A DISCURSIVE ANALYSIS OF LEARNERS' MATHEMATICAL THINKING: THE CASE OF FUNCTIONS

A research report submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, in partial fulfilment of the requirements for the degree of Master of Science.

By

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MAY, 2014

DECLARATION

I declare that this research report is my own work, except as indicated in the acknowledgements, the text and the references. It is being submitted in fulfilment of the requirement for the degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other institution.

23 May 2014

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Signed　　　　　　　　　　　　　　　　　　　　　　Date

Lizeka Constance Gcasamba

DEDICATION

To my parents,

Tony Harold Gcasamba and Nolungile Sweetness Gcasamba.

ABSTRACT

This study addressed the inherent quandary of misconceptions that impacts on performance as reported in the National Diagnostic Report of the learner performance in the National Senior Certificate (NSC) 2011 examinations. In order to gain insights into learner performance, Sfard's (2008) commognitive framework was used as a theoretical lens to examine learners' mathematical thinking about functions. This study first described components of function concept from a constructivist perspective and further redescribed these in discursive terms i.e. from a focus on learners' use of terminology to words/word use; from representations to visual mediators; from competencies to routines; and from concept definition to endorsed narratives.

Data was collected through written tests and interviews of Grade 11 learners in one of the multilingual schools in one Province in South Africa. The research approach was first quantitative .Twenty six learners were given tasks involving functions that would highlight their errors. The study then moved to an interpretive qualitative approach based on Sfard's commognitive theory. The qualitative study had five participants. A multiple methods strategy of data collection was employed during this stage: learners' interview transcripts, written work and researcher's field notes.

The quantitative study confirms that learners were making errors on functions. The analysis of the qualitative study revealed that learners' discourse included a combination of colloquial and mathematical discourse as expected. Interestingly, while all the features of mathematical discourse were present in learners' mathematical discourse, their routines and words were linked to errors.

Keywords: Commognition, mathematical discourse, functions, errors

ACKOWLEDGEMENTS

Okokuqala, Qamata, iinceba zakho zimi ngonaphakade. I remain humbled to the divine intelligence which never relents in guiding the seeking mind. I owe my success to this study to ***uThixo*** who sustained me throughout this intellectual inquiry even when the odds were not in my favour.

I am indebted to my supervisor Professor Jill Adler whose expertise, humility and warmth made me feel safe throughout this journey. I would also like to thank her for her financial support towards my studies and for conference attendance. The conferences helped me to grow academically and interacted with other researchers. The time spent in these conferences has been absolutely worthwhile. I am blessed to have had the honour of working with such an exceptional person.

I am grateful to Dr Erlina Ronda for her intellectual contributions and for the time she invested in the supervision and reading of my work.

My profound gratitude also goes to Dr Essien for all his academic support and for his ongoing interest in my progress.

I would also like to thank the school at which the research was conducted, and learners who were willing participants.

I am eternally grateful to my children Nomzingisi and Hlumela for their patience while I was battling with the work and in the process having had to neglect them.

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Chapter 1 IDENTIFICATION OF THE RESEARCH PROBLEM

1.1 Problem statement

The performance of South African learners in schools is of great concern, especially in Mathematics. In the National Diagnostic Report of the learner performance on National Senior Certificate (NSC) 2011[1] examinations, it is noted that:

> *While some candidates performed excellently in this paper [paper 1 [2]], many performed poorly. Many of the errors made in answering this paper have their origins in a poor understanding of the basics and foundational competencies taught in the earlier grades....(DoE, 2012, p. 99)*

In the extract above, reference is made to errors and the fact that they originate from previous learning. For constructivists , the source of errors is misconceptions (Nesher, 1987; Olivier, 1996). Misconceptions result from the efforts learners make as they try to make sense of mathematics (Olivier, 1996). In other words, errors provide evidence that learners are thinking (Confrey & Smith, 1991; Smith, DiSessa, & Roschelle, 1993). According to Hufferd-Ackles, Fuson, and Gamoran Sherrin (2004), learners' thinking in mathematics is even more evident when the learners are encouraged to explain their thinking, defend and justify their mathematical ideas. They further suggest that learners' errors can be used as a source for mathematical ideas. This is supported by Ryan and Williams (2007), who propose that learners should be engaged in a discussion about their reasoning in relation to the errors they make in order to understand their thinking. Learners' explanations can be through any kind of communication, such as written or verbal forms. A study of learner errors then needs to be framed by a theory that links thinking and communication.

This study employs the communicational framework that relates thinking as a special case of an activity of communicating (Sfard, 2007b). In developing her theory of mathematical thinking and learning, Sfard (2008) identifies and describes five "quandaries" of mathematical thinking that persist, despite the long history of research in this area . The five quandaries include numerical thinking, abstraction (and transfer), misconceptions, learning disability and understanding. According to Sriraman (2009), Sfard's theory resolves these

[1] The empirical work for this study was carried out in 2012 and so the results that were relevant were the 2011 results.
[2] Paper 1 includes assessment questions on functions and algebra.

quandaries and further helps in explaining why learners have difficulty learning mathematics (Sriraman & Nardi, 2013). The quandary of interest in this study is misconceptions.

1.2 Focus of the study

Function has been identified as one of the concepts examined in maths at matric level where learners have displayed errors and misconceptions as reported in the National Diagnostic Report (DoE, 2012). In this document, it was reported that questions assessing the function concept were poorly answered. A number of reasons were documented: (i) learners could not interpret different representations of functions and failed to obtain information from the given graphs; (ii) learners were unable to convert flexibly between verbal, symbolic and graphical representations of the functions; (iii) learners demonstrated poor mathematical vocabulary; (iv) learners were struggling with algebraic calculations; and (v) learners did not understand the definition of function. It is suggested in this report that these difficulties resulted in errors, which later impacted on performance.

Learner difficulties with functions and their representations are not unique to South Africa. Others have reported learner difficulty in linking graphical and tabular forms of representations to algebraic forms of functions (Brenner et al., 1997). Furthermore, research indicates that these difficulties may lead to errors (Even, 1998; Ryan & Williams, 2007). Booth (1988, p. 20) argues that '*one way of trying to find out what makes algebra difficult is to identify the kinds of errors students commonly make in algebra and then to investigate the reasons for these errors*'. An investigation into learner strategies and related errors, when dealing with tasks related to functions, can illuminate learner difficulties and their thinking more generally. This, in turn, can provide insight into learner performance.

1.3 Purpose of study

The primary motivation for this study is the persistent difficulty that learners experience with the function concept. The following factors were the motivation to embark on this study: (a) The National Diagnostic Report (DoE, 2012) of the learner performance mentioned above; (b) my experience of working as a high school mathematics tutor in South Africa, where I saw most high school learners (Grades 11-12) experience difficulties when solving tasks on functions; (c) my knowledge of existing research related to learners' difficulties when dealing with the concept of function; (d) a pre-pilot study based on function tasks that I conducted at the start of this project where my focus was to investigate if, and then in what form, the errors suggested in literature were prevalent in learners' work. The investigation showed that the errors existed and some were similar to those suggested in literature, but they were more prolific and the learners were able to give explanations of their workings. The pre-pilot study also helped me to refine the research instruments, and details are provided in Chapter 4. Lastly, many studies have explored learners' thinking in mathematics in various ways. The approach in these studies towards errors and misconceptions has been informed largely by theories of cognition and particularly constructivism. I am interested in what a discursive approach to learners' thinking might bring.

1.4 Objectives

The study investigated Grade 11 learners' thinking when solving tasks on algebraic functions[3].

1.5 Critical questions

In order to gain deeper insights into learners' thinking about, and making sense of functions, this exploratory study will try to answer the following research questions:

1. What common errors do learners make when completing tasks involving algebraic functions?
2. What features of mathematical discourse (word use, routines, visual mediators and narratives) are evident in the learners' discourse?
3. In what ways, if at all, are these features linked to learners' errors?

1.6 Conclusion

This chapter has highlighted the vexing difficulties that have inhibited learner performance in mathematics assessments. I then pointed out a key element of poor performance, that is, errors. I went on further to discuss links between errors and thinking, as summarised in the diagram below:

[3] In this study, I use the descriptor algebraic functions to refer to all the families of functions in the grade 11 mathematics curriculum i.e. linear, quadratic, hyperbola and exponential functions.

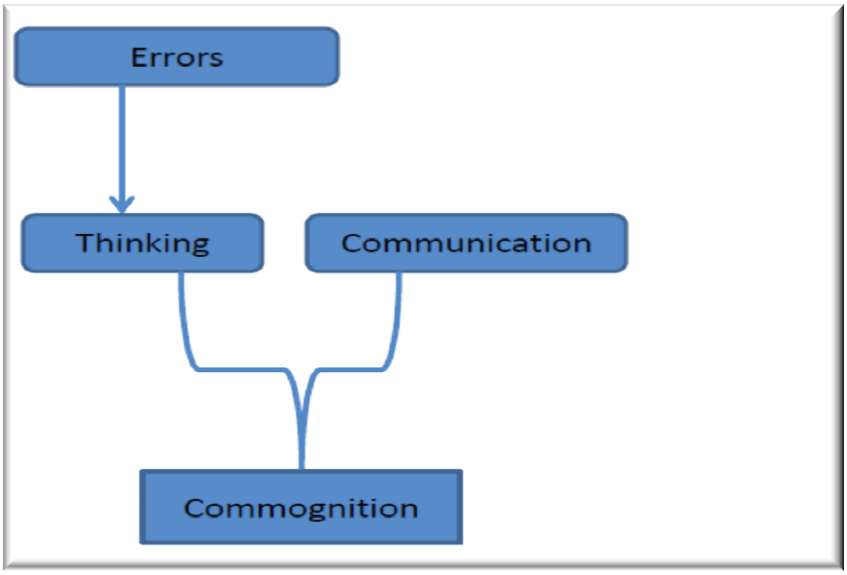


Figure 1:1 Link between errors and thinking

I then suggested a discursive approach to an investigation of learners' thinking. This study adopts Sfard's (2008) commognitive approach to thinking, which is explained in detail in Chapter 2 as a theoretical framing for the study. Link between errors and thinking

1.7 Outline of the research report

This chapter gave an overview of this study. In Chapter 2, I position this study within the commognitive [4] framework (Sfard, 2008). In addition, I describe the commognitive framework in detail, including a description of each of its four key mathematical discourse features. Chapter 3 outlines the literature reviewed that further locates and supports this study. The research design, research contexts and the method of data collection used in the study are described in Chapter 4. Chapter 5 addresses the first research question, by providing analyses and findings from the test used in this study. Chapters 6 and 7 deal with the analysis of the interview in terms of the commognitive framework. Finally, Chapter 8 provides a discussion of the findings of the interviews and Chapter 9 summarises the study's contribution to mathematics education research, its limitations and suggestions for future research.

[4] Although this framework has been described and used in other publications (e.g., Ben-Yehuda, Lavy, Linchevski, & Sfard, 2005; Sfard, 2007), Sfard (2008) will be used as the primary reference throughout this study because it represents Sfard's most elaborated rationale for and detailed description of commognition.

Chapter 2 THEORETICAL FRAMEWORK

2.1 Introduction

This study and its questions are framed by Sfard's (2008) theory of commognition. I thus start with a discussion of this theory. The purpose of an appropriate theoretical framework is that, it allows the study to be reformulated so that illuminating explanations and concepts can be brought to bear on the observations and the results of the study. A theoretical framework is also integral to the coherence of the data analysis process.

2.2 Defining commognition theory

Sfard (2008) defines commognition in terms of two key concepts: thinking and communication. She considers cognitive processes and interpersonal communication as facets of the same phenomenon. Within the commognitive framework, thinking is described as an individualised communication (communication with oneself). This individualised communication is referred to as interpersonal (Sfard, 2007b; Vygotsky, 1978) and it does not have to be audible or verbal. It is dialogical in nature and understood as a conversation with oneself. Sfard (2007b) adds that this interpersonal communication involves an action of having conversations with oneself. In colloquial talk it can be expressed as '*communicating one's thoughts*' or '*putting thoughts in words*' (Sfard, 2006, p. 9).

Forms of communication include written language, spoken language, physical objects and artefacts used for discursive purposes. In education studies, different types of communication that bring people together while at the same time excluding others are referred to as discourses. Sfard (2008, p. 93) refers to the term discourse as '*different types of communication set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors'*. In other words, a discourse is characterised by keywords or vocabulary and the way these keywords are used; mediating tools that are visual devices that people use to help themselves while communicating; and by a form of repetitive actions which are rule-regulated in terms of that discourse.

According to Sfard (2008), any discourse has colloquial and literate parts, whether these are in the physical sciences, languages, or social sciences. She distinguishes between two types of discourses as follows:

Colloquial discourses[5]

Colloquial discourses are those that develop from everyday talk. They are non-specialised everyday discourses (Sfard, 2008). They are sometimes referred to as spontaneous (Vygotsky, 1978), because they are common and familiar (D. Kotsopoulos, 2007) and these develop through experience or result from repetitive actions (Sfard, 2007b).

Colloquial discourses are mediated by: material objects in our everyday lives that can be easily scanned with our eyes; and images of concrete objects that can be seen and also physically manipulated for demonstration purposes (Sfard, 2007b). For example, in mathematics we can use rulers and/or gestures with our hands to show slope.

The words that are used in a colloquial discourse are those used in everyday language. Colloquial words can be found in mathematical discourses, but are used with different meanings. Such words are often referred to as words with multiple meanings (Zevenbergen, 2000). For example, the word 'function' has colloquial meanings (e.g. a *gathering* or a *role*) and literate meanings in mathematics discourse (e.g. *its formal definition*). Patkin (2011) and D. Kotsopoulos (2007) argue that, when learners bring colloquial words into mathematics communication, they sometimes make errors, suggesting that colloquial discourse may be 'harmful' in mathematical learning. This view has been argued against by, for example, Pirie (1998), who suggested that colloquial discourse can be used to produce legitimate mathematical answers. Others who supported this argument have argued that colloquial discourse can be a resource for conceptual development from children's prior knowledge (Adams, 2003; Moschkovich, 2003; Vygotsky, 1978).

In the light of the above, the debate about colloquial discourse is not resolved. Some think it is a problem, whilst others think it is not. For me, it is going to be interesting for my study to see how the learners use colloquial and literate discourses.

[5] The learners' colloquial discourse does not necessarily refer to everyday use of concrete objects or everyday language. It (colloquial discourse) is referring to mathematical ideas expressed more informally.

Literate discourses

Literate discourses refer to specialised discourses. School discourses like Biology discourse, Physics discourse, English literature discourse and Mathematics discourse are literate discourses (Sfard, 2008). Unlike spontaneously acquired colloquial (everyday) discourses, literate (school) discourses require deliberate teaching (Sfard & Cole, 2002). The literate discourse of interest in this study is the school mathematics discourse.

A key feature that has emerged in this discussion is that any discourse (including mathematics) has both colloquial and literate parts to it. Sfard links these parts through mathematical learning and notes that '*mathematical discourse learned in school is a modification of children's everyday discourses, learning mathematics may be seen as transforming these spontaneously learned colloquial discourses…*' (Sfard, 2007b, p. 573). From a commognitive perspective, mathematical learning is a change in participation in mathematical discourse. In other words, learning in mathematics means modifying one's present discourse so that it resembles the properties of the discourse practised by the mathematical community (Sfard, 2007a).

In Sfard's terms, the mathematical discourse develops from the colloquial discourse, which is an important starting point. To develop mathematical discourse requires a fundamental change in the discourse practices. Thus, investigating how learners modify their everyday talk towards that of literate discourse could help to gain insights into how they learn mathematics.

2.3 Mathematical discourse

Sfard (2008), presents four features of mathematical discourse: word use, visual mediators, routines and narratives. Following is a description of each of these features.

2.3.1 Word use

One defining feature of a discourse is its words (vocabulary). A discourse counts as mathematical if it features mathematical words. In mathematics, vocabulary refers to words that pertain to quantities or shapes. In contrast to colloquial discourse, words in mathematics are highly specialised. There are many words that learners meet in a mathematics classroom context that also appear in non-specialised colloquial discourse (e.g. the word '*function*'), but these words take on different meanings when used in mathematical discourse.

The meanings of mathematical words are generally used and shared by participants within a mathematics discourse (Sfard, 2007b). For example the word *parabola* signifies a graphical representation of a quadratic function and this meaning is shared amongst participants in the mathematical (functions) discourse. In contrast, the word 'root' is used in a number of ways in everyday discourse, but has a literate or specialised meaning in mathematics and in function discourse, a meaning shared again within mathematics discourse practice. Of particular interest are the ways in which the words are used. Word use is an all-important matter because, '*being tantamount to what others call word meaning, it is responsible for how the user sees the world, and it is one of the distinctive characteristics of discourses*' (Sfard, 2005, p.245). In particular, a learner's word use distinguishes different discourses, which is crucial in this study.

Sfard (2008) categorises word use into a four stage model[6] such as passive driven, routine driven, phrase driven and object driven.

During the passive driven stage, an individual is first introduced to the word and cannot contribute to the conversation. For example, in the function discourse, learners name the graphs or equations based on their appearance. The process of naming the graph is an act of matching the graph with a given name. When a learner is asked for verification of why such a graph is called for example 'parabola', the course of action includes direct recognition that is self-evident. Some may use rote memorisation such as '*I learnt it at school*'.

According to Sfard (2008), routine driven word use is the early stage of word use development where learners are action-oriented and their word use is driven by routine procedures. In the routine driven stage an individual uses the word within particular discursive routines which he/she associates with tasks featuring the new word. For example, in a function discourse, word use is routine driven when naming of a graph involves not just matching a graph with a name, but referring to it with a common descriptive narrative according to some visual properties. When a learner is asked to give an explanation of why a given graph is called a 'parabola', the course of action includes direct recognition, scanning or interpretation of visual properties of that graph. A possible response would be '*it looks like*

[6] Sfard's (2008) four stage model of the development of word use, helps to gain insights into how the word use develops over time. However, in this study, learners only agreed to be interviewed once, so the focus is on description rather than development of their word use.

it has two intercepts' or using colloquial descriptions found in school mathematics discourse such as '*it is a frowning/smiling face*'.

During the phrase driven stage, an individual uses the word more flexibly with a limited number of phrases (or formal definitions).

In the final object driven stage of word use, the word is used as if it has a life of its own. The object driven use is characterised by the objectification of the word (i.e. using it as a noun). In this stage word use is driven by definition (i.e. endorsed narratives). The naming of the graph depends on its visual properties and common descriptive narrative accompanying the name of the graph (i.e. definition of the parabola). When the learner is asked why the graph is called a 'parabola', the course of action is to check the defining conditions of the graph by interpreting the global features of the graph. A possible response is that '*the graph has a one turning point (local maxima/minima), axis of symmetry*'.

An important feature of word use in mathematical discourse is objectification, which occurs through reification (replacing the talk about processes and actions with talk about objects), and alienation ('*using discursive forms that present phenomena in an impersonal way'* (Sfard, 2008, p. 295)). That is, as if they were occurring by themselves, without the participation of human beings. In her earlier work, Sfard (1991; 1992) elaborated on reification as the transition from operational to structural modes of thinking. Where the structural mode of thinking treats *'mathematical notions as if they referred to object-like entities'* (Sfard 1992, p. 60), the operational mode of thinking (processual) *'speaks about processes, algorithms and actions rather than about objects'* (Sfard, 1991, p. 4). For any mathematical object, such as function, these processes are often blended together to help the learner create the meaning and to implement the objectification process. Sfard (2008) grounds her theory in an assumption that learning mathematics is an activity of objectifying. She further argues that the change in discourse, which shows learning, is a transition from non-objectified speaking to objectified speaking.

In this study the interpretation of learners' general word use will be done in terms of descriptive categories of uses of words such as passive driven, routine driven, phrase driven and object driven. This study will further make a conclusion about learners' word use on function discourse in terms of the objectification. The objectification will be conceptualised in terms of four indicators: a combination of the processual and structural modes of thinking (Sfard 1992), flexibility in switching between different representations of the function (Nachlieli & Tabach, 2012), that is *mediational flexibility* (Ben-Yehuda, Lavy, Linchevski, & Sfard 2005, p. 203) and the ability to view different function competencies, that is, multiple routines (Ben-Yehuda et al., 2005). It is noteworthy that this discussion goes beyond word use, it is going into other features of mathematical discourse (such as routines and visual mediators) that are going to be discussed later, and it is inherent that word use involves the other features of the discourse.

Sfard (2008) refers to word use as discursive use. Where '*discursive use, in turn, means the totality of proper combinations in which the word may appear.*' (Sfard & Lavie, 2005, p. 247). These combinations include discursive routines, visual mediators and narratives.

Sfard went further and married together word and its word use in the discourse to refer to a '*concept*'[7], by operationalising the term concept to refer to a word together with its discursive uses by drawing from Vygotsky's (1987) and Wittgenstein's (1953) works.

While word use is a most important feature of mathematical discourse, in this study I am going to refer to word/word use because I am going to make a distinction between words, word & word use and word use later.

2.3.2 Visual mediators

Visual mediators are '*the providers of the images with which discussants identify the object of their talk and coordinate their communication*.'(Sfard, 2008, p. 147). In other words, visual mediators enable participants in a discourse to identify visually the objects of their discussion. This enhances mathematical communication. Visual mediators are visible objects

7 A word 'concept' appears in different forms in Sfard (2008). First a concept as word together with word use. Second, mathematical concept as an object. Lastly, formal concept definition as endorsed narrative. In this study, I had a challenge of how I use the word 'concept'. Thus, I have decided to differentiate between the concept, concept definition and formal definition of a concept. When I am referring to a concept, I am referring to a function object (e.g. intercept, gradient). In a mathematical discourse, a mathematical object constitutes "this thing" that we discuss. Sfard (2008) refers to a word 'concept' as a word together with its discursive uses. I would like to make an amendment and be more specific than Sfard and relate to the term 'concept' to learners' description or definition of concept definition, following Tall's and Vinner's (1981) lead and substitute Sfard's reference to 'concept' with the reference 'concept definition'. Thus, in this study concept definition refers to word & word use, i.e. ways in which learners define or describe a concept (object). A formal definition refers to ways in which experts define the concept, i.e. endorsed narrative (e.g. a definition of an intercept: '*the x-intercept is the point where the graph cuts the y-axis. The y intercept is the point where the graph cuts the x-axis*.' (Campbell & McPetrie, 2012, p. 375)).

that may be created or operated upon as a part of the discursive process (Sfard, 2008). For example they may be drawn, built from matchsticks or operated upon in the mind. The most common examples of visual mediators include algebraic symbols, tables, formulae, graphs, drawings, diagrams and numbers.

Sfard (2008) further suggests that visual mediators are important in establishing effective communication in that they help to create a common focal point. Tabach and Nachlieli (2011) argues that visual mediators used in communication often influence one's ideas about what is discussed, as well as the chosen discursive actions. To illustrate, when a learner is asked to determine the intercepts of a given algebraic symbolic function, the mediator that the learner chooses and uses for this task (e.g. a table of values or a graph) often dictates how the learner will complete the task or, in Sfard's terms, the *routine* chosen. For instance if a graph is chosen, it means a constructing (drawing/sketching) routine should be performed.

This discussion highlights the important role the visual mediators play in the discourse and how they are interrelated with other features of the mathematical discourse. Hopefully this discussion will also help me when attempting to answer the second research question of this study which is enquiring about features of mathematical discourse (word use, routines, visual mediators and narratives) that are evident in the learners' discourse.

Sfard (2008) proposes three categories of visual mediators: symbolic (e.g. symbolic expressions), iconic (e.g. pictures, graphs) and concrete (e.g. rulers) mediators.

Symbolic mediators

Symbolic mediators may be scanned through or used in a syntactic way (Sfard, 2008). Scanning the symbolic mediator involves an act of interpreting the global properties of the mediator. For example, the equation: $y = 2x + 1$, can be interpreted as a linear function ($y = mx + c$) with the y intercept equal to 1 ($c = 1$) and with a gradient equal to 2 ($m = 2$).

Syntactic uses of the symbolic mediator involve attending to the numeric/algebraic symbols through calculation. In other words, when calculating, the symbols are scanned and replaced by other symbols in a uniquely defined way. For example, when given an equation $y = 2x + 1$, and asked to calculate y intercept, the equation itself can be a visual mediator that is first scanned through and then x is replaced by zero (applying a defined method called the intercept method). The equation is then simplified.

Iconic mediator

Iconic mediators are visual objects (e.g. graphs, diagrams or pictures) that can also be scanned with our own eyes. Furthermore, these can be constructed (e.g. sketching the graph, diagram or picture).

Concrete mediators

Concrete mediators are objects that can be physically seen, and manipulated, such as rulers or fingers when counting. They are mostly used in colloquial discourses. According to Sfard (2008), these concrete mediators do not have to be physical objects, they can also be imagined, that is, they are '..*seen and operated upon only with the interlocutors' mind's eye*'(Sfard, 2008, p. 148).

In summary, three types of visual mediators have been discussed. These visual mediators play a very important role in function discourse by highlighting the various facets of a function. According to Nachlieli and Tabach (2012), it is important that learners display mediational diversity in order to appreciate a concept of function and this notion has a positive correlation with objectification. And for my study, this notion (mediational diversity) is linked to objectified talk.

2.3.3 Routines

Routines are repetitive patterns which are characteristic of any given discourse. Specifically, mathematical regularities can be noticed whether one is watching out for the use of mathematical words and mediators, or following the process of creating and substantiating narratives. Routines offer valuable information about what learners do and say as a course of action to justify patterns in a function discourse. Discursive routines are associated with learners' creativities when dealing with function tasks, that is, competencies used in function discourse. In this study, I am going to regard competencies such as interpretations, classifications and calculations as repetitive discursive actions.

A routine is defined as '*a set of meta-rules that describe a repetitive discursive action*' (Sfard, 2008, p. 208). Further, these meta-discursive rules determine or constrain the "how" and the "when" of discursive procedures (Sfard, 2008). The 'how' of the routine is the set of meta-rules that determine the course of action (routine or procedure). The 'when' of a routine, *'is a collection of metarules that determine or constrain, those situations in which the discursant would deem this performance as appropriate'* (Sfard, 2008, p. 208).

Both the '*how'* and '*when'* of routines play a very crucial role in mathematics. For example, knowing when to perform a certain procedure is as important as knowing how to apply that procedure. The '*how*' of the routine is mostly practiced in school mathematics discourse. The '*when*' of the routine is closely related to knowing why the procedure or action is appropriate. According to Sfard (2008), understanding why a routine works is fundamental to assessing a situation in order to decide whether or not the routine is appropriate in a particular context. In other words this can be regarded as high order thinking.

In this study, the '*how*' and the '*when*' of the routine are conceptualized in terms of three descriptive categories (applicability, flexibility and corrigibility) to explore the learners' routines. These categories[8] are adapted from Ben-Yehuda et al. (2005).

Sfard (2008) distinguishes routines in terms of their goals and mathematical objects. She divides mathematical routines into three different categories: (1) explorations, (2) deeds, and (3) rituals. Explorations are those routines whose goal is the creation of endorsed narratives about mathematical objects. Rituals are those routines whose aim is to bring social rewards and mostly address others (i.e. a teacher) and deeds inflict changes in the environment. As the study unfolded and will be evident in Chapter seven, these categories were not useful. Hence I will be referring to routines in terms of patterned ways learners have to work when dealing with functions and also in terms of function competencies as described in literature and research (e.g. interpreting and calculating).

[8] These categories are compatible with the overall assumptions of commognitive framework and will be discussed in detail in Chapter 6.

2.3.4 Narratives

Narratives are descriptions or accounts of objects. It is any written or spoken text that is used within the discourse and can be subject to endorsement, i.e. narratives can be judged as true or false. Within the commognitive framework, truth is packaged in endorsed narratives (Sfard, 2008). In mathematics, the endorsed narratives are rules generally accepted by the mathematical community and narratives that become "mathematical facts". For example, axioms, definitions and theorems are all endorsed narratives. The statements; '*the x-intercept is the point where the graph cuts the y-axis. The y intercept is the point where the graph cuts the y-axis*' are an endorsed narrative of an intercept, defining what an intercept is mathematically.

In this study, the narratives will be those utterances produced by the learners when classifying and interpreting function objects, whereas the endorsed narratives will be the definitions of different function objects that learners encounter in their mathematical classroom discourse (i.e. endorsed in the school mathematics discourse)[9].

Mathematical discourse consists of construction, recall, and substantiation narratives (Sfard, 2008). *Constructions*: These are discursive procedures resulting in new narratives. Numerical or algebraic calculations are basic types of derivations, and if performed correctly they can count as substantiations. *Memorization or recall* is the process of summoning previously endorsed or substantiated narrative whenever necessary. *Substantiations*: Involve the actions through which we decide that a narrative can be endorsed. Substantiations can be induced through prompts and questioning during interview by asking questions like 'How do you know; How did you decide; Why is that the case...' In other words, these are the narratives that produce an answer to prompting questions. Sfard and Lavie (2005) refer to these as justification to the answers of a question 'why'.

Substantiations address the ways in which decisions are made, this can be through calculations (constructions), recalling previously endorsed or substantiated narratives. Given this, it seems appropriate to examine learners' substantiating narratives[10].

[9] Endorsed narratives from the school mathematics discourse are discussed in detail in Chapter 7.
[10] More justifications about this decision will be discussed in detail in Chapter 6

To sum up, from the discussion in this section, it should be noted that the four categories of mathematics discourse (word use, visual mediator, routines and narratives) while analytically distinct, are interrelated. For example, a visual mediator (e.g. quadratic graph), has particular words associated with it (e.g. parabola), is used in routine ways (e.g. plotted/sketched) and can be described by the narrative (e.g. '*a u shaped curve with one turning point with axis of symmetry'*). The interesting part of this study is the examination of each mathematical feature and their interrelatedness, and how this connectedness contributed to learners' substantiations and modifications of their colloquial discourse.

2.4 Existing literature on commognition theory

Since Sfard's commognitive theory is relatively recent, and still under development, only a few studies have utilized this framework , and those that do exist tend to focus on the mathematical learning of younger children (Sfard 2001; Sfard, 2007b; Sfard & Lavie, 2005) or on elementary mathematics, like arithmetic and early algebra (Ben-Yehuda et al., 2005; Caspi & Sfard, 2012). There has been little work that has been done on secondary school mathematics learning from a commognitive perspective. Some reported work has been done with middle grade learners: 7, 8 and 9 (e.g. Brodie & Berger, 2010a; Kotsopoulos, Lee, Heide, & Schell, 2009; Nachlieli & Tabach, 2012). There is also reported research on the function concept from commognitive standpoint that concentrated on teaching of mathematics at tertiary level (e.g. Kim, Sfards, & Ferrini-Mundy, 2005; Tabach & Nachlieli, 2011; Viirman, 2012). From this range of studies, the papers of direct interest for my study are those of (Ben-Yehuda et al., 2005; Brodie & Berger, 2010a; Caspi & Sfard, 2012; Kim et al., 2005; Kotsopoulos et al., 2009; Nachlieli & Tabach, 2012), as together might help me gain access to commognition research, as suggested by many reviewers of Sfard's work, that empirical work done on this framework proves helpful in making sense of the theory (Forman, 2012; Sriraman, 2009) .

Ben-Yehuda et al. (2005) study was organised around the question of the degree of objectification in learners' discourse and the way in which this feature was linked to learners' arithmetical proficiency. From their study, they were able to summarise features of mathematical discourse in learners' interviews with illustrative examples, and from this developed their model of Arithmetic Discourse Profile (ADP). Although Ben-Yehuda and others work is talking about arithmetic discourse, their work gives much more substance for me in terms of methodology. This model was recontextualised and reconceptualised for the analysis of the interview for my study. Full discussion of this model appears in Chapter 6.

Brodie and Berger (2010a) developed a framework for analysing errors made by learners using the commognitive framework. They developed categories by analysing the nature of learners' errors in Grade 9 multiple choice questions. Brodie and Berger (2010a) identified four different categories of errors: errors of routines, errors of visual mediators, errors of narratives and errors of signifiers. Their work provided this study with a starting point on how to view errors from commognitive perspective with respect to various categories of mathematical discourse.

Nachlieli and Tabach (2012), studied 7th grade learners' mathematical discourses as they were working with the function concept for the first time. The results of the study showed that learners were able to participate in function's discourse but relied on routines with discursive clues from previous learning, i.e. they relied on triggers. For example, learners did not use the word 'function' in their conversations; instead relied on visual cues such as equations, graphs and lines, which were introduced to them from previous learning. These visual cues served to represent 'function'. What this suggests for my study with respect to word use is that word use in early stages can be passive driven (visual recognition or cues), and linked to previous learning. Furthermore, this paper suggests that the word use in function discourse is important for me to explore.

The aim of the paper by Kotsopoulos et al. (2009), was to investigate features of mathematical discourse (words, visual mediators, routines and endorse narratives) present in 8th grade learners' mathematics homework when working with integers. The results suggested that there may be a connection between routines and endorsed narratives. For example, the routines learners used when working with integers resulted in narratives endorsed in the mathematical community and this interaction helped learners to move forward in learning of integers. Furthermore, the disconnection between routines and endorsing narratives indicated some misconceptions or errors. Here too are pointers for my study: strong connections between routines and narratives. I will argue that these features are interconnected. I will be looking at how they (features) are interconnected and whether a disconnection between the two leads to errors or not as suggested by Kotsopoulos and others.

Kim et al. (2005), investigated the nature of tertiary students' discourse (colloquial, literate) when dealing with the concepts of infinity and limit. The results indicated that students were using colloquial discourse when defining the concept of 'infinity' and 'limit', and that this colloquial discourse did have an impact on mathematical discourse. Evidence of the impact of colloquial discourse in mathematics learning can be found in the study by Caspi and Sfard (2012). They studied the meta-discourse of arithmetic of six pairs of 7^{th} grade students as they move towards the official algebraic form of talk, and found that students' colloquial talk was full of ambiguities. At the same time, however, this spontaneous talk (colloquial) displayed some algebra-like features that may not be normally found in everyday discourses. The shared assumption implicitly present in these two studies is that the development of mathematical discourse may be influenced by colloquial discourse. To put it in other ways, the interplay between these two discourses may be a very powerful factor that moulds the development of mathematical discourse.

2.5 Conclusion

In this chapter I have described key tenets of Sfard's theory of commognition, particularly those that are directly pertinent to my study. I also reviewed research that has been done using this framework. I have focused in particular on special features of mathematical discourse: word use, routines, visual mediators and narratives. How does this discussion assist in taking my study forward, particularly in answering my research questions?

Firstly, is the importance of words and how they are used (word use). It has been reported in The National Diagnostic report (DoE, 2012), that learners' terminology deficiencies are barriers to their performance on function questions, what I would now refer to as their functions discourse.

When defining function object, words together with their discursive uses are important. Leinhardt, Zaslavsky, and Stein (1990), argue that learners make errors when they hold an inaccurate definition. Thus, I am going to be keeping a sharp eye on learners' words and word use. It has been suggested that any discourse has both colloquial and literate parts. In my study, therefore, I will investigate how learners' colloquial and literate word use relate to each other. If colloquial, is it linked to the errors they make? This will also help me to understand how word use is linked or connected to other discursive features in learners' talk. Can these various connections be linked to errors?

Secondly, is the importance of visual mediators. How is this discursive feature going to help me to answer my research questions? Again, from the National Diagnostic Report (DoE, 2012) it was reported that learners were experiencing difficulties working with different representations of functions (tables, algebraic equations, graphs), and converting between these representations that resulted in errors. I would now refer to these representations as visual mediators. Investigating learners' discursive moves when working with different mediators might help me to understand the nature of their difficulties. I might be able to answer the following questions: Which visual mediators are used by learners? How can these and their use be linked to their errors?

Thirdly, routines were also discussed. What learners do with functions, the routines they use, might help me understand the nature of their difficulties. Is it calculating, interpreting, plotting etc. and are these linked with their errors?

Finally, looking at learners narratives might help me understand how learners substantiate their discursive moves. What is the nature of their narratives? Are their substantiations endorsed in the mathematical community; in school mathematics; neither? How do these link with errors?

As a way of conclusion, this study will comment regarding general properties of learners' discourse. In this study general properties of the discourse will be conceptualised in terms of two aspects: objectification, relation between colloquial and literate discourse.

Chapter 3 LITERATURE REVIEW AND CONCEPTUAL FRAMEWORK

3.1 Introduction

In this chapter I review literature relating to functions on the one hand, and errors in mathematics learning on the other. This review informs the development of a conceptual framework that guides the study.

As will be evident in the discussion of the research literature below, much of the research on functions, as well as errors, predates Sfard's discursive approach to learning and thinking, including her own influential earlier work. I discuss this research in its own terms, and within what is mostly a constructivist paradigm, and by way of conclusion, consider this work from a discursive perspective.

3.2 The concept of function

The concept of function plays an important role throughout the mathematics curriculum. It is central to learners' ability to describe relationships of change between variables, explain parameter changes, and interpret and analyse graphs. Further, the function concept is one of the key concepts of mathematics which can easily be applied to real life situations. For example , function is an organizing idea in mathematics and science because its development allowed those in mathematics and science to solve previously untenable real life problems by representing an invariant relationship algebraically and graphically (Breidenbach, Dubinsky, Hawks, & Devilyna, 1992). There are different ways in which functions are defined and talked about. In this section, I am going to shape my discussion of research on function concept around the following themes:

- Historical evolution of the function concept.
- Function concept in mathematics education.
- Learners' conception of function concept.
- Functions and representations.
- Linking different representations.
- Functions in the curriculum.

I selected these six themes because of their emergence in literature as constructs around which a stable consensus seems to have developed regarding their importance for learners' understandings of function. When examined closely, these themes are highly related, but they nevertheless seem to provide a useful organization for entry into the issues of learning the concept of function. I will further conclude how these (themes) relate to each other and what they suggest for my study and its discursive approach.

3.2.1 Evolution of function concept

Development in algebra and geometry that took place over the centuries had a great influence on the current definition of function and the way functions are taught. This process began when Leibnitz first introduced the word function in a geometric context. This later was followed by Bernoulli in 1718, who proposed algebraic definitions expressed as equations of formula. Such a definition reads: A quantity composed in any manner of a variable and any constants (Kleiner 1989). There was no explanation offered by Bernoulli of what 'composed in any manner' meant. In 1748, Euler came up with a definition which identified function with an analytic expression[11]: '*A function of a variable quantity is an analytical expression composed in any manner from that variable quantity and numbers or constant quantities'* (Kleiner 1989, p. 284). The entire approach to functions during this time was algebraic with stress on algorithmic dependence between variables and the use of equations or formulas as representations.

The evolution of the function concept lasted for more than two centuries and can be described as a tug of war between geometric and algebraic approaches (Kleiner 1989). As a new version of the definition was introduced and discovered that, for example, its geometric definition falls short of expectations when it comes to the algebraic definition or the other way round, then that definition was rejected, and a new version reformulated. Definitions of functions evolved, each extending on the existing version until the emergence of the set theory and abstract algebra that resulted in a set theoretic definition that Bourbaki formulated in 1939. The definition reads:

> *Let E and F be two sets, which may or may not be distinct. A relation between a variable element x of E and a variable element y of F is called a functional relation in*

[11]The analytic expression involves the four operations, roots, exponentials, logarithms, trigonometry, and polynomials (Kleiner 1989).

*y if, for all $x \varepsilon E$, if there exists a unique $y \varepsilon F$ which is in the given relation with x. We give the name of function to the operation which in this way associates with every element $x \varepsilon E$ the element $y \varepsilon F$ which is in the given relation with x; y is said to be the value of the function at the element x, and the function is said to be determined by the given functional relation. Two equivalent functional relations determine the same function (*Kleiner *1989, p. 299).*

Bourbaki's definition has remained dominant in mathematics and has influenced the teaching and learning of functions at secondary schools. The idea that '*each x value has a unique y value where the set of x values is the domain and the set of y values is called the range*' is a typical informal definition similar to that of Bourbaki's definition of function. Most learners in South African secondary schools encounter this definition when they first encounter the notion of function. This is evident in the Curriculum Assessment Policy document (DoE, 2011a, p. 47):

Let A and B be non-empty sets. Any rule that assigns to each element $a \varepsilon A$ a corresponding, and uniquely determined, element $b \varepsilon B$, is a function from the set A to the set B . We commonly use a letter, such as f , to denote a function, and we write $b = f(a)$ to indicate that b is the unique element in B associated with the element a in A . We also use the notation $f : A \rightarrow B$ to emphasise that f is a function from the set A to the set B .

What can be learnt from the above discussion is that the development of the function concept has been a cyclic, prolonged process. The psychological emergence of algebraic concepts in learners seems to follow their historical evolution, and Nachlieli and Tabach (2012) stress that there is no reason to assume that those who learn functions will be spared the struggles similar to those faced by mathematicians in the past. Research in mathematics education provides much evidence for the considerable difficulty experienced by learners trying to learn the function concept. These difficulties will be discussed later in the section on 'difficulties with functions'. With respect to this study, South African learners are not immune to difficulties experienced by other learners elsewhere, and as discussed in Chapter one, this claim is substantiated by the report in the National Diagnostic Report (DoE, 2012).

3.2.2 Function concept (definition) from mathematics education

Functions are described as a unifying concept within mathematics education research (Even, 1990; Leinhardt et al., 1990). This means that functions form a single most important idea that can be found in many branches of mathematics such as algebra, trigonometry and calculus.

Within mathematics education research, the concept of function has been defined in various ways that resemble that of Dirichlet-Bourbaki definition. This is particularly noticeable in Vinner and Dreyfus (1989, p. 357) :'*Function is a correspondence between two non-empty sets that assigns to every element in the first set (the domain) exactly one element in the second set (the codomain)*'.

Many studies observed students' understanding of function by presenting questionnaires of various function situations for students to interpret (e.g. Breidenbach et al., 1992; Vinner, 1983; Vinner & Dreyfus, 1989). Typically students were asked to provide definitions for the term '*function*'. The definitions that students provided included complete and appropriate definitions for functions and partial definitions with missing parts, as well as some students being unable to provide a definition. However, what was interesting is that most students could provide a more advanced example of a function than a definition (Breidenbach et al., 1992; Vinner, 1983). Common aspects of students' definitions included the following: a function should be given by a single rule; the graph of a function should be continuous; a function should be one-to-one; function is an equation with two variables; function is a graph; function must include some algebraic formula : $f(x) = algebraic\ expression$, $y = algebraic\ expression$ (Clement, 2001).

These studies conclude that the notions that students hold about function impede students' ability to determine functionality except in very specific prototypical instances. Similar responses were recorded in the pre-pilot study I conducted at the inception of this study, for example learners said: a '*function is a graph'*; '*function is* $f(x)$'; '*function is a straight line'*; '*function is ordered pairs in Cartesian plane*(x, y)'. What emerges from research on students defining a function is that students do not refer to the formal definition, and rather define a concept of function through some of its properties, for example defining a quadratic function through the properties of its graph such as local maxima/minima, axis of symmetry etc. or by providing an example.

In summing up, the preceding discussion highlights a tension both in learners' learning and among researchers regarding what function is. And so in this study, I am going to pay attention to learners' descriptions or definitions of function objects.

In the next section, I now go on to discuss the conception of function from a psychological point of view.

3.2.3 Learners' conception of function concept

In her earlier work Sfard (1991) refers to learner understanding of the function concept in two fundamentally different ways: operationally when a mathematical concept is seen as a process and structurally when a mathematical concept is seen as an object. For example, the algebraic expression $(x - 3)$ can be seen as the process of subtracting 3 from the variable *x*. However, the algebraic expression $(x - 3)$ can be conceived of as an object because it is possible to perform actions on it and these actions transform it, like in the expression $2(x - 3) + 1$. It is well known that learners possess an operational conception of a function more strongly than an object conception of the function (Sfard, 1991). She further stresses that operational and structural conceptions of the same mathematical concept are not mutually exclusive but they are in fact complementary, and are often blended together to help the learner create meaning. The ability of learners to see a mathematical concept both as a process and as an object '*is indispensable for a deep understanding of mathematics, whatever the understanding of mathematics is*.' (Sfard, 1991, p. 5).

As a way of concluding, this study will comment on learners' mode of thinking and how this notion manifests in the way they interact with different features of mathematical discourse.

3.2.4 Function and its representations

According to Kleiner (1993), a function can be represented in various forms such as : formula, a rule, a correspondence, a relation between variables, a table of values, a graph mapping a transformation, an operation or a set of ordered pairs. DeMarois and Tall (1996) extend these categories of different representations by including the notational form (e.g. using standard function notation such as $f(x)$); a colloquial form (e.g. a learner's non-formal description of the function, such as "an input output machine". It may be spoken or written and includes metaphors or comparisons to objects and ideas that are located primarily outside formal mathematical talk); and symbolic form (e.g. using standard symbols to explicitly describe the function, such as $y = 3x + 2$). This discussion highlights the fact that function appears in different forms. A further illustration of this idea of different representations prevails in the current school curriculum where there are at least four representational systems used to study functions. These include tabular, graphical, algebraic and verbal representations. Each of the different representations provides insight into particular features of functions as highlighted by Friedlander and Tabach (2001, p. 2):

> ***Tabular*** *representations help in the exploration of co-variation between variables; and the creation of graphs;* ***algebraic*** *representations are powerful in that they provide the generalisation of the patterned relationship between variables;* ***graphical*** *representation is effective in providing a clear picture of the function, enabling its features (like intercepts) to be 'seen'; and* ***verbal*** *representation is usually linked to problem-posing and is also needed in the final interpretation of results obtained in the solution. Verbalising a given situation involves an ability to use words to accurately describe a formula, a graph or a table.*

Confrey and Smith (1991) note that different representations exhibit different properties. It is therefore important to know all different representations, because each representation emphasises and suppresses various aspects of a concept (Ainsworth, 2006). More simply, different representations emphasise various facets of a function as shown in table 3.1 below.

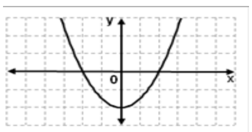
verbal	algebraic	Tabular	graphical
the square of a number	$y = x^2$	x y -2 4 -1 1 0 0 1 1 2 4	

Table 3:1 Different representations of the quadratic function

This table illustrates different representations of the quadratic function (i.e. verbal, algebraic, tabular and graphical). In order to have a comprehensive view of the quadratic function, learners need to learn to translate from one representation to another. As they move from one representation to another, they discover new aspects of the concept. Also, as they analyse the different representations, they stand a better chance to decide which representation provides better and more useful information. Essentially, they can see how these modes of representation enhance each other. In the same vein, Even (1998) argues that linking representations helps to develop generalised procedures that allow recognition of appearance of a representation in diverse forms.

Reviewed studies in this section have shown that knowledge of functions includes working in different representations. What also emerges is the important role played by these different representations, a role of being a mediating tool that can be created, and operated upon (or utilised) for the sake of communication. Sfard (2008) describes these tools as visual mediators. Drawing from Sfard's (2008) theoretical framework of mathematical discourse, the word 'visual mediator' is used in this study to mean representations. This study will investigate these visual mediators (tabular, graphical and algebraic).

3.2.5 Translating between different representations

In the previous section, an importance of linking or converting between different representations and working within each representation was highlighted. I now go on to discuss a recent framework for working with different representations. This section describes Even's (1998) model for approaching functions.

Even's framework

To work competently among different representations, Even (1998) proposed that one should think along two approaches: global and pointwise.

A global approach

According to Even (1998), a global approach entails classification of representations and interpretation of global properties. Global properties are general features of the representation such as the general shape of the graph; the behaviour of the graph; and the interval of increase or decrease (Janvier, 1981; Leinhardt et al., 1990).

One needs to be careful when using classification and interpretation, because sometimes it can be difficult to distinguish the two. Classification of representations involves the process of deciding whether a relation (graph, algebraic formula) is a function and recognizing a special kind of function among other functions (Leinhardt et al., 1990). This process involves interpreting the definition of a function and its special properties (Leinhardt et al., 1990). This classification depends on both formal definition and concept images that were developed from examples (Vinner, 1983). Interpreting representations involves making sense of the given graph or a functional equation in relation to the given context or situation and describing function or the relationship between two variables and their co-variation (Leinhardt et al., 1990).

A good example that illustrates global approach (classification and interpretation) can be seen in table 3.1 above. By scanning through these representations one can identify the graphs by interpreting the visual properties of the graph (i.e. the shape of the graph and the behaviour of the graph).

One needs to be careful when using classification and interpretation, because sometimes it can be difficult to distinguish the two. When classifying one needs to first interpret the global properties of a certain representation.

A point wise approach

Even (1998) describes a point wise approach as an interpretation of local properties (e.g. gradient, intercepts), reading discrete points point by point, construction of representations (graphs, table, algebraic); calculations and translating the graph.

Constructing involves an action of drawing or reproducing a graph in particular, by going through some procedures (Leinhardt et al., 1990), for example , plotting points from data or a table or a formula.

Translation is "*the psychological processes involved in going from one mode of representation to another; for example, from an equation to a graph*" (Janvier, 1987c, p. 27) as cited in (Brenner et al., 1997), this is sometimes reffered to as mode switching (Ben-Yehuda et al., 2005). In other words it involves an act of recognizing and matching the same function in different but equivalent representation (Leinhardt et al., 1990).

Even further adds that flexibility in using different approaches to functions guarantees flexibility in moving from one representation to another, and that both approaches are powerful and necessary in strengthening one's ability to solve problems. However, she is careful not to generalise this, because there are some instances where pointwise approach is more powerful than global approach and the other way round. For example, consider determining the y intercept of the following functions: (i) $y = (x + 2)^2 + 3$ and (ii) $y = x + 1$. In the second example, it would be easy enough to use a global approach, but would this approach (global) be effective when determining the y intercept in example (i) ?

What emerges from this section is that Even's approaches are associated with procedural skills (competencies), which consist of procedures that allow learners to connect different representations and to work within each representation.

The different competencies proposed in Even's model have been categorised as global and pointwise approches (Even,1998) as illustrated in the table below:

Global approach	Pointwise approach
Interpretation	Calculating
Classification	Translating
	Constructing

Table 3:2 Different function's competencies

When someone is engaging in a mathematical task in functions, patterns such as how one is carefully using mathematical words, or how one is following certain steps when substantiating narratives about function objects (e.g. graphs) can be observed. In fact, those repetitive patterns (routines) can be seen in almost all aspects of mathematical discourses (Sfard, 2008). In this study, the learners' repetitive patterns will be noticed through different competencies (global and pointwise) [12] such as interpreting, classifying, calculating, translating and constructing.

3.2.6 Functions in the curriculum

A curriculum framework usually outlines some operational standards that are viewed as necessary to make a particular concept meaningful to learners. Because the aim of this study was to investigate learners' mathematical thinking as they engage with the concept of

[12] In this study I am going to refer to global and pointwise approaches as 'competencies', which are redescribed in discursive terms as the 'routines'.

function in grade 11, the study considered the components of function studied at this level. The study of function is usually part of the study of algebra. In this section I will first discuss how functions are described in National Curriculum Statements (NCS) (DoE, 2003), which is now Curriculum Assessment Policy Statement (DoE, 2011a) and a Mathematics examination paper (DoE, 2011b).

The National Curriculum Statement

The National Curriculum Statement Grade 10-12 (DoE, 2003) policy document for mathematics stipulates four learning outcomes which indicate what learners are to achieve in each. According to Learning Outcome Two (LO2), stated as Functions and Algebra: '*The learner is able to investigate, analyse, describe and represent a wide range of functions and solve related problems*'.

The assessment standards which describe competencies for functional relationships for Grade 10 and Grade 11 are illustrated in table 3.3:

Description of curriculum element	Assessment Standards
1.Various types of functions	Demonstrate the ability to work with various types of functions, including those listed in the following Assessment Standard.
2.Different representations	Recognise relationships between variables in terms of numerical, graphical, verbal and symbolic representations and convert flexibly between these representations (tables, graphs, words and formulae).
3.Point-by-point plotting,	Generate as many graphs as necessary, initially by means of point-by-point plotting, supported by available technology.
4.Generalising effects of parameters	to make and test conjectures and hence to generalise the effects of the parameters a and q on the graphs of functions including: • $y = a^x + q$ • $y = ax^2 + q$ • $y = \frac{a}{x} + q$ • $y = ab^x + q;\ b > 0$
5.Properties of functions(graphs)	Identify characteristics as listed below and hence use applicable characteristics to sketch graphs of functions including those listed in 4 above (a) domain and range; (b) intercepts with the axes; (c) turning points, minima and maxima; (d) asymptotes; (e) shape and symmetry; (f) periodicity and amplitude; (g) average gradient (average rate of change) (h) intervals on which the function increases/decreases (i) the discrete or continuous nature of the graph
6.Calculations	Manipulate and solve algebraic expressions: (a) linear equations; (b) quadratic equations by factorisation; (c) exponential equations of the form $ka^x + p = m$
7.Function in context	Use mathematical models to investigate problems that arise in real-life contexts: (a) making conjectures, demonstrating and explaining their validity; (b) expressing and justifying mathematical generalisations of situations; (c) using the various representations to interpolate and extrapolate; (d) describing a situation by interpreting graphs, or drawing graphs from a description of a situation, with special description of a situation, with special focus on trends and features. Examples should include issues related to (health, social, economic, cultural, political and environmental matters.)

Table 3:3 Assessment standards for grade 10-11 (Adapted from the National Curriculum Statement grade 10-12 (DoE, 2003)

Different families of functions that are found in the curriculum include: quadratic, exponential, linear, and hyperbolic functions. The structure for studying each family of function is almost the same. It includes properties of function in different representations, translating between representations, investigating the effects of parameters on the graph of function, and applying function in real life situations.

The curriculum also emphasises the pointwise approach (Even, 1998), such as performing operations like drawing a graph (constructing) and manipulating algebraic expressions (calculating). Interpretation of global features of representations is also emphasized, for example, investigating the properties of graphs. A very important observation I noted while analysing this curriculum document (DoE, 2003), is that the formal definition of a function is only introduced in grade 12. I will come back to this point in the discussion chapter of this study.

Examination papers

Here are some examples of what is demanded in the grade 12[13] examination paper (DoE, 2011b).

QUESTION 5

Consider the function $f(x) = \dfrac{3}{x-1} - 2$.

5.1	Write down the equations of the asymptotes of f.	(2)
5.2	Calculate the intercepts of the graph of f with the axes.	(3)
5.3	Sketch the graph of f on DIAGRAM SHEET 1.	(3)
5.4	Write down the range of $y = -f(x)$.	(1)
5.5	Describe, in words, the transformation of f to g if $g(x) = \dfrac{-3}{x+1} - 2$.	(2)
		[11]

Example 3.1: Example from examination question paper

The use of both approaches is evident in the grade 12 Mathematics paper 1 (DoE, 2011b) , as illustrated in example 3.1. For example to answer 5.1 learners need to interpret global properties of the given equation in order to answer the question. To answer Question 5.2 and 5.3 learners need to calculate the local feature of the graph (i.e. intercept) and construct the graph respectively. Question 5.5 requires a translation of a given function.

[13] Note that Grade 12 exit examinations include Grade 11 work.

Common competencies that are highlighted in each of these two documents include the following: *translating* between representations; *interpreting* the different representations; *classifying*: Recognising/identifying the different properties or features of the function; *calculating* and *constructing.* These competencies are similar to those documented in mathematics education literature relevant to the concept of function (see previous section on Even's model).

Another important aspect that has emerged from these curriculum documents, is that of different families of function studied in grade 11 (e.g. linear, quadratic, exponential and hyperbola) and properties of function (e.g. intercepts, gradient etc.). Sfard (2008) classifies these as mathematical objects. According to Nachlieli and Tabach (2012, p. 11), '*Vygotsky's scientific concept translates into a formally defined mathematical object'*. Let me describe function *objects* in a bit more detail. In a mathematical discourse, a mathematical object constitutes "this thing" that we discuss. In this study, "this thing" very often is a function or different families of function (e.g. quadratic, linear etc.). Perhaps "this thing" also could be a property of a function (e.g., intercept, gradient, etc.). It is important to pay attention to the mathematical objects involved in a given discourse and thus for this study, how these too are talked about.

3.2.7 Summary of concept of function

In this section I have reviewed literature on function concept. The discussion in this section has helped me to put together the critical components that should be considered when one is trying to define functions and then redescribe these components in discursive terms.

These critical components include different domains of functions, properties of functions, different representations, and competencies. Table 3.4 gives the summary particular to grade 11 curriculum.

Components of function	**Examples**
Domains/families	Linear; Quadratic; Exponential; Hyperbola
Properties	Intercepts; Turning points; Notation; Gradient
Representations	Verbal; Table; Algebraic; Graphical
Competencies	Translating; Interpreting; Constructing; Calculating; Classifying

Table 3:4 Components of function concept

Although there are many families of functions (domains) that are required to be studied in the curriculum, this research will investigate the linear and quadratic functions, because of their prominence in secondary school. While investigating these two domains of function, focus will be on components of these two domains i.e. properties and representations, together with the competencies that are used when working with these two functions. According to Sierpinska (1994), these are critical components that should be considered when one is trying to define the function concept. Thus, I have linked these components with a discursive framework by translating them into discursive terms. The translation for these components is shown in Tables 3.5

Key components of function concept	Discursive terms
Representations	Visual mediators
Competencies	Routines
Properties	Objects
Different families/domains	Objects

Table 3:5 Discursive translation of function components

When these components are examined closely, they are highly related and seem to provide a useful organization for entry into the issues of learners' mathematical thinking with the concept of function. This study will be investigating the interaction between the learners and the function objects (intercept, gradient, linear and quadratic functions) by paying attention to learners' use of words, visual mediators used, and whether learners can move between these mediators through their routines and narratives.

Having discussed the components of function, it is time for me to talk about errors.

3.3 Errors

This study is aiming to investigate learners' errors with functions, and also seeks to explore the relationship between learners' mathematical discourses and errors. Before discussing errors with functions, it is important to explore research on errors in mathematics education more generally.

3.3.1 Errors in mathematics education research literature

Most of the work on errors adopts a constructivist perspective. From a constructivist perspective, learners do not make these errors because they do not know what to do but they make errors while making rational and meaningful efforts to cope with mathematics (Olivier, 1989; Ryan and Williams, 2007). Errors are systematic, persistent and pervasive patterns performed by learners (Nesher, 1987). Nesher (1987, p. 33) refers to a '*cluster of errors*' as misconceptions and further argues that errors arise within conceptual frameworks and are based on previously acquired knowledge.

Ryan and Williams (2007), highlight that errors are due to: modelling; prototyping; and over generalizing. Modelling refers to the way mathematics is connected to the real world. A modelling error is when a child has his/her own model of situation, in conflict with the mathematical model (experts). Prototype refers to a typical example of a concept, that is, something that serves to illustrate the typical class or model. For example, the rectangle is always seen lying flat with its base horizontal. Error as a result of overgeneralization arises when generalisations that make sense to a set of cases are inappropriately extended (Ryan & Williams, 2007). For example, a notion of '*multiplication makes bigger*' is appropriate for whole numbers and it becomes an overgeneralisation when applied to all rational numbers, thus including proper fractions or decimal fractions between 0 and 1.

In mathematics education, the discussion about errors and misconceptions is related to constructivist theory. From a constructivist approach, learners are actively involved in the process of thinking and learning, that is construction of knowledge; and in this process, error is inevitable.

Sfard's (2008) commognitive framework highlights the close relationship between thinking and communication. In all the papers I have reviewed above, none have studied errors on functions from a commognitive perspective. However Brodie and Berger (2010b), have worked on errors in general using the commognitive framework. First I want to look more carefully at errors on functions.

3.3.2 Learners' difficulties and errors with functions

Research in the 1980s and 1990s identified 'obstacles' that were seen as impeding learners' understanding of function concept. Some of these obstacles were viewed as conceptual in nature while others were not. Conceptual obstacles are those that are caused by conflicts between learners' concept image[14] and concept definition[15] (Vinner, 1983; Vinner & Dreyfus, 1989). Some difficulties are caused by misconceptions which may have been developed as a result of over-generalising an essential correct conception, or may be due to interferences from everyday knowledge (Leinhardt et al., 1990). It is important to note that when there is a difficulty, it does not necessarily mean that a misconception is the reason for the difficulty (Radatz, 1979). Rather, difficulties could imply that there is something about the task that makes it especially difficult (Leinhardt et al., 1990). It has been suggested by Booth (1988), that a productive way to investigate what makes a task difficult is to identify the common errors that learners' make.

Two of the most reported persistent difficulties in learners' understanding of function involve linking different representations and translating between different representations (Leinhardt et al., 1990; Sierpinska, 1994).

Leaners' difficulties with functions were also identified in several studies.Van der Meij and de Jong (2006) reported difficulties with properties of function such as finding the gradient, minima- maxima (turning points) and intercepts of the graph.

Even (1998), reported the difficulties with different competencies used in functions and categorised them into global and pointwise approaches. According to the report, over-emphasis on point-wise approach i.e. calculations and drawing graphs often leads to difficulties (Even, 1998). Others have reported difficulties with plotting of the graph (Janvier, 1981) and with manipulating algebraic expressions (Brenner et al., 1997). In the study conducted by Brenner et al. (1997), they reported that students, who relied too much on symbolic representation through calculation (pointwise), could not interpret the equivalent given representation (graph); they failed to make a connection between the two

[14] Concept image: The learners' concept image about the mathematical concept is the set of mental images, visual representations, or properties associated or related to a concept in the learner's mind.
[15] The learners' concept definition of mathematical concept is the definition that learners verbalise when they are asked to provide the definition of the mathematical concept

representations and therefore experienced difficulties in answering the question. It is also suggested by Even (1998) that when working with functions the use of both approaches is necessary. According to Even (1998), when a representation is attended to in a global approach literally without understanding the meaning that is conveyed by that representation, that sometimes creates problems with translation of functions from e.g. algebraic to a verbal description. For example, I observed learners recently when they were asked to classify a function y= $x/2+5$, they said it was a hyperbola, because of the fraction. This indicated that the interpretation was done literally without attending to the underlying properties of the equation.

Other studies of learners' difficulties with function concept (e.g. Breidenbach et al., 1992; Carlson, 1998; Confrey & Smith, 1991; Crawford & Scott, 2000; Leinhardt et al., 1990; Moschkovich, 1999a) offer insights into misconceptions and obstacles in understanding of function concept. Specific aspects such as the definition of a function; x intercept; notation; constant function; gradient; interpretation of graph as a picture; excessive adherence to linearity and calculations have been explored. These reports offer descriptions of learners' possible misconceptions. What follows is the brief description of each of these common threads.

Definition of function concept

According to Leinhardt et al. (1990), learners often make errors because the definition of the concept and concept image are inaccurate. They further argue that an incorrect function definition can lead to incorrect classification of a function. Typically, learners do not consult the formal definition of a concept when presented with an unfamiliar function (Vinner, 1983).

Learners' lack of understanding of a definition that lead to conflicts between students' images and their concept definition was observed in the prepilot study I conducted at the inception of this study. Learners were asked to identify the graph (in figure 3.1), and state whether the graph represented a function or not.

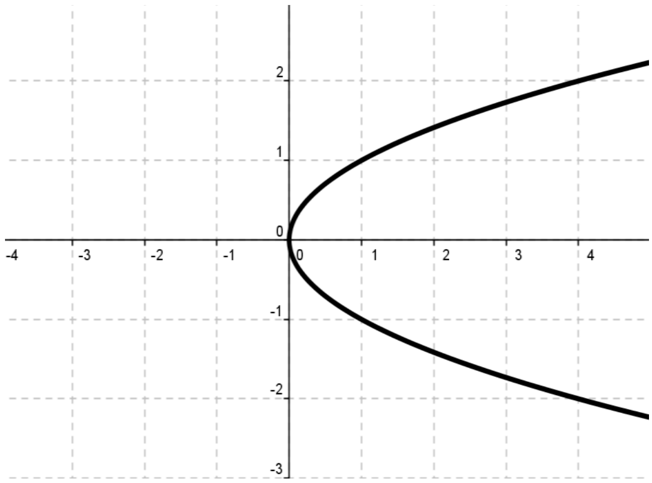


Figure 3:1 Example from prepilot study

In this example the graph was categorised as a function and a special kind of function called quadratic function. This kind of thinking could be attributed to prototypical thinking. In an investigation carried out by Tall and Bakar (1992), they reported that when the function was given in an unfamiliar way learners often classified the functions incorrectly. According to Tall and Bakar (1992), learners develop 'prototype examples' of a function or concept image, to serve as a reminder when faced with a function which looks like a prototype . For example the prototypes $y = x^2$, $y = x + c$ and $y = 1/x$, serve as cues to remember a quadratic, linear and hyperbola functions, respectively. The first sign that learners experience when working with quadratic functions is $f\ (x) = x^2$. When a different form of equation is given: $f(x) = x^2 + 9$, the interpretation is linked to the parent graph $f(x) = x^2$.

Words

The concept of function extensively borrows words in everyday language, such as, function, slope, increasing/ decreasing, limit, input/output and these can cause confusion which can lead to the formation of misconceptions (D. Kotsopoulos, 2007), because the meaning of words are different in mathematical contexts compared to their common usage (Zevenbergen, 2000). This difficulty has been widely documented especially with the use of words with multiple meaning (Adams, 2003; D. Kotsopoulos, 2007; Zevenbergen, 2000). It is clear that the term 'function' has a range of meanings in everyday speech, differing from culture to culture, often not compatible with and at any rate much less precise than a mathematical definition. Examples of these include function as purpose: "The function of the brake is to stop the car", function as event: "The function to celebrate the school's sporting victory will be held on Wednesday", function as role: "His function on the committee was to take notes", and function as mechanism: "The function of the switch is to turn on the light." Such everyday meanings of the term may impede students' development of the specific mathematical meanings.

Language can thus cause confusion which can lead to the formation of misconceptions when the meaning of words are different in mathematical contexts compared to their common usage (Dias, 2000). Hart (1981) suggests that language used in mathematics lessons is often technical and differs from learners' regular vocabulary and often needs to be redefined. For example, Hart (1981), identifies the term 'straight' as a problem for certain younger learners, a line cannot be straight if it is slanting because straight is defined as being perpendicular to the edge of the page.

***x* intercept**

Moschkovich (1999a) reported a more specific study on learners' difficulties with the x intercept. The study reported misconceptions students had with the x intercept in the domain of linear function. Students interpreted the x intercept in the equation $y = mx + c$ as either m or c in the equation. Moschkovich noted that students expected the x intercept to appear in the equation because on the graph it is as salient as the y intercept.

Notation

When learners are introduced to graphs , they are typically taught to see relationships with respect to the x and y, but when functional notation, $f(x)$, is introduced to replace y, learners experience difficulties (Van Dyke & White, 2004). Learners do not understand what $(x;\ f(x))$ represents as they are used to the numerical representation of $(1;\ -4)$. In the Park City Math Institute report, it is warned that learners who understand only one form of notation are likely to experience difficulties when dealing with a concept of functions (PCMI, 2009).

Constant functions

Confrey and Smith (1991) examined students' difficulties with constant functions. According to the report, constant functions are seen as '*monster functions'*, because one variable is missing. Leinhardt et al. (1990) add that functions that look strange or unfamiliar are often classified as non-functions. Consistent results were observed in the pre-pilot study, where learners were required to draw a function of $x = 2$. One of the learners drew a point as illustrated in figure 3.2 below. This is an indication of lack of understanding of a function as a co-variation between two variables x and y (Confrey & Smith, 1991).

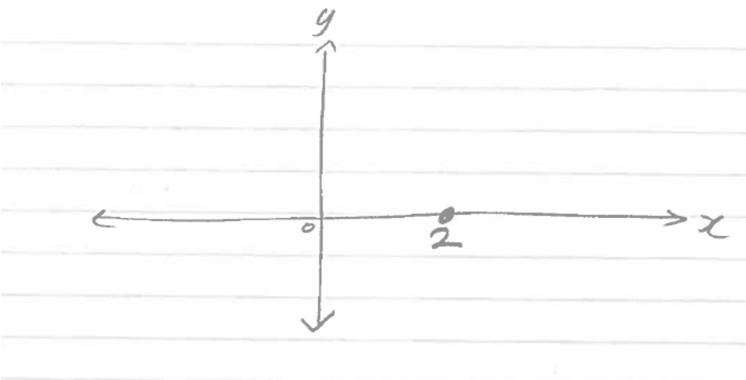


Figure 3:2 Illustration of a constant function error

Gradient/slope

Crawford and Scott (2000) examined the concept of gradient through real world applications. They found that learners could calculate gradient easily enough with the given equation but seldom understood the concept of gradient. Those who gave gradient definition mostly described it in colloquial ways. For Haapasalo (2003), difficulties with slope or gradient are rooted in previous learning. Haapasalo argues that when the gradient is first introduced in lower grades, it is introduced as a concrete slope, which means a slope is used as a general word and used in more colloquial ways. When the gradient/slope concept is introduced in more mathematical terms such as verbal, graphic or symbolic form, learners tend to experience difficulties.

Other difficulties and misconceptions with the concept of gradient have been reported to be associated with algebraic calculations (Barr, 1980) and with a notion of rate of change (Bell & Janvier, 1981; Stump, 1999).

Inability to interpret graphs (graph as a picture)

Graphical interpretation has posed challenges for many students and they have displayed numerous misconceptions. Research evidence relating to students confusing the physical aspect of lines on graphs depicting real-life situations was documented by Kieran (1993). This is similar to Janvier (1981) reference to research where students made literal interpretations of graphs such as positive gradients depicting uphill walks in a distance versus time graph. This is referred to as figurative association where the visual feature of the shape of the graph is related to the problem. Arcavi (2003) refers to this as "pictorial distraction" where visually salient information is interpreted and the underlying meanings are not considered. Clemet (1985) found that learners have an inability to see graphs as abstract representations of relationships but rather see graphs as literal pictures which are often in conflict with the correct interpretation and meaning of the graph. Hied, Zbiek, and Blume (2004) mention that learners sometimes see exponential functions as half a parabola; this observation comes from looking only at the shape rather than focusing on the important properties (gradient, intercepts, domain and range) of the graph.

Overgeneralising (Linearity)

Learners' tendencies of overgeneralising all functions to linear functions has been reported by many (Breidenbach et al., 1992; Brenner et al., 1997; Leinhardt et al., 1990; Tall & Bakar, 1992). Learners' first encounter with a function concept is through a linear function, and they tend to overgeneralise the properties they have learned in conjunction with linear functions (Leinhardt et al., 1990). Matz (1982) also identified the overgeneralisation of linear properties. Her work suggested that all errors on algebraic functions were based on properties of linear functions, which were applied in inappropriate contexts.

Algebraic calculations

Learners' misconceptions and errors when solving algebraic equations have been documented by many across different grade levels, such as Matz (1982). According to Radatz (1979) these errors are due to inadequate basic skills, for example, mathematical concepts and mathematical symbols; incorrect procedures when applying mathematical algorithms and; attempts at applying previously acquired knowledge to new situations which are irrelevant.

Construction (plotting) of graphs

Research studies confirm that learners lack graphing skills in mathematics and even in other subjects like physical science (Asli, 2001). This might be related to the order in which graphs are introduced to learners referred to as the translation process of going from one mode to another such as from table to graph sketching (Janvier, 1981). Van Doreen, De Bock, Janssens, and Verschaffel (2008) cite research examples when students often depict a fixation with the linearity concept. These include students drawing a straight line when asked to draw a graph of any function through two points and searching for straight line relationships when viewing a parabolic curve.

3.3.3 Summary of research on errors

In overview, the analysis of different studies on difficulties with function concept, shows that in mathematics education, difficulties on functions are linked to different competencies (e.g. interpreting, classifying ,calculating, constructing and translating); definition ; vocabulary; different properties of functions (x intercept and gradient) and different representations (e.g. graph, equation). These will be referred to as errors of competencies; errors of definition; errors of words; errors of properties and errors of representations respectively.

In this section I have discussed the considerable wealth of research on learners' difficulties and misconceptions with functions, most from a cognitive or constructivist perspective. Learners' difficulties and errors were considered in this study especially in designing the test questions (see chapter four) that would have the potential to draw out learners' errors as suggested by Booth (1988).

Errors provide evidence that learners are thinking (Olivier, 1996). Sfard (2008) proposed that thinking is communication. She further suggested that, in combating these errors, there is a need for a change in discourse. This suggests that investigating learners' discourse might help in understanding their thinking through their communication i.e. commognition theory. Thus, this study is going to look at learners' discourses on functions and try to see if there is any connection to the errors they make.

As it has been pointed out earlier, the research on errors from a commognitive perspective is new and under-developed , hence my reading of the literature refers to the work done from a constructivist perspective. However, Brodie and Berger (2010a) looked at errors in general from the commognition theory. The discussion on this follows, which leads to a conclusion where the conceptual framework for my study is presented.

3.4 Mathematical discourse and errors.

Sfard (2008) has retheorised her work on mathematical thinking and conceptualisation in terms of what she now calls a theory of commognition, which is based on the argument that thinking is communicating. From a commognitive perspective, errors occur when learners are participating in a different discourse from the teacher's (Sfard, 2008). That is, when learners are using different rules of a different discourse. Learners use different rules of a different discourse when they are not aware of the move needed for the new sub-discourse, and that the rules have changed. These sub-discourses are contained within the mathematics discourse; they relate to each other in different ways, some subsume others, for example a discourse on function subsumes discourses on graphs and algebraic expressions e.g. in the quadratic expression the coefficient of x^2 is positive, and in a parabola which is drawn concave up. These two can be replaced with a 'quadratic function with a minimum value (local minimum value-turning point)'.

According to Sfard, errors are narratives from the learners' perspectives which are different from those endorsed in the mathematical community. From Sfard's point of view, if we are going to interpret learner thinking more, it is not that these errors are inevitable or in need of a structural change in learners' schema, but rather, what is needed is a change in the discourse – how learners communicate functions. So, it is for this reason in my study that I will be examining errors, building on the work identified from a constructivist perspective, but through commognitive lens, that is, discourse. In this study, errors will be regarded as incorrect answers in the process of solving a mathematical problem algorithmically, procedurally or by any other method and also a deviation from what is endorsed in school mathematics discourse.

In the following sub sections, I will start by describing Brodie and Berger (2010a) model, then later redescribe the key errors that were discussed in the previous section in terms of commognitive framework.

3.4.1 Brodie and Berger (2010a) model

Brodie and Berger (2010a) developed a framework for analysing errors made by learners by drawing from Sfard's (2007, 2008) theory of mathematics as a discourse. They developed categories by analysing the nature of learners' errors in Grade 9 multiple choice ICAS[16] paper used in DIPIP[17] project. Brodie and Berger (2010a) identified four different categories of

[16] International Competitions and Assessments for schools in Australia

[17] Data Informed Practice Improvement project

errors: errors of routines, errors of visual mediators, errors of narratives and errors of signifiers.

Errors of visual mediators: According to Brodie and Berger (2010a) errors of visual mediators are caused by (a) inappropriate visual scanning that may result in the learner inferring relationships between the symbols without any appeal to the underlying mathematics; (b) inappropriate use of visual detail results: when at least one piece of information in the question is ignored when interpreting the question; and (c) difficulty with visual construction: when the learner is unable to construct a visual mediator such as drawing of the graph.

Errors of routines are caused by application of routines in an inappropriate situation-for example, use of incorrect algorithm.

Errors of narratives are referred to as narratives endorsed by the learners which are different from the mathematics community.

Errors of signifier result when learners apply incorrectly a familiar procedure in a new situation (Brodie & Berger, 2010a). For example a learner knows that in $y = x^2 + 2$ the y intercept is 2 when faced with a different function $y = (x + 1)^2 + 2$, they may interpret the y intercept as 2.

3.4.2 Summary of errors and mathematical discourse

Brodie's and Berger's work provided this study with a starting point on how to view errors from commognitive perspective with respect to various categories. However their work did not focus on function object. Therefore it does not provide the key aspects to consider when investigating the function discourse such as routines (constructing, interpreting), visual mediators (graphs, tables), and word/word use (words signifying function objects). In this study, I see myself as taking Brodie's and Berger's model, recontextualize it and extend its categories to include those discussed in the previous section (3.3.2).

In the previous section, key errors that emerged were summarised as errors related to competencies, representations, words; definitions and properties. These key aspects have been re-described in discursive terms (see Chapter 2). Competencies to routines; representations to visual mediators; properties to objects; words and concept definitions to words and words & word use. Let me speak about concept definition in more detail. A concept definition is a set of words used to specify a given concept (Tall & Vinner, 1981). Tall and Vinner (1981) distinguish between personal and formal concept definitions. Where personal concept definition refers to form of words learners use to describe or define a concept. And a formal concept definition as concept definition used by the experts. As mentioned in chapter 2 that in this study, the concept definition from learners' perspectives will be conceptualised as words and word use, and formal definition as endorsed narrative.

One of the questions this study seeks to answer is whether the learners' features of the mathematical discourse can be linked to errors they make. Brodie's and Berger's work inspired me to consider the possibility of connecting the errors from constructivist perspective (as discussed in section 3.3.3.) with discursive framework, and to translate these errors into discursive terms. The translation of the errors from constructivist perspective is shown in Table 3.6

Errors from cognitive/constructivist perspective	**Errors from commognitive perspective**
Errors of competencies	Errors of routines
Errors of representations	Errors of visual mediators
Errors of vocabulary	Errors of words
Errors of definition	Errors of word and word use
	Errors of (related to) endorsed narratives

Table 3:6 Discursive translation of errors

3.5 Conclusion

3.5.1 Conceptual framework

Miles and Huberman (1994, p. 18) defined a conceptual framework as a visual or written product, one that "*explains, either graphically or in narrative form, the main things to be studied—the key factors, concepts, or variables—and the presumed relationships among them*" (p. 18). The discussion that follows summarises the translation of key aspects of function and errors from constructivist to commognitive perspectives.

The literature reviewed show that function is a complex concept. The definition of function has evolved from the more intuitive dependence relationship between two quantities to the more abstract definition as a correspondence not only between numerical quantities but also between any two sets in general. The function concept is defined through the use of its terminology. This has been redefined as words and word use in commognition theory.

There are at least three representational modes used to present the concept of function in the grade 11 mathematics classroom: (algebraic, graphical and tabular forms). These have been redefined as visual mediators.

Being a concept, function has properties such as intercepts, gradient, turning points etc., and some familiarity with these properties is needed to appreciate the function concept. Thus, different approaches to functions were proposed by (Even, 1998). These different approaches include different competencies which can be classified as global and pointwise approaches; these have been redefined as routines. When learners are converting between these two approaches (e.g. moving from calculation to sketching a graph) and working within each approach (e.g. calculation) they experience difficulties. These difficulties may result in errors. These errors have been redefined as narratives endorsed by learners which are different to the mathematical community. Errors are evidence of learners' thinking. From commognitive perspective thinking is communication. A visual form that illustrates the connection of these key ideas is shown in the figure 3.3 below:

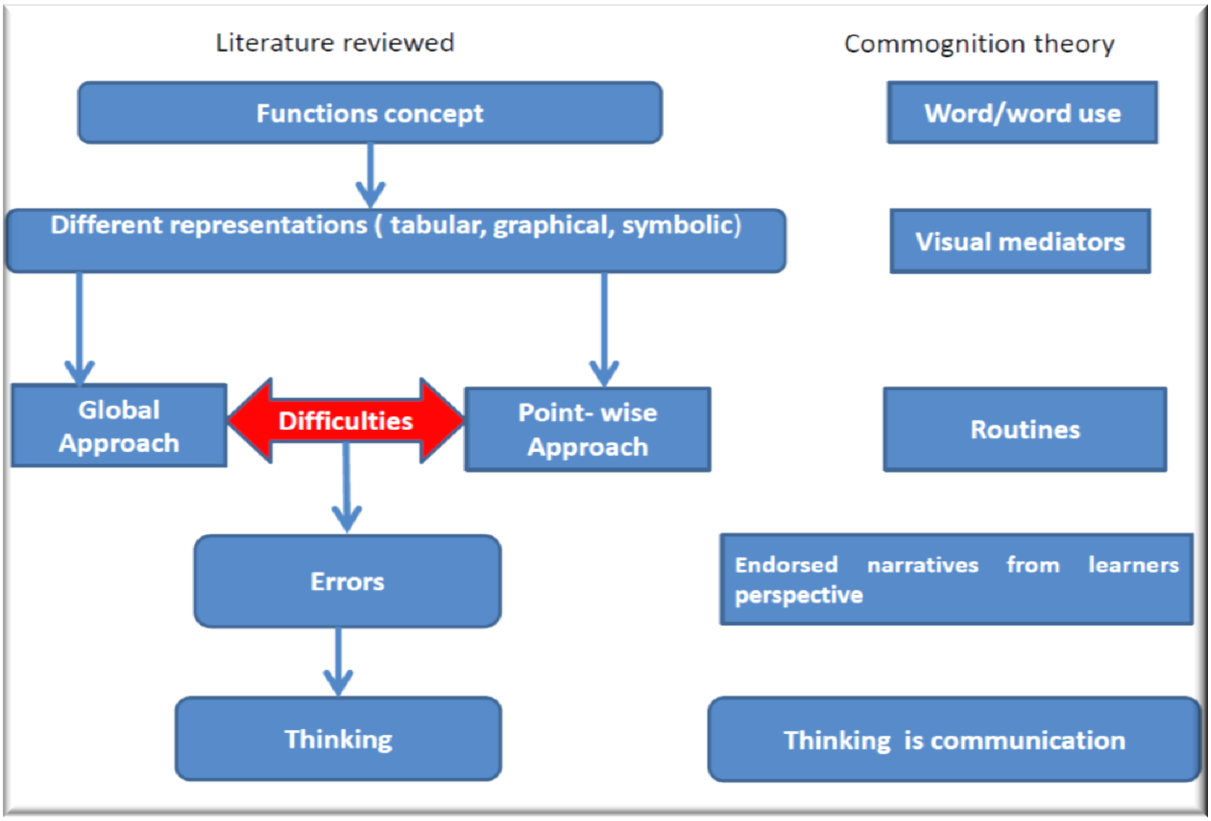


Figure 3:3 A conceptual map of the theoretical framework

The above figure illustrates the key concepts from the literature reviewed from the constructivist and commognitive perspectives.

So, how do these two orientations: constructivist and commognitive feature in my study? As told in the next chapter, I draw on constructivism for the first phase and then shift over to commognition. The review of literature from a constructivist point of view, served the purpose of formulating the design and the analysis of the test. The reviewed literature from the commognitive standpoint in Chapter 2 served to inform the construction of interview and in focus when analysing the interview. As will be evident, the errors evident in the test are analysed using the concepts on the left of framework. These are then reinterpreted for the interview and its analysis.

Chapter 4 RESEARCH DESIGN AND METHODOLOGY

4.1 Introduction

This chapter outlines the research methods for studying learners' errors when solving tasks on function and the mathematical discourses they use as they communicate their answers. The first section of this chapter provides a brief discussion on research methods employed in this study, highlighting the qualities of the interpretive qualitative research. I then introduce the commognitive approach to research and explain how it fits within an interpretive qualitative approach. Additional sections address the topics of participants, setting, data collection and instrumentation, data analysis and issues concerning rigour.

4.2 Research approach

There are two kinds of results in this study: results that indicate quantitative approach in relation to errors learners make when solving function tasks, and results in the form of analysis of the learners' discourse. The latter kind of result is dominant and one may therefore argue that this study belongs to the interpretive qualitative paradigm[18] (Ernest, 1998) , which is consistent with Sfard's (2008) commognitive approach to research. The quantitative data is not used to show statistically significant correlations, but is intended to help me identify general trends and patterns in the analysis.

I have further employed a sequential design which is characterized by the collection and analysis of quantitative data followed by the collection and analysis of qualitative data (Creswell, 2012). For example, in the quantitative phase I used a test instrument to identify and classify learners' errors and to guide the purposeful sampling of participants for a qualitative study. There are four main stages in the sequential study. The following schematic diagram shows these stages (figure 4.1).

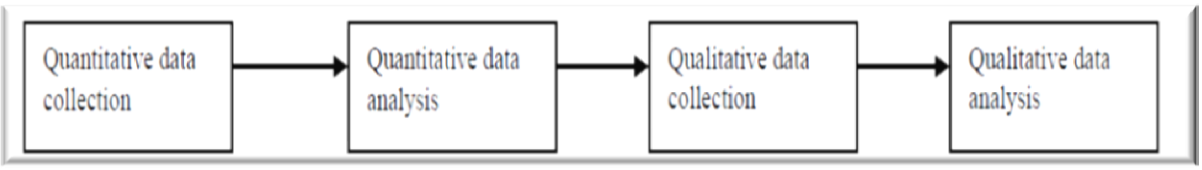


Figure 4:1 Schematic diagram representing the various stages of the design

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[18] The interpretive, the qualitative, research paradigm will be used interchangeably in the rest of this study.

Below, I will argue for the general theoretical perspective on knowledge and research of this study which is in tune with the interpretive qualitative research paradigm.

4.2.1 Interpretive qualitative paradigm

Qualitative research frequently utilizes observations and in-depth interviews. It involves descriptions in words, exploring to find what is significant in the situation. Qualitative research is more than a set of techniques and procedures. Creswell (1994, p. 2) defines a qualitative study as '*an inquiry process of understanding a social or human problem based on building a complex, holistic picture, formed with words, reporting in detail views of informants and conducted in a natural setting'*. It is a quest on how individuals construct meaning and sense of their lives (Hatch, 2002). Qualitative research is descriptive and focuses on meaning and explanations (McEwan & McEwan, 2003). Through qualitative methodology I intend to interpret learners' mathematical discourses on functions.

This interpretive standpoint serves to guide the study in providing a theoretical framework based on the interpretivist approach. Henning (2004) provides an epistemological basis (how we come to know about this world) of interpretivist philosophy that indicates that knowledge is constructed by describing peoples' actions, beliefs, values, understanding and construction of meaning. In this way, I endeavour to explore learners' discursive actions. Thick data is used mostly in qualitative research where the aim is to acquire as much meaning and interpretation of situations and contexts as possible (Henning, 2004). According to Maree (2007) , the interpretive approach encompasses a descriptive analysis with the aim of providing a deeper understanding of the social and human relationships. This is relevant to the commognition research.

Commognitive research

This study aims to interpret learners' substantiating narratives from a commognitive standpoint. Sfard (2007b) stated that any interpretive research should focus on both the 'what and how' of the human activity. Commognitive research is interpretive (Ryve, 2006) , and advocates the idea of learning as a social phenomenon of participating in the communicational activities of a distinct community. It is a discursive activity aiming at stories about the world with which to mediate and improve practice (Sfard 2012).

Sfard (2002, p. 32) is careful to point out, however, that what the best researchers can hope for is a "*convincing interpretation*" that is "*as compelling, cogent and trustworthy as possible*". Furthermore, we must regard the resulting interpretation as one of many, as a tentative and incomplete product. While the goals and interpretive stance of commognitive investigation are fairly well defined, efforts to build a strong research methodology to support this research framework are still developing. Nonetheless, Sfard (2002, p. 31) states:

> *It is clear that the proposed conceptualization of thinking implies a wide range of data-collecting strategies and can be expected to produce a rich and great diversified family of analytical methods. In addition to the already existing discourse and conversation analyses, those who work within the communicational approach to cognition have yet to construct and test their own methods of handling data, tailored according to their specific need.*

In light of the above, the commognitive approach to research is compatible with an interpretive qualitative approach and still developing. This provides my study directions as well as a scope for carving methods suited to my needs.

4.3 Setting and Participants

Commognitive approach to research as a special type of qualitative enquiry does not depend on large samples and statistical evidence on which to base claims (Opie, 2004), rather it focuses on interview (ibid), i.e. discourse, to provide rich descriptions (Merriam, 1997). For this purpose, I report a study on learners from one Grade 11 classroom, and in this sense, it is a particular group of 'case' of learners. A case study is an intensive investigation of a single unit (Opie, 2004; Schumacher, 1993). A case study looks at an enclosed system where certain features of social behaviour or activities influence the situation (Opie, 2004). Further a case study employs real people (learners and researcher) in real situations. In this study I have investigated a set of learners from one school and one particular grade: Grade 11. While I do not examine learning of these students as a class in detail, I do study their thinking employing multiple forms of data (Creswell, 2012). One of the disadvantages of focused small study such as this is that it is not generalizable (Cohen , Manion , & Morrison, 2000). The aim of this study is not to produce generalisations but to gain rich descriptions of learners' thinking and how these might be explained.

The 26 participants in this study are members of a grade 11 classroom aged between 15 and 17 in a multilingual[19] high school situated in a northern suburb in Johannesburg, South Africa. The student population in the school is predominantly Black (80%), while the remaining students are Indians & Coloureds (20%).

4.4 Selection of participants

The population should be defined while keeping in mind the objectives of the study (Opie, 2004). A sample reflects the characteristics of the population from which it is drawn. Sampling methods are classified as either probability or non-probability. Probability methods include random sampling, systematic sampling and stratified sampling. In non-probability sampling, members are selected from the population in non-random manner. These include convenience (opportunity) sampling, judgment sampling, and purposive sampling and snowball sampling. The sampling methods of interest for this study are convenience and purposive sampling. Convenience sampling is choosing a sample from those whom the researcher has an easy access (Creswell, 2012). The school was a convenience sample, because I had an access to the school since the school is one of the schools in the Wits Maths Connect project[20]. Purposive sampling, according to Schumacher (1993, p. 401) refers to '*selecting small samples of information-rich cases to study in-depth without desiring to generalize to all such cases'*.

All the 26 participants in the chosen class were given a test. All the tests were investigated and analysed in detail. I then identified ten learners based on the high number of incorrect answers in their test scripts. Thereafter, five learners were purposefully selected according to the following criteria: I chose learners who made errors on many answers in the test. Also in this category, there were learners who gave explanations (reasons) in their answers. Apart from these two main criteria, I decided to interview learners as much as possible to represent the highest categories of errors that I identified under each component under investigation. This will be discussed further in Chapter five (coding and analysis of test). Lastly, the five learners were selected according to their good communication skills and willingness to

[19] Multilingual means that learners are learning in language which is not their mother tongue and learners and teachers are speaking many different languages

[20] Wits Maths connect: is a unit at University of Witwatersrand which conducts research projects for secondary schools and also conducts Professional development

participate in the study. It was challenging to find learners who fulfilled all the above criteria together. Therefore, I relied on the help of the teacher.

4.5 Data collection: Instrumentation

4.5.1 Data collected

Two research instruments were used for this study, the test and interview schedule. The test was used as a source of data for the first question in my study. The interview schedule was used to answer the second question of the study. The following table 4.1 outlines how the test and the interview were used to help answer the first two critical questions of the study.

Research Questions	Research instruments
1. What common errors in terms of algebraic functions are made by the learners?	The test
2. Which features of mathematical discourses (word use, routines, visual mediators, and narratives) are evident in the learners' discourse?	The interview
3. In what ways, if at all, are these features linked to learners' errors?[21]	Both test and interview

Table 4:1 Alignmemt of research instruments with research questions

Let me expand further on data collection within the interview. In basic interpretive qualitative study, data is collected through interviews, observations, and document analysis (Merriam, 1997). This study included variations of all three forms of data collection within the interview process (see Table 4.2). The data collection of this study shares similarities with those used by Kieran (2002). The multiple data collection methods of the study increase its trustworthiness.

[21] This question will be answered by comparing findings from question one and two

Basic interpretive Qualitative Study	This study
Interviews	Individual interviews were conducted with 5 learners
Observations	During interview I took field notes to keep record on all observed learners’ actions
Documents	Learners' written work during interview

Table 4:2 Interpretive qualitative study data collection forms

4.6 Data collection Procedures

4.6.1 Ethical issues

Prior to conducting the research, I obtained the approval from the University of the Witwatersrand Ethics Committee (protocol number 2012ECE055). The research was conducted at a school where a Wits Maths Connect project has already gained permission to do research. Informed consent from principals and parents/guardians was obtained using relevant documentation (see Appendix A). These documents included informed invitation letters to the principal to conduct the research at the school, informed invitation letters to learners for their participation and consent forms to parents/guardians for their children participation in the study. Only learners whose parents/guardians had granted permission were tested and interviewed. Participation was voluntary and participants had the right to withdraw from the study at any time. During the reporting and discussion of data, none of the participants or the school were identified (pseudonyms were used) and participants were not judged or evaluated on their participation or non-participation. All the data that was collected had the names removed prior to analysis and reporting. By introducing myself to the learners prior to the test and the interviews, I assumed that they would feel more comfortable during the interviews by knowing that they could communicate freely with me. In the debriefing, I told the learners that if they felt uncomfortable at any stage of the interview, they had the right to withdraw.

4.6.2 Piloting

The piloting in this study had two phases. The first phase (pre-pilot) was conducted at the inception of this study, and the related results informed the questions used in the second phase (pilot of the main study).

4.6.2.1 Pre-piloting

I conducted a pre-pilot study at the beginning of this investigation because I wanted to see whether in fact learners could talk about functions. I gave a curriculum based test to some grade 11 learners to try some curriculum questions and then discussed their answers. I wanted to see whether they were comfortable to speak English and able to express their thinking. During the interaction it was evident that they were able to communicate. Through the analysis of their scripts I was able to see that they did demonstrate some errors. I then developed the test based on the pre-pilot together with the literature on functions as discussed in the chapter on literature review.

4.6.2.2 Piloting of instruments

The piloting of the instruments (test and interview) for the study was conducted in a different Grade 11 class to the ones intended for the study. Cohen et al. (2000) assert that the sample for piloting needs to be similar to the sample intended for research so that the researcher is in a position to assess and analyse the likelihood of the trends observed during pilot stage should these trends re-occur in the main study. The purpose of the pilot study is to inform the main study about the quality of the questions in the task. Also, a pilot study was helpful in choosing questions which would provide rich data about tasks in functions. The pilot study helped to indicate the task's suitability for the study in terms of clarity in the instruction, structure and content/context of the questions and whether the questions provoked mathematical thinking through different representations of functions. The execution of the pilot study gave me an opportunity as a researcher to practice the administration protocols (Cohen et al., 2000). The pilot study also helped me to remove all items that seemed to be irrelevant data for the study (Bell, 1987 cited in Opie, 2004), and to further refine the remaining questions.

After administering the test, I selected two learners randomly and conducted practice interviews with them. This exercise was for me to understand the right kind of questions to be asked and to decide on a suitable pace for interviewing learners. These interviews were tape recorded. By listening to the interviews, I decided to make some adjustments to my questioning patterns. I decided to provide the learners with more time to explain rather than me asking lengthy questions. Second, I understood that my pace was too quick and I should allow them to have more time to think and answer rather than hurriedly moving from one question to another.

During the piloting of the interviews, it became evident that the use of a video camera would enhance data and increase the credibility of the research (Sfard, 2008; Sfard & Kieran, 2001). However, learners were preparing for end of the year examinations; there was insufficient time to reapply for permission to video record learners' interviews from the ethics committee. I thus worked to capture field notes and learners' written work during the interview process.

4.6.3 Designing of the test tasks

In this study, the test was designed with the aim of collecting data relating to learners' common errors. Features such as the overall structure of the test, suitability of the items, item coherence, their appropriateness, and other features such as the face validity of the test were discussed with two subject experts and two teachers. This discussion was aimed at improving the validity of the test instrument.

The design of the test considered the two main components of linear and quadratic functions: Properties (gradient, intercepts and turning point) and representations (verbal, algebraic, graphical and table).

The next section describes the development of the test tasks used to collect learners' common errors. The first part describes the principles informing the development of the tasks. The second part describes the criteria for selecting the tasks included in the instrument; the third part presents the brief description of each of the test tasks.

Designing the test

The literature review on function in the previous chapter provided the basis for the development of the tasks. Each task was developed in such a way that highlighted learners' different approaches to functions (i.e. competencies) and categories of errors related to these competencies as suggested by Radatz (1979), who proposed that learners' errors could be assessed by following through problem solving stages, that is through examining the mechanisms used in obtaining answers (i.e. competencies used in functions in this case). The following principles informed the design and the selection of the tasks to ensure that they generated the kind of data needed to identify and describe learners' common errors with function.

The design and selection of tasks was informed by the following principles:

i. Tasks familiar to learners
ii. Variation and context of tasks

First principle is about ensuring that tasks were mathematical and familiar to the learners. The test included questions that were mathematical, familiar to the learners and clear to understand. Creswell (2012) suggests that questions should be unambiguous and easy to understand. Familiarity means that the test content was aligned with the standards of the National Curriculum Statement (DoE, 2003). To address these issues : (i) the questions in the test were adapted from literature of previous studies and redeveloped from textbooks currently used in the curriculum: Grade 11 Classroom Mathematics textbook (Laridon, 2008) and The Answer Series, 2011 (Eadie, 2011), (ii) the style of questioning was adapted from the past examination papers.

Second principle was about asking questions about the same concept in different forms. For example, an intercept concept was assessed in different ways, first from an algebraic representation and then graphical representation in question 4.3 (see question 4 in appendix B). Different competencies were assessed such as interpretation and calculations. Variation of questions has been suggested by Even (1998). The context of the tasks is about including tasks from different domains of mathematics such as calculus, trigonometry and geometry. Tasks involving different contexts were also included, for example in question 1.1 (see question 1 in Appendix B), four options were provided and one of them was a trigonometric function (i.e. option A). Even (1998) has argued that varying the questions and contexts help the learners see the function in different forms and domains.

Criteria for selecting tasks

As previously explained, the categories of errors reported in literature review chapter, National Diagnostic Report (DoE, 2012) and the pre-pilot study informed the development of tasks. Some of the tasks were adopted from mathematics textbooks used in grade 11, but the majority of the tasks were developed specifically for this study.

The same principles described in the previous section were used in selecting the tasks for inclusion in the instrument for the two main data collections.

This study involved generating data relating to learners' errors with function concept. For this, tasks that captured the categories of errors as reported in literature review were included. For example in question 1.7 (see question 1 in Appendix B), an algebraic representation together with its equivalent graphical representation were provided, and learners were asked to choose a correct answer from four options. An inclusion of a distractor in option A was done to elicit errors related to the concept of gradient. The tasks where most of the learners answered incorrectly during the pilot study were also included.

Another basis for the inclusion of the tasks in the final test for data collection was the extent to which the task allowed learners to use different approaches i.e. competencies with function such as interpretation , classification, calculation and translation (Leinhardt et al., 1990).

I made all of these judgments when selecting tasks in constant consultations with my supervisor based on results of the pilot study.

The test tasks

The test contained seven main questions with sub-questions. Each test item was developed in such a way that it captured different categories of function errors that have been reported in literature (see Chapter 3,section 3.3.2) and some that had emerged from the pre-pilot study and the related competencies. For example, in question 2.2 (see question 2 in Appendix B), learners were asked to classify the function $y = 3x^2$. The classification can be done by either interpreting the global or local properties of this algebraic representation or by constructing the graph and interpret its behaviour. The expected errors related to these competencies were also taken into account.

Table 4.3 below, provides a brief description of the tasks. The tasks are presented in more detail in the next section. The first column of the table shows the question number of the task. The second column describes the competencies involved when answering the question, i.e. the approaches that may be used (global or point-wise). The competencies may include translation, interpretation, construction, classification and calculation. The construction includes the construction of the graph or table. The third column gives a brief description of errors the task is intending to elicit.

Question		Competencies	Description of error
Q1	1.1 to 1.5	interpretation	errors related to interpretation of graphs
	1.6 to 1.7	interpretation calculation	errors related to interpretation of either local or global properties of a gradient and calculation
Q2	2.1 to 2.3	classification construction	errors related to classification of algebraic representation including interpretation (local or global properties) and terminology used when classifying
Q3	3.1 to 3.5	interpretation calculation	errors related to translation of graph to its equivalent equation including interpretation (local or global properties)
Q4	4.1	classification	errors related to interpretation (local or global properties) and calculation
	4.2	construction	
	4.3 to 4.4	interpretation calculation	
	4.5	interpretation calculation	
	4.6.1 to 4.6.2	interpretation calculation	
Q5	5.1 to 5.4	interpretation calculation	errors related to interpretation (local or global properties) and calculation
Q6	6.1	translation	errors related to translation of graph to its equivalent equation including interpretation (local or global properties)
Q7	7.1	translation	errors related to translation of verbal representation to its equivalent tabular representation including interpretation (local or global properties)
	7.2	interpretation	errors related to interpretation (local or global properties)
	7.3	classification	errors related to classification of tabular representation including interpretation (local or global properties) and terminology used when classifying
	7.4	interpretation	errors related to interpretation (local or global properties)
	7.5	interpretation calculation	errors related to interpretation (local or global properties) and calculation

Table 4:3 Classification of questions

The written responses from the test provided the data for common errors and also informed the interview schedule. Some of the questions were designed in such a way that learners could give explanations or reasons for their responses.

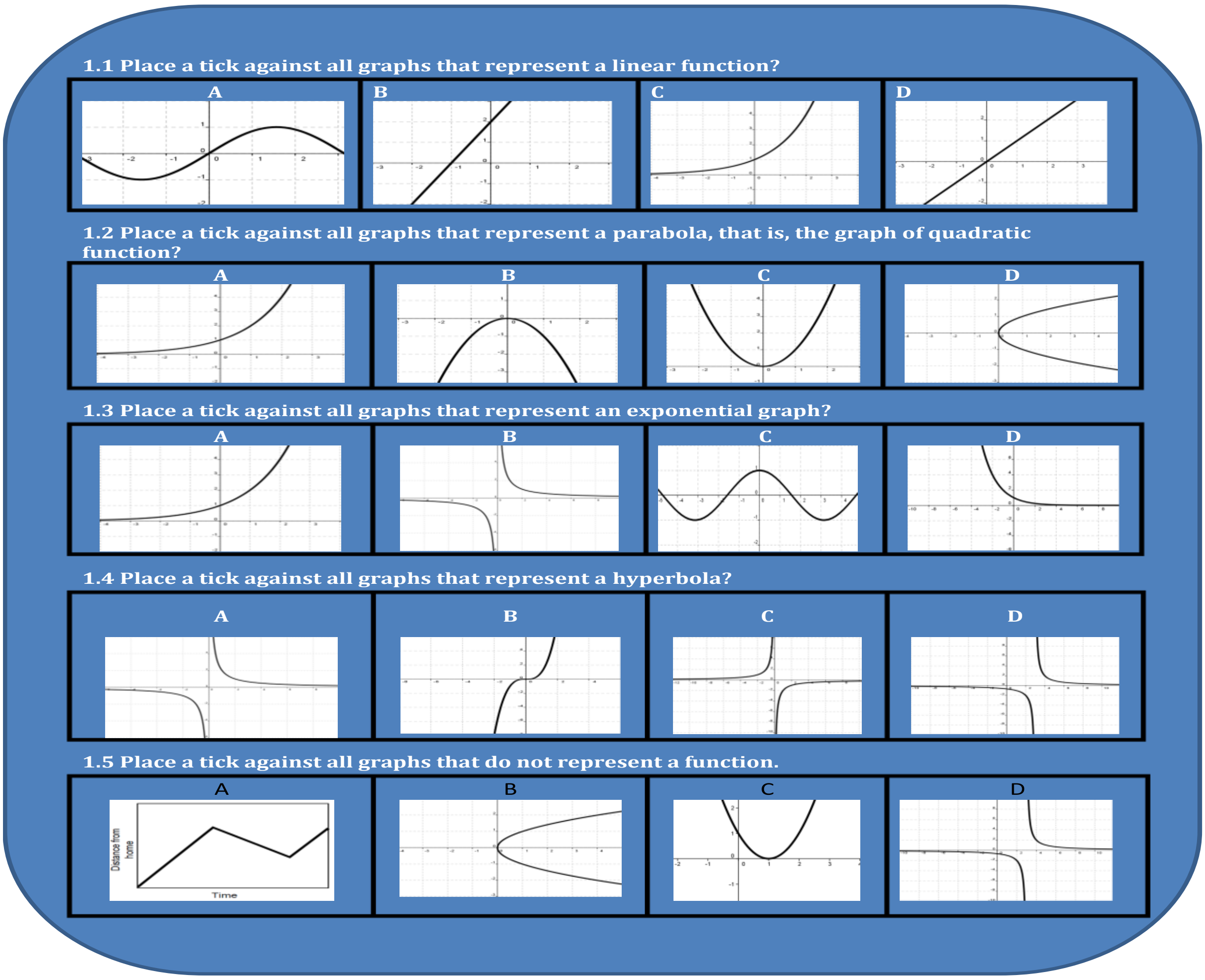
What follows is the detailed discussion of the seven questions (see Appendix B).

4.6.3.1 Discussion of the seven questions

Question 1

Question 1 was a multiple choice question divided into 7 seven sub-questions. Each question had four options to choose from i.e., A, B, C and D. All sub-questions 1.1-1.7 required only an interpretation of global features of different graphs i.e. a general shape and behaviour. At least one or two choices in each sub-question were used as distracter(s) which were designed to elicit common errors either from literature or pre-pilot study. One of the distracters was in different context (e.g. trigonometric). The shortcoming of such a form (multiple-choice) is that learners may guess the answer without real understanding.

Questions1.1-1.5



When designing questions 1.1-1.5, the intention was to test if the learners were able to identify the different functions from their graphical representations through interpreting global properties, i.e. classification. It has been pointed out that some of the questions were informed by the results of the pre-pilot study and from literature. For example, in question

1.1, the inclusion of option C (exponential graph) was informed by the results of the pre-pilot study where this function was classified as a linear function because it looks like a line.

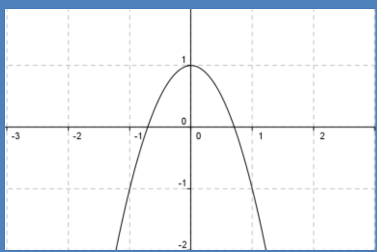
In question 1.3, option C, the cosine function was included because it is a different context and could be unfamiliar. In the same question, B was included, because it looks like an exponent and other features could be ignored i.e. two quadrants and not cutting through the y axis.

In question 1.4, options B and D were included to elicit interpretation errors. From pre-pilot study, option B was not considered as hyperbola because it cuts through the y axis.

In question 1.5, option A was included to elicit errors that result from an interpretation of the graph (e.g. graph as picture). Option B was included to elicit interpretation errors, and could be classified as a function because of origin prototype. Option C could also be ignored because of the origin prototype error (all graphs pass at the origin).

Question 1.6

1.6 The sketch below represents the graph of $y = ax^2 + 1$



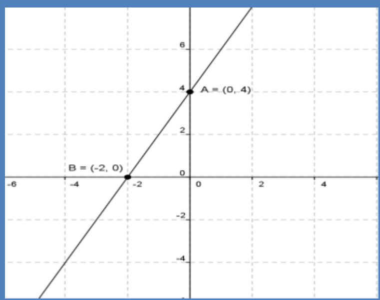
Which of the statements below is/are true, and which is/are false. Write T or F for each?

A. a **is positive**
B. a **is negative**
C. a**=-2**
D. a**=1**

Question 1.6, tested whether learners would be able to link both algebraic and graphical representations i.e. translation. This process would involve either interpreting global and local properties of the representations. Attending to global properties means interpreting the shape and the behaviour of the graph. Attending to local properties means using the co-ordinates in the graph and substituting them in the gradient formula. When this question was designed, I had an error of linearity (interpretation) in mind which has been reported in literature (Leinhardt et al., 1990) . Because the algebraic equation is in a form $y = ax^2 + 1$, it could be confused with a linear equation $y = mx + c$ and the effect of ‘a’ could be interpreted to be a gradient or slope, hence the inclusion of tasks in options A and B. Further, it has been reported in literature that a gradient is sometimes assumed to be 1 (Leinhardt et al., 1990; Moschkovich, 1999a), hence the inclusion of option D.

Question 1.7

1.7 The sketch below represents the graph of $y = px + 4$



Which of the statements below is/are true, and which is/are false. Write T or F for each ?

A. p **is positive**
B. p **is negative**
C. $p = 1$
D. $p = -1$

Question 1.7, as it has been pointed out that during the design the intention of this question was to enable learners to use either both pointwise and global approaches or one. Hence the graphical and algebraic representations were included. In a graphical representation, co-ordinates were given, in case a pointwise approach was desired. The error which the question was hoping to elicit was related to interpretation of gradient.

Question 2

You have learnt about four kinds of functions: linear, quadratic, hyperbola and exponential.

What kind of function is represented by each of the following equations, and how do you know?

2.1 $y = 3x$

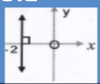
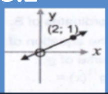
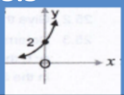
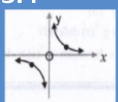
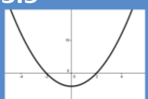
2.2 $y = 3x^2$

2.3 $y = \frac{3}{x}$

This question was designed to elicit errors related to classification. Three different functions were given in an algebraic form. Learners were required to classify each function; they could use either global or pointwise approaches or both. For example plotting a graph or interpreting global features of the given representation (algebraic). Attention was also paid on the vocabulary used to identify these functions. When answering this question, learners not only provided answers but they also justified their answers. These justifications helped me when designing an interview.

Question 3

In the table below, 5 graphs are given in the first column. Followed by the list of 5 equations in the second column. You need to match each graph with its correct equation, and give a reason for your choice.

Graphs	List of possible equations
3.1 	A. $y = \frac{2}{x}$
3.2 	B. $x = 2$
3.3 	C. $y = x^2 - 4$
3.4 	D. $y = \frac{x}{2}$
3.5 	E. $y = 2^x$

In this question, the graphical representations of different functions were given in the first column. Learners were expected to match the equivalent algebraic representation in second column. The learners may use a global approach, i.e. interpret global properties and still match the representations. The learners were asked to give justifications of their choice, which later helped me when designing the interview protocol.

Question 4

Given functions $f(x) = x^2 - 4x - 5$ and $g(x) = x - 5$

4.1 What kind of function is f and g ?

4.2 Draw f and g on the same system of axes.

4.3 What are the x intercepts f and g ?

4.4 What is the y intercept of both f and g ?

4.5 What are the co-ordinates of the turning point of f ?

4.6 Use your graph to solve for x if:

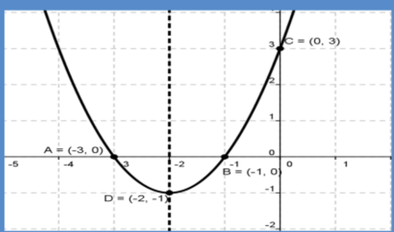
4.6.1 $x^2 - 4x - 5 = x - 5.$

4.6.2 $f(x) > 0$

Four components of linear and quadratic functions were tested in this question i.e. graphs, algebraic representations, intercept and turning point. In this question the intention during the design was to enable the learners to (i) use both global and pointwise approaches, (ii) translate between graphical and algebraic representations, (iii) and interpret the representations and (iv) vocabulary they use when classifying. The errors which the test was hoping to elicit were related to classification, construction, calculation, interpretation and translation.

Question 5

Given equation $y = x^2 + 4x + 3$ and its graph below



5.1 What is the value for $f(-2)$?

5.2 What are the x intercepts of the graph?

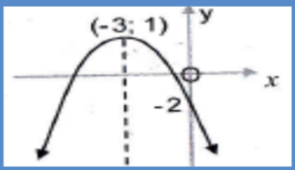
5.3 What is the y intercept of the graph?

5.4 What are the co-ordinates of the turning points?

In this question, different representations of a quadratic function were given: graphical and algebraic. This question was included to assess learners' ability to interpret global properties of the given representations. This question required use of global approach (interpreting). However, a pointwise approach could be used, but it was not necessary. The possible errors that could emerge are associated with a calculation competency (i.e. pointwise approach). Past research has shown that over emphasis on pointwise approach could result in errors (Brenner et al., 1997).

Question 6

The figure below is a parabola, with turning point (-3, 1) and y intercept (0,-2)



6.1 Determine the equation of the graph.

In this question a graphical representation of a quadratic function is given. During the design, the intention was to assess learners' ability to translate from the graph to an algebraic representation. They may use both global and pointwise approaches. This means, learners may use the global approach by scanning the graph and interpret its global features and then classify it as a quadratic function and recall standard algebraic formula [$y = ax^2 + bx + c$ or $y = a(x-p)^2 + q$], then do calculations i.e. using pointwise approach. The possible errors that could emerge are translation and calculation errors.

Question 7

The rule in this table is, 'take a number and square it'.

x	-2	-1	0	1	2	3
$f(x)$	4					9

7.1 Complete the rest of the table using the rule. The first block and the last block have been completed.

7.2 Do the values in your table represent a function? Give reasons for your answer.

7.3 If so, what is the name of the function?

7.4 If $f(x) = 1$, what can you say about the value of x ?

7.5 Give a value of $f(2)$

In this question a table was given showing a pattern of a quadratic pattern. This question was designed in such a way that it elicited errors associated with: function notation, interpretation of both global and local properties of a table and classification. In 7.1, learners could interpret global or local properties of the table and link it to a verbal representation (i.e. translation). Further in 7.3, learners were expected to classify the function and the vocabulary they used to classify was also paid attention to.

4.6.4 The interview

Interviews were chosen for the study as a data collection tool because this study sought to understand learners' thinking through the discourses that they use when working with functions. I drew from Sfard's notion of thinking as communication (Sfard, 2008). Sfard's characterization of thinking will help in explaining why learners have difficulties with the concept of function, through an analysis of emerging roles learners play in the discourse (i.e. learners' verbal communication and discursive actions). As suggested by Opie (2004), the interviews offer the opportunity to ask the question "why", to elicit explanations or justifications. It is through these justifications (narratives) that I will be able to gain insights into learners' mathematical discourses. In this study, the interviews were "*used alongside other data collection methods" with the goal of "exploring more deeply participants' perspectives on actions observed by the researcher*" (Hatch, 2002:91).

The semi structured interview protocol (questions) was developed from the learners' written textual responses from the main test. Hatch calls the questions prepared in advance of the formal interview and designed to guide the conversation "guiding questions''. The semi-structured interview was used for the purpose of observing learners' mathematical discourses when talking about functions. During the interview, guiding questions were used at the same time open and flexible discussions were done to enable the examination of mathematical discourse (Denzin, 1970 and Silverman 1993 cited in Cohen et. al. 2000). The interview also gave me an opportunity to code switch (Setati, 2005), when I saw that the learner could not explain themselves in English, since this was a multilingual class. Hatch (2002), suggests that the interviewer may deviate from prearranged text and wording of questions, in an attempt to clarify the questions. During interview sessions, the learners were asked questions from the interview protocol, they were asked to answer orally and told that they could write down their responses. Their writings and actions were observed and notes (field notes) were taken for descriptive purposes. The interviews were audiotaped and transcription was done, and all efforts to minimise interpretation during transcription was taken into consideration. Transcription of interview data will be discussed in detail in section (4.6.4.3).

4.6.4.1 Commognitive researcher and data

In a commognitive research the discourse is the principal object of attention, and it is the unit of analysis. Commognitive research places specific responsibilities on the researcher: on data collection; data production; data analysis and interpretation.

Data collection: during this period a commognitive researcher plays a role of becoming a participant observer. Sfard (2008) warns that during this process, the observer should strive to be as non-interventional as possible. The researcher should avoid enticing participants into researchers' own discourse because her actions could be interpreted as being evaluative or corrective by those she is observing. Furthermore, if the researcher refrains by keeping quiet or making any gestures, this could also be interpreted otherwise by the participants.

Data production: during this period, the data collector should conform to the principle of verbal fidelity (Sfard, 2008). This means that the data collected should include what was said and done by the participant. Sfard (2008, p. 277), says: '*The commognitive researcher is to begin her report showing what was done and said, rather than with her own story about it. Instead of revoicing the actors, she must let them speak in their own voice*.'

Data analysis and interpretation: while analysing data a principle of alternating should be observed (Sfard, 2008). This means that a commognitive researcher should play two roles, the one of becoming an insider and outsider to their own discourse. When one is the insider, she understands the contexts of the discourse, for example the language and the rules of the discourse. Sfard (2008), points out that because of this understanding a lot of possible interpretation and sense making take place. According to her, as long as the participants are adhering to the rules of the discourse, this sense making is effective. However, if the rules are not adhered to, this may leave the insider frustrated and helpless. Hence, she suggests another role to be played by the observer: the outsider. When one becomes an outsider, you remove yourself from your well-developed discourse (e.g. mathematical) and try to make sense from outside. An outsider pays special attention to what is visible; taking words out of their context. From a commognitive perspective both insider and outsider roles complement each other. A challenge for this study and analysis is to be both an insider and outsider.

4.6.4.2 Discussion of interview protocol

As already noted, the interview was semi structured and was developed based on learners' written responses on the test. The interview questions were open ended, where learners were asked to explain their thinking.

Each question belonged to one of the five components of linear and quadratic functions: gradient, table, graphs, algebraic representations and intercepts. These five components have been identified as one of the concepts learners are experiencing difficulties with as discussed in Chapter three , and it has also emerged from the test that learners were committing errors on these components. The items in these components were not mutually exclusive in the sense that an item could belong to more than one category. For example, a task assessing an intercept concept may contain the concepts of graphs, table or even algebraic representation. However, the major concept that was expected to test using the item was considered as the one that makes up that category.

I was guided by the following questions in trying to capture the features of the mathematical discourse (words, visual mediators, routines, narratives) that were used by the learners:

Question A: INTERCEPTS

The section on intercepts has three sub questions addressing different ways that an intercept can be represented i.e. Definition-verbal representation, algebraic representation and tabular representation. The inclusion of these representations is based on the way the intercept concept is defined by the experts. I have observed a principle of variation of representations (Even, 1998). This might help me identify which representation the learners are struggling with. The choice of different representations of an intercept concept is similar to that one of Moschkovich (1999a). It should be noted that the table and graphs components were also tested in this question.

Definition: Learners were asked to give a definition of an intercept concept. This question was focused on eliciting the words learners use when talking about an intercept and how these words are used. Are they using them in a colloquial way or in a literate way? Which visual mediators are they using when talking about an intercept concept? What are the competencies are they employing i.e. the routines? What is the nature of these routines? Why are they using them? What decisions are they making? How are they substantiating their actions? These are the kinds of questions I was hoping to get answers to, in order to gain insight relating to their mathematical discourse.

*Algebraic representation: L*earners were asked to determine the x and the y intercept of the function $y = 2x + 1$. This question was designed to understand learners' routines and their substantiated narratives.

Graphical representation: Different graphs of different function families were given. Learners were asked to show and describe the x and y intercepts in the following graphs:

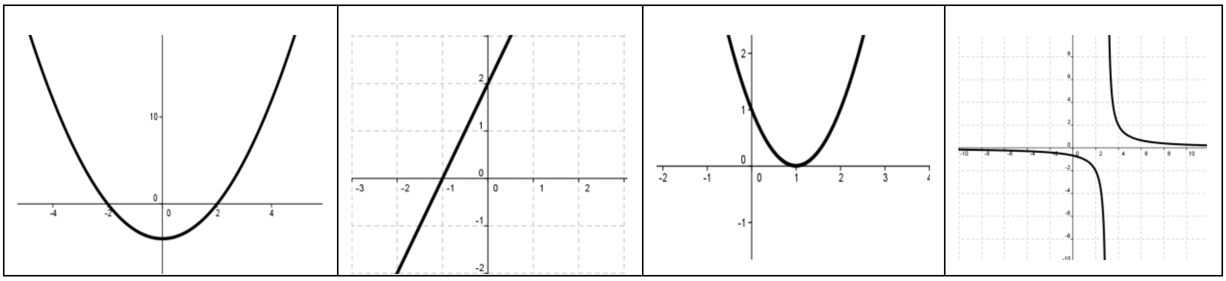


Table 4:4 Example of graphical representation of intercept

This question sought to test whether learners were able to interpret the intercepts in graphical representation. This would help me to identify their routines when identifying the intercepts. They were further asked to give reasons of their choice. Their answers have helped me to record their substantiated narratives.

Table: In this question a concept of intercept was given in a different form (set of co-ordinates). Learners were expected to interpret the intercepts (x and y) from the graph and from the table. These questions were asked in different ways, using variation as suggested by Even (1998). Below is an example of the question assessing mathematical discourses with a concept of intercept.

Example of a question:

(a) A co-ordinate (-2, 0) was given, and learners were asked to identify what does this co-ordinate represent.

(b) What are the co-ordinates of the x intercept and the y intercept in figure 4.2 below?

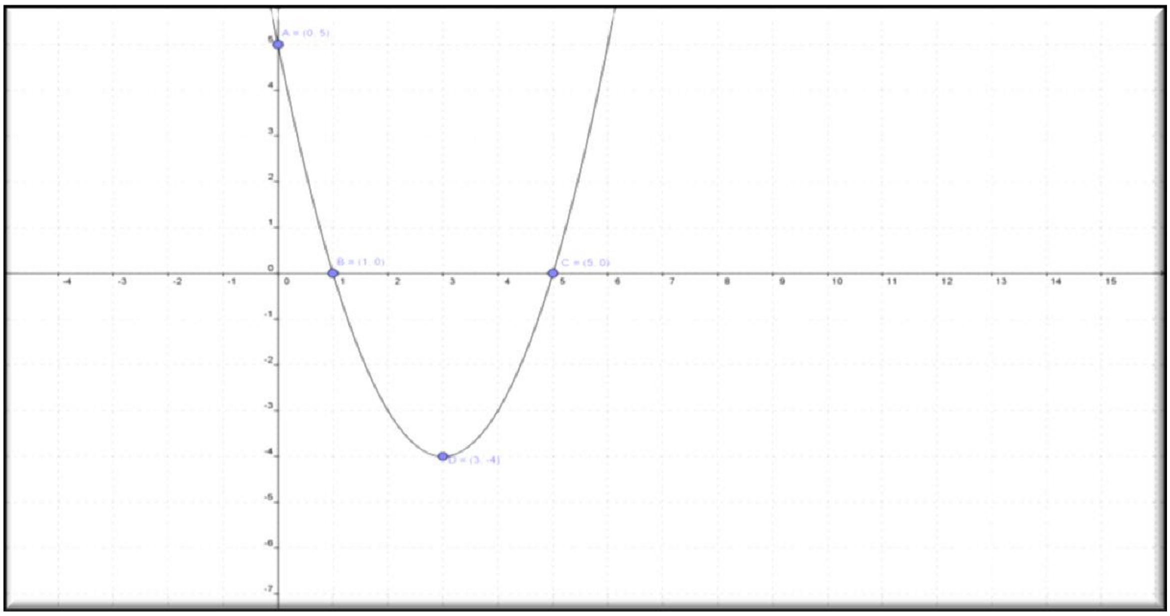


Figure 4:2 Example of graphical representation of co-ordinates

(c) Now let's say I gave you a table, showing a linear pattern (linear function):

x	-4	-3	-2	-1	0	1	2	3	4	5
y	-3	-2	-1	0	1	2	3	4	5	6

- From the table, can you tell me the value of the x intercept and the y intercept?
- Let's talk about the notation $f(x)$, what can you say about this? Can you give an example? What does it represent?
- Can you tell me from the table, what is the value of $f(0)$?
- What is this value called?

Question B: ALGEBRAIC REPRESENTATION

This question was aimed at investigating learner's discursive actions with algebraic representations of quadratic and linear functions.

Quadratic function

In this question a quadratic function $\mathrm{g(x)} = 2\mathrm{x}^2 + 5$ was given and non-examples were also included to vary the question $\mathrm{y} = \mathrm{x}^3 + 2$ and $\mathrm{y} = 3^{\mathrm{x}}$. Learners were asked to classify these functions. This question was aimed at gaining some insight into learners' discursive actions involved when attending to these visual mediators, the words they use when describing the quadratic, the routines and their substantiating narratives.

Linear function

In this question a linear function $y = 2x + 1$ was given and another linear equation where the coefficient of x was varied was also included: $\mathrm{y} = \frac{\mathrm{x}}{2} + 3$. The aim of including this equation in a fractional form served a purpose of varying the question. And a fractional mode can be confused with a hyperbola. This question was trying to gain some insight into how learners were attending to this visual mediator, the words they use when describing the linear function, the routines and the nature of these and lastly their substantiating narratives.

Question C: GRADIENT

In this question the gradient concept was tested from different domains: the linear, quadratic and a combination of linear and quadratic function domains. The rationale for including these domains when assessing the concept of the gradient is similar to Moschkovich (1999a). This question was basically testing the concept of gradient. It is the same question that was asked in the test. The number of incorrect answers was high; I was interested in understanding why learners were having difficulties with the gradient concept. How do they understand this concept? What words are they using? What are the routines and the nature of these? How are they working with different visual mediators addressing the gradient concept? And what are their substantiating narratives? Many questions were included as non-examples, to help me vary the way the gradient is seen. Below are examples of the questions:

Linear function

(i) The sketch below represents the graph of $\mathrm{y} = \mathrm{px} + 4$

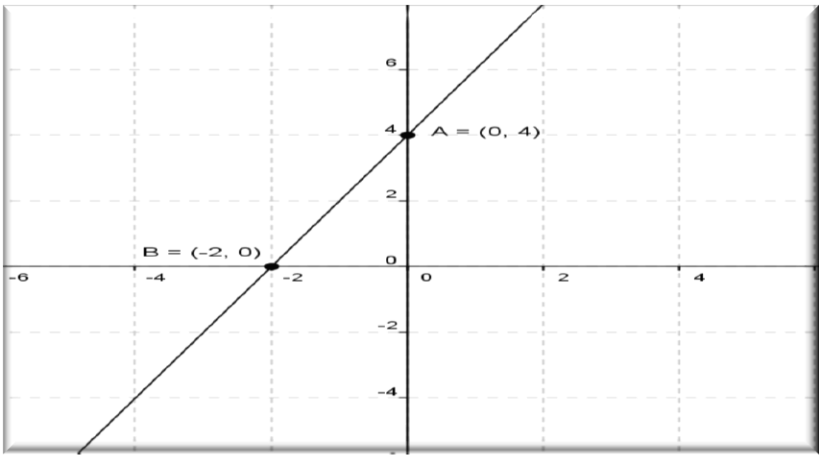


Figure 4:3 Example of a linear function

- What does p represent?
- Is the value of p positive or negative? How do you know?

Quadratic function

(ii)The sketch below represents the graph of $\mathrm{y} = \mathrm{ax}^2 + 1$

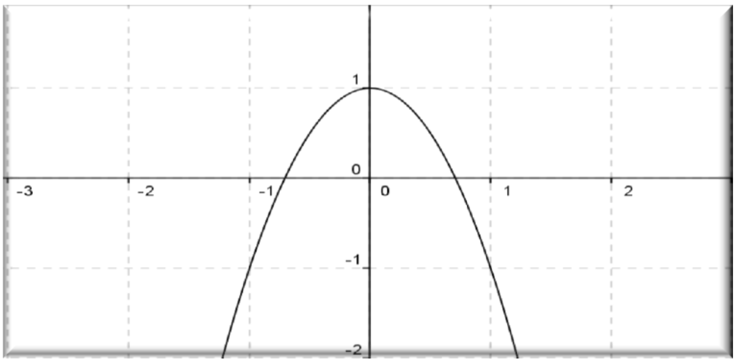


Figure 4:4 Example of a parabola

- What does a represent?

Combination of linear and quadratic functions

Let's talk about the following functions $\mathrm{f(x)} = 3\mathrm{x} + 1$ and $\mathrm{g(x)} = 2\mathrm{x}^2 + 5$

- Two Grade 12 learners were having a discussion about the two functions. One of them was saying that the gradient of $g(x)$ is 2. Is he correct?

4.6.4.3 Transcription

Sfard (2008) emphasizes that it is very important that during the transcription, interviewees' utterances are reported as uttered by the interviewee. She describes this as a principle of verbal fidelity, which minimizes the loss of meaning. For example, where a learner uses words like 'intersect' when talking about intercept, the transcriber might interpret this as intercept, whilst the learner meant 'to intercept' or 'intersect'. This means that different verbalisations may lead to different meanings. Therefore, interpretation should be minimised as much as possible (Ben-Yehuda et al., 2005; Sfard, 2008).

The transcription does not only document what the learners are saying but also what they do. This is because what learners do is an important part of the data analysis in a commognitive research (Sfard, 2008). Table 4.5, shows an example of such transcription taken from the interview transcript:

Speaker	What was said	What is done or seen
Interviewer	Okay what is it that you understand about the word intercept?	
Learner	I explain it is as a touch when I say intercept	using hands (crossing fingers)

Table 4:5 Example of transcription

All audio data from the learners' interviews were transcribed by two different transcribers to ensure accuracy, and to maintain the verbal fidelity principle of commognitive research.

4.7 Data analysis

To aid in a close examination of the data, I used multiple analysis tools. These tools allowed me to get close to the data. Tests generated data relating to common errors. The interview generated data for mathematical discourses used. The tests and interviews were analysed using two models of analysis that Hatch (2002) refers to as typological analysis and inductive analysis.

4.7.1 Typological Analysis

According to Hatch (2002), the typological analysis begins by reading through the data and then dividing the data set into elements or categories based on predetermined categories. These categories or typologies '*are generated from theory, common sense, and/or research objectives*..'(Hatch, 2002, p. 152).

The initial typologies in this study were classified as follows: test (categories of errors from literature review) and interview (using the analytical framework that was adapted from Ben-Yehuda et al. (2005)). Typological analysis has helped me to condense varied raw data into a brief summary format and to establish clear links between the research objectives and the summary findings derived from the raw data.

4.7.2 Inductive analysis

The inductive analysis 'is a systematic procedure for analysing qualitative data where the analysis is guided by specific objectives. The primary purpose of the inductive approach is to allow research findings [categories] to emerge from the frequent, dominant or significant themes inherent in raw data, without the restraints imposed by structured methodologies' (Thomas, 2003). Unlike typological analysis, categories from inductive analysis emerge from the analysis of data.

The following steps, offered by Hatch (2002, p. 162), were used to guide the analysis of the data in this study.

1. ***Read the data and identify frames of analysis***
2. ***Create domains based on semantic relationships discovered within frames of analysis***
3. ***Identify salient domains, assign them a code, and put others aside***
4. ***Reread data, refining salient domains and keeping a record of where relationships are found in the data***
5. ***Decide if your domains are supported by the data and search data for examples that do not fit with or run counter to the relationships in your domains***
6. ***Complete an analysis within domains***
7. ***Search for themes across domains***
8. ***Create a master outline expressing relationships within and among domains***
9. ***Select data excerpts to support the elements of your outline***

Figure 4:5 Steps in inductive analysis

In keeping with Hatch's model, I started typologically and then worked inductively. In other words, the initial step in the analysis was to note the frequency of the coded categories. Thereafter, I worked with the data from the test and interview questions and deduced the categories step by step. The categories were continuously revised until they closely suited the data. A detailed description of the various categories is provided in chapter five for the test and chapter six for the interview.

4.8 Considerations concerning rigour.

Rigour in research deals with '… presenting insights and conclusions that ring true to readers, educators and other researchers…' (Merriam, 1997, p. 199). The phenomenon of *rigour* has often been explained as an element of *trustworthiness* (Lincoln & Guba, 1985; Merriam, 1997; Opie, 2004) , or *validity, reliability, unbiasness* (Freeman et al, 2007).

Earlier on, I noted that this study is a qualitative study that draws in some quantitative techniques. Thus, I am going to discuss some validity notions with reference to the test from a quantitative standpoint.

4.8.1 Test

For trustworthiness, data in a research study need to have various forms of validity, for example construct validity, content validity and criterion validity. My research design has used content and constructs validity related evidence. These two forms of validity describe what is measured unlike the criterion validity which gives only criteria of how validity is determined (Fraenkel & Wallen, 1990).

Construct validity

Construct validity in this study refers to the extent to which the questions measured a theoretical construct of the study. This involved checking if the task was properly constructed so as to elicit the kind of information envisaged by the research questions through the pilot study.

Content validity

On the other hand, content validity refers to the adequacy of the questions with respect to the topic of linear and quadratic functions. In this study, experts (supervisor and colleagues) were consulted to assess the research task on its content validity. They face-validated: the tasks, content knowledge level in the questions, ordering of questions and clarity of some phrases in each question. The tasks were designed to cover enough content on the linear and quadratic functions, to give adequate information about learners' errors and mathematical thinking when solving tasks on functions.

Content analysis of the tasks also contributed to making informed judgement regarding the content validity of the tasks. My experience as a high school tutor and my content knowledge I acquired while doing my junior degree in mathematics also provided me with the necessary experience to judge the suitability of the tasks.

This study recognises the limitations of a written test. Misrepresentations and non-response are some of the serious threats to the reliability and validity of the written test. Readability, clarity of instruction, layout and length of time required to complete the test are just some of the threats to the accuracy of the research result (Cohen et al., 2000). To minimize these threats, pilot of the test was done and the tasks were revised to maximize clarity and appropriate interpretation.

4.8.2 Interview

Rigour of the interview

The commognitive research assures rigor, according to (Yackel, 2009, p. 90) as cited in (Sriraman & Nardi, 2013), '*Sfard go to great lengths to develop an approach that meets accepted standards of scientific rigour through providing operational definitions of keywords, such as thinking, communication, discourse, and mathematical object'*. What is rigorous about Sfard's defining all the concepts that she uses, is that we can have unambiguous reading and use the same definition.

During the interview, multiple data methods were used. An audio was used to capture the learners' utterances. At the same time the learners were allowed to write their responses down should they wished. These transcripts were used as part of the data. I was also taking notes.

As noted earlier, the data collected in this study was predominantly qualitative. However, I do quantify some of the data in order to identify general trends and patterns in the analysis of the discourse.

Trustworthiness and credibility

A research study is trustworthy if it is reliable, referring to the consistency of the study's findings under the same conditions. It relates to consistency of the findings if the research is repeated with the same group. To address the issue of trustworthiness all the data is viewed as an integral part of this study. This is why I have included excerpts of the transcripts during analysis to provide empirical evidence .The intention is to make the readers able to decide whether they agree with my interpretations. Further, during the interview analysis I have used the voices of the participants in describing interviews wherever possible so that the reader can draw their own conclusions.

During interviews in the study the same questions were repeatedly asked in different ways (if not understood by interviewees) to ensure some degree of trustworthiness in the responses given by the participants. I have also code switched (Setati, 2005), to ensure that the learners understood the questions asked, since this was the multilingual classroom.

In qualitative research, replicability of results cannot be guaranteed because of bias inherent in the individuals. Any given data may be represented and interpreted differently by different researchers. To ensure unbiasness, I have reported participants utterances and actions without revoicing what was uttered as suggested by (Sfard, 2008). I have tried to view the unfolding discourse in an unbiased way as possible by adopting an outsider perspective (Sfard, 2008). At the same time, I am of course aware of the fact that my mathematical knowledge makes me an insider to the discourse. However, I have specifically tried to avoid making references to what is not present in the discourse, except in contrasting the learners' discursive activities.

4.9 Conclusion

Broadly, I am working with interpretive qualitative paradigm, combined with some quantitative techniques. The quantitative methods were used in an attempt to expose Grade 11 learners' errors on function, and the qualitative methods allowed me to gain insights into the nature of mathematical discourse the learners use when engaging with a concept of function. It should be noted that the aim of this study was not to compare the errors from the test with the interview, but to first investigate learners' common errors with function components (i.e. intercept, gradient, quadratic and linear functions) through the test. Secondly, the study aimed to investigate features of learners' mathematical discourse with these components. A sequential design was used and this is characterized by the collection and analysis of quantitative data followed by the collection and analysis of qualitative data. The main research instrument in the quantitative phase was a test instrument while interviews served the purpose of the main research instrument in the qualitative phase. During this phase learners' interview transcripts, written work and researcher's filed notes were simultaneously used as multiple data sources to arrive at valid conclusions about learners mathematical discourses.

Chapter 5 CODING, ANALYSES OF TEST AND FINDINGS

5.1 Introduction

The final version of the test was administered to the 26 grade 11 learners who were the main participants of this study. The test was administered with the help of the grade 11 mathematics teacher. Later, the test papers were marked, and answers were recorded according to incorrect, correct and not attempted (see Appendix D). The results of the test helped me to identify incorrect answers; I analysed these further by coding against the developed rubric for coding errors. The coding process that was followed is explained in the next section.

5.2 Coding

5.2.1 Development of rubrics for coding learners' errors

To analyse learners' errors, I developed a rubric containing error groups. The creation of the rubric was mainly drawn from different sources such as errors reported in literature review of this study, errors from a pre- pilot study that was done at the inception of the study and from the errors reported in the National Diagnostic Report (DoE, 2012), that is, typologically (see Appendix E). New emerging categories were also added to the existing error categories and some categories were combined and renamed whenever necessary, that is, inductively Learners' incorrect answers for each component[22] under investigation were classified into error categories. For example incorrect answers in the "algebraic representation" component were grouped and each of them was given a name. Each error was categorized into only one error group. Sometimes, there was more than one error in a single answer. For a reliability check, the classification was discussed with two secondary school teachers and necessary amendments were made when there were inconsistencies. As a result of this comparison, I reduced the number of categories.

After developing a coding rubric, the coding of learners' responses followed the rubric strictly. I completed all data coding. The coding sheet was developed according to the rubric.

[22] Two main components of linear and quadratic functions including properties (gradient, intercepts and turning point) and representations (verbal, algebraic, graphical and table).

Each learner was labelled using codes L1-L26. Each coded error was put in the corresponding question (see Appendix F).

5.3 Analysis

Apart from the rubric construction, there were two other components to the quantitative analysis: the mean percentage error responses for each function component under investigation and the percentages of errors for each component. The percentages of errors for each component have helped me to explain the mean percentage error.

For each error category identified during the rubric construction, I calculated the percentage occurrence of a particular error in that category. For this, the number of learners who made this error was divided by the total number of learners (i.e. 26 learners). When the same error appeared in different questions, I calculated the percentages separately for each item. I used these percentages later to calculate the mean number of errors for each function component under investigation. The overall mean percentage for each component was obtained by calculating the average of percentage of incorrect answers (Appendix G).

The percentage of learners who provided an incorrect answer per question was recorded, as illustrated in the fourth column of table 5.1 below. The most frequent errors of different questions were recorded in the second column. The table makes the provision in the last column for displaying the mean percentage errors for each category (coded errors). An explanation regarding the values that appear in the mean percentage error column follows.

Concept	Coded errors	Question number	% of occurance	% mean error
Gradient	interpretation	1.6	92%	94%
		1.7	96%	
Table	translating	7.1	8%	47%
	interpretation	7.2	33%	
	classification	7.3	38%	
	interpretation	7.4	100%	
	interpretation	7.5	57%	
Graphs	interpretation	1.1	4%	39%
		1.2	15%	
		1.3	15%	
		1.4	4%	
		1.5	100%	
	translating	3.1	33%	
		3.2	62%	
		3.3	33%	
		3.4	33%	
		3.5	8%	
		4.6.1	59%	
		4.6.2	44%	
		5.1	38%	
		6.1	93%	
Algebraic representation	classification	2.1	8%	34%
		2.2	65%	
		2.3	8%	
	translation	4.2	52%	
Intercepts	calculations	4.3	38%	28%
		4.4	20%	
	intepretation	5.2	17%	
		5.3	36%	
Turning point	intepretation	4.5	33%	19%
	calculations	5.4	5%	

Table 5:1 Quantitative analysis of common errors

The mean percentage for each component suggest that gradient, table and graphs had the highest percentage of errors followed by algebraic representation, intercepts, and turning points.

Gradient

Learners seem to be experiencing difficulties with the gradient concept, with mean percentage of 94%. A high number of learners made errors in questions that were testing this concept, i.e. questions: 1.6 and 1.7. Difficulties with the gradient concept have been documented in the National Diagnostic Report (DoE, 2012) as well as in other past research (e.g. Haapasalo, 2003; Stump, 1999). Questions 1.6 and 1.7 were multiple choice questions which were designed to elicit interpretation errors with a gradient. These two questions seemed to be problematic and it is not clear whether learners guessed or not, since multiple-choice problems do not require learners to show their solution process. Hence these two tasks were investigated further in the interview.

Table and notation

Many questions under “table” (question 7) involved interpreting (local and global properties) of the given table using function notation and classifying the function suggested by the pattern in the table. Many unattempted questions were observed in learners’ responses. This bears evidence that learners were experiencing difficulties with interpreting a table and using function notation. Only a few obtained the correct answers. Others who have reported difficulties with function notation include Van Dyke and White (2004). In this question it is not clear whether learners were experiencing difficulties with either function notation or interpretation of the table. Hence, this question was investigated further in the interview.

Graph

Graph questions involved interpreting both local and global properties, constructing the graph, translating the graph to another form of representation i.e. algebraic. Emerging errors on this component were interpretation and translation errors. For example in question 3.1 (see figure 5.1) the given graph was associated with a linear function.

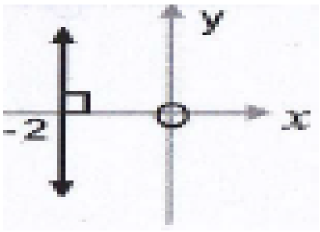


Figure 5:1 Example of interpretation of a graph

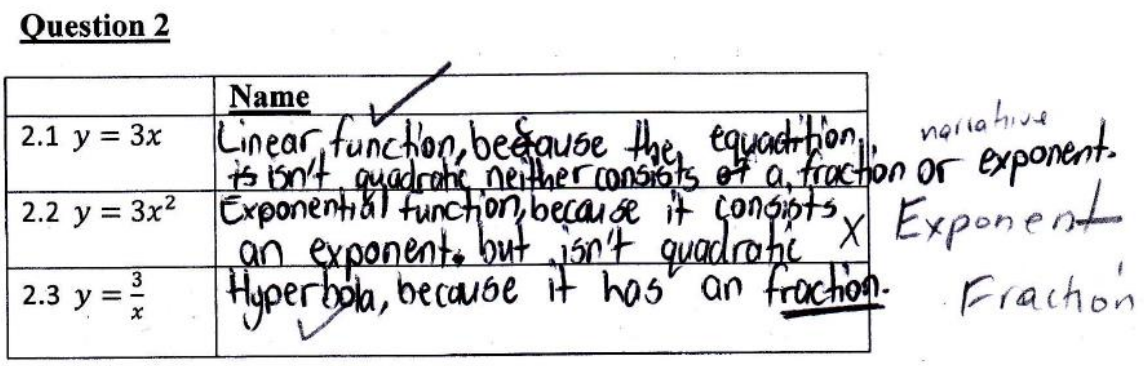
Some of the reasons offered were that the graph looks like a straight line, hence the association with a linear function.

Algebraic representation

Questions on algebraic representation involved classification tasks and translation tasks. Emerging errors on this component were translation and classification errors.

Classification

Classification errors resulted from incorrect classification of linear, quadratic and exponential functions. The underlying competencies that are involved when classifying include interpretation of global properties of the representation. For example, in Question 2, the quadratic function $y = 3x^2$ was incorrectly classified as an exponential function by 65% of learners because they saw an exponent (see figure 5.2).



Question 2

	Name	
2.1 $y = 3x$	Linear function, because the equation is isn't quadratic neither consists of a fraction or exponent.	narrative
2.2 $y = 3x^2$	Exponential function, because it consists an exponent. but isn't quadratic X	Exponent
2.3 $y = \frac{3}{x}$	Hyperbola, because it has an fraction.	Fraction

Figure 5:2 Example of classification error

Translation

Learners were expected to construct a graph from a given algebraic equation, i.e. translate. The translation was done incorrectly. For example in question 4.2 learners were required to plot the graphs of linear and quadratic functions $f(x) = x^2 - 4x - 5$ *and* $g(x) = x - 5$ on the same set of axes. In figure 5.3 below two graphs were plotted incorrectly. First a linear function where an x intercept was given as 1 and y intercept as -5. Secondly in the quadratic graph, the values of the x intercepts are -4 and +4, and y intercept is -5.

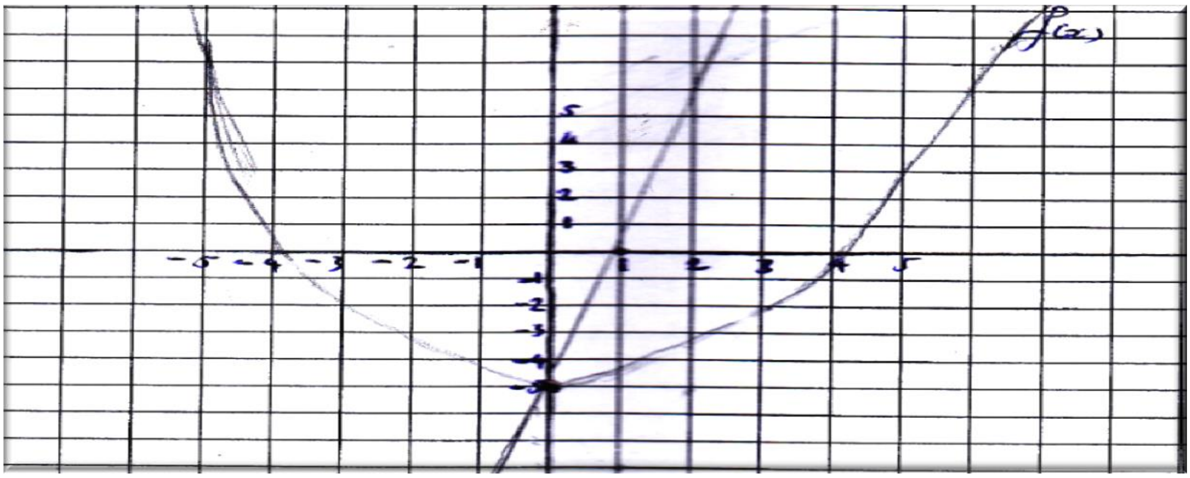


Figure 5:3 Example of translation error

Plotting the two graphs presented two types of errors: one associated with the interpretation of intercepts. Possible explanation of this could be that the x intercept is associated with the coefficient of x in the general standard form $y = mx + c$ where 'm' is associated with a value of 1, i.e gradient =1 as previously reported by many (e.g. Brenner et al., 1997; Moschkovich, 1999a). According to Brenner, this kind of thinking originates from previous learning when the linear function was first introduced. For example in the equation y= $x - 1$, we know that to get the x intercept, we let $y = 0$ which implies that $x = 1$ and for y intercept let $x = 0$ then $y = -1$. From this equation the coefficient of x is one, which is the same value as the x intercept. So this reasoning is generalised for all linear equations: the coefficient of x is the x intercept.

In the quadratic function $f(x) = x^2 - 4x - 5$, the x intercept is understood to be the coefficient of x i.e -4 and since it is known that the quadratic function has two x intercepts, the positive $(+4)$ value is included to be the other intercept. This came from the reasoning provided by one of the learners during the pre-pilot study. The y intercept is understood to be -5. This reasoning could be originating from overgeneralisation of properties of linear function as is the case when plotting the linear graph. For example, in the provided quadratic function: $f(x) = x^2 - 4x - 5$, the first term (x^2) is ignored and anything attached to a variable x is seen as an x intercept. These results are consistent with findings of Moschkovich (1999a).

Intercepts

Questions on intercepts involved understanding of intercept concept, interpreting (global and local properties). Global properties include understanding of the intercept concept. Local properties include interpretation of the algebraic equation and using that information to construct the graph. Emerging errors on this component include interpretation and calculation errors.

Calculation

Calculation errors resulted from misusing algebraic algorithms and lack of basic mathematical skills. For example in the figure 5.4 below the learner was asked to determine the y intercept of the quadratic function $f(x) = x^2 - 4x - 5$. It is clear that the learner understood the rule ($for\ y\ intercept\ let\ x = 0$). The substitution was done correctly. The problem arose when the learner multiplied the numbers. Therefore, this error is related to calculation and not to intercept concept.

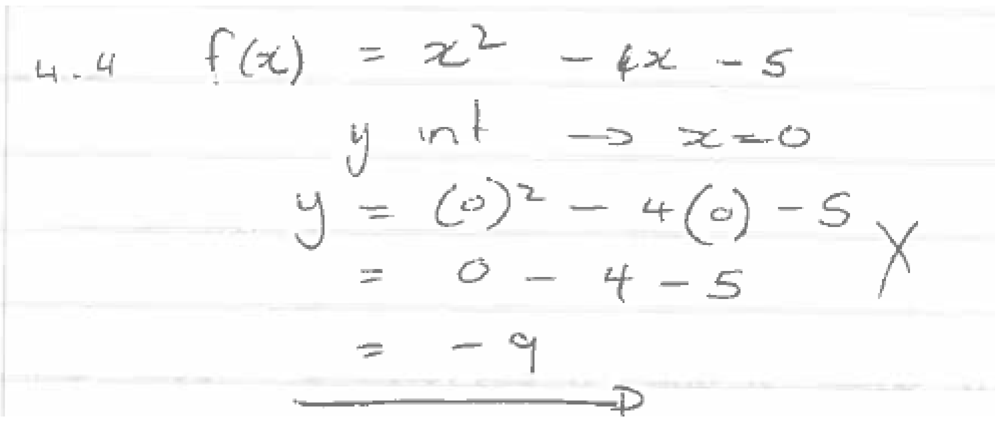


Figure 5:4 Example of a calculation error

Interpretation

In Questions 5.2 and 5.3 (see Appendix B), a quadratic graph was given together with all co-ordinates of points of intersection and turning point i.e. $(-3, 0)$; $(-2, -1)$ and $(0, 3)$. Learners were then asked to determine the coordinates of x and y intercepts. In their responses all x and y co-ordinates of the given co-ordinates were classified as x and y

intercepts respectively i.e. x intercepts were given as: $-3; -2$ and 0 and y intercepts were $0; -1$ and 3.

Turning point

Questions on turning point concept involved understanding of the turning point concept, interpreting them from the graph and calculating them from the algebraic equation. Errors that emerged were interpretation and calculation errors.

In question 4.5 (see Appendix B), learners were asked to determine the co-ordinates of the turning point of the quadratic function $f(x) = x^2 - 4x - 5$. The incorrect answers to this question presented two errors: one associated with the intercept concept and one with the turning point concept. Possible explanation of this could be that learners took the coefficient of x i.e -4 to be the x co-ordinate of the turning point and the y intercept -5 to be the y co-ordinate of the turning point. This could be attributed to the familiarity with quadratic standard form equation $y = a(x-p)^2 + q$, where co-ordinates of the turning point are (p, q), so when it comes to standard form $y = ax^2 + bx + c$, the coefficient of x (i.e $'b'$) is seen as the x co-ordinate of turning point and $'c'$ the y co-ordinate $(-b, c)$ as illustrated in figure 5.5 below:

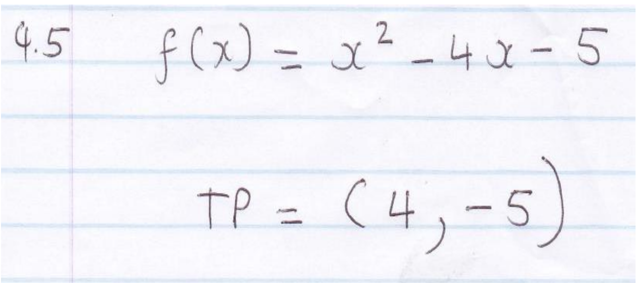


Figure 5:5 Example of intercept-turning point error

From this analysis it can be gathered that the learners' common errors include: interpretation, translation, calculation and classification. In the next section I am going to discuss these common errors.

5.4 Findings

5.4.1 Common errors

Inside each component there were clusters of errors with different percentages. The advantage of this analysis is that it provided an opportunity to separate the most frequent errors. A graphical representation of these percentages (Figure 5.6) shows some patterns in the data.

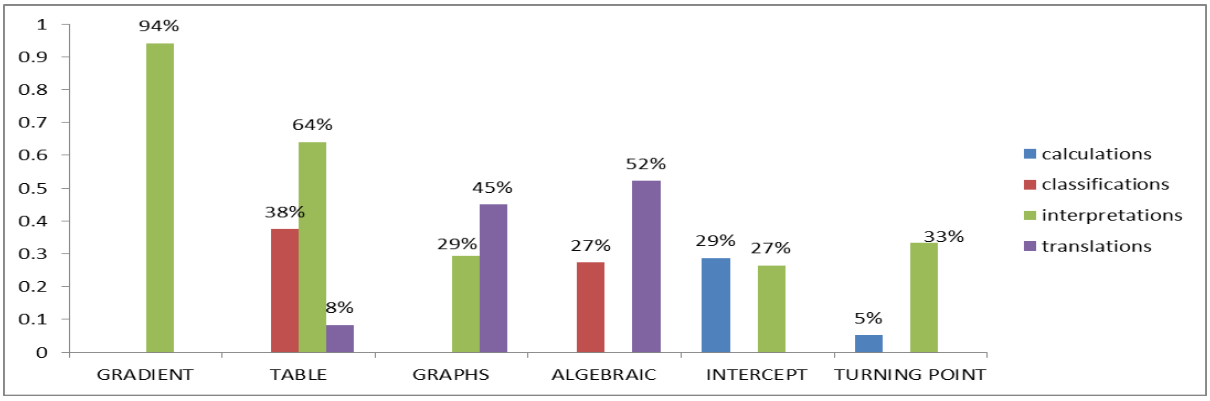


Figure 5:6 Common errors

The vertical bars represent the error types. Under gradient, only one error type was observed, i.e. the interpretation (94%). The pattern for table indicates that learners experience difficulties with interpretation (64%) and classification (38%). For graphs, the bars from the lowest to the highest represent: interpretation (29%), translation (45%). Algebraic representations evidence difficulties with translation (52%) and classification (27%). In the intercept concept recorded: calculation (29%) and interpretation (27%) errors. Turning point errors included interpretations (33%) and calculations (5%).

Interpretation errors

The results of the analysis show that interpretation competence is problematic. This was evident in all components under investigation where the highest percentage of learners who struggled with this was recorded for each component. Interpretation involves interpreting global and/or local properties of the function concept. The interpretation error resulting from interpreting global properties could be due to interpretation of the representation as a picture (Arcavi, 2003) and not attending to the underlying features of the representation. This is consistent with previous reports by Bell and Janvier (1981). Graphical interpretation has posed challenges for many learners. Empirical work that supports this notion was documented in the National Diagnostic report of Grade 12 results (DoE, 2012). Interpretations errors are linked to classification and translation errors.

Classification errors

Classification errors are associated with understanding the concept definition and interpretive skills (Leinhardt et al., 1990). The results indicate that learners seemed to be lacking concept definition and were experiencing difficulties with interpretation which resulted in incorrect classification of functions.

Translation errors

The analysis shows that learners were struggling to translate between different forms of representations, graph-algebraic (45%) and algebraic-graph (52%). Errors of this nature have been reported by Even (1998), that they are due to pointwise approach i.e. calculation.

Calculation errors

The results indicate that learners had difficulty with algebraic calculations. Errors on calculations have been widely documented in literature and also reported to be problematic in the National Diagnostic Report (DoE, 2012).

5.5 Conclusion

In this analysis it was evident that the majority of learners had difficulties with interpretation competence. This was seen in all components under investigation, with each component recording a high percentage of interpretation error. According to Booth (1988), in order for one to investigate learners' difficulties with a certain algebraic concept, an identification of common errors is necessary. The aim of this analysis was to identify the common errors learners make when working with linear and quadratic functions. The question remains: is there any relationship between learners' discursive actions and errors? In order to answer this question this study first conducted a further investigation on learners' mathematical discourses on components of linear and quadratic functions in terms of commognitive framework, and later tried to link the features of the mathematical discourse with the common errors found in this study.

Chapter 6 CODING OF INTERVIEW

6.1 Introduction

In this chapter, I present a detailed description of the analytic framework that has been used to provide lenses into words and word use, routines, visual mediators and narratives that learners used when talking about function objects[23]: intercepts, quadratic function, linear function and gradient. Further in this chapter, I describe the organisational language developed for analysis of data, and the processes through which this language emerged.

6.2 The analytical framework for analysis of interviews

6.2.1 Arithmetic Discourse Profile (ADP) analysis tool

In order to respond to the last two questions of the study, the learners' forms of communication in function's discourse when talking about components of function were interpreted in terms of the commognitive tools used by Sfard (2008). Sfard's work is still under developing. Some who have developed her work include Ben-Yehuda et al. (2005). The analytical framework for analysis of interviews in this study was adapted from the Arithmetic Discourse Profile (ADP)[24] analysis tool developed by Ben-Yehuda et al. (2005). This tool was used to analyse two primary school learners' interview discursive actions in the arithmetical discourse. The ADP contains two main dimensions divided into sub-dimensions.

These two dimensions refer to basic aspects of discourse: the subject (author) and object as they are constructed by the interviewees. The subject (author dimension) refers to the interviewee's identities, that is, endorsed stories about the person and these are not in focus in this study.

Ben-Yehuda et al. (2005) describe discursive objects through examples such as calculations, estimations, comparisons and money transactions with respect to interviewees' use of arithmetical words, mediators, routines and arithmetical endorsed narratives. For my study, discursive objects include: intercepts, gradient, quadratic and linear functions. The dimension

[23] As a reminder mathematical (function) objects are those things that are talked about (Nachlieli & Tabach, 2012). For example function components: intercept, gradient, linear and quadratic functions.
[24] In this study 'ADP' will be used to refer to arithmetic discourse profile

of interest for this study is the object dimension, because of its focus on objects of mathematical discourse.

In their model, Ben-Yehuda et al. (2005) suggested four main features of mathematical discourse. Their propositions were that learners' discursive moves could be viewed from:

(a) *Words/word use:* words together with discursive actions;

(b) *Routines*: they divided routines into three subset of meta-rules:

- applicability (focuses on how the learners are implementing the routines);
- flexibility (includes use of different routine procedures);
- corrigibility (ability to correct one's discursive procedure) .

They focused on these three properties in order to determine the 'how' and the 'when' of the routine procedure. The 'how' routine includes flexibility and corrigibility routines. According to Ben Yehuda and others, the 'how' routine performance is normally a focus of a teacher or a researcher. The 'how' routine helped me to judge the learners' discursive skills. The 'when' of the routine considers properties of routine procedure-applicability. These are procedures implemented in reactions to straightforward requests such as 'determine the intercept'.

(c) *Visual mediators* were categorised into symbolic, iconic and concrete mediators; and

(d) *Narratives*: these were categorised into substantiations, derivations and recall.

Ben-Yehuda et al. (2005) study was organised around the question of the degree of objectification[25] in learners' discourse and the way in which this feature was linked to learners' arithmetical proficiency. From their study, they were able to summarise features of mathematical discourse in learners' interviews with illustrative examples, and hence developed their model as illustrated in the table 6.1 below. It should be noted that I have recontextualised examples to reflect those of a function's discourse.

[25] Objectification is an important property of mathematical discourse, where processes and actions are replaced by talks about objects. This was discussed at length in Chapter 2.

ADP	Categories	Description	Example
Word use	Word use	Words (vocabulary) are key words used for communication purposes in a discourse. Word use refers to the ways in which participants use words in their mathematical discourse, in other words the participants' discursive actions	A word parabola signifies a graphical representation of a quadratic function and its meaning and uses is shared amongst participants in the mathematical (functions) discourse.
Visual mediators	Symbolic	These are symbolic /algebraic equations and expressions	An algebraic representation of a linear function $y = 2x + 1$
	Iconic	Visual objects that can be scanned with our own eyes.	Graphs, tables, diagrams, pictures
	Concrete	Concrete mediators are objects that can be seen or imagined Concrete mediators are objects that can be physically seen, or manipulated. They can also be imagined (i.e. through mind's eye).	Physical objects include manipulation of hands, rulers. Imagined objects include emoticons(smiley/frowning face),
Routines	Applicability	The term applicability refers to the learners' ways of matching routine procedures with tasks. In other words, the applicability conditions are rules that increase the likelihood in which the routine procedure is likely to be brought to action. These can be evoked through prompts or questioning which maybe verbal and from others or oneself.	These procedures can be observed in real life contexts (colloquial), e.g. use of concrete mediators maybe hands or fingers. At the same time they can be observed in the mathematical discourses, for example they can be performed with symbols through calculations (syntactic mediator). To illustrate, when given a quadratic function, $y = 2x^2 + 3x + 1$ and asked to provide the turning point, a learner might match a routine procedure of using the well-defined formula for the axis of symmetry $x = b/2a$, and calculate the x and the y co-ordinates of the turning point.
	Flexibility	This refers to the mathematical discourse of a person who when faced with a request for calculation, can perform more than one routine response .Flexibility also involves using different but equivalent representations (mediational flexibility)	For example, when a learner is required to determine an x intercept of the function- $f(x) = x^2 - 1$, s/he may calculate, or use a table to get the co-ordinates of the x intercepts i.e. where $y = 0$, or draw a graph to see where the graph cuts the x axis. Using different routine procedures produces different visual mediators such as table and graphs(mediational flexibility)
	Corrigibility	Corrigibility refers to the ability of assessing and correcting one's discursive performance	This action may occur for an example when a learner incorrectly classifies the function represented in the symbolic mode, and decides to switch to iconic mode through construction of a graph
Narratives	Derivation	These are discursive procedures that result in new narratives	A substantiation that resulted from routine procedures of calculating
	Substantiation	Substantiations addresses the ways in which decisions are made about whether to endorse a narrative i.e. to decide that it is true. Some substantiations are evoked through prompts and questioning during interview by asking questions like 'How do you know; how did you decide; why is that the case...'	The responses starting with, 'because', mostly indicate substantiation.
	Recall	A narrative that was endorsed through summoning previously endorsed narratives	For example prototypical examples of different functions such as quadratic $y = x^2$, hyperbola $y = 1/x$. Standard forms of different functions are also included in this category.

Table 6:1 Adapted arithmetic discourse profile (ADP)

The categories of ADP provided the language for organising the data for the interviews. It is important to note that some categories from the ADP framework were not included in the above table. I will elaborate shortly the reasons for this decision. The ADP has a number of limitations for my study since it was developed to analyse arithmetic discourse for primary school learners. As such the ADP has limitations in terms of providing tools for analysing the nature of secondary school learners' discourse. Secondly, the ADP model is limiting in providing tools for gaining entry into learners' function discourse. Hence the challenge for me as a researcher was to redevelop the tool to suit my needs for this study, that is, to develop methodological approach that would capture components of functions discourse. Doing this meant identifying complementary literature on function concept to fill the gaps. For example in providing organisational language for words/word use, routines and visual mediators, I have drawn from reviewed literature on function concepts in Chapter 3.

For my study, I am going to analyse learners' substantiating narratives for a number of reasons. Firstly, it is through learners' substantiating narratives that one can access learners' decisions as to endorse their narratives (Sfard, 2008). Secondly, according to Ben-Yehuda et al. (2005), substantiations are context sensitive and learners from different grades have different substantiating methods. Ben-Yehuda and others provide an example of the school context where the nature of mathematical discourse derivations performed by the learners are detailed enough in order to convince the teacher that the learner is familiar with different aspects of routine procedures. Further, these derivations (calculations) in the school context are normally followed by the requests (prompts). This description of substantiating narratives fits in very well in my study since the interviews were done within the school context.

6.2.2 Refining the codes for analytical tool

The adapted ADP framework was reconceptualised to include categories of emerging learners' words and word use, routines, visual mediators and substantiating narratives from the data.

Words were used in two different ways: mathematical and combination of colloquial and mathematical.

A number of routine procedures emerged from the learners' interviews. These were categorised under three meta-rules. These are: (1) *applicability*: constructing, interpreting, using method (intercept), calculating, demonstrating, using visual trigger, and comparing (to a standard form)*;* (*2) corrigibility:* correcting; and (3) *flexibility*: using multiple routines, and translating.

Different visual mediators were used: iconic (graphs, tables); symbolic (algebraic equations), and concrete mediators such as physical manipulated mediators (use of gestures with hands and fingers) and imagined mediators (use of emoticons[26]).

Narratives were substantiated through: derivation, construction[27], rule, visual[28], and recall (summoning standard form).

The adapted arithmetic discourse profile (ADP) will be referred to as the function discourse profile (FDP). The following table is the summary of relevant constructs of the FDP.

Features of mathematical discourse		Emerging features
Words/Word use	Colloquial	Combination of literate and colloquial
	Literate	Mathematical
Routines	Applicability	Calculating
		Constructing
		Using method(intercept)
		Comparing
		Interpreting
		Using visual trigger
		Demonstrating
	Corrigibility	Correcting
	Flexibility	Using multiple routines
		Translating
Visual mediators	Iconic	Scanned
		Constructed
	Symbolic	Scanned
		Syntactic
	Concrete	Manipulated
		Imagined
Narratives	Substantiation	Derivation
		Construction
		Rule
		Visual
		Recall

Table 6:2 The function discourse profile (FDP)

[26] Emoticons are metacommunicative pictorial representation of a facial expression [☺].
[27] Sfard (2008) refers to the construction as discursive procedures that result in new narratives. This study will refer to constructions as substantiated narratives that result from a constructing routine (e.g. sketching of the graph).
[28] Visual refers to a substantiation narrative that resulted from scanning the visual mediator with our own eyes (seen) or with 'mind eyes' (imagined).

The main categories in table 6.2 (highlighted columns) i.e. the first column and second column were adapted from ADP analysis tool (Ben-Yehuda et al., 2005) and analytic resources emerging from the literature reviewed in Chapter 3. The categories in the third column were emerging from learners' talk during the interview, i.e. inductively.

In the next section I will clarify the relationship between a number of technical constructs/concepts of the analytical framework and emerging categories as illustrated in table 6.2 above.

6.3 Examples of coding using data

In this section I am going to show some examples of how the interview data for this study was coded in accordance with the FDP.

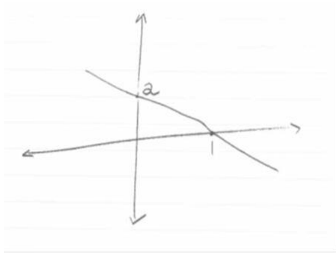
6.3.1 Words/ word use.

Two categories of word use have emerged from the learners' discourse. Learners were observed using words in mathematical ways and using a combination of mathematical and colloquial discourse.

Mathematical

To be coded mathematical, learners were observed using words in a mathematical way. This was evident through their discursive actions: routines, visual mediators and their substantiating narratives. The following transcript was chosen in order to illustrate how words were used mathematically. This extract is from the actual data that I collected.

Line	Speaker	What is said	What is done or seen
26	Billy	The intercept is a point where the graph passes the x and the y axis . the x and y axis	
27	Me	Hmmm can you give me an example	
28	Billy	Hmm y = 2 yah or x = 1 something like that	
29	Me	Okay that will be an intercept	
30	Billy	Yah	
31	Me	How would you represent in graph when you say it's a point where it cuts the x axis	

32	Billy	Let's say it's a straight line then my x is let's say it is equal to 2 and this is 1.	*drawing* 
34	Billy	And here it's a straight line	
36	Billy	Something like this	
37	Me	So what is your 1 there	
40	Billy	My 1 is the x intercept and my 2 is my y intercept	

In the above extract Billy was asked to define an intercept concept. He first gave a mathematical definition in line 26. This is an accepted definition as will be described in the next chapter (in section 7.2.1). When Billy was asked to represent his response in a different form (graph), he was observed constructing a graph (in line 32-40). This is an accepted routine (competency) used in function discourse. The constructed graph is a visual mediator also used in function discourse. Billy's discursive moves were consistent with those found in the school mathematics discourse, such as mathematical words/word use, drawing (routines), graph (visual mediator) and mathematical definition (endorsed narrative).

Combination

Learners were observed using a combination of colloquial and mathematical discourse: sometimes more colloquial, sometimes more mathematical and sometimes a good mixture of colloquial and mathematical. A typical example coded '***combination***' is shown in the extract below:

Line	Speaker	What was said	What is done or seen
38	Nhlanhla	I explain it is as a **touch** when I say intercept	using gestures by crossing fingers
39	Nhlanhla	Because its where the **graph** either **cuts** the y intercept or the x uhm intercept, **axis** I mean	
40	Me	Okay can you give me an example of that?	
41	Nhlanhla	Uhm okay on the equation of straight line, $\boldsymbol{y = mx + c}$, where $\boldsymbol{c}$**, they intercept**, it's the one touching on the **y axis** yah in a yah	
42	Me	Okay lets now you are doing grade 11 you want to explain to a grade 10 learner that when you are talking about intercept what comes to mind, how can you explain to them	

43	Nhlanhla	I could tell them that the intercept you just look at where the **graph cuts** your **Cartesian plane** when it **touches the y axis** , its where **graph meets** your y axis, that's your y intercept and on your $\boldsymbol{x}$ **axis** that's where you know graph touches x axis.Thats where you know that's your intercept	

Figure 6:1 Example of Nhlanhla's transcript[29]

Nhlanhla is a good example of where her way of speaking moves from colloquial to mathematical. In line 38-39, Nhlanhla's talk is more colloquial. She is using colloquial words such as touch and cuts and uses them in a colloquial way by showing this touching /crossing through the use of gestures by crossing her fingers. This action of crossing fingers was classified under applicability routine and was coded as '***demonstrating***'. The fingers were used to mediate her talk; and they were classified under concrete mediators and coded as ***'manipulated'*** mediators. Nhlanhla was substantiating her narrative through the visual gestures she was making with hands. This way of substantiation was coded as '***visual***'[30]. Through prompting (in line 40), Nhlanhla' goes from describing things in a colloquial way to a more mathematical way (in line 41). This was evident by using a symbolic mediator (standard form of linear function). In this line, Nhlanhla interpreted the properties of the equation, the y intercept in this instance.

Nhlanhla's discourse in line 43, demonstrates the use of a good mixture of colloquial and mathematical discourse. She used colloquial words (touch and cuts) and linked them to the graph (Cartesian plane) and interpreted the properties of the graph, that is y axis/x axis.

It is interesting to note however that what is missing from Nhlanhla's discourse is the notion of intercept as a point.

6. 3.2 Routines, visual mediators and endorsed narratives

In Chapter 2, routines, visual mediators and narratives were discussed individually. These three elements are intertwined with each other. In the following section, I will expound the

[29] Other similar tables do not have captions. I have included the caption on this transcript because I want to refer back on this transcript later.
[30] Recall that visual is the substantiation resulting from iconic and/or concrete mediator seen with our own eyes or seen with mind's eye.

method used in coding these, as well as the interrelatedness of the respective discursive actions as illustrated in table 6.3 below:

Visual mediators / Routines	Symbolic Mediator	Iconic Mediator	Concrete Mediator
Calculating	**Derivation**	**-**	**-**
Constructing	**-**	**Construction**	
Demonstrating	**-**	**-**	**Visual**
Comparing	**Recall**	**-**	**-**
Using visual trigger	**Visual**	**Visual**	**Visual**
Interpreting	**Visual**	**Visual**	**-**
Using intercept rule	**Rule**	**Rule**	**-**

Table 6:3 Matrix for interrelatedness of discursive actions

Table 6.3 provides an illustration of how applicability routines (first column) relate to visual mediators (first row) and substantiated narratives (highlighted section). The table can be interpreted as follows: when the calculating routine is applied to the symbolic mediator; this action produces a particular kind of substantiated narrative (derivation).

The following transcript will be used to explain how routines, visual mediators and substantiations were coded. This extract was chosen purposefully, since it covers a wide range of categories and will therefore serve to illustrate how I categorised the data for those categories. I will also provide the relevant indicators for these categories.

Line	Speaker	What is said	What is done/seen
43	Me	Okay let's say now you are given this function equation y = 2x + 1	Pointing at the equation y=2x+1
44	Billy	Yes	
45	Me	Neh and then how would you find the y intercept from that equation	
46	Billy	I would say let y = 0 the x intercept	
47	Me	Hmmm	
48	Billy	I would say let x=0 for let y intercept	$y = 2x + 1$ $y = 2(0) + 1$ $y = 1$
50	Billy	Then I solve it	
51	Me	Okay	
52	Billy	The y intercept right	
54	Billy	x = 0 then I have 2 times 0 plus 1	
55	Me	Hmmm	
56	Billy	Then y =1	
57	Me	Okay and is there any other method that you can use other than this one to calculate the to get your to get your x intercept.	
58	Billy	Probably the Table , table method	
59	Me	Table you so use a table method. And the y intercept	
60	Billy	I would use the same one but I will let my x = 0	
61	Me	Okay . now you said you would do a table so take me through e lento how you would do the table	
62	Billy	The table. This should be my x and this should be my y. then I would have different values	drawing a table x | -1 | 0 | 1 y | -1 | 1 | 3
64	Billy	For the x here	
65	Me	Yes	
66	Billy	Let's say my x is - 1. I will substitute -1 with + 2 here which will give me -1	
67	Me	Hmmm	
68	Billy	Yah then 0 this is zero will 1 , 1 , will be 3. No 1 this will be 3 just like that carry on. Then this will be my y intercept	
69	Me	Why y intercept	
70	Billy	Because x is zero	

In this extract, Billy was asked to determine the y intercept from the given equation $y = 2x + 1$. When Billy was responding to this question he first suggested applying the intercept method in line 46 and 48. This action was coded as '***using intercept rule'***. The resulting substantiation was coded as ***'rule'.*** In line 50-54 he suggested applying the intercept method by substituting values in the given equation and solving it. This action was categorised under applicability meta-rule and was coded '***calculating'*** routine. Attending to the symbolic mediator in this way was coded as '***syntactic***'. Furthermore, this discursive action of substantiating through calculation was coded '***derivation***'.

In line 57, Billy was prompted to provide examples, and he suggested use of a table. In line 62-68 he was observed constructing a table. This action was classified under applicability meta-rule and was coded '***constructing***'. The resulting iconic mediator (table) was coded as '***constructed***'. After constructing a table, Billy was observed interpreting the table using the intercept rule (in line 68-70). This action was coded as '***interpreting***'. All these actions were done in efforts to substantiate intercept definition. The resulting substantiation from constructing the table was coded as '***construction***', and that from interpreting the table (an iconic mediator) was coded as '***visual***'.

In this extract, Billy was observed using four different kinds of applicability routines to produce the same substantiated narrative: calculating, interpreting, constructing and using method (intercept). He was also observed switching between different mediators (from symbolic to table) i.e. mediational switching. All these actions were categorised under flexibility and were coded as '***multiple routines***' and '***translating***' routine respectively.

The above extract was not sufficient to illustrate all emerging categories such as comparing, using visual trigger and corrigibility. In the next extract I will try to address these categories.

Line	**Speaker**	**What is said**	**What is done/seen**
125	Me	Okay this one?	pointing at y=x/2 +3
126	Billy	That one is a hyperbola	
127	Me	Why	
128	Billy	No It's a straight line	
129	Me	Why is it a straight line	
130	Billy	Because as I said previously the x is to the power of 1 but on this one the gradient is half	
131	Me	Okay u have spoken about a hyperbola, what came to your mind quickly before you started to change your mind what	
132	Billy	I **saw a fraction** that's why I say it's a hyperbola	

133	Me	So when you see a fraction its err you think it's a hyperbola	
134	Billy	Yes	
135	Me	Why	
136	Billy	Because normally a hyperbola it has a the equation for hyperbola it's $a\ /x$	
137	Me	Hmmm okay	
138	Billy	So that's why I thought but in this case the x is on top not underneath	

In this task Billy was asked to classify the function $y = x/2\ + 3$. In his response he first classified the function as a hyperbola (an incorrect classification, see line 126). However in line 128-130, Billy was observed rescanning the equation and correcting himself by offering a correct classification i.e. a straight line. He also substantiated his response by suggesting properties of the linear function (in line 130). This action was categorised under corrigibility meta-rule, and was coded as '***correcting***'. When Billy was probed further to give reasons of his initial response in line 126, he was observed using a visual cue of fraction. This action was coded as ***'using visual trigger'.*** Attending to a symbolic mediator in this way was coded as **'*scanned*'.** The resulting substantiation was coded as **'*visual*'.**

An action coded as '***comparing***' can be observed in line 133-138 where Billy is comparing the given equation to a standard form of hyperbola. This standard form was somehow recalled from memory. A resulting substantiation from this kind of action (recalling standard form) was coded as '***recall'***.

In some of the transcripts, learners were observed demonstrating their responses through the use of concrete mediators. Two subcategories of concrete mediators were observed: physical manipulated and imagined objects. The concrete mediator that was coded '***physical manipulated***' is evidenced in figure 6.1 above, in line 38 where Nhlanhla is crossing fingers. The '***imagined***' concrete mediator is evidenced below:

Line	Speaker	What was said	What is done or seen
166	Billy	Mmm	
167	Me	What does 'a' represent?	

168	Billy	What	
169	Me	that diagram is exactly the same, this is an algebraic form, this is a graphical form	
170	Billy	Mmm	
171	Me	Okay, let me ask like this, is 'a' positive here?	
172	Billy	No, it's not positive**,** it's negative	
173	Me	Why is it negative	
174	Billy	Because **it's a frown**. So when its frowning it means it's a negative	

In this transcript Billy was asked to interprets the coefficient of x^2 i.e. $'a'$ in the equation $y = ax^2 + 1$. A graphical representation of this function was also given. A word 'negative' (in line 172) was associated with a graphical representation of a quadratic function (parabola). A concave down parabola triggered from memory an image of an emoticon (frowning face) normally used in everyday language of social networking (Facebook). The routine action of applying concrete mediators was coded '***demonstrating***'. And the resulting substantiations were coded '***visual***' (seen with eyes or seen with mind's eye).

6.4 Conclusion

In this chapter I have introduced the analytical framework used for analysing the mathematical discourse in this study. The presentation served three purposes. First, to introduce the analytical framework of Ben-Yehuda et al. (2005) in elaboration of the second research question. Second, to describe the organisational language developed for analysis of interview data. Third, this chapter will hopefully help the reader to better understand how I arrived at the empirical findings presented in Chapters 7 and 8.

Chapter 7 ANALYSIS OF INTERVIEW AND FINDINGS

7.1 Introduction

The aim of this study was to describe the mathematical discourses of grade 11 learners and investigate the relationship between these and learner errors. This chapter presents the results of the analysis of interview data (transcripts).The interview questions were structured around the following objects of function discourse: intercept; linear function; quadratic function and a gradient. In this chapter, I present a detailed analysis of extracts from five learners. The analysis of interviews is organised around the following analysis questions:

1. What features of mathematical discourses (i.e. words/words use, routines, visual mediators, endorsed narratives), are evident in the learners' discourse, and how can these be described?
2. Is there a connection that exists between these (features)?
3. In which way are they (features) linked to learners' errors?

The five learners interviewed in this study were given pseudonyms: Nomsa, Daniel, Nhlanhla, Jennifer and Billy. As mentioned in Chapter 4 (methodology chapter), these five learners were purposefully chosen based on the prevalence of errors in their test answers, explanations supplied in the answers and the ability to communicate. I was also hoping to see patterns of mathematical discourses from these different learners that might be linked to errors learners make without any desire to make any generalisations.

The interview was analysed by using discursive framework analysis tools (Sfard, 2008). I began the analytic work equipped with the transcript, the audio recording of the interviews, the accompanying learners' written work and my field notes.

The analysis of the interview began with transcription of data. During this process, I was guided by the principles of the interpretive judgments of a researcher (Sfard, 2008). Sfard specifically addresses the interpretive status of claims made by focusing on 'what is said' and 'what is done or seen'. 'What is said' focuses on the words the learner uses when identifying '*the object of her or his attention*' (Sfard 2000, p. 304). 'What is done/seen', is considered as what the learner is '*looking at, listening to*' when speaking (Sfard 2000, p. 304). It is also made up of '*the image a person perceives (or imagines*)' and also the '*attending procedure* *she is performing while scanning this image*' (Sfard 2000, p. 304). I made my interpretations based on these two aspects (what is said, and what is done/seen) as the clues to identify what meaning the speaker may be making. To describe what learners were saying and make consequent interpretations, I listened to the audio recording many times as I concurrently read the transcript and scrutinized the learners' work. To interpret what learners were doing (or looking at), I revisited my recorded field notes I made during the interview sessions.

Following the transcription phase, I began to interpret learners' discursive actions using the adapted analytical framework from Ben-Yehuda and colleagues now referred to as FDP (function discourse profile). This model includes the descriptions of features of 'function' mathematical discourse. The model provides the analytical tools (lenses) to access learners' function discourse (see the descriptions of the model in Chapter 6).

Before going into the main analysis, it is important that I first describe the function discourse as described in the school mathematics discourse.

7.2 Analysis of school mathematics discourse on function objects

The aim of this section is to describe the nature of school mathematics discourse in South African schools. This will be done by discussing an accepted mathematical discourse on function objects under investigation: intercept; quadratic function; linear function and the gradient. This analysis serves two purposes. Firstly, this discussion helps me when analysing the interview transcripts to judge learners' discourses against the accepted school mathematics discourse. Secondly, it serves to provide content validity.

7.2.1 Intercept[31]

In the school mathematics curriculum the intercept can be represented algebraically, or in tabular and graphical form.

[31] In this chapter and from hereon, I use concept definition to mean word and word use – because I am using Sfard. Sfard does not use the word 'concept' often and intercept in discursive terms is a discursive object. She does discuss the use of the word concept in mathematics education, and defines this as word together with word use. There is a new quandary that is coming out of this study, which will be discussed in the conclusion chapter. This new quandary is about how we use a word 'concept'. It is common in mathematics education field. And you can see my struggle in this study: is a concept as an object, what is it when we talk about it in words & word use, is that a concept definition which is not the same as how Tall and Vinner (1981) talk about it.

An x intercept in a graphical representation is the point in the Cartesian plane where the graph crosses the x axis. And the y intercept is the point in the Cartesian plane where the graph crosses the y axis.

In a tabular representation the x intercept may be interpreted as the value of the x co-ordinate in the co-ordinate pair in the table where $y = 0$ i.e $(x, 0)$. A y intercept as the value of the y co-ordinate in the co-ordinate pair in the table where $x = 0$ i.e. $(0, y)$.

In the algebraic representation, one can find the x intercept by equating $y\ to\ 0$; $x\ \ to\ 0$ for y intercept. For example, the x intercept of the linear function $f(x) = mx + c$ is found by solving the equation $f(x) = 0$, the solution is $x = -c/m$,where $m \neq 0$. The y intercept of a linear function $f(x) = mx + c$ is found by calculating the value of $f(0)$, the solution is $f(0) = c$. In the standard forms of linear function $(y = mx + c)$ and quadratic function $(y = ax^2 + bx + c)$, the y intercept can be interpreted as the value of c. However, the x intercept does not appear directly in the equations of these forms.

In school mathematics textbooks, definitions for the *x*-intercept and *y*-intercept refer to their graphic representation e.g.: '*the x-intercept is the point where the graph cuts the y-axis. The y intercept is the point where the graph cuts the x-axis*.' (Campbell & McPetrie, 2012, p. 375).

7.2.2 Quadratic function

Quadratic function is the name associated with algebraic function of the form $y = ax^2 + bx + c$ (where $a \neq 0$ and a, b and c are constants). The quadratic function can also be represented in different forms such as: verbal, algebraic, tabular and graphical. The different representations make visible various facets of a quadratic function as shown in table 7.1 below.

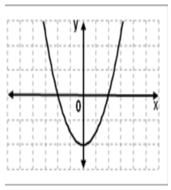
Name	**Other names**	**Verbal**	**Algebraic (general form)**	**Tabular**	**Graphical**
Quadratic function	Parabola (graph)	The function with a highest power of independent variable x equal to 2 (x^2).	$y = ax^2 + bx + c$ $y = a(x - x_1)(x - x_2)$. $y = a(x - p)^2 + q$	x y -2 4 -1 1 0 0 1 1 2 4	

Table 7:1 Different representations of a quadratic function

Other forms of representations (symbolic) are the canonical form: $y = a(x - p)^2 + q$, and the multiplicative form*:* $y = a(x - x_1)(x - x_2)$*.* The canonical form indicates the location of the parabola's turning point (p, q), while the multiplicative form discloses the location of the x-intercept (x_1; 0) and (x_2; 0) .

The graphs associated with quadratic functions are called parabolas. It is important to consider the effects of the parameter 'a', *o*n the parabola. Now, let us consider the function $y = ax^2$ to discuss this effect. Changing the value of a results in a vertical stretch of the graph of the function $y = ax^2$ (and of course the function $y = ax^2 + bx + c$*)*. The bigger the value of a *(i.e.* $a > 1$*)* the graph becomes narrower (or stretches vertically). Also, the smaller the value of value of a *(i.e.* $0 < a < 1$*)*, the graph becomes wider and shrinks vertically. Figure 7.1 illustrates:

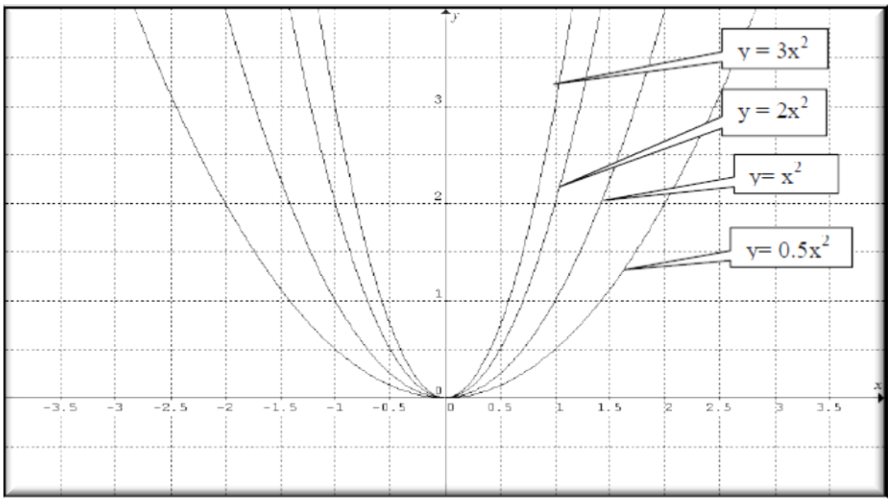


Figure 7:1 The effects of changing 'a' on a parabola

Changing the sign of 'a' also affects the shape of the graph – whether it has a maximum (concave down) or minimum (concave up). In South African context, it is normal to hear teachers and learners using emoticons such as smiley face (turning point is local minima) and sad/frowning face (turning point is local maxima) to identify the quadratic function, as illustrated in figure below:

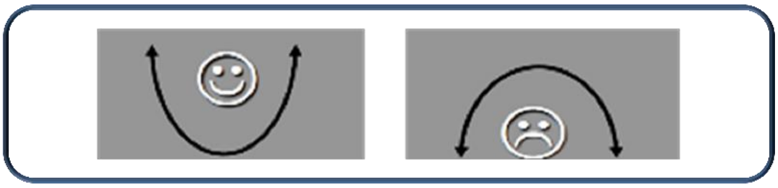


Figure 7:3 Example of emoticons describing a parabola

Other key aspects of the parabola and related parameters include: intercepts, turning point and axis of symmetry. Figure 7.4 illustrates these aspects:

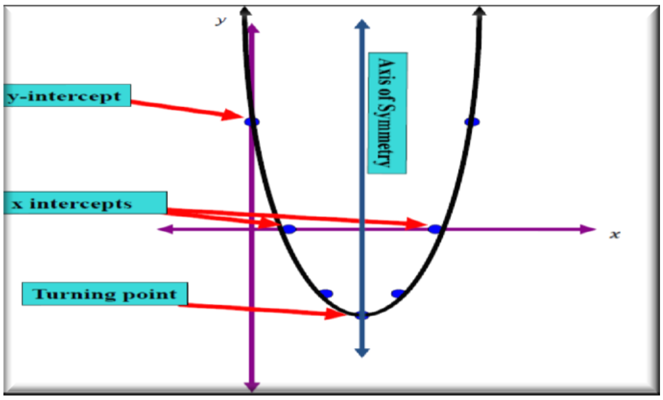


Figure 7:4 Features of a parabola

7.2.3 Linear function

The linear function can be represented in different forms such as: verbal, algebraic, tabular and graphical. The different representations highlight various facets of a linear function as shown in table 7.2 below:

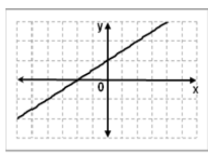
Name	Other names	Verbal	Algebraic (general form)	Tabular	Graphical
Linear function	straight line	Highest power of the independent variable is 1	$y = mx + c$ $ax + by = c$	x: -2, -1, 0, 1, 2 y: 0, 1, 2, 3, 4	

Table 7:2 Different representations of a linear function

In school mathematics textbooks the linear function is illustrated in an algebraic form of $y = mx + c$ which is the slope-intercept form of the equation of a line. And it may also be illustrated in the form $ax + by = c$, where a and b are both not zero. However in South African context in grade 11 we use the form $y = mx + c$.

The graphs associated with linear functions are called straight lines. It is important to consider the effects of the parameter 'm', on the straight line. Now, let us consider the function $y \ = mx + c$ to discuss this effect. In the figure 7.5 below, three graphs are drawn on the same set of axes: $y = x\,; y = 2x \ and \ y = \frac{1}{2}x$. 'm' changes the slope (or gradient) of the line $y = x$. The gradient of $y = 2x$ is 2. The gradient of $y = \frac{1}{2}x$ is $\frac{1}{2}$. Another way to view this is to say the line $y = x$ has experienced a vertical stretch/shrink of $'m'$.

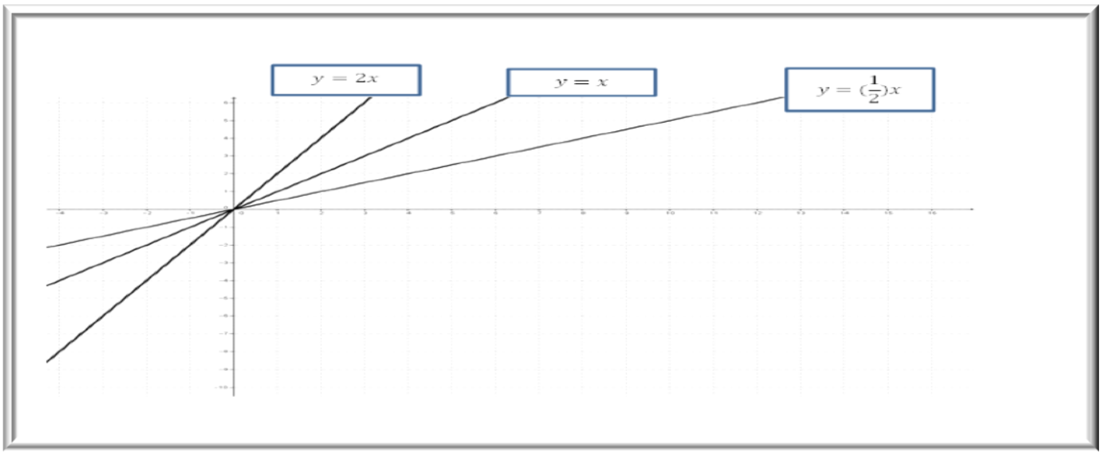


Figure 7:5 The effects of changing 'a' on a linear graph

It is important to note that 'm' in the linear function and 'a' in the quadratic function play the same role, that of vertical stretch/shrink. However it cannot be deduced that 'a' in the quadratic function is a gradient, as it is in linear function.

7.2.4 Gradient

Gradient/Slope is a fundamental function object in the high school curriculum. The gradient is sometimes referred to as a slope, and this word is mostly found in school mathematics textbooks. In South Africa, the gradient is typically introduced in Grade 8 and then reappears again in other topics such as calculus and analytic geometry.

School mathematics textbooks describe the gradient from graphical perspective as :(i) '*slope of a line as the ratio of the vertical rise to the horizontal run as you move from one point to another along the line.(ii) the measure of the steepness of a line segment*' (Campbell & McPetrie, 2012, p. 374).(iii) We also use an algebraic formula to calculate slope, $m = \frac{y_2 - y_1}{x_2 - x_1}$, where (x_1, y_1) and (x_2, y_2) are given points . A symbol of $'m'$ is mostly used in mathematical texts to represent the gradient.

The gradient can be illustrated in different representations. For example, the linear function may take the form $y = mx + b$ when represented algebraically. In this case we represent slope with parameters as $'m'$ (the coefficient of $'x'$). On the contrary, in other curved functions (quadratic, exponential, hyperbola)[32], the gradient cannot be seen straight away in the standard form as is the case with linear functions, it needs to be calculated. This highlights a very important point, that caution should be observed when interpreting a gradient in different algebraic representations of various function families.

7.2.5 Summary of school mathematics discourse

In summing up, the above discussion highlights the important features of school mathematics discourse such as words, visual mediators, routines and narratives that are endorsed.

Words: Words used to identify the objects that are unique to function discourse (e.g. intercept, gradient, quadratic and linear function).

Visual mediators: For learners to develop objectified discourse of function objects (intercept, gradient, linear and quadratic functions) they must be able to work in different representations (Nachlieli & Tabach, 2012). These representations include: graphs, equations and tables. To exemplify, one common representation found in all four function objects is the graph. Graphs are used as visual mediators for function and can be used to show features and behaviours of the different function objects under investigation.

[32] The notion of a gradient in the curved functions is only dealt with in Grade 12, when learners are introduced to Calculus.

Routines: Concerning the discussed function objects, the familiar routine from school mathematics is to examine function with table of values, equations and graphs through calculating, constructing, interpreting etc.

Narratives: narratives are descriptions of objects written or spoken and endorsed in the school mathematics discourse (formal definitions/descriptions).

In this study, I am going to use the notion of mathematical words, visual mediators, routines and narratives to describe the learners' mathematical discourse about the objects of functions discourse under investigation, and so as to what to judge against.

7.3 Main analysis

In order to study the features of mathematical discourse present in learners' discourse when talking about function objects, it was necessary for me to construct a matrix of data. By analysing each learner's features of mathematical discourse, one begins to see the considerable usage of these from words, routines, visual mediators to the way the learners substantiate their narratives. This requires being able to quantify some data otherwise it becomes too cumbersome and in a sense unreadable to accomplish. Therefore, it lends itself to tallying occurrences, thereby obtaining a picture of presence and frequency. In this sense, quantification is used to structure an overview of the data analysis. In each matrix table, in each section is an attempt to quantify per function object and the nature of each feature of mathematical discourse identified. In the first column of the matrix is the FDP. The table makes provision in the second column, for displaying the average of percentage of occurrence of each item in the FDP. The third column records the diversion[33] from the school mathematics discourse (as discussed in the previous section).

Sections (7.3.1; 7.3.2; 7.3.3 and 7.3.4) consider the four key objects under investigation: intercepts quadratic function, linear function and gradient respectively.

The discussion in each section includes:

- Features of the mathematical discourse

[33] Incorrect use

This sub-section presents features of the mathematical discourse learners have used: (i) Word and word use. (ii) Visual mediators that participants use, (iii) the routines that could be identified, and (iv) the narratives substantiated.

- Link to errors

  This sub section presents a possible link between learners' features of the mathematical discourse and learners' errors.

- Discussion and summary

  This sub-section presents the summary of the section and the key findings for the particular object under investigation.

7.3.1 Intercept

In the interview questions focused on intercepts, learners were asked to talk about the intercept from three different representations: verbal (definition), algebraic (equation), tabular (table) and graphical (graphs).

The following table shows the matrix of summary of results of learners' mathematical discourses on intercept.

FDP			Definition		Algebraic		Graphical		Tabular	
			% of occurance	Incorrect	% of occurance	Incorrect	% of occurance	Incorrect	% of occurance	Incorrect
Word use	Colloquial	Combination	60%				40%			
	Literate	Mathematical	60%				60%			
Routines	Flexibility	multiple routines	20%		40%		40%		40%	
		translating	20%		40%		40%		40%	
	Corrigibility	correcting								
	Applica-bility	constructing	60%		40%					
		interpreting	40%				100%		100%	20%
		using method			80%				80%	
		calculating			100%	20%	20%			
		demostrating	20%				20%			
		using visual trigger								
		comparing			20%		20%			
Visual mediator	Iconic	scanned	20%				100%		100%	
		constructed	60%		40%					
	Symbolic	scanned			20%					
		syntatic			100%		20%		20%	
	Concrete	manipulated								
		imagined								
Narratives	Substantiation	derivation			100%		20%		20%	
		construction	60%		40%					
		rule			80%				80%	
		visual	40%				100%		100%	
		recall			20%		20%			

Table 7:3 Categorizations of learners' features of mathematical discourse on intercept

Shown in the highlighted columns in table 7.3 are the percentages of learners coded as using a particular feature of mathematical discourse. So, for example, in table 7.3 above, the 60% in the first cell means that three out of five learners had use the combination of colloquial and literate discourses. The discussion that follows focuses on the unshaded blocks in the highlighted area of the table.

Definition: Results in table 7.3 above indicate that learners were using both mathematical and a combination of mathematical and colloquial discourse when defining the intercept. The most preferred routine was constructing a visual mediator.

Algebraic representation: Table 7.3 shows that learners preferred manipulating the symbolic mediator (algebraic representation) through calculations by applying a method (intercept) as a way of substantiating their narratives. The results in table 7.3 also show that learners had flexibly switched between different visual mediators.

Graphical representation: Results in table 7.3 indicate that all learners scanned through the graph by interpreting its local properties i.e. intercept.

Tabular representation: The Results in table 7.3 reveal that learners preferred scanning the mediator (interpreting) through the use of a method (intercept), except for one who used a different procedure.

What follows is the in-depth analysis of the data presented in table 7.3.

(i) Definition

When analysing learners' definition of the intercept I devoted my attention to the words together with discursive word use. Recall that discursive word use include routine use and visual mediator use.

In the following discussion, the learners were asked to give a definition of an intercept. The results of the analysis on the intercept show that some learners used (i) a combination of colloquial and mathematical discourse, and others (ii) mathematical discourse only. I summarise and exemplify in the table below, with regard to the given transcript:

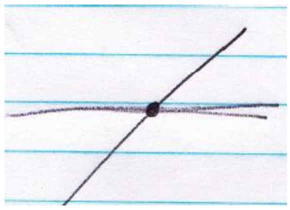
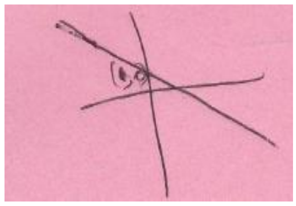
Line	Learner	What was said	What was done/seen
14	Nomsa	I can say intercept I can say maybe **crossing** something that **is to intercept**	*Drawing*
18		Maybe lets say neh I have got a line like this, yah maybe something like this I can say maybe this one intercept like on this **point** it **cuts through** this one(Drawing).	
19		Yah that is **intercepting** to **cut through** or to **go across** something like that	
38	Nhlanhla	I explain it is as a **touch** when I say intercept	using hands(crossing fingers)
39		Because its where the graph either **cuts** the y intercept or the x uhm intercept ,axis I mean	
41		Uhm okay on the equation of hyperbola where c,the y intercept, it's the one touching on the y axis yah in a yah	
43		I could tell them that the intercept you just look at where the graph **cuts** your **cartesian plane** when it **touches** the y axis , it's where graph **meets** your **y axis**, that's your y intercept and on your **x axis** that's where you know graph touches **x axis**.Thats where you know that's your intercept	
33	Jennifer	Intercept I think if you have a graph like this **parabola** it is where **it passes** through another line right? where **they intercept** with another line where **they meet** right? So on the **y axis** this is where **it intercepts** on **x axis** this is where **it intercepts**	pointing at parabola
55	Daniel	Intercept is a line that… it's where a function **passes** the line y = 0 and x = 0	
57		Ama coordinates wa kona(it's co-ordinates), where it intersects the line	
63		Intercept, angiti he graph e so, ngi funi o go bonisa, babe ngathi nayi istraight line la. gozoba ne y intercept(lets say they give you a straight line graph,there is going to be an intercept)	
69		So e bati find intercept ya lana, u zobega lona leli line, maybe izoba o one, I zoba I one… zero…(so,they say find an intercept of this line,maybe it will be one comma zero)	
26	Billy	The intercept is a point where the graph passes the x and the y axis . the x and y axis	

Figure 7:6 Examples of transcripts on definition of an intercept

Jennifer's, Nomsa's and Nhlanhla's definition had a combination of colloquial and mathematical discourse.

Consider, for example Jennifer's definition in line 33, contained narrative '*they intercept*' and '*it intercepts*'. What should be highlighted here is the use of word as if it is a verb (to intercept). This is also evident in Nomsa's definition in line 14 'crossing something that is *to intercept*'. She used synonyms like cross, cut, pass interchangeably to describe a verb 'to intercept'. Both Jennifer and Nomsa connected their colloquial word use with literate mathematical discourse by interpreting a graph and constructing two intersecting lines respectively, indicating a shift from colloquial to a literate way of defining an intercept.

Similarly, Nhlanhla has used a combination of routines from colloquial to a mathematical discourse. She connected the routines by making use of concrete mediator which is used in colloquial discourse. For example, in line 38, Nhlanhla used the hands by crossing the fingers to describe the word intercept through the word use 'touch' and 'cuts' from everyday language. An action of crossing fingers was used to illustrate the '*intersection*' of two lines. This action was further matched with interpretation of graphical representation. What should be highlighted in her discourse is that, although she started off by describing the intercept in colloquial ways, her discourse shifted to a more mathematical discourse. This is an important prerequisite for mathematical learning.

From the analysis of the three learners, I claim that learners' word use was routine driven. They were observed doing something through their discursive routines from crossing fingers, interpreting and constructing lines. They substantiated their narratives through visual mediators (i.e. graphs). I also further claim that the three learners had used a combination of colloquial and mathematical discourse and were observed shifting their substantiating narratives from colloquial to a more mathematical way.

On the other hand, Billy and Daniel described the word intercept in mathematical way only. In other words they used features of mathematical discourse only. Billy gave an intercept definition that is endorsed in the school mathematics discourse. Consider for example Billy in line 26, '*The intercept is a point where the graph passes the x and the y axis . The x and y axis*'. Daniel on the other hand used the word intercept by connecting with a visual mediator (graph), he also situated his definition by constructing the graph in the Cartesian plane and interpreting the properties of the graphical representation.

The routines in Billy's and Daniel's discourse were used as they substantiated the narratives about the intercept definition: ' *intercept', 'the graph cuts your Cartesian plane, when it touches the y axis*', etc. From the analysis of two learners, I can conclude that Billy substantiated his narratives by associating the word 'intercept' with phrases (narratives) endorsed in the school mathematical discourse. In other words, his word use was 'phrase driven'. Daniel, substantiated his narratives by connecting word intercept with iconic mediators through the use of routines associated with the word intercept (as constructing and interpreting). In other words his word use was routine driven.

There was evidence of a strong connection between the learners' mathematical discourse features (words, routines, visual mediators and substantiated narratives). For example, those who were using a word in mathematical way, constructed graph in the Cartesian plane to illustrate the point of intersection with the x and the y axis. Those who used a combination of colloquial and mathematical ways demonstrated that intersection, through crossing of fingers and connected that through interpretation of a graphical representation. These actions produced visual mediators, which were then used to substantiate their use of word, '*intercept'*. This connection indicated the intertwinement between features of mathematical discourse.

It is important to highlight the response from Nomsa (see line 14-19). Her response provided some clues about a possible connection between features of mathematical discourse and errors. In the excerpt, Nomsa provided a definition with a combination of colloquial and mathematical discourse. She started off by describing an intercept in a colloquial way in line 14 using words such as crossing and using intercept as if it is a verb. She moved to a more mathematical way (in line 18-19). She connected the colloquial words with visual mediators (two intersecting lines) by performing a mathematical routine of drawing. These discursive actions were done in efforts of substantiating a narrative '*crossing something, i.e. to intercept*', in other words to illustrate the point of intersection. However this definition is not endorsed in the function discourse of intercept, i.e. some piece of information is missing such as a ***point in the Cartesian plane*** *where a graph is* **cutting the x and y axis**. These are critical features when defining the intercept from the graphical perspective. Because of this missing information there is a disconnection between the definition of an intercept (endorsed narrative) and Nomsa's substantiating narratives. Such a disconnection results in errors (Kotsopoulos et al., 2009).

(ii) Algebraic

The symbolic mediators (algebraic representation) are most common type of mediators used in function discourse. These can either be interpreted or referred to as syntactic. When the learners were given an algebraic representation to determine the intercepts they were observed attending to the symbolic mediator through a syntactic mode calculation. During this process, the symbols were scanned and replaced by other symbols. All learners were observed calculating the intercept using a well-defined rule except for one learner (Nomsa) who used a different method. For example, when Billy was computing an intercept in line 46-

56 (see excerpt below), his primary visual mediator was symbolic as he used algebra based routine such as substitution using a method (to find y intercept, let $x = 0$).

Line	Speaker	What is said	What is done or seen
46	Billy	I would say let y = 0 the x intercept	
47	Me	Hmmm	
48	Billy	I would say let x=0 for let y intercept	$y = 2x + 1$ $y = 2(0) + 1$ $y = 1$
50	Billy	Then I solve it	
51	Me	Okay	
52	Billy	The y intercept right	
54	Billy	x = 0 then I have 2 times 0 plus 1	
55	Me	Hmmm	
56	Billy	Then y =1	

The response from Nomsa again provided some clues on possible connection with errors. One thing to be noted in Nomsa's routine procedure of calculating and constructing resulted in an error. While trying to do the calculation she substituted the x value with '*any number*' and she chose -1 instead of zero, and obtained an answer of $y = -1$, which she referred to as the y intercept. The chosen value led to a narrative (y intercept) which is not endorsed in the mathematical community. An extract below illustrates Nomsa's discursive moves.

Line	Speaker	What is said	What is done/seen
28	Me	How would you go on to find that x value that you say its an x intercept?	
29	Nomsa	You can substitute with any number.	
30	Nomsa	Like maybe you can say you can use numbers from uhm on the line right	
31	Me	Hmmm	
32	Nomsa	Maybe 1 to 4	
33	Nomsa	From 0 to 4	
34	Nomsa	And then -1 to -4	
35	Nomsa	And then those numbers you can substitute with x to get the value of uhm x neh	
36	Me	Do you want to show me what you are saying?	
37	Nomsa	Yah can use maybe -1 I can substitute -1 in the value of x and then I will say (writing),uhm 2(-1) + 1 ,-1 this is the value of y	Calculating $y = 2x + 1$ $y = 2(-1) + 1$ $y = -1$
38	Me	What is that? Is that the y intercept ?	
39	Nomsa	Yes	

Nomsa's substantiating narratives (derivation in line 37) could be linked to insufficient knowledge relating to definition of the intercept (i.e. endorsed narrative). She substituted x with '*any number'*, which is in contrast with the rule used in the school mathematics discourse, where for y intercept, the $'x\ value\ is\ substituted\ with\ zero'$ (intercept method). In other words, her narrative is not endorsed by the school mathematical community, hence an error (Sfard, 2008). Brodie and Berger (2010b), classify this error as error of signifier: a word substitute signifies substitution of any number without paying attention to the context of an intercept object. These errors were evident in her test responses and the test of the others. For interest, figure 7.7 illustrates this type of error.

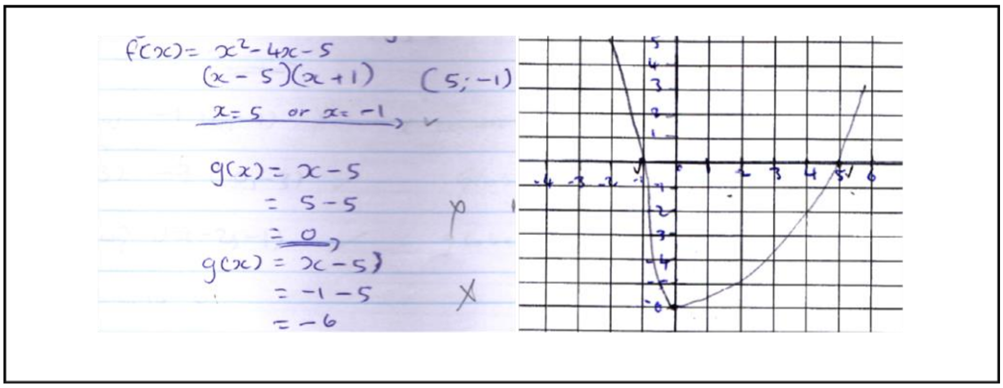


Figure 7:7 Example of error related to the definition of intercept

What the figure above shows is that the learner does not understand the intercept rule. The x intercepts were first calculated ($x = 5$ and -1). These x values were then substituted in another equation in an effort to calculating the y intercept. It seems to this learner that any x value is good enough to calculate the y intercept, the notion of substituting the x variable with zero (intercept method) is missing. The learner went further to represent the answer in a graph (as illustrated in figure 7.6).

On the contrary, the other learners who have used the intercept method (intercept) provided appropriate answers. Their responses are endorsed in the school mathematics discourse.

(iii) Graphical

When learners were asked to determine the $x\ or\ y$ intercept from different graphical representations, they were observed interpreting the visual properties of the graphs and the behaviour of the graphs. This was evident in their utterances, for example when talking about intercepts of the different functions. Nhlanhla (in the following excerpt) interpreted salient properties of the graphs.

Line	Speaker	What is said	What is seen/done
78	Nhlanhla	And there is only **one x intercept** because it's a **straight line ,**so it's gonna be 1 and my y intercept is -6	Looking at given graphs
79	Me	Okay for a straight line the x intercept and the y intercept is only 1	
80	Nhlanhla	Yah	
81	Me	Okay	
82	Nhlanhla	In the here the x intercept they have to be **2** because it's a parabola, it's gona **touch** the x intercept **twice**	

On the other hand, Billy was observed interpreting the general behaviour of the graph, as evidenced in the following excerpt.

Line	Speaker	What is said	What is seen/done
91	Me	why did you have to point those as your x intercepts	Looking at given graphs
92	Billy	Mainly because they cut the x axis the y axis no , no the x axis	

The analyses of the two excerpts above indicate that learners demonstrated routines used in function discourse i.e. interpreting both global and local properties of the graph, yet the notion of intercept as a point is still missing. This seems to suggest that the learners understood the intercept as the intersection with the axis only (x or y axis).

It can be concluded that Billy's and Nhlanhla's course of actions consisted of visual recognition, "by looking at it", recalling, and using what they remembered as, "the properties of it". To put it simpler, their word use was passive driven, and no formal definition (endorsed narrative) was evident in their discourse.

(iv) Tabular

A tabular representation was given. All five learners preferred substantiating their narratives by interpreting the given visual mediator through the use of the well-defined rule on intercept (intercept method), as illustrated in the excerpt below.

<table>
<tr><th>Line</th><th>Speaker</th><th>What is said</th><th>What is seen/done</th></tr>
<tr><td>125</td><td>Nhlanhla</td><td>The x intercept?</td><td>Looking at the given table
<table><tr><td>x</td><td>-4</td><td>-3</td><td>-2</td><td>-1</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td></tr><tr><td>y</td><td>-3</td><td>-2</td><td>-1</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td></tr></table></td></tr>
<tr><td>126</td><td>Me</td><td>Yes</td><td></td></tr>
<tr><td>127</td><td>Nhlanhla</td><td>Its negative 1 comma zero $(-1\ ;0)$</td><td></td></tr>
<tr><td>128</td><td>Me</td><td>Why</td><td></td></tr>
<tr><td>129</td><td>Nhlanhla</td><td>Because that's where my y is 0</td><td></td></tr>
<tr><td>132</td><td>Me</td><td>And the y intercept</td><td></td></tr>
<tr><td>133</td><td>Nhlanhla</td><td>Its zero is to one
$(\ 0\ ;\ 1)$</td><td></td></tr>
</table>

Summary of intercept analysis

An intercept was assessed from three perspectives: verbal (definition), algebraic, graphical and tabular.

Definition:

The analysis of the definition section brought three main findings. Firstly, it produced evidence showing that learners' discourse had a combination of colloquial and mathematical discourse. And this combination was more mathematical. Some of the learners had shifted their discourse from colloquial to a more literate discourse. A very important point to be highlighted is that the mathematical definition was done from a graphical perspective except for one piece of information that was missing: the notion of intercept as a **point** (in the Cartesian plane where a graph is cutting the x and y axis). It seems these learners were interpreting the intercept as an intersection of two lines, which differs from the endorsed narrative in school mathematics, hence an error.

Secondly, most of the learners' word use was linked to their discursive routines. This is an indication of routine driven word use. In other words, learners described the intercept by applying certain procedures such as constructing graphs, using gestures. Routine driven word use is associated with processual mode of thinking and hence not objectified.

Thirdly, learners' substantiating narratives short with lack of formal definition give some clues on learners' errors. As seen in Nomsa's discourse when calculating the y intercepts, her derivation was missing an endorsed rule specified in the school mathematics discourse. Further regarding her word use when defining the intercept, her definition had some piece of information missing (the notion of an intercept as the point).

Algebraic:

The analysis of the data indicates that learners reacted to the symbolic mediator by applying a well-defined rule (intercept method). The errors that emerged could be linked to learners' insufficient knowledge relating to the definition of an intercept (i.e. endorsed narrative), as seen with Nomsa's calculating routines.

Graphical:

The analysis suggests that learners interpret the salient features of the graph. Further, the analysis suggests that learners view the intercept as the '*intersection*' of a graph with the axes (x /y axis). The notion of intercept as a point is missing in the interpretation. This notion is very critical when defining an intercept from a graphical standpoint. A description without the notion of 'a point' leads to a disconnection between learners' substantiating narratives and narratives endorsed within the school mathematics discourse. This disconnection leads to errors.

Tabular:

Learners interpreted the table by using a rule (intercept method).

7.3.2 Quadratic function

In this question learners were required to classify the given quadratic function $g(x) = 2x^2 + 5$. Non examples were also included $y = x^3 + 2$ and $y = 3^x$. The following table shows the matrix of summary of results of learners' mathematical discourses on quadratic function.

FDP			Quadratic	
			% occurance	Incorrect
Word use	Colloquial	Combination		
	Literate	Mathematical	100%	
Routines	Flexibility	multiple routines		
		translating		
	Corrigibility	correcting	20%	
	Applica-bility	constructing		
		interpreting	20%	
		using method	20%	
		calculating	20%	20%
		demostrating		
		using visual trigger	60%	
		comparing	60%	
Visual mediator	Iconic	scanned	20%	
		constructed		
	Symbolic	scanned	80%	
		sγntatic	20%	
	Concrete	manipulated		
		imagined		
Narratives	Substantiation	derivation	20%	
		construction		
		rule	60%	
		visual	60%	
		recall	60%	

Table 7:4 Categorizations of learners' features of mathematical discourse on quadratic function

Data in the unshaded blocks in the highlighted area of the table 7.4 show that all five learners have used mathematical words when classifying the quadratic function. This is shown in the first cell with 100%. The preferred routines are visual trigger (60%) and comparing (60%).

What follows is an in-depth analysis of the data in table 7.4.

The results show that when learners were asked to identify the algebraic function $g(x) = 2x^2 + 5$, they provided mathematical words to identify the function. Nomsa and Daniel used the term ‘quadratic’ to identify this function. On the other hand, Jennifer and Billy used a term parabola. The term parabola is a keyword used to identify a quadratic graph in the function discourse. This highlights learners’ reliance on graphical representation when defining function objects. This was also observed when they were defining an intercept in the previous section.

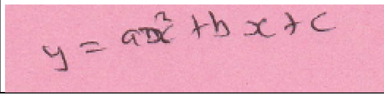
The discussion of learners’ discursive actions will incorporate the learners’ routine procedures. All learners were interpreting the global and local features of the equation: properties and behaviour of the equation.

Billy, Nhlanhla and Nomsa, interpreted the equation in a certain way. The analysis of their actions reveals that their utterances of the word ‘quadratic’ occurred when they saw an exponent of 2. In other words learners reacted to the algebraic equation by interpreting the prominent visual information i.e. exponent of 2. This is evident in excerpts below:

Line	Speaker	What was said	What is done/seen
110	Billy	Because the x^2 has **exponent 2**	Scanning the equation $g(x) = 2x^2 + 5$
153	Nhlanhla	Yah because the highest exponent it has an **exponent of 2**	
137	Nomsa	Because it has got an **exponent** the x has got an exponent of 2	

From the excerpts above, it seems learners substantiated the narrative (quadratic) through a narrative ‘because the x^2 $has\ exponent\ of\ 2$’. In a traditional mathematics classroom, quadratic function is introduced by writing the standard form of quadratic function ($y = ax^2 + bx + c$). It is normal to hear a teacher emphasizing on the highest power of variable as 2 whenever referring to the quadratic equation. The exponent of 2 is used as a visual trigger or a pictorial distraction (Arcavi, 2003). From this analysis, I conclude that the three learners’ word use was passive driven. That is, the word (quadratic) is matched with prominent visual information that is matched with prototypic example of the quadratic function (what they remembered as the properties of the quadratic function). In Sfard's language, learners' substantiating narratives are linked to previously endorsed narratives which are recalled from memory, and here their recall is incorrect.

On the other hand, Daniel and Jennifer reacted to the equation by matching the visual properties of the equations to the general standard form of quadratic function ($y = ax^2 + bx + c$). This is shown in the excerpts below:

Line	Speaker	What was said	What is done/seen
219	Daniel	Because **it looks** like a quadratic equation	Scanning the equation $g(x) = 2x^2 + 5$
220	Me	Which one is a quadratic equation?	
221	Daniel	$y = ax^2 + bx + c$(writing)	

Line	Speaker	What was said	What is done/seen
185	Jennifer	It's a parabola	Scanning the equation $g(x) = 2x^2 + 5$
186	Me	Why	
187	Jennifer	Because it's fit into the general equation of the parabola	

Daniel's and Jennifer's substantiating narratives are also linked to the passive driven word use. They are relying on visual recognition of what looks like a quadratic function, this is evident in Daniel's excerpt (line 219), where he says, 'Because ***it looks*** *like a quadratic equation'*. Their substantiating narratives are triggered by the visual appearance and matching the equation to its prototypic standard form which is somehow recalled from memory. This is seen in line 187 of Jennifer's utterances '*because it fits into the general equation of the parabola*'.

The response from Nhlanhla (see excerpt below) provided some clues about possible connection to errors. Nhlanhla incorrectly identified the function by referring to it as a hyperbola. It is interesting that Nhlanhla admitted that she had a problem in differentiating between words hyperbola and parabola. This is evident in turn 200 where she says: '*I get confused with hyperbola and parabola'*.

Line	Speaker	What is said	What is done/seen
199	Me	Hmmm , can you help me understand what is a hyperbola?	
200	Nhlanhla	I get confused with hyperbola and parabola	

The analysis of an episode below reveals that Nhlanhla understands the properties of the given function but confuses the words. The following episode illustrates her discourse on hyperbola and parabola.

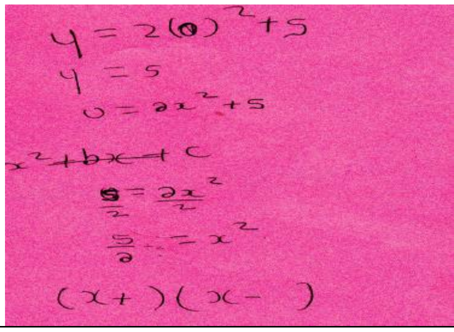
Line	Speaker	What is said	What is done/seen
148	Me	Now let's talk about this quadratic mmm we are talking about this function neh ok. $g(x)$ is equal to $2x^2 + 5$ ok. What is the name of this function what do we call this function	Scanning the equation $g(x) = 2x^2 + 5$
149	Nhlanhla	It's a **hyperbola**	
150	Me	Why is it a hyperbola	
151	Nhlanhla	Because the , it can also be an **exponential** graph	
152	Me	Hmmm	
153	Nhlanhla	Yah because the highest exponent it has an **exponent of 2**	
154	Me	Yes	
155	Nhlanhla	Err and also this one can be a **hyperbola , parabola** because it doesn't have the second form of x	
156	Nhlanhla	Is in a form of ax^2 and it doesn't have bx	
157	Me	So it can take 2 forms that's what you are trying to say	
158	Nhlanhla	No this is the equation it's an exponential	
159	Me	It's an **exponential**	
160	Nhlanhla	Yes	
161	Me	Then you said the reason was	
162	Nhlanhla	Because the equation doesn't have it's in a form of $\boldsymbol{ax^2 + bx + c}$	
164	Me	Ok if it is in that form	

164	Nhlanhla	It's gona be a **hyperbola**	
165	Me	It is a hyperbola	
166	Nhlanhla	Yah hyperbola	
167	Me	**What is a hyperbola**	
168	Nhlanhla	**It's in a form of** $\boldsymbol{ax^2 + bx + c}$	
169	Me	Ok that one is a hyperbola	

The above excerpt shows how Nhlanhla named the quadratic equation $g(x) = 2x^2 + 5$ and her discursive actions (word use). Nhlanhla used the word hyperbola while naming the quadratic function. What is interesting in her substantiations is the narrative 'the highest exponent of the variable was 2'.This narrative served as an endorsement of her identification of the function (hyperbola). This keyword is not generally used and shared by participants within a function discourse to identify a quadratic function[34]. Hence, it is classified as an error. In other words there is a disconnection between a word and an endorsed narrative (school mathematics discourse). It is interesting to see that Nhlanhla's discursive actions were consistent with those used in most mathematics classrooms, which are regarded as expert's (teacher's) endorsed narratives. For example when Nhlanhla saw the exponent of 2 she matched the given equation with the general quadratic equation $y = ax^2 + bx + c,$ to substantiate her answer. It is normal to hear a teacher endorsing a narrative on quadratic function in this way. Nomsa went further to substantiate her narrative in line 191 by asserting that '*if you have 2 intercepts then it's a hyperbola*'. In line 195 of the excerpt below Nhlanhla was observed calculating by applying an intercept method (a well-defined rule in the function discourse). The following excerpt illustrates her discursive procedure:

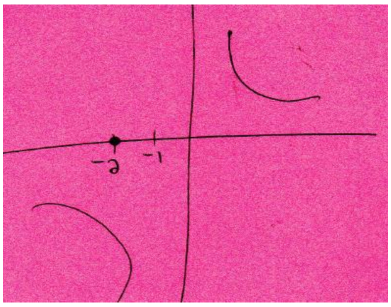
Line	Speaker	What is said	What is done/seen
191	Nhlanhla	If I get **2 x intercept** then it means it's a **hyperbola**	
192	Me	Hmmm	
193	Nhlanhla	And if I get **one** maybe is a **straight line** or some other graph	

[34] What is used is the keyword 'quadratic function' sometimes 'parabola' as noted earlier.

194	Me	Ok proceed.	
195	Nhlanhla		

The analysis of Nhlanhla's discourse on quadratic function indicates the use of more than one routine response (flexibility): interpreting properties of the graph and calculating in an effort to substantiate the same narrative (quadratic function). However the derivation did not function in concert with an endorsed procedure of basic algebraic calculations (factorization). This disconnection resulted in error. Although Nhlanhla has demonstrated flexibility in her discursive actions in an effort to substantiating the same narrative, one of the substantiations resulted in error.

Nomsa went further to describe the parabola (hyperbola). The following excerpt captures her description:

Line	Speaker	What is said	What is done/seen
202	Nhlanhla	But the other one is a **parabola** which has lines on two different quadrants	
203	Me	Hmmm	
204	Nhlanhla	Then a hyperbola it's got a **smiley** face or a **sad** face	
205	Me	What is a smiley face and a sad face	
206	Nhlanhla	if your 'a' on ax^2 is negative	
207	Nhlanhla	The graph is gonna face down	
208	Me	Hmmm	

209	Nhlanhla	So it's gonna be a sad face	
210	Me	What do we call that kind of a graph	
211	Nhlanhla	Hyperbola	
212	Me	Hyperbola	
213	Nhlanhla	Yes	
214	Me	And then a parabola	
215	Nhlanhla	It lies on two different quadrant	
216	Me	So that one is a parabola	
217	Nhlanhla	Yes	

The above excerpt shows how Nhlanhla used the keyword parabola to identify the hyperbolic function. She substantiated her narrative (parabola) by performing a routine procedure of constructing the iconic mediator (graph). These discursive actions are well defined procedures in school mathematics discourse when talking about the hyperbolic function.

Nhlanhla had further interpreted the graphical representation of the quadratic function by using colloquial terms. This is evident in line 204 where she says: *'Then a hyperbola, it's got a smiley face or a sad face*'. These are colloquial words (emoticons) that one would find in school mathematics discourse when interpreting the graphical representation of the parabola.

From this analysis, it is evident that Nhlanhla's word use is objectified. In other words, through naming each function (hyperbola and parabola) by the defining conditions of each function and interpreting both the global and local properties of each function. Further, her interpretations were driven by the properties and common descriptive narratives accompanying each function (although she confused the words, as it has been alluded to). Objectified word use is associated with a blend of processual and structural mode of thinking. The processual thinking has been heavily demonstrated by Nhlanhla through her routines. Having said this, I therefore claim that Nhlanhla's discursive actions (from word use, routines and visual mediators) suggest that she objectified the properties of the hyperbola and parabola functions (Ben-Yehuda et al., 2005). However she confused the words. The notion of objectification was evident in her flexible transitions between mediators, in her ability to use many different routines, in her ability to translate colloquial mediators into mathematical and in her combination of processual and structural thinking. Table 7.5 below summarises her discursive actions on the two functions.

	$y = 2x^2 + 5$	$y = \frac{a}{x}$
Words	Hyperbola	Parabola
Routine	Interpreting Calculating	Construction
Visual mediator	Symbolic scanned Symbolic syntactic Concrete mediator(smiley face)	Iconic constructed
Substantiating narrative	Derivation Rule Visual(emoticons)	Construction Visual(iconic mediators)

Table 7:5 An example of Nhlanhla's discursive actions

Although Nhlanhla seemed to have objectified the properties of each function (parabola and hyperbola), it must be warned that her confusion with words when identifying the two functions could mean that she was likely to make an error, especially during assessments (examination). For example when asked an examination question through a verbal representation only, e.g. 'what are the properties of the hyperbola?' or asked through multiple choice questions. Nhlanhla's confusion with words suggests that not enough time was allocated to discussing use of function terminology during instruction as alluded to in the National Diagnostic Report (DoE, 2012).

Summary of quadratic function analysis

In summary, the analysis of this section indicated that learners' word use was passive driven which is associated with matching the word with prominent visual features of the equation i.e. learners were observed interpreting the quadratic function by relying on visual triggers (exponent 2), which is used to remind them of a quadratic function. From this, I claim that passive word use is associated with visual triggers. No errors could be linked to visual triggers. However, words used by learners to classify the quadratic function could provide clues on learners' classification errors, as seen with Nhlanhla. At least one learner (Nhlanhla) has demonstrated an objectified word use.

7.3.3 Linear

In this question learners were required to classify (identify) two functions $y = 2x + 1$ and $y = \frac{x}{2} + 3$.

The following table shows the matrix of summary of results of learners' mathematical discourses on linear function.

FDP			y=2x+1		y=x/2+3	
			% of occurance	incorrect	% of occurance	incorrect
Word use	Colloquial	Combination				
	Literate	Mathematical	**100%**	**20%**	**100%**	**80%**
Routines	Flexibility	multiple routines				
		translating				
	Corrigibility	correcting	20%		20%	
	Applica-bility	constructing			20%	
		interpreting	**100%**			
		using method				
		calculating			20%	
		demostrating				
		using visual trigger			**80%**	
		comparing	**80%**	20%	60%	
Visual mediator	Iconic	scanned				
		constructed			20%	
	Symbolic	scanned	100%		60%	
		syntatic			40%	
	Concrete	manipulated			40%	100%
		imagined				
Narratives	Substantiation	derivation			20%	
		construction				
		rule				
		visual			80%	
		recall	100%		40%	

Table 7:6 Categorizations of learners' features of mathematical discourse on linear function

$y = 2x + 1$: The unshaded blocks in the highlighted area in table 7.6 show that all five learners' word use was mathematical (100%) except for one learner (20%) whose word use did not match the experts'. All learners preferred routine was interpreting (100%) the global properties of the mediator and about 80% of learners compared the visual mediator with the standard formula.

$y = \frac{x}{2} + 3$: The results in table 7.6 indicate that all learners' word use was mathematical. However about 80% of the learners, had used the words in a different way from that of the expert. The most preferred routine was use of visual triggers.

What follows is the report on features of mathematical discourses used by the learners.

(i) $\boldsymbol{y = 2x + 1}$

All learners have used mathematical descriptions of the given function. What should be highlighted here is that all learners except for one have used correct words to identify the given function. Those who offered correct identification were seen interpreting global properties of the equation by comparing it to a general standard form of linear function.

Nomsa who provided an incorrect identification compared the equation to an incorrect standard formula. This is evident in the excerpt below:

Line	Speaker	What is said	What is done/said
153	Me	Okay Now let's talk about this function $y = 2x + 1$ what is the name of this function	
154	Nomsa	It's a parabola	Scanning equation $y = 2x + 1$
155	Me	$y = 2x + 1$ why do you think it's a parabola	
156	Nomsa	Uhm because uhm this equation(writing)	$y=(x-p)+q$
157	Nomsa	No,no,no this is a straight line graph it's not a parabola it's a straight line graph	
158	Me	Why , why , why is it a straight line	
159	Me	What makes you think it's a straight line	
160	Nomsa	Because of the equation $y = mx + c$ yah , yah I get it	

Nomsa, in the excerpt above, was seen matching the given iconic mediator to the standard form $y = (x - p) + q$ to substantiate her answer. However this matching routine i.e. comparing was not in concert with the narrative the learner was trying to substantiate because it was not a well-defined standard form in the functions discourse for linear functions. In other words Nomsa's substantiating narrative was different from experts (endorsed narratives), hence an incorrect identification. In this case Nomsa's discursive action could be linked to errors associated with interpretation. It seems the standard form offered is confused with a quadratic standard form $y = (x - p)^2 + q$ but '2' in the exponent of $(x - p)$ is missing. It should be noted that when Nomsa realised that she was working with a straight line and not a parabola, she corrected herself by offering a general form of a linear equation, $y = mx + c$ (see line 157-160). This discursive action demonstrates a property of corrigibility (ability to correct one's discursive action).

From this analysis, I claim that all the five learners' word use was passive driven. Their substantiating narratives were driven by the visual properties of the given function. I further claim that disconnections between learners' substantiating narratives and endorsed narratives led to errors, as seen in Nomsa's discourse.

(ii) $y = \frac{x}{2} + 3$

Nomsa, Jennifer, Billy and Nhlanhla[35] offered an incorrect description of the function: hyperbola, as evidenced in the following excerpt:

Line	Speaker	What was said	What is done
125	Me	Okay this one?	pointing at $y = x/2 + 3$
126	Billy	That one is a **hyperbola**	
127	Me	Why	
128	Billy	No It's a straight line	
129	Me	Why is it a straight line	
130	Billy	Because as I said previously the x is to the power of 1 but on this one the gradient is half	

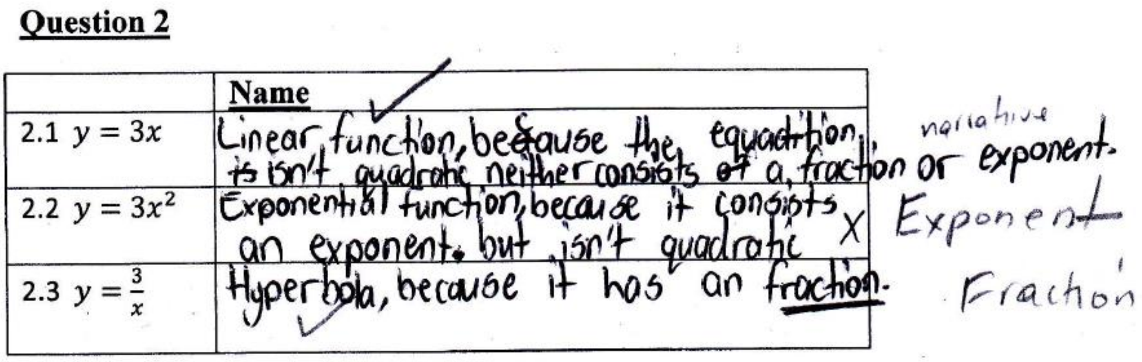
[35] As previously discussed Nhlanhla says parabola when referring to the hyperbola.

131	Me	Okay you have spoken about a hyperbola. What came to your mind quickly before you started to change your mind?.	
132	Billy	I **saw a fraction** that's why I say it's a hyperbola	
133	Me	So when you see a fraction its err you think it's a hyperbola	
134	Billy	Yes	
135	Me	Why	
136	Billy	Because normally a hyperbola it has a the equation for hyperbola it's a/x	
204	Me	Let's say is in another function $y = x/2 + 3$ what will be this	
205	Jennifer	Ok $y = x/a + q$ yah I think this is ahh **hyperbola**	$y = \frac{x}{a} + q$
206	Me	Why	
207	Jennifer	Because the equation is similar to that	
254	Me	Ooh okay now let's say you are given this function neh $y = x/2 + 3$	
255	Nhlanhla	Hmmm	
256	Me	What is this function called?	
257	Nhlanhla	Hmmm I think this can be a parabola(**hyperbola**)	
258	Me	Why a parabola	

259	Nhlanhla	Hmmm because x is in the denominator	
161	Me	Let’s talk about this one $y = x/2 + 3$	
162	Nomsa	Hmmm	
163	Me	What is the name of this function	
164	Nomsa	Uhm **Parabola or hyperbola** I am not sure which one but then it has got this uhm in uhm this one neh. I think a parabola what’s that equation again. $x + 1$ I think p/q I am not sure	
165	Nomsa	Mmmm so between a parabola	
166	Me	So I am saying $y = x/2 + 3$, what makes you think it’s a parabola	
167	Nomsa	It is it’s a **fraction**	

In the excerpts above the three learners were observed using words such as fraction, x in denominator and the words were linked to a routine procedure of comparing the given equation to a standard form of a hyperbola ($y = a/x$). Hyperbola is a specialised mathematical word used in function discourse to identify a hyperbolic function. It seems the learners were relying on what they saw. That is, their word use was driven by visual cues. This is evident in Billy’s utterances in line 132. His response was triggered by the fraction he saw. This led him to summon a prototype example of a standard form of a hyperbola from memory (a/x).

The results of the above analysis suggest that learners substantiated their narrative through visual interpretation and recalling of a prototypic example of a standard form of the hyperbola. From this, I claim that there is a disconnection between learners’ substantiated narrative with the endorsed narrative of a linear function in the school mathematics discourse. Such a disconnection results in errors. There were many errors of this nature in the test, which suggests that others were doing it. This analysis helped me to explain learners' interpretation errors. An example of such errors is illustrated in figure 7.8 below:


Question 2

	Name	
2.1 $y = 3x$	Linear function, because the equation is isn't quadratic neither consists of a fraction or exponent.	narrative
2.2 $y = 3x^2$	Exponential function, because it consists an exponent. but isn't quadratic X	Exponent
2.3 $y = \frac{3}{x}$	Hyperbola, because it has an fraction.	Fraction

Figure 7:8 Example of use of visual triggers

Daniel on the other hand gave a correct identification of a given equation. He was observed attending to the equation through calculation (in line 255 below). In other words he substantiated his narrative through derivation. His derivation matched the endorsed narrative of the school mathematics discourse.

Line	Speaker	What is said	What is done/seen
250	Me	Okay now let's talk about this function neh. Let's talk about $y = x/2 + 3$	
251	Daniel	Hmmm	
252	Me	What is this function called	
253	Daniel	It's a linear function	
254	Me	Why	
255	Daniel	Because if you remove the denominator and multiply everything by 2 you gona get this equation $(2y = x + 3)$	$y = \frac{x}{2} + 3$ $2y = x + 6$

Summary of linear function analysis

Four of the five learners were interpreting the prominent information appearing in the equation (fraction) and matching it with a prototypic example of a given function. To them, the 'fraction' signified the hyperbola. They seem to be applying a prototypic example (signifier) of a hyperbolic function in a different situation. Such an action sometimes results in errors (Brodie & Berger, 2010a). It can be concluded that a disconnection between learners' substantiating narratives and what is endorsed in the school mathematics discourse result in errors. However, the analysis of the data has indicated that a connection between the two, result in correct substantiations, as seen with Daniel. This finding confirms that errors result from substantiating narratives that were not paired with endorsed narratives (Kotsopoulos et al., 2009).

Learners' reaction to the question on interpreting the linear function (algebraic) by relying on visual trigger routine could be linked to their errors (interpretation errors).

7.3.4 Gradient

In this question, learners were required to talk about the gradient from a linear function ($y = px + 4$) and a quadratic function ($y = ax^2 + p$) domains.

The following table shows the matrix of summary of results of learners' mathematical discourses on gradient.

AFD			y=px+4		y=ax^2+p		g(x) = 2x^2 + 5	
			% of occurance	Incorrect	% of occurance	Incorrect	% of occurance	Incorrect
Word use	Colloquial	Combination			**80%**	20%	40%	
	Literate	Mathematical	**100%**		20%		80%	60%
Routines	Flexibility	multiple routines						
		translating						
	Corrigibility	correcting						
	Applica-bility	constructing			20%			
		interpreting	**80%**		40%		80%	
		using method						
		calculating	60%		20%		20%	20%
		demostrating	**100%**		**60%**			
		visual trigger						
		comparing	60%				40%	
Visual mediator	Iconic	scanned			40%			
		constructed			20%			
		scanned	80%		20%		60%	
	Symbolic	syntatic	40%		20%			
	Concrete	manipulated	40%		60%			
		imagined						
Narratives	Substantiation	derivation	60%		20%		20%	20%
		construction			20%			
		rule	20%		40%		80%	20%
		visual	100%		100%			
		recall	80%					

Table 7:7 categorizations of learners' features of mathematical discourse on gradient

$y = px + 4$: The data in the unshaded blocks in the highlighted area of the table show that all learners' word use was mathematical. So, for example, in table 7.7 above, the 100% in the first cell means that all five learners had use literate (i.e. mathematical discourses). The preferred routine was demonstrating properties of the given equation through manipulation of concrete mediators, followed by interpreting routine.

$y = ax^2 + p$: The results indicate that 80% of learners had used the word gradient by combining both mathematical and colloquial discourse. The most preferred routine was demonstrating through manipulation of concrete mediators.

What follows is the report on features of mathematical discourses used by the learners.

$(\boldsymbol{i})\ \boldsymbol{y = px + 4}$

When learners were asked to identify the coefficient of x i.e 'p' in the given symbolic mediator $y = px + 4$, they used mathematical words such as gradient and coefficient of x.

Those who used the word gradient were observed scanning the equation, and substantiated their narratives as illustrated in the excerpt below:

Line	**Speaker**	**What was said**
142	Billy	'because we were taught before that any value of the variable before x is the gradient of the graph'
245	Jennifer	'because that's how we were taught'

In the above excerpts, Billy's and Jennifer's word use was driven by recalling the narrative endorsed from previous learning (i.e. passive driven). This is evident in their substantiating narratives: 'we were taught' and 'that's how were taught'.

Others were observed interpreting the equation and comparing it to the standard formula of the linear function. This is evidenced in the following excerpts:

Line	**Speaker**	**What was said**	**What is done/seen**

274	Nhlanhla	Yah , yah what makes give you that idea it is the gradient of the line it's in a form of $mx + c$	Scanning equation $y = px + 4$
269	Daniel	Because I think it's the same equation $mx + c$	

Nhlanhla's and Daniel's word use was driven by visual recognition of the salient features of the given equation and then compared with the standard form of linear function (i.e. passive driven). It is normal that the parameter 'm' is interpreted as a gradient in the general standard formula of a linear function (Moschkovich, 1999a).

Nomsa on the other hand used a keyword coefficient of $'x'$. Nomsa was observed interpreting the given symbolic mediator. It seems she interpreted the function as if it was any algebraic expression when she responded. When probed further on what the coefficient represented, she responded as follows:

Line	Speaker	What was said	What is done/seen
174	Me	Yes it is a coefficient what is that coefficient, what does it stand in for? Which information is it giving you	
175	Nomsa	Whether the graph is gonna be ***positive or negative***	
176	Me	Is going to be negative or positive	
177	Nomsa	Hmmm	
178	Me	So now can you tell if the graph is going to be positive or negative?	
179	Nomsa	The coefficient is positive	

In the above excerpt Nomsa responded by interpreting the general behaviour of the graph ('*is going to be negative or negative'*). The analysis of the excerpt above first indicates that Nomsa was aware of the properties of $'p'$ in a functions' discourse, for example she was able to link properties of 'p' with the graph. Further this could mean she did not have enough functions' vocabulary at that point to describe $'p'$ e.g. gradient. This again suggests the terminology deficiency.

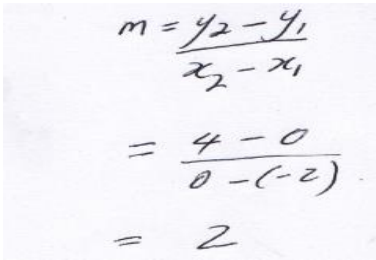
Learners were asked further to decide whether 'p' was positive or negative. There was evidence of use of a combination of colloquial and mathematical discourses. For example, learners were observed using concrete mediation (using hands) and syntactic mediation (calculations). The use of hands indicates a colloquial discourse; the calculation is more of a mathematical discourse.

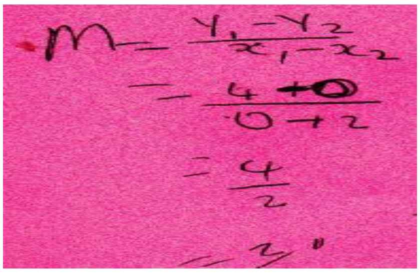
All learners were noticed demonstrating their answers using their hands. Their actions correspond to their answers, positive or negative, as illustrated in the excerpt below:

Line	Speaker	What was said	What is done or seen
149	Me	So now when you look on the graph this is a representation of this is the value , the is p negative or positive	
150	Billy	It's positive	
151	Me	Why positive	
152	Billy	Because the graph is moving from left to right that is a positive slope	using hands

The excerpt above illustrates that Billy was substantiating the narrative 'positive' from visual interpretation of the graph (graph is moving from left to right), this movement is demonstrated through the use of hands.

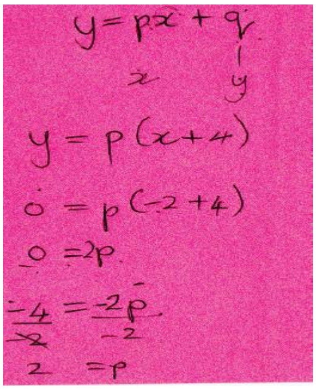
The narrative '*positive*' was also substantiated through calculation. This was evident in Nhlanhla, Billy and Jennifer's discursive actions. For example, Billy (line 154-164) and Nhlanhla (line 276-277) had applied the symbolic formula in a well-defined way specific to the function discourse (gradient formula). The answer that resulted in the calculating routine corresponded with the narrative they were trying to endorse (positive).

Line	Speaker	What is said	What is done/seen
153	Me	Ooh okay that's the only way? Is there any other way from the given information that can help you determine whether p is positive or not?	
154	Billy	I can calculate it.	
155	Me	Calculate	
156	Billy	As in the formula of the gradient	
157	Me	Hmmm	
158	Billy	$y_2 - y_1$	
160	Billy	Over $x_2 - x_1$	
161	Me	hmmm	
162	Billy	Then I substitute my values which 4-0/0 – (-2)	
163	Me	Hmmm	
164	Billy	Where by I will get 2 a positive 2	

Line	Speaker	What is said	What is done/seen
275	Me	Now on this p neh is this p negative or positive	
276	Nhlanhla		
277	Nhlanhla	Yah , yah its positive	

From this analysis, I conclude that learners' word use is routine driven. That is, they were matching the narrative 'positive' through interpretation of visual properties of the graph. They were also applying a calculating routine using a formula associated with the word 'gradient'.

Jennifer's response provided some clues about a possible connection between features of mathematical discourse and errors. She suggested a different calculation procedure. This is shown below:

Line	Speaker	What was said	What is done or seen
266	Jennifer	uhm think I would start substituting	
268	Jennifer	One of the points on the equation	
269	Me	Yes	
270	Jennifer	To find 'p'	
271	Me	Ok can you please do that?	
272	Jennifer	Ok I would say $y = p\,x + q$ wow $y = px + 4$ right ?	
273	Me	Mmmm	
274	Jennifer	Ok then where x is where y is 0	
275	Me	Mm	
276	Jennifer	$x = 2 + 4$,which is going to be 0 $p - 4$ is 2 which is going to be p then 0 or	
278	Jennifer	All the way I wrote it down compared to this ooh	
279	Me	Mmmm	
280	Jennifer	Ok $p \ -2 \ +4$ oh yes, yes uhm so is going to be 2p and then this is going to be is that to be -2p. This is going to be to the other side -4 then we say divided by 2 , divided by 2 by -2 . which is going to be $2 = p$	
281	Me	Oh so you would substitute that formula to confirm	

In the excerpt above, Jennifer opted to attend to the equation in a syntactic way, by replacing x and y variables with 2 and 0 respectively. This procedure was in contrast with general procedure used when calculating a gradient as used by others. Jennifer's substantiating narratives (i.e. derivation) was not consistent with the endorsed procedure in the school mathematical community (gradient formula), hence the error. Jennifer's discursive procedure suggests that she was not aware of well-defined rules (gradient formula) applied when calculating the gradient. This suggests gaps in knowledge relating to gradient definition. Jennifer's routine calculation can be linked to the errors. I therefore claim that, errors are resulting from a disconnection between learners' substantiating narratives with the endorsed narratives in the school mathematics discourse.

$(ii)\ y = ax^2 + p$

In this task the learners were given both the symbolic (equation) and iconic mediator (graph). They were asked to interpret properties of $'a'$ the coefficient of x^2. There was an evidence of use of a combination of colloquial and mathematical discourses. For example learners were observed using the combination of colloquial and mathematical words. These words were linked to word use. They were observed scanning the iconic mediator to substantiate their narrative. This is evident in the following excerpt:

Line	Speaker	What is said	What is done/seen
167	Me	What does 'a' represent?	
168	Billy	What	
169	Me	that diagram is exactly the same, this is an algebraic form, this is a graphical form	
170	Billy	Mmm	
171	Me	Okay …let me ask like this, is 'a' positive here?	
172	Billy	No, it's not positive**, it's negative**	Scanning the graph
173	Me	Why is it negative	
174	Billy	Because **it's a frown**. So when its frowning it means it's a negative	
175	Billy	Because as w**e have read** in the past last year grade 10, **if the parabola is in a shape of a frown facing upside down**	
176	Billy	**Its negative, the 'a' is negative**	

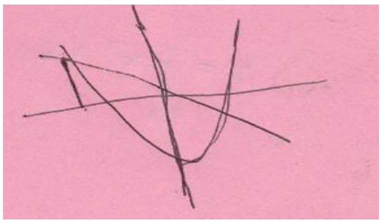
In the above excerpt, Billy scanned the iconic mediator (graph) and offered a narrative '*it's negative*'. This narrative '*negative*' is associated with the colloquial global properties of the iconic mediator (graph) i.e. '*frowning', facing upside down'*. Others have also used these colloquial properties such as '*look up or down*', for example Jennifer in line 286 ('a' represent a parabola. I think it represents uhm if the graph is gonna *look up or down***)** and Nomsa in line 193 (It's *looking down*).

Line	Speaker	What is said	What is done/seen
286	Jennifer	a' represent a parabola, I think it represents uhm if the graph is gona l**ook up or down**	
293	Jennifer	if 'a' is positive + ,graph will look up, if 'a' is negative then it will be like this (**look down)**	pointing at the graph

Line	Speaker	What is said	What is done/seen
188	Me	Okay now let's say that you are given this graph neh. This one is $y = ax^2 + p$ and together with this one. This is algebraic and this is graphical. What is the name of this graph	pointing at the graph
189	Nomsa	This one	
190	Me	Hmmm	
191	Nomsa	Uhm this is a parabola	
192	Me	Why it's a parabola	
193	Nomsa	It's **looking down**	

On the other hand, Daniel described 'a' in both colloquial and mathematical ways by using words like: wideness, inclination, spread, opening. He further used his hands to demonstrate wideness. He also drew a graph to try to illustrate the wideness (283-304).

Line	**Speaker**	**What is said**	**What is done/seen**
283	Daniel	Mmmm in think the **wideness of a parabola**	Using the hands
286	Me	Okay now if someone said its representing a gradient are they correct or not?	
287	Daniel	They are correct I think	

288	Me	If they say a represent a gradient why	
289	Daniel	Because gradient has something to do with the wideness the graph	
291	Daniel	The gradient is how ***eish*** let me draw the graph. This is a straight line and the gradient is how ***eish*** I can't explain it	
292	Me	You can speak in isiZulu	
293	Daniel	Even in Zulu I cant	
294	Me	Okay show me by your hands	
295	Daniel	How that	
297	Daniel	How the graph	
298	Me	You can use your hands	
299	Daniel	Sort of like **inclination**	showing inclination with hands
301	Daniel	That how yah but now the hyperbola it's like this you know so I think since it also has the same I think it is also moving **spreading** it has to mean that a is gradient	Using the hands
302	Me	Ooh okay so you saying its showing the inclination	
303	Daniel	Hmmm	
304	Daniel	How ,the graph **is opening**	

In the above excerpt, Daniel is demonstrating his flexibility i.e. switching between mediators: demonstrating with hands (concrete), and constructing a graph (iconic). It is evident that

Daniel was short or struggling with words to describe effect of 'a', hence the mode switching. Daniel's frustration is evident in line 289-291, where he is using a word '*eish*'. This is a colloquial word used in everyday language to express frustration. This suggests that limited vocabularies were used in classroom to describe the co-efficient of x^2 in a quadratic function. This is confirmed in Billy's talk:

Line	Speak	What was said	What is done/seen
172	Billy	I am not sure what it's called but it represents the graph that is **wide, open or slender**	Using hands
174	Billy	Because it's a frown. So when its frowning it means it's a negative	
175	Billy	Because as we have read in the past last year grade 10, if the parabola is in a shape of a frown facing upside down	
176	Billy	Its negative, the 'a' is negative	

In the excerpt above, Billy described the effect of 'a' using colloquial words like frown, wide, open or slender. These words were connected with the graph. Further, the analysis of Billy's discursive actions shows the use of a combination of colloquial and mathematical discourse. Further, it illuminates the fact that learners prefer interpreting the definition of gradient from a graphical representation. This is consistent with the results of the analysis on intercept object. The combination of the two discourses helped the learners to substantiate their narratives in ways that almost resemble the narratives endorsed in the school mathematics discourse, particularly from a graphical representation perspective. However, no specialised mathematical words were used to describe the coefficient of x^2. The results of this analysis are not surprising, because in mathematics literature there is no specific word for the coefficient of x^2 in the quadratic equation ($y = ax^2 + bx + c$), although there is one in the coefficient of x in the linear equation ($y = mx + c$) i.e the gradient. Traditionally, linear functions and solving quadratic equations are considered prerequisites for quadratic functions. In the mathematics textbooks 'a' and 'm' are sometimes described as representing the wideness of the graph for example ('stretch or shrink'). Because of this, it is highly likely that the gradient will be associated with the coefficient of x^2. As evident in the following:

Line	Speaker	What is said	What is done/seen
308	Me	Ooh okay now let's say you are given these graphical those algebraic representations they say $f(x) = 3x + 1$ neh and also $g(x) = 2x^2 + 1$. The f(x) is a linear and the g(x) is a parabola. Now there are two learners grade 12, they are busy discussing about this graph these two functions and then one of them is saying that the gradient f $g(x) = 2$ is he correct by saying so?	
309	Daniel	Yes	
310	Me	Why	
311	Daniel	Because since if I that err the eish	
313	Daniel	For this graph neh $2x^2 + 5$	
314	Me	Hmmm	
315	Daniel	$2x$ is a…………	
317	Daniel	And since I said that 'a' is also a wideness and something to do with the gradient it means that 2 is also the gradient	

In the above excerpt, Daniel was asked to evaluate whether the coefficient of x^2, '2' in this case was a gradient or not. He associated the coefficient of x^2 with a gradient. The substantiated narrative ('*gradient of* $g(x) = 2x^2 + 5$ *is 2*') is not consistent with the endorsed narrative in the school mathematics discourse, hence an error. Possible explanation for this could be overgeneralisation of properties of linear function (Moschkovich, 1999a).This overgeneralisation can be linked back to the way the slope is used in classrooms and textbooks. From the analysis of Daniel's discourse about the gradient, I conclude that his word use was routine driven. Daniel associated his word use with all different discursive routines and flexibly converting between different visual mediators from concrete to iconic mediators.

Summary of analysis of the gradient

In this section learners were required to talk about the gradient from two different function domains: linear function ($y = px + 4$) and quadratic function ($y = ax^2 + p$ and $g(x) = 2x^2 + 5$).

Linear function ($y = px + 4$)

The analyses of learners' discourse on algebraic equation $y = px + 4$ show that learners demonstrated flexibility in converting between different representations and using different routines. This flexibility was also observed as the linked features of the colloquial and mathematical discourses. For example they were observed mode switching between equivalent representations i.e. using different mediator that is symbolic through calculation and hands (concrete mediator) as they were trying to justify that the gradient was positive (substantiating narrative). What should be noted is that learners shifted from colloquial to literate substantiating narratives with no ambiguities. What has also emerged in this analysis is that poor formal definition of gradient (endorsed narratives) are linked to errors, as seen in Jennifer's case.

Quadratic function ($y = ax^2 + p$ and $g(x) = 2x^2 + 5$)

Learners were talking about 'a' the coefficient of x^2 by using a combination of colloquial and mathematical discourse. The results indicate that the learners could not differentiate between a word gradient and a coefficient of x^2 in the quadratic function. It seems the coefficient of x^2 and x signified the gradient or slope. There are a number of possible explanations for this. First, it could be the generalisation of the role of the coefficients of x and x^2 in both the linear and the quadratic functions in the school mathematic discourse: the role of vertical stretch or shrink. Secondly, it is normal to hear learners associating the word gradient with words like coefficient (of variable $x \text{ or } x^2$), inclination, stretch, shrink and wideness. In school mathematics discourse there is no unique keyword used to identify the coefficient of x^2, hence the confusion. The words can be linked to errors associated with interpretation of the gradient. The conclusion reached in this analysis raises a very important question, 'what is the endorsing narrative associated with the coefficient of x^2 in the quadratic equation e.g. $y = 2x^2 + 5$? I must admit that learners are only introduced to the notion of gradient with the quadratic functions only in grade 12. In this study, the questions assessing gradient in quadratic functions were only introduced with an aim of assessing how learners were talking about the gradient from different functions.

Another important observation to be noted in the way learners described the gradient from both linear and quadratic functions is that their word use was routine driven and no formal definition was present in their talk. They reacted to the request to define the gradient by applying the routine procedures. Their discursive actions were consistent with those of the intercept. This suggests that learners were action oriented. Which implies that their mode of thinking is more processual (Sfard 1992).

What needs to be highlighted is that learners' definitions (descriptions) were done from a graphical perspective. In other words, learners' leaned on one mediating tool.

7.4 Conclusion

This chapter has presented the findings of the interview analysis of function objects: intercept quadratic function, linear function and the gradient. The analysis provided microscopic views of learners' mathematical thinking through the use of features of mathematical discourse in answering the guiding questions of this analysis. The results of the analysis indicated firstly that all features of mathematical discourse were present in the learners' discourse. Secondly, the connection existed between these features. However there is a considerable number of cases where the learners committed errors (mostly linked to visual cues and words). My study is aiming at linking these errors in relation to learners' features of mathematical discourse. The next chapter includes a comprehensive synthesis of the findings as they relate to the research questions of this study.

Chapter 8 DISCUSSION

8.1 Introduction

In this chapter, based on the results presented in Chapter 7, I will discuss how the proposed analytical framework contributed to the analysis of the learners' mathematical discourses. The discussion will be presented in four sections: in the first section, I provide a brief summary of the study. In the second section, I present the discussion of the main research findings related to the questions that guided my analysis. The questions were enquiring about the features of mathematical discourse (i.e. word use, routine, visual mediators and narratives) present in learners' (function) discourse and how they were related to errors. In the third section, the connection of main findings is made with the first research question. The first question was about learners' common errors made when completing tasks involving algebraic functions. The last section provides concluding remarks of the chapter.

8.2 Summary of the study

This study examined four function objects found in grade 11 curriculum; intercept; linear function; quadratic function and the gradient through the lens of discourse. The setting for the study was a group of grade 11 learners in a multilingual classroom in Johannesburg. A test was administered to 26 grade 11 learners and five learners were purposively sampled for the interview. Interview data was captured using audio recorder. Additional data collected included learners' written responses during interview.

Data analysis included the transcription of all five interviews. The analysis of the interview was done using an analytical framework adapted from Ben-Yehuda et al. (2005). Typological and inductive analysis was used to analyse the interview data. Overall, the interview findings were presented qualitatively. However, I do quantify some of the data in order to identify general trends and patterns in the analysis.

I have chosen five different learners whose test responses exhibited many errors so that I could draw from their discourses' rich possibilities of being able to link their discourses with errors. I am using these five leaners to illustrate in an exploratory way so as to generate insights and hypothesis and not to make general claims. In this chapter, when I am referring to the word 'learners', to indicate the discourses of the five learners as discussed in the previous chapter.

The purpose of this study was to explore a discursive framework as an analytic tool to describe learners' mathematical thinking through the analysis of their mathematical (function) discourse. In order to gain insights into learners' mathematical thinking on functions, this study carefully examined the key features of mathematical discourse: words and word use, visual mediators, routines and narratives substantiated, structured by four function objects linked to learner error.

8.3 Discussion of main findings

The discussion engaged with in this section is concentrated on answering the last two critical questions that support this study. After some elaboration of Ben Yehuda's and others analytical framework, I re-expressed the critical questions in the form of analysis questions.

Critical questions	Analysis questions
1. What features of mathematical discourse (word use, routines, visual mediators, and narratives) are evident in the learners' discourse? 2. In what ways, if at all, are these features linked to learners' errors?	1. What features of mathematical discourses (i.e. words/words use, routines, visual mediators, narratives), are evident in the learners' discourse, and how can these be described? 2. Is there a connection that exists between these (features)? 3. In which way are they (features) linked to learners' errors?

Presented below is the summary of the analyses in the form of responses to the analysis questions.

The first of the questions inquired about the distinct features of the mathematical discourse present in the learners' discourse. The four key mathematical features highlighted in the discursive framework are: word/word use; routines; visual mediators and narratives. It is important to mention that these features are inextricably linked together so much so that investigating one means (implicitly or explicitly) investigating the other. In my case, even though I had originally set out to investigate each feature of mathematical discourse separately, through the analysis say for example of learners' substantiating narratives, I found that substantiated narratives contained words and word use, routines and visual mediators. So this meant that I could not discuss one feature without missing the other feature, as evident in the next sections. In the conclusion chapter, I am going to comment on how this aspect has expanded the scope of my study.

On the basis of the analyses of learners' interviews, I claim the following:

- *Words/word use*: Learners' word use had the characteristics of colloquial and mathematical discourse. Learners strongly connected their word use to visual mediators (graphs). Learners associated their word use by linking routines through all kinds of discursive routines (i.e. routine driven) and visual cues (i.e. passive driven). There was also a mismatch in word use between some of the words and how they are used in mathematical discourse.
- *Routines*: Learners were observed applying three different routines: calculating, constructing and interpreting. They have also demonstrated an ability to use different routines and convert between different but equivalent visual mediators in an effort to endorse the same narrative, demonstrating some flexibility. At the same time learners' mathematical discourse were guided by visual cues (visual triggers).
- *Visual mediators*: Although most of the questions were introduced through visual mediators, i.e. graphs and equations, learners demonstrated use of all three types of visual mediators (symbolic, graphical and concrete). However, the graphical representations were the most preferred ones.
- *Substantiated narratives*: Learners substantiated their narratives through discursive routine procedures i.e. they preferred applying certain procedures to substantiate their narratives.

The second question inquired about the connection between learners' discursive actions. I claim that the learners' discursive actions are intertwined together to produce substantiated narratives. For example in a case where the learner is trying to provide a definition of an intercept, learners strongly connected their word use, routines and narratives to visual mediators (graphs). The study further revealed that the disconnection between any of the features resulted in errors.

The final question was concerned with exploring the possible link that could be made between features of mathematical discourse and errors. I found that words, routines (calculations and visual triggers) and some of visual mediators were all linked to errors (i.e. producing inappropriate [36] narratives). Further, I claimed that errors were linked to insufficient knowledge relating to formal definitions which resulted in disconnection between learners' substantiating narratives and what is endorsed in the school mathematics discourse.

In what follows, I am going to discuss the nature of features of learners' mathematical discourse, and highlight any connection between errors that might exist.

8.4 Research question two: Features of mathematical discourse

8.4.1. Words/word use

In this study the word/word use was investigated mostly in relation to the intercept and gradient objects of the function discourse. The analyses in this study show the following:

(a) *Combination of colloquial and mathematical words*. Learners have demonstrated combined use of colloquial and mathematical words .This was noticeable when learners were talking about the gradient, quadratic function and intercept. For example gradient (slope, wideness, shrink); parabolas (smiley/frowning face) and intercept (to intercept). These findings support the ideas of other researchers (e.g. Adams, 2003; Moschkovich, 2003; Vygotsky 1993), who suggest that mathematical talk includes both colloquial and mathematical words.

It is interesting to note that learners have used a combination of colloquial and mathematical words and made efforts to shift their discourse (combination) towards a more literate mathematical discourse. Jennifer, Nomsa and Nhlanhla in the analysis of the intercept and mainly Billy in the gradient analysis, provide an empirical evidence of this claim.

[36] Inappropriate narratives refer to narratives not endorsed in the school mathematics discourse.

Such a shift is an indication of mathematical learning (Moschkovich, 2003). According to Sfard (2008), mathematical learning involves a shift from a colloquial discourse to a more objectified talk. However, there was no clear indication of objectification in learners' mathematical discourse. It is important to note that I did not set out to focus on objectification, but somehow there were some instances in the learners' discourse that provided some clues to the notion of objectification such as use of multiple routines and mediational switching. This was seen in Nhlanhla's discursive moves in the quadratic function analysis where she was talking about the 'parabola' and the 'hyperbola'

(b) *The results have further suggested a disconnection between word and word use*. Such a disconnection led to inappropriate substantiated narratives. For example in Nhlanhla's discourse on hyperbola where she confused a word hyperbola with parabola (see section 7.3.2) and in gradient discourse of several learners where a coefficient of x^2 in $y = 2x^2 + 5$ was identified as a gradient (see Chapter 7, section 7.3.4). It is worth noting that even though learners have used incorrect words, learners' word use or substantiating narratives when using these words were close to those endorsed in the school mathematical discourse. The disconnection between words and word use resulted in incorrect definition of the function objects (errors). Sfard (2008) posits that word and its discursive use should function in concert with each other. Discursive use includes routine use, visual mediator use and substantiated narratives. These elements of discursive use are interrelated (Sfard, 2008). Once there is a discord, errors result (Kotsopoulos et al., 2009).

(c) *Word use was related to visual trigger (routines).* In other words, the word use was determined by visual scanning which was dependent on visual cues. This was evident in the linear and quadratic function's analyses. In quadratic function analysis we saw Billy, Nhlanhla and Nomsa demonstrating reliance on visual cues. This is suggestive of word use being passive driven. According to Sfard (2008), passive driven word use is the early stage of word use development where learners are using the words through visual recognition of properties they associate with the word (or prototypic example matched with the word). She further stresses the notion of moving towards the objectified use of a word. This move involves a blending of talking about processes and objectified talks of objects (Sfard, 2008).

Another important finding relating to reliance on visual cues is that they are associated with errors observed across the test responses. This was observed from Nomsa, Jennifer, Billy and

Nhlanhla where they confused the form of the function $y = x/2 + 3$, by relying on visual cues. This is discussed in detail in the following section on 'routines'.

(d) The learners' definition (*substantiating narratives*) was mediated by graphs. This was highlighted in the intercept and gradient analyses. It seems learners are leaning on graphs when expressing their ideas. Researchers (e.g. Confrey & Smith, 1991; Even, 1998; Sfard, 2008) allude to the notion of mediational flexibility (use of different mediators such as graphs, equations, tables in case of function) and warn of dangers of leaning on one mediator.

It also emerged that no formal abstract definition was included in the substantiating narratives. This was observed in the intercept and gradient analyses. In the intercept analysis, none of the learners offered the formal definition except for one, Billy. Billy was seen in line 26 (section 7.3.1; figure 7.5) defining the intercept by using a phrase endorsed in the school mathematics discourse. In both analyses of intercept and gradient, learners' word use was routine driven. This finding resonates with research findings of Pettersson, Stadler, and Tambour (2013). This finding is not unexpected since the concept of a function in the school curricular is introduced through different representations i.e. graphs, equations, and tables (Van Dyke & White, 2004). According to Nachlieli and Tabach (2012), the abstract definition of a function during instruction appears towards the end of the secondary schooling. This is true for the South African curriculum where the concept of intercept for example is introduced through different representations such as algebraic and graphical representation. Further, in the National Curriculum Statement (DoE, 2003, p. 6), it is specified that the formal definition of the function should be introduced only at a later stage, in Grade 12:

> *Introduce a more formal definition of a function and extend Grade 11 work on the relationships between variables in terms of numerical, graphical, verbal and symbolic representations of functions and convert flexibly between these representations (tables, graphs, words and formulae).*

In summing up, throughout the analysis of word use, it can be concluded that the word use was associated with a process. In other words learners were action oriented. They were looking for something to do instead of giving a formal definition, i.e. routine driven. For example, when asked to define an intercept concept they were observed connecting whatever they said with actions performed on graphs, equations and hands rather than about the properties of the intercept or even the formal definition of the intercept. This claim was also evident in the gradient concept analyses. This finding is in agreement with Kim et al. (2005) findings which showed that students were action oriented when defining a concept of infinity.

This finding further suggests that learners could comfortably provide definition through some routine procedures. However, this does not suggest that learners objectified the concept (deep mathematical understanding). According to Sfard (1991), learners' ability to see a mathematical concept both as a process and as an object – and so objectifying is necessary for a deep mathematical understanding of that concept.

These results reveal that the main difficulty that learners experienced was originating from poor access to appropriate mathematical words and reliance on visual cues. This was a challenge for some learners when presenting the formal definition which resulted in errors. Nevertheless, I would like to acknowledge learners' ability to implement mathematical routines, different kinds of mediators to present their ideas and to transform their colloquial talk towards the mathematical talk. For this I can claim that their word use was more mathematical and had features of school mathematics discourse with one piece of information sometimes exact.

The results of the analyses of word use have further provided an important finding that can be linked to errors of definition. Others have reported errors on definition, but were not distinguishing between words and word use. The strength of this study is that it has been able to put under microscope the errors related to definition (as reported in the test), and discovered that these are related to the words learners use.

I have identified inappropriate keywords as errors of words. These results extend the work of Brodie and Berger (2010a) who have dealt with errors from a discursive perspective incorporating errors of routines, errors of visual mediators and errors of calculations.

In this study, I was able to identify errors of words in learners' mathematical discourse, bringing to researchers and teachers some illustrations and hypotheses about the impact of words in learners' mathematical thinking. It may be possible that these results are due to the lack of use of terminology during instruction, as reported in the National Diagnostic Report

(DoE, 2012). According to Patkin (2011), insufficient mathematics vocabulary may inhibit conceptual understanding, which creates difficulty and later result in errors. This was evident in Billy's talk about gradient. He was struggling to find words to describe the coefficient of x^2, and resorted to use a gradient to describe this. The important thing I have pulled out through discursive lenses is the attention to language. How learners use words and word use is very critical in a multilingual classroom. Drawing from Nhlanhla's results, I noted that when words are confused (e.g. hyperbola versus parabola), there are serious consequences.

Having dealt with words/word use, I now turn to another feature of mathematics discourse: routines.

8.4.2 Routines

In this section, I attend to the procedures implemented by the learners in reaction to the requests such as: define or determine an intercept, classify different functions and interpret a gradient. That is, mathematical regularities that were noticed through learners' word use; mediator use or processes of creating mediators and substantiating narratives. While doing this, I consider properties of routine performances such as flexibility, applicability and corrigibility. These correspond to Ben-Yehuda et al. (2005) categorization of routines.

Different types of routines were identified and were consistent with those used in function discourse. In the discussion that follows, I describe and discuss the predominant routines enacted or described by the learners.

Applicability refers to the learners' ways of matching routine procedures with tasks. These can be observed through different competencies such as calculating, interpreting etc. This study has found that learners attended to the symbolic mediator through visual cues (visual triggers). Visual trigger involves visual scanning and interpreting the mediator by relying on visual cues, for example, in the case of quadratic equation in section 7.3.2. Billy, Nhlanhla and Nomsa were observed interpreting prominent visual properties of an equation ($y = 2x^2 + 5$, exponent of the variable equal to 2), which is the visual cue associated with the quadratic function. These findings are consistent with those of Nachlieli and Tabach (2012), who found that learners were able to participate in the function discourse, but they relied on routines with discursive cues created from previous learning.

Overreliance on visual cues can be strongly linked to the errors (Tall & Bakar, 1992). This was evident in the linear function analysis where Billy, Nhlanhla and Nomsa classified the function $y = x/2 + 3$, as hyperbolic. One possible explanations for this result could be due to the interpretation that was made without making sense of the representation, which resulted in errors of visual scanning (Brodie & Berger, 2010a). It seems the learners were relying on visual cues, a fraction in this instance, which triggered a hyperbola. A fraction is associated with the hyperbola, because a prototypic standard form of the hyperbola has a numerator and a denominator ($y = a/x$). Again this creation of prototype (Tall & Bakar, 1992) has been overgeneralised for all functions with a fraction which resulted in error (Ryan & Williams, 2007). In Sfard's language, the visual cue (fraction) signifies a hyperbolic function and is applied in different situations which lead to learners' substantiating narratives being in discord with the endorsed narrative (quadratic function in this case), hence an error.

Calculating routines were demonstrated by the learners and were used sparingly. Mostly, calculations were used to serve as a confirmation of substantiated narratives. This was evidenced in the gradient analysis where Billy, Nhlanhla and Jennifer were seen confirming the slope of the graph through calculations by employing the gradient formula.

An important observation I made was that most of the learners' calculations resulted in incorrect derivations. And these can be linked to errors. This study results have shown that errors related to calculations are due to insufficient formal definition (endorsed narratives), as seen in these cases of Nomsa where she was calculating the y intercept, and Jennifer, when calculating the gradient. This brings to fore the power of microscopic lenses of the discursive framework. Some have reported calculation errors to be due to inadequate basic skills (Radatz, 1979), the discursive framework has gone deeper and magnified the calculation errors .

The flexibility and corrigibility routines are in focus when commenting about the general properties of learners' routine performance (Ben-Yehuda et al., 2005).

Flexibility: This involves applying different routine procedures such as calculating, constructing, scanning etc., and also using different but equivalent representations (translating), in efforts to substantiate the same narrative. The results indicate that flexibility routines were demonstrated. This was evident when learners were defining a gradient concept, where we saw Billy's, Nhlanhla's and Jennifer's use of calculating, constructing and interpreting routines. This strengthened their ability to substantiate their narratives and provide appropriate responses. This finding corroborates ideas of the power of using different approaches as advocated by Even (1998). She further argues that use of different routines guarantees mode switching i.e. convert from one representation to another. This idea was confirmed by this study's findings (as discussed in section 7.3.4). Translating from one presentation to another helped the learners to express their ideas about the same object. Billy and Daniel gradient discourse provided evidence where they were observed moving from one representation to another. And they were able to explain different aspects of the gradient highlighted by each representation. Translating between representations helps to develop generalised procedures that allow recognition of appearance of a representation in diverse representations (Even, 1998; Sfard, 2008).

Corrigibility: Corrigibility routine was demonstrated by a few. Corrigibility involves an ability to correct and evaluate one's discursive actions. Drawing from Billy and Nomsa's results relating to the linear function, it indicates that using different routine procedures helped in correcting errors. This is consistent with Ben-Yehuda et al. (2005) ideas which suggest that an ability to mode switch helps in correcting errors.

8.4.3 Visual mediators

Recall, according to Sfard (2008), visual mediators are visible objects that may be created or operated upon as a part of the discursive process. For instance, relating to the intercept and gradient analysis, learners' discourse was mediated by a combination of visual mediators and words (mathematical and colloquial). This helped them to co-ordinate their communication. This supports (Ryve, Nilsson, & Pettersson, 2013) findings where visual mediators and words became the focus of leaners' attention and helped in establishment of effective communication.

The participants of this study used different forms of mediators to communicate their thinking namely: iconic, algebraic and concrete. Some have used a combination of the three demonstrating 'mediational flexibility'(Ben-Yehuda et al., 2005). This was shown in quadratic function analysis where Nhlanhla was using a combination of concrete, symbolic and graphical visual mediators. Similarly, in the gradient analysis Billy and Daniel were observed using a combination of the three. According to Tabach and Nachlieli (2011), it is important that learners display mediational diversity (flexibility of using different visual mediators) in order to appreciate a concept of function.

Iconic mediator: learners substantiated their narratives from a graphical perspective (as discussed in the previous section on word use)

Symbolic mediators were scanned and manipulated through calculations. The scanning was done when trying to classify a function. The scanning which was done without interpreting the underlying meaning resulted in errors (as discussed in the previous section on routines). Some of the calculations resulted in errors.

8.4.4 Substantiating narratives

Narratives are discussed under three categories namely; substantiations, derivations and recall.

Substantiations: During interviews, the situations that helped in eliciting learners substantiating rules were created. I made numerous requirements for substantiations and those that arose when learners responded to questions such as ‘how would you explain that, ‘why is that’. In these instances, learners responded with a word ‘because’ - this was seen throughout the interview data. This finding supports the ideas of Ben-Yehuda et al. (2005) suggesting that the ability to answer such questions can be seen as an activity of substantiation. According to Ben-Yehuda et al. (2005), the routine procedure is also a prerequisite of substantiation. In other words, one can observe learners’ substantiations through their discursive routines. It has been noted that the routines in learners’ discourses were used as they substantiated narratives about intercept or gradient definition. For example they substantiated their narratives about the intercept object through routines such as drawing and interpreting graphs and doing calculations (using equations). This finding also suggests a processual mode of thinking, that is, learners were looking for things to do when substantiating their narratives, as opposed to substantiating through definitions i.e. narratives endorsed in the school mathematics discourse. Substantiations were mostly visual. In other words the narratives were strongly connected to visual mediators (graphs in this case). This finding supports the findings of Pettersson et al. (2013).

Another main finding in this study is that the learners' narratives moved from colloquial substantiations to a more mathematical substantiations. This finding supports Sfard's (2008) ideas that the mathematical discourse develops from colloquial mathematical discourse; which is an important starting point, and to develop mathematical discourse requires a fundamental change in the discourse practices. The question this finding is raising is whether this change is objectified or moving towards objectified talk. The answer to this question is beyond the scope of this study.

Recall: This is an ability to summon previously endorsed narrative. The results of this study did show that learners were recalling previously endorsed narratives (prototypic examples of a function) and these were linked to errors. For example in the case of visual cues of exponent of 2, this was created as a reminder to a quadratic function ($y = x^2$) and 'fraction' in relation to linear function. I have already argued that when these prototypical examples are applied in an inappropriate situation, this resulted in errors. Learners' substantiating narratives did not function in concert with the endorsed narratives.

Derivation: Algebraic calculations were demonstrated in ways to substantiate narratives. Algebraic calculations were dealt with extensively in the preceding discussion on calculating routines. However, it should be noted that such derivations provided some explanations of learners' errors, as argued in the next section.

What has been illuminated from the discussion of the three different categories of narratives is that recall and derivation are acts of substantiation. More detail is provided in the conclusion chapter.

8.5 Research question three: Link of discursive actions to errors.

In this section, I link the discursive actions to errors in general, then to test errors.

8.5.1 Errors in general

What can be gathered from the preceding discussion is that words, visual trigger routines, calculating routines are all linked to errors. Further, the findings of this study suggest that a disconnection between words and word use leads to errors. And also the disconnection between endorsed narratives and substantiating narratives leads to errors.

Errors of words: These included words that were used erroneously such as hyperbola versus parabola, coefficient of x^2 versus gradient and intercept versus intersect. These cannot be explained through reviewed literature on vocabulary such as words with multiple meanings in everyday and mathematical language (Adams, 2003; D. Kotsopoulos, 2007; Zevenbergen, 2000). In case of hyperbola/parabola, these words have similar ends ('bola') but they cannot be classified as homophones[37]. On the other hand, in the case of the coefficient of x^2, one cannot strongly claim that the errors of words are associated with everyday language because in mathematics literature there is no specific word for the coefficient of x^2 in the quadratic equation ($y = ax^2 + bx + c$), although there is one in the coefficient of x in the linear equation ($y = mx + c$) i.e the gradient. Traditionally linear functions and quadratic equations are considered prerequisites for quadratic functions. In the mathematics textbooks 'a' and 'm' are sometimes described as representing the wideness of the graph for example ('stretch or a shrink'). Because of this, it is highly likely that the gradient will be associated with the coefficient of x^2.

Errors of visual trigger: This has been discussed in detail in the section on routines. Several possible explanations for this are offered by Brodie and Berger (2010a): Inappropriate visual scanning that may result in the learner inferring relationships between the symbols without any appeal to the underlying mathematics; and inappropriate use of visual detail that results when at least one piece of information in the question is ignored when interpreting the question.

[37] Homophones are words that sound similar. Words such as sum/some and whole/hole along with words which are slightly different in sound such as off/of, sixty/sixteen and tens/tenths are possible hiccups in Mathematics. These words tend to pose problems for learners particularly when learners are not very fluent in English.

Errors of narratives: these are the errors that resulted from the disconnection between endorsed narratives and learners' substantiating narratives. This was explained in detail in preceding section on narratives.

This study was unable to demonstrate that substantiated narratives from colloquial discourse lead to errors as argued by Patkin (2011). Instead it has emerged that learners substantiate their narratives through a combination of colloquial and mathematical discourses and this combination was more mathematical and almost similar to those endorsed in school mathematics discourse with some piece of information missing if not exact.

8.5.2 Test errors

The errors that emerged support the theoretical arguments I have engaged with, in Chapter 3, where I made a link between Brodie and Berger (2010a) categories of errors with errors from constructivist/cognitive perspectives. This study has further extended Brodie's and Berger's categories of errors by including the errors of words, reclassifying error of routines category by including visual triggers and calculating errors, and lastly errors of narratives have been re-categorised into two groups: substantiations not endorsed in the school mathematics discourse and insufficient endorsed narratives (definition). A summary of errors from the three sources (Brodie and Berger (2010a), the empirical data and from the test) is presented in a tabular form shown in table 8.1:

Features of mathematical discourse	**Errors from commognitive perspective**		**Errors from cognitive/constructivist perspective**
	(Brodie & Berger, 2010a)	**Errors from empirical data**	**Errors from the test**
Words/word use		**Words**	Classification
Routines	Inappropriate visual scan	**Visual trigger**	interpretation
	visual construction		construction
	Signifier	**Calculating**	calculation
		Visual trigger	interpretation
Narratives endorsed in the mathematics discourse	Narratives not endorsed in the mathematics discourse	**Substantiations not endorsed in the school mathematics discourse**	Incorrect answers or verbalisations resulting from any of the competencies.
		Insufficient formal definition (endorsed narratives)	Calculation

Table 8:1 Linking of features of mathematical discourse to errors

The errors that emerged from the test analysis were associated with classification, interpretation, translating and calculating competencies. The discussion on learners' mathematical discourses leads one to link words and visual trigger routines to errors associated with *classification competency*. This link may be explained by the fact that scanning of visual properties are necessary for classification of representations (Kieren, 1990). It is through the words that we name the objects of our talk (Sfard, 2008). In addition, classification depends on both formal definition and properties of the function (Leinhardt et al., 1990; Vinner, 1983). From a commognition perspective, this formal definition is broken down between word and word use, a disconnection between the two results in errors (Sfard, 2008). Scanning involves interpretation of a representation, when done through visual appearance without attending to underlying features of the representation i.e. relying on visual cues (triggers), errors of visual scanning result (Brodie & Berger, 2010a). Literal interpretation of representations has been linked to curricula which have been accused of providing little opportunities for interpretation of representations (Bell & Janvier, 1981).

Calculating routines have been linked to errors associated with algebraic calculations. This study has unpacked the errors of calculations and distinguished between two sources of errors associated with calculations: inappropriate calculation procedures and insufficient formal definition (endorsed narratives).

8.6 Conclusion

In this chapter, I provided a discussion of the results and analyses of the interviews in an attempt to answer my last two research questions. The most interesting results to emerge from the data are that everything seems to point to the significance of words and word use and learners' errors with the function components under investigation were minimal.

In the concluding chapter, I will reflect back on the study, my three research questions, the results of the study, and the theoretical and analytical frameworks used to analyse the data, and link those with the problem of performance addressed in chapter one. I discuss whether commognition theory was able to help me gain insights on learners' performance. In addition, I discuss the limitations of the study and make suggestions about how the study may lead to further investigation.

Chapter 9 CONCLUSION

9.1 Introduction

It is time to ask whether the commognitive framework has fulfilled my intentions and proved itself as a conceptual lens to investigate mathematical thinking. The study was set out to investigate Grade 11 learners' thinking when solving tasks in algebraic functions through commognitive framework. The research questions that this study was seeking to answer were as follows:

1. What common errors do learners make when completing tasks involving algebraic functions?
2. What features of mathematical discourse (word use, routines, visual mediators, and endorsed narratives) are evident in the learners' discourse?
3. In what ways, if at all, are these features linked to learners' errors?

By analyzing the test results, I concluded that the errors on function components under investigation were prolific and were related to classification; interpretation, translation and calculation competencies. The study went further to examine these function components in detail through discursive analysis. The results of this were discussed in detail in the previous chapter and are not repeated in this chapter. What is discussed in this chapter is the summary of main arguments (findings) of this study. I also discuss how these findings address the problem of performance as indicated in Chapter one; the strengths and the limitations of this study, and what are the doors this study has opened for future research.

9.2 The main argument of this study

In the previous chapter, I discussed salient features of mathematical discourse on function objects in learners' discourse. The question now is; what are the central arguments suggested by the study? Although the sample for my study was too small to allow for generalisations, findings in this study may serve as a basis for making judgments about the properties of the function discourses of the other 26 learners and so I can develop some hypothesis questions for further study. In what follows, I am going to make a number of observations regarding the general properties of learners' function discourse

The analyses of learners' discourse shed light relating to learners' abilities to participate[38] in the function discourse. This was evident by the presence of all the features of mathematical discourse. In Chapter Two, I indicated that for Sfard (2008), the four features of mathematical discourse are inextricably intertwined. This intertwinement in my study was evident in the fact that in the discussion of each feature, all the other features were present and were interacting with each other.

This study has shown that the learners' mathematical discourse were error prone and less polished than those of the school mathematical discourse-indicating lack of objectified talk.

Learners avoided speaking about formal definition and kept their discourse at the level of routine procedures. The definitions seemed to play no role in learners' substantiating narratives. Definitions are endorsed narratives (mathematical facts) that describe the objects. This study has shown that learners kept their discourse at a level of processes. This finding led me to think that instruction should reconsider the place of definitions in the process of learning if learners are to be fluent and competent enough in the discourse of mathematics. What this means in practice however is very complex because we know as teachers if we give the learners the formal definition, that may not work. But the whole issue of how we build the words and word use in instruction is very important.

The use of visual triggers was largely demonstrated, but it was cue based, i.e. interpretation of visual mediators was based on their appearance without paying attention to underlying properties of the visual mediators (graph or equation). The visual triggers are prototypic examples that learners were sensitized to thorough previous learning. The data has shown that use of these resulted in errors. This result highlights importance of emphasising on global approach of functions (Even, 1998). This approach addresses important aspects of function discourse, i.e. classification and interpretation.

In light of the above, the learners' current discourse can be described as passive driven and routine driven. According to Sfard (2008), in order for learners to speak in an objectified

[38] In this context, 'participate' refers to taking part in a mathematical discourse using features of the mathematics discourse. I am not using Sfard's description of 'participating in a discourse' where she associates participating with a metaphor that views learning "as a process of becoming a member of a certain community" (Sfard,1998), as an increasing ability to participate meaningfully in a particular social context.

way, their discourse must stand in two legs, that of processes and that of objects. Since the five learners' function discourse tend to be mainly processual, the implications for instruction is that these learners need to be ushered towards the objectified version of the discourse.

In the analysis, it was evidenced how visual mediators, especially graphs, function as a way of establishing a common focal point and mediational tool. The study has further shown that graphs were the most preferred visual mediator to mediate communication and seemed to have provided learners with the focal point when talking about function objects. However, in order for learners to appreciate the concept of function they need to engage with different forms of function representations. As I mentioned earlier, for learners to be fluent and competent enough on the discourse, mediational diversity should be an additional catalyst for objectification.

The function discourse of the five learners may be described as a combination of colloquial and mathematical discourse, and so, consistent with Sfard's (2008) assertions that any discourse has colloquial and literate parts. Learners' discourse indicated a shift from colloquial discourse to a more mathematical discourse. In those contexts learners shifted their routines, word use and narratives to substantiate their narratives. It has been strongly argued that learners' discourse (combination of colloquial and mathematical discourse) was not linked to errors.

Language of function terminology has played part in learners' difficulties with functions. And in the analysis it was evident that some of the learners are using words erroneously, and they seem to know what they are talking about. They use hyperbola when they mean parabola because both sound the same. They use the word intersect to mean intercept. The word gradient is used erroneously to identify the coefficient of x squared in the quadratic function ($y = ax^2 + bx + c$). However, the way these words are used resembles that of school mathematics discourse, with some piece of information missing. Words were linked to classification errors. The results of this study substantiate the importance of getting these words clear and we should make a distinction around them. Teachers need to be cognizant of those language specific features of the discourse that seem to hinder learners' performance. Thus to support meaningful learning of functions, teachers may wish to deliberately capitalize on the existing interplay between learners' colloquial talk and literate mathematical discourse on functions.

The notion of commognition proved useful in an attempt to understand difficulties experienced by learners with the concept of function. In this study a commognitive analysis revealed that, at least some of the difficulties may stem from lack of function terminology, overreliance on visual cues and poor access to endorsed narratives (formal definition). The results of this study have implications for instruction. Teachers need to play a role in helping learners to change the discourse and to develop learners' mathematical discourse to the level of the experts' mathematical discourse and the complexity of which should not be undermined.

9.2 How do these findings address a problem of performance?

In my introductory remarks in Chapter one, I highlighted difficulties learners often experience with the concept of function as reported in the National Diagnostic Report (DoE, 2012). These difficulties were associated with interpretation of representations (graphs in particular), converting between different representations, vocabulary, algebraic calculations and function concept definition. This study redefined these from the commognitive perspective as: interpretation into routine procedure, converting between representations into mediational switching, vocabulary into words, algebraic calculations into routines and concept definition into words & word use.

This research study has tried to explain underlying reasons for these difficulties from a commognitive perspective. To aid in the discussion of underlying reasons of difficulties, I employ a diagram consisting of three columns (see Figure 9.1).

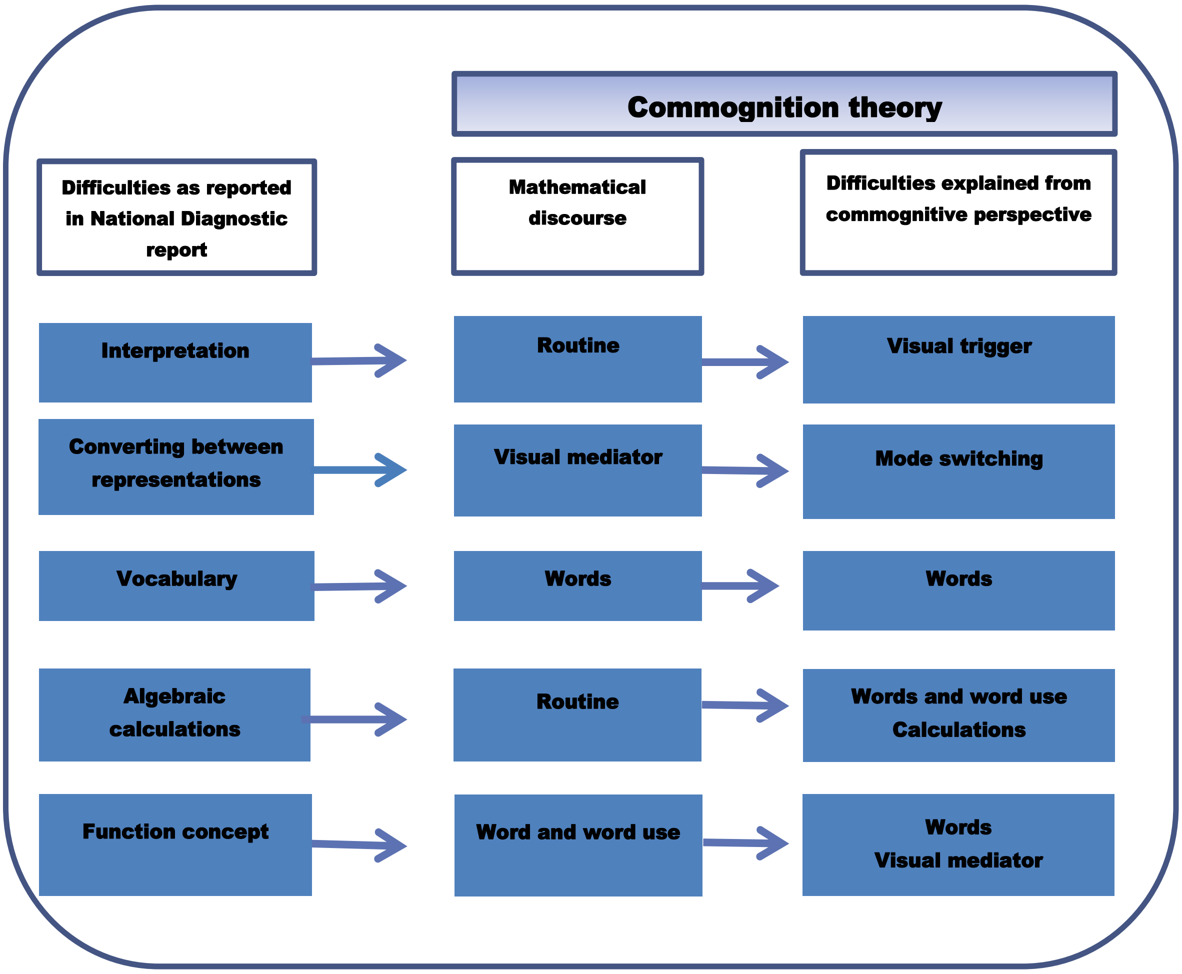


Figure 9:1 Link between research problem and commognitive framework

The first column represents difficulties as reported in National Diagnostic Report (DoE, 2012). The second column refers to the features of the mathematical discourse. The last column represents the difficulties explained from commognitive perspective. The diagram is used to discuss the relationship between learners' difficulties with function and commognitive research and to illustrate how commognition theory has helped me to gain insights on performance as reported in the National Diagnostic Report (DoE, 2012).

Difficulties with: (i) interpretation are due to relying on visual triggers. (ii) Converting between different representations are due to the lack of flexibility in using different routines. Converting involves use of different approaches (Even, 1998) and these were redefined to refer to routines. These routines include: calculating, interpreting, constructing, translating and classifying. Flexibility between routines allows for movement between representations i.e. mode switching (Ben-Yehuda et al., 2005). Failure to engage with these routines explains the difficulty to move between different representations; (iii) vocabulary is due firstly to reliance on visual triggers, which are applied without paying attention to underlying properties of the representation. Secondly, they are due to lack of vocabulary used in functions discourse; (iv) algebraic calculations are due to incorrect calculating procedures and also to insufficient formal definition; and (v) function concept definition are due to incorrect words used to identify functions and relying on defining the function concept from a graphical representation. It is not enough to stay within an understanding strongly leaned on one representation. According to Nachlieli and Tabach (2012), in order to objectify a function concept, learners need to be competent in the discourses on different representations.

As claimed earlier, Sfard's commognition enabled me to open up learners' mathematical thinking through their mathematical discourse. In what follows, I provide the strengths and limitations of Sfard's framework.

9.3 Strengths and weaknesses of commognitive framework

The strength of Sfard's (2008) framework helped to unpack the 'concept definition' by breaking it into words and word use, something others have not been able to do. Secondly, it elaborated on the routines of the discourse by highlighting the communication breakages in learners' discourses. According to Sfard (2008), routines are amongst the elements of mathematical discourse that remain tacit in learners' discourse. The exploration of discourse with a focus on routines together with substantiations they produced enabled the identification of patterns and disconnections in learners' discourse.

What was gained from using this framework is that the combination of colloquial and mathematical discourse can help strengthen learners' objectification of mathematical objects.

Commognitive framework helped me to see various emerging roles the learners play and how they interact with the objects in the discourse. It further allowed rich descriptions of learners' discourses, which could not be explored by other approaches.

Sfard has provided scientific rigour with operational definitions of keywords of discourses. However, her work was not easy to read with too many categorization and descriptions of concepts, sometimes overlapping one another. And the boundaries between these are blurred.

Take for example the case of iconic and symbolic mediators' categorizations. Sfard (2008) differentiates between three visual mediators: iconic, symbolic and concrete. At the same time symbolic mediators include iconic mediators (see Sfard (2008, p. 148):'*symbolic artifacts include icons, such as conventional or individually designed diagrams*'). This indicates some kind of overlapping. In another example she differentiates between three forms of endorsed narratives: construction, substantiation and recall. One can argue that constructing and recalling involve acts of substantiating. Recalling is part of one's every discursive action, for example when deriving or substantiating, remembering previously endorsed narrative is involved.

It has been reported in several reviews that Sfard's work resolves four quandaries that have been in existence around mathematical thinking. One of them is misconceptions. Misconceptions from a constructivist perspective are linked to prior knowledge or learning. In the National Diagnostic Report (DoE, 2012), learners' poor performance was reported to be linked to previous learning. Is there anything new that commognition theory has explained about misconceptions? Some of the learners' errors were linked to visual triggers (prototypes) previously endorsed narratives. This notion is consistent with the theory on misconceptions and how they impact on performance. Sfard provides a new concept to describe these i.e. 'previously endorsed narratives'. Furthermore, commognition theory provided microscopic lenses, and allowed me to see further and more deeply what is usually invisible. It further made me see logic in discursive actions that appeared to be nonsensical.

A new quandary emerged from this study, and relates to the word 'concept'. While using Sfard's theory, I needed to deal with Sfard's notion of the word 'concept'. Sfard refers to the concept as words together with word use. Is it the concept definition or the object? Or is it the form of words and word uses that learners use to describe or define the concept. Sfard classifies the formal definition as the endorsed narrative (definition which is accepted by the mathematical community at large). However she makes no clear distinction between the formal definition and what she refers to concept (word & word use), as done by others such as Tall and Vinner (1981). According to Tall and Vinner (1981), concept definition is a form of words used to describe that concept. They go further and categorise concept definition into personal definition and formal definition of the concept. To them a personal definition refers to the learners' reconstruction of the concept definition (learners' words they use when explaining the concept definition). And the formal definition is the concept definition which is accepted by the mathematical community at large. Tall's and Vinner's definition of the 'concept definition' is helpful because it provides a clear description from two perspectives, that of learners and that of experts, something missing in Sfard's work. This led to some tension when using the word 'concept' in this study. There is still more work that needs to be done in refining some of the keywords used in commognitive theory.

9.4 Implications and Recommendations

According to Sfard, learners make errors because they don't know the next move to make. She advocates on a change of discourse. As I have already mentioned, this study results have implications for instruction. Instruction should pay attention to terminology, visual literacy (interpretation) and pay attention to previous learning in order to help learners change their discourse. The findings of this study have important implication for integration of colloquial discourse during instruction, in curriculum documents: textbooks, examination question papers and policy statements.

9.5 Further research

In discussing possible continuations of the research conducted in this study, I choose to focus on four themes: research design on mathematical discourses; analytical approaches for studying mathematical discourses; research on teachers' mathematical discourses; and design research on mathematics curriculum (i.e. mathematics education and curriculum documents)

9.5.1 Research design

In this study, learners were individually interviewed. The interview transcripts offered a large amount and variety of information. Even so, I wish that this was even more so – for two reasons. Firstly, as noted earlier that during the interviews it became apparent that the use of video recordings was necessary to enable me to revisit past events (Sfard, 2008) and capture all the participants' actions without relying solely on note-taking. In many moments of the study of the transcripts, I had to go backward and forward to understand learners' object of attention (what the learner is doing or looking at). It is highly possible that some of learners' actions could have been missed, although notes were taken this could not guarantee capturing all the learners' actions. The video recordings could have opened up a different way of revisiting the interview in real time. Unfortunately for me, ethical clearance for video recordings was not requested. In hindsight, I regret not complementing my data with video recordings.

Secondly, as I analysed the data, I time and again felt that there could have been some obstructions from my side through probing and also through the way the questions were structured. The use of prompts and probes that allowed learners to explain and elaborate produced rich data, but might have affected the learners' substantiations. For example, most of the time participants substantiated their narratives when they were prompted by asking short questions like 'Why?' When the question was asked through a visual mediator, it determined which routine learners would choose. Sfard (2008, p. 214) also points out that '*verbal prompts, such as questions or requests are often regarded in school as holding the exclusive responsibility for students' choices of discursive procedure'*. This means that the way the question is asked determines the choice of discursive procedure; i.e. word use, routine, mediator, endorsed narratives.

Lessons drawn from this experience with respect to future work includes letting participants talk with one another (Ryve, 2006; Sfard & Kieran, 2001) in order to avoid enticing learners in my own discourse as a researcher. Further, I could have used a video to capture all learners' discursive moves, as is done by many in the commognitive research (Ben-Yehuda et al., 2005; Ryve et al., 2013; Sfard & Kieran, 2001).

9.5.2 Analytical framework

As previously argued in this study, discursive research tools on mathematical thinking are still under developing, and need more attention (Sfard & Kieran, 2001). This study has contributed to that development by redeveloping Ben-Yehuda et al. (2005) ADP analytical framework. This framework was originally developed to analyse elementary school learners' arithmetical discourse. I reconceptualised this tool to fit function discourse for secondary school learners. I therefore suggest that further research continues into secondary mathematics topics.

9.5.3 Mathematical discourse of teachers

The findings of this study indicate that learners make errors and this is due to disconnection between their substantiating narratives (discursive actions) and endorsed narratives. From a commognitive perspective, errors occur when learners are participating in a different discourse from the teacher's (as representatives of the mathematical community) and that they are not aware of the next move and that the rules have changed. This has implication on teachers, to help learners change their discourse. Regarding the findings related to the words and word use, an implication to the words used during instruction was alluded to. In the light of these implications, it would be interesting to carry out a research study focusing on teachers' mathematical discourse and how they support learners' participation in the function discourse.

9.5.4 Research on mathematics curriculum

If we view colloquial discourse as an important feature of mathematical learning (Moschkovich, 1999b; Sfard, 2008), there are also implications that are closely related to the goals of the South African mathematics curriculum. This study shows that the use of a combination of colloquial and mathematical discourses result in a more mathematical discourse with few pieces of information missing. It would be interesting to see how much of colloquial discourse is supported in the curriculum (including documents and pedagogy).

9.6 Limitations

The type of questions on the test and interview might not be totally representative of function concept. Although the analysis tool suggested by Sfard (2008) takes into account important features of learners' discursive moves, such as, what is said and what is done, my use of this tool suggests the need to strengthen it. The study was limited to audio recordings and note-takings. It has been alluded to throughout this study that the credibility of the research would have increased if I had an opportunity to video-record the interviews, as suggested by (Sfard, 2008; Sfard & Kieran, 2001).

9.7 My reflections

The scope of the study turned out to be much larger than had initially been anticipated. When I consider all features of mathematical discourse: word use, routines, visual mediators and narratives, I regard word use as all important, revealing facts concerning learners' discursive moves (i.e. routines, visual mediators and narratives). In this study, learners' word use provided significant information about all the other features of mathematical discourse. Moreover, a careful analysis of learners' word use in function discourse would also shed light on the properties of the mathematical discourse. Therefore, if any other features should be considered when investigating learners' mathematical discourse, I recommend 'word use' to be the main feature to be investigated.

9.8 Concluding remarks

Inherent problem of performance still exists. Discursive analysis produces descriptions of what it is learners do, and not what is 'wrong', enabling in a context where deficit discourses on learners and teachers in mathematics prevail. This is not to hide away from poor performance, rather to emphasise how important discursive actions are hence implications for teaching and learning.

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APPENDIX A: Letters seeking permission

INFORMATION LETTER TO LEARNERS

UNIVERSITY OF THE WITWATERSRAND

MATHEMATICS RESEARCH PROJECT

13th April 2012

Dear Learner,

My name is Lizeka Gcasamba. I am currently doing my MSc degree in Mathematics Education. As part of my degree I am doing a study investigating learners' mathematical thinking when solving tasks involving functions.

Your school principal has given me permission to send you this letter of invitation to participate in this research study on mathematical thinking.

Learners who agree to participate in the study will answer a (written) task questionnaire and will be tape recorded in one hour session three times in the month of July/August 2012.These recorded interview sessions will take place after school. The focus in these tape recordings and the task questionnaire will be on the mathematical thinking when solving tasks on functions.

I intend to protect your anonymity and confidentiality. Your name(s) will not be used in the final report of this research study. I will remove any reference to personal information that will allow someone to guess your identity.

Remember that you are not obliged to participate. Should you require any further information do not hesitate to contact me on my telephone number as below.

Yours faithfully,

Lizeka Gcasamba

Cell: 071 178 3673
email: lizeka.gcasamba@wits.ac.za

CONSENT FORM FOR LEARNERS

UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG

MATHEMATICALTHINKING RESEARCH

Researchers Details: Ms Lizeka Gcasamba
Email: lizeka.gcasamba@wits.ac.za
Cell : 071 178 3673

Supervisor Details: Professor J. Adler
Email : jill.adler@wits.ac.za
Tel(w): 011- 717 3413
Fax : 011- 717 3109

Consent **form for learners** participating in the study.

I, ... agree to participate in the research study named above, particulars of which (i.e. problem solving task and interviews) have been explained to me. A written information letter has been given to me to keep.

I, therefore, give consent to the following:

- Tape Recording of the interview in which my voice will be part of the tape recorded text.

  Yes ☐ No ☐

- The possible future use of tape-recorded text for teaching purposes.

  Yes ☐ No ☐

....
Signature of participant Date

....
Signature of witness Date

....
Signature of teacher/researcher Date

INFORMATION LETTER TO PARENTS

UNIVERSITY OF THE WITWATERSRAND

MATHEMATICS RESEARCH PROJECT

13th April 2012

Dear PARENT(S),

My name is Lizeka Gcasamba. I am currently doing my MSc degree in Mathematics Education. As part of my studies I am doing a study investigating learners' thinking when solving tasks involving functions.

Your child's school principal has given me permission to send you this letter of invitation to participate in this research study on mathematical thinking. Learners whose parents agree that they participate in the study will answer a (written) task questionnaire and will be tape recorded in one hour session three times in the month of July/August 2012.These recorded interview sessions will take place after school. The focus in these tape recordings and problem solving written responses will be how is the mathematical thinking when working with functions is promoted to facilitate learning.

I intend to protect the learners' anonymity and confidentiality. Their name(s) will not be used in the final report of this research study. I will remove any reference to personal information that might allow someone to guess the learners identity.

Be informed that your child is not obliged to participate (i.e. participation is voluntary). Should you require any further information do not hesitate to contact me on my telephone number as below.

If you agree that your child be part of this research study, please complete the consent form attached by signing on the spaces provided and return it to me.

Yours faithfully,

Lizeka Gcasamba

Cell: 071 178 3673
email: lizeka.gcasamba@wits.ac.za

CONSENT FORM FOR THE PARENTS

UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG

MATHEMATICAL THINKING RESEARCH

Researchers Details: Ms Lizeka Gcasamba
Email: lizeka.gcasamba@wits.ac.za
Cell : 071 178 3673

Supervisor Details: Professor J. Adler
Email : jill.adler@wits.ac.za
Tel(w): 011- 717 3413
Fax : 011- 717 3109

Consent form **for the parents**.

I, ... agree that my child participate in the research study named above, particulars of which (i.e. details of problem solving task and interviews) have been explained to me. A written information letter has been given to me to keep.

I, therefore, give consent to the following:

- Tape Recording of the interview in which the voice of my child will be part of the tape recorded text.

  Yes □ No □

- The possible future use of tape-recorded text for teaching purposes.

  Yes □ No □

....	....
Signature of the Parent(s)	Date
....	....
Signature of witness	Date
....	....
Signature of teacher/researcher	Date

APPENDIX B: Test

TEST ON ALGEBRAIC FUNCTIONS

NAME .... GRADE…………………

AIM OF THE STUDY:
Research in Education has shown that some learners have difficulties when solving tasks on functions. I want to find out how you solve tasks on functions and what strategies you use to solve these tasks. This will help me to understand some of your difficulties so that I can be able to find ways of helping you and other learners with mathematics learning.

WHAT YOU NEED TO DO

1. Write your name as indicated above .

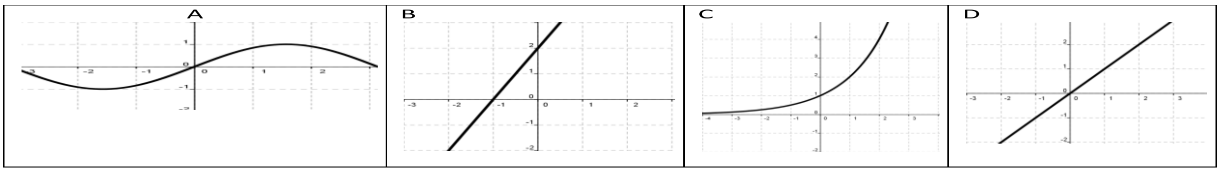
2. The paper consists of ...5... pages and ...7... questions

3. Attempt all the questions in the test. Show all the necessary workings and reasoning in the answer sheet provided and on the writing paper provided.

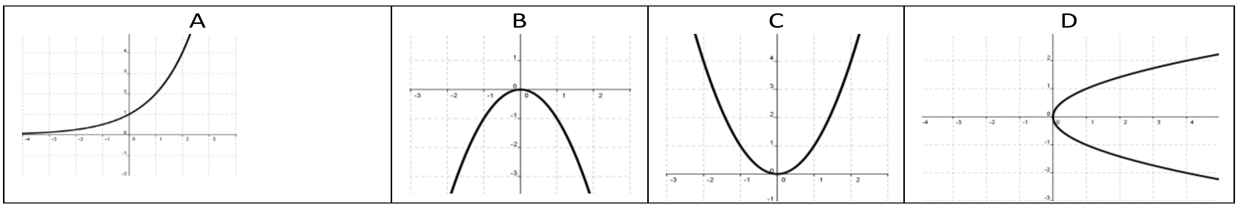
Thank you for participating in this activity & research

<u>Question1</u>

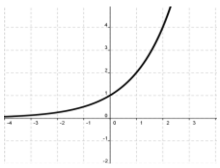
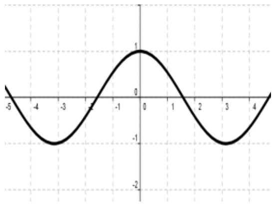
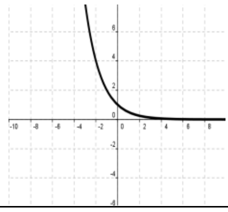
1.1 Place a tick against all graphs that represent a linear function?



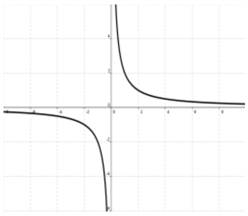
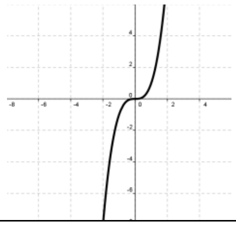
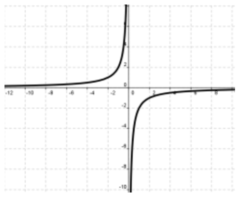
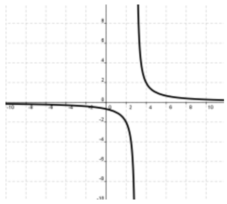
1.2 Place a tick against all graphs that represent a parabola, that is, the graph of quadratic function?



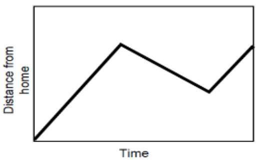
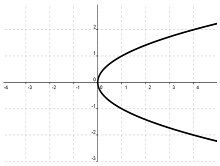
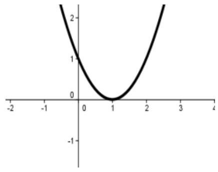
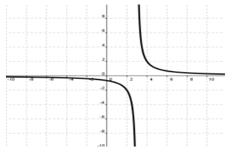
1.3 Place a tick against all graphs that represent an exponential graph?

A	B	C	D
			

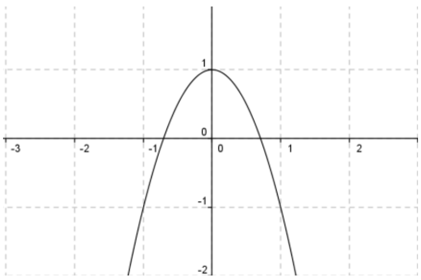
1.4 Place a tick against all graphs that represent a hyperbola?

A	B	C	D
			

1.5 Place a tick against all graphs that do not represent a function.

A	B	C	D
			

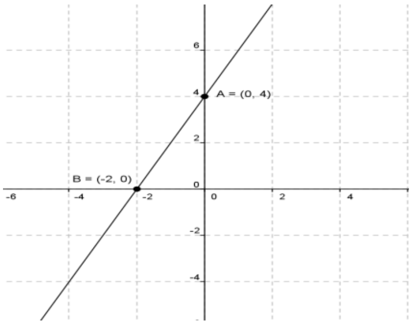
1.6 The sketch below represents the graph of $y = ax^2 + 1$



Which of the statements below is/are true, and which is/are false. Write T or F for each?

A. a is positive
B. a is negative
C. a=-2
D. a=1

1.7 The sketch below represents the graph of $y = px + 4$



Which of the statements below are is/true, and which is/are false. Write T or F for each ?

A. p is positive
B. p is negative
C. $p = 1$
D. $p = -1$

Question 2

You have learnt about four kinds of functions: linear, quadratic, hyperbola and exponential.

What kind of function is represented by each of the following equations, and how do you know?

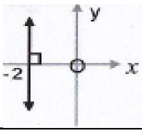
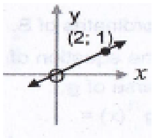
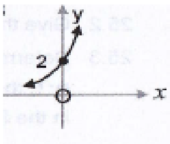
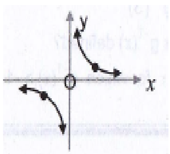
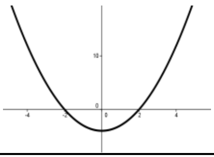
2.1 $y = 3x$

2.2 $y = 3x^2$

2.3 $y = \frac{3}{x}$

Question 3

In the table below, 5 graphs are given in the first column. Followed by the list of 5 equations in the second column. You need to match each graph with its correct equation, and give a reason for your choice.

<table>
<tr><th>Graphs</th><th>List of possible equations</th></tr>
<tr><td>3.1

</td><td>A. $y = \frac{2}{x}$</td></tr>
<tr><td>3.2

</td><td>B. $x = 2$</td></tr>
<tr><td>3.3

</td><td>C. $y = x^2 - 4$</td></tr>
<tr><td>3.4

</td><td>D. $y = \frac{x}{2}$</td></tr>
<tr><td>3.5
</td><td>E. $y = 2^x$</td></tr>
</table>

Question 4

Given functions $f(x) = x^2 - 4x - 5$ and $g(x) = x - 5$

4.1 What kind of function is f and g ?

4.2 Draw f and g on the same system of axes.

4.3 What are the x intercepts f and g ?

4.4 What is the y intercept of both f and g ?

4.5 What are the co-ordinates of the turning point of f ?

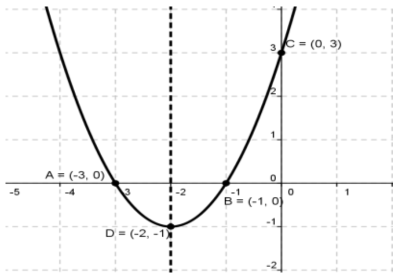
4.6 Use your graph to solve for x if:

4.6.1 $x^2 - 4x - 5 = x - 5.$

4.6.2 $f(x) > 0$

Question 5

Given equation $y = x^2 + 4x + 3$ and its graph below



5.1 What is the value for $f(-2)$?

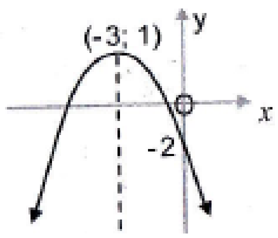
5.2 What are the x intercepts of the graph?

5.3 What is the y intercept of the graph?

5.4 What are the co-ordinates of the turning points?

Question 6

The figure below is a parabola, with turning point (-3, 1) and *y* intercept (0,-2)



6.1 Determine the equation of the graph.

Question 7

The rule in this table is, 'take a number and square it'.

7.1 Complete the rest of the table using the rule. The first block and the last block have been completed.

x	**-2**	**-1**	**0**	**1**	**2**	**3**
f(x)	**4**					**9**

7.2 Do the values in your table represent a function? Give reasons for your answer.

7.3 If so, what is the name of the function?

7.4 If $f(x) = 1$, what can you say about the value of x ?

7.5 Give a value of $f(2)$

The End.Thank You.

APPENDIX C: Interview protocol

A: INTERCEPTS:

What is your understanding of the word intercept? Or how can you explain to a grade 10 learner, a concept of an intercept?

Can you explain in any other way?
Can you give me an example?

They might explain it:

(i) using terms like x intercepts, y intercepts, coordinates(this will help me to see the **words** they are using)
(ii) They might draw it(**visual mediator**)
(iii) Maybe explain using algebraic equation(**visual mediator**)
(iv) How they are explaining/answering ii & iii will help me with **routines**
(v) All the four above will help me with **narratives**

Follow up questions:

- will be done depending on the words they have used i.e. intercepts(x or y),coordinates etc.
- if they could not give their understanding of the concept(intercepts) at all

<u>x/y Intercepts:</u>

(a) If I gave you a function $y=2x+1$

How would you find an x intercepts/ y intercept?

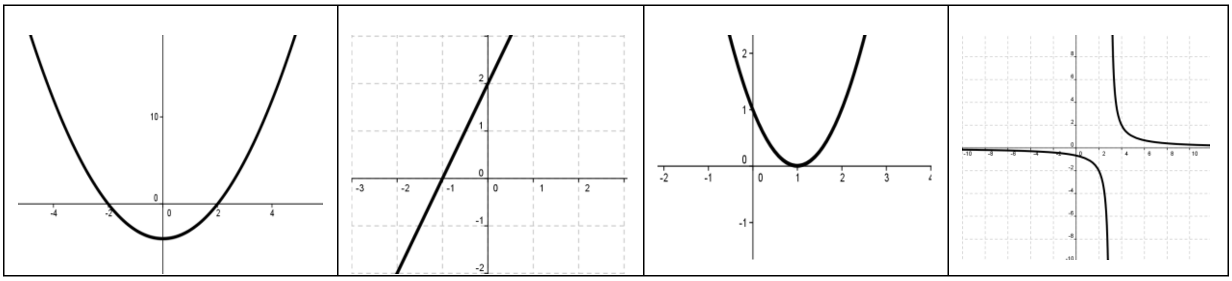
Why are you using that method/way?

Could you find the intercepts by using any other method than the one you have used?

How they answer this question will help me with the (routine, visual mediator)

Why they have chosen to answer it in a certain way, will help with (narrative, routine)

(b) If I drew the following graphs in the set of axes:



- Can you please show me the x and y intercepts in each graph?

(c) Given the point (-2,0),what does it represents? What can you talk about when you see this point?

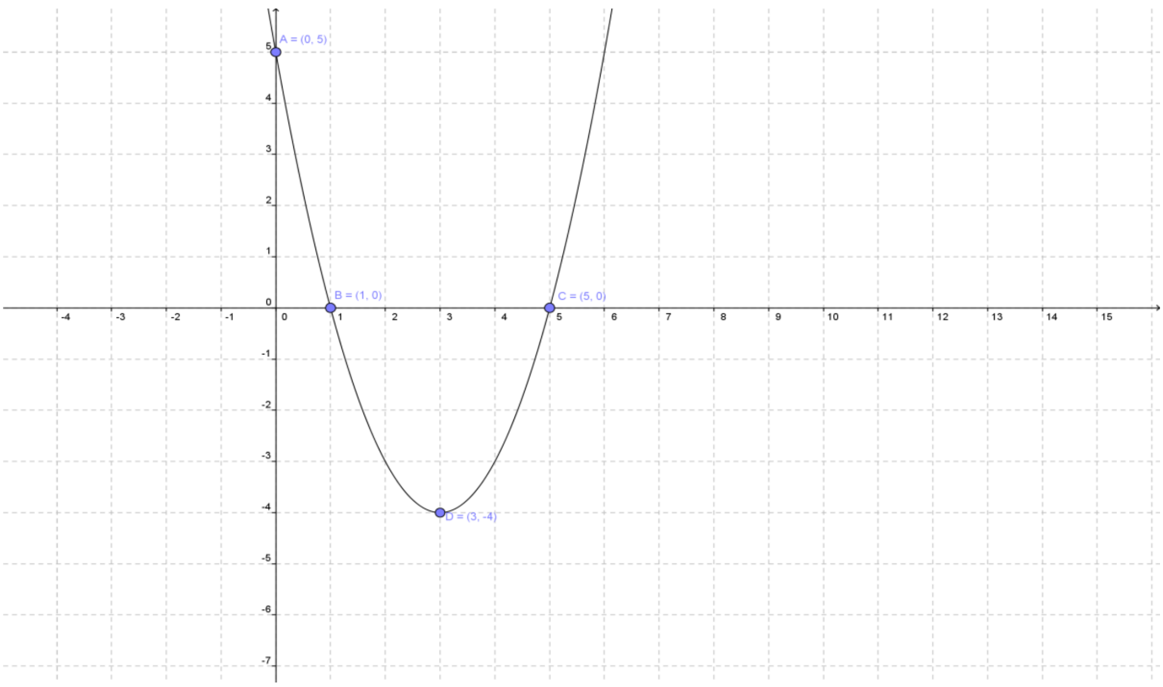
Maybe they are going to say co-ordinates (words).

Then, I would ask if they can show it in a different form, give an example or maybe draw (visual mediator)

Why have you decided to do that?

How, why, where are they drawing it (**routine, narrative**)

(b) What are the co-ordinates of the x intercept and the y intercept in figure below?



(c) Now let's say I gave you a table, showing a linear pattern (linear function):

x	-4	-3	-2	-1	0	1	2	3	4	5
y	-3	-2	-1	0	1	2	3	4	5	6

- From the table, can you tell me what is the x intercept and the y intercept?
- How do you know?

(d) Let's talk about the notation f(x), what can you say about this? Can you give an example? What does it represent?

- Can you tell me from the table, what is the value of f (0)?
- What is this value called?

B. ALGEBRAIC REPRESENTATION

Quadratic

(a) What is this function called $g(x) = 2x^2 + 5$?

They might say exponential (**words**)

Why exponential (**narratives, routine**)

Which other forms/ways can help you to identify this function (**visual mediator**)

Follow up:

What is this function $y = x^3 + 2$?

What is this function $y = 3^x$?

Linear

(b) What is the name of this function y=2x+1

What distinguishes it from other functions?

Let's say you are given a function $y = \frac{x}{2} + 3$. What is the name of this function?

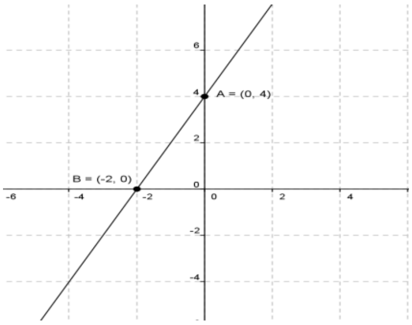
They might say hyperbola, division (**words)**

How do you know (**narratives, routines, visual mediator**)

C. -GRADIENT

Linear

(a) The sketch below represents the graph of $y = px + 4$



What does p represents

They might say gradient (**words**)

Why, how do you know (**narrative, routine?**)

Maybe they will explain through a form y=mx+c (**visual mediator**)

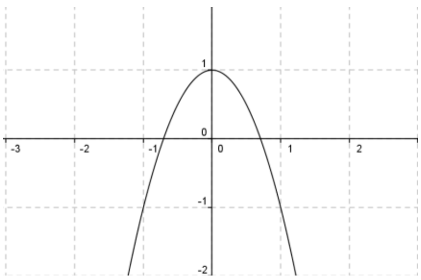
Follow up:

Is the value of p positive or negative? How do you know?

Are they going to answer using the diagram (**visual mediator**) or calculation (**routine?**)

Quadratic

(b) The sketch below represents the graph of $y = ax^2 + 1$



What does a represent?

How, why do you know that (routine, visual mediator, narrative)

Can they see if this is a different from gradient?

Combination of linear and quadratic functions

Let's talk about the following functions $f(x) = 3x + 1$ and $g(x) = 2x^2 + 5$

- Two of Grade 12 learners were having a discussion about the two functions. One of them was saying that the gradient of $g(x)$ is 2. Is he correct?

End!!!!!!!!!!!!!!!!!!!!!!!!

APPENDIX D: Example of the results of the test

questions	correct	incorrect	no attempt
1.1	25	1	0
1.2	22	4	0
1.3	22	4	0
1.4	25	1	0
1.5	0	26	0
1.6(a)	1	24	1
1.6(b)	2	22	2
1.7	2	21	3
2.1	22	2	2
2.2	9	17	0
2.3	22	2	2
3.1	14	5	6
3.2	7	14	5
3.3	13	8	5
3.4	15	6	5
3.5	20	2	4
4.1	24	0	2
4.2(a)	13	10	3
4.2(b)	13	10	3
4.3	10	3	13
4.4	16	4	6
4.5	13	5	8
4.6.1	8	10	8
4.6.2	5	4	17
5.1	9	6	11
5.2	15	3	8
5.3	7	1	15
5.4	18	1	7
6.1	1	14	11
7.1	11	1	14
7.2	0	11	15
7.3	0	7	19
7.4	0	10	16
7.5	0	26	0

In table above, the first column shows the labelling of questions; second column the number of learners who have answered correctly, column 3 is the number of learners who answered incorrectly, column 4 is the number of learners who did not attempt to answer the question.

APPENDIX E: Rubric of error categories

Categories of errors	Errors from literature	Errors from National Diagnostic report	Errors from prepilot
Algebraic calculations	√	√	√
Translation	√	√	√
Vocabulary/vocabulary use	√	√	√
Interpretation	√	√	√
Function concept	√		√
Classification	√		√
Constant functions	√		√
Plotting/Scaling	√		√
Function notation	√	√	
Intercept concept		√	√
Co-ordinates			√
Linearity	√		
Gradient	√		

Rubric of categories of errors

√-means present

APPENDIX F: Coding sheet

Learners	4.2		
	Graph	Answer	Error
L1	quadratic	incorrect	algebraic calculation
	linear	correct	none
L2	quadratic	incorrect	constructing(scaling/labelling)
	linear	incorrect	constructing(scaling/labelling)
L3	quadratic	correct	none
	linear	correct	none
L4	quadratic	correct	none
	linear	correct	none
L5	quadratic	correct	none
	linear	incorrect	algebraic calculation
L6	quadratic	correct	none
	linear	correct	none
L7	quadratic	correct	none
	linear	correct	none
L8	quadratic	correct	none
	linear	correct	none
L9	quadratic	no attempt	none
	linear	no attempt	none
L10	quadratic	incorrect	constructing(scaling/labelling)
	linear	incorrect	constructing(scaling/labelling)
L11	quadratic	incorrect	algebraic calculation
	linear	incorrect	algebraic calculation
L12	quadratic	incorrect	algebraic calculation
	linear	incorrect	algebraic calculation
L13	quadratic	correct	algebraic calculation
	linear	correct	algebraic calculation
L14	quadratic	no attempt	none
	linear	no attempt	none
L15	quadratic	incorrect	constructing(scaling/labelling)
	linear	incorrect	constructing(scaling/labelling)
L16	quadratic	correct	none
	linear	correct	none
L17	quadratic	incorrect	constructing(scaling/labelling)
	linear	incorrect	constructing(scaling/labelling)
L18	quadratic	no attempt	none
	linear	no attempt	none
L19	quadratic	incorrect	constructing(scaling/labelling)
	linear	incorrect	constructing(scaling/labelling)
L20	quadratic	incorrect	algebraic calculation
	linear	incorrect	algebraic calculation
L21	quadratic	incorrect	constructing(scaling/labelling)
	linear	correct	none
L22	quadratic	correct	none
	linear	correct	none
L23	quadratic	incorrect	algebraic calculation
	linear	incorrect	algebraic calculation
L24	quadratic	incorrect	algebraic calculation
	linear	incorrect	constructing(scaling/labelling)
L25	quadratic	correct	none
	linear	correct	none
L26	quadratic	correct	none
	linear	correct	none

Each learner was labelled using codes L1-L26

APPENDIX G: Mean percentage

Mean percentage of incorrect responses for "gradient"

Component	Coded errors	Question number	% of incorrect answers	% mean error
Gradient	interpretation	1.6	92%	94%
		1.7	96%	

APPENDIX H: Ethics clearance

Wits School of Education

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Tel: +27 11 717-3064 Fax: +27 11 717-3100 E-mail: enquiries@educ.wits.ac.za Website: www.wits.ac.za

Student Number: 9300068M

Protocol Number: 2012ECE055

Date: 24-Oct-2012

Dear Lizeka Gcasamba

Application for Ethics Clearance: Master of Science

Thank you very much for your ethics application. The Ethics Committee in Education of the Faculty of Humanities, acting on behalf of the Senate has considered your application for ethics clearance for your proposal entitled:

FUNCTIONS AND THEIR REPRESENTATIONS: AN INVESTIGATION OF LEANERS' MATHEMATICAL THINKING WHEN SOLVING ALGEBRAIC FUNCTIONS THROUGH A DISCURSIVE ANALYSIS OF ERRORS

The committee recently met and I am pleased to inform you that clearance was granted. The committee was delighted about the ways in which you have taken care of and given consideration to the ethical dimensions of your research project. Congratulations to you and your supervisor!

Please use the above protocol number in all correspondence to the relevant research parties (schools, parents, learners etc.) and include it in your research report or project on the title page.

The Protocol Number above should be submitted to the Graduate Studies in Education Committee upon submission of your final research report.

All the best with your research project.

Yours sincerely

M Mabeta

Matsie Mabeta
Wits School of Education

011 717 3416

Cc Supervisor: Prof. J Adler