CHAPTER 1 INTRODUCTION TO THE STUDY

1.1 INTRODUCTION

Mathematical reasoning is one of the important aspects of the new curriculum in South Africa (DoE, 2002). To partake in this activity learners are expected to actively examine, conjecture, make valid justifications, generalize and present arguments for mathematical solutions realised in problem solving. Educators are therefore expected to promote and incorporate these practices during mathematics lessons from early stages.

This study is designed to investigate grade 9 learners' mathematical reasoning when generalising from number patterns. The study will be conducted in one grade 9 class. The class consists of 29 learners of mixed ability (13 girls and 16 boys) at a former model C school in Johannesburg, South Africa. I will work in a qualitative paradigm using a case study approach. Research literature related to mathematical reasoning within problem solving has been reviewed in order to shape and inform the study. The theoretical framework of commognition, developed by Sfard (2008) which defines "thinking as communicating" will be used in the study. A questionnaire-based task with items on number patterns is designed to be used as the main data gathering instrument and from which a task-based interview will be developed. Six learners purposely drawn from the main sample (see section 3.3.1) will be the focal-sample for the task-based interviews. Data collected through use of the task and interview will be analysed mainly using qualitative approaches informed by the theoretical framework the study is using.

1.2 BACKGROUND/CONTEXT

Recently, there have been curriculum shifts in mathematics in many countries that are driven by national policies, and influenced by research. There is advocacy for a transition from traditional ways of teaching that emphasize mastery of procedures and algorithms, to approaches which also forster mathematical reasoning in learners or learning mathematics as a set of practices e.g. justifying, representing, generalising etc. It is believed that these new approaches to teaching mathematics would enable learners to re-invent mathematical ideas, procedures and algorithms when forgotten hence serve to aid retention of the learnt facts. These curriculum innovations are also found in South Africa. Brodie & Pournara (2005) argue that these shifts are signalled by terms such as educator, facilitator and mediator within the South African curriculum. According to Brodie these terms indicate new roles for teachers and learners. The South African Department of Education, aligning itself with these curriculum shifts, advocates that learners should investigate patterns and relationships in order to “develop mathematical thinking skills such as generalizing, explaining, justifying, representing ..., predicting and describing” (DoE, 2002:63).

In South Africa, algebraic reasoning in the General Education and Training band (GET) curriculum is very limited in scope and yet the learning achieved in Mathematics in the GET band has to provide the basis for the demands of Mathematics in the Further Education and Training (FET) phase. This situation creates a need to explore how mathematical reasoning evolves from algebraic reasoning in the GET band. The essentials of Numeracy developed in the GET band are taken further into working in more symbolic ways (algebraic) in the FET band. In so doing, the emphasis on contexts and integration within Mathematics and across the curriculum is maintained where mathematical modelling which largely constitutes algebraic generalization becomes more prominent. The GET engagement with shape, space and measurements becomes more formalised, and the methods and uses of statistics and chance are dealt in greater depth in the FET. How Mathematics can contribute to an understanding of financial issues is taken beyond dealing with budgets as in GET band.

The revised national curriculum statement (RNCS) put in place to strengthen Curriculum 2005 (C2005) is the first National Curriculum Statement (NCS) that emphases integration of mathematical reasoning within mathematics and across other learning areas (DoE, RNCS: 2005). The C2005 is an Outcomes-Based Education (OBE) curriculum established in 1997. Therefore, it is imperative that teachers and policy makers understand how learners reason mathematically in the context of the NCS demands. It is against this background that I decided to explore learners’ algebraic reasoning when generalizing. In particular I wish to explore the representations and routines the grade 9 learners use when engaging with a task on number patterns and also the difficulties they experience. The questionnaire-based task being used in the study contains questions set in different representations (forms) of number patterns. Some questions, are formulated in terms of numbers only (e.g. table of values), some in terms of pictures/diagrams and some as word problems. Table 1 provides a summary of how the Mathematics Learning Outcomes (LO’s) of the GET and FET bands are linked:

Table 1: How the Mathematics Learning Outcomes of GET and FET bands are linked.

LEARNING OUTCOME	GET (RNCS) PHASE	FET PHASE
1	Number and Number Relationships	Number and number Relationships
2	Patterns, Functions and Algebra	Functions and algebra
3	Shape and Space	Shape, Space and measurement
4	Measurements	Data Handling and Probability
5	Data Handling and Probability	

1.3 RESEARCH PROBLEM

1.3.1 STATEMENT OF PURPOSE

The study sets out to investigate grade 9 learners' algebraic reasoning during problem solving involving number patterns. This study is therefore located in the broad area of mathematical knowledge for learning. In order to gain deeper insights into the learners' reasoning when generalizing, I first analysed the written work of the 29 participants followed by the interview responses from the target six learners drawn from the main sample. The following specific research questions guided the study:

1.3.2 RESEARCH QUESTIONS

1. What routines (*strategies*) and visual mediators do grade 9 learners use when engaging in a task on number patterns?

2. How do these grade 9 learners explain orally their thought processes in problem solving involving number patterns?

1. 3. 3 NUMBER PATTERNS

The study investigates learners' algebraic reasoning when generalizing in one algebra topic "Number Patterns". The study of number patterns has become an integral component across all grades of the South African school Mathematics curriculum (DoE, 2002:2003b). Number pattern activities in the Senior Phase (grades 7 - 9) are essentially an extension of the Intermediate Phase with the expectation that learners at this level can "use algebra and algebraic processes in their description of these patterns" (DoE, 2003a:39). In the FET Phase learners explore sequences, series and real-life mathematical problems which develop and require an ability to generalize. The literature review undertaken to inform the study suggests that linear sequences is an appropriate topic to elicit rich data for generalization processes. However, it is not clear how the learners' routines influence their generalization in number patterns. This study explored the routines and mediating tools the grade 9 learners' use when engaging in the task on numeric and geometric patterns.

1.3.4 LIMITATIONS OF THE STUDY

The limitations of the study include:

- Learners' written responses from the task and the oral responses from the follow up task-based interview are unique to the learners' experiences. As such the findings of the study cannot be generalized to all grade 9 learners in South Africa or to the ability levels that the six learners represent.

- The validity of the study's findings cannot be guaranteed should another person replicate [conduct] the study under similar conditions. This may be partly due to unintentional biases of the researcher who is also a key participant and a mathematics educator. Although it is possible that the researcher may deliberately choose what to see or hear, as a researcher I tried as much as possible to be objective in my analyses with the guidance of the research questions.

1.4 RATIONALE OF THE STUDY

My focus in the study was on learners' algebraic reasoning when generalizing from number patterns. Generalizing and justification activities of algebra afford us opportunities to express relationships between quantities. Such activities provide further opportunities that help us move beyond empirical arguments of arithmetic into making algebraic generalizations and justifications. The reasons for my choice of the number patterns topic in algebra are based on the following factors: my experience of working as a high school mathematics educator in South Africa whereby I see most high school learners (grades 8 – 12) experience difficulties to reason algebraically even after being introduced to algebra, my awareness of the current curricular developments both locally and internationally, and also my knowledge of existing literature and current research related to learners' algebraic thinking particularly when generalizing in number patterns.

The high visibility of pattern-based activities in recent school mathematics materials points to the increased value placed on teaching patterns as a productive way to introduce algebra (Driscoll, 1999:10). As early as kindergarten, learners see different coloured geometric shapes often sequenced in orderly manner from which they are asked to predict the next few shapes. Then later, when number concept is taught these learners begin to recognize and describe the various number patterns. The habit of thinking that causes one to look beyond a perceived pattern to wonder what "always works" for any pattern's rule of generality is a feature of a broader capacity of mathematical thinking. Generalization is a thinking process that applies throughout mathematics.

1.4.1 ALGEBRA WITHIN THE SOUTH AFRICAN SCHOOL CURRICULUM

In South African classrooms, the focus of algebra in the new curriculum has moved away from a focus on symbolism, rote learning and manipulative skills to the link between procedural and conceptual understandings. Procedural and conceptual understandings are essential for the development of problem solving abilities and routines that bring about progress in the learning of mathematics, in particular algebra. In the RNCS for the learning area mathematics (DoE, 2001: 19), the LO on "Number Patterns, Functions and Algebra" stipulates that a learner must be able "to recognize, describe and represent patterns and relationships, and solve problems using algebraic language and skills" (DoE, 2003a: 37). There is no specific statement that emphasises learning algebra skills in their own right; rather the emphasis is on using algebra for the purpose of problem solving and communication in mathematics. This suggests that mathematics is about learning to communicate mathematically within a discourse. For example, the learner needs to become fluent with vocabulary and symbols, ways of speaking, writing and presenting the mathematical arguments. Within the South African school curriculum, algebra is made up of "simplifying expressions, factorizing, solving equations, functions and graphs, variables, word problems and others" (DoE, 2003a : 37). At GET phase in particular, school algebra consists of "number patterns, solving linear equations, graphs, algebraic expressions and algebraic vocabulary" (DoE, 2002: 75-79).

1.4.2 PERSONAL EXPERIENCE AND CURRENT CURRICULAR REFORMS

From my personal teaching experience in the mathematics classrooms, learners in the GET and FET bands continue to grapple with algebraic concepts. As a result they perform poorly in both internal and external mathematics examinations. Also, educational research locally and internationally has shown that learners have difficulties in carrying out problem solving that requires the use of algebraic expressions (e.g. Herscovics & Linchevski, 1994; Sfard & Linchevski, 1994, 1995; Bernarz & Janvier, 1996; Bernarz, 2001). There are also major curriculum shifts happening internationally in relation to the way school algebra is conceived and its purpose in mathematics curricular in general (e.g. Pimm, 1982; Arcavi, 1995; Wheeler, 1996; MacGregor & Stacey, 1997;Malati, 1997; Brodie, 2005). Nonetheless, from the literature reviewed, not much has been reported on algebraic reasoning at GET phase from a South African perspective (Brodie, 2005). This study is intended to contribute to filling this gap in the South African mathematics education. Hopefully the findings will benefit the current formal and informal debates in the mathematics education research community.

The study is expected to inform me as a teacher about learners' thinking and the difficulties that learners encounter when engaged in a task on number patterns. In this chapter I have described the importance of reasoning in mathematics particularly when generalizing from number patterns and how the new curriculum in South Africa emphases the notion through its Learning Outcomes and Assessment Standards-AS (DoE, 2003b).

1.5 THE STRUCTURE OF THE REPORT

1.5.1 CHAPTER 1

This research report consists of seven chapters. In this chapter I have discussed among other things: the background of the study, the purpose of the study and its research questions, the rationale, my intended contribution to mathematics education literature, specifically with reference to the link between generalizing from number patterns and algebra.

1.5.2 CHAPTER 2

In Chapter 2 I discuss the theoretical perspective and the literature review that helped in shaping and informing the study. The theoretical framework is based on the notions of *thinking* and *communicating* in a mathematical discourse. This theoretical framework of commognition is adapted from Sfard (2008). Sfard's perspective of commognition is used to explain how a participant of a discourse communicates with others or with himself by verbal means or otherwise. The other key feature of Chapter 2 is the literature review, which highlights important aspects drawn from other studies.

1.5.3 CHAPTER 3

In Chapter 3 I discuss the study's design and methods used in data collection and analysis. The rationale for the choice of the methods and their appropriateness is explained to indicate how the study was conceptualised. An outline of key issues related to ethical compliance is highlighted. Similarly, issues of validity and reliability are discussed.

1.5.4 CHAPTER 4

Chapter 4 presents the analytical framework (AF) the study is using. The AF provides a description of how data were analyzed and interpreted. The AF draws on previous literature and previous studies in the area of generalization and is elaborated so that it is consistent with the theory of commognition.

1.5.5 CHAPTER 5

Chapter 5 presents the analyses of quantitative and qualitative data that was collected through the questionnaire based task. The problem solving routines and solution representations (forms of mediation) that learners demonstrated in the written task when generalizing in number patterns are highlighted and discussed quantitatively. Also, the problem solving routines and solution representations that learners discussed in the task-based interviews with the researcher are analysed (qualitatively).. The possible implications of learners' routines, representations and oral justifications to generalization of patterns are also highlighted.

1.5.6 CHAPTER 6

Chapter 6 presents a discussion about pedagogical implications of issues and trends that mainly emerged in the results and findings reported in chapter 5. The explanations and interpretations of the results were drawn from the literature reviewed, theoretical assumptions (see Chapter 2) where appropriate and from my own experience as a researcher and a classroom practitioner.

1.5.7 CHAPTER 7

In chapter 7, the final chapter of the report, conclusions are reached about learners' reasoning routines in the GET phase when generalizing from number patterns. This was done in view of summarising the findings of the study. The reflections made highlight the experiences and knowledge I have gained while working on this investigation. The weaknesses of the study are discussed and highlighted in terms of limitations of a qualitative research study.

CHAPTER 2 THEORETICAL FRAMEWORK AND LITERATURE REVIEW

2.1 INTRODUCTION

The introduction of the new curriculum following the advent of democracy in South Africa brought changes in mathematics teaching and learning in the classrooms. These changes are consistent with the view that mathematics should be viewed as a key subject among other learning areas (DoE, 2002; 2003). The shifts in the curriculum are in line with the communication theory of cognition (Sfard, 2008) which describes thinking as an activity of communication and learning mathematics as an initiation into a specific type of a discourse.

The purpose of an appropriate theoretical framework is that it allows the study to be reformulated so that illuminating explanations and concepts can be brought to bear on the observations and the results of the study. A theoretical framework is also an integral to the coherence of the data analysis process. In this research I will broadly work within a commognitivist's perspective as outlined in the following section.

2.1.1 COMMOGNITION PERSPECTIVE

According to Sfard (2007) educational studies traditionally conceptualise learning as the "acquisition of knowledge" where knowledge consists of entities such as *ideas* or *concepts*. In contrast, participationists define learning as a form of transformation in what and how people are doing in patterned human processes both as an individual and as a collective. They conceptualise developmental transformations as changes in what and how people are doing instead of inquiring about personal acquisition. Participationists further claim that patterned collective activities are developmentally prior to those of the individual (Sfard, 2008). In a commognitive perspective (ibid.), learning is rooted in the participationist assumption that all uniquely human skills are products of individualization of historically established collective activities. For example, children develop the ability to speak and read by slowly changing from interlocutors to independent performers (Vygotsky, 1987). According to Sfard (2007) learning occurs when there is change and development of a discourse. A discourse is a type of communication that draws some individuals together, while excluding others. Sfard (2008) contends that learning is about individualizing a collective discourse through "participation".

In my study, the commognition theory provides a lens to explain the learners' reasoning when they participate in mathematical discourses, in particular algebraic generalization in number patterns.

2.1.2 DEFINING COMMOGNITION THEORY

Sfard (2008:65) defines commognition in terms of two key concepts: '**thinking**' and '**communication**'. Briefly, commognition refers to *thinking* as *communicating*. Sfard argues that thinking as individualized communication, and learning as becoming adept in historically established discourses is rooted in the participationist's assumption that all uniquely human skills are products of the individualization of established activities (Sfard, 2007). As an inherently individual activity, thinking is no different to other uniquely human abilities. Thinking can therefore be defined as the individualized form of the activity of communicating. In self-communication, a person does not necessarily communicate with herself audibly or visibly; nor does it have to be in words. Such type of communication is dialogical in nature and it involves acts of informing oneself, arguing, asking questions and in fact waiting for one's own responses (ibid.).

The commognitive definition of *thinking* according to Sfard implies that communication is both interpersonal and intrapersonal (individual cognition). On the other hand the commognitive definition of *communication* refers to a performed patterned collective activity that mediates and coordinates other activities of the collective (Sfard, 2008). There are four basic tenets of commognition and these are: thinking as an individualized (interpersonal) communication, mathematics as a form of discourse, learning mathematics as a changing discourse and commognitive conflict as a source of mathematical learning.

2.1.3 BRIEF DESCRIPTION OF THE COMMOGNITIVE TENETS

Thinking as individualization: This refers to a process of communicating between a person and herself or himself. This self-communication does not have to be in any way audible or visible, nor does it have to be in words. The notion "thinking" is in fact dialogical in nature thus it involves acts of informing ourselves, arguing, asking questions, and waiting for our own responses. The commognitive definition of thinking implies that argumentation does not have to be inter-personal; rather it can take place within a person and hence becomes an act of communication in itself.

Mathematics as a form of discourse: Mathematics is a distinct discourse with its own vocabulary that contains, for example, words that refer to numbers and operations on numbers and to geometric shapes.

Learning mathematics as a change in discourse: Learning mathematics means modifying one's present discourse so that it acquires the properties of the discourse practiced by the mathematical community. Such change can be attained by extending the vocabulary, developing new routines or by producing and endorsing new narratives. The terms "routines" and "endorsed narratives" are briefly defined in section 2.1.4. In a commognitive perspective, enquiring what the participants of a study have yet to learn is equivalent to asking about the required transformations in their ways of communicating mathematical concepts and ideas.

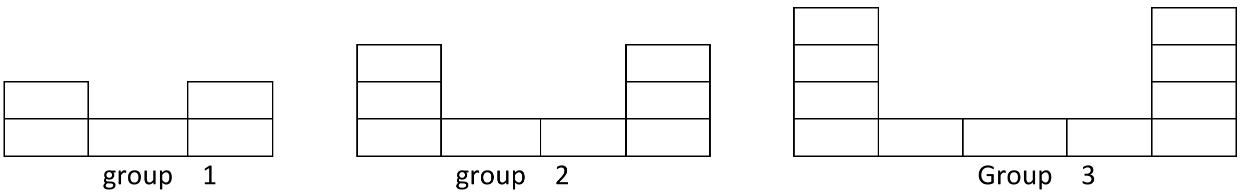
Commognitive conflict: This refers to a situation that arises when interlocutors participating in inconsistent discourses try to communicate with one another. Inconsistent or incommensurable discourses differ in their use of words and in the rules of substantiation as they do not share criteria for deciding the endorsement or rejection of a given narrative. Commognitive conflict may be interpersonal or intrapersonal.

In this study the theory of commognition is situated within a Mathematical discourse focussing on learning where learning is a "participation" activity that involves social interactions. Sfard (2008) regards learning as participating in discourses. Through this participation, learning mathematics means modifying one's present discourse so that it acquires the properties of the discourse practiced by the mathematical community. As a result the learner develops new vocabulary and routines and is able to endorse certain narratives. The commognitists view learning as the "change" that comes when substantial features of a discourse have been transformed and such "change" may occur due to conflict created by two discourses within a discourse (Ben-Zvi & Sfard, 2009:5-6). Identifying modifications in the discourse can help teachers measure the learners' developmental changes that result in learning. I will draw specifically on the second commognitive tenet, *mathematical discourse* on thinking (Sfard, 2008) by focusing on the features or properties that make a discourse count as mathematical.

2.1.4 MATHEMATICS AS A DISCOURSE

In education studies, different types of communications that bring people together while at the same time excluding others are referred to as ***discourses***. There exist different discourses each with its own characteristics. For example, a mathematical discourse is comprised of many sub-discourses, such as algebraic and geometric discourses. The mathematical discourses are made distinct by specific features or properties: their key words or vocabulary and the way these keywords are used; the visual mediators with which participants of a discourse identify the object of their talk and coordinate their communication, and by the form and outcomes of their processes (the routines and endorsed narratives they produce respectively). A brief discussion on how I intend to use Sfard's notion of mathematical discourse (Sfard, 2008: 133-4) is given in the following section. I draw on question 2 of the questionnaire based task of this study to illustrate.

Figure 1; Tile - Pattern:



1. Draw tile-group number 5. Explain in words what group number 5 would look like and how many tiles are in the group?
2. How many tiles are in each of the following group number:
   a) Group 6
   b) Group 7
   c) Group 10
   d) Group 50
3. How many tiles are in the n^{th} group of tiles?

In this question, the use of visual mediation may result in the observation that each group of tiles consists of tiles that are placed in the middle at the bottom (bottom-centre) with an associated number of tiles on each of the vertical sides. This approach uses iconic representations. Thus for n^{th} figure there are $3n$ tiles with extra two tiles on each side. This generates an expression $3n$ +2.

- ***Vocabulary***: These are key words used for communication purposes in a discourse. A discourse counts as mathematical if it features mathematical words. In mathematics, vocabulary refers to words that signify quantities or shapes and their meanings are generally used and shared by participants within a mathematics discourse. Word use is very important because what the user is able to say about the world and what s/he sees is informed by the word meaning.

  To be considered as mathematical, a discourse has to contain words that count as mathematical. For example, in algebraic discourse the use of some words may differ from their use in an arithmetic discourse or in everyday discourse. In this study I am not analyzing *word use* specifically as a distinct feature or property of a mathematical discourse. Rather my focus is on the *routines*, *mediators* and *narratives* the participants use to communicate their reasoning when generalizing from the task on number patterns.

  An example of vocabulary or word use in generalization

  In Mathematics, operational words such as: *addition* and *multiplication* may imply *increasing* in some instances whereas *subtraction* and *division* may imply a *decrease.* The interpretation to such words depends on the context in which they are used. For example, the tiles *increase* by 3 in number per group all the time in question 2 from the questionnaire – based task used in the study.

- ***Mediators***: These are visible objects that are operated upon as a part of the process of communication. Visual mediators enable participants of a discourse to identify objects of their talk; this enables communication. For example, meanings of symbols are made visible through a well organized syntax called mathematical language. In a mathematical discourse, operations on visual mediators such as symbolic artefacts becomes automated and embodied. The most common examples of visual mediators include mathematical formulae, graphs, drawings, diagrams, numbers and verbal explanation. The mediators are generally viewed as a part and parcel of the act of communication and in particular of the reasoning processes.

An example of a mediator - use in generalization

Drawing on question 2: conversion of the pictorial pattern representation of tiles in multiple or other forms such as:

i) Verbal explanation: the tiles increase by three for every change in position or group.

ii) Numeric: 5; 8; 11; ....; .....;..... = 3x1+2; 3x2+ 2; 3x3+2; ......;.......

iii) Symbolic or algebraic: $T_n = 3 \text{ x } n + 2$ where n is group number and T_n is the number of tiles in that group.

- ***Narratives***: This refers to any sequence of utterances framed as a description of objects, of relations between objects or of the process with or by objects subject to endorsement or rejection with the help of discourse-specific substantiation procedures (Sfard, 2007). In a mathematical discourse, endorsed narratives are often labelled as true because they describe a true state of affairs. Examples of endorsed narratives include mathematical theorems, definitions, proofs etc.

Examples of narratives in generalization

The rules or formulae that learners generate for the n^{th} term and other specific calculated values for the sub-questions of question 2 are examples of narratives. The examples of mediators listed i) – iii) above are utterances subject to endorsement or rejection. As such, they may be regarded as narratives.

- ***Routines***: These are repetitive patterns characteristic of a discourse. For example, mathematical regularities can be noticed whether one is watching the use of mathematical words and mediators or following the process of creating and substantiating narratives about numbers or geometrical shapes. The strategies the participants of this study used to communicate their thinking when generalizing from number patterns are referred to as “routines” (see “analytical framework” chapter 4).

An example of routines in generalization

> The learner's transformation of pictorial patterns to a number or symbolic representation either recursively or explicitly with the aim of deriving a generalization, is a routine. A different routine may be when; the learner makes use of other definitions or formulae to generate new rules. For example the use of a general linear formula; $T_n = a + (n-1)*d$ to generate or derive a mathematical rule such as $T_n = 3 \times n + 2$ for the n^{th} term in the tile problem 2.

In the context of the envisaged study, I use the discourse characteristics to analyse learners' written or oral responses in the following manner. As learners engage in problem solving of pictorial, diagrammatic or numeric problems that lead to generalizing I, as a researcher, will be looking for the kind of *words/vocabulary* the learners use. I will consider how these words relate to the mathematics embedded in the number patterns. And I will specifically analyse routines, i.e. strategies, used by the learners in solving the problems. I will further analyse the symbolic artefacts (*visual mediators*) that each learner uses or creates when generalizing, for example, mathematical formulae or graphs. Finally, I will endorse or reject the learners' representation of their routines in terms of whether they concur with the accepted school mathematics narratives or not.

2.1.5 SUMMARY OF COMMOGNITIVE PERSPECTIVE:

A participationist:

- regards thinking as developing from patterned collective activity.
- regards thinking as an internalized and individualized form of communication.
- defines mathematics as a discourse which has its own register and means of communication governed by some rules involving words, visual mediators, routines and narratives. Mastery of these rules results in the learner becoming a successful participant in mathematics discourse.

The theory of commognition has the following basic principles called commognitive tenetes:

1. Thinking as individualization of communication.

   The commognitive definition of thinking implies that argumentation does not have to be inter-personal; rather it can take place within a person and hence becomes an act of communication in itself.

2. Mathematics as a form of discourse.

   Commognition is defined as thinking and interpersonal communication and this happens according to certain established rules. According to Sfard (2008) these rules involve a particular use of words or vocabulary, visual mediators, narratives and routines.

3. Learning mathematics as changing discourse.

   Commognitivists define learning mathematics as individualizing mathematics through mathematical communication both with others and oneself. Learning of an individual student or the class as a whole can be measured by identifying the extent to which rules of a mathematics discourse are used in communication by the participant(s).

4. Commognitive conflict as a source of learning.

   According to Sfard (2007:575) learning occurs as a result of commognitive conflict (change of discourse). She defines this conflict as a “situation in which communication is hindered by the fact that different discussants are acting according to different meta-rules through the use of the same words but in different ways” (Sfard, 2007:576). The change of discourse is viewed in terms of two types of learning: object-level learning and meta-level learning. Object-level refers to learning that expresses itself through expansion of the existing discourse by extending the vocabulary; constructing new routines and producing new endorsed narratives. On the other hand, meta-level learning involves change in meta-rules of the discourse for example defining a word or identifying geometric figures will now be done in a different and unfamiliar way.

In the following section I highlight the literature review that has contributed to shaping the study.

2.2 LITERATURE REVIEW

The purpose of a literature review is to show the relationship between research at hand and previous related research studies. In fact, a literature review reveals how the research at hand can make a unique contribution. As such, the researcher has a responsibility to indicate the connections between current research and the previous studies, as well as the gaps and weaknesses of previous studies (Vital & Jansen, 1997). A review of the previous studies also provides a researcher with useful lenses for identifying methods and techniques of gathering information which could be helpful when conducting the current research study. In light of the above, I reviewed the literature relating to the three main aspects of this study and these are:

- Literature relating to number patterns.
- Literature relating to generalizations.
- Literature relating to algebraic thinking when generalizing.

In this section, I define what number patterns and generalizations are and how they are connected in mathematics particularly in the algebra discourse.

2.2.1 NUMBER PATTERNS

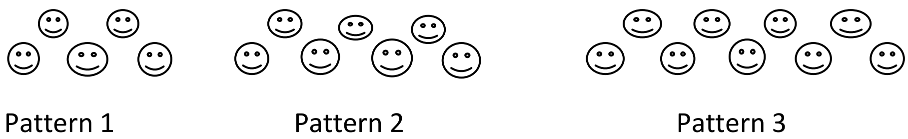
Formal school algebra often begins with instructions about manipulation of literal expressions in equations and functions. The beginning of the algebraic reasoning is generally aimed at identifying fundamental relationships among situations [quantities] and expressing them using correct algebraic symbolism (Kieran, 1990). Formal school algebra recommends introducing algebra through numeric and geometric patterns (Lee, 1996; Macgregor & Stacey, 1995).

Words such as “generalization” and “number patterns” are worth highlighting in this study. Algebra provides a language in which to express generalities precisely and concisely, to the extent that these generalities can be manipulated in order to facilitate reasoning. Generalization and patterns are two interlinked notions and difficult to separate from each other in a context. The underlying structure of the pattern sequence is often used to generalize the number and arrangement of the terms for any step in the pattern. A *number pattern* is a sequence of numbers in which there is a well-defined rule for calculating each number from the previous numbers or from its position in the sequence. In a geometric number pattern, the numbers relate to a sequence of geometrical figures in which each figure is derived from the previous figure by some well-defined procedure. A number or geometric number pattern is linear if each number is obtained by adding a constant difference to the previous number or, equivalently, if each number is a linear function of its position in the sequence.

Several types of patterns or sequences are identified in previous research for example: *Repeating patterns* are patterns with an easily recognizable repeating cycle of elements (Zazkis, 2002:2). An example of a repeating pattern is the linear pattern, where an element of the pattern is duplicated vertically, horizontally or obliquely e.g. KLMKLMKLM..... Another form of repeating pattern is the cyclic pattern in which there is no beginning or ending point. Students' abilities to generalize number patterns rely on their observations and understanding of "units of repeat" within a cycle of patterns.

Growing patterns are patterns in which the number of components of an element in a sequence increase or decrease systematically. Growing patterns are commonly used for pattern generalizations. It may be most effective to introduce students first to repeating patterns and then to growing patterns (Papic, 2007:5-6). Papic argues that experience with repeating and growing patterns can develop functional thinking when students move beyond simple data sets to seeking relationships between data sets. Generalizing repeating patterns from the unit of repeat of any orientation provides opportunity to further generalize even more difficult patterns such as growing patterns. The following pictorial pattern representation is an example of a growing pattern.

Figure 2: An example of a growing pattern – pictorial representation



2.2.2 ALGEBRAIC GENERALIZATION

The process of *generalization* of patterns in mathematics introduces learners to algebra and algebraic reasoning. Carraher (2007:2) says that "functions in mathematics are generally introduced through the use of algebraic expressions; however, this option is not viable for most young learners who are not familiar with algebraic notations". The study of number patterns for elementary school learners could be a viable means of developing an understanding of algebraic concepts. When learners observe and understand how to continue a pattern and then generalize that pattern, they have demonstrated algebraic reasoning skills. Warren and Cooper (2006) argue that while not all elementary school students are able to formally express a pattern generalization, the approach of introducing algebra as the study and generalization of patterns lays the foundation for learners to reach the next level of understanding in algebra and to use formal, symbolic representations to describe patterns. Warren and Cooper (2006) concur with Papic that experience with repeating and growing patterns can develop functional thinking when students move beyond simple data sets to seeking relationships between data sets. Generalizing patterns has been described as "one of the roots of, and routes into, algebra" (Zazkis & Liljedahl, 2002:4)

The idea of algebraic generalization is being able to express a pattern algebraically, to notice and state the commonality that can be generalized to all terms in a given sequence (Radford, 2006). However, many research studies done on students of different age group define "generalization" differently. See Table 2 below.

TABLE 2: DEFINITIONS OF GENERALIZATION

Age group studied	Definitions of generalization and author
Pre-kindergarten (4-6 year olds)	Patterning is an essential skill in early mathematics learning... This includes the ability to identify and describe attributes of objects and similarities and differences between them (Papic, 2007:8)
3rd graders aged 9	Mathematical generalization involves a claim that some property or technique holds for a large set of mathematical objects or conditions (Carraher, 2008:3)
6th graders aged 11	*Figural* generalization focuses on visualization of the pattern and how it changes with each transformation. *Numerical* generalization mainly focuses on numbers in the transformation of patterns; it may involve trial and error and finite difference approaches (Becker, 2006)
Junior & High school	Generalizing a pattern algebraically rests on ones ability to grasp a commonality among elements of a sequence S. And also the learner needs to be aware that this commonality applies to all elements of S. Finally the learner must be able to generate a rule that provides a direct expression for any element of S (Radford, 2006:4). Radford further describe generalization as factual, contextual or symbolic.
Pre-service teachers	Generalization is constructed through abstraction of the essential invariants. The abstracted qualities are relations among objects, rather than objects themselves (Zazkis. 2002 : 381)

Several types of generalization exist. Dorfler (1999:90) differentiates between empirical generalization and theoretical generalization. Drawing on Dorfler's work, Zazkis and Liljedahl (2002) defines empirical generalization as making generalizations based on common features or common qualities of objects or terms in a pattern. In contrast theoretical generalization is "intentional and extensional whereby essential invariants are identified and generalization is constructed through abstraction of these essential invariants" (Zazkis & Liljedahl, 2002:3). Carraher (2007) also distinguishes between empirical and theoretical generalizations by looking at aspects of reasoning involved in each. According to Harel and Tall generalization is "applying a given argument in a broader context". They focus on three types of generalizations: expansive generalization, reconstructive generalization, and disjunctive generalizations (cited by Zazkis & Liljedahl, 2002:3-4). Becker discusses two types of generalizations: numerical generalizations, which mainly focus on numbers in the transformation of patterns and figural generalizations, which focus on visualization of the pattern and how it changes with each transformation (Becker, 2006:2). Radford (2006) differentiates between what he calls algebraic generalizations and arithmetic generalizations and presents the generalization process as consisting of three elements. The first element in the generalization process is "noticing commonality in some given particular terms." (Radford, 2006:14) The second element in the generalization process is "formation of a general concept by generalizing the noticed commonality to all the terms of the number sequence." (Zazkis & Liljedahl, 2002: 3-4). The third element of the process involves use of the generalization to provide "an expression for any term in the sequence" (Carraher, 2007:2). He further argues that in order to complete the first two components of generalization process students must perform an arithmetic generalization, and it is only when the third element/component becomes part of the process that we have an algebraic generalization.

Many problems faced by students in generalizing number patterns are not necessarily linked to recognizing a pattern but rather in creating a useful algebraic pattern or rule about the pattern. Students are often able to see a pattern and can continue a pattern but grapple to create a formula for it. Becker (2006) discusses a research study involving grade six students and their abilities to generalize patterns algebraically. The students in this study were able to generalize a pattern and justify their approaches only after the researcher employed some intervention measures. Carraher (2007:5) suggests that elementary phase and intermediate high school students learn to make generalizations in situations involving physical quantities.

From all these studies conducted by several researchers, it seems that patterns and generalization is a key topic which is included in the mathematics curriculum at all instructional levels. The National Council of Teachers of Mathematics (NCTM) supports the inclusion of algebra in the K-12 curriculum, with a great emphasis on "students' learning to make generalizations about patterns" (Carraher, 2007:2). While instruction in the elementary and intermediate grades focuses on identification and justification of number patterns, the approach to introducing patterns prepares the students for a deeper understanding of patterns and the ability to algebraically generalize number patterns.

In an attempt to seek more information on students' understanding of mathematical number patterns, other studies (Macgregor & Stacey, 1993; Orton & Orton, 1994, 1996; Stacey, 1989; Swafford & Langrall, 2000) have explored learners' understanding of linear geometric patterns with students aged 9 -14 and have identified quite a number of strategies corresponding to various levels of understanding the patterns. For example, Stacey (1989:150) showed students a pattern of ladder-like figures built with matches and asked the students to determine the number of matches that would be needed to build ladders with 20 and 100 rungs. The general rule to this problem context was found to be $3n + 2$ for a ladder with n rungs. Stacey identified several different approaches/strategies that students used when working in such number patterns (e.g. counting, difference, whole-object and linear methods).

- **A global picture of generalization**

Drawing from Radford (2006), algebra is made up of variables, the use of letters, and the use of basic operational signs (i.e. rules for sign-use) and discovery of their shared qualities. Learners generalize number sequences in different ways for example, trial and error method where different symbols are used until some rules or formulae are generated. Another approach that learners use to generalize in mathematics is to identify and make conjectures about how each number of the pattern problem is related to the next number in that pattern (add or difference method). All this takes the form of mental imaginations which later turn into formulae. The generated formula basically becomes a general representation satisfying all the shared properties of the pattern or sequence for generalization. However, such a formula or rule has to be expressed in the correct and acceptable algebraic notation (endorsed narratives) within a specific context [domain].

Stages of generalization

Several studies on generalization (Radford, 2006; Papic, 2007; Carraher, 2007 etc.) suggest different stages through which students move as they generalize from number patterns. These stages are achieved through different methods. These stages together with other aspects of generalization, as posited by different educational researchers, are summarized in Table 3.

TABLE 3: ASPECTS OF GENERALIZATION

AUTHOR	**CHARACTERISTICS OF GENERALIZATION**
Zazkis, 2002.	• Generalization defined as a single formula • The unit of repeat is key to generalization • Equivalent expressions is key to generalizing • Algebraic symbols describes generalizations • The position of every element can be determined from generalization.
Becker & Rivera, 2006.	• Seeing patterns additively. • Generalizing through listing and visualizing. • Generalizing tasks visually and numerically through direct proportion. • Predisposition towards generalizing formulae. • Reverse operations. ***NB: These are important characteristics observed in my study's data.***
AUTHOR	**STAGES OF GENERALIZATION**
Radford, 2006.	• Noticing commonality of elements. • Forming a general concept. • Generating a rule which gives an expression for each element in the sequence. • Justifying equivalence
Carraher, 2007.	• Building an idea of the pattern and noticing similarities or differences of the elements in a sequence. • Visualization.
Papic, 2007.	• Observing linear, cyclic and growing patterns. • Locate an element that repeats itself in the pattern. • The repeating portion becomes the common element of the pattern. • Justification of the discovered unit or portion of repeat in the pattern.

The literature review reveals a number of similarities on the notion of generalization. The main similarity was the issue of prevalence of several and different types of generalizations that learners make. Radford (2006) argues that students use layers of algebraic generality which are in the form of factual, contextual or symbolic generalization (see section 2.2.2. and Table 2). According to Radford students who use factual, contextual or symbolic generalization demonstrate true algebraic generalization (theoretical) unlike those that use empirical methods.

On the other hand Becker & Rivera (2006) categorised students' approaches to generalization as figural and numerical. Figural generalization is similar to theoretical generalization while numerical generalization is comparable to empirical generalization. The studies referred to above are similar in the following ways:

- In all the studies, linear, growing and repeating patterns were used.
- Small groups of participants were asked to generalize and justify their findings when solving number pattern problems.
- The studies reported on students' routines when justifying generalizations.
- Emphasis on the importance of early development of the ability to generalize patterns.
- All studies were based on the assumption that pattern recognition is the foundation of introducing and building algebraic skills and reasoning.
- All studies explicitly recommend the study of patterns as an introduction to formal school algebra.

Besides having similarities, these studies also differed in many ways starting from their definition of the term "generalization", the age group researched (see table 2), length and type of research carried out etc. The longer studies included that of Papic which was based on a six month intervention programme with 53 preschoolers and Radford, whose research was a longitudinal study of several teachers' students over a period of six years. On the other hand, Carraher observed 15 grade three learners across two lessons; Zazkis used 36 pre-service teachers on one task over a period of two weeks and Becker studied 29 grade 6 learners over a period of six weeks.

Most researchers discussed algebra as a discourse that relates to students' understanding and representation of generalization of the number patterns while focusing on different aspects of generalization process. Radford (2006) identifies a process he calls "objectification" which actually involves different levels of generality. He argues that the levels of generality that students achieve are reflected in their representation and communication of the pattern. Contrary to Radford, Carraher focussed on student approaches to a task and noted that formal representations of a pattern as a function in closed form should be the ultimate goal for algebra teachers. "Generalization is not merely about rigorous inference. There is an important role for conjecture in mathematical generalization. Functions can also be introduced in situations where students are encouraged to make conjectures" Carraher (2007:4).

All these studies made some mention of the importance of using different representations in generalizations. Radford said that students' representations depended heavily on their social environment, that the semiotic gesture a student uses is impacted by their culture (Radford, 2006:13). Carraher noted that students can use natural language, line segments, function tables, Cartesian graphs, and algebraic notation to express generality (Carraher, 2008:6). Becker mentioned students' predisposition to generating formulae when trying to generalize patterns although they seem to experience difficulty in finding alternative generalizations as well as determining equivalence between formulae (Becker & Rivera, 2006:6).

Carraher indicates that two major types of student approaches to a generalization task were prevalent: recursive or as a function in closed form. Students who used a recursive approach either focused on the difference from one iteration to the next. For example, $f(n) - f(n\text{-}1)$, or they focused on the difference between the result of an iteration of the function and the number of the iteration e.g., $f(n) - n$ (Carraher, 2008:16-17).

Papic (2007) focussed on three different types of patterns rather than different types of generalization. Her research methods include observing students work on pattern tasks and observing children's' use of patterns during play. She describes two main types of patterns: repeating patterns (e.g. linear, cyclic and hopscotch) and growing patterns (Papic, 2007:9). My study uses both the linear numeric and linear geometric patterns (see Appendix A). These numeric and geometric patterns are referred as "linear" because they all translate into linear algebraic generalizations of the form $y = \mathrm{m}\,x + \mathrm{c}$ or $\mathrm{T}_n = \mathrm{a} + (n\text{-}1)\,\mathrm{d}$.
Drawing on the literature reviewed it is imperative that mathematics instructional programs should include attention to patterns, functions, symbols, and models so that all learners:

- understands various types of patterns and functional relationships.
- use symbolic forms to represent and analyse mathematical situations and structures.
- use mathematical models and analyse change in both real and abstract contexts.

2.2.3 ALGEBRAIC THINKING

Algebraic thinking refers to the use of tools and symbols as mediators (multiple representations) to analyze different mathematical contexts. At an elementary level this takes many forms such as extending pictorial and number patterns, doing and undoing, understanding equivalence, solving for unknown and writing a generalization for a pattern (Carpenter, Fraenkel, & Levi, 2003; Kaput, 2000; NCTM, 2000). I summarise these mathematical skills as:

- Extracting information from a given context.
- Representing the extracted information mathematically using visual mediators such as verbal explanations, diagrams, tables, graphs, algebraic representations.
- Interpreting and applying the mathematical findings such as solutions from solving equations for the unknowns, testing the conjectures and identifying the functional relationships in a context.

The use of number patterns in this investigative process is an example of exploring learners' algebraic reasoning. My study is using a task based questionnaire which elicit learners' skills such as pattern seeking (extract information), pattern recognition (mathematical analysis) and generalization which involve interpretation and application of pattern *seeking* and *recognition* skills.

2.2.3.1 REASONING THROUGH GENERALIZATION

In many countries "pattern" recognition and "generalization" are considered fundamental to the development of mathematical reasoning. Hence they have become important components of mathematics curricular reform. However, "number pattern" is not central to any mathematics curricular. Most of the current curricular are organised into parallel strands, for example, number system, space, measurement, data and statistics, functions and algebra (see Table 1). Such curricular structure or organisation does not explicitly encourage teachers to see common processes of patterns across the strands or to make important connections between strands or learning outcomes. The RNCS (2005) in South African classrooms emphasises the importance of generalisation in some of the specific outcomes (SO: 2, 9 & 10).

A 2007 study commissioned by Mathematics Learning and Teaching Initiative (MALATI) in Cape Town in South Africa reported on grade 7 learners' ability to handle algebraic generalisation problems. The learners were drawn from eight schools of which seven were situated in traditional black townships. The focus of the study was on moments when the learners grapple with deciding about the validity of their generalisations. The Malati study showed that most learners' justification methods were not valid because the learners themselves were not aware of the role of context-information in the process of generalisation (see Sasman, Linchevski, Olivier & Liebenberg, 1998, 1999). Also, learners used almost exclusively numerical strategies when calculating; they neglected drawings and favoured recursive methods and made several mistakes related to the use of direct proportion. The study does not elicit what strategies dominated in the grade 7 learners' reasoning processes when generalising and what could have changed if a different pedagogical context was used. Hence this is seen as a gap.

My study focuses on learners' use of different generalization strategies [routines] in different representations. Also, the study explores how different contexts in which patterns appear resonate differently with different learners. In some previous research on mathematical reasoning and algebraic generalisation (e.g. Orton & Orton, 1994, 1996; MacGregor & Stacey, 1993) much about learners' difficulties is reported (see literature in the following section). However, there is not enough information in these reports to enable teachers in the practice to understand how the learners reflect on their strategies and solutions when engaged in problem solving and generalisation. Some of these difficulties arise from learners' inability to operate spontaneously with or on the unknowns (*algebra- arithmetic cognitive gap*), the nature and operational structure and reification process of algebraic symbols/objects (*conceptual development*), and difficulties in using representations and meaning comprehension (*errors and misconceptions*) related to algebraic thinking. Issues or aspects relating to students' difficulties with generalization are explored briefly in my study with a view to seeing whether new trends emerge from the data collected specifically for this study.

2.2.3.2 LEARNERS' THINKING PROCESSES IN GENERALIZATION

The use of patterns to probe and promote generalisation is seen by many as a pre-algebraic activity (e.g. Mason et al., 1985; Mason, 1996, Lee, 1996). Recent research approaches to the study of algebra have focussed on investigating patterns. The search for regularities in different mathematical contexts, the use of variables and symbols to represent patterns and generalisations are key components of school mathematics curricular in many countries ranging from pre-kindergarten to secondary schooling. Mason (1996) noted that school algebra is traditionally centred on numbers and functions of numbers. Observing and reasoning about patterns in number sequences is an opportunity for learners to experience the process of mathematical generalisation. Yet at the same time, a number of researchers, including Radford (2000) and Noss *et al* (1997), point to the difficulties students encounter in shifting from pattern spotting to structural understanding. Students often tend to base their conclusions on superficial or incidental patterns they observe in the sequence, rather than on arguments referring to its structure. Although the use of structural reasoning increases modestly with age, Küchemann & Hoyles (2005) note that empirical reasoning remains widespread.

Stacey (1989) carried out an investigation focusing on the generalisation of linear patterns with students aged between 9 and 13 years old. A significant number of the participants used an erroneous direct proportion method in their attempt to generalise. Stacey also reported some inconsistencies in the students' strategies in problems that could readily be solved by use of a drawing or recursive methods. Stacey concluded in her findings that drawing had a major influence on the students' approaches, although she did not explore the theme any further. Also Garcia Cruz & Martinon (1997) conducted a study with 14 - 16 years old students. The study aimed at analysing the way the students validate results and on ascertaining if they favoured numerical or geometric analytical strategies. The findings of the research showed that drawings or pictures played a double role in the process of abstracting and generalizing. Drawings represented a setting for students who used the visual strategies in order to achieve generalisation and, on the other hand, acted as a means of checking the validity reasoning of those students who opted for numerical analysis approaches. Mason et al. (2005) promotes the use of the strategy of "Watch What You Do" as learners draw further cases of patterns and attend to how they naturally draw the patterns efficiently. Each efficient drawing method offers a potential generalization when expressed as an instruction on how to draw or represent the pattern. What become clear from the different studies is that learners' arithmetic incompetence and their fixation on recursive methods when generalizing are obstacles to successful algebraic generalisation. Hence a cognitive gap between arithmetic and algebra is created [exists].

Carraher et, al. (2000) also reported that algebraic reasoning skills and algebraic representations are not foreign concepts in primary school mathematics curricular. In particular they note that algebraic reasoning is embedded in arithmetic exploration in the early grades. They argue in their work that ascribing late emergence of algebraic reasoning to developmental constraints is incorrect; they further argue that algebraic thinking enters the school curriculum too late. They suggest that this could be the main cause of learners' subsequent difficulties in algebra.

2.2.3.3 COGNITIVE GAP BETWEEN ARITHMETIC AND ALGEBRA

Reflecting on the literature and my own classroom experience, arithmetic and algebra can be viewed from two perspectives. Separation of algebra and arithmetic creates the gap that I suggest is the main cause of learners' difficulty with algebra. Arithmetic is perceived as the study of numerical quantities and relationships existing between them through carrying out operations on these quantities. Arithmetic does not contain as many symbols as algebra does (in the form of variables) and yet it is fundamental in the study of algebra course. It should be noted that the four basic operations of arithmetic (+,-, / and x) are key too in the algebraic processes. Wheeler (1996) argues that algebra is the completion of arithmetic characterised by abstractions and generalisations.

Early research in mathematics education described the algebra and arithmetic relationship as being *dichotomous* because there is a perception that their tasks are different in structure and the purpose they serve (Carraher et. al, 2006; Herscovics et. al, 1994). Arithmetic was seen in early studies as dealing with operations that involved numerals and focused on the evaluated result while algebra was seen as dealing with generalised numbers in the form of variables and functions (Carraher et. al, 2000). The mathematics taught in schools at present is characterised by work with numerals in primary grades and work with variables in the middle and higher grades. That is, arithmetic and algebra are kept separate. Many text book authors seem to take very little or no consideration at all about linking algebraic and arithmetic concepts as early as possible across all grades. I suggest that these are the main factors that influence learners to think that algebra is merely doing manipulations by following algorithms and some rules and that algebraic reasoning is disconnected from arithmetic reasoning. The arithmetic and algebra gap widens as learners continue acting on or with symbols without really making sense of their activities when working on arithmetic and algebraic sequenced instructions separately. Lima et, al. (2008) described this as the absence of symbol-embodiment. From my experience of teaching high school mathematics, learners' conceptual and procedural intuitions about numeric and symbolic representations are very limited and shallow suggesting a reason why they still experience difficulties with algebra in the upper grades. Kaput (2000) supports the argument by attributing learners' difficulties with algebra to the delay in starting algebra until high school coupled with detachment of algebra from other fields and from within mathematics.

There is also a *symbiotic perspective* which refers to coexistence of algebra and arithmetic and as argued by researchers, this signals an approach which can reduce the levels at which learners struggle to make sense of algebra (e.g. Davis 1971-2, Polanski 1991, Carraher et. al, 2000; Herskovits et. al, 1994; Kaput 2000 and Mason 1996). Recent research arguments echo the call for early algebraic instruction in which algebraic reasoning skills become part of the primary school mathematics curricular focus. Bodanski (1991) found in his study that algebraic thinking embedded within the usual activities of arithmetic is possible to teach at the beginning of the basic course of education and can lead to narrowing the cognitive gap. Also, difficulties with algebra can be minimised through a slow transition from arithmetic to algebra itself (Carraher et. al, 2000). In support of this Kaput (2000) advocates that the algebraification of arithmetic will help integrate algebraic reasoning into the entire mathematics curriculum. What stands out for me is that the difficulties learners experience with algebra may be lessened if teachers in the classroom take the symbiotic perspective when teaching algebra at all levels of schooling. This may enable learners to operate on numerals and symbolism that will trigger reasoning at an early age. For example, mathematical statements such as $T_n = 3n + c$ and $354 + 530 = \square$ are in fact both algebraic in nature; yet most school textbooks present the two as algebra and arithmetic respectively. A child's inability to spontaneously act with or on the unknown in any of these two examples indicates existence of the cognitive gap.

2.3 CONCLUSION

In conclusion, what is apparent from most of the studies is the fact that authors agree that generalization of number patterns is the path towards algebraic reasoning. However, this route can be difficult to follow as most students seem to grapple with connecting patterns to algebra. Students experience no problems when identifying a pattern but rather have problems in perceiving an algebraically useful pattern (Zazkis & Liljedahl, 2002: 382). The literature review on research findings in algebraic generalization from patterns reveals that most studies were done in developed countries (i.e. first world countries) and very little has been reported about third world countries, particularly those in Sub-Saharan Africa. This literature is silent on "how" and "to what extent" do the learners particularly at elementary and junior high school grades reason mathematically when generalizing from number patterns. Hence I decided on this study.

In this Chapter, I have discussed the literature review with focus on literature relating to mathematical patterns, generalizations and algebraic thinking (reasoning) in which the definitions and process entailed by each of these aspects have been described for this study. Literature relating to algebraic thinking focusing on generalization strategies and research findings linked to number patterns has also been reviewed for this study. Furthermore, a theoretical perspective that frames the basis for explanations for this study has been discussed. In Chapter 3, I discuss the research design and methodology that facilitated the process of data collection and analysis.

CHAPTER 3 RESEARCH DESIGN AND METHODOLOGY

3.1 INTRODUCTION

In this chapter, I first outline the research approach and design methods. I then discuss issues of validity and reliability. I also discuss ethical issues related to the study.

3.2 RESEARCH APPROACH

The study is informed by the commognitive approach which focuses on thinking as communicating in a Mathematical Discourse. The learners' mathematical strategies [routines] and how they present and communicate their answers when generalizing from number patterns are explored**.** According to Schumacher and McMillan (1993), a sound research design provides results that can be relied upon. The research design explains the plan used to select the participants, characteristics of the research site and data collection procedures employed to answer the research questions. Leedy (1989) defines research methodology as an operational framework within which facts are placed such that their meanings may be seen more clearly. I highlight and justify my choice of data analysis approach (i.e. research methodology) while reflecting on the guiding research questions, theoretical framework and the problem that has informed the study.

This study aims to explore and interpret in detail the learners' spoken and written responses-explanations using both qualitative and quantitative approaches respectively. The spoken explanations refer to the learners' comments during the task-based interviews while the written responses refer to written answers to the task-based questionnaire. Creswell defines a qualitative study as "an inquiry process of understanding a social or human problem based on building a complex, holistic picture, formed with words, reporting in detail views of informants and conducted in a natural setting". On the other hand, he defines quantitative study as "an inquiry into a social or human problem, based on testing a theory consisting of variables that can be measured with numbers analysed statistically in order to verify the predicted generalizations about a theory" (Creswell, 1994:2 cited in Leedy, 1997). Leedy (1997) further points out the essential features of the two approaches. A quantitative approach relies on deductive analysis where data is reduced and presented in numeric form whereas a qualitative approach uses inductive methods of analysis i.e. narratives are constructed from data.

3.3 RESEARCH METHODOLOGY

3.3.1 CASE STUDY

This is a case study consisting of 29 grade 9 learners, ages 14 to 16. The study was conducted in a former model C school situated in a northern suburb in Johannesburg, South Africa. The student population in the school is predominantly Black (50%), while the remaining students are Indians & Coloureds (30%), Whites (19%) and Chinese (1%).

The case study design is chosen because of its flexibility and adaptability (Schumacher & McMillan, 1993). Opie (2004) describes a case study as an in-depth study of a single instance. A case study looks at an enclosed system where certain features of social behaviour or activities influence the situation. Opie (2004) further argues that a case study employs real people (learners & researcher) in real situations. This is a case study because I investigate a set of pupils from one school and one particular grade 9 class. All the participants in this class were given a task-based questionnaire (see section 3.3.2.1). I then identified six learners at different levels of ability in generalizing whose work I investigated and analysed in detail. I also interviewed these students. The six learners were purposefully sampled according to their general performance (abilities) in class and their ability to speak. The learners' abilities varied from high, medium to low ability. Purposive sampling, according to Schumacher and McMillan (1993: 401) refers to "selecting small samples of information-rich cases to study in-depth without desiring to generalise to all such cases". This study employed primarily a qualitative approach (see 3.2); a method of data collection that focuses on a small scale sample and elicits quality information is called a qualitative method (Schumacher & McMillan, 1993; Fraenkel & Wallen, 1990). Some quantitative data was also collected and forms the background to this study (see Chapter 5).

3.3.2 RESEARCH INSTRUMENTS

3.3.2.1 THE QUESTIONNAIRE-BASED TASK

Qualitative approaches usually use various forms of collecting data. This study is no exception as a mathematics task on number patterns and the questionnaire-based task was used to collect data. Literature based on several studies in education research have revealed that the quality of the data gathering tools such as questionnaires, observations, interviews and tests, are key (see Fraenkel & Wallen 1990, Schumacher & McMillan 1993, Sanders

1995, Oppenheim 1996, Bell 1999 and Cohen et. al. 2000). In this study, I used a questionnaire-based task on number patterns. The layout of the question items in the task provided enough workspace for the participants to write both their solutions as well as an explanation of their thinking around the solution. The learners' written explanations served three purposes. Firstly, it was a necessary research requirement in order to accurately categorise the adopted solution routines. Secondly, it was hoped that writing would scaffold the participant's process of reflection (Kaput, 1991). Thirdly, it was hoped that verbal reasoning in natural language, as opposed to mathematical abstraction, would lead learners to symbolic formulation or representation of the general terms.

The questionnaire-based task design is adapted from literature of previous studies and re-developed from Grade 9 Classroom Mathematics textbook (Laridon et. al. (2006), Malati (2007), and Healy and Hoyles (1999)), to provide evidence of difficulties, strengths and weaknesses experienced by learners when solving problems and to reveal what causes such weaknesses or strengths (Cohen et. al. 2000). It is hoped that the questionnaire task used in this study elicits the learners' reasoning and the challenges they encounter when solving algebraic problems. Cohen et. al. (2000) and Fraser (1991) concur that a task of this format is useful in that it determines how much the learners know about the mathematics content. The task consists of two sections with items that are numeric and diagrammatic. The first section (see appendix A) was used in this study and consists of four questions leading to linear generalizations. The second section was used for other research purposes and consists of five questions that led to quadratic generalizations. The "number patterns topic" is chosen because much of the literature in mathematics education (see Arcavi, 1995; Wheeler, 1996; MacGregor & Stacey, 1997; Chick & Kendal, 2004 etc.) report that number patterns in school mathematics is key for introducing algebra.

The written responses from the questionnaire-based task provided quantitative data and also to informed the interviews. I made analytic notes on each of the 29 participant's written response in order to keep track of their routines. It is from the routines that one may be informed of the learners' reasoning skills when generalizing from patterns. The learners' representations, meanings and routines were all classified into categories from the written responses (see details in chapter 4).

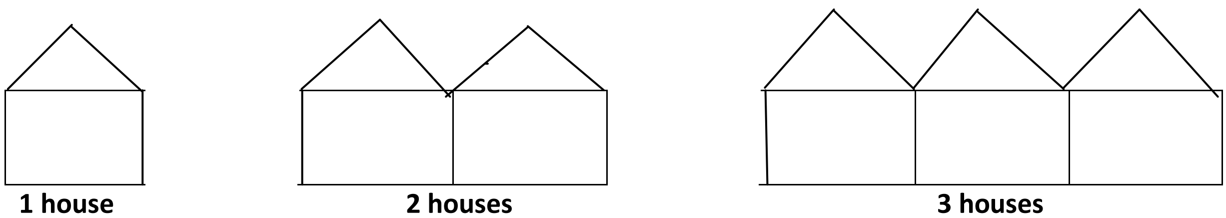
3.3.2.2 DISCUSSION OF THE FOUR QUESTIONS IN THE QUESTIONNAIRE BASEDTASK

Various pattern questions in both numeric and pictorial contexts were drawn from the relevant literature, e.g. Stacey (1989), Orton (1997), Healy and Hoyles (1999) and Malati (2007), and adapted to the needs of this study. For example, from each number pattern question, participants were required to provide numerical values for the next, 10^{th}, 50^{th} terms as well as a written explanation of their reasoning on each questions' solution process. The participants were also asked to provide an algebraic expression for the n^{th} term. Each question was constructed in such a way that the learners worked sequentially from small to larger terms. It was hoped that this would encourage participants to closely examine the properties and general relations in a given pattern context. The use of notation systems within this process was viewed as "contributing to the organization of the person's thinking processes" (Kaput, 1991:54). In this study, the learners' representations should reveal such processes.

In Questions 1 and 2, pictorial patterns were presented using three consecutive terms.

<u>QUESTION 1</u>

A contractor is asked to build a new set of townhouses in attached clusters of different sizes. He created a plan for one, two and three house clusters as shown in the following page. The builder used computer software to draw and generate line segments used to represent the houses.

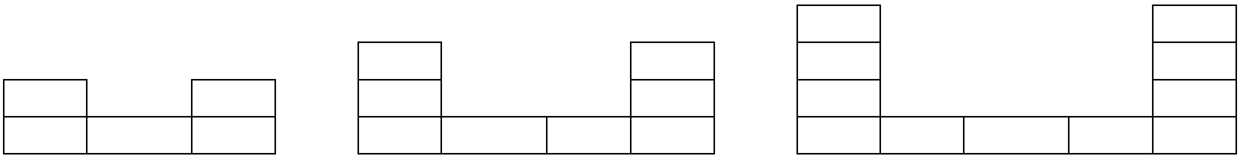


1. How many line segments are needed to draw or create a plan for:
   - i) 4 houses
   - ii) 10 houses
   - iii) 50 houses

2. Generate a formula or rule for the above pattern (i.e. nth term)

QUESTION 2

The figures below show three groups of tiles:



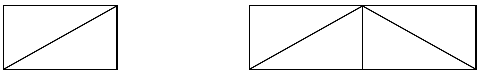
Group 1 **Group 2** **Group 3**

1. Draw tile-group number 5. Explain in words what group number 5 would look like and how many tiles are in the group?
2. How many tiles are in each of the following group number:
   e) Group 6
   f) Group 7
   g) Group 10
   h) Group 50
3. How many tiles are in the n^{th} group of tiles?

In Question 3, numeric patterns were presented in both pictorial and tabular forms.

QUESTION 3

The United Artists Studio makes geometric decorative borders or motifs for picture frames. One motif is made from a pattern consisting of 4 dots and five lines joining these dots as shown diagrammatically and in table form below.



Number of motif (*m*)	1	2	3	4	20	30	63	**m**
Number of dots (*d*)	4							
Number of lines (*l*)	5							

1. Extend and draw the next two motif's designs.
2. Copy and complete the above table for the rest of the motifs' designs or pattern.
3. What is the mathematical relationship between:
   i) *m* and *d*.
   ii) *m* and *l*.
4. Will the points representing each of the relationships in i) and ii) lie on a straight line when plotted on the graph?

In Question 4, a numeric patter was presented as a simple sequence of numbers.

QUESTION 4

Study the number patterns and answer the questions that follow:

a) 5; 12; 19; ___; _____; _____;.....

b) 35; 30; 25; _____; _____; ___ ; ....

1. Determine in each case;
   i) the next three terms in the pattern.
   ii) the 10^{th} term.
   iii) the 50^{th} term.
2. Generate a formula or rule for the n^{th} term

The literature review undertaken to inform this research suggests that linear sequences would be most appropriate in terms of eliciting rich data at all levels of the pattern generalisation process. As such, the choice of the linear numeric sequences as well as the various pictorial contexts was an attempt to provide sufficient variety in the verbal and visual reasoning skills likely to be found in the group of 29 Grade 9 participants of the study. The questions in the task also provide means of assessing participants' knowledge and their ability to apply this knowledge to new situations.

The use of a written task has advantages. Some advantages as outlined by Tuckman (1978) and Sax (1968) are of particular necessity for this study. For example,

- Learner's written responses with procedures can be checked by the researcher thereby informing the researcher about aspects of the task that might not have been identified when designing the task.
- The questionnaire items are relatively easy to conduct compared to interviews which are time consuming to conduct. The questionnaire provides immediate data that can be categorized and analyzed.
- It is not easy to omit learners' responses when analysing written work compared to interviews where errors may occur in both the recordings and the transcriptions. There is a also possibility of addition, misinterpreting or omitting the interviewees' statements in the recording or transcription.

Apart from the advantages, a questionnaire task as a research instrument has its own limitations. These include:

- The difficulty in setting questions that will measure the intended outcome.
- The coding process of the open-ended questions is challenging in that the researcher is required to notice important trends coming up in the data. For example, interesting patterns, surprising or unexpected instances, and any apparent inconsistencies or contradictions (Cohen et. al. 2000)
- Questionnaires may not be completed or returned and there may be very little or no written responses to analyse.

Some of these challenges and limitations informed the study in that ways of minimising them were sought in advance. For example, the questions in the questionnaire were subjected to expert scrutiny that involved colleagues in the mathematics department at the school where the study was conducted and also my supervisor. After the review, the questionnaire items were then piloted to further check whether the study's three questions would be answered.

3.3.2.3 PILOTING THE QUESTIONNAIRE-BASED TASK

The piloting of the main instrument for the study was conducted in a different grade 9 class to the one intended for the study. The two grade 9 classes were from the same school. Cohen et. al. (2000) asserts that the sample for piloting needs to be similar to the sample intended for research so that the researcher is in a position to assess and analyze the likelihood of the trends observed during pilot stage, should these trends re-occur in the main study. The purpose of the pilot study was to inform the main study about the quality of the questions in the task. Also, a pilot study became very helpful in choosing questions which would provide rich data about generalization. The pilot study helped to indicate the task's suitability for the study in terms of clarity in the instruction, structure and content/context of the questions and whether the questions really provoked mathematical reasoning. The execution of the pilot study gave me opportunity as a researcher to practice the administration protocols (Cohen et. al 2000; Wallen & Fraenkel 1990).

The pilot studies should also help get rid of all the items that seem to provide irrelevant data for the study (Bell, 1987 cited in Opie, 2004). In my study, certain questions were removed in the main task as a result of the pilot. Spelling and other minor errors noticed during piloting of the instrument were corrected. The amount of time spent in piloting the task informed me about how much time I needed for the main study.

3.3.2.4 THE QUESTIONNAIRE TASK-BASED INTERVIEW

The task-based interview was developed from the learners' written textual responses from the main questionnaire-based task. The interviews were administered to 6 target learners selected from the 29 participants according to their abilities (see section 3.3.1) that is, 2 high achieving, 2 medium achieving and 2 low achieving learners. An interview session with each participant was recorded using a voice recorder and lasted for about 45 - 60 minutes maximum. A day prior to the interview appointment, each participant was given back his or her written response sheets to the questionnaire task. During the interview, and through continuous probing, learners were expected to reflect on their answers and the processes they went through when doing the written task-based questionnaire.

The task-based interview is used for two purposes. Firstly, observing participants' mathematical behaviour when they generalizing from number patterns. And secondly, to draw some inferences about the learner's meanings and representations, cognitive processes, knowledge structures or changes in these in the course of the interview (Goldin, 1997:40). I (at best) was able to observe the six target participants' verbal and spoken explanations as captured on the voice recorder during the interview sessions.

Interviews were chosen for the study as a data collecting tool because the research requires the raw opinion of the respondents on their written responses. As Opie (2004) indicates, the interviews offered the opportunity to ask the question "why". In this study, task-based interview offered me the opportunity to deviate from prearranged text and wording of questions. During an interview, open and flexible discussions should enable the examination of matters and issues not fully covered in the other data collection schedules (Denzin, 1970 & Silverman 1993 cited in Cohen et. al. 2000). I also made notes during the course of the interview about key ideas or issues that emerged from learners' spoken explanations. I did this in order to understand and establish their reasoning. Later, I transcribed each recorded interview from the voice recorder.

The following table 4, outlines how the questionnaire task and the questionnaire task-based interview were used to help answer the two critical questions of the study.

TABLE 4: RESEARCH QUESTION INSTRUMENT USED

1. What routines (*strategies*) and visual mediators do grade 9 learners use when engaging in a task on number patterns?	The task-based questionnaire and interview
2. How do these grade 9 learners explain their thought processes when generalizing in number patterns?	The task-based interview

3.3.2.5 THE TRANSCRIPTIONS

A transcript is intended to be an exact account of what was said in an interview. In this study, the interviews were recorded using a voice recorder and then later transcribed onto the computer. They were transcribed several times to increase accuracy. The section on analytic framework provides details regarding how the transcripts were analysed (see chapter 4).

3.4 VALIDITY AND RELIABILITY IN QUALITATIVE RESEARCH

3.4.1 VALIDITY & RELIABILITY

The strength of a research study lies in its validity and reliability - a concern that any person embarking on an investigation would have to address. An example of such a concern was echoed succinctly by Bosk (1979: 193) cited by Maxwell (1992) when he said “All fieldwork done by a single fieldworker invites the question, why should we believe it?” Such a question made me aware about the dangers of not explaining and supporting my objectivity in the study. In this qualitative research **trustworthiness** is used to address issues of validity and reliability. Trustworthiness, according to Lincoln and Guba involves credibility, transferability, dependability and confirmability (Opie, 2004:71). These four concepts are extensions, or adaptations of the traditional categories used in quantitative research of internal validity, external validity, reliability and objectivity.

3.4.2 TRUSTWORTHINESS

Opie (2004:68 -72) discussed trustworthiness and credibility as indicators of goodness of research:

> The basic issue in relation to trustworthiness is simple: How can an enquirer persuade his or her audiences (including self) that the findings of an enquiry are worth paying attention to, worth taking account of? What arguments can be mounted, what criteria invoked, what questions asked, that would be persuasive on this issue?
> (Lincoln and Guba, 1985:290 cited in Opie, 2004: 70)

A research study is **trustworthy** if it is reliable, referring to the consistency of the study's findings under the same conditions. It relates to consistency of the findings if the research is repeated with the same group. Reliability is viewed as synonymous with dependability that indicates the extent to which results can be regarded as stable (Lincoln and Guba cited in Opie, 2004). In qualitative research, replicability of results cannot be guaranteed because of bias inherent in the individuals. Any given data may be represented and interpreted differently by different researchers. During interviews in the study the same questions were repeatedly asked in different ways (if not understood by interviewees) to ensure some degree of trustworthiness in the responses the participants gave. Other possible factors that may have affected objectivity and the outcome of the study to some extent are as described in detail in the following section.

3.4.2.1 THE INSTRUMENTS

For trustworthiness, data in a research study need to have various forms of validity, for example construct validity, content validity and criterion validity. My research design used content and construct validity related evidence. These two forms of validity describe what is measured unlike the criterion validity which gives only criteria of how validity is determined (Fraenkel & Wallen, 1990).

Construct validity in this study refers to the extent to which the questions measured a theoretical construct of the study. This involved checking if the task was properly constructed so as to elicit the kind of information envisaged by the research questions through the pilot study. On the other hand, content validity refers to the adequacy of the questions with respect to the topic number patterns. In this study, experts (supervisor and colleagues) were consulted to assess the research task on its content validity. They face-validated the task by critiquing for example, the content knowledge level in the questions, ordering of questions and clarity of some phrases in each question.

The task was designed to cover enough content on the number patterns. Enough content gave adequate information about learners' knowledge, meanings and algebraic reasoning when generalizing in patterns.

3.4.2.2 USE OF TAPES

Audio recording of the interviews with the learners was done using a voice recorder. Gestures and facial expression conveying feelings of learners during interviews were not taken into account as the study focused mainly on learners' verbal and spoken explanations when generalizing. These aspects could be a limitation of the study. However, for the purpose of maintaining the scope and purpose in terms of the study's three questions this strict adherence had to be made.

3.4.3 DATA INTERPRETATION

The written task and interview data were transcribed, analysed and then interpreted. Also, audio tapes and interview transcripts were audited by colleagues for accuracy. The interpretation of data was guided by the theoretical framework of commognition along with the notion of mathematical reasoning as operationalized in the RNCS and NCS documents (DoE, 2003a:39; DoE, 2003b:18).

3.4.4 RESEARCHER EFFECT

My presence as a teacher in the study may have an effect on the behaviour of the participants especially during interviews. Fraenkel & Wallen (1990) contend that unexpected observers are likely to detract learners from concentrating on the task at hand. To avoid this from happening in this study, no observer was allowed into the interview venue.

3.4.5 RESEARCHER BIAS

Being a researcher and at the same time a teacher may have affected what I said and actually did in this study. In the interviews, I recorded, transcribed all the four questions then decided to analyze only questions 1, 2 and 4 because they provided rich data. The preparation of interview guiding questions from the learners' written responses helped me as a researcher minimise such bias.

3.5 ETHICAL ISSUES

3.5.1 ETHICS OF THE STUDY

Both the pilot and the main instruments were conducted under consideration of the ethical issues in research. The purpose of the research was clearly explained to all the participants. Before commencing the research study, I applied for ethical clearance from both the Gauteng department of Education (GDE) and the Ethics Clearance Committee at the University of the Witwatersrand. In these applications, I made it clear that the rights of individuals participating in the research were to be respected and protected. The research was conducted at a school where I work as a mathematics educator. This enabled me to have easy access to the school and the participants. Ethical - Consent letters were sent to the school principal and to parents whose learners participated in the research study requesting their permission. In the letters, I disclosed the aim and reasons for the study and how the learners and teachers would benefit from it. I indicated in the consent letters sent to all parties that participation was voluntary.

The problem of social power as supported by Opie (2004) may hinder accessibility of the respondents to the findings of this study. As a result all respondents and the principal were assured that the study's findings will be made known to them. Also, all un/used data would be kept in safe place once the report is written (for approximately 5 years) and thereafter destroyed.

3.5.2 CONFIDENTIALITY

Confidentiality throughout the study was adhered to and I used pseudonyms (L1, L2, etc.) to keep anonymity of the participants. During the interviews, I tried not to offend the participants.

3.6 CONCLUSION

In this chapter I have described and motivated my research design, instruments and data collection procedures. Issues of trustworthiness in qualitative research and how this relates to this study were also discussed in the chapter. The outline of an analytical framework and analyses of the data follow in the next two chapters of the study respectively.

CHAPTER 4 ANALYTIC FRAMEWORK - DATA ANALYSIS

4.1 INTRODUCTION

In this chapter, I present a detailed description of the AF that was used to provide a lens into the routines and mediators that students used when solving specified tasks.

4.2 CLASSIFICATION OF LEARNERS' ROUTINES

My classification approach of the learners' strategies for generalizing in mathematical task involving number patterns is derived from Becker and Rivera (2003). Lannin (2006) used similar categorization. The adapted generalization framework was later reconceptualised and interpreted in terms of Sfard's perspective of a mathematical discourse, using her theory of Commognition (see section 2.2.1). During the reconceptualization process, the emerging learners' strategies and methods of working from the written text and interview responses were categorised as *routines*. Over 15 different strategies/methods (i.e. routines) were identified from the learners' written and interview responses. These include: counting, recursive, guess, connecting, relate to pattern, solve equation, analyse structural, work backwards/reverse, functional, uses expression, visual (iconic), addition, multiplication, division, proportioning, substitution, etc. The learners used different representations such as symbols, formulae, graphs, pictures and drawings to communicate their thinking when responding to the questionnaire task. In the commognitive framework these are referred to as *visual mediators*.

During the analyses of the collected data my foci were on pattern generalization routines, representations (mediators) and the interpretations and meanings the learners hold. The meanings are inferred from spoken explanations as well as written responses to communicate mathematical reasoning. This communication is in the context of an algebraic discourse. For the purpose of monitoring the trends in the data, "learner routines" (LR) and "learner visual mediators" (LVM) are two main classifications of participants' work in this study. The LR and LVM each entail various forms of learners' knowledge content, solution representations, and meaning all of which contribute to pattern generalization. The learners' forms of communication in an algebraic discourse when generalizing from the number patterns is interpreted in terms of the commognitive tools as used by Sfard (2008) in order to respond to the three questions of the study. In this study, the learners routines identified from the data were classified into three main categories as in Table 5 below: *Numerical*, *Figural* and *Pragmatic generalizations* (Becker & Rivera (2003). The LR classifications together with LVMs classifications were used in both quantitative and qualitative analyses of the data.

Table 5: The classified categories and sub-categories of the *learners' routines* (LR) and *learners' Visual Mediators* (LVM).

LR Category	**LR** Sub-category	Further **LR** sub-categories	**L V M**			
			iconic	numeric	Symbolic/words	Table/graph
Numerical generalization	Recursive (Re)	Counting (Co) Chunking (Cu) Guess & check (G)				
	Explicit (Ex)	Structural analysis (SA) Work backwards (Wb) also called reversibility Solve equation (Se).				
Figural generalization	Proportioning (P) Structural analysis (SA)					
Pragmatic generalization	Relational (R)					
Unclassified Others	Unclassified	Unclassified				

The systematic coding and classification of data was used for both the written questionnaire-based task and the related interviews.

Category Abbreviations: **Re**: Recursive, **Co**: Counting, **Cu**: Chunking, **Ex**: Explicit, **SA**: Structural analysis, **R**: Relational, **P**: Proportion, **Wb**: Work backwards, **Se**: Solve equation, and **G**: Guess. SEE THE USE OF THESE IN QRASS SUMMARY PER QUESTION IN APPENDICES B - E.

4.3 DETAILED DESCRIPTIONS OF ROUTINES: CATEGORIES AND SUB – CATEGORIES

From the literature reviewed, there has been very little consistency in the naming of strategies that learners use in problem solving involving number patterns. The basic procedural descriptions of different strategies are largely similar while strategies' nomenclature is different (Stacey, 1989; English & Warren, 1998; Healy and Hoyles, 1999; Orton & Orton, 1999; Bishop, 2000; Lannin, 2003, 2005 & 2006 etc.). I have in this study tried to consolidate these similar but differently named categories into single categories.

4.3.1 THE ROUTINES: CATEGORIES

The method of classification is a key component of this study. The routines are interlinked and I give a description of each routine in the following section. Later on (in Section 4.5) I describe the indicators for each of these routines.

- **Numerical generalization routines**

Learners use numerical generalization routines when they employ some form of trial and error techniques without sense of what the coefficients or constants in the linear pattern represent. With numeric generalization routines, learners also demonstrate a lack of representational fluency in terms of formal conventions in algebra discourse. Furthermore variables are merely used as placeholders in linear number patterns. Numeric generalization routines consist of recursive or explicit methods. The recursive category includes counting, chunking, and guessing etc. whereas the explicit category extends to structural analysis of the pattern problem.

- **Figural generalization routines**

In this category, learners employ perceptual similarity routines in that they focus on the relationships between or among pictorial terms. This category is further classified into "proportioning" and "structural analysis" routines of working in number patterns.

- **Pragmatic generalization routines**

In this category, learners use both numerical and figural routines when generalizing. The learners identify patterns within number sequences with specific attributes (properties); they are also able to identify relationships between the terms in the pattern. Pragmatic generalization routines also require the use of formal notational conventions and algebraic representations.

4.3.2 SUB-CATEGORIES OF ROUTINES

The routines that the participants adopted when determining the next, 10^{th}, 20^{th} or the 50^{th} terms in a pattern were further analysed and classified into one of several sub- categories, some of which (explicit and recursive categories) are further sub-divided. In this section I sometimes use the pictorial and numeric patterns from section A of the questionnaire task to exemplify the categorizations.

4.3.2.1 NUMERIC GENERALIZATION ROUTINE: ROUTINES USING RECURSION

This involves writing or constructing a term based on the previous term or terms within the number pattern, for example, $f(n+1) = f(n) + d$. The learners tend to focus on the relationship between successive numbers in the pattern. The learners do not justify their generalizations non-inductively or in some other valid way. The learners frequently employ trial and error methods as a numerical routine with no sense of what the parameters in particular rule represent. Furthermore, some of their numerical methods contain fallacies and contradictions, and they seem to be object-oriented in the sense that the generalization rules they develop tend to be justified solely in terms of how well the formulae fit the limited information they have examined (Becker & Rivera 2005).

Further refinements of recursive routines may sometimes be identified. For example, counting, chunking, and guessing routines are examples of the sub-categories of recursive generalization routines.

- **Counting**

  This is a type of recursion strategy and involves constructing a model or re/drawing a pictorial representation of a number pattern and physically counting the desired structure/object. The terms in a sequence or pattern are determined consecutively through addition of a constant difference every time to the previous term(s).

Example:

In question 4b, the 10^{th} term could be calculated by adding (in a recursive manner) six lots of the constant difference (-5) to 4^{th} term (20).

Thus, $T_{10} = 20 – 5 – 5 – 5 – 5 – 5 – 5 = - 10$

- **Chunking**

This is a method similar to the counting approach; here chunks of the constant difference are added to a given term rather than recursively adding a constant difference to the previous term.

Example:

The 20^{th} term in question 1 can be determined by chunking the ten constant differences of five. Thus $T_{50} = 51+10x5 = 51+50 = 101$.

- **Guess work**

With guess work, a rule or formula is presented although the learner does not really know why such a rule works (he will also be unable to provide a proof or a justification).

4.3.2.2 NUMERIC GENERALIZATION ROUTINE: EXPLICIT ROUTINES

This is a method whereby a general formula or rule is first derived for the n^{th} term. The desired term or position number of a given term is then calculated directly from this formula using substitution in the n^{th} term. For an explicit rule, a general relationship is developed between the input values and their corresponding output values. Routines may be classified as explicit regardless of whether the generated rule is correct or incorrect.

Example:

The explicit rule for the n^{th} term in question 2 is $T_n = 3n + 2$. This can be equally expressed as: $T_n = 2n + (n + 2)$, $T_n = 5+ (n -1)3$, $T = 5 + (3 \text{ x } n – 3)$. All these expressions are algebraically equivalent.

Further refinements of explicit routines may sometimes be identified. For example, the work backwards (reversibility), structural analysis and solve equation routines are examples of the sub-categories of explicit generalization routines.

- **Solving equation**

In this routine, the learner formulates a general rule directly from the relationship of variables inherent in the pattern. Basic principles of solving an equation are applied in this context.

Example:

The learners recognize that multiplication and division are inverse operations to each other and so are addition and subtraction. And also, the learners are aware that the order of operation matters in a generated rule or formula such as $T_n = a + (n - 1) \times d$ for the n^{th} term.

- **Work backwards as a means of justification**

Another form of explicit routine is the work-backwards routine. This involves the use of the generated rules of formulae to determine either the position of the term (independent variable) or the term itself (dependent variable).

Example:

From question 4a, the position of 768 in the pattern would be determined as follows:

Since $T_n = 768$ then,

$T_n = 7n - 2$, substituting $T_n = 768$ to solve for n;

$7n - 2 = 768$

$7n = 768+2$

$7n = 770$

$n = 770 \div 7 = 110.$

This means, the term 768 is at position number 110 in the number pattern problem given in question 4a. This routine involves working backwards in contrast to most of the questions in the task which require finding the term with change in position.

4.3.2.3 FIGURAL GENERALIZATION ROUTINES

Participants who use predominantly figural generalization are capable of justifying their generalizations non-inductively and in other valid ways due, in part, to the manner in which they are able to connect their symbols and variables to the patterns that generate the figures. The learners seem to focus on the relation by seeing sequences of figural cues as possessing invariant structures and thus, are necessarily constructed in particular ways (Becker & Rivera, 2005).

Further refinements of figural routines may sometimes be identified. For example, the structural analysis routine is an example of the sub-category of figural generalization routines.

- **Proportioning**

This refers to the use of a constant rate of change as a factor for multiplication followed by addition or subtraction of a certain number to adjust the expression until the rule holds for a given pattern. Awareness of some relation between the number and its position exists and such a relation is expressed as a single multiplication.

For example, $y = 3x$ or $T_n = 5n$.

- **Structural analysis**

This is an example of an explicit routine in which the learner sees a structure in the pictorial representation of the pattern and works back and forth to determine the term or the term's position.

4.3.2.4 PRAGMATIC GENERALIZATION ROUTINES

The learners who employ figural generalization routine are more likely to generalize number patterns to an explicit generality rule. Note the terms "predominantly numerical" and "predominantly figural" in the sections 4.3.2.2 & 4.3.2.3 imply the possibility that some learners manifest pragmatic modes for expressing generality (Becker and Rivera, 2005); that is, the learners' generalization abilities reflect a capacity for employing both numerical and figural routines. Proficiency in translating between various representations is an essential component for the development of algebraic reasoning when expressing generality. For example, learners demonstrate proficiency when they frequently make connections between pictorial terms as in questions 1, 2 and 3 of the questionnaire task through verbal descriptions or explanations, numeric or symbolic representations while solving and generalizing pattern problems.

- **Relational**

The use of a relational routine falls under both numeric and figural forms of generalizations (pragmatic) routines. Relational routines focus on the relationship between the pictures or numbers in the pattern and their position in the entire sequence.

Example:

In question 3 the position number or number of motifs (n) is an input and the number of dots (Dn) or lines (Ln) in the motifs at position (n) are the outputs. See the tabular representation below. The table represented as figure 3 below uses structural analysis of the motifs.

Figure 3: Tabular representation only

Number of motifs (n)	1	2	3	4	10	20	50	n
Number of Dots (Dn)	4	6	8	10	22	42	102	$2n + 2$
Number of Lines (Ln)	5	9	13	17	41	81	201	$4n + 1$

In my study, data instances revealing numeric generalization routines far exceed other routines. As such the focus of my analyses is numerical generalizations. This does not mean that the other trends and routines from the data were not given attention - there was also interplay among the numeric, figural and pragmatic forms of generalizations thus some learners (high ability) used most of the routines featured in the three categories of generalization within a pattern problem (see table 5).

- **Unclassified others**

These are routines which are neither numerical, figural nor pragmatic routines. These routines do not fit into the categories of routines classified in table 5. For example, “I don’t know” responses could not be categorised elsewhere.

4.4 ASPECTS OF GENERALIZATIONS

The three main categories classified above reveal two aspects of generalization that may lead to fostering learners’ algebraic reasoning. These are ***globalizing*** and ***extending*** (Driscoll, 1999:94), see results section in Chapter 6.

4.4.1 GLOBAL GENERALIZATION

Global generalization refers to thinking about what is “always true” in considering mathematical rules and relations. Friedlander and Hershkowitz (1997) suggest a way to scaffold activities to encourage globalizing, progressing from what they call a “working generalization” to an ‘explicit generalization”. Globalizing activity is further discussed in Discussion of Results chapter.

4.4.2 EXTENDING OR LOCAL GENERALIZATION

Extending also known as local generalizing implies pursuing mathematical procedures beyond an algebraic result itself. The learners locally generalize by following the lines of further inquiry suggested by a particular mathematical result. Geometric patterns such as in questions 1, 2 and 3 are common features in most current curricular materials and can be used as a context in which to solicit and extend learners’ thinking in mathematics. Lee (1996:95) summarizes a set of student routines with geometric patterns by reporting, “seeing a useful pattern... seemed to be more problematic than simply ‘simply seeing a pattern’. Perceptual agility seemed to be key: being able to see several patterns and willing to abandon those that do not prove useful”.

4.5 INDICATORS OF CATEGORIES

Since the researcher’s task (myself in this case) is to categorise learner activities into one or more of the above routines, and since this categorization necessarily involves inference (from learners’ activities or written text) it is necessary to present indicators for these categorisations.

4.5.1 NUMERICAL GENERALIZATION

The learners use numerical generalizations if they employ trial and error (guess) techniques without a sense of what coefficients or constants in a linear pattern represent. With respect to the questionnaire-based task items, if learners generalize by making a table or writing down a formula, checking that it works on one or two terms and then moving on to the next question without justifying their activities, these responses were classified as numerical generalisation. Another key indicator of numeric generalization is when the learners investigate relationships between the output values only (recursive) or relationships between inputs and output values.

Example 1 of recursive and explicit rules

Using the cluster house problem as an exemplar, learners are reasoning recursively if they explore consecutive output values to determine the number of line segments required per cluster. Noticing that the number of line segments increases by 5 each time a new house is added will mean that the learner is able to configure particular instances leading to a general relationship. A realization of the relationship between the output values 6; 11; 16; 21; . . . . . . etc. as a numeric relationship of adding 5 relates to the iconic representation of the pattern. And that relationship enables a learner to draw a general conclusion about the recursive relationship.

Similarly, if a learner considers counting the number of line segments per house s/he may notice that it contains three lines on top, one at the bottom and two on the sides. Adding one more drawing of a house results in five more line segments and not 6 because one line on the side is shared. Such a conjecture allows the learner to realize that the strategy could be applied for any number of houses. This result is in the development of an explicit rule, $T_n = 5n + 1$ where T_n is the number of line segments and n is the number or position of the houses. In this case, we say the learner began reasoning by recognizing an iconic relationship and applied the reasoning to a general case of (generalization) through the use of explicit rules.
In other instances, the learner can connect recursive and explicit rules. For example, each time the number or position of houses increases by 1 as a result of a house being added, the number of the line segments increases by 5. This lead to a development of a rule such as
$T_n = a + (n\text{-}1)\ d$ i.e. $T_n = 6 + 5(n\text{-}1)$ which simplifies to $T_n = 5n + 1$.

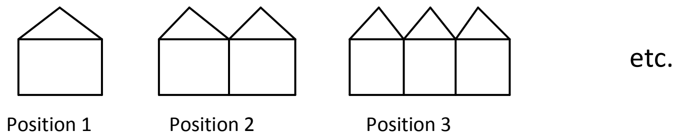
4.5.2 FIGURAL GENERALIZATION

Figural generalization is indicated if the learner employs perceptual similarity routines which focus on the relationship among pictorial or diagrammatic terms. See Example 2 below. With figural generalization, variables are treated not only as placeholders but also with a functional relationship perspective. For a response to be classified as figural generalization, the learner treats the pattern spotting activity as trivial. S/he makes empirical decisions based only on what the pictures and table tells them without looking at the relationship of the picture position number and its precise number of lines, dots or tiles.

Example 2 of figural approach:

In figural generalization, students use pictures or tables to deduce numeric relationships. For example, in question 1, the tabulated values are each determined consecutively from its proceeding term either by addition or subtraction of a constant difference. On the other hand, a pictorial context would mean a counting method realized by drawing a simple diagram of the required term and then counting the individual elements afterwards. See the tabular and pictorial representations below:

Figure 4: i) **FIGURAL OR PICTORIA REPRESENTATION**



ii) **TABLE REPRESENTATION**

Cluster position	1	2	3	4	5	n
Number of lines	6	11	16	21	26	$5n+1 = T_n$

4.5.3 PRAGMATIC GENERALIZATION

The learners employ both numerical and figural routines when generalizing, when they see patterns of numbers as having both *attributes* (properties) of and r*elationships to each other*. This implies that learners identify and observe the structural attributes from any form of pattern representation that results in generalization. Conjecturing a functional relationship using both the numerical and figural methods in a pattern is a key indicator of a student demonstrating pragmatic generalization. A pragmatic generalization routine mainly focuses on learner's formal conventions and algebraic representations. The learner's representational fluency in this category refers to their symbol, word, table, iconic or numeric use and interpretation (i.e. meanings they make of) when generating a rule.

4.6 MATHEMATICAL VISUALIZATION

As defined in chapter 2 in terms of commognition perspective, "visual mediators provide the images with which the discussants identify the object of their talk and coordinate their communication" Sfard (2008:147). Visual mediation is a process that embraces the use of visual mediators for communication purposes in a discourse. The communication of algebraic generalizations takes different forms, for example; symbolic, iconic, gestures and verbal (verbal includes written words and spoken words).

4.6.1 VISUAL MEDIATORS

With respect to the visual mediation category, the following sub-categories were seen to be the main forms of learners' representations: iconic, numeric, symbolic, words and tables (See table 6). Although the visual mediation category was mostly applied to the analyses of the interview data, various visual mediators were apparent in the participants' written representations of solutions. They were thus also useful in the quantitative analyses. I highlight these mediators in the following section.

4.6.2 BRIEF DESCRIPTION OF VISUAL MEDIATORS

Pictorial representation

Traditionally, *pictures* or *diagrams* and other visual representations have been considered an essential component of mathematics curricula as they facilitate learners' insight and understanding of mathematical concepts. Most learners find it easy to associate the context with the real life setting when modelled either using diagrams or pictures**.**

Numbers

The data collected reveal that numeric responses are the most common type of mediators used by participants in the study to communicate their ideas and thinking. This could imply that the learners feel comfortable in the use of numbers to verify their solutions.

Verbal explanation

This is a very common type of mediator and widely used in ordinary conversation by participants of a discourse. The verbal explanation expresses one's reasoning. Hence, it could be considered as the basic mediator for the participants of a mathematical discourse to communicate their mathematical ideas and thinking.

Algebraic representation

Algebraic symbols are visual mediators created and used in a mathematics discourse. Algebra is conventionally communicated through symbolic representations.

4.6.3 VISUAL MEDIATORS AND THEIR INDICATORS

The following table summarises the types of visual mediators in my study.. The columns relating to iconic and numeric mediators are adapted from Hoyles and Healey's (1999) categories of generalization strategies whereas the last two columns are my own theoretical elaborations. As can be seen in table 6, I have only categorised LVM with regard to the numerical generalization routines. This is because, in my research, numeric generalization routines predominate.

TABLE 6: SUMMARY OF LVM

Numeric Generalization Routines	**Learner's Visual Mediators**			
	Iconic	**Numeric**	**Table/graph**	**Symbols/words**
Recursive or Counting Routines	The learner describes a relationship that occurs in terms of the visual relationship between consecutive values of the independent variable. For example, in Question 1 five line segments must be added for each additional house.	The learners notice a number pattern in the results for consecutive values of the independent variable. For example, the number of line segments goes up by 5 each time the number of houses increases by 1.	Tabulate numeric terms recursively from either numeric or geometric patterns.	Express the pattern properties in words or using symbols (the use of verbal explanation when finding next term).
Explicit Routines	An explicit rule is constructed based on a visual representation of the situation. For example, there are 6 lines for 1^{st} position house and a common difference of 5 for the consecutive terms. So multiplying 5 by position number and adding 1 gives a generalized number of the line segments.	The learners guess an explicit rule based on a numeric pattern in the output values using proportioning method. For example, since the number of lines in 1 house is 6 multiplying number of houses by 6 and + or – some number will give the number of line segments at any position in the pattern.	Express number patterns in a table or graphical form	Use words to express the nature of a sequence.
Chunking	A recursive rule is established based on a relationship established in the diagram. . For example if one cluster house has 6 lines then 5 cluster houses would have 6+5(4) line segments.	The learner builds on a recursive pattern by using a table of values. Example as under iconic representation.		Mainly expressing verbally the process of chunking based on the diagrams.
Work backward (reversibility)				Explaining in own words the use of a general rule to find position of a term in a pattern
Structural analysis				Learners explain in own words the structural attributes or properties in a pattern.

As will later be discussing visualization of the pictorial structure of the terms in each pattern has influence on how learners make justifications about their own generalizations. The visual mediators that were used for example in questions 1 and 2 have been described in detail in chapter 5. There we see the diverse use of different LVM when generalizing from number patterns.

4.7 CONCLUSION

In this chapter, I have described the analytical framework used in the study's data analysis process. The categories and sub-categories of LRs and LVMs are the key analytic tools in connecting the study's findings to the theoretical perspective of commognition and the previous investigations on number patterns. I do not claim that the analytic framework the study is using captures every possible activity related to generalization. For example, an activity that is not adequately captured in the analytical framework is "visual reasoning" in generalization. The question design of the task is limited to linear generalization. In the chapter 5 that follows, I provide the quantitative and qualitative data analyses of the study respectively.

CHAPTER 5 ANALYSIS AND DATA INTERPRETATION

INTRODUCTION

My study centres around the following two questions:

- What routines (*strategies)* and visual mediators do grade 9 learners use when engaging in a task on number patterns*?*
- How do these grade 9 learners explain orally their thought processes in problem solving involving number patterns?

In this chapter, I present the overall quantitative and qualitative analyses of the data collected using the written task based questionnaires and task based interview responses respectively. The learners' written work and interview responses were analysed both quantitatively and qualitatively in accordance with the tools developed in the preceding analytical framework (see chapter 4). The participants' routines (Sfard, 2008) in this study refer to the learners' repetitive actions, procedures or methods used to understand and solve the numeric-geometric pattern problem(s). In order to explain the learners' written text and verbal responses, explanations were drawn from the past literature and the theoretical framework (see Chapter 2) the study is using. My own intuition as a classroom practitioner also contributed to my research role in this study.

5.1 QUANTITATIVE ANALYSIS

In the following section, I analyse and discuss the results of all the participants' task-based written responses. The focus here is on the participants' use of routines and mediators (Sfard, 2008) within the different generalization categories proposed in the analytical framework (see table 5).

5.1.1 LEARNERS' GENERALIZATION APPROACHES

Generalizations in mathematics and algebra in particular are expressed in various ways. These include verbal (i.e. written and spoken words) or symbolic syntax which translate sometimes to the same or different meaning. For example L23 approached question 1 in two different ways: He first applied both recursive and explicit methods to generate the rule $T_n = 5n + 1$. A different learner used a different approach to the same question (i.e. Question 1) by considering the base of each set of cluster houses as one line segment. This was an approach that led the learner to generate a rule $T_n = 3n + 3$. (Only the first rule is, of course, correct).

Most learners were able to express the given sequence from a particular scenario through a general rule by means of a functional relationship without realizing that they were making use of independent and dependent variable-concepts in the context. See an example of globalization by tabular method in discussion of results section.

5.1.2 LEARNERS' WRITTEN TEXT RESPONSES

In this section, I discuss the results and analyses of the written text responses (*quantitative data*) to the questionnaire-based task. To give clarity to each stage of the analysis, learners' results are summarised in tables classified into routine categories. The first four geometric-numeric linear pattern questions from section A of the questionnaire task were analyzed in this study because the learners provided rich and enough written text data compared to the responses to questions from section B. All the learners' responses to items 1 – 4 were carefully classified into categories. This classification was based on the specific routines (strategies) the learners used, followed by levels of completion for each item within the question and an overall justification of the generalization established during the interviews. The results of the written text-response analyses are summarised on the Question Response Analysis Summary Sheet (QRASS) adapted from Samson and Schaefer (2007). See appendices B to E. These summary sheets contain level descriptors (LD) and task level attainment (TLA); these categories are both explained in the following section. The summary sheets give an overview of the results of my study which I use as a basis for my discussion later.

5.1.2.1 LEVEL DESCRIPTORS (LD)

The learners written or verbal responses to the generalization item in each question i.e. questions 1 to 4, were classified into "**levels**" with numbers assigned ranging from **1** to **4** with the exception of question 4 whose level descriptors were **1** to **3**. A LD (Samson and Schaefer (2007). refers to question classification-stages represented by number point system **1**, **2**, **3** and **4** (see tables 8 & 9). These numerical numbers are assigned according to sub-question's level of difficulty for example, **1** is assigned to any attempt to determine or calculate the next two terms of a pattern and **4** for determining the n^{th} term or general rule. It should be noted that the numbers 1 to 4 do not in any way represent any hierarchy or marks achieved. I did not start from level zero in the analysis as zero would mean the learners did not attempt the questions. The learner's ability on each question was judged on the total score of these levels. The code TLA: Total Level Attainment represented in the 7^{th} column of each question's QRASS in appendices B to E became the score board to help determine how the learners faired in the questionnaire task.

5.1.2.2 LEARNERS' CHOICES OF ROUTINES AND MEDIATORS

The learners' written responses during the problem solving session on number patterns include the following mediators: symbols or pictures (iconic), numbers, verbal explanations, tables and graphs. The routines chosen by these learners when responding to the questionnaire task varied according to the content and context embedded in each question. Written responses to all the questions were carefully scrutinized and analyzed along <u>two</u> main dimensions: **routines** (my emphasis) and learners' type of **mediators** (see QRASS appendices B-E and table 10).

The number of responses to the 25 items spread across the four questions of the questionnaire based task by the 29 participants was 787; this was because some had given more than one form of solution representation. The actual number of expected answer-responses is 725 i.e. (25 x 29). The total number of routines (from 787 responses) identified in the participants' responses across the three main categories of generalization (See Tables 5 & 7)is 1450 and this number is double the number of the expected responses because some learners used more than one routine within a particular stage of each sub-question in the questionnaire based task. There were also 10 instances of a question not having been attempted at a particular stage, for which no analysis or characterization was possible (see the QRASS appendices).There were more participants' responses to the numeric decontextualized pattern in Questions 4 than in any of the other three questions in the task. This could mean that the participants were more familiar with the arithmetic situations than others.

Each response was carefully analysed and assigned into one of three routine categories described in section 4.3.1 that is, *Numerical*, *Figural* and *Pragmatic* generalizations. Table 7 summarises the use of the three routine categories over the first four questions in the questionnaire based task.

TABLE 7: OVERALL ROUTINE UTILIZATION BY 29 PARTICIPANTS IN THE STUDY.

ROUTINE CATEGORIES AND SUB-CATEGORIES	**(Level - Descriptors) LEVEL(S) ATTAINED**					
	1	**2**	**3**	**4**	Totals	Total %
Numeric: Recursive:	Next terms	10^{th} or 20^{th}	50^{th} term	n^{th} term	Routines	
counting,	499	267	6	0	772	53.3
guessing & check	12	13	3	1	29	2.0
chunking	15	8	0	0	23	1.6
Explicit: Solve equation Structural analysis Work backwards (Reversibility)	79	210	237	43	569	39.3
Figural: Proportioning Structural analysis	8	5	2	2	17	1.2
Pragmatic: Relational						
	0	0	0	20	20	1.4
	0	0	0	13	13	0.9
Unclassified others	**1**	**2**	**4**	**0**	**7**	**0,5**
TOTAL	**614**	**505**	**252**	**79**	**1450**	**100**

A number sequence can be generated either by using a general formula, where the independent variable represents the position of a term, or by relating one term recursively to the previous term in the sequence. Use of a recursive approach tends to emphasize local and specific aspects of the relationship, while an explicit formula reflects the relationship in a general way.

With respect to Table 7, the learners' procedures were dominated by numerical generalization which accounted for about 96 % (recursive + explicit). The figural, pragmatic and unclassified routines accounted for about 4%. However, the total number of routines identified from the data exceeded the expected number of responses from the entire sample because some participants used multiple routines within and across level(s). There were also a few instances where some sub-questions were not attempted at a particular level and it became very difficult to analyse (unclassified others). Table 7 summarises the routines that consistently and frequently appeared in the data. Table 7 also shows that 56.9% of the learners' overall numeric generalization routines were recursive while about 39.3% were from explicit approaches. Most of the learners' routines under numeric category were recorded in the first three TLA. The figural and pragmatic routines accounted for 1.2 % and 2.3 % of the overall routines respectively. The unclassified routines accounted for only 0.5 % of the entire data and were not categorized for analysis purposes.

TABLE 8: LEARNERS' OVERALL NUMERICAL GENERALIZATION ROUTINES

NOTE: RE: Recursive routines EX: Explicit routines

TLA's	**Level 1**		**Level 2**		**Level 3**		**Level 4**		**Total**	
QUESTION NUMBER	**Next terms**	**%**	**10^{th} or 20^{th} term**	**%**	**50^{th} term Or more**	**%**	**n^{th} term**	**%**	**Average %**	
1 .Three consecutive pictorial terms of cluster houses with a numerical value of independent variable given.	*RE* *EX*	*62* *25*	*RE* *EX*	*35* *55*	*RE* *EX*	*0* *70*	*RE* *EX/RE*	0 72	49 56	
2. Three consecutive pictorial terms of groups of tiles with a numerical independent variable given.	*RE* *EX*	*70* *18*	*RE* *EX*	*39* *48*	*RE* *EX*	*0* *75*	*RE* *EX/RE*	0 86	55 57	
3. Two consecutive pictorial terms of decorative motifs with partly tabulated dependent and independent variables indicated.	*RE* *EX*	*55* *22*	*RE* *EX*	*20* *51*	*RE* *EX*	*0* *60*	*RE* *EX/RE*	0 68	38 50	
4. Three consecutive purely numeric terms with a dependent variable indicated as in questions 1, 2, 3 and 4.	*RE* *EX*	*88* *10*	*RE* EX	79 26	RE *EX*	0,5 45	RE *EX/RE*	0 53	83 36	TOTAL
Overall %	***RE***	***69***		***43***		***0,5***		***0***	***56***	***812***
Overall %	***EX***	***15***		***36***		***50***		***55***	***39***	***566***
Figural, Pragmatic & unclassified others									***5***	***72***
OVERALL ROUTINES										***1450***

Since the numeric generalization routines far outnumber the Figural and Pragmatic generalizations, this report focuses on the former. Drawing on the participants' numeric routines classified in table 7, I present a further analysis of learners' numeric generalization routines against level descriptors that the participants of the study used and achieved respectively in responding to the four questions of the questionnaire-based task as in Table 8.

In table 8, the recursive and explicit routines dominate the numerical calculation of the next terms in each of the four questions from the questionnaire task. At level **1** there were more recursive routines (69 %) compared to explicit routines (15 %), at level **2** recursive methods dropped to 43% while explicit methods rose to 36 % from 15 %. And level **3** was dominated by explicit methods as most learners found the numbers increasingly difficult to work with recursively. At level 4, only the participants who employed explicit methods were able to generate a rule for the n^{th} term. In total, the recursive methods and accounted for 56 % of routines used while explicit methods accounted for 39 %.

The explicit routines largely involved making use of rules derived from a numerical or pictorial representation and expressed in terms of T_n. The symbol *n* is the term's position in a sequence or pattern - the independent variable, while T_n refers to the term itself (dependent variable). Numeric generalization approaches in total accounts for almost 96 % (see table 7) of all solution methods revealed at the four levels (levels 1-4). The findings resonate with what Lannin (2004:217) describes as being almost a natural tendency for learners to follow recursive more than explicit approaches when generalizing from linear sequences where the common difference is simply added to the previous or existing term.

Reflecting on the 29 learners' written answers, fewer learners generalized correctly at levels 3 and 4 as compared to levels 1 and 2 across all the four questions in the questionnaire based task.. Analysis of the task based interview responses also show that the participant's TLA or scores are attributed to the nature and context of the number pattern question (see qualitative analysis). For example, the participants seemed to struggle less when presented with sequences or patterns consisting of consecutive numeric terms (see question 4a) compared with the pictorial and tabular pattern representations. All the questions set in section A were linear-geometric patterns of the form $y = mx + c$. Concerning linear patterns, Stacey (1989) distinguishes between "near generalization" tasks in which finding the next pattern or element can be achieved by counting, drawing, or constructing a table, and "far generalization" tasks, in which finding a pattern requires an understanding of the general rule.

5.1.3 LEARNERS' USE OF MEDIATORS

The written responses to all the questions from the task were carefully analysed and categorised in terms of routines and how they were visually mediated (i.e. forms of representations) or spoken of. During the analysis of the generalization routines, it emerged that different types of arguments and mediators were used by the 29 participants of the study to communicate their ideas and thinking. The mediators are summarized in table 9 against the number of participants who used them as follows:

TABLE 9: SUMMARY OF THE USE OF MEDIATORS BY LEARNERS

Types of mediator used (only)	Number of learners (n = 29)
Pictorial representations only e.g. diagrams, tables or graphs	1
Verbal (words) explanation only	4
Number representation only	2
Algebraic (Symbolic) representation only	0
Pictorial + Number representations	9
Verbal + Number representations	7
Algebraic + Number representations	4
All the different mediators above	2

Table 9 indicates the following: Some participants of the study were able to communicate their responses using only one form of mediator (representation). For example, only one participant extended the pattern by drawing further pictures while none used symbolic representation only throughout the task. Most participants (24) used some numeric methods in their responses. There are only two participants who showed a strong understanding of multiple representations of solutions when generalizing in number patterns (L23 & L28). It is also interesting to note that the number of participants who used a combination of pictures and numbers to communicate their thinking is more than any other mediator category and this could indicate some interesting trends. In the context of this study such an "association" is neither conclusive nor is it linked to the focus of the study.

THE QUESTIONS

The objective of questions 1 & 2 was to investigate learners' ability to generalize from patterns represented in diagrams and pictures format. My task was to identify and classify the routines (strategies) that learners employed when working on these questions. In the context of reading from diagrams or pictures in a pattern, there are several opportunities that may lead to a generalization. It is interesting to note that most learners derived numeric sequences from the three pictorial terms in both questions in a similar manner. For example, the learners counted the number of items in each pictorial or figural term to generate a numeric sequence. The numeric sequence may have prompted learners to find the common difference among the terms in order to determine the next term in both questions. The inclusion of either a dependent or independent variable seemed to influence learners work more towards explicit methods of generalization.

The highest number of learners demonstrating recursive routines occurs in question 4. In this question, two sequences each with three consecutive numeric terms of an arithmetic sequence are given. The learners in the study experienced difficulties in generating the correct rules or formulae in the patterns. Levels **1** and **2** in question 4 contributed the highest scores (TLA) with respect to recursive routines compared to the other questions in the questionnaire based task responses (see Table 8). The use of the common difference between the consecutive terms seemed to have influenced learners to adopt counting methods which were largely classified and categorized in table 7 as recursive numeric generalizations.

There were two interesting learner related behaviours worth noting with regard to the nature of the responses to question 3 and 4 b), These are: when the two or three consecutive pictorial terms in a sequence are presented in a tabular format which reveals the independent variable, for example, the motifs' position as in question 3, the learners' use of recursive methods decreased slightly.

In the formula $T_n = a + (n - 1)\, d$, "T_n" represents the term in a sequence, "a" represents the first term of the sequence, "n" the position of a term in a sequence and d the common difference or constant difference between any two consecutive terms. In the corresponding formula $y = mx + c$, m is the rate of change which gives the gradient and c is the y – intercept.

This may be attributed to the presence of both dependent (number of lines or dots in motifs) and independent (position of motifs) variables which might influence some learners to conjecture a relationship between the two. Such a relational functional approach resulted in the use of explicit procedures rather than recursive routines. Similarly, in question 4 b) the responses show a drop in learners' tendency to generalize a decreasing sequence compared to the other increasing sequences. Most learners made use of substitution approach into the general expression $T_n = a + (n - 1)\, d$ in order to determine both the numerical terms and the general formula. In my personal capacity as a mathematics educator I became curious to find out how some of the participants knew about the formula $T_n = a + (n - 1)\, d$. In reply, some learners indicated that they learnt the formula together with an equivalent formula of the form $y = mx + c$ at mathematics extra lessons sessions which are run at the school every afternoon. On the other hand, other learners indicated that they learnt the use of such formulae from grade 10 and 11 friends during study sessions. Most participants of the study had difficulty in noticing that the common difference in a decreasing pattern was minus 5 (i.e. – 5). As a result they ended up substituting +5 into the general expression. Other learners managed to conjecture *d*, the common difference as "- 5" but when substituting carried out incorrect manipulations.

From the first question (see appendix A) about line segments used to design the cluster houses, the context itself requires a learner to decide how s/he can continue with such a picture pattern. One could possibly suggest drawing some more of these pictures, first in order to clarify the general rule visually so as to become aware of the counting technique. By structurally analysing the cluster houses' design from one position to another, the pictorial pattern is equivalent to showing the first three terms of the linear sequence 6; 11; 16 ; ……… with a general formula $T_n = 5n + 1$. From this background, typical questions requiring learners to determine for example the next, 10^{th}, 20^{th} and 100^{th} terms as well as an expression for the n^{th} term were asked. If a learner verbalises how the picture sequence extends then s/he will have accomplished what I regarded as the first attempt and procedure towards generalization in this study.

Apart from being a mere visual representations of a numeric patterns (see Questions 1, 2,& 3 in appendix A), number sequences presented in a pictorial context allow for a potentially deeper appreciation of underlying structure of the pattern, as such context allows for both a greater depth and scope of interpretation. Clausen asserts that working directly from a pictorial context is often preferable to an algebraic treatment derived from purely numerical patterns. The use of a pictorial context is also deemed "safer" as it limits the chance that "irrelevant number patterns will mislead one into assuming the truth of an invalid generalization" (Clausen, 1992:18).

The responses of learners who wrote and attempted all 4 questions are summarised in Table 10. I have taken into account all the responses of sub-questions within the main question when capturing number of learners falling into multiple response levels. For example, Question 1 includes 1.1 i), ii), iii) and 1.2. Similarly, Question 2 includes 2.1 i), ii) 2.2 i), ii), iii), iv) and 2.3 etc.

TABLE 10: ABSOLUTE AND PERCENTAGE RESPONSES TO QUESTIONS 1 - 4 FROM THE QUESTIONNAIRE BASED TASK

Level (TLA)	**1**	**2**	**3**	**4**	**5**
Question number	Learners who made an attempt	Learners who got incorrect numerical answer and incorrect formula	Learners who got correct numerical answer but no formula provided	Learners who got correct numerical answer but incorrect formula	Learners who got correct numerical answer & provided an appropriate formula
Q1	29	3	2	10	14
%	100	10,2	6,8	34,4	48,6
Q2	29	0	1	8	20
%	100	0	3,4	27,7	68,9
Q3	29	1	9	8	11
%	100	3,4	31,0	27,5	37,7
Q4 a)	29	1	2	12	14
%	100	3,4	6,7	41,5	48,6
Q4 b)	29	1	12	7	9
%	100	3,4	41,5	24,1	31,6
Total %	100	4,1	17,9	31,1	46,9

Drawing on the quantitative data in table 10, all 29 learners attempted to answer all questions. Nearly half of the learners (14 out of 29 i.e. 48.6 %) generalized correctly from questions 1 and 4a using the pragmatic approach of generalization (i.e. numeric + figural methods). In question 4b approximately 31.6 % of the learners were able to generalize correctly, i.e. $T_n = -5 \times n + 40$. In Question 2, about 69 % of the learners generated the correct formula for n^{th} term while the lowest performance was in question 3 where 37.7 % generated an appropriate formula and got a correct numeric answer. The drop in learner achievement in this question is attributed to multiple representations of the pattern problem. The learners grappled with pictorial-tabular relations at the same time. This was revealed during the interviews where some learners indicated that they would be more comfortable if the pattern problem was pictorial only or in the form of a table. Overall, about 47 % of the participants in the study correctly generalized from the number patterns.

Analysis of each participant's solutions revealed that learners L2, L9 and L19 provided the most incorrect responses to the questionnaire items and accounted for 10.2 % of the entire sample. Examples of L2, L9 and L19 written responses to question 1 were as follows:

TABLE 11: EXAMPLES OF SPECIFIC LEARNERS'S SOLUTIONS

LEARNER CODE: QUESTION NO:	**L2**	**L9**	**L19**
1.1.	6+ 6 + 6+6 = 24	L_4 = 4 x 6 -1 = 23	T_4 = 6 + 4 houses = 10
1.2 i)	h_4 = 24 , h_6 = 6x6=36 So 10 houses = h_{10} Hence h_{10} = 24 + 36 = 60	L_{10} = 10x6 – 1 = 59	T_{10} = 6 + 10 houses = 16
ii)	h_{50} = 50 x6 = 300	L_{50} = 5 x L10 -1	T_{50} = 6 + 50 = 56
1.3	h = x *6	L_n = 6 (n) - 1	n^{th} term = 6 + n

Drawing from table 11 the formulae generated by the three learners, L2, L9 and L 19 signal the consequences of these learners not carefully observing and identifying common relations in a given pattern and/or paying insufficient attention to what is asked in a given question. These learners were distracted during the entire process of finding a solution. As a teacher, I find this a common scenario among the learners in the mathematics lessons: finding an answer is a priority and of less importance is the procedure and steps followed in getting the final answer. The detailed analysis of each learner per question is given in the appendices B - E.

5.1.4 SUMMARY OF PREVIOUS DISCUSSION

The quantitative analysis presented in this chapter shows that learners have a shared approach that favours numeric generalization routines. And many of the routines identified in the present study are consistent with the results reported in the previous research studies on number patterns. What featured in this study as recursive and explicit routines correspond to what Stacey's (1989) calls counting, difference, whole object and linear methods. The recursive routines further correspond to Orton and Orton's (1994) recursive methods.

Numeric generalization routines and other routines as classified and categorized in Table 8 have been reported in literature of the studies that used a set of similar number pattern questions to the ones in appendix A. For example, learners such as L23 & L28 who were able to predict the next few terms or further numbers in a pattern were also able to express their procedure in algebraic form. Being able to recognize the equivalent expression and use algebra in solving problems associated with number pattern is an indication of a student's ability to reason mathematically (algebraic representation).

However, what the quantitative analysis has not been able to show is the nature of learners' mathematical discourse in terms of their use of the specific routines in conjunction with word use, mediators and narratives. It is possible that some students were simply performing learned procedures (rote learning). The next section presents a qualitative analysis of the interview data where learners' mathematical discourse is explored further.

5.2 QUALITATIVE ANALYSIS

The notions of generalisation, justification and proof are intricately interwoven. Generalisation, by its very nature, cannot be separated from justification/proof and is seen as a critical component of the algebraic reasoning process. The types of generalisation activities in this study include those presented in pictorial contexts, thus allowing for a possible connection to a referential context that has the potential of enhancing the generalisation process. The central role of justification within the context of this study is seen as *communication* of mathematical understanding, and students' approaches of their generalizations are seen to provide "… a window to view their understanding of the general nature of their rules" (Lannin, 2005:251).

This section presents a qualitative analysis of learners' solution routines regardless of whether the answers are correct or not. Such an analysis is deemed to provide pertinent information on the learners' mathematical reasoning.

I used the interview probes so as to encourage the learners to tell and say more. Other techniques I used in the elaboration probing include: gently nodding my head as the interviewee talked, softly voicing "yeah" very often and sometimes I remained silent but very attentive. Probing questions included: 'Tell me more about that?' and 'Can you give me an example of what you are talking about?'. The six learners interviewed in the study were coded as L5, L9, L15, L17, L23, and L28. The choice of these learners is discussed in detail in methodology chapter (see section 3.3). Table 12 shows the QRASS - a summary of the responses of the learners interviewed. It can be seen that some learners (L23, L28) coped very well with the questionnaire, others (L5, L9) had many difficulties with the questions, and yet others (L15, L17) were in between. The learners who coped well with the task belonged to the high ability group, those that experienced difficulties with the questionnaire based task were from the low ability group and then the last two learners who were classified as medium ability, their performance lie in between (see sections 3.3 and 5.2.2). The average percentage scores for each group (high, low and medium) as shown in table 12 in the row denoted √**: %** are; 89 %, 41 % and 67 % respectively.

TABLE 12: SUMMARY OF THE QUESTIONNAIRE RESPONSE SUMMARY SHEET FROM 6 INTERVIEWEES

KEY: √: Correct answer **√χ: Partly correct answer**

χ: No attempt at all ***: Incorrect answer**

L : LOW ABILITY **H** : HIGH ABILITY **M** : MEDIUM ABILITY

Q & Lnr Code	L5		L9		L23		L28		L17		L15		Total X	Total √	√X & *	G-total
1.1 i)	√		√		√		√		√		√		0	7	0	7
1.1ii)	√		√		√		√		√		√		0	7	0	7
1.1iii)	X		*		√		√		√		X		3	3	1	7
1.2	X		X		√		√		√		√		2	5	0	7
2.1 i)	√		√		√		√		√		√		0	7	0	7
2.1ii)	X		X		√		X		X		X		5	2	0	7
2.2i)	√		√		√		√		√		√		0	7	0	7
2.2ii)	√		√		√		√		√		√		0	7	0	7
2.2iii)	√		√		√		√		√		√		0	7	0	7
2.2iv)	*		*		√		√		√		X		1	4	2	7
2.3	X		X		√		√		√		√		2	5	0	7
3.1	√χ		√χ		√		√		√		√		1	4	2	7
3.2i)	*		X		√		√		√		√		1	4	2	7
3.2ii)	*		X		√		√		√		√		1	4	2	7
3.3i)	√		√		√		√		√		X		1	5	1	7
3.3ii)	*		*		χ		X		X		X		4	0	3	7
3.3iii)	*		*		χ		X		X		X		4	0	3	7
4 a) i)	√		√		√		√		√		√		0	7	0	7
ii)	√χ		√		√		√		√		√		0	6	1	7
iii)	X		√		√		√		√		√		1	6	0	7
4 b) i)	√		√		√		√		√		√		0	7	0	7
ii)	√χ		√χ		√		√		X		X		3	2	2	7
iii)	X		X		√		√		X		X		5	2	0	7
√ : %	9	39 **L**	10	43 **L**	21	91 **H**	20	87 **H**	16	69 **M**	15	65 **M**				
X : %	6	26	7	31	2	9	3	13	7	31	8	35				
* : %	5	22	4	17	0	0	0	0	0	0	0	0				
√χ: %	3	13	2	9	0	0	0	0	0	0	0	0				
TOTAL	**23**	**100**	**23**	**100**	**23**	**100**	**23**	**100**	**23**	**100**	**23**	**100**	**34**	**108**	**19**	**161**

5.2.1 INTERVIEW APPROACH

The questionnaire task based interviews were set to follow up learners' reasoning and how they justify orally their written responses to questions in the questionnaire based task. The interviews were conducted formally. A formal interview (Patton, 2002:342), otherwise known as the semi-structured interview (Cohen and Mansion, 1994:273), is an open-ended approach to interviewing in which questions flow from the immediate context. During the interviews participants were asked to provide an oral explanation either in the form of an addition or subtraction to their previously written explanations. These interviews took place soon after my analysis of the mathematical responses to the questionnaire based task had been completed. The purpose of these interviews was to provide the research participants with the opportunity to further explain or expand on their written responses. The checking process of an individual participant constitutes a form of external validation (Lewis and Ritchie, 2003:276). Since the main objective of these interviews was to allow for an accurate categorisation of the adopted solution routines, only learners' written questionnaire responses were taken into account. In addition, interviews were restricted to those specific cases where a participant's written articulation of their mental reasoning was either ambiguous or required illumination or elaboration by oral explanation. The interview targeted participants were allocated or divided into three groups (see section 3.3.1 or 5.2.2 below) under methodology. The next section outlines each participant's generalization approach to the questionnaire task.

5.2.2 DESCRIPTION OF THE INTERVIEWED PARTICIPANTS

As highlighted in section 3.3.1, the choice of the six learners from the sample of 29 is entirely based on the general knowledge, performance (abilities) in class and their ability to speak. In the following section I, further give a brief description for each of the six participants in their respective ability groups.

HIGH ABILITY GROUP

L23 relied on formulating a general rule first by conjecturing first term and common difference in the visual pattern in order to determine 10^{th}, 20^{th} and 50^{th} terms in the sequences. He worked backwards and provided justifications for each step in his procedures.

L28 mostly relied on figural – numeric computations (analysis) and use of a general linear formula to generate a functional relation or rule. She justified her solutions as well by demonstrating a good understanding of working backwards (reversibility).

MEDIUM ABILITY GROUP

L17 relied on formulating the general rule, $a + (n\text{-}1)\ d$, first. She then used this rule to answer the rest of the questions.

L15 tabulated the dependent variables against the independent variables. She then used a numeric generalization by relying on counting methods for smaller terms and then using the expression $y = ax+b$ *or* $y = mx + c$ to generate a linear rule to determine 50^{th} term and to check the others.

LOW ABILITY GROUP

L5 relied on extending the pattern either pictorially or numerically (recursive) and hardly managed to get the correct general rule.

L9 relied on numeric counting (recursive) and could not properly use symbols to formulate a general rule (lacked verbal and algebraic representation skill).

INTERVIEW ITEMS

The interview items were drawn from Section A of the questionnaire task. And the actual transcripts analyses are drawn from questions 1, 2 and 4 only. I chose these questions because they are contextually and structurally different yet the content embedded in them led to similar type of generalizations. The other reason is that most participants in the study completed answering these questions (i.e. 1, 2 and 4). A comprehensive summary (QRASS) of the six interviewees showing the distribution of responses to the four questions that I analysed from the questionnaire task is shown in table 12 in section 5.2 It is interesting to note that each participant from the target groups had made an attempt to answer all questions. 67 % of the responses provided by these learners were correct. 25 responses out of 36 among the six target participants show the correct generalization of the n^{th} term from each question. This is approximately 69% of the interviewees and almost 15 % of correct generalizations in the entire study sample (29 participants).

The next section provides a comparative analysis of numeric routines and forms of mediators used in quantitative and qualitative approaches respectively.

5.2.3 LEARNERS' GENERALIZATION ROUTINES

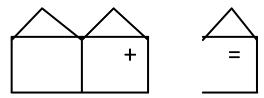
I categorised the learners' interview responses according to their use of routines and mediators. I present this analysis in a table format adapted from Healy & Hoyles (1999) - See table 13. Learners' use of visual mediators is directly linked to the routines they adopt when generalizing from the number pattern in problem solving. An example of the dominant routines and mediators analysed from the learners work is shown in tables 13 and 14: The complete discussion on how the learners' routines relate to their use of visual mediators is represented in discussion section of results in chapter 6.

TABLE 13: SUMMARY OF LEARNERS' ROUTINES VERSUS MEDIATORS IN GENERALIZATION

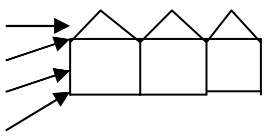
Numeric Generalization Routines	**MEDIATORS**			
	ICONIC	**LEARNERS**	**NUMERIC**	**LEARNERS**
Recursive Rule: counting and guessing	The learner describes a relationship that occurs in the visual relation of the situation between consecutive values of the independent variable. For example, in Q1 each additional house adds 5 line segments and 1 line segment becomes common to two houses.	L5 , L9, L17	The learners notice a number pattern in the results for consecutive values of the dependent variable. For example the number of line segments goes up by 5 each time when number of houses increases by 1 each time.	L5, l9, L15, L17, L23, L28
Recursive Rule: Chunking	A recursive rule is established based on a relationship established in the diagram, adding a unit onto known values of the desired attribute. For example if one cluster house has 6 lines then 5 cluster houses would have 6+5(5) line segments.	L15, L17	The learner builds on a recursive pattern by referring to a table of values, building a unit onto known values of the desired attributes. Example is under iconic representation.	L5, L17, L23
Explicit Rule	An explicit rule is constructed based on a visual representation of the situation by connecting to a counting technique. For example, in Q1 there are 6 line segments for 1st position house and a common difference of 5 for the consecutive terms. So multiplying 5 by position number then adding 1 gives a generalized number of the line segments.	L23, L28, L15	The learners guess an explicit rule based on a numeric pattern in the output values. They reason that since the number of tiles in the first group is 5 and 8 in the second group then multiplying number of tiles' group by 3 and add 2 results in number of tiles. Hence I did not endorse the routine. While others applied proportional routine for example, if group 1 has 5 tiles then group 10 will have 5x10 = 50 tiles	L15, L23, L28

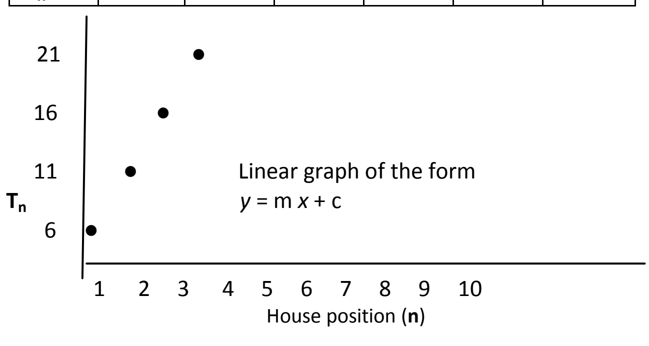
(Adapted from Healy & Hoyles, 1999)

TABLE 14: EXAMPLES OF MEDIATORS USED BY THE PARTICIPANTS

<table>
<tr><th>SYMBOLIC MEDIATOR(S)
i. e. numbers and algebraic representations</th><th>ICONIC MEDIATOR(S)
i.e. numbers, verbal, tables, graphs and diagrams/pictures</th></tr>
<tr><td>Counting:
Learners L5, L9, L17 counted individual pattern items in an unstructured approach.</td><td>Visualization-Eidetic:
L15, L23, L28 mostly focussed on perceptual (visual) pattern analysis rather than on mathematical properties.</td></tr>
<tr><td>Operate on terms:
The learners (L5, L9, L15, L17) carried out their calculations using known terms in the pattern in order to find the target term or number. For example;
6; 11; 16; ....
Thus 16+5 = 21, 21+5= 26, 26+5= 31 etc.</td><td>Combine pictures or diagrams (structural analysis):
Learners chunked the known terms to obtain the next others.


Position 2 position 3</td></tr>
<tr><td>Operate on Difference:
The learners' calculations were based on the numerical difference between any two consecutive terms. An example is very similar to operating on a term or number above.</td><td>Chunking (Inter – term analysis): L15, L17 & L23.
From the diagrams in questions 1, 2 and 3, some learners opted for a different visual analysis. For example, the cluster houses' plan in position 2 contains 9 line segments compared to varying the addition of a house with 1 line segment less than a house at position 1 every time.</td></tr>
<tr><td>Operate on variables:
Learners' calculations based on relationship between dependent and independent variables.
For example, L23 & L28's written response to question 4 part b) is: $T_n = 40 – 5xn$
At position 50, n = 50; $T_{50} = 40 – 5x50$
$= 40 – 250$
$= - 210$</td><td>Chunking (Intra – term analysis):
Partitioning of line segments within a pictorial term in a pattern or sequence.

set of 6 lines
set of 3 lines
set of 4 lines
set of 3 lines

Example: L15, L17, L23</td></tr>
<tr><th>NUMERIC MEDIATOR(S)</th><th>TABLE & GRAPH MEDIATOR(S)</th></tr>
<tr><td>This involves all the routines in which a numeric number structure is used to define a term. An example drawn from L15's work.

<table>
<tr><th>n</th><th>T_n</th><th>T_n</th></tr>
<tr><td>1</td><td>6</td><td>1 + (1x5)</td></tr>
<tr><td>2</td><td>6+5</td><td>1 + (2x5)</td></tr>
<tr><td>3</td><td>6+5+5</td><td>1 + (3x5)</td></tr>
<tr><td>4</td><td>6+5+5+5</td><td>1 + (4x5)</td></tr>
<tr><td>etc.</td><td>.............</td><td>..............</td></tr>
<tr><td></td><td>................</td><td>1+(nx5)</td></tr>
</table></td><td>L19 and 28 analysed question 1 as:
<table>
<tr><th>House(s)/position (n)</th><th>Total line segments (Tn)</th></tr>
<tr><td>1</td><td>6</td></tr>
<tr><td>2</td><td>11</td></tr>
<tr><td>3</td><td>16</td></tr>
<tr><td>etc.</td><td></td></tr>
</table>
L28 constructed table for Question 1 as:
<table>
<tr><th>n</th><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>Etc.</th></tr>
<tr><td>T_n</td><td>6</td><td>11</td><td>16</td><td>21</td><td>26</td><td>......</td></tr>
</table>

</td></tr>
</table>

5.2.4 LEARNER - INTERVIEW ANALYSIS

In the following section I provide a detailed analysis of extracts from six learners' interview transcripts. Only transcripts of questions 1, 2 and 4 are reported and presented in this study. Learners' responses (written & spoken) to question number 3 did not provide rich data to analyse. Hence I do not analyse the response to this question.

5.2.4.1 LOW ABILITY LEARNERS

NOTE: Interpretations in the Transcriptions and Episodes:
R represents the researcher; **L5** represents learner coded as number 5;
[ ] : Pause ; **{ }**: Learner counts or reads loud.

EPISODE 1 FOR L 5

Episode 1_{L5}: Learner 5 response to question cluster houses' pattern (Question 1)

Speaker	What was said	Properties of mathematical discourse with respect to routines, mediators, words and narratives.
3 R	Take me through your written responses, how did you start with the cluster house – problem?	
4 L5	I figured the number of line segments on a given set of cluster houses by counting say at position one, two and so on. And then I realized that the pictures gave a similar number representation to: 6, 11, 17 , ... , by removing one line segment on one house to get to the next and then added five more line segments that's how many more lines I would need any way in the next position. So I did the same thing for all the questions that followed in order to get number of line segments.	*Mediator*: diagrams and numbers *Routine*: counting
5 R	Show me how you determined (calculated) the number of line segments for 10 or 20 houses?	
6 L5	You mean like 2, 3, 4.... Until ten and twenty? Ok, I think I figured out 51 lines in 10 houses and then use this to find the line segments now in 50 houses. So multiplying 51 by 2 gives 102 line segments for 20 cluster houses.	*Routines*: counting and guessing *Mediator*: diagram and numbers
7 R	Would you use a different approach for 50 houses?	
8 L5	Yes, but I think needs more time to find another method. So yeah!,.... You just do the same method as for 10 or 20 houses above.	*Routines*: counting *Mediators*: Numbers
9 R	Say more about that....	
10 L5	I mean if 10 houses = 51 lines, 20 houses = 102 line segments, then 40 houses will give 102 multiply by 2 = 204 lines. So 5o houses will be (10 + 40) houses = (51 + 204) line segments.	*Routines*: Proportioning *Mediator*: diagrams and numbers

EPISODE 2 FOR L 5

Episode 2 $_{L5}$: Learner 5 response to question cluster houses' pattern (Question 2)

Speaker	What was said	Properties of mathematical discourse with respect to routines, mediators and narratives.
61 R	How did you do this question?	
62 L5	I looked at figure … 1 and counted these blocks. The same thing with fig. 2 until figure number 4.	*Mediators*: diagrams *Routines*: counting & chunking
63 R	Why did you have to do that?	
64 L5	I think to look for any pattern that can help in the working this out.	*Mediator*; verbal explanation
65 R	How does that help you in finding the solutions to the problem?	
66 L5	Can find a formula. Here my formula is 3 x figure number plus or minus 2 and can be used to get the amount of blocks in any tile group or figure	*Routines*: structural analysis *Narrative*: formula rejected
67 R	Did you write all what you have said in your written responses?	
68 L5	No, No, No I only wrote answers and not really showing the formula nx3 or 3xn + 2 I think so.	*Narratives* : rejected *Mediator*: symbols
69 R	At first you wrote nx5 as the formula but now you say its nx3+2. Explain this difference.	
70 L5	Yeah, when I looked at figure 1, there are 5 tiles. Then I said this should be 1x5 = 5 where 1 is the figure or tile group number but I think this can be true for group 2.	*Mediators*: diagrams, Verbal explanation and number. *Routines*: structural analysis
71 R	Will this work for all the figures or tile groups?	
72 L5	Yes it does work. This can be by formula calculation or counting the tiles in each group if the pictures are extended……	*Mediators*: verbal explanation *Routines*: structural. analysis
73 R	How would find the group number when you have 218 tiles	
74 L5	Make an equation nx3=218 and then work out the problem for a group position n like in cluster houses' problem.	*Mediator*: symbol and numbers Verbal explanation *Routines*: proportioning *Narratives*: rejected

EPISODE 3 FOR L 5

Episode 3_{L5}: Learner 5 response to questions on numeric pattern (Question 4)

Speaker	What was said	Properties of mathematical discourse with respect to routines, mediators and narratives.
138 R	Can you take me through your response first. And then tell me what you would do if there were 50 numbers and want to find the 50^{th} term or number.	
139 L5	With 50 terms given as in part iii), I would probably, [......], I mean what I could have done is, ..., just look at 5 as for first number and then adding seven to second gives 12 at position 2. I will keep on doing this until I have the 50^{th} term.	*Routines*: counting *Mediator*: number, verbal explanation. *Narratives*: 50^{th} term-endorsed
140 R	Why are you adding seven?	
141 L5	I can see that the numbers in the pattern are changing by seven in 4a and 5 in 4b	*Mediators*: numbers *Routines*: counting
142 R	Okay, are you talking about 4b as well?	
143 L5	Yeah. All these are similar sequences because they use numerical terms and I have to always start from the first number and see how to get the second or third, ... etc.	*Mediators*: numbers
144 R	Will you keep adding the five until 50^{th} term in 4b?	
145 L5	Not really. But on the first question 50^{th} term yes. I will add five but the pattern is decreasing, and it's confusing.	*Routines*: counting
146 R	Explain more about this addition of a number five.	
147 L5	I mean 50^{th} house = 50 + (-5) = 55 + 5 = 55.	*Routines*: counting, add *Mediator*: numbers *Narratives*: rejected
148 R	Why do you have minus five added to 50?	
149 L5	50 is the position of my lager value and minus 5 is what I will add ever time since the pattern is a decreasing one. So in 4a, the 50^{th} term is 50 + (7) = 57.	*Routines*: counting *Mediators*: numbers, verbal explanation. *Narratives*: rejected
150 R	Does this method always work for large values?	
151 L5	I think so. It should work as well for larger position numbers.	*Words*
152 R	What will be the nth term in 4a and 4b?	
153 L5	In question 4a, its n + (7) = 5 + 7 and in question 4b I found it to be n + (-5) = n + 5 or n –5.	*Mediators*: number, symbols *Narratives*: formula rejected *Routine*: counting
154 R	Is the expression (n + 5) the same as (n – 5)?	
155 L5	Yes, n stands for position of the term am looking for and 5 or minus 5 is just what I need to keep adding in order to get the next terms.	*Routines*: recursive counting *Mediator*: verbal explanation *Narrative*: rejected

5.2.4.1 (i) LEARNER L5: RESEARCHER'S COMMENTS

The analysis of episodes 1 - 3 shows that different types of visual mediators were used by learner L5 to communicate her ideas. These mediators include numbers, diagrams, algebraic representation and verbal explanations. The learner 's use of the mediators provided a means to understand the thinking process the learner engaged in when generalizing from the number pattern task.

Learner L5 mostly used numbers as a form of visual mediators to communicate her thinking and ideas. Such an approach resulted to numeric generalization (see episodes 1, lines 4, 6, 10 and episode 3, lines 147-9). On the other hand the learner used both verbal and algebraic representations to explain and communicate her ideas respectively (for example, see line 66 of episode 2, line 139 of episode 3). The algebraic expressions the learner is using are incorrect narratives as they do not generalize the number patterns in context (see line 153 of episode 3). The numeric representations were derived from diagrams drawn by the learner to extend a given geometric pattern before analyzing. The use of diagrams or pictures in this case provided access to numerical generalization.

A noteworthy finding in learner L5's use of verbal explanation is a mismatch of the numeric and algebraic representations when establishing a generalization. Also the analyses show that her use of the verbal explanation helped in supporting as well as interpreting the use of other mediators when representing their generalization. Learner L5 used recursive methods when generalizing from the number pattern task. These methods include counting and guessing and fall under numeric generalization category (episode 1, line 10; episode 3, line 147). This learner applied the incorrect narratives in most of her responses (see the n^{th} term formulae, episodes 2 & 3, lines 74, 135 respectively). The learner has misconceptions about the formulae that can be used to derive general rules for any given pattern. This is manifested in the way she incorrectly situated the operational signs + or/and – among the algebraic symbols (see line 74 of episode 2).

EPISODE 4 FOR L 9

Episode 4_{L9}: Learner response to question on cluster houses' pattern (Question 1)

Speaker	What was said	Properties of mathematical discourse with respect to routines, mediators and narratives
1 R	Tell me how you responded to this question.	
2 L9	When I read the question I…, found it is asking how many line segments are needed to construct 4, 10 or even 50 houses. And I think I answered these questions after observing what was going on with given pictures of cluster houses.	*Routines*: structural analysis *Mediators*: numbers, diagrams and verbal explanation
3 R	What did you find or see?	
4 L9	I found, […….], but I think these were incorrect. Am supposed to work out the formula first.	*Mediator*: verbal explanation
5 R	Do you mean you had to use a different method?	
6 L9	Yes	
7 R	Explain the method please.	
8 L9	I just ,…. I think counted the lines in 4 and 10 houses but I guessed for 50 houses	*Mediators*: numbers and diagrams *Routines*: counting and guess
9 R	Can you think of any formula or rule to help do this?	
10 L9	In grade 8 we learnt the formula Tn = a -(n-1)d at maths clinic but am not sure if the bracket has – or + ,…. may be its Tn = a-(n+1)d or a(n-1)d.	*Routines*: explicit *Narratives*: rejected
11 R	How do you know that you really have to use these expressions?	
12 L9	This is like a basic formula for finding equations for number patterns.	*Mediator*: verbal explanation *Narratives*: rejected
13 R	Explain more about your formula then and do you use it.	
14 L9	The symbols have each a meaning. For example, 'a' is first term, 'd' is the difference and 'n' is position of the term in a pattern. Knowing this expression helps in finding any pattern's general rule.	*Narratives*: symbol meaning endorsed *Mediators*: symbol, formula *Routines*: structural analysis

EPISODE 5 FOR L 9

Episode 5_{L9}: Learner response to question on Tiles pattern (Question2)

Speaker	What was said	Properties of mathematical discourse with respect to routines, mediators and narratives
31 R	Can you write an expression or formula for T_n?	
32 L9	Am not really sure what T_n is.	*Mediator*: symbols
33 R	It is a way to refer to nth term or number of tiles in this tiles pattern problem. See, we called our first number T_1, and T_2 the second number etc., so we can call the 50^{th} number T_{50}, and n can be any number.	*Mediator:* verbal explanation symbols *Narratives*: endorsed the explanation
34 L9	Oh! I see, so I think the number will..... Yeah! ...., depend on the number.	
35 R	What expression did you write for T_{50} or T_n?	
36 L9	I first counted all the tiles in the groups as drawn in the question here. So got 5, 8, 11, 14, and it means T_{50} = 5+8+11+14+17+ 20+ .... + Until 50 times.	*Mediators*: numbers *Routines*: counting, chunking
37 R	Is that for T_{50} only?	
38 L9	Yes, but for Tn I'm not sure. Or do you mean n+5 times n+8?	*Routines*: counting *Narratives*: rejected
39 R	How do you express your observation of the pattern in words?	
40 L9	I see first 5 tiles in figure 1, 8 tiles in figure 2 and then 11 in figure 3 etc.	*Mediators*: diagrams, verbal explanation
41 R	So what mathematical rule or formula can you come up with what you have said?	
42 L9	You mean a formula ......, I think its some number or numbers times 3 plus 2 or 3 minus 2 every time.	*Narratives*: rejected *Routines*: recursive and explicit

EPISODE 6 FOR L 9

Episode 6_{L9}: Learner 9 response to questions on numeric pattern (Question 4)

Speaker	What was said	Properties of mathematical discourse with respect to routines, mediators and narratives
94 R	How did you find the 10th, 20th and 50th numbers in question 4?	
95 L9	I extended the numbers in the pattern.	*Routine*: counting & extending
96 R	Ok. But how did you know that at position six the number is 40?	
97 L9	You see here ...., (pointing a pattern), from first number 5 to next 12 we add 7, and from 12 to 19 we do the same thing. So every time I add a seven gives me the next number *x.	*Mediators*: numbers, verbal explanation *Routines*: counting
98 R	So how then, did you find 20th or 50th numbers?	
99 L9	I wrote down all the numbers until twenty and the same thing until I have fifty of them. For example; it's easy once you know what to add: 5, 12, 19, 26, 33, 40, 47, 54, 61, 68, 75, 82, 89, 96, 103, 110, 117, 124, 131, 138, 145, 152, ..., etc.	*Routines*: counting, guessing *Narratives*: endorsed *Mediators*: numbers
100 R	How do you explain in your own words the general rule for this pattern, I mean the nth term expression?	
101 L9	I think it is; if you have a number or term, every time you times it by 7 then you still need to add or minus something else. So here,......, I need to minus 2.	*Routine*: recursive and explicit *Narrative*: endorsed *Mediator*: numbers and verbal
102 R	Will that always work with a number pattern like this?	
103 L9	I think so, [......], but I'm not very sure.	*Narrative*: rejected
104 R	What will be your formula for the n^{th} term?	
105 L9	This one is difficult for me because of n. But if I know the number represented by n, I can try to think........ ..	*Narrative*: rejected

5.2.4.1 (ii) LEARNER L9: RESEARCHER'S COMMENTS

Learner L9 appeared to have used verbal explanation, numbers, algebraic expressions and numbers as forms of visual mediators to communicate her thinking as she responded to the number pattern task. The verbal explanation and number representation appeared most frequently in her dialogues throughout the task.

Algebraic symbols are visual mediators which participants of a mathematical discourse use to access and communicate their thinking. In her approach to derive an algebraic representation for the $n^{\mathbf{th}}$ term, she uses her prior knowledge on patterns especially the use of the general formula to derive a new generalization (episode 4, line 10). The use of algebraic symbolism or representations proved to be the most challenging for most of the participants in the study. For example, the results from the L5 and L9 analyses show that none were successful in using algebraic symbols to mediate their communication when generalizing from numeric and geometric patterns in the task. Learner L9 is not very sure of the operational signs in the formula and she ended up using the incorrect expression which I rejected (un endorsed narrative – Episode 4, line 10). Learner L9 appeared to grapple with what T_n means when asked to deduce or generalize the pattern in terms of the nth term (episode 5 lines 36-9).

The narratives L9 is using can be seen during manipulation of numbers and other forms of representations (mediators). The incorrect recall of previously learned concepts in grade 8 is an indication of some memorization of mathematical rules with limited understanding resulting in incorrect narratives. The learner failed to verbally explain and interpret the formula for the nth term (Episode 6, Line 101). The learner has misconceptions about the formulae that can be used to derive general rules for any given pattern. This is manifested in the way she incorrectly situated the operational signs + or/and – among the algebraic symbols (see line 10 of episode 4).

From the dialogue with learner L9, the analysis shows that she mostly used recursive methods (routines) under numeric generalization. These include, counting, chunking, structural analysis and guessing (see lines 2, 8, 36, 97 of episodes 4, 5 and 6 respectively).

5.2.4.2 MEDIUM ABILITY LEARNERS

EPISODE 13 FOR L 15

Episode 13_{L15}: Learner 15 response to questions on cluster houses' pattern (Question 1)

Speaker	What was said	Properties of mathematical discourse with respect to routines, mediators and narratives
23 R	Take me through the rules you generated in questions 1, 3 and 4.	
24 L15	In question 1, I have two equivalent rule and these are: $h_n = 5(n)+1$ and $(1- x) + 6x = y$.	*Narratives*: endorsed *Mediator*: symbol, verbal explanation
25 R	What do you call the letters as used in the rules above?	
26 L15	Va*riables I think*. But in this case, I will call them *unknowns* because they can represent any number.	*Words:* variables and unknowns
27 R	How would find the position if you are given a term in a sequence?	
28 L15	Ok. You mean if there are 103 line segments.	*Mediator*: number
29 R	Yes. How do you find or work out the position.	
30 L15	I will make $y = 103$ and then solve for x.	*Routines*: solve equation *Mediator*: symbols, numbers
31 R	What does that give you?	
32 L15	Its $(1-x) + 6x = 103$, $1+ 5x =103$, $5x = 103 – 1 = 102$ hence $5x÷5 = 102÷5$, $x = 20,4$	*Routines*: solve equation, chunking *Narratives*: endorse formula, reject interpretation *Mediator*: symbols, numbers
33 R	What does $x = 20,4$ mean to you?	
34 L15	It's the answer to the problem when there are 103 line segments.	*Narratives*: rejected *Mediator*: verbal explanation.

EPISODE 14 FOR L 15

Episode 14_{L15}: Learner15 response to questions on numeric pattern (Question 2)

Speaker	What was said	Properties of the Mathematical discourse with respect to routines, mediators and narratives
29 R	Explain your approach to this question.	
30 L15	[….], then she reads the entire Q.	
31 R	What do you notice?	
32 L15	From figures 1-4, I see already a pattern such as 4; 6; 8 ; ..... for grey tiles and also another for white tiles. You multiply the figure number by 2 in order to get grey tiles and sometimes use of a formula would be suitable in this case.	*Mediators*: diagram, numbers *Routines*: proportioning
35 R	What formulae did you generate?	
36 L15	G_n = n x 2: Grey tiles and G_n = n + 2 : White tiles where G_n is number of tiles per group and n is the group number or position.	*Narratives*: endorse formula *Mediator*: symbols *Routine*: structural analysis
37 R	How many tiles did you find in group 10?	
42 L15	This is what I did: Group 10 = G_n (w) + G_n (g) = nx2 + n+2 = 10x2 + 10+2 = 20 + 12 = 32	*Routines*: explicit *Mediator*: symbol, number *Narratives*: endorsed
43 R	Is it possible to use one formula?	
44 L15	Yes it is possible.	
45 R	How do you do that?	
46 L15	Combine the formulae above and simplify, I mean Gn(w) and Gn(g)	*Routines*: explicit, structural *Mediator*: verbal explanation
47 R	Say more......	
48 L15	If Gn = nx2 and Gn = n = 2 then G_{total} = (nx2) + (n+2) = 3n +2	*Routines*: explicit, structural *Narratives*: endorsed formulae *Mediator*: symbols

EPISODE 15 FOR L 15

Episode 15_{L15}: Learner 15 response to questions on cluster houses' pattern (Question 4)

Speaker	What was said	Properties of mathematical discourse with respect to routines, mediators and narratives
121 R	Take me through how you did this question.	
122 L15	Ok. Here it is slightly different from the previous questions. This is straight forward and easy because we have numeric patterns in a) as well as in b).	*Mediators*: verbal explanation
123 R	How did you find the next three terms?	
124 L15	There is a common difference of 7 in part a) so its easy to see what formula is needed. But also one can only continue the pattern by adding 7 every time to get the next number. Thus 5; 12; 19; 26; 33; 40 and 35; 30; 25; 20; 15 ; 10 respectively.	*Routines*: counting *Mediator*: numbers, verbal explanation
131 R	What were your 10^{th} and 50^{th} terms in these patterns?	
132 L15	First I had to find a formula or rule for each pattern.	*Mediator:* verbal explanation
133 R	How did you come up with the formula?	
134 L15	The common difference was 7, so I had to put 7 aside and then multiply the numbers for the position of the number by 7 and had to either add or subtract a certain number for you to get to the number you are looking for.	*Routines*: counting, chunking. *Mediator*: verbal explanation
139 R	So what formulae did you generate?	
140 L15	Tn = 7xn – 2 in 4a) and Tn = 40 – 5xn in 4b). I got these expressions using the technique above.	*Routines*: explicit *Narratives*: endorsed *Mediator:* symbols

5.2.4.2 (i) LEARNER L15: RESEARCHER'S COMMENTS

Learner L15 initially used recursive methods under numeric generalization routine in questions 1 to 3. She developed a chunking approach from the recursive approach. This gave way to explicit rules such as $h_n = 5(n) + 1$ or/and $(1- x) + 6x = y$ for question 1 where n, x are positions of terms, h_n and y are the terms in the pattern (Episode 13, Line 24). The learner relied on number and algebraic representations to mediate her communication and thinking (Episode 13, Line 32, episode 14, lines 32, 36, 42 and 48). The numbers as a type of mediators were derived from the contextual (diagrams) representation of the patterns in questions 1 to 3 (Episode 14, Line 32). Verbal explanation became the main tool for providing and justifying mathematical validity of the generated rules (see episode 13 lines 26-30).

In an attempt to determine the position of the term, L15 carried out the correct manipulation procedure but failed to validate her solution (Episode 13, Line 32-34). When she equated the number 103 to the rule (1-x) + 6x her solution for x was 20,4; this did not matter really apart from being the answer to the question. In the context, the symbol x represented the position of cluster houses. As such this should be a whole number and not a decimal number. The learner's algebraic manipulation resulted in an incorrect narrative (solution when working backwards see Episode 13 line 33.

The learner is able to manipulate/work out the value of a term or its position but does not seem to fully understand the meaning of the solutions in terms of the context of the problem.

The analyses of L15's interview responses from the three extracts leads me to infer the following:

- L15 demonstrated appropriate mathematical computational skills as required by the task.
- L15 mostly approached the questions using numeric analysis and reasoned deductively when it came to generating a rule. She was able to use mathematical notation and variable representations adequately.
- L15 communicated mathematically although she could not always provide detailed explanations to justify her routines and solutions.
- L15 reasoned deductively in her routines and relied on concrete examples to support her ideas and thinking.
- She also reasoned proportionally when extending patterns.

EPISODE 16 FOR L 17

Episode 16_{L17}: Learner17 response to questions on cluster houses' pattern (Question 1)

Speaker	What was said	**Properties of mathematical discourse with respect to routines, mediators and narratives**
3 R	Take me through your responses in Question 1	
4 L17	Ok Sir. I looked at the paper and for the 1^{st} house I counted with my fingers all the line segments so I got 6; 11; 16;	*Mediators*: diagrams, numbers *Routines*: structural analysis
5 R	What did you do next.	
6 L17	I made up a formula when I saw there is a pattern for the number of line segments.	*Mediator*: verbal explanation
7 R	Explain more ……	
8 L17	My formula is Tn = 6 + (n-1)x5	*Narratives*: formula endorsed *Mediators*: symbols
9 R	What do the symbols Tn and n stand for in your formula?	
10 L17	Tn is the term, 6 is the first term in the pattern, n-1 is the position and 5 is the common difference among the numbers in a pattern.	*Mediators*: number & symbol
11 R	How many line segments are in 4 houses.	
12 L17	From my formula Tn = 6 + (n-1)x5 will be T_4 = 6 + (4-1)x5 = 21	*Narratives*: formula endorsed *Routines:* solve equation *Mediator*: symbol
25 R	Say more, ……..	
26 L17	I mean, you will have to extend the pattern that long or else draw the pictures of the cluster house further and this takes too much time.	*Routines*: counting, extending *Mediators:* diagrams
27 R	Is taking time a problem....?	
28 L17	Yes. You see from 6; 11; 16; 21;..... carries on. And once you have a position number 251 then it means you write down all the numbers until the pattern reaches 151. Its going to be very difficult ... but as long as you understand it in that way......	*Routines*: counting *Mediator*: numbers
29 R	What about number of line segments in n houses?	
30 L17	This question is asking the nth term, and my formula with n in it is the answer. So Tn = 6 + (n-1)x5 represents line segments in n houses.	*Narratives*: endorsed *Mediator*: symbols

EPISODE 17 FOR L 17

Episode 17_{L17}: Learner 17 response to questions on tiles pattern (Question 2)

Speaker	What was said	Properties of mathematical discourse with respect to routines, mediators and narratives
31 R	Take me through your response to this question.	
32 L17	Ok. fig 1 has 5 tiles; fig 2 has 8 tiles and fig 3 has 11 tiles I did the counting using my fingers to see if there is a pattern such as this that every time I go to next figure the common difference was 3.	*Mediators:* number & picture *Routines* : counting, structural analysis
33 R	How did this help you?	
34 L17	Yes Sir, like in fig 5 I just used my fingers because adding 3 always say from fig 3 = 11; then I knew there will be 15 tiles..... [ ]. No! Its .... { counts fingers 11, 12 ,13 , 14}. Yes its 14 Sir. And then from there I just kept on adding 3 this time I had not yet written down the formula.	*Routines*: Recursive, explicit , structural analysis
35 R	Describe the pattern briefly.	
36 17	You see... the horizontal tiles at the bottom are 7... each time they increase 1 more tile at the base and the columns each time also increase by 1 tile. So when the base is 7 the top tiles are 5. This look like 3x1+2; 3x2+2; 3x3+2 e.	*Mediators*: diagram, numbers *Routines*: counting, chunking, explicit
37 R	How many tiles did you find for 6^{th} and 7^{th} groups?	
38 L17	20 and 23 tiles	*Mediators* : number, pictures
39 R	Explain more, …….	
40 L17	I fig 5there are 17 tiles, then its 17 +3 and 20+3 in the next two fig.	*Routines*: counting, explicit, structural analysis *mediator*: numbers

EPISODE 18 FOR L 17

Episode 18_{L17}: Learner 17 response to questions on numeric pattern (Question4)

Speaker	What was said	Properties of mathematical discourse with respect to routines, mediators and narratives
87 R	How did you determine the next three terms?	
88 L17	I used a formula.	*Narrative* – formula endorsed
89 R	Which formula is it?	
90 L17	$T_n = a + (n-1)xd$; $T_n = 35 + (n-1)x-5$; this simplifies to $T_n = 40 - 5n$.	*Narratives* : formulae endorsed *Mediators* : Symbols
91 R	What was the formula for 4a?	
92 L17	Must I write it also.	
93 R	Yes.	
94 L17	Ok. Its $T_n = a + (n-1)xd$ where a = 5 and d = 7 then $T_n = 7n-2$.	*Narratives* : formula endorsed *Mediators*: numbers and symbols
95 R	Would you generate the formulae without starting from $T_n = a + (n-1)d$?	
96 L17	This is a basic expression and one needs to know otherwise you will have difficulties.	*Mediator*: verbal explanation
97 R	Where is this basic formula or expression coming from?	
98 L17	My mathematics teacher at grade 8 taught me and also at maths extra lessons, I think we did something like this.	*Routines* :explicit *Narratives* : formula *Mediator* : symbols
99 R	What if you had forgotten?	
100 L17	I don’t think it would be easy to work out these numbers.	*Mediator:* verbal explanation

5.2.4.2 (ii) LEARNER L17: RESEARCHER'S COMMENTS

Learner L17 demonstrated recursive and partly explicit methods when answering the questions in the pattern task. The use of fingers when counting the number of line segments in the position cluster houses resulted in a numeric representation of the pattern (episode 16, line 4; episode 17, line 34). Using prior (see episode 18, line 98) knowledge L17 decided to generate rules or general expressions for the n^{th} term from the formula $T_n = a + (n-1)*d$ (see lines 95-6 of episode 18).

The use of previous knowledge became key for L17 in her response to the pattern tasks. The rules (algebraic) generated in each question were correct (i.e. endorsed narratives). For example; in episode 18 lines 96-100, the learner (L17) believed that one could not find solutions to these pattern questions if s/he does not know or has forgotten some rule that their grade 8 mathematics teachers taught them.

The learner used numbers, verbal explanations, diagrams and formula (algebraic) representations as mediators to communicate her ideas and thinking (for example; episode 16 lines 4, 8, 12, 28 and 30). L17 seemed to be more concerned with the use of recursive methods (see episode 16 lines 26, 28 and episode 17 line 34) in each pattern context and quickly jumps to an explicit rule learnt in the previous grade 8 and at extra mathematics lessons (Episode 18, lines 90 and 96). The learner's response in episode 18, line 100 seems to indicate that he was concerned about how much time it would take to think about other approaches to calculate larger terms in the patterns and not necessarily to what extent the derived rule (generalization) could generate correct values. The analyses of L17's interview responses from the three extracts leads me to infer the following:

- L1 demonstrated partial understanding of necessary computational skills as required by the task.
- L15 mostly approached the questions using numeric analysis to find a few terms. He was able to use mathematical notation and variable representations to communicate his thinking.
- L15 communicated mathematically although, he could not frequently provide detailed explanations to justify her routines and solutions.

5.2.4.3 HIGH ABILITY LEARNERS

EPISODE 7 FOR L 23

Episode 7_{L23}: Learner 23 response to question cluster houses' pattern (Question 1)

Speaker	What was said	Properties of mathematical discourse with respect to routines, mediators and narratives
9 R	I see you read there are three houses at position 3. So, can you please explain why you are multiplying by the number of houses or house position by 5?	
10 L23	If you look at the first house, the number of line segments can be expressed as 5 x 1 plus some number. Similarly, for two houses, its 5 x 2 plus the same number am adding in the first picture. So the numeric pattern constructed from the pictorial representation 6; 11; 16; 21; ...... ,can be expressed as 5x1 + (1); 5x2 + (1); 5x3 + (1); 5x4 + (1); where the number added each time is 1.	*Routines*: verbal explanation, structural anal and counting *Narratives*: endorsed *Mediators*: numbers
11 R	I would like to understand your method here. Why are you adding 1?	
12 L23	Because, in the first house there 1, 2, 3, 4, 5, and 6 (pointing at the picture) line segments use. So I figured out that 5 x 1 house gives 5 lines but I have counted 6, this means my answer is one line missing. By adding 1 to 5 makes it 6 line segments altogether. So my expression looks like 5x1 + 1. And this seem to generate a rule like Tn = 5 x n + 1 where Tn is term or line segments and n is position of cluster houses.	*Mediators:* diagrams, numbers, symbols. *Narratives:* endorsed *Routines*: structural anal and counting
89 R	From question 1, Can you tell me how you would determine the position number when you are given 201 line segments?	
90 L23	I will first write down my formula Tn = 5n +1. And if 201 is given as number of line segment then my Tn = 201 so I just can calculate n.	*Narrative*: endorsed *Routines*: solve equation *Mediator*: symbol, number
91 R	How would you really do that? Show me please.	
92 L23	Solving a formula above for n; so 5n + 1 = 201 after substitution. Therefore 5n = 201 – 1 and n = 40, the position with 40 cluster houses containing 210 line segments.	*Routines*: counting, solve equation *Mediator*: symbol, number
93 R	Will this always work?	
94 L23	Yes	*Narrative:* needs clarity hence rejected
95 R	Explain to me more as to how will this workout ...	
96 L23	I did check some few more examples by means of substitution methods in the Tn expression above.	*Narratives*: endorsed
97 R	How many examples are enough for checking if the rule is correct?	
98 L23	I think, [….] two or three are enough?	*Narrative*: reject
99 R	Explain what you are going to do in order to determine position for 21 houses.	
100 L23	Similarly, 21 = 5 x n + 1. And then 5 times n is 20. Therefore position n = 4. ...., by solving an equation in the form of Tn expression.	*Routines*: solve equation *Narratives*: endorsed

EPISODE 8 FOR L 23

Episode 8_{L23}: Learner 23 response to questions on tiles pattern (Question 2)

Speaker	What was said	Properties of mathematical discourse with respect to routines, mediators and narratives
35 R	Explain how you did this question.	
36 L23	In the question, they have given us the tiles in different groups or positions and want us to find number of tiles in certain groups which are not represented in the diagrams here….	*Mediators*: diagrams, verbal explanation
37 R	How then did you do that?	
38 L23	In fig 1 there are 5 tiles; fig 2 has 8; fig 3 = 11 and fig 4 = 14…., {he redraws the tiles and count aloud}…, so I see the pattern as: 5; 8; 11; 14; ……………	*Mediators*: diagrams, numbers *Routines*: structural analysis, counting.
39 R	What does this mean to you?	
40 L23	It's a pattern with the common difference of 3 between any two terms or numbers.	*Routine*: counting, difference method.
41 R	Can this pattern carry on?	
42 L23	Yes it will and its probably a direct proportion	*Narrative*: rejected
43 R	How many tiles were in group 5?	
44 L23	I said T_n=3xn+2 and since its starting at 5 tiles and a difference of 3 then got this formula.	*Narratives*: formula endorsed *Mediator:* symbols, numbers *Routine*: structural analysis
45 R	What the Tn expression for?	
46 L23	It is a formula that generates the pattern 5; 8; 11; 14 ; ……..So in group 5, there are 17 tiles.	*Narratives*: formula endorsed *Mediator*: numbers
47 R	In which group can you find 29 tiles?	
48 L23	I would use the formula or carry on extending the pattern until I reach 29 tiles.	*Routines:* counting, extending. *Mediator:* verbal explanation, and numbers
49 R	Explain more on this…..	
50 L23	Otherwise I would work out using reverse or backward like I did with question 1. Only that this time am using a different equation T_n = 3xn+2. So Tn = 29 tiles meaning that 29 = 3xn + 2, working out for n gives n=9	*Routines:* reversibility, solve equation *Narratives*: endorsed *Mediators*: symbols
51 R	What does 9 mean or stand for?	
52 L23	Tile group number or position for 29 tiles.	*Narrative*: endorsed

EPISODE 9 FOR L 23

Episode 9_{L23}: Learner 23 response to questions on numerical pattern (Question 4)

Speaker	What was said	Properties of mathematical discourse with respect to routines, mediators and narratives
91 R	How did you do question 4?	
92 L23	In 4a, I found there is 7 as a common difference between any two terms and chose to continue this pattern using repetitive addition of 7. The next three terms were: 26; 33 and 40. For part 4b) I realized that the terms in the sequence are decreasing by 5 hence the difference became minus 5. Hence the next three terms were 20; 15; 10.	*Routines*: counting, difference method *Mediator*: verbal explanation & numbers *Narrative:* endorsed
93 R	Did you solve these problems differently?	
94 L23	Yes I also used the formula to check if my solutions are valid.	*Narratives*: endorse use of formula *Routine*: Solve to justify
95 R	What formulae are these and how are you generating them?	
96 L23	Like in questions 1 and 2 or generating the rules using Tn = a+(n-1)d expression where “a” is first term, “n” is position and “d” is the difference value between terms in the pattern. For part a) Tn = 7xn -2 and in b) Tn = -5xn + 40	*Narratives*: formula endorsed *Routines*: explicit methods *Mediator*: symbols
97 R	How do you know that these are correct formulae?	
98 L23	Again these are like generalized expressions or rules, which mean they work in most cases.	*Narratives*: endorse generalized *Mediator:* verbal explanation
99 R	What are the terms on 10^{th} and 50^{th} positions?	
100 L23	Substituting n=10 or 50 in the above formulae I got 68 and 348 for question 4a whereas for n= 10 or 50 gives – 10 and – 210 for question 4b respectively.	*Narratives*: endorsed *Routines*: solve equation *Mediator:* symbols, numbers
101 R	Does this approach always work?	
102 L23	Yes, it does work mostly.	
103 R	What terms did you get for the nth positions in these two patterns?	
104 L23	The nth term is given in each case by T_n expression always and this means in 4a, T_n=7xn-2 and in 4b, T_n= -5xn+40 respectively.	*Narratives*: endorse the n^{th} rule *Mediator*: Symbols *Routines*: recursive and explicit
105 R	What position would you get – 500 in 4b)?	
106 L23	That will be 2500 + …, [….], If I substitute T_n = -500 into the expression or rule will be ,….. -5(-500) +40.	*Routines*: solve equation *Mediator:* numbers and symbol *Narrative*: endorsed formulae
107 R	Will that give you position?	
108 L23	No! No! No! that will be like am solving for the term. I think we have to work backwards since we know the term this time and not position.	*Routines*: solving equation *Mediator*: verbal explanation
109 R	So what will that position will that be?	
110 L23	Since T_n= -5xn +40 then – 500 = - 5xn +40 which is -500-40 = - 5n and this gives – 540/- 5 = 108. Therefore the position is 108.	*Narratives*: endorsed *Mediator*: symbols, and numbers *Routines:* solve equation

5.2.4.3 (i) LEARNER L23: RESEARCHER'S COMMENTS

Learner L23 used different types of visual mediators when communicating his thinking while generalizing from the numeric and geometric pattern task. The mediators which L23 used include numbers, picture/diagrams, algebraic symbolism and verbal explanations (for example; episode 8, lines 92, 96, 100 and episode 9, lines 38, 44, 50). Learner L23 employed all three categories of generalization routines when generalizing from the number pattern task. At any stage the learner was able switch from recursive to explicit methods or proportion and relational approach. The recursive and explicit approaches within numeric generalization dominated in both his algebraic representations and verbal explanations (See episode 7 lines 10 & 90, episode 8 lines 38, 40, 44 & 92). Learner L23's description of question 1 (Episode 7, Line 10) indicates that in order to get the next number of line segments, a number is added to the previous set or position of cluster houses. In the next step (Episode 7 Line 12) he justifies the need to combine the relationship between the position number and difference in line segments between position houses. Eventually, Learner L23 manages to generate the rules for finding larger terms and, in particular, nth terms (episode 7, line 1; episode 8, line 44, and episode 9, line 96). It is worth noting that the learner further indicates that these rules can also be derived from the general expression $T_n = a + (n - 1)\, d$ in the case of linear numeric or geometric patterns (episode 9, line 96).

In moving from picture or diagram representations into numeric pattern representations for example in questions 1 to 3 (Episode 7, Line 12 and episode 8, line 36), a figural generalization routine was used. And it is the use of the two main routines, numeric and figural, that counts as pragmatic. Learner L23's use of mediators and routines led to mathematically valid endorsed narratives which resulted in correct generalization.

Reflecting on the analyses of L23's interview responses from the three extracts above, I infer that L23 has demonstrated the following:

- L23 has a strong foundation of mathematical knowledge such as basic skills, algorithms, conceptual knowledge.
- L23 can abstract mathematical ideas and representations. For example, he was able to demonstrate abstract reasoning and used variables correctly to represent different mathematical terms.
- L23 is able to reason inductively by recognizing, extending and generalizing patterns. He is able to conjecture and justify his explanations. Inductive reasoning is when a sequence of individual information is generalized into a conclusion that is related to

those pieces of information. Thus by recognizing the trend one should have the ability to draw conclusion about the next term in a sequence.

- L23 communicated mathematical ideas in a logical manner. He could use the mathematical notations in multiple ways.

EPISODE 10 FOR L 28

Episode 10 $_{L28}$: Learner 28 response to question cluster houses' pattern (Question 1)

Speaker	What was said	Properties of mathematical discourse with respect to routines, mediators and narratives
11 R	Why are you working with house-positions and the number of line segments?	
12 L28	First I changed the diagrams given into numbers. So these are now terms each on its specific position for example, when n=1 T_n= 6; n=2, T_n=11 and n=3, T_n=16 .... And working as we do with linear graph or equation n=x and Tn=y. I found that gradient = 5 and substituting this into $y = mx+c$ when n=1 (x), Tn =6 (y) then c=1.	*Mediators*: diagram, graph, tables, numbers and symbols *Routines*: counting, chunking & explicit. *Narratives*: endorse formulae
13 R	Explain more on how you used the values from these calculations.	
14 L28	In this situation, it means $y = mx + c$ is the same as Tn = 5n + 1.	*Narratives*: endorsed *Mediator*: symbols
15 R	Are the two expressions equal?	
16 L28	I would not say it in that way. I think the "same" here means the give the same line graph on the Cartesian plane.	*Mediator*: verbal explanation
17 R	So can you please explain how you used your general rule Tn = 5n+1?	
18 L28	I counted the line segments at position n = say, 1, 2, 3, 4, lines in a rectangle and 5, 6 extra two lines that are making the roof of the house. This help to make sure if my formula is correct (justification). I did the same for position n=2.	*Mediator*: diagram, numbers. *Routines:* structural analysis *Narrative*: endorse justification
19 R	Okay, how many lines are at position 2?	
20 L28	Three verticals and four horizontal lines plus another four as the roofs making eleven line segments by addition.	*Mediators*: diagrams *Routine*: structural analysis
21 R	Is 11 the number of line segments for two houses?	
22 L28	Yes, I checked this by substituting into the formula as well.	*Routines*: solve by substitution
23 R	Did you follow this procedure for 50 houses or had used a different one?	
24 L28	No, 50 is a big number. I just used my formula without checking it because this worked for n=1 & 2. A formula is very good for working with large numbers since you cannot draw 50 pictures of houses in order to count the number of line segments in them.	*Narratives*: reject *Routines*: solve by substitution *Mediators:* symbol and number
25 R	If you consider the numeric pattern of n and Tn above, why is the number of line segments going up by 5?	
26 L28	Every time the position number increases by one, you are actually adding a house by five lines more than the previous. For example if I add a house on n=3, this last line (pointing to the picture) will be shared with the extra house so I just need 5 lines to complete the diagram or picture.	*Routines*: structural analysis *Mediators*: diagrams, symbol numbers and verbal explanation.

EPISODE 11 FOR L 28

Episode 11_{L28}: Learner 28 response to questions on tiles pattern (Question 2)

Speaker	What was said	Properties of mathematical discourse with respect to routines, mediators and narratives
55 R	Explain you method how did you get 17 tiles?	
56 L28	This is what I did. You see, … every time you put 1 tile on the base, you also put two tiles one on side or ends of the group, yeah!	*Routines*: structural analysis *Mediator*: diagrams and verbal explanation.
57 R	Do you mean a tile on each end of the base tiles?	
58 L28	No! you put here…. on [ ] I mean the columns. Like for this tile on the base you need to add one more on each side …. Also one here and one here.	*Mediators*: verbal explanation diagrams *Routines*: structural analysis
59 R	How many tile are there in group 6?	
60 L28	Sir, I came up with a formula to work out this apart from counting the tiles one by one.	*Narratives*: use of formula endorsed. *Routines*: diagram and structural analysis
61 R	What formula is it?	
62 L28	It's Tn = 3xn + 2	*Narratives*: endorse formula *Mediator*: symbols
63 R	How did you formulate this 3xn + 2 expressions?	
64 L28	The difference between the tiles from one group to another is 3, and then you always add 2 on the columns that is why I have plus two in the formula.	*Routines*: structural analysis *Mediator*: verbal explanation
65 R	Do you always have to add two?	
66 L28	In this question yes. You Add 0ne tile here,,, [ ] and one tile here also,.. [ ]. So its always two tiles more. This is like 3x1+2; 3x2+2; 3x3+2; 3x4+2; e.t.c. which gives 5; 8; 11; 14 as a pattern anyway.	*Mediators*: diagram, number *Routines*: structural analysis, counting and explicit
67 R	Can this approach work when finding number of tiles in the 7^{th} group and any other group?	
68 L28	Yes because the formula generalizes the numbers in this tiles pattern	*Narratives*: endorsed
81 R	How would work out the 50^{th} group of tiles?	
82 L28	I will draw extending the given pattern as from the first four pictures. But sir, I think it's a waste of time. Imagine the position or the group is 5000 eeeeish!	*Mediators*: diagram, numbers *Routines*: counting, struct. A
83 R	Explain more why you say it's a waste of time?	
84 L28	Because there are lots of tiles to draw. It is better to use a formula Tn = 5000 = 3xn + 2 and solve the equation backwards for n.	*Narratives*: endorse formula *Routines:* solving equation

EPISODE 12 FOR L28

Episode 12_{L28}: Learner 28 response to questions on numeric pattern (Question 4)

Speaker	What was said	Properties of mathematical discourse with respect to routines, mediators and narratives
155 R	Do you really understand the rules you have generated:	
156 L28	Yes I do understand this work and all the formulas.	*Narrative*: understanding
157 R	Explain in detail to me especially the letters you are using.	
158 L28	Ok. In question 1, the formula I found is $T_n = 5n + 1$ from the general linear expression y = mx + c or alternatively $T_n = a+(n-)*d$ *which I learnt from maths clinic*. T_n is a symbol representing a "term" or "number" and *n* is the position of any number in a sequence and d id the common difference between terms in the pattern.	*Narratives*: endorse formulae *Routine*: pragmatic generally *Mediator*: verbal explanation, and symbols
159 R	Do these meanings also apply in your general expression?	
160 L28	Yes, and "*a*" represents the first term or number in the pattern whereas "*d*" is the difference between any two consecutive numbers.	*Narrative*: formula interpretation *Mediator*: symbols
161 R	Do these letters have specific names in mathematics or algebra?	
162 L28	I think, I would call *a* and *d* unknowns because the represent fixed numbers in a sequence while T_n and *n* are variables because they are dependent and independent variables respectively.	*Mediator*: verbal explanation and symbols
163 R	So you are saying an unknown is specific and fixed, ...	
164 L28	Yes, but variables change every time.	
165 R	Explain more about this change.	
166 L28	For example, in question 4a, $T_n = 7n – 2$. So when position $n = 2$, $T_2 = 12$ while if n changes to 10 then $T_{10} = 68$, [……], this is how I did work out 10^{th} term of the number pattern.	*Narratives*: endorsed formulae *Routines*: explicit methods *Mediators*: symbol and numbers
167 R	How did determine the 50^{th} term in 4b?	
168 L28	I used the same method of solving T_{10} but….., this time my formula is $T_n = - 5xn+40$ so this gave, […..] the answer of $T_{50} = - 210$.	*Routines:* solve equation *Narratives*: endorse *Mediator*: symbol and number
169 R	What position would you find -1200 as a term in this pattern?	
170 L28	Eeeeh! That's too far. But I think, […..] using the same idea of working out an equation for position n in Tn expression helps. So my Tn = -1200 which means -1200 = - 5xn + 40 and this is -5n= -1200-40 so -1240/-5 will give n=248. It will be a problem if one does not know how to generate a formula in pattern problems.	*Routines*: solve equation *Narratives*: endorsed *Mediator*: verbal explanation, numbers and symbols.

5.2.4.3 (ii) LEARNER L28: RESEARCHER'S COMMENTS

Learner L28 used different types of visual mediators when communicating her thinking while generalizing from the numeric and geometric pattern task. The mediators which L28 used include numbers, picture/diagrams, tables, algebraic symbolism and verbal explanations (for example, episode10, lines12, 14, 18, 24; episode 11, lines 56, 62, 66, 84 and episode 12, lines 158, 162, 172).

Learner L28 approached the task on number pattern in a similar way to L23 except that her recursive and explicit methods focused on the relationship between the term and its position in the pattern. In doing this, the formula $Tn = a + (n-1)d$ is expressed in terms of the equation $y = mx + c$ (Episode10 , line 12). This approach fell into the category of numeric generalization routine. The move from picture or diagram representations into numeric pattern representations, for example, in questions 1 to 3, (episode 10, lines 12, 18 and 20) indicated a figural generalization routine. And it is the use of the two main routines, numeric and figural, that counts as pragmatic.

Having established the numeric pattern recursively or explicitly, L28 derived or generated her generalization expression in terms of the general formula related to a linear equation, i.e. $y = mx + c$ (episode 10, line 12). From the use of a general equation $y = mx + c$, alternatively, $T_n = a + (n-1)*d$, L28 managed to derive the correct generalization expressions in the pattern task (see episode 10, line 14, episode 11, line 62 and episode 12, lines 166 & 168). Similar to Learner L23, Learner L28's use of the visual mediators and routines led to mathematically valid endorsed narratives that resulted in correct generalizations.

Reflecting on the analyses of L28's interview responses from the three extracts above, I infer that L28 has demonstrated the following:

- L28 has a strong foundation of mathematical knowledge such as basic skills, algorithms, conceptual knowledge.
- L28 can abstract mathematical ideas and representations. For example, she was able to demonstrate abstract reasoning and used variables correctly to represent different mathematical terms.
- L28 is able to reason inductively by recognizing, extending and generalizing patterns. She is able to conjecture and justify her explanations.

- L28 is also able to reason deductively as she identified necessary conditions for any generalized rule to hold (i.e. domains of applicability). Deductive reasoning is used when a generalization relating to pieces of information is known and a conclusion about a specific piece of information is desired or can be drawn.
- L28 communicated mathematical ideas in a logical manner. She could comfortably use the mathematical notations in multiple ways.

5.2.5 DISCUSSION OF THE SIX LEARNERS INTERVIEWED

The learners L23, L28 were classified as high ability based on their responses to every question of the written task. The two learners are characterized by their abilities in identifying, applying and verifying generalization routines correctly in any problem context in order to generate the correct rules. The verification process involves justifying the entire approach and solution with respect to the problem context.

The ability to make proper choice of generalization routines for each question in the task showed that these learners can model most algebraic problems at GET phase. Having demonstrated three generalization routine categories (see table 5 in chapter 4) in their written and oral responses, the two learners' ability to use multiple routines provided them with various options to best responses to particular instances. L23 and L28's ability to describe the various facets of the rules for the n^{th} term indicates their strong thinking and understanding of the generality of rules. For example, they were able to interpret the symbols and their operational signs in the rule or general formula such as $T_n = a + (n-1)*d$.

The two learners seem to understand that the rules model the pattern context on the basis of which quantity changes and which quantity stays the same (i.e. in/dependent variables). Hence they were able to deduce how the pattern continues. For example, L23 understood that the rule $T_n = 5n+1$ is derived visually and directly from the pattern context. He saw that each additional cluster-house added to each consecutive position of the cluster-houses require adding 5 line segments and this was represented in the rule by multiples of 5. The plus 1 in the rule was verified by conjecturing a number to add or subtract to the multiples of 5 that would result in the pattern 6, 11, 16, 21, ….., ……

The learners L15 and L17 belonged to the medium ability group. Their analyses were characterized by their flexibility in understanding and use of recursive approaches such as counting or chunking coupled with the explicit use of the ideas learnt previously (i.e. use of the formulae $y = mx+c$ or $T_n = a+(n-1)*d$). The learners' responses throughout the task used numeric generalization routines. It is interesting to note that besides mostly deriving correct rules to generalize the pattern problems, the learners still grappled with linking their rules to the context setting. Such instances provoke a dilemma as to why their derivation procedures continued providing rules that worked with different input or output values for each number range for x = 20,4 for y =103 (episode 13 lines 32, 34). The learner L15 carried out the manipulation procedures correctly but with less understanding as demonstrated by how she worked backwards.

The learners L5 and L9 were drawn from the low ability group. These learners were mainly characterized by their inability to provide a complete generalization using at least two forms of representations (multiple representations of responses). The learners seemed to have a fair understanding of the numeric generalization routine, particularly recursive methods, as their written and oral responses employed no explicit reasoning. Both learners struggled to generate or establish any form of the correct algebraic generalization in terms of the n^{th} term as they seem to perceive that the rules should work for particular cases and not in general (see episode 1 line 10 and episode 5 line 36). It is also interesting to note that L5 managed to provide a correct verbal generalization in episode 2 line 66 but later grappled with the correct algebraic representation as observed in lines 69 and 70 of episode 2.

The analyses of the interviews indicate that one-third of the participants (L23 & L28) are able to employ the three generalization routines correctly when generalizing from the number patterns. These routines include numeric, figural and pragmatic routines (see table 5 in chapter 4) with further subcategories, for example *recursive*, *explicit, proportioning, relational* and *guessing, chunking, counting*, structural analysis, work backwards methods respectively. These learners seemed to have a strong understanding of multiple representations of generality rules within or across contexts in task.

5.3 SUMMARY OF THE ANALYSES AND FINDINGS

In this chapter, I discussed the results and analyses of written and interview responses. The theoretical and analytic framework provided me with tools for analyzing the written and interview responses. Through the analysis of the questionnaire task-based interviews a further refinement of the analytic framework emerged with a focus on LR (see table 5). Three of the four properties of a mathematical discourse i.e. *routines*; *mediators and narratives* were explored in the learners' responses to the questionnaire based task.

A further examination of the three properties of the mathematical discourse provided insight into learners' abilities to communicate their reasoning when generalizing from number patterns. The analyses also show that the learners' manipulations within and across different representations was the main challenge. The learners' use of routines and mediators to produce correct mathematical narratives when generalizing in number patterns depends on their ability to reason mathematically. The absence of any of the distinct features of a mathematical discourse makes it difficult for participants of a discourse to communicate their thinking.

The overall results show that the participants used recursive methods (i.e. counting and chunking), a type of numeric generalization routine, to represent and express their algebraic reasoning more than they employed the other methods (routines) combined. In particular the high and medium ability target participants (L15& L17, L23 & L28) demonstrated that they can use the recursive and explicit routines interchangeably in a problem context. In contrast the explicit routines were almost non-existent in the lower ability participants. In the next chapter, I provided further discussion of the results and findings of this study.

CHAPTER 6 DISCUSSION OF RESULTS AND FINDINGS:

INTRODUCTION

This study set out to explore grade 9 learners' algebraic reasoning as they generalize from numeric- geometric number patterns. It focused mainly on the routines that the grade 9 learners employ, and how they represent their responses when communicating ideas and thinking (i.e. mediators and narratives) when generalizing from number patterns. In this report, I have organized and discussed the participants' responses according to a commognitive perspective. In the analyses, I examined the participants' routines, mediators and narratives as the main features of a discourse using the analytic tools developed for the study (see chapter 4).

6.1 PARTICIPNTS' COMMON ROUTINES

Many of the learners' routines identified in the present study have been found in previous research studies on number patterns. The generalization routines that I have categorized as numeric (i.e. recursive and explicit methods) corresponds to what Stacey's (1989) and Orton and Orton's (1994) categorize as counting, difference, whole object, and linear methods. What I have categorized as figural and pragmatic generalizations corresponds to recursive and functional strategies as identified by Swafford and Langrall (2000). The findings of the study show that the participants' routines do not come into their minds in a straightforward manner regardless of the clues embedded in the number pattern context they engage in. For example, participants who are able to predict further numbers in the pattern tend to express their manipulations algebraically, to recognize equivalent expressions, and to use algebraic rules in solving problems associated with the number patterns.

The routines that the participants of the study employed mostly fell into the category of *numeric* generalization which includes *recursive* and *explicit* routines. The learner was considered to approach the task recursively when s/he demonstrates counting, guess and check or chunking methods. Most learners were able to determine the next few terms recursively but not the rule for the n^{th} term. As most learners used recursive-counting methods at some stage, it is important to take advantage of the learners' abilities to employ recursive thinking to communicate ideas and work to expand this into algebraic reasoning.

Current studies show that learners' strategies (routines) can provide useful information for classroom pedagogy. For example, learners use different routines when they generalize linear situations (Healey & Hoyles, 1999, Stacey, 1989, Swafford & Langrall, 2000 and Lannin 2001). It is not surprising that *recursive* approaches dominated other routines when learners generalize from number patterns. The findings also relate to other studies' (e.g. Macgregor and Stacey, 1993; Hershkowitz et.al 2002) which reveal learners' tendencies to generalize recursively rather than to use explicit approaches. An interesting observation is that most learners who relied on recursive methods did not attempt to determine a 50th term or they gave incorrect responses. There were also a few instances in which recursive and explicit methods were employed simultaneously at some levels; this often resulted in correct responses (see interview analyses). The success of some learners (e.g. L23 & L28) in generalizing number patterns can be partly attributed to the nature of the questions in the task. All the questions in section A were linear-geometric patterns of the form y = mx + c while section B of the questionnaire task (not part of this study) consisted of non-linear pattern questions. Concerning linear patterns, Stacey (1989) distinguishes between what she calls "near generalization" tasks in which finding the next pattern or elements can be achieved by counting, drawing or tabulating and "far generalization" tasks, in which finding a pattern requires an understanding of the general rule.

However, it is difficult to predict from these results if the learners would show the same or similar generalization routines when they are presented with non consecutive figural or pictorial terms in a pattern instead of the three consecutive terms as in the study's questionnaire-based task. Although this scenario is not the focus of this study it might be interesting to explore or learn from other studies on issues associated with instructional design. What is not clear also is how these generalization routines affect the development of algebraic reasoning (see algebraic thinking in section 2.2.3).

6.2 PARTICIPANTS' COMMONLY USED MEDIATORS

Representation is the display of the mathematical relationships using pictures, graphs and symbols. The symbolic representation for example may take one of the following forms: tables, algebraic expressions, formulae, numbers or verbal (written or spoken words). Use of different mathematical representation includes the following aspects: i) interpretation of the mathematical relationships presented in any of the forms cited above; ii) linking or matching of different representations of the same mathematical relationship. For example matching the pictorial, tabular, graphical, verbal explanation and a rule such as $T_n = 5*n+1$ in question 1 of the task; iii) the creation of multiple representations of the same relationships and iv) recognizing how a change in one representation affects the other with the same relationship. An understanding of multiple representations of mathematical concepts is one of the skills of algebraic reasoning. Drawing on the theoretical framework of mathematical discourse, the word "*mediators*" is used in the study to mean *representations*. The participants of this study made use of different forms of mediators to communicate their thinking when generalizing from number patterns (see table 9).

The participants from the high ability group described how they made sense of pictorial representations translated into numeric representations before attempting to generalize symbolically. The learners acknowledged that numeric representation of patterns is useful and assists their understanding. In their responses to questions 1, 2 and 4, learners' routines interact with endorsed narratives when generalizing. Also, the high ability learners attempted to find the relationship between the terms and their position in terms of n and T_n.

The analyses of the medium and low ability participants reveal important aspects. Their verbal algebraic reasoning was characterized by an interaction between routines and mediators which produced narratives that I either endorsed or rejected. On the other hand, inadequate algebraic reasoning demonstrated by the low ability learners was consistently linked to inappropriate knowledge transfer between representations when generalizing.

According to Sfard (2007:571) "visual mediators are means [with] which participants of a discourse use to identify the object of their talk and co-ordinate their communication". For Sfard, all forms of communication are visually mediated regardless of whether the mediators are obviously present or not; in a discourse they are always used by participants to create narratives. The use of mediators such as verbal explanation, numbers, diagrams or graphs (examples of iconic mediators), tables, algebraic expressions and formulae were observed in the participants' responses. These mediators formed part of the multiple representations the learners used to communicate with in the discourse on *algebraic generalization* from number patterns. The *numbers* were the most common form of mediators (24) used to communicate their thinking, followed by verbal explanation (11). Diagrams, graphs, tables and algebraic representations were used the least (see table 9 in section 5.1.3). The numeric (number) mediation comes as no surprise because most of the participants' mathematical manipulations were classified under numeric routines of generalization. This may be an indication that the verbal skills develop along with numeric and symbolic reasoning. This result has huge implications for the teaching strategies for algebra in schools. The learners' use of informal methods such as verbal explanations for finding solutions during problem solving creates an in-depth understanding of the mathematical concepts.

On the other hand the participants of the study found extending patterns numerically easier than making symbolic representations of the same number sequences. For example, an extract represented in episode 17, reveals an interesting observation: L17 initially derived his rules for patterns from a general rule of the form $T_n = a + (n\text{-}1)*d$ for any linear number pattern. When asked to argue for (justify) the formula $T_n = 3n+2$ in question 2, another reasoning skill was realised (Episode 17, lines 34 & 36). Specifically L17's visual mediation of the tile pattern helped him to access the recursive and explicit routines. This shows a functional relationship in terms of in/dependent variables. In the context of the tile problem, the independent variable is the tile group or position number (n) while the dependent variable is the number of tiles per group or position (T_n). The participant failed to see the functional relation when he identified the pattern 3x1+2; 3x2+2; 3x3+2; .... ; ..... to mean the same pattern as 5; 8; 11; ..... ; ....; ...... His visual mediation and routine processes engaged in the tile problem could not translate into the mathematical narrative $T_n = 3\text{x}n + 2$ and yet this is equivalent to the numeric representation shown above.

Many learners grappled with generating rules using the symbols where some had difficulty with which letters or symbols to use and their interpretation. The purpose of using algebraic representation was perceived by many participants of the study as a means to perform some manipulations to get an answer. Such perception implies that the use of algebra as a means of expressing generality was non-existent in their routines. The analysis of the participants' forms of representations has taught me that the use of different representations in mathematics is not as important as linking and translating the representations within and across contexts correctly. Making connections between different mathematical representations is an issue of conceptual understanding and was not fully explored in this study. For example, in episode 11, line 66, L28 justifies how she generated $T_n = 3n + 2$, a rule of generality for the tiles problem in the task. Through the use of recursive and explicit routines mediated by numeric and algebraic representations, L28 was able to produce the correct narratives. She also demonstrates her relational/functional reasoning by recognizing in/dependent variables in play when carrying out manipulations. She reckons that the representations $T_n = 3n + 2$, 3x1 + 2; 3x2 + 2; 3x3 + 2; .... and 5; 8; 11; ...... are equivalent expressions as they mean the same number pattern.

The results on the use of mediators are in line with Lannin's (2005) findings that suggest many learners' inability to work simultaneously with multiple representations. The study by Gagatsis and Elia (2004) found that pupils tend to experience difficulties in transferring information across contexts rather than within contexts. For example, the numeric pattern in question 4 was found to be tackled with greater success by some learners compared to the corresponding patterns in questions 1, 2 and 3. During the analysis of the learners' data, in many instances the transition from arithmetic to algebra was observed to be either too abrupt or nonexistent probably due to the fact that these participants were more familiar with manipulating numeric representations from school mathematics textbooks than other forms of representation. The mathematics curricular and textbooks from early grades arguably need to take a symbiotic approach to arithmetic and algebra. Separating the algebra and arithmetic makes it more difficult for the learners to learn algebra especially in later grades. A broader conception of algebra emphasises the development of algebraic reasoning and not just being skilled in algebraic procedures. Development of algebraic reasoning is generally reflected in the students' ability to generate, represent, and justify generalizations about fundamental properties of arithmetic (Carpenter et al., 2003).

6.3 LEARNERS' ALGEBRAIC REASONING

In a mathematical Discourse, algebra continues to be a manipulable language for expressing generalities concerning relationships. What is key about this language is that it does capture and expresses the idea of structure itself. So whenever participants in a discourse express a generality about numbers they are actually describing and capturing some structure. For example, the structure of all odd numbers can be expressed as $2n+1$ for any integer n (Radford, 2000). In expressing the generality, participants of a mathematical discourse turn their attention to properties which characterize numbers, such as properties of 0 and 1, distributive, associative, commutative laws and the presence or absence of both additive and multiplicative inverses. On the other hand, the counting of objects, pictures, diagrams making up structured arrays representing numbers themselves reveal generalizations expressed in algebraic language. It is therefore no coincidence that the participants of this study were engaged in communicating through this language to explicitly generate correct rules (generalizations).

Expressing generality is an example of *reasoning aspects* (see section 2.2.3) required in algebra and is characterized by one's ability to operate on, and with, unknowns or variables (Carpenter, Franke & Levi, 2003; Kaput, 2000; NCTM, 2000). There are two types of reasoning the participants in the study demonstrated: inductive reasoning and deductive reasoning. The overall analyses show that most learners reasoned inductively when generalizing in number patterns. The fact that most participants of this study continued using familiar ways of operating based on numbers is an indication of their inability to make a transition from arithmetic to algebraic thinking. In algebra, variables are used in several ways. They may represent:

- A specific unknown number(s) such as in $15n + 5 = 50$, $n^2 + 5n + 6 = 0$ or a varying quantity that relates to another e.g. $y = 5x + 5$ or $x \longrightarrow 8x - 5$.
- A generalization that can take on values of a set of numbers as in $x + (-x) = 0$ and/or $3n - 7$ and an object or equivalence, for example, 1m = x cm or x F=1Y i.e. x feet = 1 yard.

This finding is consistent with literature reviewed. For example, Stacey and Macgregor (2000) found that many students solve problems by calculating with known numbers and working towards the answer, instead of constructing and using equations (i.e. generalities) as statements of equivalence relating known and unknowns (Stacey & Macgregor, 2000). The introduction of a well designed mathematics curriculum that aims at strengthening the learners' arithmetic problem solving skills while providing opportunities for the growth of algebraic thinking may be a way forward to narrow the algebra – arithmetic gap (see symbiotic approach in section 2.2.3.3). Also, giving the learners plenty of opportunities to work with visual symbols that are not letter symbols can help smooth the transition to algebraic reasoning. For example, using pictorial symbols to represent quantities first then moving on to the use of letter symbols as unknowns or variables.

Consistent with prior research the overall findings are similar to the studies conducted on both students and elementary school pre-service teachers' understanding of algebraic generalizations. The studies reveal that although most participants, who were able to solve specific cases numerically by extending the patterns, had considerable difficulty when it comes to determining an algebraic rule (MacGregor & Stacey, 1997; Zazkis & Liljedal, 2002; Hallagan, Rule and Carlson 2009:201-206). Such methods limit learners from accessing the general structure of all elements in the pattern context.

As outlined in the theoretical framework (section 2.1.3), a "narrative" is any text, spoken or written that is framed as a description of objects or of relations between objects or activities with or by objects and is subject to endorsement or rejection i.e. objects or activities that can be labelled as true or false (Sfard, 2007:572). Endorsed narratives which are also known as "factual statements" refer to the narratives labelled as true and acceptable by the members of the mathematical community. Being able to generalize from number patterns depends on the correct use of routines and narratives that lead to a desired conclusion. The mathematical definitions, rules (axioms), theorems or propositions and formulae are examples of narratives. The results of this study indicate that most correct narratives (endorsed) were demonstrated and used by the high ability participants and partly by the medium ability participants (L23, L28, L15 & L17) in support of each step of generalization (see episode 14, lines 36, 42 & 48; episode 9, lines 96, 104 & 110). On the other hand the lower ability participants grappled with using correct narratives in their responses. For example, it was observed that L9 claimed to derive the general rule for the n^{th} term from the formulae n*5 and $T_{\boldsymbol{n}}$ = a-(n+1)d or a(n-1)d respectively (episode 4, line 10). But none of these algebraic expressions could lead to a correct generality to the patterns in the task, hence I rejected them. Inadequate understanding of how to use mediators and routines when generalizing from number patterns appeared to be the cause of the learners' production of incorrect mathematical narratives (Episode 2, lines 69, 74 & 153; episode 4, line 10).

6.4 OVERVIEW OF LEARNERS' DIFFICULTIES IN RELATION TO THE TASK

Throughout the study, learners found it difficult to express generalities in appropriate language (endorsed narratives) using appropriate icons, words and symbols (visual mediators). This is because most learners in the study only learn to use appropriate symbolic notations in contexts which have meaning for them (Mason, Burton & Stacey, 1982). Mathematical symbols carry abstract meanings (Pirie, 1998) and sometimes involve unknown numbers. These symbols are often ambiguous to learners and yet have to be precisely connected to the language of representation based on the problem context.

Therefore, regardless of the abstract nature of symbols in mathematics the learners still need to translate the mathematics into correct language of representations (see algebraic language in section 6.3). Being able to continue any given pattern can be taken as an understanding of the repeating patterns. On the other hand, being able to describe the “general” aspect of the pattern can be seen as a means to the solution of the pattern. From the cohort of the interviewed learners only three (i.e. L23, L28 and L15) used the correct general expressions that could satisfy the n^{th} term and position n for any term in the patterns.

Some sequences in the questionnaire based task were interpreted as repetition or growing patterns in both visual and numerical contexts. The three most frequent cases happened to be question 1, 2 and 3 with visual-pictorial terms in their sequences and question 4 with consecutive numeric terms in the sequences. Learners relied on recursive methods without paying attention to the independent variable (position number) in these sequences. The majority of these learners achieved better results/scores (TLA see QRASS Appendices) on questions involving numerical sequences than on those involving visual sequences. They presented very low scores in completing the motifs’ sequence whose nature was purely visual and partly tabulated.

The first two questions may have caused some difficulties possibly due to the pentagonal shape of the cluster house design in 1 and the simultaneous variation of the width and height of the tile-group. Question 3 of the questionnaire task was the most difficult for most learners to solve. Learners identified only the dots irrespective of the number of lines in motifs at any given position. The presence of the table of values may possibly have influenced them to pay less attention to detail and they hence used, inappropriately, a direct proportion method.

Drawing on episodes 1 and 2, learner L5 was able to continue with the patterns in the context (form) given and the later into numeric form. By means of counting, L5 determined a pictorial representation of the cluster houses by generating a numeric pattern 6; 11; 17; …. ;……; …… (see line 4 of episode 1). The participant further claimed that with this approach, he could calculate the number of line segments for any set of houses. It does indicate that for L5, a search for an easier solution at first was not in his approach. But later he realized that there should be some sort of “formula” that could help solve the problem quickly (Episode 2, line 66). After persisting, the learner noticed that each tile – figure starts with 2 plus (figure position number) multiples of 3 (Episode 2, lines 68 & 74).

Learner L5 experienced difficulties in finding solutions to question 4 a), b). I wondered what made this learner fail to comprehend the numeric form of a pattern and to deduce some general expression as was done in questions 1 and 2. He did eventually manage to see the repeating pattern in 4a) and 4b) as increasing by 7 (line 139) and decreasing by 5 respectively. There are two things I noticed in the conversation with L5. The solutions to a tile problem could be found from the generality (Episode 2, line 66) made by attending to the first element in each unit of repeat. In this approach, L5 introduced algebraic symbolism which was absent in his solution in question 1. What is noticed here is a common tendency among the participants to look for an exact solution or a general solution. The search for a solution implies looking for a single formula that determines the location or position for any given number. As such, the participants' failure to generate such general expressions or formula (i.e. generalization) left them feeling inadequate. A common opinion among peers and students is that mathematics consists of individual work devoted to solving tasks with only one correct answer (Schoenfeld, 1992). Embracing such opinion will imply that mathematical understanding equals skills to use algorithms and getting the right answer in problem solving.

Some of the learners' difficulties that resonated from their engaging with the task on number patterns are summarised in the following five points:

- Students have difficulty with Algebra for one of the same reasons they have difficulty with arithmetic – an inability to translate pattern contexts into solvable mathematical symbols (generality rules). An inability to integrate arithmetic and algebraic knowledge in the number pattern context is due to a partial understanding of either of the two (arithmetic – algebra gap).

- Students with low performance abilities grapple to distinguish between relevant and irrelevant information; they have difficulty with multiple translation and representation problem situations.

- Algebraic translation and representation involves assigning variables, noting constants, representing relationships among variables and assigning numerical values to variables.

- Abstraction – the use of symbols to represent numbers and other values is seen a difficult process. For example, learners struggle with manipulations (numeric) to solve linear equations or functional relations.
- The assumptions that many learners are familiar with basic mathematical vocabulary and operations are not in line with the reality. Many learners are not fluent with numbers. For example, learners view the + and – as an operation as opposed to a sign,
- Also, previously learned mathematical concepts are inappropriately applied to the task at hand. For example, proportion and substitution were inappropriately applied in calculating the position of a any given term (episode 1, line 6; episode 2, line 74).

Misconceptions are flawed beliefs which students hold. Misconceptions in mathematics are conceptual structures in the learner's mind which make sense to the learner but are not consistent with the socially sanctioned body of mathematical knowledge; they are incorrect features (narratives) of student knowledge that are repeatable and explicit. A major cause of misconceptions is the lack of understanding of mathematical concepts. Such misunderstanding may then lead to the formation of misconceptions and false generalizations, which in turn hinder the learning of mathematics if they are not properly dealt with. Misconceptions denote a line of thinking which surface from deeper levels of errors (Nesher, 1987). Resnick (1984) also argues that the more incomplete the students' knowledge base, the greater the likelihood that the students will generate incorrect mathematical inferences and narratives. Many misconceptions are deep rooted, robust and resistant to change. , Osei (1998) described his students' misconceptions as "very strongly held" and "not easily changed by classroom instruction. This was evident in Sasman et al (1998)'s research results on functions; they found that after all the activities they gave, a multiplication misconception was persistent and remained obstinate.

In my study, there were some *misconceptions* from L15 and L17 which presumably emanated from learners' previous experiences with linear equations. I see from their discourse that they always solved equations from left to right and never vice-versa. Although the two learners recognized that their approach relates to that of solving for the unknown in the general expression, they continued to disregard the inverse operations when finding the position n of a given term in sequence or pattern (thus, they could not work in reverse with the general expression). This reversibility method of finding a solution for the unknowns works relatively well for any question about the position of a term using the expression or rule for the n^{th} term.

For L15, it is interesting to note that the approach of working out T_n as unknown was not much of an issue but rather if given the value of the term say 500, the learner grappled to find the value of *n* (i.e. position of the term 500). For example, drawing on episode 13, line 26, L15 seems to think that a variable and an unknown mean the same thing as they are merely symbols or letters that represent any number. In this episode, the learner does not make a correct mathematical distinction between the two and it is such misconceptions that lead to unendorsed mathematical narratives. What is worth noting here is that most of the participants' generalization routines were inadequately interacting with the narratives necessary for communicating in mathematical discourse.

6.5 LEARNERS' MATHEMATICAL COMMUNICATION

The study draws on a commognitive perspective which views communication (an interpersonal activity) and thinking (an intrapersonal activity) as different forms of the same phenomenon (Sfard, 2008). Communication has been a key throughout this study. The participants were tasked to communicate their ideas first in writing and then orally through interviews with the researcher to explain their mathematical thought processes behind each pattern question. The aim was to foster students' use of mathematical language by having them define number patterns in terms of some relationships while using different forms of representations.

The notion of *language fluency* either for ordinary communication or communication in a discourse emerges from the analyses made in chapter 5. This study is situated within a mathematics discourse and is guided by the theory of commognition. The learners participating in the study use English language for communicating and learning in the classroom whereas on the other hand mathematics as a discourse has its own mathematical language (register). This scenario resulted in many learners being placed between a rock and a hard place as English language is not the first language of most participants. A mathematical register develops when language is used to express mathematical ideas and meanings (Pimm, 1991). This implies that participants of a mathematical discourse create and use correct narratives to communicate their ideas and thinking. Mathematics register is the way in which mathematical ideas and concepts have to be communicated. The language used in the mathematics register serves different purposes as opposed to the ordinary language for example, mathematical *words* and *instructions* in both verbal and symbolic representation are mathematics specified and often unfamiliar to most learners for example, words like volume other than “capacity”, function meaning an equation other than a “party” and symbols like – i.e. a subtraction sign other than a dash.

Boulet (2007) suggests that language plays an important role in the mathematics classroom. He further argues that fluency in a language opens up the whole world to mathematics learners since they need mathematical language to clarify and justify their ideas and procedures. In addition to being responsible for creating the opportunities for learners to engage in discussions, exploring, negotiating, and sharing knowledge (Manouchehri & Enderson, 1999) teacher’s own use of language in the mathematics classroom serves as an important example of effective communication. Manouchehri and Enderson (1999) further claim that spoken language is in large part responsible for problems in the teaching and learning of mathematics holds in the sense that most of our learners are second language learners so if language of communication in classes is not clear and understood then problems in the subject arise. Teacher’s knowledge in the mathematical discourse is very important. It helps them identify language problems in their learners and settle the difficulties. Boulet (2007) again argues that learners must be driven into ways of developing a mathematical language so that they can be able to read, interpret and describe mathematical notations themselves and also define and understand mathematical terms. From the findings of this study, I concur with Boulet that the learners need to master the language aspects of a discourse. Since the study set out to explore learners’ algebraic reasoning when generalizing number patterns, it was necessary that the participants possess a knowledge base or language aspects of numbers, patterns and generalization (see sections 2.2.1and 2.2.2).

One strategy that can be used to initiate and motivate learners into communicating mathematically is by encouraging them to write down their talk or thoughts about mathematics and thereafter work on the informal to make it formal. A second way could be taking the informal spoken language, formalise it and then put it in writing as formal written language. In this case learners rehearse more formal spoken language skills through play. Jaworski (1995) suggests that, where the focus is on mathematical language, learners may be asked to say what they have seen, maybe on a poster a picture or a diagram, to the rest of the class, under the constraints of no pointing or no touching. Similar approach was adopted in this study during the interview sessions with the participants except that the participants were responding to the researcher’s questions. This helps to focus the challenge on to the language being used to retell the story of the picture, poster or diagram. A third way could be through investigations where learners are to report back their findings to the class using formal language; this may help learners to become more formal in the use of language in public and their work may also become more structured.

In this study, the participants' mathematical communication resonates with the theoretical framework of commognition. Sfard (2008) combined the two terms communication and cognition into "commognition" meaning that thinking is communication and learning mathematics is to modify and extend one's discourse. Classroom mathematics discourse is a tool which helps learners to understand mathematical concepts through communication and effective engagement in mathematical activities. The communication for instance could be through written work or classroom interactions. Mathematics discourse is important for professional growth and helps the participants for example, teachers and learners to take charge of their own mathematical learning. A discourse makes thinking public and creates opportunities for learners to engage in arguments, negotiate meaning and agreements. By expressing their ideas learners clarify their own understanding.

6.6 COMPARISON OF INTERVIEWED STUDENTS

Almost 60% of the participants (entire sample – 29 learners) were able to apply their conceptual understanding of implicit numeric rules to predict the next terms' positions in the pattern, without necessarily being able to articulate the general rule (generalization). A few learners (5) included a constant at first as they directly worked from the general expression $T_n = a + (n-1)\,d$ when formulating their own rules. Later these 5 learners experienced difficulty when the numbers they were working with increased. They resorted to doubling methods (say, 10^{th} term is two times 5^{th} term) i.e. they inappropriately assumed direct proportionality between terms (Orton & Orton, 1998). The two learners in the low ability target group did not recognize that a constant need to be included even in their numeric explicit rules. Surprisingly, they were able to apply their incomplete rules correctly to most of the next few terms of the patterns in the questionnaire based task. The target learners from both medium and high ability groups chose expressions based on operating with unknown terms when generalizing from number patterns and also adopted their own syntax in the form of symbolic language to communicate their generalizations. These findings resonate with the notion that learners bring numerous resources to mathematical situations and the approaches they choose to apply to any given problem are influenced by a variety of factors for example, prior knowledge, social context, problem context and nature or quality of instructional design (Carraher 2007:5).

It is also striking to note how comparable the written and oral/verbal generalizations of the interviewed participants were. Analysis show that learners found it easier to informally verbalise the generalization rather than to provide a formal written response. For example, in question 2, some orally gave descriptions such as “tiles increasing by 2 every time”, “number changes at the bottom and the stairs” (i.e. sides- columns), “the change is group times 2 and plus 2 tiles”. Such a range of oral responses demonstrated that learners were able to verbalise their generalization. However, many grappled in expressing their thoughts in written form.

Reversibility reasoning was another challenge that the target learners faced during the interview. Each learner was presented with the number of line segments or tiles and asked to determine what position these represented. Such questions tested their ability to work backward (reversibility) where one needs to equate the correct rule of generality or formula with a given number (see episode 2 line 74) and then solve the equation in terms of position. Unfortunately the majority found this very difficult; perhaps this was because they could not link their generalizations to properties of equations and also some learners seem to have limited understanding of number patterns. For example, L9 started investigating the pattern (episode 4, lines 2 & 4) and then later decided to rely on her prior knowledge of using a formula

$T_n = a-(n+1)*d$ learnt (Episode 4, line 10) in grade 8 the previous year. Reflecting on the formula above, the learner experienced difficulty in assigning the + or – operational signs among the symbols a, n, and d. As a result the approach to generalize the solution or check which formulae would be applicable came to a dead end. Similarly, learner L5 eventually abandoned her formula building routine and opted to continue attending to counting patterns such as addition of 5 (Episode 1, line 4) and addition of 7 (Episode 3, line 153. Manson argues that students rush into building equations involving unknowns with little knowledge about how these unknowns are linked to the pattern problem or context (Mason 1996:75). Questions on patterns seek algorithms that lead to generalization and not a mere algebraic manipulation. I make a similar observation. Learner L9’s symbols in the formula learnt in her previous grade were not helpful as the task consists of pattern seeking questions and not just algebraic expressions. The letters *a*, *n* and *d* are used as symbols. Generally, the initial observations in the learners' verbal responses (i.e. written and spoken), show that the constant difference between numbers or terms in a pattern were realized with little problem (visually mediated). Thus similar to L9, most learners looked along the visual pattern and linked the terms in the pattern with their positions. Ironically, many participants were led to give partial solutions to questions without being thoughtful of multiplicative or additive operations necessary to establish or generate correct generalization rules.

Having examined *high ability* learners' (L23 & L28) work both from the written task and the interview sessions, it is clear that they have strong foundation of mathematical knowledge which includes basic skills and algorithms. The two learners demonstrated that they can appropriately apply such diverse skills to various representations of number patterns. L23 and L28 consistently used variables and symbols in their work and in so doing reasoned both deductively and inductively by recognizing, extending and generalizing the patterns. During the interview, L23 identified the necessary condition under which the variables or symbols make sense. For example, he reasoned deductively that both the positions and terms in the sequences must be whole numbers. The learners in this group further demonstrated multiple approaches to problem solving in number patterns and used their results to verify the solutions. The learners L23 and L28 in fact showed the potential to reason in several modes such as inductive, deductive, proportioning and geometric visualization "see researcher comments" in sections 5.2.4.3 i and ii) respectively. And also, L23 and L28 were able to effectively communicate their mathematical ideas when solving mathematical problems.

On the other hand, the analyses of the *medium ability* target group show some understanding of basic mathematical computational skills, ideas and concepts based on problem context. The learners L15, L17 have a facility for numerical representations and demonstrated this in a variety of pattern representations. They reasoned deductively in their mathematical solutions and used specific examples to justify their thought processes. They were able to recognize and extend patterns, partly reasoned proportionally and communicated mathematically though with less detail than the high ability group.

The analyses of the *low ability* group (L5 & L9) show that the learners grappled with understanding the problem context, and with communicating their responses orally and in written form. Most of their responses to parts of the four questions in the study's questionnaire task were found by using counting/recursive approaches. The learners in this group experienced difficulty in using notations or symbolism; hence they could not establish general rules for the n^{th} terms.

During the analysis of the target group learners' responses, it became clear that some learners generally understood what they were doing and that some aspects of mathematical reasoning were used in support of their thinking. However, as a reader of their responses (both written & interview), I was mandated to make subtle inferences across the three groups while making links to the overall nature of responses of the 29 participants. The trend that emerged from the data therefore suggests that there is an urgent need in our classroom practices to guide learners of all abilities to communicate their mathematical processes through identifying, recognizing, conjecturing and justifying generalizations from the number patterns. This trend confirms the Commognitive perspective's rightful place in this study.

6.7 SUMMARY & CONCLUSION

In this chapter, I discussed the results and analyses of the study's written and interview responses. The results show that learners displayed various forms of representations such as numbers, pictures, diagrams, graphs, tables, verbal explanations and algebraic symbolism while using different routines to generalize number patterns.

The results also show that most learners are unable to apply generalization routines correctly as they constantly grapple to link and translate different mathematical representations within a pattern-problem context. This trend was more prevalent in the learners' written work and was evidenced by some of the difficulties in the interviews especially when the participants were asked to justify their responses.

Overall, many participants' use of *routines* and *mediators* on the number pattern task were paired inadequately hence producing incorrect narratives. The notions of routines and visual mediators are intertwined and this could be problematic for the participants of this study. Since this is not the focus of my study, this aspect could be explored through further [future] research studies. The next chapter details the conclusion, limitations, implications or recommendations of this investigation on the grade 9 learners' algebraic reasoning when generalizing from number patterns.

CHAPTER 7 CONCLUSIONS, REFLECTIONS AND LIMITATATIONS

7.1 INTRODUCTION

In this chapter, I present my reflections on the entire study. I discuss lessons learnt from the data collection process and conclusions drawn from the findings of the study. Algebra is the area of mathematics within which the study was conducted with a particular focus on the topic “number patterns”. I was interested in the topic because having taught the GET and FET mathematics, I am aware that most high school learners grapple with the section. And I thought by studying the section one would be able to identify learners’ strengths and weaknesses when introduced to algebra through number patterns. Algebra as a branch of mathematics functions as a language for all other branches of mathematics. This study set out to investigate the GET learners’ mathematical (algebraic) reasoning - routines when generalizing from number patterns. It focussed mainly on the i) learners’ communication when explaining their thinking, ii) mathematical representations the learners use when generalizing and iii) generalization routines the learners use when engaged in a task on number patterns.

In mathematics education, debates and research are generally informed by theories of learning. This study was no exception. I adopted a theoretical framework of Commognition by Sfard (2008) to investigate the learners’ reasoning through generalization of patterns. According to Commognitive perspective, thinking is a special activity or form of communication (see chapter 2). While using the theory I drew on “mathematics as a discourse” one of the four commognitive tenets to explain the grade 9 learners’ communication and thinking when they engaged in a number pattern task. There are four properties that a mathematical discourse can be identified with: i) vocabulary or word use, ii) routines, iii) mediator, iv) narratives (Ben-Yehuda, et. al., 2005; Sfard, 2008). I focussed on analysing the latter three aspects (ii, iii and iv) of the learners’ mathematical discourse in my study.

This is a case study which employed a task-based written and interview questionnaire as a means of collecting data. I examined and analyzed the collected data in order to answer the two critical questions below. The analytical framework conceptualised in chapter 4 provided necessary tools to analyse the participants’ responses (data). During the analysis, the main focus was on the participants’ use of routines and mediating tools in the form of representations that were used to create narratives to communicate their mathematical (algebraic) reasoning.

The two questions that guided the research study are as follows:

3. What <u>routines</u> (*strategies*) and <u>visual mediators</u> do grade 9 learners use when engaging in a task on number patterns?
4. How do these grade 9 learners explain orally their thought processes in problem solving involving number patterns?

7.2 DETAILED FINDINGS

Mathematics is a useful subject as it helps in making predictions, and number patterns are all about prediction. Working with number patterns leads directly to the concept of functions in mathematics i.e. a formal description of the relationship among different quantities. Recognizing patterns is an important problem-solving skill used to generalize what one sees into a broader solution. I provide the responses to the two critical research questions for the study in the following section.

1. **What <u>routines</u> (*strategies*) and <u>visual mediators</u> do grade 9 learners use when engaging in a task on number patterns?**

The results and findings of the study revealed several forms of generalization routines and visual mediators that the participants of the study used when engaged in a task on number patterns. From a Commognitive perspective, the vocabulary, routines and mediators are the properties of any mathematical discourse that any learner needs to acquire and internalize if s/he is to be a participant in a mathematical discourse. The results show that many participants implemented routines and mediators that were largely unrelated and inadequate to create endorsed narratives. The two frameworks (see chapters 2 & 4) provided me with lenses to identify the learners' sources of difficulty as observed during the analyses of both written and oral data (see analysis and discussion of results sections).

The learners demonstrated high ability levels in making verbal and numeric representations (50%) when engaged in problem solving involving number patterns. The participants of the study largely found it easier to informally verbalise their generalization than providing a formal proof or written response. The other forms of mediators were found to be in the following proportions; 30% (symbolic), 18% (tables & graphs) and 2% pictorial (table 10). The verbal mediators consisted of written words, spoken words and algebraic symbolism.

The use of tables, graphs, pictures and diagrams were referred to as iconic (mediators) representations (see table 14) and featured the least in the learners' forms of mediation. The analyses of the data also reveal that learners used different routines when engaging in generalization from number patterns. The participants of the study used numeric generalization routines the most than figural, pragmatic or the two combined. The numeric generalization routines were classified into recursive and explicit routines.

The high ability learners used both recursive and explicit generalization routines in and across the questionnaire-based task items and further provided correct rules, and offered valid contextual justification. And also the medium ability learners used mainly recursive and partially explicit routines in which they provided correct rules derived from $T_n = a + (n\text{-}1)\ d$, and offered empirical justifications. On the other hand the low ability learners grappled to analyse whether the recursive or explicit routines could be used hence were unable to provide or derive correct rules and justification. The generalization routines that were classified in this study are categorised and presented in the following table 15:

TABLE 15: LEARNERS' GENERALIZATION ROUTINES

Numeric routines	**Figural routines**	**Pragmatic routines**
Counting Skip counting chunking Add strategy Linear or explicit Recursive explicit multiply solve	Structure analysis Proportioning Counting: Add and difference methods (see counting in table 14)	Relational

TABLE 16: CHOICE OF GENERALIZATION ROUTINE BY LEARNER ABILITY

Low Ability Learners: These used only numeric routines (L5 and L9)	**Medium Ability Learners: These mostly used numeric routines (L15 and L17)**	**High Ability Learners: These used all the three classified routines (L23 and L28)**
Recursive rule, counting, skip counting, guessing with no checking	Recursive rule, chunking, explicit, solve equations, guess and check.	Relational approach plus Numeric and Figural routines

Most participants (29) of the study continued using familiar ways of operating based on numerical manipulations when generalizing in number patterns, an indication of their inability to make transition from arithmetic to algebraic reasoning. This is consistent (see section 6.3) with Stacey et, al. (2000) who argued that many students tend to solve mathematical problems by calculating with known numbers first and then work towards the answer instead of constructing and use equations as statements of equivalence relating to known and unknown quantities. This scenario can be corrected in order to narrow the gap (see section 2.2.3.3) once teachers and learners begin to employ the symbiotic approach in the classroom pedagogies. Although most participants successfully solved specific cases numerically by extending the patterns, still they had considerable difficulty in generating rules that translate into correct generalizations (endorsed narratives).

Also, based on the descriptive findings from the grade 9 learners' written responses, many used various forms of visual mediators (forms of representation). These are verbal, tables, graphical, pictorial and numbers and symbolic (algebraic) representations (see table 10). Drawing from the theoretical framework of commognition, the learners' representations were referred to as 'mediating tools' for communicating in a mathematics discourse. In many instances the high ability learners used routines and visual mediators that produced correct mathematical narratives. And the medium ability learners partly linked their numeric generalization routines and mediators to produce endorsed narratives (see episode 14, lines 36, 42 and 48; episode 9, lines 96, 104 and 110). On other hand the low ability learners had difficulty to link their recursive generalization routines with visual mediators hence I rejected most of their narratives (see episode 2, lines 69, 74 and 153; episode 4, lines 10). The high and medium ability participants mostly chose to work with expressions by operating on the unknowns in order to derive general rules. These learners (H & M) were able to articulate and use their own syntax in the form of symbolic language to communicate their reasoning when generalizing in number patterns. The reversibility reasoning (work backwards) is seen to be quite a challenge for most of the six participants interviewed (for example L5, L9, L15 and L17).

Teachers can support the development of symbol sense in their learners by being aware of the obstacles the learners face as they work with symbols in the mathematics classrooms. The use of visual mediators (see table 14) can help to ease the transition to using letter symbols. Driscoll (1999:122) argues that "it is more natural to represent a quantity with a picture rather than a letter symbol". For example, when comparing prices of a cap and an umbrella, it may make more sense to represent the price of the cap with a picture of a cap and the price of an umbrella with a picture of an umbrella rather than the prices of these quantities with letter symbols. The learners can indeed make a smooth transition to symbolism if they are first introduced to the use of pictorial symbols to represent quantities before moving to the use of letter symbols (i.e. unknowns and variables). The results of the study show that most participants preferred the use of numbers as mediators. While it is important to be cautious about rushing learners into using symbolic representation, it is equally necessary to look for opportunities that guide learners towards using symbolic expression when their work shows signs of readiness. For example, most participants of the study generated a handful of numerical examples that seem to reveal a consistent underlying process (see section 6.2), which in turn, lends itself to a smooth transition to symbolic (algebraic) representation.

2. **How do these grade 9 learners explain orally their thought processes in problem solving involving number patterns?**

From the qualitative analyses of the six participants interviewed, the main focus was on how they articulate when justifying both their written and spoken responses hence the results show that;

- Most learners' use and interpretation of the mediators to communicate their reasoning about the task is inadequate. The notion of mathematical language and Language of Learning and Teaching (LoLT) emerges as a potential area of enquiry.

- Some learners indicated that they are solving the task problems using the mathematical knowledge from previous year's work or as learnt at extra tuition sessions, for example use of formula $y = \mathrm{m}x + \mathrm{c}$.

- Learners indicated that they are just using the rules of solving equations.

- Some learners focus on output and input relationship of variables, however, they grappled a lot to apply such functional relationship to questions that require reversibility as a form of reasoning (work backwards to find a *position* $(\boldsymbol{n})$ instead of a *term* $(\boldsymbol{T_n})$) in a given number pattern.

Most of the interviewed (L5, L9, L15 and L1&) participants' oral [spoken] explanations were inadequate to support the kind of the mathematical narratives they produced hence I did not endorse such narratives. And also the participants' articulation of the reversibility reasoning seemed to be problematic.

7.3 REFLECTIONS ON THE STUDY

I have learnt a lot in conducting this research. For example,

- Research can be a guiding tool on classroom pedagogy i.e. teaching and learning.

- There are lessons that can be drawn from the findings such as nature of the instruments and the level of complexity of the items (questions) in terms of the cognitive demands.

- Establishing a good rapport with the participants is helpful in encouraging learners to express their thinking when probed with question about the study.

- Piloting the instrument helped to get the refined questionnaire-based task for the study.
- Conducting the interviews was not an easy task. From it I have learnt that eye contact is very importance when interviewing people and coming up with right probing questions that can elicit rich data from the respondent is a must have skill.

- As an experienced educator and a practising researcher in mathematics education, I found it challenging during the period of the investigation to carry out the roles of the researcher on the very learners I teach and vice versa.

The list above shows key areas that I can draw lessons from about conducting a research study in the broad field of mathematics education. The last point is very important to me because it highlights critical moments of two practices which I need to improve on in my future research endeavours.

I found it interesting to note that some of the participants' written responses and comments made during the study showed that they found the work challenging, while on the other hand, in a personal communication with some participants outside the research they reckon to have enjoyed the work because the questionnaire-based task tested some exploratory skills such as identifying, recognizing and conjecturing contexts. Such learners' personal opinion shared in our ordinary conversation made me wonder as to what that means first, in the study and secondly in education studies.

7.4 LIMITATIONS OF THE STUDY

The findings and results are limited to the 29 participants of one grade 9 class from a particular former model C school and as such cannot be generalized. Thus, although any general trends or patterns observed in the course of this study are only relevant to the group of 29 research participants who took part in the study, such "generalisations" could be broadened or increasingly refined by future research involving further samples from the larger population.

Obviously the use of a particular topic in this case "number patterns" to elicit learners' reasoning when generalizing impacted the routines, narratives and forms of mediators the participants of the study used to communicate their ideas and thinking.

In the context of the South African schooling system, the mathematics curriculum is designed using English language, mathematics school text books are written and published in English, English is the LoLT in mathematics and other learning areas too. And yet many of our learners use/see English as second, third or even 4th language for communication. It is at this notion that I argue that use of the theoretical framework of commognition is limited as a tool for understanding how learners think.

7.5 IMPLICATIONS OF THE STUDY

The results of this study provide a guide for both the curriculum and classroom instruction design on the use of number patterns for introducing algebra in the junior high school grades (GET and FET). The emphasis of the RNCS and C2005 on investigation as a pedagogical approach to number pattern generalization, as well as its requirement that learners explore number patterns and hence "make conjectures and generalisations" as well as "provide explanations, justifications and attempt to prove conjectures" (DoE, 2003b:18), has important pedagogical implications for classroom practitioners. It is within this pedagogical context that this study finds practical significance. The results of the present study give strong support to the notion of students' generalization from number patterns by investigating the routines and forms of representations they adopt when solving pattern generalisation tasks both pictorial and numeric in nature.

Also I argue in support of previous studies on generalization as a means of introducing algebraic thinking through patterns, that it is not any easy approach to follow as most learners grapple to make connections between the pattern and algebra. Zazkis and Liljedahl (2002:382) explain that "students do not generally have a problem with seeing the pattern but in perceiving an algebraically useful pattern". This means that when the participants of the study see a pattern in a certain way, it becomes difficult for them to depart or move away from their initial perception hence they cannot generate algebraic symbolism from the pattern itself.

From a commognitive perspective, what is central in the study is the participants' ability to use the four properties of a mathematical discourse when generalizing from patterns. The students' generalization in the context of this study is seen as communication of mathematical understanding, and the analytic processes used proved to be highly successful in providing a window of understanding into each pupil's rules of generality.

On the other hand, the design and use of instructional tasks that promote learners' use of multiple generalization routines and mediators to communicate their mathematical thinking should be considered by mathematics teachers. This can be done for example; when teachers let their students discover patterns on their own and ask guiding questions that move them from specific to general. The processes by which students discover the general rules for numeric-geometric patterns provide opportunities for them to expand their reasoning skills and access basic conceptual knowledge of functions.

7.6 RECOMMENDATIONS

The present study focused on a mixed ability group of grade 9 learners from a former model C school. It would be interesting to repeat this study with either a high or low ability group of participants under similar research protocols, this time using a task that include patterns leading to quadratic and exponential functions. However, drawing on the findings of a pilot study, the lower ability students would probably grapple with articulating a written explanation of their reasoning. It would be equally interesting to repeat the present study with only a high ability groups of learners but with a mixed selection of pattern questions in the task. This would probably further highlight the trend on how participants make use of the features or properties of a mathematical discourse to communicate thinking through generalizations as identified in this study. In addition, this would add further insight into the complex interplay between learners' use of routines and mediators to create narratives that are acceptable in a mathematical discourse.

Research studies in future need to further investigate learners' ability to reason mathematically by integrating algebra with other mathematical content areas (topics). Alternatively, as the routines and mediators investigated in this study seem to be intertwined further research studies that can explore which generalization routines and mediators successfully lead students to developing general algebraic ways of thinking are needed.

7.7 CONCLUSIONS

According to Driscoll (1999), students learn to generalize patterns at an early age, from elementary shapes to number patterns later on. Recognizing patterns is very important to algebraic reasoning since the main role of patterns focuses on helping students recognize commonalities, constructing rules of generality, representing these rules using correct symbols, moving from recursive reasoning to explicit reasoning and providing justifications on the generalizations. Early practice of these skills may provide access to routines that will help students in upper grades of schooling with algebraic reasoning.

The study of pattern has become an integral component across all grades of the South African school Mathematics curriculum (DoE, 2002, 2003b). The results of this study strongly support the notion that question design can play a critical role in influencing pupils' choice of strategy and level of attainment when solving pattern generalisation tasks. Furthermore, this study identified several generalization strategies (routines) and representations the learners used to demonstrate their reasoning when working on a task on number patterns. The teacher's understanding of the learners' ability to generalize from various contexts of number patterns, using different mathematical skills (i.e. representations, generalization routines) has the potential to impact positively on the classroom pedagogies.

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APPENDICES:

APPENDIX A1: QUESTIONNAIRE BASED TASK ON NUMBER PATTERNS

NAME .... (L1; L2 etc) AGE .... GENDER ....

AIM OF THE STUDY:

Research in Education has shown that some learners have difficulties in carrying out problem solving in mathematics questions. We want to find out how you solve problems and what mathematical reasoning you are engaging in. This will help to understand some of your difficulties so that we are able to find ways of helping you and other learners.

WHAT YOU NEED TO DO

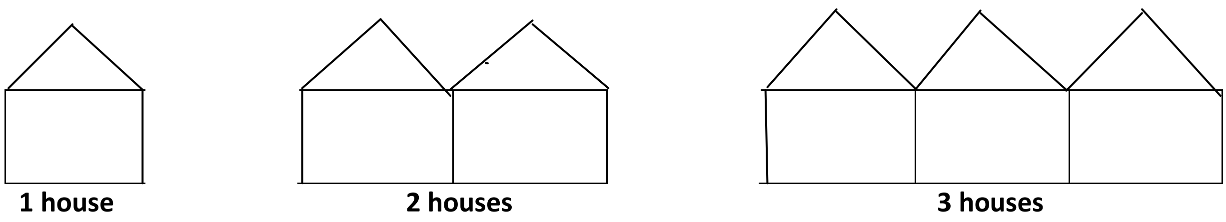
1. Write your name as L1 or L3 for Learner number 1 or 3, etc.
2. The paper consists of ...... pages and ...... questions each with sub-questions
3. Attempt all the questions in the questionnaire task. Show all the necessary working and reasoning in the spaces provided or on rough paper provided. Make sure your name is on each script used e.g. L4.

Thank you for participating in this activity & research

SECTION A

QUESTION 1

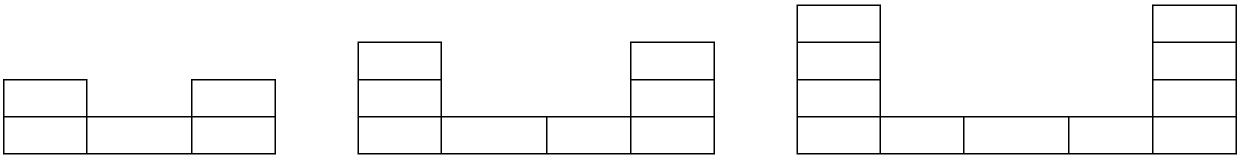
A contractor is asked to build a new set of townhouses in attached clusters of different sizes. He created a plan for one, two and three house clusters as shown below. The builder used computer software to draw and generate line segments used to represent the houses.



3. How many line segments are needed to draw or create a plan for:
   - iv) 4 houses
   - v) 10 houses
   - vi) 50 houses
4. Generate a formula or rule for the above pattern (i.e. n^{th} term)

QUESTION 2

The figures below show three groups of tiles:



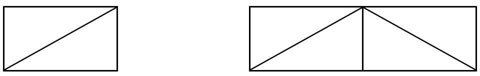
Group 1 **Group 2** **Group 3**

4. Draw tile-group number 5. Explain in words what group number 5 would look like and how many tiles are in the group?
5. How many tiles are in each of the following group number:
   i) Group 6
   j) Group 7
   k) Group 10
   l) Group 50
6. How many tiles are in the n^{th} group of tiles?

In Question 3, numeric patterns were presented in both pictorial and tabular forms.

QUESTION 3

The United Artists Studio makes geometric decorative borders or motifs for picture frames. One motif is made from a pattern consisting of 4 dots and five lines joining these dots as shown diagrammatically and in table form below.



Number of motif (*m*)	1	2	3	4	20	30	63	**m**
Number of dots (*d*)	4							
Number of lines (*l*)	5							

5. Extend and draw the next two motif's designs.
6. Copy and complete the above table for the rest of the motifs' designs or pattern.
7. What is the mathematical relationship between:
   iii) *m* and *d*. ii) *m* and *l*.
8. Will the points representing each of the relationships in i) and ii) lie on a straight line when plotted on the graph?

In Question 4, numeric patterns were presented as a simple sequence of numbers in the question 4 below:

QUESTION 4

Study the number patterns and answer the questions that follow:

a) 5; 12; 19; ___ ; ____ ; ____ ; ....

b) 35; 30; 25; ____; ____; ___ ; ...

3. Determine in each case;
   iv) the next three terms in the pattern.
   v) the 10^{th} term.
   vi) the 50^{th} term.
4. Generate a formula or rule for the n^{th} term

SECTION B

Q 1. Bacteria in a test-tube doubles in every hour. If initially there are 2 bacteria in the test-tube, how many bacteria will there be after:
i) 3 hours ii) 5 hours iii) 10 hours iv) n hours.

Q2. Margaret uses small square blocks to build pictures that form a pattern. The table below shows the number of the blocks she needs to build a particular picture.

Picture number	Number of square blocks
1	1
3	9
4	16
5	25
6	36

a) Determine the number of blocks that will be used in:
i) 11^{th} picture ii) 25^{th} Picture iii) 60^{th} picture iv) n^{th} picture

Q 3. State whether the given statement below is Always True, Sometimes True or Never True, for $x > 0$. Give reason(s) for your answer:

As x increases, 10^x increases.

Q 4. Choose the correct answer in each case; **Always True**, **Sometimes True** or **Never True,** for $x > 0$. Give reason(s) why you say so.

1) $(1/n)^x$ decreases as x increases if $n > 1$

2) $(1/n)^x$ decreases as x increases if $0 < n < 1$

Q 5. Determine the solution to each of the following, explain your reasoning:

1) For which value(s) of x will $4^x = 5^x$

2) For which value(s) of x will $3^4 = 9^x$

Sources: *Number pattern questions Adapted from GET- Learning Outcomes (DoE, 2006. pp. 30 – 31) and from MALATI (2007). Exponents' Questions Adapted from Laridon et al. (2006) Classroom Mathematics Grade 9.*

APPENDIX A2: (i)

INFORMATION LETTER TO LEARNERS

UNIVERSITY OF THE WITWATERSRAND

MATHEMATICS RESEARCH PROJECT

19th June 2009

Dear Learner,

My name is Williams Ndlovu. I am currently doing my MSc degree in Mathematics Education. As part of my degree I am doing a study investigating learners' mathematical reasoning when generalizing in problem solving involving number patterns.

Your school principal has given me permission to send you this letter of invitation to participate in this research study on mathematical reasoning.

Learners who agree to participate in the study will answer a (written) task questionnaire and will be tape recorded in one hour session three times in the month of July/August 2009. These recorded interview sessions will take place after school. The focus in these tape recordings and the task questionnaire will be on the mathematical reasoning when generalizing in number patterns.

I intend to protect your anonymity and confidentiality. Your name(s) will not be used in the final report of this research study. I will remove any reference to personal information that might allow someone to guess your identity.

Remember that you are not obliged to participate. Should you require any further information do not hesitate to contact me on my telephone number as below.

Yours faithfully,

Williams Ndlovu

Cell: 082 758 7491

email: wndlovu2005@yahoo.com

APPENDIX A2: (ii)

CONSENT FORM FOR LEARNERS

UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG

MATHEMATICS REASONING RESEARCH

Researchers Details: Mr W Ndlovu
Email: wndlovu2005@yahoo.com
Cell : 082 758 7491

Supervisor Details: Professor M. Berger
Email : margot.berger@wits.ac.za
Tel(w): 011- 717 3411
Fax : 0865535614
Cell: 082 3444994

Consent **form for learners** participating in the study.

I, ... agree to participate in the research study named above, particulars of which (i.e. problem solving task and interviews) have been explained to me. A written information letter has been given to me to keep.

I, therefore, give consent to the following:

- Tape Recording of the interview in which my voice will be part of the tape recorded text.

  Yes ☐ No ☐

- The possible future use of tape-recorded text for teaching purposes.

  Yes ☐ No ☐

....
Signature of participant Date

....
Signature of witness Date

....
Signature of teacher/researcher Date

APPENDIX A2: (iii)

INFORMATION LETTER TO PARENTS

UNIVERSITY OF THE WITWATERSRAND

MATHEMATICS RESEARCH PROJECT

19th June 2009

Dear PARENT(S),

My name is Williams Ndlovu. I am currently doing my MSc degree in Mathematics Education. As part of my studies I am doing a study investigating learners' mathematical reasoning when generalizing in problem solving involving number patterns.

Your child's school principal has given me permission to send you this letter of invitation to participate in this research study on mathematical reasoning.

Learners whose parents agree that they participate in the study will answer a (written) task questionnaire and will be tape recorded in one hour session three times in the month of July/August 2009. These recorded interview sessions will take place after school. The focus in these tape recordings and problem solving written responses will be how is the mathematical reasoning when generalizing in number patterns is promoted to facilitate learning.

I intend to protect the learners' anonymity and confidentiality. Their name(s) will not be used in the final report of this research study. I will remove any reference to personal information that might allow someone to guess the learners identity.

Be informed that your child is not obliged to participate (i.e. participation is voluntary). Should you require any further information do not hesitate to contact me on my telephone number as below.

If you agree that your child be part of this research study, please complete the consent form attached by signing on the spaces provided and return it to me.

Yours faithfully,

Williams Ndlovu

Cell: 082 758 7491

email: wndlovu2005@yahoo.com

APPENDIX A2: (iv)

CONSENT FORM FOR THE PARENTS

UNIVERSITY OF THE WITWATERSRAN, JOHANNESBURG

MATHEMATICS REASONING RESEARCH

Researchers Details: Mr W Ndlovu
Email : wndlovu2005@yahoo.com
Cell : 082 758 7491

Supervisor Details: Professor M. Berger
Email : margotberger@wits.ac.za
Tel(w): 011 - 7173411
Fax : 0865535614
Cell: 082 3444994

Consent form **for the parents**.

I, ... agree that my child participate in the research study named above, particulars of which (i.e. details of problem solving task and interviews) have been explained to me. A written information letter has been given to me to keep.

I, therefore, give consent to the following:

- Tape Recording of the interview in which the voice of my child will be part of the tape recorded text.

  Yes ☐ No ☐

- The possible future use of tape-recorded text for teaching purposes.

  Yes ☐ No ☐

....
Signature of the Parent(s) Date

....
Signature of witness Date

....
Signature of teacher/researcher Date

APPENDIX A2: (v)

LETTER TO THE SCHOOL PRINCIPAL

UNIVERSITY OF THE WITWATERSRAND
MATHEMATICS RESEARCH PROJECT

8th June 2009

Dear Madam.

My name is Williams Ndlovu. I am currently doing my MSc degree in Mathematics Education. As part my studies I am doing a study investigating learners' mathematical reasoning when generalizing in problem solving involving number patterns.
I am requesting you as the Principal of the school to give me permission to work with learners from one grade 9 class in your school in this research. Should you allow them to participate, they would be asked to contribute in two ways. First, allow them to participate in an open task on problem solving from number patterns and then 6 learners will be selected according to their abilities from the total grade 9 sample to further participate in semi-structured interview which will focus on their written responses to the problem solving task. With your permission the interviews will be tape recorded to ensure that I can make accurate record of what these participants say and do. Once the tape is transcribed, the 6 participants will each be provided with a copy of the transcript to verify that the information they provided is accurate.
I intend to protect their anonymity and the confidentiality of their responses as far as possible. Their names and contact details will be kept in a separate file from any data that they may supply. This will enable linking of data by name. The participants will only be referred by pseudonyms should any publication emerge from this research study. I will do my best to remove any references to personal information that might lead to identification of any participant. However, it should be noted that the study involves a small sample of the participants; it is possible though that someone may still be able to identify some participants. If, however, for some reason they would like their names to be used in publications, the participants will need to make written request to me. Once the research is completed, a brief summary of the findings will be available to school and participants. It is also possible that findings will be presented at academic conferences and hopefully published in national and international academic journals.

Please be advised that the participation of your school in this research study is voluntary. Should you wish to withdraw at any stage, or withdraw any unprocessed data you have supplied, you are free to do so without prejudice. Your decision to participate or not, or to withdraw, will be completely independent of your dealings with the University of the Witwatersrand.

Please indicate that you have read and understood this information by signing the accompanying consent form and return it to me as soon as possible. Should you require any further information do not hesitate to contact me – my telephone numbers are below.

Yours faithfully,

Williams Ndlovu - Cell: 082 758 7491
Email: wndlovu2005@yahoo.com

APPENDIX A2: (vi)

CONSENT FORM FOR THE PRINCIPAL

UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG

MATHEMATICS REASONING RESEARCH

Researchers Details: Mr W Ndlovu
Email : wndlovu2005@yahoo.com
Cell : 082 758 7491

Supervisor Detail : Professor M. Berger
Email : margot.berger@wits.ac.za
Tel(w): 011- 7173411
Fax: 0865535614
Cell: 0823444994

Consent form **for the principal**.

I, ... agree that the school can participate in the research study named above, particulars of which (i.e. details of problem solving task and interviews) have been explained to me. A written information letter has been given to me to keep.

I, therefore, give consent to the following:

- Tape Recording of the interview in which the voice of the participating learners will be part of the tape recorded text.

  Yes ☐ No ☐

- The possible future use of tape-recorded text for teaching purposes.

  Yes ☐ No ☐

- The participants being interviewed at some point during the research.

  Yes ☐ No ☐

- Tape recording of the participants' interview sessions with the researcher.

  Yes ☐ No ☐

Signature of the Principal Date

Signature of witness Date

Signature of teacher/researcher Date

APPENDIX A2: (vii)

University of the Witwatersrand
Private Bag 3
P.0. Box Wits, 2050
Johannesburg
9th June, 2009.

ATT: Ebrahim Farista
Gauteng Department of Education, Room 911
111 Commissioner Street,
Johannesburg
Cc: Nomvula Ubisi: Room 910

Dear Sir/Madam;

RE: Application for conducting a research study at Greenside High School

I am currently a Master of Science student registered with the University of the Witwatersrand in the school of Education in Johannesburg. In partial fulfilment of the requirements for the degree of Master of Science, I am required to submit a research report.

To this effect, I apply for permission to undertake a research study at Greenside High School in your district. The purpose of the study is to ***investigate Grade 9 learners' mathematical reasoning when generalizing in problem solving involving number patterns***.

The significance of the study is that it will inform curriculum planning for teacher education; inform better ways to support teachers' effective classroom practices when introducing algebra and also benefit the on-going debates in mathematics education research community. About 30 students (14 - 16 years) are required to participate in the study by engaging with a problem solving task on number patterns. This will be followed by answering interview questions.

Thank you for your anticipated cooperation in this regard.

Yours Faithfully,

Williams C. Ndlovu

Student Number: 0616576G

Cell: 0827587491; email: wndlovu2005@yahoo.com

APPENDIX B: ANALYSIS OF QUESTION 1: (QRASS)

	Levels	Category				Routine	Category	
Learner Code	1: next term i.e. 4^{th} term	2: 10^{th} & 50^{th} term	3: The n^{th} term	Overall stages	TLA (6)	Routine for few terms	Routine for 10^{th} & 50^{th}	General formula for the n^{th} term
L1	√	√ √	√	3	6	Re	G	Hn = 1 + 5n
L2	√	X X	X	1	1	Co	Co	6N + 5
L3	√	√ √	√	3	6	Ex/P	Ex	Tn = 5n + 1
L4	√	√ √	√	3	6	Re/Co	Ex/SA	Ln = 5(n) + 1
L5	√	√ X	X	1 ½	2 ½	Co	Co	-
L6	√	X *	√	2	4	Co	G	-
L7	√	√ √	√	3	6	Co	Co	6 + (n – 1) x 5 = n^{th}
L8	√	√ √	√	3	6	EX/ R	Ex/SA	(5 x n) + 1
L9	√	√ *	X	1 ½	2 ½	Re/Co	G	5 x n – 1
L10	√	√ √	X	2 ½	5	Re/Cu	Re/Co	-
L11	√	√ X	√	2 ½	5	Re	Ex	4n + (n +1)
L12	√	√ √	√	3	6	Re	Ex	4x + (x + 1) = y
L13	√	X X	X	1	1	G	G	-
L14	√	X X	X	1	1	Co	G	-
L15	√	√ X	√	2 ½	5	Co	Ex	(1 - x) + (x * 6)
L16	√	√ X	X	2 ½	5 ½	Re	Re	-

L17	√	√ √	√	4	6	Re	Ex/Se	tn = 1 + 5n
L18	√	√ X	X	1 ½	1 ½	G	-	5n – 1
L19	√	√ √	√	3	6	Co	Cu	tn = 6 + (x – 1) x 5
L20	√	√ √	√	3	6	Co	Ex/Se	Tn = 5n + 1
L21	√	X X	X	1	1	Co	-	2n
L22	√	√ χ	X	2 ½	5	G	G	6n – (n – 1)
L23	√	√ √	√	3	6	Ex/Co/R	Ex/SA/Se	Tn = 5 x n + 1
L24	√	√ √	X	2 ½	5 ½	G	G	-
L25	√	√ √	√	3	6	Re/Co	Ex/P	(5 x n) + 1
L26	√	√ X	X	1 ½	2	Re	-	-
L27	√	√ X	√	2 ½	5 ½	Ex/R	Ex	5 x (n – 1) + 6 = tn
L28	√	√ √	√	4	6	Ex/SA/R	Ex/SA/R	Tn = 6 + (5n – 1x5)
L29	√	√ χ	X	1 ½	2	Co	-	5n
% √	100	83 48	56					
% X	0	17 45	44					
% *	0	0 7	0					

NOTE: EX: Explicit
SA: Structural analysis
CO: Counting
RE : Recursive
G : Guess

APPENDIX C: ANALYSIS OF QUESTION 2: (QRASS)

Categor (TLA)	Levels		Category				Routine	Category	
Learner Code	1: Draw 5th group tiles	2: 6th, 7th & 10th terms	3: 5oth term	4: The nth group of tiles	Overall stage descr	TLA (10)	Routine for 5th group tiles	Routine for 6th, 7th & 10th terms	General formula for the nth term
L1	√ √	√	X	X	2	3	Re	G	2n + 3
L2	√ χ	√	X	X	1 ½	2 ½	Co	Co	-
L3	√ χ	√	√	√	3 ½	9 ½	Ex/R	Ex	3n + 2
L4	√ √	√	√	√	4	10	Re/Co	Ex/SA	(2n + 2) + n
L5	√ χ	√	X	X	1 ½	2 ½	Co	Co	Tn = 3 tiles + 2
L6	√ √	√	√	X	3	5 ½	Co	G	3n
L7	√ χ	√	√	X	3	5 ½	Co	Co	2n + (2)
L8	√ √	√	√	√	4	10	EX/R	Ex/SA	N x 3 + 2
L9	√ X	√	*	X	1 ½	2 ½	Re/Co	G	-
L10	√ √	√	√	X	3	5 ½	Re/Cu	Re/Co	3n + t = Tn
L11	√ χ	√	*	√	2 ½	6 ½	Re	Ex	4x – (x – 2) = y
L12	√ √	√	√	√	4	10	Re	Ex	N (2) + 2
L13	√ √	√	*	√	3	5 ½	Co	Cu	-
L14	√ √	√	√	√	4	10	Co/Ex	Ex	5 + 3(t -1)
L15	√ X	√	X	√	2 ½	6 ½	Co	Ex/P	Tn = 2n + (n + 2)
L16	√ χ	√	√	√	3 ½	9 ½	Re	Re	2(N + 1) = T

L17	√ x	√	√	√	3 ½	9 ½	Re	Ex	5t – 2 – 2(t – 2)
L18	√ √	√	√	√	4	10	Ex	Ex/Co	5 + (t – 1) x 3
L19	√ √	√	X	X	2	3	Co	Cu	-
L20	√ √	√	X	X	4	10	Co	Ex	N x 4 – (N – 2)
L21	√ √	√	X	X	2	3	Co	G	-
L22	√ √	√	*	*	2	3	G	G	-
L23	√ √	√	√	√	4	10	Ex/Co/Se	Ex/SA/w	Tn = n x 3 + 2
L24	√ χ	√	*	*	2	3	G	G	-
L25	√ √	√	√	√	4	10	Re/Co	Ex	tn = (n + 1) x 2 + n
L26	√ x	√	√	√	3 1/2	9 ½	Ex	Ex/P	5 + (n – 1) x 3
L27	√ √	√	√	√	4	10	Ex/P	Ex	3 (t – 1) + 5
L28	√ √	√	√	√	4	10	Ex/SA	Ex/SA/R	Tn = 3 x n +2
L29	√ χ	√	X	X	1 ½	2 ½	Co	G	-
	100 59	100	65	56					
% x	0 41	0	21	37					
% *	0 0	0	14	7					

NOTE: EX: Explicit
SA: Structural analysis
CO: Counting
RE : Recursive
G : Guess

APPENDIX D: ANALYSIS OF QUESTION 3: (QRASS)

Categories (TLA)	Level	Category						Routine		Categor	**Samples of n^{th} term**
Learner Code	1: Tabulate the next motifs'	2: 20, 30^{th} & 63^{rd} terms	3: Are these relationships non/linear?	4: Relations of *n* vs. *d* and *n* vs. *l*	5: The n^{th} position of the motifs'	Overall stage descriptor	TLA (15)	Routines for stages 1 & 2	Routines for the relations	Routine for non/linear relationships	General formula for the n^{th} term
L1	√	√ x	√	√	√	5	15	RE	EX	ES	2n+2; 1+4n
L2	√	√ *	√	X	√	4	11	CO	RE	EX	T=2n+2; 4n+1
L3	√	√ *	√	√	√	5	15	RE	EX	EX/SA	D=2n+2; L=4n+1
L4	√	√ X	√	√	√	5	15	CU	RE	EX	Tn=2n+2; Tn=4n+1
L5	X	* x	√	*	*	1	3	CO	G	-	-
L6	√	√ √	X	X	√	3	8	RE/E	SA	EX	(n-1)2+4; 4xn+1
L7	√	√ *	X	X	√	3	8	CO	RE	EX	M(2) +2 & 4M+1
L8	√	√ √	X	X	√	3	8	RE	G	G	4+(n-1)2;5+(n-1)4
L9	x	* x	√	*	*	1	3	RE	-	G	-
L10	√	√ *	*	*	√	3	8	Co/Cu	RE	EX	D 2(n+1); L=4n+1
L11	√	X x	X	*	*	1	1	CO	G	-	-
L12	√	X x	X	*	*	1	1	CO	G	-	-
L13	√	X x	X	*	*	1	1	CO	G	G	-
L14	√	X *	X	*	*	1	1	CU	-	G	-
L15	√	√ √	X	X	√	3	8	CO	EX/w	EX/cu	tn=2n+2; 4n+1
L16	√	X x	*	*	*	1	1	-	-		-
L17	√	√ √	√	X	√	3	11	RE/CO	EX	REEX	2d+2 & 4L +1
L18	√	X x	√	X	*	2	4	CO	-	-	-

L19	√	√ *	√	*	√	4	11	EX	RE	EX	Tn= 2xn+2; 4xn+1
L20	√	√ X	√	*	√	4	11	CO	EX/P	EX/RE	Nx4 + 1; 2(N+1)
L21	√	√ x	X	√	√	4	12	EX	SA	EX	4(x) +1; 2*x + 2
L22	X	X *	X	X	X	0	0	-	-	-	-
L23	√	√ √	√	X	√	3	11	SA	EX/P	EX/SA	d=2n+2; L=4n+1
L24	√	X x	*	*	*	1	1	EX	SA	CO/EX	-
L25	√	√ X	√	√	√	5	15	EX	EX	EX	2xn + 2; 4xn + 1
L26	√	X *	X	X	X	1	1	-	-	-	d = 2n; lines = 4n
L27	√	* x	*	*	*	1	1	-	-	-	-
L28	√	√ √	√	X	√	3	11	EX	EX/SA	EX/RE	2n+2 & 4n+1
L29	√	X x	*	*	X	1	1	CO	-	-	2n +3; 5+4n
% √	90	55	45	17	55						
% X	10	34	38	38	10						
% *	0	11	17	45	35						

NOTE: EX: Explicit
SA: Structural analysis
CO: Counting
RE : Recursive
G : Guess

APPENDIX E: ANALYSIS OF QUESTION 4a: (QRASS)

Categories		Levels	Category			Routine	Category	
Learner Code	1: next 3 terms	2: 10^{th} & 50^{th} term	3: The n^{th} term	Overall stage descriptor	TLA (6)	Routine for next 3 terms	Routine for 10^{th} & 50^{th} term	General formula for the n^{th} term
L1	√	√ √	x	2 ½	3	Co	Ex	7n – 2
L2	√	√ χ	√	2 ½	5	Co	Co/Ex	(n x 7) – 2
L3	√	√ χ	√	2 ½	5	G	Cu	N(7) -2
L4	√	√ √	X	2 ½	3	Co	G	-
L5	√	√ X	X	1 ½	2	Co	G	-
L6	√	√ √	√	4	6	Re/Co	Ex	7n-2 = tn
L7	√	√ X	√	2 ½	5	Co	Cu	n^{th} x 7 – 2
L8	√	√ X	X	1 ½	2	Co	G	N + 2
L9	√	√ x	X	1 ½	2	Co	G	-
L10	√	√ √	X	2 ½	3	Co	Co/G	7n
L11	√	√ √	X	4	6	Re	Ex/Re	(N x 7)+ 2
L12	√	√ √	√	4	6	Re/Co	Ex/Re	(7n) – 2
L13	√	X X	X	1	1	Co	Co	(7n) + 5
L14	√	χ χ	X	1	1	Co	Co	N + 7
L15	√	√ √	√	4	6	Re/Co	Ex/P	t = 5 + (n – 1)*7
L16	√	χ χ	X	1	1	Co	G/Cu	7xn x2

L17	√	√ √	√	4	6	Co	Cu/R	4(n-1) + 3n +2
L18	√	√ X	√	2 ½	5	Re	Ex	5n + (2n – 2)
L19	√	√ √	X	2 ½	3	Co/Cu	Ex	N(7) – 2
L20	√	X X	X	1	1	G	G	(nx5) – 2
L21	√	√ X	X	1 ½	2	Co	G	-
L22	√	√ √	√	4	6	Re/Co	Ex	Tn = 7n -2
L23	√	√ √	√	4	6	Cu/Co/Sa	Ex/SA	5 + (n – 1) x7
L24	√	√ √	√	4	6	RE	EX	7(n-1) + 5
L25	√	√ X	√	2 ½	5	Co	Co/EX	3(3n-2)-2(n-2)
L26	√	√ √	√	4	6	Cu/Re	Co	(n-1)x7 +5
L27	√	√ √	√	4	6	Re	Ex	N x 7 – 2
L28	√	√ √	√	4	6	Cu/P	R/Ex/SA	Tn = 7n - 2
L29	√	√ χ	X	1 ½	2	Co	G	-
% √	100	86 51	52					
% X	0	14 49	48					
% *	0	0 0	0					

NOTE: EX: Explicit
CO: Counting
G : Guess
SA: Structural analysis
RE : Recursive
R : Relational approach

APPENDIX E: ANALYSIS OF QUESTION 4b: (QRASS)

Categories		Levels	Category			Routine	Category	
Learner Code	1: next 3 terms in the pattern	2: 10^{th} & 50^{th} term	3: The n^{th} term of the pattern	Overall level descriptor	TLA (6)	Routine for the next 3 terms	Routine for 10^{th} & 50^{th} term	General formula for the n^{th} term
L1	√	√ χ	√	2 ½	4 ½	EX	EX	5(-n) + 40 = tn
L2	√	√ χ	√	2 ½	4 ½	RE/CU	EX	Tn = - 5n + 40
L3	√	√ χ	X	1 ½	2	CO	G	Tn = position term – 5
L4	√	√ χ	X	1 ½	2	RE	CO	- 5 x n x 35 = tn
L5	√	√ X	X	1 ½	2	CO	G	Tn = - 5n
L6	√	√ χ	√	2 ½	4 ½	CO	EX	35+(n-1)x(-5) =n^{th} term
L7	√	√ χ	√	2 ½	4 ½	EX/CO	EX/CU	35 – 5x(n-1) = Tn
L8	√	√ χ	√	2 ½	4 ½	RE	EX	5(8-n)
L9	√	X X	X	1	1	CO	G	t = - 5 + 40
L10	√	√ *	*	1 ½	2	CO	G	-
L11	√	√ *	*	1 ½	2	G	G	-
L12	√	√ √	X	2	3	CO	CU/EX	35 + N (-5)
L13	√	√ χ	*	1 ½	2	CO	G	-
L14	√	√ χ	*	1 ½	2	CO	RE/G	-
L15	√	√ X	√	2 ½	4 ½	RE/CO	RE/CO	5 x (8 – n) = n^{th} term
L16	√	√ √	√	4	6	RE/CO	EX/SA	
L17	√	√ √	X	2	3	CO	EX	N x(-5) + 35

L18	√	√ *	*	1 ½	2	CO	CO	-
L19	√	√ *	*	1 ½	2	RE	-	-
L20	√	√ *	*	1 ½	2	CO	-	-
L21	√	√ *	*	1 ½	2	CO	-	-
L22	√	√ √	X	2	3	EX	RE/EX	Tn =n (- 5) +35
L23	√	√ √	√	3	6	EX/R	EX/SA/Se	T =(- 5) x n + 40
L24	√	√ *	*	1 ½	2	CO	-	-
L25	√	√ *	*	1 ½	2	G	G	-
L26	√	√ *	*	1 ½	2	CO	-	-
L27	√	√ √	X	2	3	CO	RE/G	Nth term = -5n – (+ 40)
L28	√	√ √	√	3	6	EX/R	EX/SA/Se	Tn = 40 - 5 n
L29	√	√ *	*	1 ½	2	RE	-	-
% √	100	97 24	31					
% **X**	0	3 42	28					
% *	0	0 34	41					

NOTE: EX: Explicit, SA: Structural analysis, CO: Counting, RE : Recursive, G : Guess, R : Relational

APPENDIX F: INTERVIEW TRANSCRIPTIONS FROM THE SIX PARTICIPANTS

<u>Interpretations in the Transcriptions and Episodes</u>:

R represents the researcher; **L5** represents learner coded as number 5;

[ ] : Pause ; **{ }**: Learner counts or reads loud; .... : Pattern continues.

TRANSCRIPT 1: (Learner 5: L5)

1 R: Welcome to this follow-up interview on the research questionnaire you responded to recently.

2 L5: Thank you Sir.

3 R : Take me through your written responses, how did you start with the cluster house – problem?

4 L5: I figured the number of line segments on a given set of cluster houses by counting say at position one, two and so on. And then I realized that the pictures gave a similar number representation to; 6, 11, 17 , ... , by removing one line segment on one house to get to the next and then added five more line segments that's how many more lines I would need any way in the next position. So I did the same thing for all the questions that followed in order to get number of line segments.

5 R : Show me how you determine the number of line segments for 10 or 20 houses?

6 L5: You mean like 2, 3 , 4 .... until ten and twenty? Ok, I think I figured out 51 lines in 10 Houses and then use this to find the line segments now in 50 houses. So multiplying 51 by 2 gives 102 line segments for 20 cluster houses.

7 R : Would you use a different approach for 50 houses?

8 L5: Yes, but I think needs more time to find another method. So eeeish!,.... You just do the

Same method as for 10 or 20 houses above.

9 R : Say more about that....

10 L5: I mean if 10 houses = 51 lines, 20 houses = 102 line segments, then 40 houses will give 102 multiply by 2 = 204 lines. So 5o houses will be (10 + 40) houses = (51 + 204) line segments.

11 R: First, I would like if you can take me through your solutions to questions 1 -4 starting with question 1.

12 L5: At first I looked at Q1 with cluster houses in three different positions. And saw that in the first position, there is one house made of 6 line segments followed by 2 houses with 11 lines and three houses having 16 lines. This gave me number pattern: 6; 11; 16; ....

13 R: What does this pattern mean to you?

14 L5: It gave me a kind of an idea to think of working out a formula to represent this pattern. And Yes! to help calculating lines for next set of cluster houses.

15 R: Explain more on what you did.

16 L5: I tried first to work with a formula Tn = n x 6 – 1. Aaah! But when I tried it to find number of line segments in first house when n = 1. It didn’t work out.

17 R: Did you stop or carried on?

18 L5: No! I went to second figure and made n = 2 in the same formula and it worked well. But then I had to try n = 3 and it did not work this time and found that I actually had to subtract twice than the first time.

19 R: What did that tell you.

20 L5: Something is wrong with my formula or substitution.

21 R: Why did this approach not work?

22 L5: That is why I had to carry on investigating more especially about the formula I am using.

23 R: Did further investigation help you.

24 L5: Yes, this formula Tn = n x 6 – 1 So I did it in a different way now I remember.

25 R : Take me through your different way.

26 L5: I figured the number of line segments on a given set of cluster houses by counting say at position one, two and so on. And then I realized that the pictures gave a similar number representation to; 6, 11, 17 , ... , by removing one line segment on one house to get to the next and then added five more line segments that’s how many more lines I would need any way in the next position. So I did the same thing for all the questions that followed in order to get number of line segments.

27 R: Show me how you determine the number of line segments for 10 or 20 houses?

28 L5 : You mean like 2, 3 , 4 .... until ten and twenty? Ok, I think I figured out 51 lines in 10 Houses and then use this to find the line segments now in 50 houses. So multiplying 51 by 2 gives 102 line segments for 20 cluster houses.

29 R: Would you use a different approach for 50 houses?

30 L5: Yes, but I think needs more time to find another method. So eeeish!,.... You just do the same method as for 10 or 20 houses above.

31 R: Say more about that....

32 L5: I mean if 10 houses = 51 lines, 20 houses = 102 line segments, then 40 houses will give 102 multiply by 2 = 204 lines. So 5o houses will be (10 + 40) houses = (51 + 204) line segments.

33 R : Did this new formula work?

34 L5: When I checked the formula I tried it out on the first house and it worked. Then on the second house, it also worked. I did the same with the third house. In each diagram of the houses, the formula worked.

35 R: What conclusion do you make/draw from this.

36 L5: I concluded that in order to find the number of line segments one had to use Tn = 5 x n + 1 and yaaah! That’s it.

37 R: How did you do the 4^{th} house(s)?

38 L5: Ok. I said n which is 4 and substituted for an amount Tn and got 20 + 1= 21.
39 R: How do you write that?
40 L5: Its Tn = 5xn +1 = 5x4 + 1 = 21 lines.
41 R: What do the symbols in your/this formula represent?
42 L5: Sir, Tn is term or number and n is its position.
43 R: What did you do on the question about 50 houses?
44 L5: Also like n=4 but this time n=50. This gives Tn = 5xn +1 = 5x50+1= 251.
45 R: What is this 251 for?
46 L5: Number of line segments in 50 houses.
47 R: If you have number of line segments, would be able to position of the cluster houses?
48 L5: Position! Yes Sir.
49 R: How would you do that?
50 L5: Suppose my lines are 251, in order to get position – n... of the houses I think one need to work like going back.
51 R: Explain more...
52 L5: I will make 251 = n x 5 + 1 and solve for n instead of finding Tn.
53 R: What answer can you get in this case.
54 L5: This is what can be done:

.... her working went like;....

Tn = n x 5 + 1
251 = n x 5 + 1
251 – 1 = n x 5
250 / 5 = (n x 5) / 5
250 / 5 = n
50 = n i.e. 50th position gives 251 line segments.

55 R: What have you learnt from doing this ?
56 L5: Ooooh! I see it's the same question in the questionnaire only to work backwards. So you can get answers quickly to these questions if one uses a formula method.
57 R: What if you difficulties in finding a formula, how would you do these questions?
58 L5: By counting line segments and number of houses every time
59 R: How different is that really say if 500 houses?
60 L5: No! 500! That's a huge number. I need just to use the formula than drawing pictures of houses up to 50. This will take half a day and will need more pages.

QUESTION 2

61 R: How did you do this question?
62 L5: I looked at fig 1 and counted the blocks or tiles. Then I went to fig 2 and did the same until figures 3 and 4.
63 R: Why did you have to do that?
64 L5: To see if there is a pattern that I can work out a formula.
65 R: Did counting tiles per group helped find a formula?
66 L5: Yes, my formula is 3 x 1 gives 2 and this becomes n x 3 + 2 to get the amount of blocks in a group.
67 R: Is what you are saying shown in your solutions?
68 L5: No. I only wrote answers and not showing the formula Tn = n x 3 +2.
69 R: In this formula you have n x 3 but in the previous one you had n x 5, Why these are different?
70 L5: Yeah! When I looked at fig 1 = 5, blocks, fig 2 = 8 but 5 + 3 = 8. Then decided to keep adding 3 to get the next term or number.
71 R: Will this work for figures 1 – 4?
72 L5: Yes it does work.

73 R: Where does the coefficient 3 of n in the formula come from?

74 L5: In fact it comes from the difference in blocks between any two positions.

75 R: Is the number in the first formula $T_n = 5n + 1$ the difference?

76 L5: Ooooh! Am not sure about that.

77 R: Would you go back to find out or verify?

78 L5: Ok. In fig 1 = 6; fig 2 = 11 and fig 3 = 16; .... Eeeeyah! 5 is the difference. So 3 and 5 are kind of numbers related to the difference in the patterns.

79 R: Did you learn something having answered this questionnaire?

80 L5: Yes.

81 R: Tell me more....

82 L5: In order to get a term or a number, I would have to multiply n by the difference between the figures or tiles from each figure and think about other number to + or – afterwards.

83 R: How many tiles are in group 10?

84 L5: 32

85 R: How did you find 32 tiles?

86 L5: I drew more pictures and counted the tiles in my drawings.
(I said, $T_n = 3n + 2$ and make n = 10 to get $3 \times 10 + 2 = 30 + 2 = 32$)

87 R: Can this pattern carry on?

88 L5: Yes it can.

89 R: Explain more....

90 L5: Because the number of groups increase and also number of tiles is increasing.

91 R: What would you do if you have 602 tiles?

92 L5: Since $n \times 3 + 2 = 602$ then $3n = 602 – 2 = 600$. Therefore $n = 600/3 = 200$

93 R: Explain what this 200 is for?

94 L5: Yaaaah! I see, 200 is actually the group number.

95 R: What other lesson have learnt from this exercise?

96 L5: I think what is coming out clearly is.... if you have...[ ] ... a specific pattern and so there will be a specific difference which you can use to make a formula.

97 R: What else (method) could you have done to answer this question?

98 L5: Counting the blocks or tiles and using the formula. Yes, drawing more pictures. But a formula is the best and easy because counting or drawing would take more time.

QUESTION 3

83 R: Take me through this question.

84 L5: Thanks. They asked us to work out the number of dots and lines used to make a motif. So I first looked at the difference of the dots and fig 1 had 4; fig 2 = 6 and I immediately saw the formula coming.

85 R: What formula was this?

86 L5: Its $T_n = n \times 2 + 2$

87 R: Explain the 2's in this formula.

88 L5: The 2 multiplying n is the difference and the other 2 I decided to add on to get the next term as the numbers are increasing. I should have subtracted if decreasing ********

89 R: Did you use this formula to complete the table of values?

90 L5: Yes, to get the number of dots and a different formula to get the number of lines.

91 R: Explain how you got the formula for the number of lines.

92 L5: First my two formulas are $T_n = 2n + 2$ and $T_n = 4n + 1$; Ok in fig 1 = 5 lines; fig 2 = 8 lines then yaah! Sorry fig 2 had 9 and not 8. Then I said n which is the position multiply by 4 add 1 got $T_n = n \times 4 +1$.

93 R: How did you use these formulas in completing the table?

94 L5: You see in this table, they gave us the number of motifs' e.g. motif 1 I said its $1 \times 2 + 2$ gives dots and $1 \times 4 + 1 = 5$ gives lines. So this will look like this:

DOTS: 1x2+2; 2x2+2; 3x2+2; 4x2+2; 5x2+2; 6x2 +2; 7x2+2; ....
LINES: 1x4+1;2x4+1; 3x4+1; 4x4+1; 5x4+1 ; 6x4+1 ; 7x 4+1; ....

95 R: Did you find it easier completing the table?
96 L5: At first it was not difficult and did not require the use of formula but as the position of motifs' grew bigger or increases needed to use a formula.
97 R: Would you still complete the table without a formula?
98 L5: Yes.
100 R: Explain how...
101 L5: I would count until I get to that number of motifs' but takes quite some time than using a formula.
102 R: What would be the motif position if you had 100 dots?
103 L5: Isn't this same thing of working backwards?
104 R: You decide about the method?
105 L5: OK. You would say n x 2 + 2 = 100

$$
\begin{aligned}
n \times 2 &= 100 - 2 \\
(n \times 2)/2 &= 98 / 2 \\
n &= 49^{th} \text{ position}
\end{aligned}
$$

106 R: Suppose you have 801 line. What position motifs' could this be?
107 L5: I will use a similar approach to the above. ....She works out using 801 = 4 x n + 1 and found ... , n= 200
108 R: How easy do you find this method?
109 L5: Its easy when you practice or understand. But I think am using how to use operations like in equations.
110 R: How did you know the use of operations with equations?
111 L5: We learnt in class and these are mathematical rules.
112 R: Do you know the rules very well?
113 L5: Yes. I know them BUT I don't know how to explain to somebody but I can use them correctly.
114 R: What kind of graph would you get if these values are plotted on the x-y axes?
115 L5: A straight line, because they are both increasing.
116 R: Did you get a straight line graph?
117 L5: No
118 R: Why are they not straight lines?
119 L5: It's hard, I really don't know.
120 R: What mathematical relationship is shown or represented in your graphs?
121 L5: I said they are both increasing, therefore DIRECTLY proportional.
122 R: Explain more...
123 L5: Like for example, if the lines were increasing but dots decreased then this would be INVERSE proportion.

QUESTION 4

124 R: How did you answer this question?
125 L5: We are now given numbers and not pattern through pictures as in questions 1- 3.
126 R: In what way did that help you?
127 L5: I saw immediately the difference between any two terms is the same.
128 R: What were your next three terms?
129 L5: I worked the difference in the numbers first and then use it to find other numbers.
130 R: How did you do that?
131 L5: You have to use a formula or if not then you have to count the numbers.
132 R: Explain more....
133 L5: Yeah! By counting you just say it starts off with 5 so 5+7 = 12; 12+7=19; 19+7=96 . No!

No! No! its actually 26.

134 R: How did you get 96?

135 L5: No Sir, It was just a mistake.

136 R: Let's go back how you do the counting method.

137 L5: Yes, you carry on and on adding 7 to get the next numbers but a formula would be easier.

138 R: Can you take me through your response first. And then tell me what you would do if there were 50 numbers and want to find the 50^{th} term or number.

139 L5: With 50 terms given as in part iii), I would probably....... I mean what I could have done is, ..., just look at 5 as for first number and then adding seven to second gives 12 at position 2, I will keep on doing this until I have the 50^{th} term.

140 R: Why adding seven?

141 L5: I can see that the numbers in the pattern are changing by seven in 4a and 5 in 4b

142 R: Okay, are you talking about 4b as well?

143 L5: Yeah. All these are similar sequences because they use numerical terms and I have to always start from the first number and see how to get the second or third, ... etc.

144 R: Will you keep adding the five until 50^{th} term in 4b?

145 L5: Not really. But on the first question 50^{th} term yes. I will add five but the pattern is decreasing, and its confusing.

146 R: Explain more about this adding of five

147 L5: I mean 50^{th} house = 50 + (-5) = 55 + 5 = 55

148 R: Why do you have minus five added to 50?

149 L5: 50 is the position of my lager value and minus 5 is what I will add ever time since the pattern is a decreasing one. So in 4a, the 50^{th} term is 50 + (7) = 57

150 R: Does this method always work for large values?

151 L5: I think so. It should work as well for larger position numbers.

152 R: What will be the nth term in 4a and 4b?

153 L5: In 4a, its n + (7) = 5 + 7 and in 4b I found it to be n + (-5) = n + 5 or n – 5.

154 R: Is (n + 5) the same as (n – 5)?

155 L5: Yes, n stands for position of the term am looking for and 5 or minus 5 is just what I need to keep adding in order to get the next terms.

138 R: Did you really use a formula here?

139 L5: Yes I did. My formula was Tn = 7 x n – 2 and I checked when I got the answers to the questions.

140 R: Can you show how you checked.

141 L5: I did this; T1 = 7x(1) -2 = 5; T2 = 7x(2)-2 = 12; T3 = 7x3(3)= 19 So my next three terms were: T4 = 7x(4)=26; T5=7x(5)=33 and T6=7x(6) = 40

142 R: What would you do or say when the number do not correspond to those in a pattern as you do the checking?

143 L5: Then either my substitution is wrong or formula.

144 R: What is the term on nth position?

145 L5: Is the same formula Tn = 7n – 2.

146 R: What position could you get for the term 208?

147 L5: This would be 7xn – 2 = 208. Therefore, 7n = 210 and n=30

148 R: Are sure with your calculation for n=30?

149 L5: Very much so.

150 R: What can you if you got n = 30,76?

151 L5: Mistake in the calculation or position is not there. May be the pattern has changed.

152 R: Are saying it is possible to get the value of position n = 30,76?

153 L5: No am not saying that.

154 R: Explain more.....

155 L5: Because, then there is just,..... [ ] I don't know how to explain this mathematically.

Ooooh! But now I think the position needs to be a whole number... and one cannot get a position with a remainder.

156 R: How did you do the last part.

157 L5: I found it difficult because the numbers are decreasing so I did not know what to do with the difference of 5 between the numbers in this pattern.

158 R: Did you write down any answer?

159 L5: I was thinking about the formulas to be n + 7 and n – 5 or n+ 5.

160 R: Your General Comment about this work.

161 L5: I think its much easier to work with a formula on number patterns than other methods. The questions can be in different formats but if you know how to get a formula its like you working in algebra now.

162 R: Why Algebra?

163 L5: Because Algebra is helping us to calculate or find the nth term or position in a number pattern. So the general formula is algebra. You see, Tn and n are unknowns.

164 R: Thank you very much for your time.

165 L5: Thank you Sir.

TRANSCRIPT 3 (L28: DK)

1 R: Welcome to the follow-up interview about the research questionnaire on number patterns

2 L28: Thank you.

3 R: Take me though your responses starting with question 1

4 L28: In this question I noticed that every time when I added another house, it increases by 5 line segments.

5 R: What do you mean adding another house every time.

6 L28: You know what sir, a house has [ ], I don't understand the question you are asking sir.

7 R: What do you ...... Repeated the question.

8 L28: Ok. In one house there 6 line segments and with two houses its 11 and three houses its 16 lines. As you see from that you gte 6; 11; 16; ...... as a pattern where 5 is an increase every time.

9 R: What does this mean to you?

10 L28: Its a number pattern which can be worked out using house positions and number of line Segments, ...

11 R: Why are you working with house-positions and the number of line segments?

12 L28: First I changed the diagrams given into numbers. So these are now terms each on its Specific position foe example, when n=1 Tn= 6; n=2 Tn=11 and n=3 tn=16 .... And working as we do with linear graph or equation n=x and Tn=y. I found that $\Delta n/\Delta Tn = 5$ and substituting this into y = mx+c when n=1 (x), Tn =6 (y) then c=1

13 R: Explain more on how you used the values from these calculations.

14 L28: In this situation, it means y = mx + c is the same as Tn = 5n + 1.

15 R: Are the two expressions equal?

16 L28: I would not say it in that way. I think the "same" here means the give the same line graph on the Cartesian plane.

17 R: So talk to me how you used your general rule Tn = 5n+1?

18 L28: I counted the line segments at position n = say, 1, 2, 3, 4, lines in a rectangle and 5, 6 extra two lines that are making the roof of the house. This help to make sure if my formula is correct (justification). I did the same for position n=2.

19 R: Okay, how many lines are at position 2?

20 L28: Three verticals and four horizontal lines plus another four as the roofs making eleven line segments by addition.
21 R: Is 11 the number of line segments for two houses?
22 L28: Yes, I checked this by the formula as well.
23 R: Did you follow this procedure for 50 houses?
24 L28: No, 50 is a big number. I just used my formula without checking it because this worked for n=1 & 2. A formula is very good for working with large numbers since you cannot draw 50 pictures of houses in order to count the number of line segments in them.
25 R: If you consider the numeric pattern of n and Tn above, why is the number of line segments going up by 5?
26 L28: Every time the position number increases by one, you are actually adding a house by five lines more than the previous. For example if I add a house on n=3, this last line (pointing to the picture) will be shared with the extra house so I just need 5 lines to complete the diagram or picture.
11 R: Explain about your solution from the first part of the question.
12 L28: With 4 houses, its 21 lines.... for 10 houses its 51 lines and 50 houses will have 251 line segments.
13 R: How do you get 21; 51; 251?
14 L28: I used a general formula to do this.
15 R: How did generate this formula you are talking about?
16 L28: No method really.... [ ]. I just came up with a formula anyway.
17 R: How did you decide that this is the formula I will use in this question?
18 L28: Fine. Yaaaah! I looked at the line segments of these cluster houses and saw the common difference is 5 and I then multiplied 5 by n, n being the number of house and added 1,there was my formula.
19 R: Is that what you call a general formula?
29 L28: Yes but it can be written clearly as 5n + 1
21 R: Is 5n + 1 a general formula/
22 L28: Yes, used to find how many line segments say are in 3 or 5 houses.
23 R: How did you use this formula?
24 L28: Lets say there are 2 houses, then 2 x 5 + 1 because the number of houses is represented by n and 5 is the difference then you add 1 line segment.
25 R: Where is the formula from your answer script?
26 L28: Here it is, ... {She points where the formula was written}. It is 5n + 1.
27 R: You are saying 5n + 1 but I see Tn = 5n +1 on your page.......
28 L28: Oh, yaah! This Tn just stand for term 4.
29 R: What do tou mean by term 4?
30 L28: Yes, its just number of houses.
31 R: What is your real formula in this case?
32 L28: Its Tn = 5n +1
33 R: Did you work with this formula in answering these questions?
34 L28: Yes I did.
35 R: How did you use the formula again?
36 L28: I use it to answer the following questions.
37 R: Please show how you found line segments for 50 houses.
38 L28: I multiplied 50 by 5 and added 1 to get 251 lines
39 R: What do you call that method in maths?
40 L28: Oh Yes Sir, Its substitution.
41 R: Will this pattern continue?
42 L28: Yes it will until we have n number of houses.
43 R: How would work out the position if you are give 501 line segments?

44 L28: ....[ ] I think I can find the number of houses made up of these 501 lines.
45 R: How would you do that?
46 L28: Tn = 501 and substituting this into the formula I can solve for n which is the position.
47 R: Explain more.....
48 L28: My formula remember was Tn = 5n + 1, so if n = 501 then;

$$\begin{aligned} 501 &= 5n + 1 \\ 501 - 1 &= 5n \\ 500 &= 5n \\ 500/5 &= 5n/5 \\ 100 &= n \end{aligned}$$

49 R: what have learnt from working on these questions?
50 L28: I think, with number pattern problems you must always remember to look for the common difference between any two numbers or terms before deciding on the formula.

QUESTION 2

52 L28: .... She reads the whole question... 51 R: Take me through this question as well.
53 R: How did you find these answers to this question?
54 L28: Its 17 tiles
55 R: Explain your method, how did you get 17 tiles?
56 L28: This is what I did. You see ... every 1 tile you put on the base, you also put 1 on each side or end of the group ... [ ].
57 R: Do you mean 1 tile from each side of the base?
58 L28: No, you put here ... [ ] I mean the columns. Like for this tile on the base you add 1 more on each side ... also 1 here and 1 here.
59 R: How many tiles are there in group 6?
60 L28: Sir I came up with a general formula to work out this apart from counting one by one.
61 R: What was the formula?
62 L28: Tn = 3n + 2
63 R: How did you formulate this 3 x n +2 expression?
64 L28: The common difference between the tiles from one group to another is 3 and then you always add 2 tiles on the columns, that is why I have plus 2 in the formula.
65 R: Do you always have to add 2?
66 L28: In this question Yes. You add 1 tile here ... { } and 1 tile here... { }. So its always two tiles more.
67 R: Can this approach work when finding tiles in the 7^{th} group and any other group?
68 L28: Yes because the formula generalizes the numbers in this tile pattern.
69 R: What did you find for this?
70 L28: 23 Tiles
71 R: Explain more ......
72 L28: You actually multiply 7 which is n x 3 + 2.
73 R: How does that look like mathematically?
74 L28: Ok. Its ... Tn = 3n + 2, so T_7 = 3x7 + 2 = 21+2 = 23 tiles
75 R: What about the 10^{th} group of tiles?
76 L28: I used the same formula and found 32.
77 R: Do you have other methods for working on number patterns' questions?
78 L28: Yes
79 R: say more....

80 L28: One can actually draw the tiles one group at a time until say any position.

81 R: How would do that for 50th group of tiles.

82 L28: I will draw extending the pattern given from the first four pictures. But sir, I think its a waste of time. Imagine position or group is 5000, Eeeeeish!

83 R: Explain more why you say its a waste of time?

84 L28: Because there are lots of tiles to draw. Its better and easy to use a formula Tn = 5000 = 3xn + 2 and solves this equation backward for n.

85 R: What is your comment about the general formula in number patterns?

86 L28: Its easier to work with formulae because you just substitute to get what you are looking for whereas drawing, takes up too much time. .... And also you can make a lot of mistakes, if you miss out only 1 tile, then your entire drawing is wrong.

87 R: What is meant by finding tiles in the nth group?

88 L28: It means find the general formula or to find Tn.

89 R: It appears that you answered this last question before any of the first few questions. Explain why did you do that?

90 L28: I find it easier to find formula first because if I had done up to 50 tiles only to find the formula at the end is gonna take up time.

91 R: Your comment before we can move to the next question.

92 L28: Yes, finding a general formula first whenever there is a pattern is very important.

QUESTION 3

93 R: In your own words how did you understand this question?

94 L28: ... The learners read the entire question statement word by word.

95 R: What do you understand from that?

96 L28: I think this is about motifs; as drawn from which a table of values showing a motif number or position versus the number of dots or lines making up the motif's.

97 R: Take me through how you completed this table.

98 L28: First I looked at the Dots, you see when you add 1 motif, the number of Dots increases by 2 but the lines increases by 4.

99 R: Is that a method you used to fill the table?

100 L28: Yes. But I did put this information in a formula form first.

101 R: What formula?

102 L28: For Dots its 2n + 2 and for Lines its 4n + 1

103 R: Did you use other methods beside these formulae?

104 L28: Yes

105 R: Explain more....

106 L28: I mean I used the formula in the last column for nth motifs' position.

107 R: What about the first gaps in the table?

108 L28: No I did the table as well but only few numbers I could just to get the pattern for my general formula.

109 R: How helpful is that approach?

110 L28: Its good because I then used the formula for larger numbers e.g. 63 and 120.

111 R: What is the mathematical relationship between motifs; position (n) and number of dots(d)?

112 L28: These are DIRECTLY Proportional.

113 R: Explain more.....

114 L28: This means both n and d variables increases.... you see when n goes up, d also increases.

115 R: What about n and l (lines) or lines vs dots?

116 L28: All are directly proportional to each other.

117 R: How do represent such mathematical relationships.

118 L28: You draw graphs.

119 R: Say more about graphs....

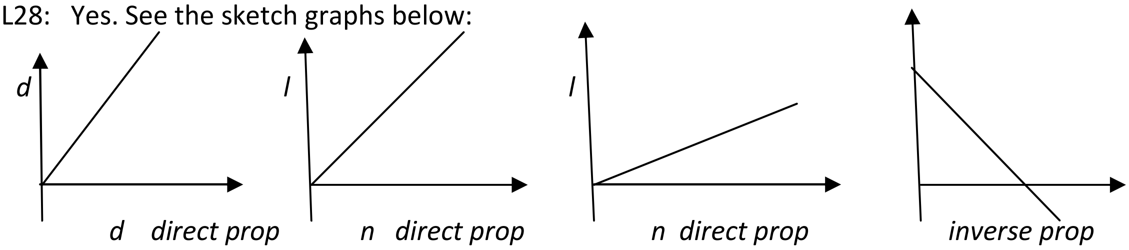
120 L28: Yes, a line graph is what you get. You see if a line is going from top right on the x-y axes to bottom left its direct proportion. But if its coming from top left to bottom right then its Inverse relationship.

121 R: What do you mean by inverse?

122 L28: The two quantities change differently lets say when one is increasing the other one decreases.

123 R: Can you give examples to this.

124 L28: Yes. See the sketch graphs below:



125 R: What do you learn from such/these sketch graphs?

126 L28: I think they are all linear graphs and therefore can generate linear equations.

127 R: Say more ....

128 L28: When we are doing statistics, we use line graphs and also on the Cartesian plane in algebra to find the gradient of the line through given points.

129 R: How does this relate to the concept of gradient.

130 L28: You see Tn = 2n + 2 is a form of the linear equation like y mx + c. So if the line was exponential then the formula would be different.

131 R: Could you have generated a formula from the graph if you did not have the table of values or the motifs?

132 L28: Yes

133 R: Explain more........

134 L28: you can find the gradient of the line by using any two points on this line say m = change in y divide by change in x and substitute into y = mx + c:

135 R: How would you relate number of dots and lines or position motifs in terms of y = mx + c?

136 L28: Sir its simple, you see y = mx + c and Tn = 2n +2 or tn = 4n + 1; Tn = y and c = 2 or 1 and also m = 2 or 4. These are all linear equations.

QUESTIO N 4

137 R: How did you find the next three terms in this question?

138 L28: I noticed the common difference between any two terms in the pattern to be 7

139 R: what did that mean to you?

140 L28: The pattern was increasing every time by 7. So the next three terms are: 26; 33; 40

141 R: Take me through the second pattern as well.

142 L28: Ok. The common difference was -5 i.e. minus 5.

143 R: Why the difference is negative this time?

144 L28: Because the numbers in the pattern are going down.

145 R: Say more......

146 L28: The terms are decreasing.

147 R: How did you work ut the 10^{th} and 50^{th} terms?

148 L28: I made a general formula for each pattern. From first pattern its Tn = 7n – 2.

149 R: Explain why 7n this time and not 5n, 3n or 2n as in previous questions?

150 L28: 7 is common difference and Tn is the term while n is the position of the term.

151 R: Where does -2 in the formula come from?

152 L28: Because when you multiply 7 by n the position number, there is 2 extra so I have to subtract the expression by 2. The common difference is stil 7 and if you add 2 you get which is 2 more than 7

153 R: What about the formula for the second sequence, What formula did you come up with?

154 L28: I used .... [ ], there is a formula they showed us to use in grade 8. Ok, I think its Tn = a + (n- 1)d where a if first term n-1 is common difference and Tn if the term. manipulating this gives me a formula Tn = -5 x n + 40 which is Tn = 40 – 5n

155 R: Do you really understand these formulae or rules you have generated?

156 L28: Yes I do understand this work.

157 R: Explain in detail to me especially the symbols you are using in your rules.

155 R: Did you use this formula when coming up with the nth terms?

156 L28: No

157 R: Explain more...

158 L28: Because I could not remember it very well.

159 R: If someone does not know how to generate these formulae, how could they do these questions?

160 L28: Finding a common difference in the pattern is a good start.

161 R: Which other methods would you recommend?

162 L28: Coming up with a formula only.

163 R: Why?

164 L28: I just see if there is a pattern in a given set of numbers or objects and if its addition or subtraction... then take it from there.

165 R: Do you have any question?

166 L28: No, because I ask my questions in a maths lesson in class if I did not understand something.

167 R: Thank you for attending this interview.

168 L28: Thank you.

TRANSCRIPT 5 (L17: MM)

1 R: Welcome to this interview.

2 L17: Thank you Sir.

3 R: Take me through your responses in Q1

4 L17: Ok Sir. I looked at the paper and for the 1st house I counted with my fingers all the line segments so I got 6; 11; 16; ....

5 R: What did you do next.

6 L17: I made up a formula when I saw there is a pattern for the number of line segments.

7 R: Explain more.....

8 L17: My formula is $T_n = 6 + (n-1)x5$

9 R: What does Tn and n mean in your formula?

10 L17: Tn is the term, 6 is the first term in the pattern, n-1 is the position and 5 is the common difference among the numbers in a pattern.

11 R: How many line segments for 4 houses?

12 L17: From my formula Tn = 6 + (n-1)x5 will be $T_4 = 6 + (4-1)x5 = 21$

13 R: What about 10 houses?
14 L17: Using the same method and substitution I got 51
15 R: How would you determine the line segment for 11^{th} position of houses?
16 L17: If 10 house I found 51 the 11 houses would be 51+5 = 56
17 R: Why adding 5?
18 L17: Its the same as doing substitution for n =11 into the formula. But because I knew line segments for 10 houses so 11 needs just to add 5.
19 R: What did you find for 50 houses?
20 L17: Tn = 6 + (n-1)x5 = 6 + (50 -1) x5 = 251
21 R: Can you work out the same solution using a different method?
22 L17: Yes, but a formula is much easier by means of substitution.
23 R: What do you suggest if others experience difficulty in generating a formula?
24 L17: You can use you hands to count or like .... [ ] but the Tn formula works faster. But if you don't understand then its going to be very difficult for you to work on such problems.
25 R: Explain more...

26 L17: I mean, you will have to extend the pattern that long or else draw the pictures of the cluster house further and this takes too much time.
27 R: Is taking time a problem....?
28 L17: Yes. You see from 6; 11; 16; 21; ..... carries on. And once you have a position number 251 then it means you write down all the numbers until the pattern reaches 151. Its going to be very difficult ... but as long as you understand it in that way......
29 R: What about number of line segments in n houses?
30 L17: This question is asking the nth term, and my formula with n in it is the answer. So Tn = 6 + (n-1)x5 represents line segments in n houses.

QUESTION 2

31 R: Take me through your response to this question.
32 L17: Ok. fig 1 has 5 tiles; fig 2 has 8 tiles and fig 3 has 11 tiles I did the counting using my fingers to see if there is a pattern such as this that every time I go to next figure the common difference was 3.
33 R: How did this help you?
34 L17: Yes Sir, like in fig 5 I just used my fingers because adding 3 always say from fig 3 = 11; then I knew there will be 15 tiles..... [ ]. No! Its .... { counts fingers 11, 12 ,13 , 14}. Yes its 14 sir. And then from there I just kept on adding 3 this time I had not yet written down the formula.
35 R: Describe the pattern briefly.
36 L17: You see... the horizontal tiles at the bottom are 7... each time they increase 1 more tile at the base and the columns each time also increase by 1 tile. So when the base is 7 the top tiles are 5.
37 R: How many tiles did you find for 6^{th} and 7^{th} groups?
38 L17: 20 and 23 tiles.
39 R: Explain more...
40 L17: I fig 5 – 17 tiles, then its 17 +3 and 20+3.

41 R: How would you find the group number if given the number of tiles in a group?
42 L17: Is this for Q2 or Q1?
43 R: This question is about position and could be asked in any number pattern. Say in Q1

you have 51 line segments. Would you be able to find the position of the houses?

44 L17: Yes.

45 R: Explain more....

46 L17: I Think I would be able with the formula as I know the first term so Tn = a + (n-1)d

47 R: Is this formula the same as Tn = 6 + 9n-1)x5

48 L17: Yes because Tn if the same term, a = 6 and n-1 is the position, d is the difference and n number of term.

49 R: So, how do you work out 51 line segments?

50 L17: In this case my Tn = 51 so I will do 51 = 6 + (n -1)x5 and solving for n the answer is n = 10

51 R: What is n = 10 standing for?

52 L17: Its the position of houses containing 51 line segments. Eeeeeish! You caught me by surprise with this question yooh!.

53 R: What do you mean?

54 L17: This is working backwards and didn’t think I will be able to do it. And there was no such question in the questionnaire you administered to us.

QUESTION 3

55 R: Take me through your solutions in this question.

56 L17: The first row was given to as number of motifs and then in 2^{nd} and 3^{rd} rows I realised that ... [ ].

57 R: How did you determine the number of dots in the 2^{nd} row of the table?

58 L17: Ok. For n = 1 its 4 dots and n =2 its 6 dots. This is like adding 2 dots every time.

59 R: How did the pattern of Dots look like?

60 L17: Its 4; 6; 8; 10 ; 12; ....

61 R: What was the number of dots in the nth position of motifs?

62 L17: Its Tn = 2 + 2n

63 R: How did you find the term 2n in your formula?

64 L17: Eeeeish! I cannot remember what I did Sir.

65 R: What about the number of lines in the third row?

66 L17: For n =1 there 5 line and n = 2 had 9 lines, ... [ ].

67 R: Do you notice anything from this?

68 L17: Yes Sir. Then if n = 3 it will be 13 lines.

69 R: How do you get 13

70 L17: I saw the common difference between 5 and 9 was 4. By adding 4 to 9 gives the third term.

71 R: How many lines would you get from 500^{th} position?

72 L17: It will be Tn = 1 + 4 x 500 = 2001

73 R: How different could you have approached this problem?

74 L17: I would carry on with the formula because drawing the motifs takes time. I cannot afford to draw 500 motifs of boxes.

75 R: Why do you insist the use of formula?

76 L17: No matter how big the numbers in a pattern could be the formula simplifies working with.

77 R: Which two formulae have you been using in this problem?

78 L17: Tn = 2 + 2n for the DOTS and Tn = 1 +4n for the Lines

79 R: How different are the two formulae?

80 L17: The first represents a set of even numbers in the patter. And the second represents the set of odd numbers in a pattern.

QUESTION 4

81 R: Take me through this question.
82 L17: ... She re-writes 5; 12; 19; ...., Ok from 5 to 12 and 12 to 19 there is a difference of 7 to get to the next number or term.
83 R: What are the next three term?
84 L17: 26; 33; 40
85 R: What about from the other pattern in part b)?
86 L17: The difference in this number set is 5 and each time the numbers are subtracting 5

87 R: How did you determine the next three terms?
88 L17: I used a formula.
89 R: Which formula is this.
90 L17: Tn = a + (n-1)xd; Tn = 35 + (n-1)x-5; this simplifies to Tn = 40 - 5n. ********
91 R: What was the formula for part a)
92 L17: Must I write also?
93 R: Yes.
94 L17: Ok. Its Tn = a + (n-1)xd where a = 5 and d = 7 then Tn = 7n =2
95 R: Would you generate the formulae without starting from Tn = a + (n-1)d?

96 L17: This is a basic expression one needs to know otherwise you will have difficulties.

97 R: Where is this basic formula or expression coming from?
98 L17: My mathematics teacher at grade 8 taught me.
99 R: What if you had forgotten?
100 L17: I don't think it would be easy to work out these numbers.

101 R: Do you have any comment having worked on these number pattern problems?
102 L17 Eeeeyah! From Q1 I, ... firstly when I looked at the houses I learnt how to notice a pattern and then think about the formula. The rest of the questions worked well when I used this approach.
103 R: What part/section of mathematics would you associate with these questions?
104 L17: Algebra
105 R: Explain more....
106 M: The numbers in the pattern are being generalized into a formula. And we are also substituting to find some unknowns.
107 R: Do you have anything to say in this interview.
108 L17: Not really Sir. But I did enjoy working on these questions and even during this interview. It was good that it was not for marks but just something to help you think. If this task was a test I do not think I would have passed however, it was somehow easier because we got more time to do it.
109 R: Thank you very much for your participation in this exercise.
110 L17: It is a pleasure Sir.

TRANSCRIPT 6 (L15: NC)

1 R : Welcome to this afternoon interview , ...
2 L15 : Thank you Sir.
3 R : Would you please take me through your response to question 1
4 L15 : We are supposed to find number of line segments in the houses which are a design

of the new set of town houses clustered at different positions.

5 R: What did you notice about these cluster houses?

6 L15: The 1st house has 6 lines , 2nd house has 11 lines and the 3rd house, 16 line segments. This gives a number pattern of the form: 6; 11; 16; ....

7 R: Say more about the pattern.....

8 L15: The pattern has 5 as the common difference between any two terms.

9 R: How does it help you knowing 5 as the difference?

10 L15: To answer questions about 4th and 5th houses.

11 R: What answers did you get?

12 L15: 21 and 26 respectively.

13 R: How did you get these numbers?

14 L15: This pattern results into Hn = 5xn + 1 as a mathematical general formula.

15 R: Why deciding on using the formula.

16 L15: Its easy to work on such problems and also there questions did not specify what method to use.

17 R: How many line segments are there for 10 house?

18 L15: 51line segments are in these 10 cluster houses.

19 R: Explain more....

20 L15: 10 x 5 + 1 = 50 + 1 = 51. Actually in my answer script I said hx = (1-x) + x * 6 because it still make sense to me and results into the same answer anyway.

21 R: Can you give an example.

22 L15: The two formulae are: Hn = 5n + 1 and hx = (1-x) + x* 6 where x = n =position.

$H_{10} = 5 \times 10 + 1$ $\quad$ $hx = (1-10) + 10*6$

$= 50 + 1$ $\quad$ $= -9 + 60$

$= 51$ $\quad$ $= 5$

23 R: What can you conclude from these examples?

24 L15: Hn = 5n + 1 and hx = (1-x0 + 6*x are both general formulae for the same pattern expressed differently but can be used to answer the same questions.

25 R: What was the secret behind you approach to these questions?

26 L15: Observe the pattern, look for similarities or differences and the work with these findings to generate a formula where possible.

27 R: Say more.....

28 L15: I mean some patterns may not fit into any formula while some does.

QUESTION 2

29 R: Explain your approach to this question.

30 L15: ... she reads the entire Q...

31 R: What do you notice?

32 L15: From figures 1-4, I see already a pattern such as 4; 6; 8 ; ..... for grey tiles and also another for white tiles. You multiply the figure number by 2 in order to get grey tiles and sometime use of a formula would be suitable in this case.

33 R: How did you use such information.

34 L15: I came up with a formula representing this information.

35 R: What formulae did you generate?

36 L15: Gn = n x 2 : Grey tiles and Gn = n + 2 : White tiles where Gn is number of tiles per group and n is the group number or position.

37 R: How many tiles did you find in group 10?

38 L15: This is asking both (Grey and white Tiles) altogether.

39 R: So what did you do?
40 L15: I used each one of the two formulae and added the answers.
41 R: Explain more....

42 L15: This is what I did: Group 10 = Gn (w) + Gn (g)
= nx2 + n+2
= 10x2 + 10+2 = 20 + 12 = 12
43 R: Is it possible to use one formula?
44 L15: Yes it is possible.
45 R: How do you do that?
46 L15: Combine the formulae above and simplify.
47 R: Say more......
48 L15: If Gn = nx2 and Gn = n = 2 then G_{total} = nx2 + n+2 = 3n +2

49 R: Can this formula result into 32 tiles for 10^{th} group?
50 L15: Yes. 3x10 + 2 =32
51 R: This is fantastic findings. Isn't it?
52 L15: Yes it is.

QUESTION 3

53 R: How did you work out this Q.
54 L15: For the number of line you just had to,.....[ ]. You could first draw for and another geometric motif say 2^{nd} and 3r and from there find the differences or similarities in the pattern to help find a formula.
55 R: Describe the motifs as given in this question.
56 L15: The 1^{st} motif has 5 lines and 4 dots, 2^{nd} motif has 9 lines and 6 dots
57 R: With such information at hand, would you predict the number of lines and dots in 3^{rd} and 4^{th} motifs'?
58 L15: Yes
59 R: How could you do that?
60 L15: Because from 1^{st} and 2^{nd} motifs' the number of dots differ by 2 then 2 will be part of your formula. And then you need to try working with the numbers to find the constant number.
61 R: Say more....
62 L15: The constant number in this case is 2.
63 R: Why do you call 2 a constant?
64 L15: Because there is a common difference of 2 between the dots in the motifs'.
65 R: What formula are talking about in this case?
66 L15: 2n + 2
67 R: Is 2n + 2 you general formula?
68 L15: No, the complete formula is Dn = 2n +2 where Dn = number of dots and n = is position.
69 R: What formula did you generate for the number of lines in motifs'?
70 L15: Ln = 4n + 1 and 4 is common difference and 1 is constant ******
71 R: How many dots and lines are there in position 63.
72 L15: Substituting into the formulae; for n = 63 then D_{63} = 128 and L_{63} = 253
73 R: How would you determine the motifs' position if you know the number of dots or lines?
74 L15: Suppose I have 60 dots, then I would find the position by solving Dn = 2n +2 for n as we do when working with equations in algebra.
75 R: Explain more....
76 L15: Let Dn = 60 therefore 60 = 2n+2, 2n = 60 – 2 giving n = 29

77 R: How would you work when there are 201 line in the motifs'?

78 L15: It's the same method of working backwards but with the other formula this time.

79 R: Explain more.....

80 L15: Assume Ln = 201 and solve by substitution now my formula will be 201 = 4n + 1 i.e. n = 50

81 R: Can you think of you own question either in terms of dots or lines and try to determine the motifs' position.

82 L15: I would say: What position do we find the motifs' with 48 lines.

83 R: Ok. How would solve that?

84 L15: 48 = 4n + 1 so 4n = 48 – 1 and n = 47/4 = 11.75

85 R: What is this 11.75 for?

86 L15: It is the position since we are solving for n

87 R: Are you satisfied with this answer?

88 L15: No! No! No! It does not make sense in this context.

89 R: Why?

90 L15: You see Sir, the position number cannot be say three and half or this 11.75 which is a decimal fraction/number. It has to be a whole number in order to make sense.

91 R: So what do you learn from this question?

92 L15: It is false...[ ] may be I did not think carefully when making up this question. ********

93 R: What do you mean the question is false.

94 L15: It cannot be solved.... [ ]. Ooooh! You see 4n +1 in an odd number and not an even number like 48.

95 R: Explain more.....

96 L15: 4n + 1 from the formula about lines in motifs' is odd. So the number of lines should always be odd then we can solve and find a good answer.

97 R: what is a good answer?

98 L15: I mean the solution to a problem that make sense ************* (sense making?)

99 R: What have learnt on the use of the general formula you generated?

100 L15: I n the 1st formula the number of dots even in the table are even hence the formula must be an even expression. While the 2nd one should be odd.

101 R: How can you tell that the formula expression is an even or odd.

102 L15: Even number means "being multiple of 2" and odd number has "1 or itself as multiples"

103 R: Carry on with the response about lessons learnt out of this activity.

104 L15: Yaaah! One can use the formula to solve for position number of a term in a pattern, plot graphs or even generate a pattern if you have the formula first.

105 R: What do you understand by the position of the motifs'?

106 L15: In this context, the position number also represents the number of motifs' in that position. E.g. At position 1 you have 1 motif; position 2 you have 2 motifs' and so on..... so n stand for two quantities.

107 R: What kind of graph did you get for plotting value in row 2 against those in row 1or row 3 versus row 1.

108 L15: Straight lines

109 R: What mathematical relationship is there between these quantities?

110 L15: They are all DIRECT Proportions.

111 R: Explain more.....

112 L15: The quantities all increase and follow a linear pattern on the graph and are not INVERSELY related.

113 R: What is INVERSE relationship?

114 L15: When two quantities are compared in such a way that when one is increasing then the other is at the same time decreasing.... that is inverse or indirect proportion.

115 R: Your comment about this question.

116 L15: I think there are many things that we have learnt in mathematics and all link to number

patterns. For example, geometric shapes, linear equations, linear graphs on x-y axes, gradients, scatter plot and many more graphs in statistics etc..... [ ]

117 R: How does this work link to the concept of gradient?

118 L15: You see for example Ln = 4n + 1 is of the form y mx +c where y = Ln; m = 4 and c = 1 that is why Ln = 4n + 1 gives a straight line on the Cartesian plane *******

119 R: What about scatter plots in Statistics.

120 L15: Scatter plots come from two different quantities of data compared on x and y axes to find the line of BEST FIT between the data. Its the same as comparing motif position and number of lines in that motif. ******* And if you observe carefully, you start from pictures, drawings, numbers and then all of sudden you are doing algebra working with formulae.

QUESTION 4

121 R: Take me through how you did on this this Question

122 L15: Ok. Here its slightly different from the previous questions. This is straight forward because we have numeric patterns in both a) and b).

123 R: How did you find the next three terms?

124 L15: There is a common difference of 7 in part a) so its easy to see what formula is needed. But also one can only continue the pattern by adding 7 every time to get the next number.

125 R: say more....

126 L15: The numbers are

127 R: Which method did you use.

128 L15: Both

129 R: Why both?

130 L15: The formula to check the one of always adding 7 and vice versa.

131 R: What were your 10th and 50th terms?

132 L15: I first had to find a formula or rule for each pattern.

133 R: How did you come up with a formula here?

134 L15: The common difference was 7, so I had to put 7 aside and then multiply the numbers For the position of the number by 7 and had to either add or subtract a certain number for you to get to the number you are looking for.

135 R: Did you opt for adding or subtracting a number?

136 L15: I subtracted.

137 R: Why subtraction?

138 L15: Because when I multiplied position 1 by 7 = 7 and the only way I could get to 5 was by subtracting.

139 R: So what formula did you generate?

140 L15: Tn = 7n -2; n is position of the number and Tn is actually the number on position n and Tn = 40 – 5xn. I got these expressions using the technique above ***

141 R: Take me through part b)

142 L15: .... The learner reads the pattern as: 35; 30; 25; 20; 15 and 10.

143 R: How did you find the numbers after 25?

144 L15: The numbers form a pattern and were decreasing by 5 every time. So I kept on

subtracting 5 from 25 to get the other numbers.

145 R: What are you 10^{th} and 50^{th} terms?

146 L15: I found .... [ ], the general formula first and it was -5n + 40.

147 R: How did you find this?

148 L15: Since the numbers are decreasing by 5, then the 5 is negative and then if you multiply the 1^{st} term where n = 1 by -5 obviously you get -5. So for you to get to the number that is on position 1 you had to add 40 that gives 35. I also tried the other numbers as a matter of checking the formula is right.

149 R: Can one say these patterns in a) and b) carry on and on?

150 L15: Yes Sir even though for b) where it decreases by 5, as it goes further the numbers will go negative.

151 R: Your general views about the exercise on number patterns?

152 L15: Number patterns is just about knowing what you are doing enjoying it because you need to be clever enough to find the difference or similarities and see how the numbers follow each other. But if you don't you cannot come up with the general formula.

153 R: Could you have found the solutions differently other than use of formula?

154 L15: Not really, it will result into the same thing even though steps will change.

155 R: Would you have presented your methods and solution in a different manner?

156 L15: It depends on how the question is phrased and presented as well.

157 R: Say more....

158 L15: You with numbers in a table, you can represent the numbers differently by a graph or even draw diagram.

159 R: Thank you for you participation in this interviews.

160 L15 You are welcome.

TRANSCRIPTION 7 (L23: FM)

1 R Welcome F to this follow-up interview.

2 L23: Thank you.

3 R: Would mind taking me through your response to Q1

4 L23: Not at all Sir. I understand that you had make it in accordance to the visuals given. I see three different sets of cluster houses made up of different amounts of line segments.

5 R: How different are the line segments in these houses?

6 L23: In 1^{st} house I see 6 lines; 2^{nd} house has 11 and 3^{rd} house its 16 line. That is what I meant by different.

7 R: Ok. Carry on.

8 L23: And then I looked at the questions which require us finding the number of line segments in 4, 10 and 50 sets of houses from the initial pictures or generate a formula. Position house number and the difference or increase of line segments between consecutive terms is very important.

9 R : I see you read there three houses at position 3. , can you explain why you are multiplying by the number of houses or house position by 5?

10 L23: If you look at the first house, the number of line segments can be expressed as 5 x 1 plus Some number. Similarly, for two houses, its 5 x 2 plus the same number am adding in the first picture. So the numeric pattern constructed from the pictorial representation 6; 11; 16; 21; ......, can be expressed as 5x1 + (1); 5x2 + (1); 5x3 + (1); 5x4 + (1); ....., where the number added each time is 1.

11 R : I would like to understand your method here. Why are you adding 1?

12 L23: Because, in the first house there 1, 2, 3, 4, 5, and 6 (pointing at the picture) line segments use. So I figured out that 5 x 1 house gives 5 lines but I have counted 6, this means my answer is one line missing. By adding 1 to 5 makes it 6 line segments altogether. So my expression looks like 5x1 + 1.

13 R: What is it that you actually did.

14 L23: The cluster houses give a pattern such as 6; 11; 16; ..... in terms of the line segments they are talking about. And formulated a formula for this pattern using Tn = a + (n-1)d where Tn = term; a = 1^{st} term; n = position of term and d = difference between the terms. So my Tn = 6 + (n-1) x 5 which simplified to Tn = 5n +1 as my general formula.

15 R: What is the Tn = a + (n-1)d?

16 L23: It is a basic formula that one can use to see if there is a linear relationship between quantities in any given pattern or not and it always work if the pattern is indeed linear.

17 R: What if it does not work?

18 L23: It could mean an exponential or other mathematical relationship or none at all.

19 R: Is there another method of doing this?

20 L23: Yes using the table could do which will eventual lead you to a formula.

21 R: Which methods were you using in you response?

22 L23: The formula but I can try the table method as well.

23 R: How will that look like?

24 L23: In could draw two rows in which one row is house position and the other number of lines and even draw a graph from these values in the table.

25 R: How did the formula help you?

26 L23: I substituted n = 4; 10; 50 into Tn = 5n + 1 to get the number of line segments.

27 R: What did you get?

28 L23: 21; 51 and 251

29 R: What about n houses?

30 L23: It is the same as the general formula Tn = 5n + 1

31 R: If you have say 501 line segments, what would be the position of the cluster houses?

32 L23: You could still use the formula but in a different manner like working backwards I think.

33 R: Explain more....

34 L23: I think you could say .... [ ], Tn = 501

35 R: Say more......

36 L23: Substituting Tn = 501 into the general formula to work out the position n i.e. 501 = 5n+1, n = 100

37 R: What is n = 100 for?

38 L23: It is for both the position of the houses and the number of houses at that position.

....

.... etc

89 R: From question 1, Can you tell me how you would determine the position number when you are given 201 line segments?

90 L23: I will first write down my formula Tn = 5n +1. And if 201 is given as number of line segment then my Tn = 201 so I just can calculate n.

91 R: How would really do that? Show me please.

92 L23: Solving a formula above for n; so 5n + 1 = 201 after substitution. Therefore 5n = 201 – 1 And n = 40, the position with 40 cluster houses containing 210 line segments.

93 R: Will this always work.

94 L23: Yes.

95 R: Explain to me more as to how will this workout, ...

96 L23 I did check some few more examples by means of substitution methods.

97 R: How much is enough examples for checking?

98 L23: Two or three are enough.
99 R: Explain, what are you going to do in order to determine position for 21 houses.
100 L23: Similarly, 21 =l 5 x n + 1. And then 5 times n is 20. Therefore position n = 4. ...., To – Cont.

QUESTION 2

35 R: Explain how you did this question/
36 L23: In this question they have given us the tiles in different groups or positions and want us to find the number of tiles in certain groups which are not represented in diagrams here.
37 R: How did you do that?
38 L23: In fig 1 there are 5 tiles; fig 2 has 8; fig 3 = 11 and fig 4= 14 ... { he redraw the tiles and count them aloud} ..., so I see the pattern 5; 8; 11; 14; ....
39 R: What does this mean to you.
40 L23: Its a pattern with a common difference of 3 between any two numbers or terms.
41 R: Can this pattern carry on?
42 L23: Yes it can and its probably a direct proportion.
43 R: How many tiles were in group 5?
44 L23: I said Tn = 3n+2 and since its starting at 5 and the difference is 3 then got my formula.
45 R: What is Tn = 3n+2 for?
46 L23: It is a formula that generates the pattern 5; 8; 11; 14 ..... etc. So in group 5 there are 17 tiles.
47 R: In what group could you find 29 tiles?
48 L23: I would use the formula or carry on extending the pattern until I reach 29 tiles.
49 R: Explain more on this.....
50 L23: Otherwise, I would work out in reverse way like I did with one question you asked in Q1 i.e. Tn = 3n +2 the 29 = 3n +2 hence n=9
51 R: What does 9 mean or stand for?
52 L23: Tile group number or position for 29 tiles.

53 R: What is the best approach to these questions?
54 L23: I think using a formula is much easier than carrying on with pictures or ... [ ]
55 R: Explain more....
56 L23: The meaning and purpose behind is the same, however other methods are time consuming.

QUESTION 3

58 R: Take me through you response again here.
59 L23: They gave us motif' and different motifs' make up a pattern consisting of Dots and Lines. So we had to find how many dots or lines in relationship to position or number of motifs'
60 R: Did you find it difficult completing the table?
61 L23: No, It was very easy.
62 R: How did you do it?
63 L23: I used both the table method and formula as the numbers got bigger and bigger.
64 R: Which method would you like to discuss first?
65 L23: I think the table approach.
66 R: Say more.....
67 L23: .... the learner redraw motifs' and table of values as in the questionnaire *****
" And therefore, needed us to complete the gaps for DOTS and LINES
68 R: Do those numbers follow a pattern?
69 L23: Yes. They go like this: DOTS: 4; 6; 8; ....

LINES: 5; 9; 13; ......

70 R: How did this observation help you?
71 L23: It was easy for me to determine a few more numbers in these patterns.
72 R: Say more....
73 L23: I found out that there will be 8 dots and 13 lines in 3^{rd} position and I could make a formula.
74 R: Tell me what formulae were you thinking about.
75 L23: For the dots: 2xn +2 and Lines: 4xn +1
76 R: How do you know that these formulae are true.
77 L23: I used substitution of the position numbers say n=1; 2; 3; 4; 5 etc into these formulas and indeed I managed to get the numbers in the table.
78 R: Does this always work?
79 L23: Yes, because the formula I am developing becomes the generalisation in this context.
80 R: Did you solve this in a different way?
81 L23: I could use ratio and proportionality method and even by graph.
82 R: Is that all.
84 L23: Sometimes by guessing. But this is not reliable most of the time.
85 R: How do the motifs' position, number of dots and lines compare or relate?
86 L23: They are directly proportional to each other. Each quantity increases.
87 R: How do you present such relationship?
88 L23: Sketching a graph on x-y axes will be ok.

35 R: Can you tell me how you started question 3.
36 L23: Okay, you see questions 1, 2, 3 are pretty the same. Each starts with pictures except question 3 which has also a table of values showing kind of these (pointing in column of the table) three things.
37 R: What are these things?
38 L23: I call these variables because these are actually the things that are changing in the motifs pattern.
39 R: Yeah, but what are these things you call variables?
40 L23: Oh! You mean *n, d* and *l*; n is position of a motif, d is number of dots found in motifs' corners, and l is the number of lines used in constructing the motifs. And you see, these can change if I add or subtract motifs from any given picture. Sir, I can also represent question 1 or 2 in tables.
41 R: How can that be?
42 L23: Okay, instead of three rows for *n, d* and *l*; I will have a table with two rows with *n* and *Tn* (i.e. line segments) for question 1 and also *n* and *Tn* (for number of tiles).
43 R: So, how would you complete your cells (boxes) in the table?
44 L23: It's very easy once you have generated the formula, then you just carry out substitution to find the value missing in each box of the table. Or simply by counting lines, tiles or dots from picture sequences but it could be hard when you have many pictures.

QUESTION 4

89 R: Take me through your responses to this section and what were thinking.
90 L23: We are given a few numbers that seem to form a pattern and the question require us to analyse further and come up with the next three terms in each case/part.

91 R: How did you do part 4) ?
92 L23: In 4a, I found there is 7 as a common difference between any two terms and chose to continue pattern using therepetitive addition of 7. The next three terms are 26; 33; 40.

And for part b) i realized that the terms in the sequence are decreasing by 5 hence the difference became – 5. Using this -5 the next three terms were found to be 20; 15; 10.

93 R: Did you solve these problems differently?

94 L23: Yes. I also used formulas to check if my solutions are valid.

95 R: What formulas are these and how are you generating them?

96 L23: Like in questions 1 and 2 or generating from Tn – a + (n-1)d expression. For part a) Tn = 7n -2 and part b) Tn = -5n +40

97 R: How do you know that these formulae are correct?

98 L23: Again these are generalized expressions or rules, which means they work in most cases.

99 R: What are your terms in 10th and 50th positions:

100 L23: Substituting n = 10 and n= 50 in these formulas above, I got 68 and 348 for a) and also -10 and -210 for part b).

101 R: Does this approach always work?

102 L23: Yes it does work mostly.

103 R: What term do you get at nth position for the two patterns?

104 L23: The nth term is given by Tn always and in this case its Tn = 7n – 2 and Tn = -5n +40

105 R: What position would you get -500 in the second pattern?

106 L23: That will be 2500 + ... [ ] if I substitute -5(-500) + 40.

107 R: Will that give you position?

108 L23: No! No! That will be like am solving for the term. I think we have to work backwards since we know the number this time and not position.

109 R: So what position will that be?

110 L23: Since Tn = -5n + 40 then -500 = -5n + 40 which is -500-40 = -5n, n= -540/-5 = 108. Therefore the position is 108.

111 R: What have learnt in doing such questions?

112 L23: One finds it easy if you work with a formula but you must be in apposition of learning the basics of generating these formulas.

113 R: What part or section of maths would you associate the work on number patterns?

114 L23: Algebra and mainly equations.

115 R: Why algebra?

116 L23: Because we are starting with totally a different context of numbers only to represent such information using a general formula. That is generalisation and of course algebra is defined as generalised arithmetic so its algebra.

117 R: Thanking you for participating in this exercise.

118 L23: Thank you sir.

TRANSCRIPTION 8 (L9: DebK)

1 R: Tell me how you responded to this Question.

2 L9 : When I read the question I found it is asking how many line segments are needed to construct 4, 10, or 50 houses. I think I answered the questions after observing what was going on with the given pictures of cluster houses.

3 R: What did you find or see?

4 L9 : I found ...., but I think these were wrong am supposed to work out the formula first.

5 R: Do you mean you had to use a different method?

6 L9 : Yes

7 R: Explain the method please.

8 L9: I just ,...... I think counted the lines for 4 and 10 houses but I took a guess for 50 houses.

9 R: Can you think of any formula or rule to help do this?

10 L9: In grade 8 we learnt a formule Tn = a+(n-1)d in maths clinic but am not sure if the bracket as – or +,.... may be its Tn = a-(n+1)d or a(n-1)d.
11 R: How did you know that you got to these expressions?
12 L9: This is a basic formula for finding equations for number patterns.
13 R: Explain more about your basic formula.
14 L9: The symbols have meanings e.g. a = 1^{st} term, d = common difference and n = position of a term in a pattern. Knowing this expression helps in finding any pattern's general rule.

15 R: What was your answer for 4 houses?
16 L9: 21 lines.
17 R: Explain more.....
18 L9: Substituting x = 4 in the formula 5x + 1. So 5*4 +1 = 21
19 R: What about 10 and 50 houses.
20 L9: Again by substitution 5*10 +1 = 51 and 5*50 +1 = 251 line segments.
21 R: How did you approach these questions?
22 L9: I looked at the number pattern for any similarities or differences in order to decide on what my formula could be.

QUESTION 2

23 R: What was your approach to this Question.
24 L9: I noticed that every time you draw the tiles, the base tiles increase by a specific number as well as the tiles by the sides-columns.
25 R: How many tiles is this pattern starting with?
26 L9: In group 1, there are 5 tiles. Group 2 has 8 and also Group 3 has 11. This form the number pattern 5; 8; 11; ....
27 R: What would be you next three terms of this pattern?
28 L9: These are 17; 20; 23. It appears every time you add 3 to get the next term.
29 R: Did you solve this in a different way?
30 L9: I used a formula.

31 R: Can you write an expression or formula for T_n?
32 L9: Am not really sure what T_n is.*
33 R: It is a way to refer to nth term or number of tiles in this tiles pattern problem. See, we called our first number T_1, and T_2 the second number etc., so we can call the 50^{th} number T_{50}, and n can be any number.
34 L9: Oh! I see, so the number will depend on the number.
35 R: What expression did you write for T_{50} or T_n?
36 L9: I first counted all the tiles in the groups as drawn in the question here. So got 5, 8, 11, 14, ..., and it means T_{50} = 5+8+11+14+17+ 20+ .... + until 50 times. **
37 R: Is that for T_{50} only?
38 L9: Yes, but for Tn I'm not sure. Or do you mean n+5 times n+8 ? ***
39 R: How do you express your observation of the pattern in words?
40 L9: I see first 5 tiles in figure 1, 8 tiles in figure 2 and then 11 in figure 3 etc.
41 R: So what mathematical rule or formula can you come up with what you have said?
42 L9: You mean a formula ......, I think its some number or numbers times 3 plus 2 or 3 plusing 2 every time. ****

31 R: What formula did you use?
32 L9: It is 3n +1
33 R: How do you know this is true?

34 L9: I check by substitution of n = 1, 2, 3, etc to see if I can still get the terms I know from the above Pattern.
35 R: Give an example how this could be done.
36 L9: The first term is n = 1. Therefore the formula becomes 3 x 1 + 1 = 4
37 R: Is your answer correct?
38 L9: No. This is wrong answer. In group 1 the number of tiles is 5 and NOT 4 so the plus 1 in the formula should be + 2. And the correct formula is now 3n +2.
39 R: Do you need to check again?
40 L9: Yes to make sure. For example; If n = 1 then 3 x 1 +2 = 5 and n = 3; then 3 x 3 + 2 = 11
41 R: Does this always work?
42 L9: Yes it is working in this context. May be one needs to check more numbers to be sure.
43 R: How many tiles did you get for the 10^{th} and 50^{th} groups?
44 L9: 3x10 + 2 = 32 and 3x50 + 2 = 252 tiles
45 R: How would you find the group position for 5000 tiles?
46 L9: I would use my formula still because this is a big number and using a formula is a bit easy.
47 R: How would do that?
48 L9: Its 3x500 + 2 = 1500+2 =1502.
49 R: Could solve this in a different way?
50 L9: I would not know how to. May be drawing the tiles would help somehow.
51 R: What have you learnt in doing this exercise?
52 L9: I am using the general formula to find the numbers.
53 R: What section of maths could this work belong?
54 L9: Arithmetic, geometry and its all about algebra.
55 R: What do you mean its all about algebra?
56 L9: Because in algebra the teachers always tell us to work and solve for x and other symbols called unknowns.
57 R: Your views or any comment about the interview and the work on number patterns.
58 L9: The work in mathematics is basically linked. ***************
59 R: Explain more.....
60 L9: Like maths is linked to other things in real world and also here number patterns problems are linked to algebra.

QUESTION 3

61 R: How did you find this compared to the other two.
62 L9: It was not easy and so I only completed the table of values
62 R: How did you work out your values for the dots and lines of motifs'?
64 L9: The dots increase by 2 every time you move position and the lines change by 4 at the same time.
65 R: How do you know that information?
66 L9: I counted the dots and lines in the motifs one by one.
67 R: Did you solve this in a different way?
68 L9: Not really, but may be putting the numbers in the last column as unknowns.
69 R: How would you know that doing that is correct?
70 L9: Because its the nth position of the motifs and n can be any number so we use letters for that.
71 R: Why letters this time when all along you were calculating real numbers in your table?
72 L9: I mean for the position n it is too general and my expression mean that.
73 R: So, what numbers did you come up with?
74 L9: Number of dots: 4; 6; 8; 10; 12; 14; 16 ,........
Number of lines: 5; 9; 13; 17; 21; ....

75 R: How did you find these answers for positions 63 and 120?
76 L9: For position 63, I extended each pattern by the difference numbers between the terms and put the numbers in groups of 10 or 15 but for 120 position I just took a guess.******
77 R: How sure would you be with a guess?
78 L9: You never know, you could be right or wrong as its a matter of taking a chance.
79 R: What are your numbers in the last column?
80 L9: 2 x + 2 for the dots and 4 x + 1 for the lines where my x represents n, the motifs' position.
81 R: Does this always work for nth position.
82 L9: I think so.
83 R: Take me through the other questions.
84 L9: I found this confusing so I decided not to do until I take my time to think about. So I moved on to question number 4.

QUESTION 4

85 R: What were you thinking about this question?
86 L9: It looked very straight forward at first but part b) I realized that its not easy.
87 R: Why did you that part b) is difficult?
88 L9: You see all other pattern problems in this questionnaire have increasing values and all of sudden in this one the numbers are going down so I thought there is a mistake.
89 R: Did you solve it in a different way?
90 L9: I did not solve it especially coming up with the formula.
91 R: What were the next three terms in each case?
92 L9: For the 1st pattern I got 26; 33; 40 and the in the 2nd pattern: 20; 15; 10 but am not sure.
93 R: Did you work out the 10th , 50th and nth terms from these patterns.

94 R : How did you find the 10th , 20th and 50th numbers in question 4?
95 L9 : I extended the numbers in the pattern.
96 R : Ok. But how did you know that at position six the number is 40?
97 L9 : You see here ...., (pointing a pattern), from first number 5 to next 12 we add 7, and from 12 to 19 we do the same thing. So every time I add a seven gives me the next number.
98 R : So how did find 20th or 50th numbers?
99 L9 : I wrote down all the numbers until twenty and the same thing until I have fifty of them. For example; it's easy once you know what to add: 5, 12, 19, 26, 33, 40, 47, 54, 61, 68, 75, 82, 89, 96, 103, 110, 117, 124, 131, 138, 145, 152, ..., etc.
100 R : How do explain in your own words the general rule for this pattern, I mean the nth term expression?
101 L9 : I think it is; if you a number or term, every time you times it by 7 then you still need to add or minus something else. So here,......, I need to minus 2.
102 R : Will that always work with patterns?
103 L9 : I think so but I'm not very sure.
104 R : What will be your formula for the nth term?
105 L9 : This one is difficult for me because of n. But if I know the number represented by n, I can try to think........ . q

94 L9: I found a formula for the first pattern which is 7 x – 2 but I cannot explain properly the method of getting it. And used it to find the answers to this question.

95 R: What numbers did you find?
96 L9: I substituted $x = 10$ and $x = 50$ into $7x – 2$ and my answers are 68 and 348.
97 R: Can this always work?
98 L9: I think so because we are using a formula.
99 R: What numbers did you find from the second pattern?
100 L9: They were negatives so I thought it must wrong. Aren't we working with positive numbers in these patterns? So why the answer is now negative integer?
101 R: What do you think?
102 L9: I left this part because its confusing.
103 R: What is confusing here?
104 L9: The numbers are not in the same pattern as those in 4a) and you see, they are from top..... its like a decreasing pattern which I have not done before.*******
105 R: So you cannot think of any method of doing this?
106 L9: Sir, the difference is – 5 and not +5, so how can you have minus difference? Is minus sign not mean the difference already......, it does not make sense I think.
107 R: Do you have any comment about this work.
108 L9: This work Sir, makes us think and not just calculate answers fast, fast. It is good exercise to work on.
109 R: Thank you very much for your participation in this research.
110 L9: Thank you too, Sir.

______________THE END________________