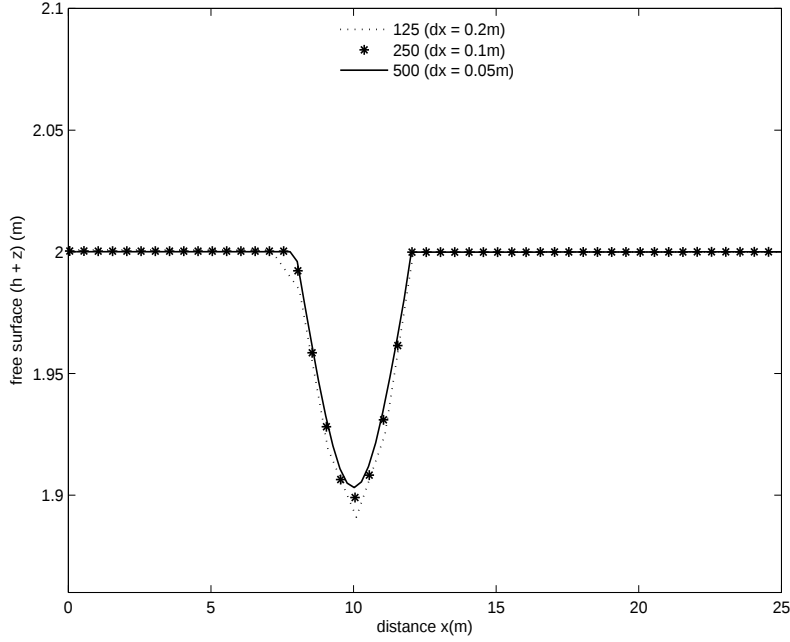


Figure 5.2: Steady subcritical flow over a hump: Free water surface using different lattices.

The value of $g = 0.009$ was chosen when comparing numerical results between different lattice sizes. There was little difference found, refer to Figure 5.2. The results further indicate that, when lattice sizes become smaller better results are obtained, i.e the results of $\Delta x = 0.1$ m and $\Delta x = 0.05$ m are almost the same, but there is a small difference between $\Delta x = 0.2$ m and $\Delta x = 0.1$ m. Hence the results of $\Delta x = 0.05$ m are preferred, since results based on $\Delta x \leq 0.1$ m provide better and accurate solutions.

We have tested the accuracy of the approach by comparing the computed steady water surface with the analytical solution as depicted in Figure 5.3, showing an excellent agreement. The L^2 - error norm was used to verify our results, defined as

$$\|\mathbf{C}\|_{L^2} = \sqrt{\frac{\sum_{ij} |C^n - \tilde{C}(x_i, y_j, t_n)|^2}{\sum_{ij} |\tilde{C}(x_i, y_j, t_n)|^2}}, \quad (5.3)$$

where C^n is the computed LB solution and $\tilde{C}(x_i, y_j, t_n)$ is the computed analytical solution,

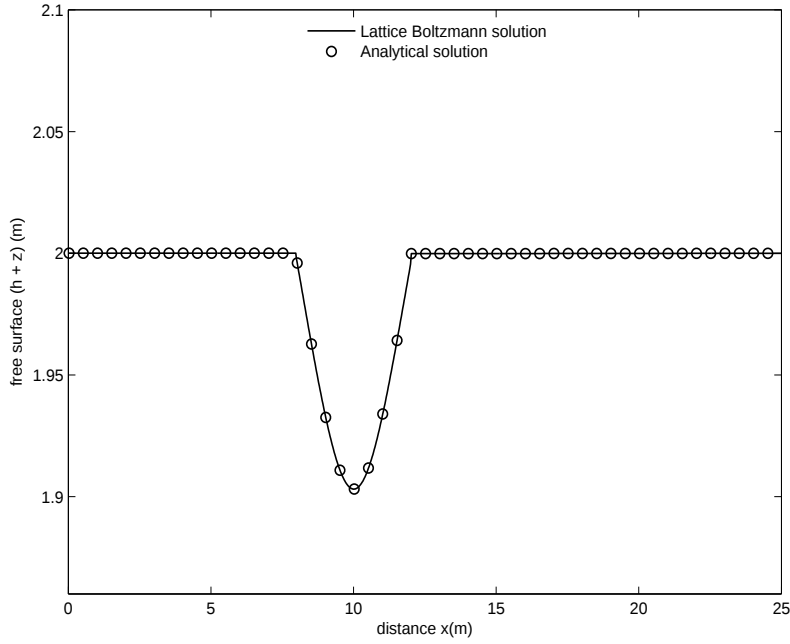


Figure 5.3: Steady subcritical flow over a hump: Comparison of the free water surface.

respectively, at time t_n and lattice point (x_i, y_j) . It was found that, the comparison of the computed LB solution with the analytical solution indicates that the relative error for the water depth is 0.325 %. To test the conservative property of the model, the numerical solution of the discharge was done and is depicted in Figure 5.4. The relative error was about 0.18 %. This suggests that the model is conservative and accurate. Note that, the above results were based on $\Delta x = 0.05$ m lattice size.

To check if Proposition 4 holds, the parameter λ was varied from -6 to 14 using $g = \frac{1}{3e}$ on different lattice sizes, refer to Table 5.2. It is interesting to note that when the value of λ increases in magnitude, leads to unstable results. We are unable to offer the explanation of why when varying the value of λ gave no advantage in the convergence of the solution. It was shown in [11] that when distribution functions changes signs, leads to a stable equilibrium distribution function for the SWEs.

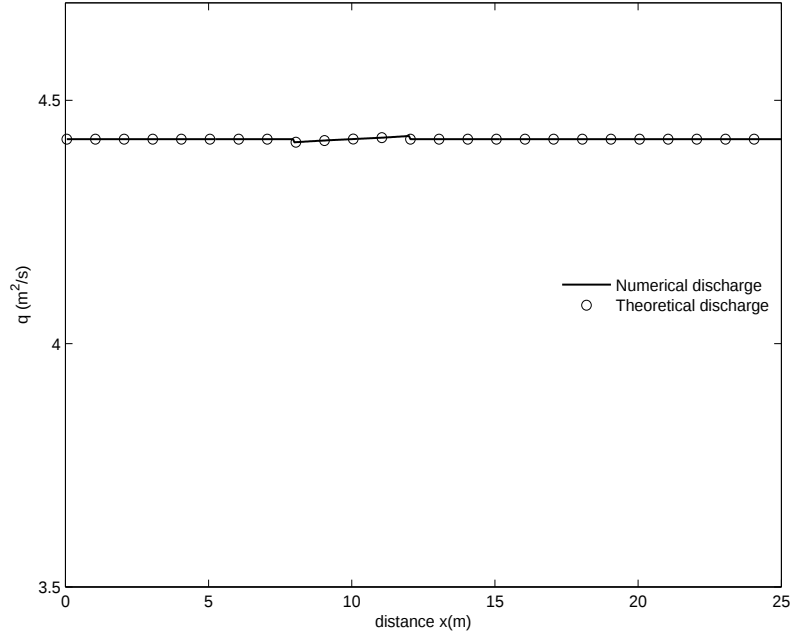


Figure 5.4: Steady subcritical flow over a hump: Comparison of discharge.

5.2.2 Example 2: Tidal wave flow

Consider a 1D problem of a tidal wave in a channel. It is a test problem used by Bermudez and Vázquez [3] to verify an upwind discretization of the bed slope source. The bed topography is defined by (refer to Figure 5.5)

$$H(x) = 50.5 - \frac{40x}{L} + 10\sin\left(\pi\left(\frac{4x}{L} - \frac{1}{2}\right)\right), \quad (5.4)$$

where $L = 14$ km is the length of the channel and $H(x)$ is the partial depth between a fixed reference level and the bed surface, giving $z_b = H(0) - H(x)$.

The initial conditions for the water height and velocity are

$$h(x, 0) = H(x) \quad (5.5)$$

and

$$u(x, 0) = 0. \quad (5.6)$$

At the inflow and outflow of the channel, respectively, we define

$$h(0, t) = H(0) + 4 - 4\sin\left(\pi\left(\frac{4t}{86400} - \frac{1}{2}\right)\right) \quad (5.7)$$

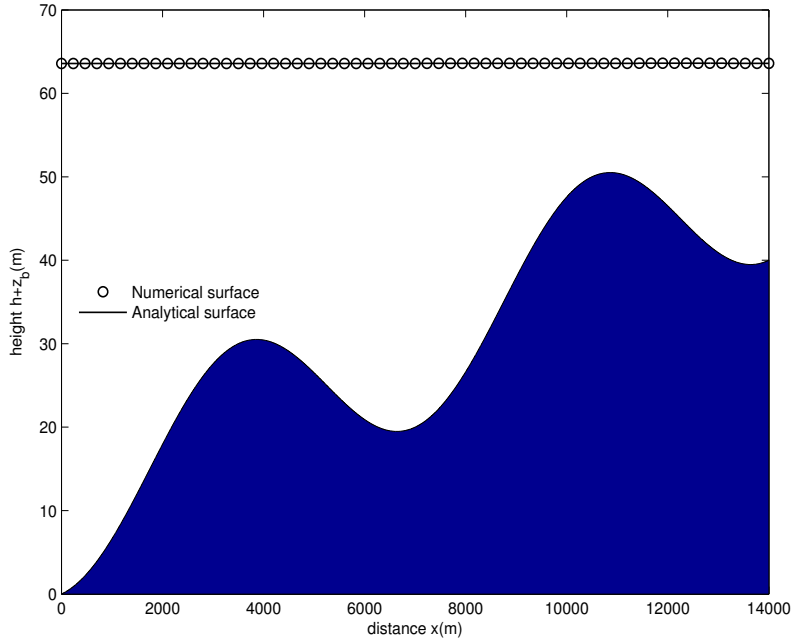


Figure 5.5: Numerical and analytical free surface for the tidal wave flow at time $t = 9117.5$ s.

and

$$u(L, t) = 0. \quad (5.8)$$

With reference to Bermudez and Vázquez [3], the asymptotic analytical solution for this test example is given by

$$h(x, t) = H(x) + 4 - 4\sin\left(\pi\left(\frac{4t}{86400} - \frac{1}{2}\right)\right) \quad (5.9)$$

and

$$h(x, t) = \frac{(x - 14000)\pi}{5400h(x, t)} \cos\left(\pi\left(\frac{4t}{86400} - \frac{1}{2}\right)\right). \quad (5.10)$$

The D2Q9 velocity model is used with f^{eq} defined by Equation (4.36). The value of the gravitational acceleration used is between 0 and $\frac{3}{5}$, i.e $g \in (0, \frac{3}{5})$ and $\lambda = 1$. We choose to use Proposition 4 since we have shown in Example 1 that, the two equilibrium distribution functions in Equations (2.26) and (2.35) behave the same when modelling shallow water flows. Similarly, we used a 2D code to produce the numerical results for a 1D problem. Periodic boundary conditions were used in the upper and lower walls.

First we need to discuss the *time accuracy* of the algorithm. For a time periodic flow with period T , the time accuracy is of order k when the relation

$$\epsilon = \frac{1}{T} \sum_t \sqrt{\frac{\sum_{\vec{r}} |\vec{u}(\vec{r}, t) - \vec{v}(\vec{r}, t)|^2}{N^d}} < \frac{\kappa}{N^k}, \quad (5.11)$$

holds [26]. Here, $\vec{u}$ is the numerical solution, $\vec{v}$ the analytical solution, N is the spatial resolution in each dimension, d the number of dimensions and κ is a constant. From Equation (5.11), k is given by

$$\log \epsilon < \log \kappa - k \log N. \quad (5.12)$$

Note that k determines the order of accuracy of the algorithm. It has been shown that if boundaries are neglected, the LB method is of second order accuracy, i.e $k \approx 2$ [9]. If the analytical solution of the flow is known, then we can compare it with the numerical solution. The methodology used in Example 1 will be used in this example. The L^2 - error norm is used as defined in Equation (5.3).

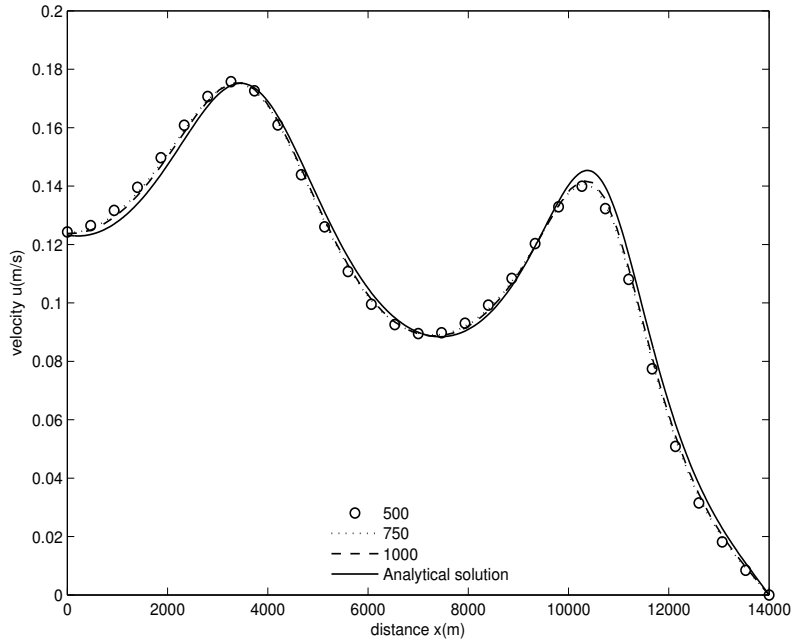


Figure 5.6: Numerical and analytical free surface for the tidal wave flow at time $t = 9117.5$ s.

Lattice size (m)	L^2 - error norm
$\Delta x = 7$	5.27×10^{-2}
$\Delta x = 14$	6.39×10^{-2}
$\Delta x = 28$	6.68×10^{-2}

Table 5.3: Comparison of numerical and analytical solutions using L^2 - error norm.

Three uniform lattices with 500×50 , 750×50 and 1000×50 , which corresponds to $\Delta x = 28$ m, $\Delta x = 14$ m and $\Delta x = 7$ m, respectively, were used. For the numerical computation, we used $\tau = 0.6$ and $e = 200$ m/s. Similar to Example 1, the value of λ was varied between -4 and 7 with $g = \frac{1}{3e}$. We observed that the algorithm converged when $t = 9117.5$ s. To quantify the results obtained, we compared the asymptotic results in Equations (5.9) and (5.10) with the computed solution. Figure 5.6 shows a comparison of the numerical solutions with the analytical solution at $t = 9117.5$ s, where $g = 0.0017$ ($\frac{1}{3e}$) was used. From Figure 5.6, it was not clear to make remarks on the advantage of decreasing the lattice sizes. It was verified using relative error by comparing numerical solutions with analytical solutions. It was found that when using $\Delta x = 7$ m gave better results, refer to Table 5.3. We can safely conclude that, when lattice size is decreased then accurate results are obtained. Similar behavior has been observed in Example 1. For the water depth it was found that the relative error was about 2 %. The numerical and analytical solutions for the free surface is depicted in Figure 5.5.

5.2.3 Example 3: Flow over a sudden-expansion channel

In this example, we demonstrate that shallow water equations can be used to simulate a recirculation in shallow water flows. Now consider a two-dimensional (2D) flow over a channel with a symmetric sudden-expansion. The channel expansion ratio is 3:1 with a channel expansion of 3 m wide and 4 m long. The entrance of the channel is 1m wide and 2m long, refer to Figure 5.7. Unlike in Example 1, there is no bed slope. Friction at the bottom is neglected.

For numerical computations, the D2Q9 velocity model is used with f^{eq} defined by Equation (4.36). The structure of the grid contains 120×60 lattice points with, $\Delta x =$

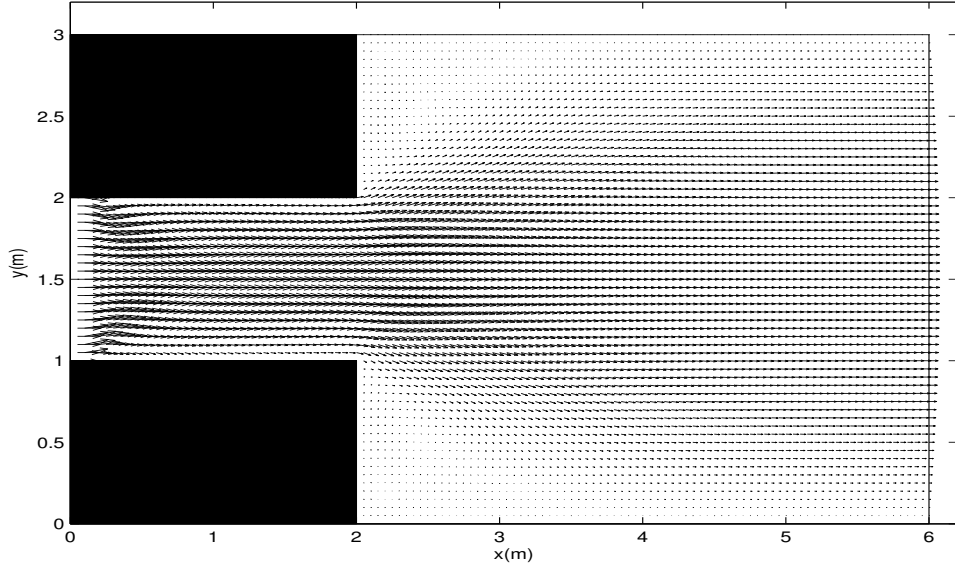


Figure 5.7: Sudden-expansion channel: velocity field, where $g = 0.15$.

$\Delta y = 0.05$ m, $\Delta t = 0.025$ s and $\tau = 1$. The speed of the lattice e is given by $e = \Delta x / \Delta t$. The following conditions are imposed on the channel boundaries, the water level $h = 0.16$ m is used at the out flow boundary, zero gradient depth is specified together with the discharge $q = 0.032$ m^3/s at the inflow boundary and velocity $v = 0$ is imposed at the inflow.

Gravity (g)	Number of iterations
0.001	21645
0.08	13432
0.15	11123
0.23	31373
0.3	—
0.5	—

Table 5.4: The summary of gravity values for $\lambda = 1$ on a 120×60 grid.

Different values of g between 0.001 and 0.5 were used in the computation with the parameter $\lambda = 1$ fixed. The steady state solution was reached using the convergence criterion in Equation (5.2), refer to Table 5.4 for the number of iterations taken to reach the

steady state. We observed that when the value of g was varied in the interval $0 < g < \frac{3}{5e}$, the algorithm converged. The value of g was fixed and the parameter λ was varied between -2 and 12, which lead to similar observations experienced in Example 1, refer to Table 5.5. The Figure 5.7, shows the velocity field with bit of circulating flows on both sides of the channel. From this observation, we can conclude that SWEs are capable of simulating circulations that occurs in shallow water flows.

Values of (λ)	Number of iterations
-2	23039
4	23039
7	23039
12	—

Table 5.5: The summary of λ values using $g = 0.1667$ ($g = \frac{1}{3e}$).

In conclusion, we have used three examples in this chapter to test the LB method. It can be concluded that the LB method performs well for both steady and time-dependent problems. In addition the theoretical results in Chapter 4 have also been tested. The numerical results agree with the theory hence, one can conclude that the stability structure is a good tool for designing the LB method. Further, on the time-dependent problems on the unsteady problem, excellent and accurate results are obtained without requiring special treatment on the source terms or complicated upwind discretization of the gradient fluxes.

Chapter 6

Conclusion and Future Research

The objective of this dissertation is to investigate the stability of the lattice Boltzmann (LB) method applied to shallow water equations (SWEs). To achieve this objective, structural stability was adopted as the main study for the resolution of the studied models.

We used a stability structure defined in [1] to investigate the stability of the LB equations which are currently being applied to simulate SWEs. The models we have chosen to work with were two-dimensional (2D) and had sufficient symmetry, which is a dominant requirement for the recovery of SWEs [34]. The popular two-dimensional nine velocities (D2Q9) lattice pattern was preferred since it is easier to use in numerical computations.

The most successful part of this dissertation was the analytical results. From the analytical tests, we found out that when certain parameters of the models are adjusted, lead to stable results. Numerous numerical tests were done to verify this.

The models used in this dissertation were found to be stable and their consistency has not been proven as it was not within the scope of this work. Hence, an important aspect that requires further research is to find the consistency of the models. In general, it is known that a consistent and stable lattice Boltzmann model implies convergence.

Appendix A

Computing Equilibrium Distribution Function

The D2Q5 equilibrium distribution function for the SWEs is given by

$$f_i^{eq} = \begin{cases} h - \frac{gh^2}{e^2} \\ \frac{gh^2}{4e^2} + \frac{h\xi_i \mathbf{u}}{2e^2}, & i \neq 0 \end{cases}$$

which contains no $O(u^2)$ terms. The equilibrium distribution functions satisfy

$$h = \sum_{i=0}^4 f_i^{eq}, \quad h\mathbf{u} = \sum_{i=0}^4 \xi_i f_i^{eq} \quad (\text{A.1.1})$$

Now computing, at $i = 0$

$$\begin{aligned} \frac{\partial f_i^{eq}}{\partial f_j} &= \frac{\partial}{\partial f_j} \left(h - \frac{gh^2}{e^2} \right) \\ &= \frac{\partial}{\partial f_j} (h) - \frac{\partial}{\partial f_j} \left(\frac{gh^2}{e^2} \right) \\ &= 1 - \frac{2gh}{e^2} \end{aligned}$$

and at $i \neq 0$

$$\begin{aligned}
 \frac{\partial f_i^{eq}}{\partial f_j} &= \frac{\partial}{\partial f_j} \left(\frac{gh^2}{4e^2} + \frac{h\xi_i \mathbf{u}}{2e^2} \right) \\
 &= \frac{\partial}{\partial f_j} \frac{g}{4e^2} (h^2) - \frac{\partial}{\partial f_j} \frac{\xi_i}{2e^2} (h\mathbf{u}) \\
 &= \frac{gh}{2e^2} + \frac{1}{2} \xi_j \xi_i
 \end{aligned}$$

Therefore

$$\frac{\partial f_i^{eq}}{\partial f_j} = \begin{cases} 1 - \frac{2gh}{e^2} \\ \frac{gh}{2e^2} + \frac{1}{2} \xi_j \xi_i, & i \neq 0 \end{cases} \quad (\text{A.1.2})$$

Appendix B

Projection Matrix

To prove that a matrix $\mathbf{A}$ is a projection matrix, we need to show that

$$\mathbf{A} = \mathbf{A}^2.$$

In linear Algebra, multiplication of matrices can be viewed as a binary operation that takes a pair of matrices and produce another matrix. This is as follows; the product of two matrices an $m \times n$ matrix $\mathbf{A}$ and $n \times p$ matrix $\mathbf{B}$ is an $m \times p$ matrix $\mathbf{AB}$ whose entries are given by

$$[AB]_{ij} = \sum_{k=0}^N A_{i,k} B_{k,j}.$$

Now given

$$\frac{\partial f_i^{eq}(h, \mathbf{u})}{\partial f_j} = \begin{cases} 1 - \frac{2gh}{e^2} \\ \frac{gh}{2e^2} + \frac{\xi_j \xi_i}{2e^2}, & i \neq 0, \end{cases} \quad (\text{B.0.1})$$

We need to show that

$$[f_{f_i}^{eq}(h, \mathbf{u})]^2 = [f_{f_i}^{eq}(h, \mathbf{u})].$$

Note that

$$\sum_{k=1}^4 \xi_k = 0, \quad \sum_{k=1}^4 \xi_k^2 = 2.$$

If

$$\begin{aligned} A^2 &= \sum_{k=0}^4 A_{i,k} A_{k,j} \\ &= \sum_{k=0}^0 A_{i,k} A_{k,j} + \sum_{k=1}^4 A_{i,k} A_{k,j} \end{aligned}$$

We need to show that

$$1 - \frac{2gh}{e^2} = \left(1 - \frac{2gh}{e^2}\right) \left(1 - \frac{2gh}{e^2}\right) + \left(1 - \frac{2gh}{e^2}\right) \sum_{k=1}^4 \left(\frac{gh}{2e^2} + \frac{1}{2} \xi_j \xi_k\right) \quad (\text{B.0.2})$$

and

$$\begin{aligned} \sum_{k=1}^4 \left(\frac{gh}{2e^2} + \frac{1}{2} \xi_j \xi_k\right) &= \sum_{k=1}^4 \left(\frac{gh}{2e^2} + \frac{1}{2} \xi_j \xi_k\right) \left(1 - \frac{2gh}{e^2}\right) \\ &\quad + \sum_{k=1}^4 \left(\frac{gh}{2e^2} + \frac{1}{2} \xi_j \xi_k\right) \sum_{k=1}^4 \left(\frac{gh}{2e^2} + \frac{1}{2} \xi_j \xi_k\right) \end{aligned} \quad (\text{B.0.3})$$

In Equation (B.0.1) if we take $g = 1/2$ and $e = 1$, using Equation (A.1.1) then

$$\frac{\partial f_i^{eq}}{\partial f_j} = \begin{bmatrix} -4 & -4 & -4 & -4 & -4 \\ 1.25 & 1.75 & 1.25 & 0.75 & 1.25 \\ 1.25 & 1.25 & 1.75 & 1.25 & 0.75 \\ 1.25 & 0.75 & 1.25 & 1.75 & 1.25 \\ 1.25 & 1.25 & 0.75 & 1.25 & 1.75 \end{bmatrix}$$

Therefore

$$[f_{f_i}^{eq}(h, \mathbf{u})]^2 = [f_{f_i}^{eq}(h, \mathbf{u})]$$

Appendix C

Symmetric Positive Definite Matrix

In mathematics, a symmetric positive definite matrix is defined as a matrix which is in many ways comparable to a positive real number. For example, an $n \times n$ matrix $\mathbf{A}$ is said to be symmetric positive definite if $\mathbf{z}^T \mathbf{A} \mathbf{z} > 0$ for all non-zero vectors $\mathbf{z}$ with real entries $\mathbf{z} \in \mathbb{R}^n$ where $\mathbf{z}^T$ denotes the transpose of $\mathbf{z}$.

Now take:

$$E_0 = \text{diag} \left[\frac{1}{1-2g}, \frac{2}{g} \mathbf{I}_4 \right] \quad (\text{C.0.1})$$

which can be written in the form

$$E_0 = \begin{bmatrix} \frac{1}{1-2g} & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{g} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{g} & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{g} & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{g} \end{bmatrix}.$$

For E_0 to be symmetric positive definite matrix then

$$0 < g < \frac{1}{2}.$$

Appendix D

Computing Rank

Consider the following equation:

$$\frac{\partial f_i^{eq}(1, \mathbf{u})}{\partial f_j} = \begin{cases} \frac{1}{3} \\ \frac{1}{6} + \frac{\xi_j \xi_i}{2}, \quad i \neq 0 \end{cases}. \quad (\text{D.1.1})$$

The above Equation (D.1.1), can be written in the form

$$\frac{\partial f_i^{eq}(1, \mathbf{u})}{\partial f_j} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \frac{-1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \frac{-1}{3} \\ \frac{1}{6} & \frac{-1}{3} & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{-1}{3} & \frac{1}{6} & \frac{2}{3} \end{bmatrix}$$

The above matrix $\frac{\partial f_i^{eq}(1, \mathbf{u})}{\partial f_j}$ can be reduced to the echelon form

$$\frac{\partial f_i^{eq}(1, \mathbf{u})}{\partial f_j} = \begin{bmatrix} 1 & 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there are 3 nonzero rows in the reduced form matrix, it indicates that the maximum number of linearly independent rows is 3; hence the rank is 3.

Appendix E

Using a parameter outside the required range

Using the equation below, then

$$\frac{\partial f_i^{eq}(1,0)}{\partial f_j} = \begin{cases} \frac{(8+\lambda)}{9} - \frac{(4+\lambda)}{3}g, & i = 0 \\ \frac{(1-\lambda)}{18} + \frac{(1+\lambda)}{6}g + \frac{1}{3}\xi_j\xi_i, & 1 \leq i \leq 4 \\ \frac{(\lambda-1)}{36} + \frac{(2-\lambda)}{12}g + \frac{1}{12}\xi_j\xi_i, & 5 \leq i \leq 8. \end{cases} \quad (\text{E.0.1})$$

We want to find E_0 such that

$$E_0 \left[\frac{\partial f_i^{eq}(1,0)}{\partial f_j} \right] = R \quad (\text{E.0.2})$$

where R is a symmetric matrix. The square matrix E_0 must be symmetric and positive definite. Choose

$$E_0 = \text{diag} \left[\left(\frac{9}{8+\lambda-12g-3\lambda g} \right), \left(\frac{18}{1-\lambda+3g+3\lambda g} \right) \mathbf{I}_4, \left(\frac{36}{\lambda-1+6g-3\lambda g} \right) \mathbf{I}_4 \right]. \quad (\text{E.0.3})$$

From the above choice of E_0 , we need to choose the parameters g and λ such that E_0 is positive definite. Substituting equation (E.0.3) into (E.0.2) gives

$$R = \gamma \xi^T \xi + E_0 \Psi_{ij}, \quad (\text{E.0.4})$$

where γ is a constant and Ψ_{ij} is an $(N \times N)$ matrix defined as follows

$$\Psi_{ij} = \begin{cases} \frac{(8 + \lambda - 12g - 3\lambda g)}{9}, & \text{for } i = 0 \\ \frac{(1 - \lambda + 3g + 3\lambda g)}{18}, & \text{for } i = \{1, \dots, 4\} \\ \frac{(\lambda - 1 + 6g - 3\lambda g)}{36}, & \text{for } i = \{5, \dots, 8\}. \end{cases}$$

For R to be symmetric, the following relationship is found using the second term in the above equation (4.41)

$$\frac{6}{1 + 3g - \lambda + 3\lambda g} = \frac{3}{-1 + 6g + \lambda - 3\lambda g}, \quad (\text{E.0.5})$$

giving the following

$$\lambda = 1.$$

For the stability structure [1] to hold, $g \in (0, \frac{3}{5})$.

Now take $g = 1$ and $\lambda = 1$, for example, and substitute these parameters into (E.0.1) giving

$$\frac{\partial f_i^{eq}(1, 0)}{\partial f_j} = \begin{cases} -\frac{2}{3}, & i = 0 \\ \frac{1}{3} + \frac{1}{3}\xi_j\xi_i, & 1 \leq i \leq 4 \\ \frac{1}{12} + \frac{1}{12}\xi_j\xi_i, & 5 \leq i \leq 8, \end{cases} \quad (\text{E.0.6})$$

(**NB:** the value of g is outside the required range.) The above equation (E.0.3) becomes

$$E_0 = 3 \text{diag} \left[-\frac{1}{2}, \mathbf{I}_4, 4\mathbf{I}_4 \right]. \quad (\text{E.0.7})$$

Since we need E_0 to be symmetric and positive definite, the above equation E.0.7 does not meet the requirements. Therefore, the stability structure 1 in Chapter 4 does not hold.

Bibliography

- [1] Banda M. K, Yong W. A and Klar A. *A stability notion for lattice Boltzmann equations*. Siam J. Sci. Compt. Vol 27 2006; **6**: 2098-2111.
- [2] Bhatnagar P, Gross E.P, and Krook M.K. *A model for collision processes in gases: I. small amplitude processes in charged and neutral one component system*. Phys. Rev., **94**: 511 - 525, 1954.
- [3] Bermudez A and Vázquez M.E. *Upwind methods for hyperbolic conservation laws with source terms*. Computers and Fluids; **23**: 1049 - 1071, 1994.
- [4] Borthwick A.G.L and Akponasa G.A. *Reservoir flow prediction by contra-variant shallow water equations*. J. Hydr. Eng. Div, ASCE, **123(5)**: 432-439, 1997.
- [5] Bouchut F. *Constructions of the BGK models with a family of kinetic entropies for a given system of conservation laws*. J. Stat. Phys, **95**: 113 - 170, 1999.
- [6] Casulli V. *Semi-implicit finite difference methods for two-dimensional shallow water equations*. Journal of Computational Physics, **86**: 56 - 74, 1990.
- [7] Cercignani C, Illner R and Pulverenti M. *The Mathematical Theory of Dilute Gases*, Appl. Math. Sci 1994. 106.
- [8] Chen J-Z and Shi Z-K. *Solutions of 2D shallow water equations by genuinely multidimensional semi-discrete central scheme*. Journal of Hydrodynamics, **18(4)**: 436-442, 2006.
- [9] Chen S and Doolen D.G. *Lattice Boltzmann method for fluid flows*. Annual review of fluid mechanics, **30**: 329 - 364, 1988.

-
- [10] Cooperstein B.N. *Elementary Linear Algebra: How to do it, Why it works*. University of California, Santa Cruz, 2004.
 - [11] Dellar P.J. *Non-hydrodynamic modes and a priori construction of shallow water lattice Boltzmann equations*. Physical Review E (Statistical Nonlinear Soft and Matter Physics) 2002; **65**: 036309
 - [12] Eberly D. *Stability Analysis for Systems of Differential Equations*. Geometric Tools, LLC, 1998.
 - [13] Falconer R.A. *An introduction to nearly-horizontal flows*. Chapman and Hall, London, 1993. In Coastal, Estuarial and Harbour Engineer's Refence Book. Editors: Abbott M. B and Price W. A.
 - [14] Featherstone R.E and Nalluri C. *Civil Engineering Hydraulics*. Blackwell Science, Oxford, 3rd edition, 1995.
 - [15] Frisch U, d'Humieres D, Hasslacher B, Lallemand P, Pomeau Y, and Rivet J.P. *Lattice gas hydrodynamics in two and three dimensions*. Complex Systems, **1**: 649 - 707, 1987.
 - [16] Frisch U, Hasslacher B and Pomeau Y. *Lattice-gas automata for the Navier-Stokes equation*. Phys.Rev. Lett, **56****14**: 1505 - 1508, 1986.
 - [17] Goutal N and Maurel F, Proceedings of the second Workshop on Dam-break Wave Simulations, HE-43/97/016/**B**
 - [18] Hardy J, O. de Pazzis, and Pomeau Y. *Molecular dynamics of a classical lattice gas: Transport properties and time correlation functions* . Phys.Rev., **A13**: 1949 - 1961, 1976.
 - [19] Harris S. *An Intro to the Theory of the Boltzmann Equation*. Holt, Rienhart and Winston Inc, New York, 1971.
 - [20] Higuera F.J and Jimenez J. *Boltzmann approach to lattice gas simulations*. Europhys. Lett., **9**(**7**): 663 - 668, 1989.

-
- [21] Jin S and Xin Z. *The relaxation schemes for systems of conservation laws in arbitrary space dimensions*. Comm. Pure Appl. Math. , **48**: 235 - 277, 1995.
- [22] Junk M and Yong W. A. *Rigorous Navier-Stokes limit of the lattice Boltzmann equation*. Asymptotic Anal: 2003; **35(2)**: 165-185.
- [23] Koelman J.M.V.A. *A simple lattice Boltzmann scheme for Navier-Stokes fluid flow*. Europhys lett., **15**: 603 - 607, 1991.
- [24] Kuipers J and Vreugdenhil C.B. *Calculations of two-dimensional horizontal flow, part 1*. Technical report, Delft Hydraulics Laboratory, The Netherlands, 1973.
- [25] Lallemand P and Luo L-S. *Theory of lattice Boltzmann method: Dispersion, dissipation, isotropy, Galilean invariance, and stability*. Phys. Rev, E; **61**: 6546-6562, 2000.
- [26] Viggen E.M. *The Lattice Boltzmann Method with Applications in Acoustics*. Thesis , <http://www.lbmethode.org/media/hosted-doc:viggen-masters-09.pdf>.
- [27] McGruirk J.J and Rodi W. *A depth-average mathematical model for the field of the near side discharges into open-channel flow*. J. Fluid Mech., **86**: 761 - 781, 1978.
- [28] Mei R, Luo L-S, Lallemand P and d’Humières D. *Consistent initial conditions for lattice Boltzmann simulations*. Computers and Fluids; **35**: 855 - 862, 2006.
- [29] Moslehian, Mohammad Sal; Rowland, Todd; and Weisstein, Eric W. *Projection Matrix*. From **MathWorld**—A Wolfram Web Resource. <http://mathworld.wolfram.com/ProjectionMatrix.html>
- [30] Noble D.R, Chen S, Georgiadis G,J, and Buckius R.O. *A consistent hydrodynamic boundary condition for the lattice Boltzmann method*. Physics of Fluids, **7**: 203 - 209, 1995.
- [31] Patankar S.V. *Recent development in computational heat transfer*. Heat transfer, **110**: 1037 - 1045, 1998.

-
- [32] Qian Y.H. *Lattice gas and lattice kinetic theory applied to the Navier-Stokes equations*. PhD thesis University of Marie Curie, Paris, 1990.
- [33] Rothman D.H and Zaleski S. *Lattice-Gas Cellular Automata*. Lattice-Gas Cellular Automata, 1997.
- [34] Salmon R. *The lattice Boltzmann method as a basis for ocean circulation modelling*, J. Marine Res. **57**: 503 - 535, 1999.
- [35] Salmon R. *Lattice Boltzmann solutions of the three-dimensional planetary geostrophic equations*, J. Marine Res. **57**, 847(1999).
- [36] Skordos P.A. *Initial and boundary conditions for the lattice Boltzmann Method*. Physical Review E **48**: 4823 - 4842, 1993.
- [37] Sokolnikoff I. S and Redheffer R. M. *Mathematics of Physics and Modern Engineering*. McGraw-Hill, New York, USA, 2 edition, 1966.
- [38] Spivak M. *Calculus*. Benjamin W. A Inc, New York, USA, 1967.
- [39] Stansby P.K and Zhou J.G. *Shallow-water flow solver with non-hydrostatic pressure: 2D vertical problems*. Int. J.Num. Fluids, 28:541-563, 1998.
- [40] Sterling J. D and Chen S. *Stability analysis of lattice Boltzmann methods*. J. Compt. Phys; **123**: 196-206, 1996.
- [41] Strang G. *Introduction to linear algebra, 3rd Edition*. Wellesley-Cambridge Press, 2003.
- [42] Toro E.F. *Riemann problems and the WAF method for solving 2-D shallow water equations*. Philosophical Transactions of the Royal Society of London, Series A; **338**: 43 - 68, 1992.
- [43] Toro E.F. *Riemann Solvers and Numerical Methods for Fluid Dynamics: A Practical Introduction*. Springer, Heidelberg, Germany, 2nd edition, 1999.

-
- [44] Vázquez-Cendón ME. *Improved treatment of source terms in the upwind schemes for shallow water equations in the channels with irregular geometry*. Journal of Computational Physics; **148**: 497 - 526, 1999.
- [45] Wolf-Gladrow D.A. *Lattice-Gas Cellular Automata and Lattice Boltzmann Models*. Springer, Verlag Berlin Heidelberg, Germany, 2000.
- [46] Yong W. A. *Basic aspects of hyperbolic relaxation systems*. Freistühler (ed) et al. Advanced in the Theory of shock waves. Boston, MA: *Birkhäuser*. Prog. Nonlinear Differ. Equ. Appl. **47** (2001), 259-305.
- [47] Yong A. W and Luo L. S. *Non-existence of H theorems for athermal lattice Boltzmann models with polynomial equilibria*. Phys. Rev. E 2003; **67**, 051105.
- [48] Zhang F. *Matrix Theory: Basic results and techniques*. Springer-Verlag, New York Inc, 1999.
- [49] Zhou J.G. *A lattice Boltzmann model for shallow water equations*. Comp. Meth. Appl. Mech. Eng., **191(32)**: 3527 - 3539, 2002.
- [50] Zhou J.G. *An elastic-collision scheme for lattice Boltzmann methods*. International Journal of Modern Physics C, **12(3)**: 387:401,2001.
- [51] Zhou J.G. *Lattice Boltzmann methods for shallow water flows*. Springer-Verlag, Berlin, Germany, 2004.
- [52] Zhou J.G. . *Velocity-depth coupling in shallow water flows*. J. Hydr.Eng., **121(10)**: 717-724,1995.
- [53] Zhong L, Feng S and Gao S. *Wind-driven ocean circulation in shallow water lattice Boltzmann model*. Advances in Atmospheric Sciences; **22**: 170 - 175, 2005.
- [54] Zou Q, Hou S, Chen S and Doolen G. *An improved incompressible lattice Boltzmann model for time-independent flows*. J. Stat. Phys; **81**: 35 - 49, 1995.
- [55] Zou Q, He X. *On pressure and velocity boundary condition for the lattice Boltzmann BGK model*. Physics of Fluids; **35(6)**: 1591-1598, 2002.