A Comparison of Numerical Methods to Solve Fractional Partial Differential Equations

MSc Dissertation

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I declare that this dissertation is my own unaided work. It is being submitted for the degree of Masters of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination at any other university.

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Abstract

A comparison of two numerical methods - finite difference and Adomian decomposition method (ADM) - to solve a variety of fractional partial differential equations that occur in finance are investigated. These fractional partial differential equations fall into the class of Lévy models. They are known as the Finite Moment Log Stable (FMLS), CGMY and the extended Koponen (KoBol) models. Convergence criteria for these models under the numerical methods are studied. ADM fails to accurately price a claim written on these models. However, the finite difference scheme works well for the FMLS and KoBol models.

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Dedications

I dedicate this MSc dissertation to my parents, brother and sister in thanks for all the love and support they have given me.

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Chapter 1

Introduction

Undoubtedly the most well known model for pricing financial derivatives is the Black-Merton-Scholes model. This model is relatively simplistic and as such, has some drawbacks when it comes to pricing financial derivatives. The major drawback, as will be discussed later in this chapter, is its inability to accurately price out the money instruments.

To counter this, a large number of other models have been introduced. These models are more complicated however have the advantage of being able to more accurately price financial instruments. This thesis looks at three such models: Finite Moment Log Stable (FMLS), CGMY (named after its authors: Carr, Geman, Madan, Yor) and KoBol (extended Koponen) Models.

To price a financial instrument that follow any of these models, it is required to solve a fractional partial differential equation (FPDE). This thesis looks at and compares two of the most popular methods for solve FPDEs, namely the finite difference scheme and the Adomian decomposition method (ADM).

This chapter outlines three main concepts that this thesis deals with: financial derivatives, Lévy processes and fractional partial differential equations. These concepts will be discussed individually in detail in later chapters.

1.1 Financial Derivatives

There lies an immense amount of interest and literature on the pricing of financial derivatives. A financial derivative is an instrument whose price depends on, or is derived from, the value of another asset [1]. Often, this underlying asset is a stock.

In modern times, the number of variations of financial derivatives available is practically limitless. Among the most common are the European call and European put options:

Let S_t be the value of a stock at time t. Denote a European call option by $c[S_t, T-t, K]$ which gives the holder the right, but not the obligation to buy S_t for a predetermined price known as the strike price, K, set at t at some future date T [1]. Denote a European put option by $p[S_t, T-t, K]$ which gives the holder the right, but not the obligation to sell S_t for a predetermined price known as the strike price, K, set at t at some future date T [1].

The concept of financial derivatives is not new. There are references that show that they have roots dating to the time of the ancient Greek civilization where farmers used derivatives to lock onto a price to sell their crops at a future date [2]. Thus, regardless of market conditions, Greek farmers were able to ensure that their crops were sold for a set price.

While there remains some historical debate as to the exact date of the creation of financial derivatives, it is well accepted that the first attempt at modern derivative pricing began with the work of Charles Castelli [3] published in 1877. Castelli's book was a general introduction to concepts such as hedging and speculative trading, but it lacked mathematical rigor. Twenty three years later, Louis Bachelier offered the earliest known analytical method for option pricing in a work entitled *Théorie de la spéculation* [4]. This work was a step in the right direction. However he modeled the changes of stock prices via Brownian motion - a model that allows stock prices to become negative and produced option prices that exceeded the price of the underlying stock.

Following that, a number of other papers extended Bacheliers work, most notably by Paul Samuelson [5], Richard Kruizenga [6], A. James Boness [7] and Osborne [8]. Osborne's contribution was important as he was the first to introduce the concept of modeling stock price movements using geometric Brownian motion. He justified his approach based on the Weber-Fechner law, which states that people perceive the intensity of stimuli on a log scale rather than a linear scale. But it wasn't until 1973 with the introduction of the Black-Merton-Scholes equation that a significant breakthrough in the subject of option pricing was put forward.

1.2 The Black-Merton-Scholes World

In 1969, Fischer Black and Myron Scholes got an idea that would change the world of finance forever. The central idea of their paper revolved around the discovery that one did not need to estimate the expected return of a stock in order to price an option written on that stock. By changing measure, one could use the risk free rate instead and perfectly price the option under the proposed model [1, 9]. By 1973 Black and Scholes, with the help of Robert C. Merton, had written and proposed a first draft of their to-be famous paper to the Journal of Political Economy for publication, but it was rejected. A second rejection occurred at the hands of the Review of Economics and Statistics. However with the help of Merton, Miller and Eugene Fama they were able to review and modify the paper so that eventually it was accepted by the Journal of Political Economy in 1973 [9]. Both Robert Merton and Myron Scholes were awarded a Nobel Prize in economics in 1997. Fischer Black had died of throat cancer in 1994 and was thus not eligible for the award. It is perhaps a cruel twist of fate that the highly influential figure of Merton is often absent from the naming of the Black-Merton-Scholes equation, as his input was an invaluable contribution to the work of Black and Scholes.

Since then, the impact that the paper has had on modern economy has been great. Despite the shortcomings of the Black-Merton-Scholes pricing framework, which will soon be discussed, it remains up until today as the most influential and most widely used method to price options.

This groundbreaking work assumed that the stock price, S_t, followed a log-normal distribution (geometric Brownian motion) [1, 9]. That is:

$$dS_t = \mu S_t dt + \sigma S_t dZ_t, \tag{1.2. 1}$$

where $\mu > 0$, the drift, is the expected continuously compounded rate of return from holding S_t; $\sigma > 0$ is the volatility of returns from holding S_t; and dZ_t is the Brownian motion increment which is assumed to have a Gaussian distribution. That is $dZ_t \sim N(0, dt)$. Note that using Ito's formula (see appendix A) (1.2. 1) could alternatively be written as

$$d\ln(S_t) = (\mu - \frac{1}{2}\sigma)dt + \sigma dZ_t. \tag{1.2. 2}$$

Under the assumption of (1.2. 1), Black, Merton and Scholes derived a partial differential equation (PDE) to solve for the value of a financial derivative $V(S, t)$ depending on an underlying asset S at time t to be [1, 9]:

$$rV(S,t) = \frac{\partial V(S,t)}{\partial t} + rS\frac{\partial V(S,t)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2}, \tag{1.2. 3}$$

where r is the risk-free rate of interest.

The fact that we can replace the unknown μ by the market observable r forms the crux of the Black-Merton-Scholes equations and allows for dynamic hedging to occur. Equivalent to (1.2. 3), using $x = \ln(S_t)$ the PDE can be written as:

$$rV(x,t) = \frac{\partial V(x,t)}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V(x,t)}{\partial x^2} + (r - \frac{1}{2}\sigma^2)\frac{\partial V(x,t)}{\partial x}. \tag{1.2. 4}$$

The beauty of the Black-Merton-Scholes model lies in its simplicity. It is possible to find closed form solutions for the price of a European call or European put option [1, 9].

1.3 Introduction to Lévy Processes

When first introduced, the Black-Merton-Scholes model was blindly accepted. Their model wasn't just *a* model, it was *the* model for pricing derivatives. However, the East Asian and Russian financial crisis of 1997 exposed the weaknesses of the model. The likelihood of an extreme movement in the stock price is higher in the real world than the Black-Merton-Scholes model suggests. Further, the model assumes that the path that a stock price follows is continuous whereas in reality stock prices can jump. Scholes and Merton suffered greatly when this event happened, as their company, LTCM, lost 4.6 billion dollars. Today, this incident is frequently quoted in texts as one of the prime examples of the importance of risk management in financial institutions [10].

To overcome this, a number of more complicated models have been proposed. One possibility is to assume a two factor model. Such models have the same form as (1.2. 1) but allow σ to follow a stochastic process [11]. The other popular alternative, on which this thesis focuses, is to assume a model of the Lévy family for the evolution of the stock price [12, 13, 14]. These models are also known as jump processes.

This thesis deals with three particular Lévy models which have been proposed in the literature, namely the FMLS, CGMY and KoBol models [14].

1.4 Introduction to Fractional Partial Differential Equations

Recall that in the Black-Merton-Scholes case, the solution to value a derivative $V(x, t)$ required the solving of a partial differential equation given by (1.2.

4). However, in the case of the above mentioned Lévy models, the solution to $V(x,t)$ is given by a fractional partial differential equation (FPDE)[14].

The use of FPDEs has applications in finance [15, 16, 17] and in other areas. In recent years, they have been used to describe a number of processes in many fields including fluid mechanics, biology, hydrodynamics, solid state physics and optical fibers [18, 19, 20, 21, 22]. Such processes are related to equations where diffusion occurs slower or more rapidly that under normal conditions. Fractional partial differential equations are an extension of fractional calculus where the order of the derivative is allowed to take on non-integer values.

1.5 Numerical Approximations to Solve FPDEs

In general, it is not possible to find an exact solution to an FPDE. However, there are a variety of numerical methods that can be used to get approximate solutions. This thesis considers two such methods: the finite difference scheme and ADM.

The finite difference method extends the classical finite difference scheme for PDEs to fractional order [23, 24]. Adomian decomposition method solves FPDEs via a power series expansion [25, 26]. Both of these methods have advantages and disadvantages, which will be explored in this thesis.

This thesis will delve into each of the aforementioned concepts in detail. While other papers look at solving FPDEs using a specific approximation, this paper will compare two of the most well known methods for solving FPDEs. Furthermore, the thesis focuses both on the models and the numerical methods used to solve them. Most of the literature focuses on either one or the other, meaning that this thesis gives a clear, complete and detailed picture of the work - ideal for any practitioners in the field who are looking to incorporate new models into their derivative pricing methodologies.

The thesis is organised as follows: Chapter 2 sets up the mathematical framework for the thesis. The concepts of fractional calculus, Lévy processes and the FPDEs that this thesis deals with are explored. The numerical methods used in this thesis to solve the FPDEs (finite difference schemes and ADM) are derived. Chapter 3 explores the numerical solution obtained when solving the required FPDEs. Chapter 4 contains concluding remarks.

Chapter 2

Mathematical Framework

This chapter looks at the details of the mathematics required in this thesis. Fractional calculus and Lévy processes are discussed. The specific Lévy processes that this thesis deals with are then discussed as well as using Fourier transformations to get an FPDE to solve for the price of a financial derivative. Finally, the numerical methods used to solve these FPDEs are derived in detail.

For details on stochastic calculus the reader is referred to appendix A of this thesis.

2.1 Fractional Calculus

Recall that $V(x,t)$ represents the value of a financial derivative depending on $x = \ln(S)$ at time t, where S is the underlying asset. The solution to $V(x,t)$ when using one of the mentioned Lévy models is given by a fractional partial differential equation. In this section, the basic concepts of fractional calculus and the fractional derivative is outlined. For a detailed analysis of the subject see [15, 27].

The origins of fractional calculus date back to the same time as the invention of classical calculus. Fractional calculus generalises the concept of differentiation and integration a step furthermore by allowing non-integer order.

The idea was first raised by Leibniz in 1695 when he wrote a letter to L'Hospital where he said: *'Can the meaning of derivatives with integer order to be generalized to derivatives with non-integer orders?'* To this L'Hospital replied with a question of his own: *'What if the order will be $\frac{1}{2}$?'* To this, Leibniz said: *'It will lead to a paradox, from which one day useful consequences will be drawn.'* - [30].

This was indeed the case. The concept of a fractional derivative is not intuitive. The operator $d^{1/2}/dx^{1/2}$ is difficult to quantify. However, the growth in the field of fractional calculus has been vast with many mathematicians contributing to its growth. Most notably Riemann, Liouville, Caputo, Letnikov, Grunwald and Weyl - see [15, 28]. In fact, the fractional derivative is termed the Riemann-Liouville derivative after its major contributors.

As we shall see, fractional calculus has strong links with Fourier transformations. Grunwalds formula allows for the linearisation of fractional derivative, making it easier to work with in some cases.

What follows is an outline of how the fractional derivative is defined. In fact, we start out the definition of this operator using the integral. Let l be an integer. Then the l^{th} order integration of a function $f(x)$ is given by

$$f^{[l]}(x) = \int_a^x dx_1 \int_a^{x_1} dx_2 ... \int_a^{x_{l-1}} dx_l f(x_l), \tag{2.1. 1}$$

where a is a constant. By using Cauchy's formula for repeated integration, equation (2.1. 1) can be written as

$$f^{[l]}(x) = \frac{1}{(l-1)!} \int_a^x \frac{f(y)}{(x-y)^{1-l}} dy. \tag{2.1. 2}$$

To extend (2.1. 2) to a non-integer order, γ, the Γ function that generalizes the factorial function for non-integer values is invoked to obtain

$$ {}_aJ_x^{\gamma} f(x) = f^{[\gamma]}(x) = \frac{1}{\Gamma(\gamma)} \int_a^x \frac{f(y)}{(x-y)^{1-\gamma}} dy. \tag{2.1. 3}$$

Equation (2.1. 3) is known as the left Riemann-Liouville fractional integral of order γ. The left Riemann-Liouville fractional derivative is defined by

$$ {}_aD_x^{\gamma} f(x) = \frac{\partial^{\gamma} f(x)}{\partial_+ x^{\gamma}} = \frac{\partial^{\xi}}{\partial x^{\xi}} f^{[\xi-\gamma]} = \frac{1}{\Gamma(\xi-\gamma)} \frac{\partial^{\xi}}{\partial x^{\xi}} \int_a^x \frac{f(y)}{(x-y)^{\gamma+1-\xi}} dy, \tag{2.1. 4}$$

where ξ is the smallest integer greater than γ. The right sided Riemann-Liouville derivative is obtained by switching the limits of integration in (2.1. 1):

$$ {}_xD_b^{\gamma} f(x) = \frac{\partial^{\gamma} f(x)}{\partial_- x^{\gamma}} = \frac{1}{\Gamma(\xi-\gamma)} \frac{\partial^{\xi}}{\partial x^{\xi}} \int_x^b \frac{f(y)}{(x-y)^{\gamma+1-\xi}} dy, \tag{2.1. 5}$$

where b is a constant. These definitions are easily extended to the case where $a = -\infty$ and $b = \infty$. The fact that we have defined two fractional derivatives, left and right, seems unintuitive. However, it is simply a directional component in the same way that with classical calculus we can define the left and right sided derivative of a function.

To better understand the mechanics of Adomian decomposition which will later be discussed, it is useful to introduce some properties of the operators J and D. Let α', β' > 0, $m' - 1 < \alpha' \leqslant m'$ and $\gamma' > -1$. Then:

$$ {}_0J_x^{\alpha'}\,{}_0J_x^{\beta'} f(x) = {}_0J_x^{\alpha'+\beta'} f(x) \tag{2.1. 6}$$

$$ {}_0J_x^{\alpha'} x^{\gamma'} = \frac{\Gamma(\gamma'+1)}{\Gamma(\gamma'+\alpha'+1)} x^{\gamma'+\alpha'} \tag{2.1. 7}$$

$$ {}_0J_x^{\alpha'}\,{}_0D_x^{\alpha'} f(x) = f(x) - \sum_{k=0}^{m'-1} \frac{f^k(0)}{k!} x^k \tag{2.1. 8}$$

2.2 The Lévy Process

Recall that geometric Brownian motion fails to capture some essential dynamics of real stock price changes. As a result, many alternative models have been proposed in an attempt to better price financial derivatives. These models can be subdivided into two categories: Parametric models and non-parametric models. For a detailed description of Lévy processes, the reader is referred to [12, 13].

In parametric models, we specify the functional form of the underlying asset. For example, the Black-Merton-Scholes model is parametric as we assume that the stock changes according to geometric Brownian motion. These models have three classifications. They either assume a continuous diffusion process with deterministic volatility, allow volatility to be a stochastic process of its own, or assume a model with jumps. The latter is the case with Lévy models.

Another alternative has been to create a mapping principle that switches between distributions [31].

The non-parametric approach consists of extrapolating the model from the market data [32]. The name given to this method is the expansion method as one infers the different terms of the expansion and can then reconstitute the distribution.

The parametric method is by far better documented and well studied in the literature. Most notably, Lévy processes have been very successful in capturing the dynamics of stock price changes.

Lévy processes have long been used in finance. In fact, Brownian motion falls into the class of Lévy processes. Recall that this was the model proposed by Bachelier in 1900. However, the first attempt to model stock price data using an exponential non-Gaussian process was put forward by Mandelbrot [33] in 1963. He noticed that the log of relative price changes showed evidence of a long tailed distribution. He thus proposed that the Brownian motion component should be replaced by an *α-stable Lévy motion* with $\alpha < 2$. This creates a model where stock prices change only by jumps. The parameter α plays an important role in Lévy processes. The case $1 < \alpha < 2$ represents superdiffusion, where particles spread faster than in the classical case. The case $0 < \alpha < 1$ represents subdiffusion, where particles spread slower than in the classical case. For a detailed description of the parameter α and its role in diffusive processes the reader is referred to [34].

A few years later in 1967, Press proposed an exponential Lévy process that was non-stable [35]. He used a Brownian motion component and independently a compound Poisson process with independent normally distributed jumps to model log-stock price changes.

Since then, a whole host of other Lévy models have been proposed. Perhaps the most popular one not examined in this thesis is the *variance gamma* model introduced by Madan and Seneta [36] in 1990. This model is a pure jump process.

What follows is a mathematical definition of the Lévy process.

Definition: let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space, that satisfies the usual condition.

A cádlág, adapted, real valued stochastic process $L = (L_t)_{t>0}$ with $L_0 = 0$ is called a Lévy process if the following conditions hold:

(i) L has independent increments. That is, $L_t - L_s$ is independent of F_s for any $0 \leqslant s < t \leqslant T$.

(ii) L has stationary increments. That is, for any $s, t \geqslant 0$ the distribution of $L_{t+s} - L_t$ does not depend on t.

(iii) L is stochastically continuous. That is, for every $t, \epsilon > 0 : \lim_{s \to t} P(|L_t - L_s| > \epsilon) = 0$.

Brownian motion is a Lévy process. In fact, it is the only Lévy process with continuous sample paths.

The family of Lévy processes is huge, however, there is a compact characterisation. That is, a time dependent random variable X_t is a Lévy process if and only if it has independent stationary increments with a log characteristic function given by the Lévy-Khintchine representation:

$$\ln \mathbb{E}[e^{i\eta X_t}] = t\psi(\eta) = \kappa i t\eta - \frac{1}{2}\sigma^2 t\eta^2 + t\int_{\mathbb{R}-\{0\}} (e^{i\eta x} - 1 - i\eta H(x))W(dx), \tag{2.2. 9}$$

where $\kappa \in \mathbb{R}$, $\sigma \geqslant 0$, $H(x)$ is a truncation function and the Lévy measure W satisfies:

$$\int_{\mathbb{R}} \min(1, x^2)W(dx) < \infty. \tag{2.2. 10}$$

The function, ψ is called the characteristic exponent of the Lévy process and is important when using Fourier transformations. A Lévy process can be seen as a combination of three processes. A drift component, a Brownian motion process and a jump component. These three aspects are completely determined by the Lévy-Khintchine triplet: (κ, σ^2, W). The measure, W, is responsible for the frequency and magnitude of the jumps. If W can be written as $W(dx) = w(x)d(x)$ then $w(x)$ is called the Lévy density.

Recall that Black-Merton-Scholes assumed that S_t followed (1.2. 2). that is:

$$d\ln(S_t) = (\mu - \frac{1}{2}\sigma)dt + \sigma dZ_t, \tag{2.2. 11}$$

and that the main finding of Black, Merton and Scholes was that one did not need to estimate μ to price a derivative. In fact, under a suitable change of measure, one could replace μ with the market observable risk-free rate r. Mathematically,

$$d\ln(S_t) = (r - \frac{1}{2}\sigma)dt + \sigma dZ_t^{\mathbb{Q}}, \tag{2.2. 12}$$

where $dZ_t^{\mathbb{Q}}$ is the increment of the Brownian motion under what is called the *risk neutral measure*.

When pricing under a Lévy model, the Brownian motion component is replaced with a Lévy component. So the resulting stochastic differential equation becomes:

$$d\ln(S_t) = \mu dt + dL_t^{\mathbb{P}}, \tag{2.2. 13}$$

where the superscript $\mathbb{P}$ of $dL_t^{\mathbb{P}}$ indicates that we are using a measure where μ needs to be estimated from the real world. This is often called the *real world measure* or the *historical measure*. As with the Black-Merton-Scholes case, it is now necessary to change to the risk neutral measure. However, this model is an incomplete market model. An incomplete market allows for different prices of an asset in different states of the world. In the Black-Merton-Scholes world, the market model was complete and thus there was one unique risk neutral measure. However, in the case of the Lévy model this is not the case.

A variety of ways have been proposed to pick the risk neutral measure in the Lévy world case including the minimal measure, the Esscher measure, the variance optimal measure and the minimal entropy measure - see [37, 38]. This thesis assumes an approach whereby a change of measure keeps the stock price process within the Lévy family of models. Under this assumption, the risk neutral log-stock price process becomes:

$$d\ln(S_t) = (r - v)dt + dL_t^{\mathbb{Q}}, \tag{2.2. 14}$$

where v is the convexity adjustment that plays a similar role to the $\frac{1}{2}\sigma^2$ term in (2.2. 12) where the Black-Merton-Scholes framework was used.

Geometric Brownian motion is a Lévy process and can be retrieved by the above Lévy framework by choosing the Lévy-Khintchine triplet:(0,σ^2,0).

2.3 Fourier Transformations

The connection between Fourier transforms and Lévy processes rests on the following result: if the log-stock process follows (2.2. 14) where the characteristic exponent of the Lévy process L_t is given by $\psi(\eta)$ then the Fourier transform,

$$f(\eta) = \int_{-\infty}^{\infty} e^{i\eta x} f(x)dx = F\{f(x)\}, \tag{2.3. 15}$$

of the value of a European option $V(\eta, T)$ written on a stock S_t satisfies:

$$\frac{\partial V(\eta, t)}{\partial t} = (r + i\eta(r - v) - \psi(-\eta))V(\eta, t), \tag{2.3. 16}$$

with boundary condition $V(\eta, T) = \Pi(\eta, T)$.

Equation (2.3. 16) holds for all Lévy processes with finite exponential moments - i.e. ψ exists - see [14] for a proof of this result. This result will be used in the next section where we apply the Fourier transform to 3 specific models to obtain the FPDE to solve for the price of a European derivative, $V(x, t)$.

2.4 The FMLS, CGMY and KoBol Processes

The FMLS, CGMY and KoBol models all fall into the class of Lévy models. We focus on the convergence of these models in this thesis under different numerical approximations. If we assume any of these models for our stock price movement and then use the Fourier transform given by (2.3. 16), we end up with a FPDE which must be solved to get $V(x, t)$.

These Lévy processes followed after the significant work of Mandelbrot who was discussed in earlier sections. Based on his work, Koponen, Boyarchenko and Levendorskiî, introduced the log-stable (LS) process. However, the LS model had infinite variance which made it mathematically difficult to work with. As a result it was proposed to modify the process by introducing a dampening effect in the tails. This ensured finite moments and gave rise to the KoBol process. Finally, Carr, Geman, Madan and Yor proposed the popular CGMY model which has become a widely accepted and used model for equity pricing [13, 14, 39].

For an LS (log-stable) process, the Lévy density is given by:

$$w_{LS}(x) = \begin{cases} Dq|x|^{-1-\alpha} & x < 0 \\ Dpx^{-1-\alpha} & x > 0 \end{cases}, \qquad (2.4.\ 17)$$

where $D > 0$, $p, q \in [-1, 1]$, $p + q = 1$ and $0 < \alpha \leqslant 2$. As already stated above, his process has infinite variance making it mathematically difficult to work with. With $p = 0$ and $q = 1$ we can transform the process into what is known as the FMLS process. This process is interesting as it can only incur downward jumps. This is easily seen by looking at the Lévy density at $p = 0$ and $q = 1$. By applying the Fourier transformation given by (2.3. 16) the resulting FPDE for the price of a European claim is given by

$$\frac{\partial V(x,t)}{\partial t} + (r + \frac{1}{2}\sigma^{\alpha}\sec(\frac{\alpha\pi}{2}))\frac{\partial V(x,t)}{\partial x} - \frac{1}{2}\sigma^{\alpha}\sec(\frac{\alpha\pi}{2})\frac{\partial^{\alpha} V(x,t)}{\partial_{+} x^{\alpha}} = rV(x,t), \tag{2.4. 18}$$

for $\alpha \neq 1$. By truncating the tail of the LS distribution we obtain two more tractable Lévy processes, the first one being the CGMY process.

The Lévy density for the CGMY process is given by:

$$w_{CGMY}(x) = \begin{cases} C\frac{e^{-G|x|}}{|x|^{1+Y}} & x < 0 \\ C\frac{e^{-M|x|}}{|x|^{1+Y}} & x > 0 \end{cases}, \tag{2.4. 19}$$

where $C > 0$, $G \geqslant 0$, $M \geqslant 0$ and $Y \leqslant 2$. The parameter C is a measure of the overall activity level while G and M control the exponential decay of the left and right tails. The distribution is symmetric when $G = M$.

The resulting FPDE for the price of a European claim is then given by:

$$\frac{\partial V(x,t)}{\partial t} + (r-v)\frac{\partial V(x,t)}{\partial x} + C\Gamma(-Y)e^{Mx}\frac{\partial^{Y}(e^{-Mx}V(x,t))}{\partial_{-} x^{Y}} + C\Gamma(-Y)e^{-Gx}\frac{\partial^{Y}(e^{Gx}V(x,t))}{\partial_{+} x^{Y}}$$

$$= (r + C\Gamma(-Y)[M^{Y} + G^{Y}])V(x,t), \tag{2.4. 20}$$

where $v = C\Gamma(Y)[(M-1)^{Y} - M^{Y} + (G+1)^{Y} - G^{Y}]$.

The second process that results from truncating the tails of the LS distribution is the KoBol process. The Lévy density is given by

$$w_{KoBol}(x) = \begin{cases} Dq|x|^{-1-\alpha}e^{-\lambda|x|} & x < 0 \\ Dpx^{-1-\alpha}e^{-\lambda x} & x > 0 \end{cases}, \qquad (2.4.\ 21)$$

where $D > 0$, $\lambda > 0$, $p, q \in [-1, 1]$, $p + q = 1$ and $0 < \alpha \leqslant 2$. The parameter λ controls the decay of the exponent. p and q control the skewness, and D is a measure of the overall activity level. The resulting FPDE for the price of a European claim is then given by:

$$\frac{\partial V(x,t)}{\partial t} + (r - v - \lambda^{\alpha}(q-p))\frac{\partial V(x,t)}{\partial x} + \frac{1}{2}\sigma^{\alpha}[pe^{\lambda x}\frac{\partial^{\alpha}(e^{-\lambda x}V(x,t))}{\partial_{-}x^{\alpha}} + qe^{-\lambda x}\frac{\partial^{\alpha}(e^{\lambda x}V(x,t))}{\partial_{+}x^{\alpha}}]$$

$$= (r + \frac{1}{2}\sigma^{\alpha}\lambda^{\alpha})V(x,t) \qquad (2.4.\ 22)$$

where $v = \frac{1}{2}\sigma^{\alpha}[p(\lambda-1)^{\alpha} + q(\lambda+1)^{\alpha} - \lambda^{\alpha} - \alpha\lambda^{\alpha-1}(q-p)]$.

There is no explicit solution to any of the resulting FPDEs. Hence a numerical method must be applied to get a solution. This thesis will consider and compare the finite difference scheme and the Adomian decomposition method.

2.5 Numerical Methods to Solve FPDEs

FPDEs have become an increasingly popular way of modeling a real world process in many fields including finance. The past three decades have been particularly interesting with FPDEs playing an important role in modeling processes in Fluid Flow, Solute Transport or Dynamical Processes in Self-Similar and Porous Structures, Diffusive Transport, Material Viscoelastic Theory, Electromagnetic Theory, Dynamics of Earthquakes, Control Theory of Dynamical Systems, Optics and Signal Processing, Bio-Sciences, Economics, Geology, Astrophysics, Probability and Statistics and Chemical Physics to name a few [18]. The fact that so many fields have benefited from the increasing popularity of FPDEs goes to show that despite their unintuitive mathematical definition, many natural real life processes do in fact follow such models.

FPDEs allow us to model processes where diffusion occurs at rates that normal derivatives cannot capture. Another key element to FPDEs is that they can model processes which require *long memories in times.* This key attribute is lacking in normal diffusion models.

As FPDEs have played a more meaningful role in mathematics over the past few decades, it has been necessary to come up with methods to solve them. In general it is not possible to find the exact solution of an FPDE. However a number of useful and efficient methods have been proposed in the literature.

This thesis looks at two such methods to solve FPDEs in detail. The finite difference scheme and the Adomian decomposition method.

2.5.1 The Finite Difference Scheme

Finite difference schemes are a powerful and intuitive way to solve PDEs. In essence, we discretise the differential operators d/dx and d/dt and move forward (or backward) in time in small steps to solve for a PDE at a specified time.

Note first that all the FPDEs derived in the previous section are of the form:

$$\frac{\partial V(x,t)}{\partial t}+A\frac{\partial V(x,t)}{\partial x}+B(x)\frac{\partial^{\alpha}(f(x)V(x,t))}{\partial_{+}x^{\alpha}}+C(x)\frac{\partial^{\alpha}(h(x)V(x,t))}{\partial_{-}x^{\alpha}}+DV(x,t)=0 \tag{2.5. 23}$$

This thesis uses a forward in time implicit finite difference scheme to solve

(2.5. 23). We can then specifically solve (2.4. 18), (2.4. 20) and (2.4. 22).

The first step is to define the usual framework when dealing with PDEs. That is, for $L \leqslant x \leqslant R$ and a finite time interval $[0, T]$. The terminal condition for the FPDE is given by $V(x,T) = s(x)$. In Section 4, this scheme and ADM will be used to price a European call option using the derived FPDE. In the case of a European call, $s(x) = \max(e^x - K, 0)$ where K is the strike price. Further, the boundary conditions are $V(L,t) = 0$ and $V(R,t) = b(R,t)$. Discretise the time interval $[0, T]$ into m equal pieces of size Δt so that $m\Delta t = T$. Discretise the interval $[L, R]$ into $n+1$ equal pieces of size Δx so that $n\Delta x = R$. To abbreviate notation, let $V_{i,j} = V(L+i\Delta x, j\Delta t)$ where $i = 0, 1, 2, ..., n$ and $j = 1, 2, ..., m$. The same notation will be used for any function that is either a function of x or t. Note also that as per the convergence criteria derived by Meerschaert, Tadjeran and Scheffler, we require $1 \leqslant \alpha \leqslant 2$ for convergence when dealing with the methods provided for the finite difference scheme [23].

The first two terms and the last term are easily discretised as they have no fractional component and thus can be treated as normal. The time derivative is thus discretised as

$$\frac{\partial V(x,t)}{\partial t}\bigg]_{x=L+i\Delta x, t=j\Delta t} \approx \frac{V_{i,j+1} - V_{i,j}}{\Delta t}. \tag{2.5. 24}$$

The second term in (2.5. 23) is discretised via

$$A\frac{\partial V(x,t)}{\partial x}\bigg]_{x=L+i\Delta x, t=j\Delta t} \approx A\frac{V_{i+1,j} - V_{i-1,j}}{2\Delta x}, \tag{2.5. 25}$$

and the last term is simply $DV_{i,j}$.

What is left now is to discretise the two fractional derivatives. The methodologies used to do so are taken from Tadjeran, Meerschaert and Scheffler from their relatively new paper, where they use a Crank-Nicholson type scheme with a local truncation error of $O((\Delta t)^2) + O(\Delta x)$ that solves a right sided FPDE [23]. Another paper by Tadjeran and Meerschaert then extends this to the left sided FPDE [24].

A result of fundamental importance to the finite difference scheme to solve an FPDE is the left-shifted and right-shifted Grunwald formula. The right-shifted Grunwald formula is given by

$$\frac{\partial^{\alpha}(f(x)V(x,t))}{\partial_{+}x^{\alpha}} = \frac{1}{\Gamma(-\alpha)} \lim_{N\to\infty} \frac{1}{q_{(1)}^{\alpha}} \sum_{k=0}^{N} \frac{\Gamma(k-\alpha)}{\Gamma(k+1)} f(x-(k-1)q_{(1)})V(x-(k-1)q_{(1)},t), \tag{2.5. 26}$$

where N is a positive integer and $q_{(1)} = \frac{x-L}{N}$. Equation (2.5. 26) is of importance in fractional calculus as it allows us to linearise the fractional derivative. This result is used to estimate the spatial α-order fractional derivative. Analogously, the left shifted Grunwald formula is defined by

$$\frac{\partial^{\alpha}(h(x)V(x,t))}{\partial_{-}x^{\alpha}} = \frac{1}{\Gamma(-\alpha)} \lim_{N\to\infty} \frac{1}{q_{(2)}^{\alpha}} \sum_{k=0}^{N} \frac{\Gamma(k-\alpha)}{\Gamma(k+1)} h(x+(k-1)q_{(2)})V(x+(k-1)q_{(2)},t), \tag{2.5. 27}$$

where N is a positive integer and $q_{(2)} = \frac{R-x}{N}$.

It is convenient to replace the Γ functions in (2.5. 26) and (2.5. 27) by the 'normalized' Grunwald weights:

$$g_{\alpha,k} = (-1)^k \frac{(\alpha)(\alpha-1)...(\alpha-k+1)}{k!} \tag{2.5. 28}$$

for $k = 1, 2, 3, ...$ and $g_{\alpha,0} = 1$.

The next step is to discretise the Grunwald formula by defining the right and left fractional partial differential operators, respectively.

$$\delta^{+}_{\alpha,x}(f_i V_{i,j}) = \frac{1}{(\Delta x)^{\alpha}} \sum_{k=0}^{i+1} g_{\alpha,k} f_{i-k+1} V_{i-k+1,j} \tag{2.5. 29}$$

and

$$\delta^{-}_{\alpha,x}(h_i V_{i,j}) = \frac{1}{(\Delta x)^{\alpha}}. \sum_{k=0}^{n-i+1} g_{\alpha,k} h_{i+k-1} V_{i+k-1,j} \tag{2.5. 30}$$

Notice that (2.5. 29) is simply a discretised version of (2.5. 26) and (2.5. 27) is simply a discretised version of (2.5. 29).

All the terms of (2.5. 23) can now be discretised. Using all the pieces derived above, a Crank-Nicholson type scheme to solve (2.5. 23) is obtained for $i = 1, 2, 3, ..., n-1$:

$$\begin{aligned} &\frac{V_{i,j+1} - V_{i,j}}{-\Delta t} + A\frac{V_{i+1,j} - V_{i-1,j}}{2\Delta x} + \frac{B_i}{2}[\delta^{+}_{\alpha,x}(f_i V_{i,j+1}) + \delta^{+}_{\alpha,x}(f_i V_{i,j})] + \\ &\quad \frac{C_i}{2}[\delta^{-}_{\alpha,x}(h_i V_{i,j+1}) + \delta^{-}_{\alpha,x}(h_i V_{i,j})] + DV_{i,j} = 0 \end{aligned} \tag{2.5. 31}$$

The negative in front of Δt arises as we have a terminal boundary condition but are using a forward in time scheme; hence the direction needs to be reversed. The boundary conditions imply that $V_{0,j+1} = 0$, $V_{n,j+1} = b_{n,j+1}$. Rearranging and writing in matrix form yields:

$$(I - P - Q)\underline{V}_{j+1} = (A + P + Q)\underline{V}_j + z_{j+1} \tag{2.5. 32}$$

where I is an $n + 1 \times n + 1$ identity matrix,

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ g_{\alpha,2}f_0\eta_1 & g_{\alpha,1}f_1\eta_1 & g_{\alpha,0}f_2\eta_1 & 0 & \ldots & 0 & 0 \\ g_{\alpha,3}f_0\eta_2 & g_{\alpha,2}f_1\eta_2 & g_{\alpha,1}f_2\eta_2 & g_{\alpha,0}f_3\eta_2 & \cdots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ g_{\alpha,n-1}f_0\eta_{n-2} & g_{\alpha,n-2}f_1\eta_{n-2} & g_{\alpha,n-3}f_3\eta_{n-1} & g_{\alpha,n-4}f_4\eta_{n-1} & \cdots & g_{\alpha,0}f_0\eta_{n-2} & 0 \\ g_{\alpha,n}f_0\eta_{n-1} & g_{\alpha,n-1}f_1\eta_{n-1} & g_{\alpha,n-2}f_2\eta_{n-1} & g_{\alpha,n}f_3\eta_{n-1} & \cdots & g_{\alpha,1}f_{n-1}\eta_{n-1} & g_{\alpha,0}f_n\eta_{n-1} \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix} \tag{2.5. 33}$$

with $\eta_i = \frac{B_i \Delta t}{2(\Delta x)^\alpha}$,

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & ... & 0 & 0 \\ g_{\alpha,0}h_0\theta_1 & g_{\alpha,1}h_1\theta_1 & ... & g_{\alpha,n-3}h_{n-3}\theta_1 & g_{\alpha,n-2}h_{n-2}\theta_1 & g_{\alpha,n-1}h_{n-1}\theta_1 & g_{\alpha,n}h_n\theta_1 \\ 0 & g_{\alpha,0}h_1\theta_2 & ... & g_{\alpha,n-4}h_{n-3}\theta_2 & g_{\alpha,n-3}h_{n-2}\theta_2 & g_{\alpha,n-2}h_{n-1}\theta_2 & g_{\alpha,n-1}h_n\theta_2 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & ... & g_{\alpha,0}h_{n-3}\theta_{n-2} & g_{\alpha,1}h_{n-2}\theta_{n-2} & g_{\alpha,2}h_{n-1}\theta_{n-2} & g_{\alpha,3}h_n\theta_{n-2} \\ 0 & 0 & ... & 0 & g_{\alpha,0}h_{n-2}\theta_{n-1} & g_{\alpha,1}h_{n-1}\theta_{n-1} & g_{\alpha,2}h_n\theta_{n-1} \\ 0 & 0 & 0 & 0 & ... & 0 & 0 \end{bmatrix} \tag{2.5. 34}$$

with $\theta_i = \frac{C_i \Delta t}{2(\Delta x)^\alpha}$,

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \\ -\frac{A\Delta t}{2\Delta x} & 1 + D\Delta t & \frac{A\Delta t}{2\Delta x} & 0 & ... & 0 & 0 & 0 & 0 \\ 0 & -\frac{A\Delta t}{2\Delta x} & 1 + D\Delta t & \frac{A\Delta t}{2\Delta x} & ... & 0 & 0 & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... & & \\ 0 & 0 & 0 & 0 & ... & -\frac{A\Delta t}{2\Delta x} & 1 + D\Delta t & \frac{A\Delta t}{2\Delta x} & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & -\frac{A\Delta t}{2\Delta x} & 1 + D\Delta t & \frac{A\Delta t}{2\Delta x} \\ 0 & 0 & 0 & 0 & ... & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.5. 35}$$

and

$$z_{j+1} = [0, 0, 0, ..., 0, 0, b_{n,j+1}]' \quad (2.5.\ 36)$$

Subject to $V_{i,0} = s_i$. Provided $(I - P - Q)$ is invertible,

$$\underline{V}_{j+1} = (I - P - Q)^{-1}[(A + P + Q)\underline{V}_j + z_{j+1}] \quad (2.5.\ 37)$$

solves (2.5. 23).

2.5.2 Adomian Decomposition Method

Recently a large amount of interest has been placed on Adomian decomposition to solve a variety of mathematical problems. These include problems in physics, turning point problems, boundary value problems, algebraic equations and many other areas of applied mathematics [40, 41]. The method was introduced in the 1980s by Adomian [42].

The method involves splitting an equation into linear and non-linear parts, and then decomposing the solution into an infinite series. This series has to be truncated for practical purposes but by adding more terms it is possible to get arbitrarily close to the exact solution in a specific domain.

The convergence of the series has been thoroughly tested for a variety of problems by many authors. They have been found to converge quickly and with high accuracy on a specific domain - see for example [40, 41]. Further, ADM often yields to exact solution.

This thesis will outline the method in general and will then look at solving the problem for an equation of the form (2.5. 23).

Consider the following nonlinear equation:

$$ {}_0D_t^{\alpha'} V(x,t) + LV(x,t) + NV(x,t) = g(x,t), \tag{2.5. 38}$$

with $t > 0$ where L is a linear operator, N is a nonlinear operator and $g(x,t)$ is a source term. ${}_0D_t^{\alpha'}$ is the Riemann-Liouville fractional derivative defined in the previous section. Using property (2.5. 30) and applying ${}_0J_t^{\alpha'}$ to both sides of (2.5. 38) yields:

$$ V(x,t) = V(x,0) + {}_0J_t^{\alpha'} g(x,t) - {}_0J_t^{\alpha'}[LV(x,t) + NV(x,t)], \tag{2.5. 39}$$

provided $0 < \alpha' \leqslant 1$. The case $\alpha' > 1$ is easily incorporated by using (2.5. 38), however in the case of (2.5. 23) we require $\alpha' = 1$ as the time component of the FPDEs given by (2.4. 18), (2.4. 20) and (2.4. 22) are all of order 1.

ADM requires that the solution of $V(x,t)$ be decomposed into an infinite series as

$$V(x,t) = \sum_{n=0}^{\infty} V_n(x,t). \tag{2.5. 40}$$

The difficulty lies in calculating the nonlinear component in (2.5. 39). This is done via

$$NV = \sum_{n=0}^{\infty} A_n. \tag{2.5. 41}$$

where A_n are the so-called Adomian polynomials which were constructed by Adomian [42]. The Adomian polynomial A_n can be used to calculate all forms of nonlinearity and are given by

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N(\sum_{c=0}^{n} \lambda^c V_c)\right]_{\lambda=0}. \tag{2.5. 42}$$

By substituting (2.5. 40) and (2.5. 41) into both sides of (2.2. 12) the solution of $V(x,t)$ in terms of ADM is given by:

$$\sum_{n=0}^{\infty} V_n(x,t) = V(x,0) + {}_0J_t^{\alpha'} g(x,t) - {}_0J_t^{\alpha'} [L(\sum_{n=0}^{\infty} V_n(x,t)) + \sum_{n=0}^{\infty} A_n]. \quad (2.5.\ 43)$$

From this equation, the iterates are calculated in the following recursive way:

$$V_0(x,t) = V(x,0) + {}_0J_t^{\alpha'} g(x,t),$$

$$V_1(x,t) = -{}_0J_t^{\alpha'} [LV_0 + A_0],$$

$$V_2(x,t) = -{}_0J_t^{\alpha'} [LV_1 + A_1],$$

$$\vdots$$

$$V_n(x,t) = -{}_0J_t^{\alpha'} [LV_{n-1} + A_{n-1}]. \quad (2.5.\ 44)$$

Finally the approximate solution of $V(x,t)$ denoted by $\phi_n(x,t)$ is then given by:

$$\phi_n(x,t) = \sum_{k=0}^{n} V_k(x,t) \tag{2.5. 45}$$

Now to apply the ADM derived above to (2.5. 23). Note first that in the case of (2.5. 23) we solve the FPDE subject to a given *terminal condition* $V(x,T) = s(x) = \max(e^x - K, 0)$. This is easily incorporated into the ADM formalisation provided above. As shall be seen, the Grunwald formula will once again be vital to estimate the fractional derivate. First, multiply both sides of (2.5. 23) by $-{}_TJ_t^1$ to obtain:

$$V(x,t) = V(x,T) - ({}_TJ_t^1)\Big[A\frac{\partial V(x,t)}{\partial x} + B(x)\frac{\partial^\alpha(f(x)V(x,t))}{\partial_+ x^\alpha} + C(x)\frac{\partial^\alpha(h(x)V(x,t))}{\partial_- x^\alpha}$$
$$+DV(x,t) = 0\Big], \tag{2.5. 46}$$

$$\Rightarrow V(x,t) = V(x,T) + \int_t^T \Big[A\frac{\partial V(x,t)}{\partial x} + B(x)\frac{\partial^\alpha(f(x)V(x,t))}{\partial_+ x^\alpha} + C(x)\frac{\partial^\alpha(h(x)V(x,t))}{\partial_- x^\alpha}$$
$$+DV(x,t) = 0\Big]dt. \tag{2.5. 47}$$

The next step is to recall that the right and left sided Grunwald formulas given by (2.5. 26) and (2.5. 27) can be used to calculate the right and left fractional derivatives. For practical purposes it is not possible to allow the summations of the right and left sided Grunwald formula to go to infinity so they must be truncated at some 'large' integers $N1$ and $N2$, respectively.

Using this and decomposing the solution of $V(x,t)$ as per the requirements of ADM, the solution of $V(x,t)$ is then given by:

$$\begin{aligned} V(x,t) = \sum_{n=0}^{\infty} V_n(x,t) = s(x) + \int_t^T \Big[A \sum_{n=0}^{\infty} \frac{\partial V_n(x,t)}{\partial x} \\ + \frac{B(x)}{q_{(1)}^{\alpha}} \sum_{k=0}^{N1} g_{\alpha,k} f(x-(k-1)q_{(1)}) \sum_{n=0}^{\infty} V_n(x-(k-1)q_{(1)},t) \\ + \frac{C(x)}{q_{(2)}^{\alpha}} \sum_{k=0}^{N2} g_{\alpha,k} h(x+(k-1)q_{(2)}) \sum_{n=0}^{\infty} V_n(x+(k-1)q_{(2)},t) + D \sum_{n=0}^{\infty} V_n(x,t) \Big] dt. \end{aligned} \tag{2.5. 48}$$

The final step is to calculate V_n recursively as:

$$V_0(x,t) = s(x)$$

$$V_1(x,t) = \int_t^T \Big[A \frac{\partial s(x)}{\partial x} + Ds(x)$$

$$+\frac{B(x)}{q_{(1)}^{\alpha}}\sum_{k_1=0}^{N1} g_{\alpha,k_1} f(x-(k_1-1)q_{(1)})s(x-(k_1-1)q_{(1)})+$$

$$\frac{C(x)}{q_{(2)}^{\alpha}}\sum_{k_1=0}^{N2} g_{\alpha,k_1} h(x+(k_1-1)q_{(2)})s(x+(k_1-1)q_{(2)})\Big]dt,$$

$$V_2(x,t) = \int_t^T \Big[A\frac{\partial V_1(x,t)}{\partial x} + DV_1(x,t)$$

$$+\frac{B(x)}{q_{(1)}^{\alpha}}\sum_{k_2=0}^{N1} g_{\alpha,k_2} f(x-(k_2-1)q_{(1)})V_1(x-(k_2-1)q_{(1)},t)+$$

$$\frac{C(x)}{q_{(2)}^{\alpha}}\sum_{k_2=0}^{N2} g_{\alpha,k_2} h(x+(k_2-1)q_{(2)})V_1(x+(k_1-1)q_{(2)},t)\Big]dt,$$

$$\vdots$$

$$V_n(x,t) = \int_t^T \Big[A\frac{\partial V_{n-1}(x,t)}{\partial x} + DV_{n-1}(x,t)$$

$$+\frac{B(x)}{q_{(1)}^{\alpha}}\sum_{k_n=0}^{N1} g_{\alpha,k_n} f(x-(k_n-1)q_{(1)})V_{n-1}(x-(k_n-1)q_{(1)},t)+$$

$$\frac{C(x)}{q_{(2)}^{\alpha}}\sum_{k_n=0}^{N2} g_{\alpha,k_n} h(x+(k_n-1)q_{(2)})V_{n-1}(x+(k_n-1)q_{(2)},t)\Big]dt. \qquad (2.5.\ 49)$$

As ADM decomposes the series into an infinite sum, it is necessary to truncate the series at some point N^*. Thus, the approximate solution to solve (2.5. 23) via ADM is given by:

$$\phi_{N^*}(x,t) = \sum_{k=0}^{N^*} V_k(x,t), \tag{2.5. 50}$$

where V_k is calculated according to (2.5. 49). The size of N^* which guarantees accurate solutions varies depending on the problem. Practitioners are best advised to use a measure such as the relative or absolute error to determine convergence. Another important factor to consider is computational power as time for convergence of this method can grow exponentially with N^*.

This chapter derived the pure mathematical framework for pricing a derivative under three specific Lévy models using FPDEs. However, when using numerical methods it is important to look at how well they work for a specific problem, where they converge and what their advantages and disadvantages are. The next chapter deals with this difficult problem.

Chapter 3

Application of Numerical Methods to the Derived FPDEs

3.1 Introduction

This section looks at applying the finite difference method and ADM to the FPDEs derived in Section 2.4. The numerical methods are used to solve a simple European call option written on a stock S_t.

3.2 No Arbitrage Bounds

Recall that a European call option gives the owner the right, but not the obligation to buy a unit of stock, S_t, for a pre-specified amount, K, known as the strike price. The payoff at maturity, T, and hence the terminal boundary condition is $\max(S_T - K, 0)$ or $\max(e^x - K, 0)$ for $x = \ln S_t$. The price of a European call option at time t is denoted by $c[S_t, T-t; K]$.

The reason for picking such a well studied derivative, as opposed to a more exotic structure, is that the European call option is a well understood financial product. When working with a numerical method it is difficult, if not impossible, to work out how good the solution is. When working with a well known object such as a European call, the task of doing so becomes easier. Firstly, one can compare the solution to the closed form Black-Merton-Scholes solution. Secondly, there are no-arbitrage bounds that must hold when pricing an option. If these bounds are violated, then there has been a mistake in pricing. These no arbitrage bounds are independent of the pricing model used.

The principle of no arbitrage states that for every terminal payoff that is positive with some probability, but is always nonnegative, has a positive price [1]. In simpler terms, assuming no arbitrage insures that there are no *free lunches* in that one cannot start out with no money and construct a portfolio to make riskless money. In reality arbitrage opportunities do exist, however supporters of the no arbitrage principle will argue that these doors of opportunity are open for very small periods of time so that they may as well be ignored.

The no arbitrage bounds set by such an option:

$$c[S_0, T; K] \leqslant S_0, \tag{3.2. 1}$$

and

$$c[S_0, T; K] \geqslant \max(0, S_0 - Ke^{-rT}). \tag{3.2. 2}$$

It is easy to see why (3.2. 1) and (3.2. 2) must hold under the no arbitrage principle.

If (3.2. 1) were false then one could buy the share and sell the call option. The total cost for doing this is $c[S_0, T; K] - S_0 \geqslant 0$. If the owner of the call chooses to exercise at time T, the net value is $S_T - (S_T - K) > 0$. If there is no exercise at time T, then the total value of the portfolio is $S_T \geqslant 0$. In either case, an arbitrage would occur.

To show that (3.2. 2) must hold under the no arbitrage principle, consider two portfolios. The first one: long a forward contract with delivery price equal to K written on S_0. Then the initial value of this contract at time zero is $S_0 - Ke^{-rT}$.

The second portfolio: buy a European call option with strike price K at a cost $c[S_0, T; K]$. At time T, if $S_T < K$, the first portfolio is worth $S_T - K < 0$ and the second portfolio is worth 0.

If $S_T > K$, then the first portfolio is worth $S_T - K > 0$ and the second portfolio is also worth $S_T - K > 0$. The second portfolio dominates the first portfolio in all states of the world, thus $c[S_0, T; K] > S_0 - Ke^{-rT}$ in a non arbitrage world.

Lastly, note that $c[S_0, T; K] > 0$ as a European call option is always pays out at least nothing, its value must always remain positive.

If any of the above inequalities does not hold then there is a problem with the numerical method and convergence has failed.

Convergence of the European call is now tested via the finite difference scheme and the Adomian decomposition method. The parameters chosen for the option are $r = 0.1$, $T = 1/2$ and $K = 80$. The computer used to run the code for both the finite difference scheme and ADM used a 2.2GHz duo processor, with 2GB RAM.

3.3 Convergence Under the Finite Difference Scheme

For the finite difference scheme, $b(R,t) = e^R - Ke^{-r(T-t)}$. The following graphs were generated using Matlab, chosen for its powerful vector handling functionality.

There are three measures of error used in this section. The maximum absolute error between the second last solution vector and the last solution vector - mathematically, $\max |\underline{V}_{\frac{T}{\Delta t}-1} - \underline{V}_{\frac{T}{\Delta t}}|$. This measure is not a solution convergence measure, as the scheme could converge but to the wrong solution.

It can however be used to measure the erraticness of the solution. Erratic solutions would exhibit large differences between consecutive time steps.

The maximum absolute error between the final solution vector, $\underline{V}_{\frac{T}{\Delta t}}$ and the Black-Merton-Scholes solution evaluated at time T for $i\Delta x$, $i = 0, 1, 2, ..., n$, and the maximum error between the final solution vector, $\underline{V}_{\frac{T}{\Delta t}}$ and the arbitrage bound $\max(e^{L+i\Delta x} - Ke^{-rT}, 0)$, $i = 0, 1, 2, ..., n$. In this thesis we will refer to these errors as $MAXABSERROR\_MODELNAME$, $MAXABSERROR\_MODELNAME\_BMS$ and $MAXABSERROR\_MODELNAME\_ARBBOUND$, respectively, where $MODELNAME = FMLS, KoBol, CGMY$. Note that the use relative error measures are impossible as there are a large number of 0 entries when the call option is far out the money.

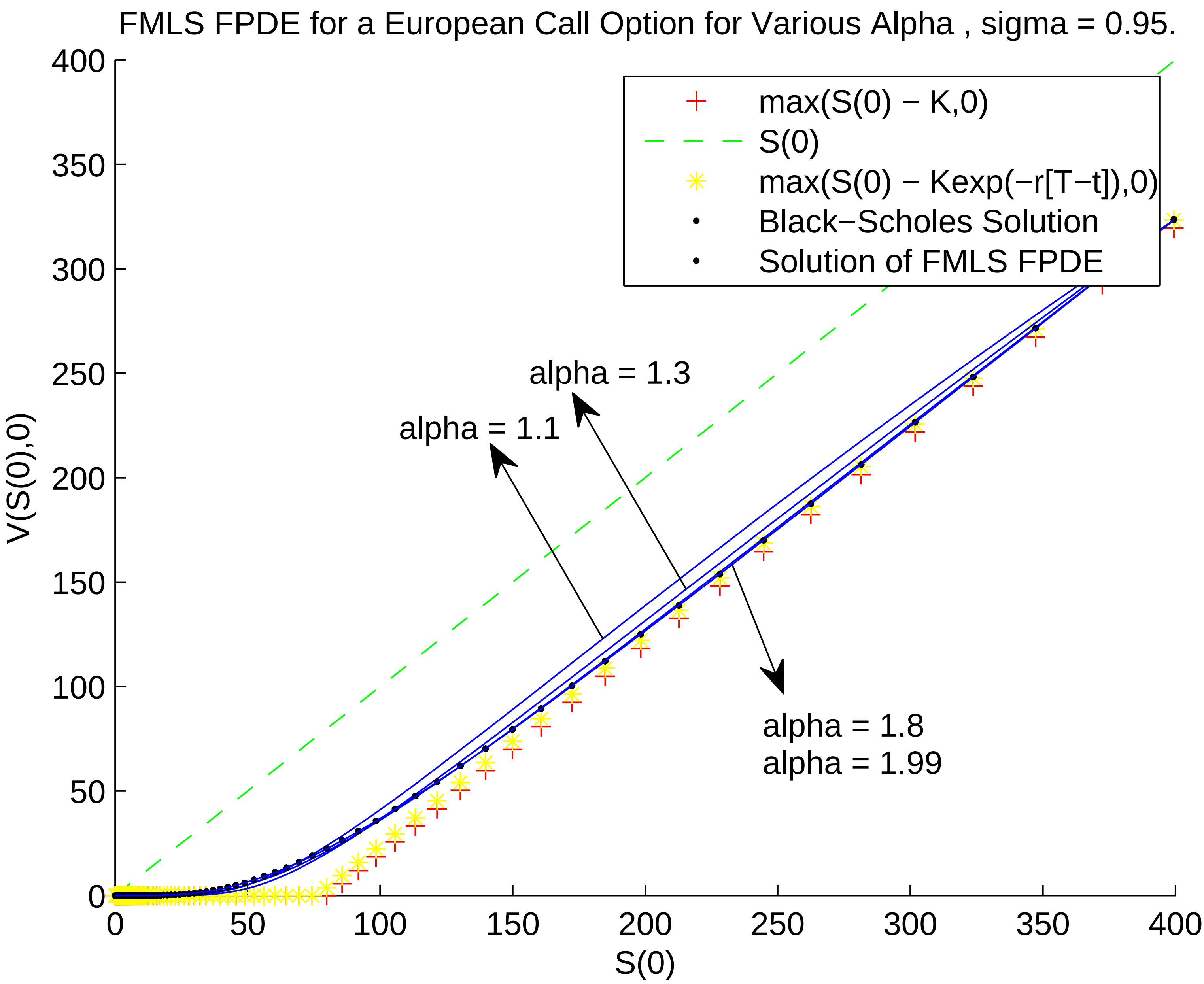

Figure 3.1: Finite Difference, FMLS FPDE, $T = 0.5$, $r = 0.1$, $K = 80$, $\sigma = 0.95$, various α.

As Figure 3.1 shows, $\alpha \to 2 \Rightarrow$ the FMLS FPDE converges to the Black-Merton-Scholes PDE and (2.4. 18) becomes the Black-Merton-Scholes PDE. In fact, they are almost equal when $\alpha = 1.8$. Furthermore we get the following error measure outputs:

Error Measure for FMLS FPDE with varying α		
α	$MAXABSERROR\_FMLS$	$MAXABSERROR\_FMLS\_BMS$
1.1	0.067	12.58
1.3	0.051	5.24
1.8	0.048	1.5
1.99	0.049	0.26

As can be seen for the output of $MAXABSERROR\_FMLS$, in all cases the FMLS FPDE converged to a solution as we get values that are small. Further, as expected the difference between the Black-Merton-Scholes solution and the FMLS solution is close to zero when $\alpha = 1.99$ and increases as α decreases.

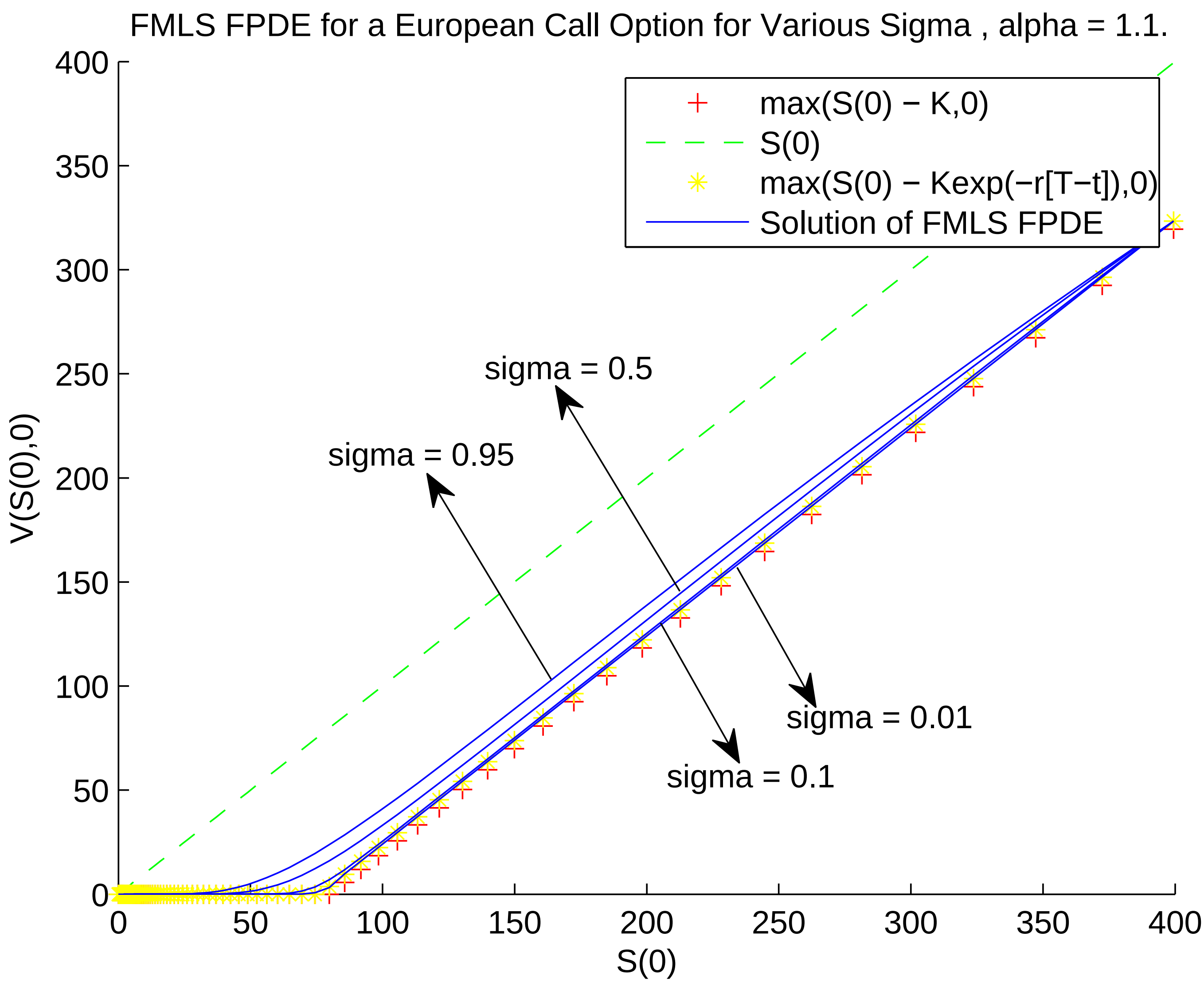

Figure 3.2: Finite Difference, FMLS FPDE, $T = 0.5$, $r = 0.1$, $K = 80$, $\alpha = 1.1$, various σ.

The following summarises the error measures:

Error Measure for FMLS FPDE with varying σ		
σ	$MAXABSERROR\_FMLS$	$MAXABSERROR\_FMLS\_ARBBOUND$
0.01	0.017	0.65
0.1	0.022	3.44
0.5	0.046	12.33
0.95	0.050	20.01

As Figure 3.2 illustrates, as σ decreases, the price of both the Black-Merton-Scholes solution and the FMLS solution tend to $\max(S_0 - Ke^{-r(T-t)}, 0)$. Buying an in-the-money call will be similar in price to a forward with delivery price K and an out the money call will have no value. When σ is large, both solutions get larger to account for the extra uncertainty. The scheme has good convergence to a solution for different values σ as we again obtain maximum absolute errors that are small.

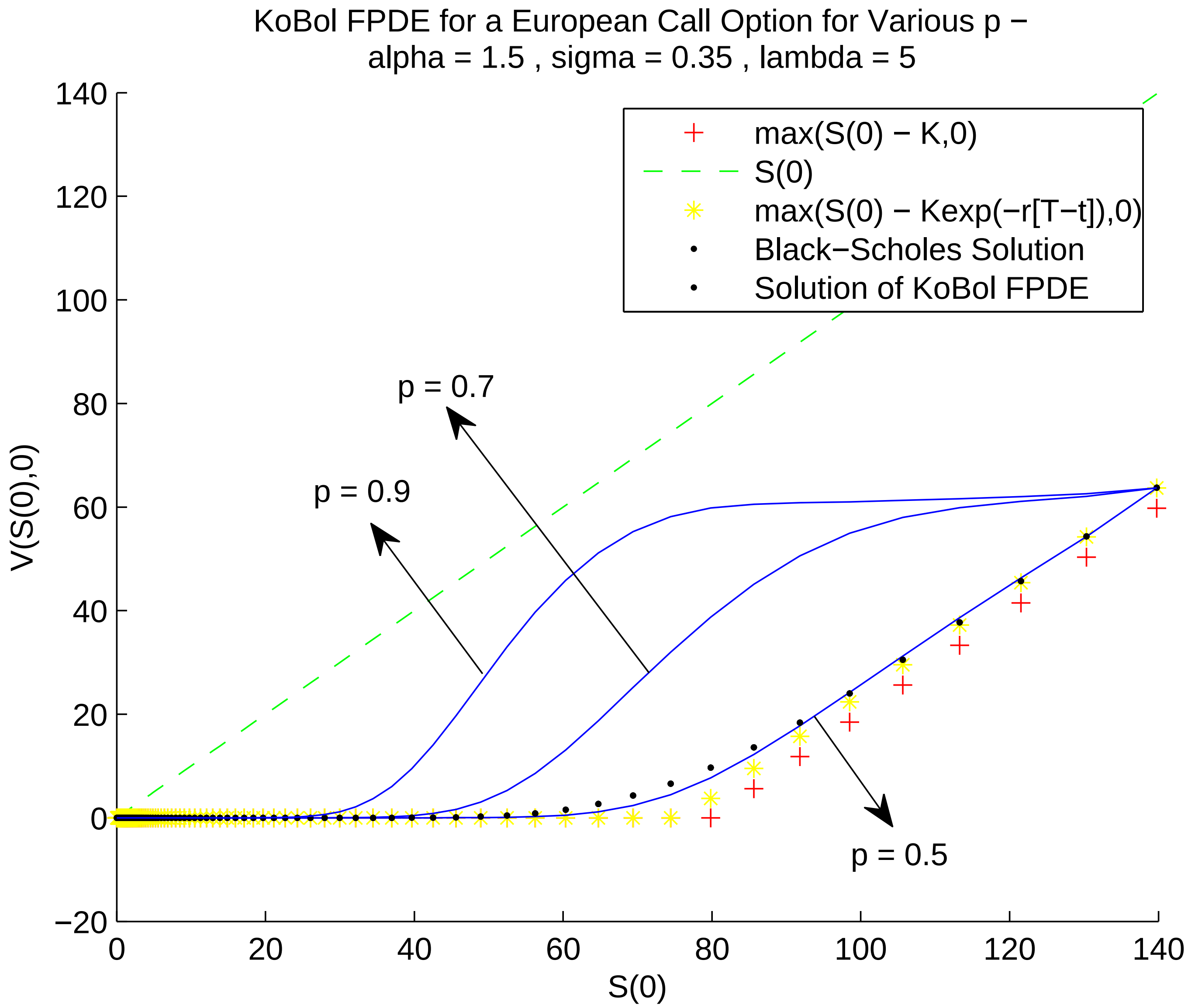

Figure 3.3: Finite Difference, KoBol FPDE, $T = 0.5$, $r = 0.1$, $K = 80$, $\alpha = 1.5$, $\sigma = 0.35$, $\lambda = 5$, various p.

Error Measure for KpBol FPDE with varying p		
p	$MAXABSERROR\_KoBol$	$MAXABSERROR\_KoBol\_BMS$
0.5	0.000024	2.15
0.7	0.00016	32.18
0.9	0.00030	51.52

The parameter p controls the skewness of the KoBol density, with $p = 1 \Rightarrow$ the distribution is maximally skewed to the right. We see this effect in Figure 3.3. As the error measure summary above shows, as p increases so the solution of the KoBol deviates further from the Black-Merton-Scholes solution because of larger added skewness. The scheme converged to a solution as the error measure $MAXABSERROR\_KoBol$ shows.

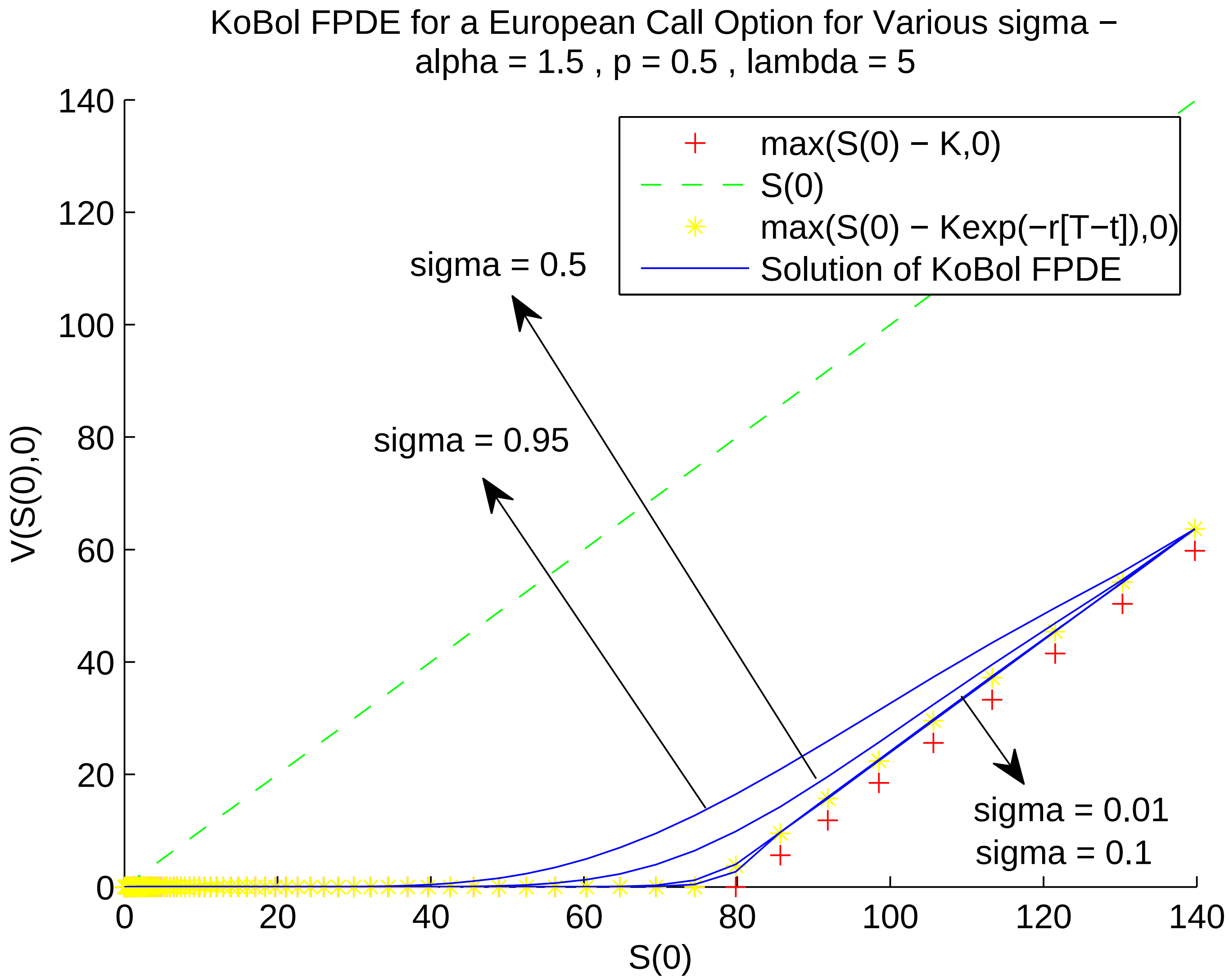

Figure 3.4: Finite Difference, KoBol FPDE, $T = 0.5$, $r = 0.1$, $K = 80$, $\alpha = 1.5$, $\lambda = 3$, $p = 0.5$, various σ.

Error Measure for KoBol FPDE with varying σ		
σ	$MAXABSERROR\_KoBol$	$MAXABSERROR\_KoBol\_ARBBOUND$
0.01	0.000017	1.06
0.1	0.000016	1.20
0.5	0.000030	6.45
0.95	0.000049	12.74

The parameter σ has the same effect on the KoBol FPDE as with the FMLS FPDE as Figure 3.4 and the summary above show.

The KoBol FPDE has good convergence bounds. It explodes when we choose 'extreme' values for σ or λ. The scheme failed to converge for $\lambda < 0.5$, $\lambda > 15$, $\sigma < 0.05$ and $\sigma > 10$. This restriction is not too large a hindrance, as regular markets would not require such extreme inputs. In regular markets for example $0.2 < \sigma < 0.5$ is usual.

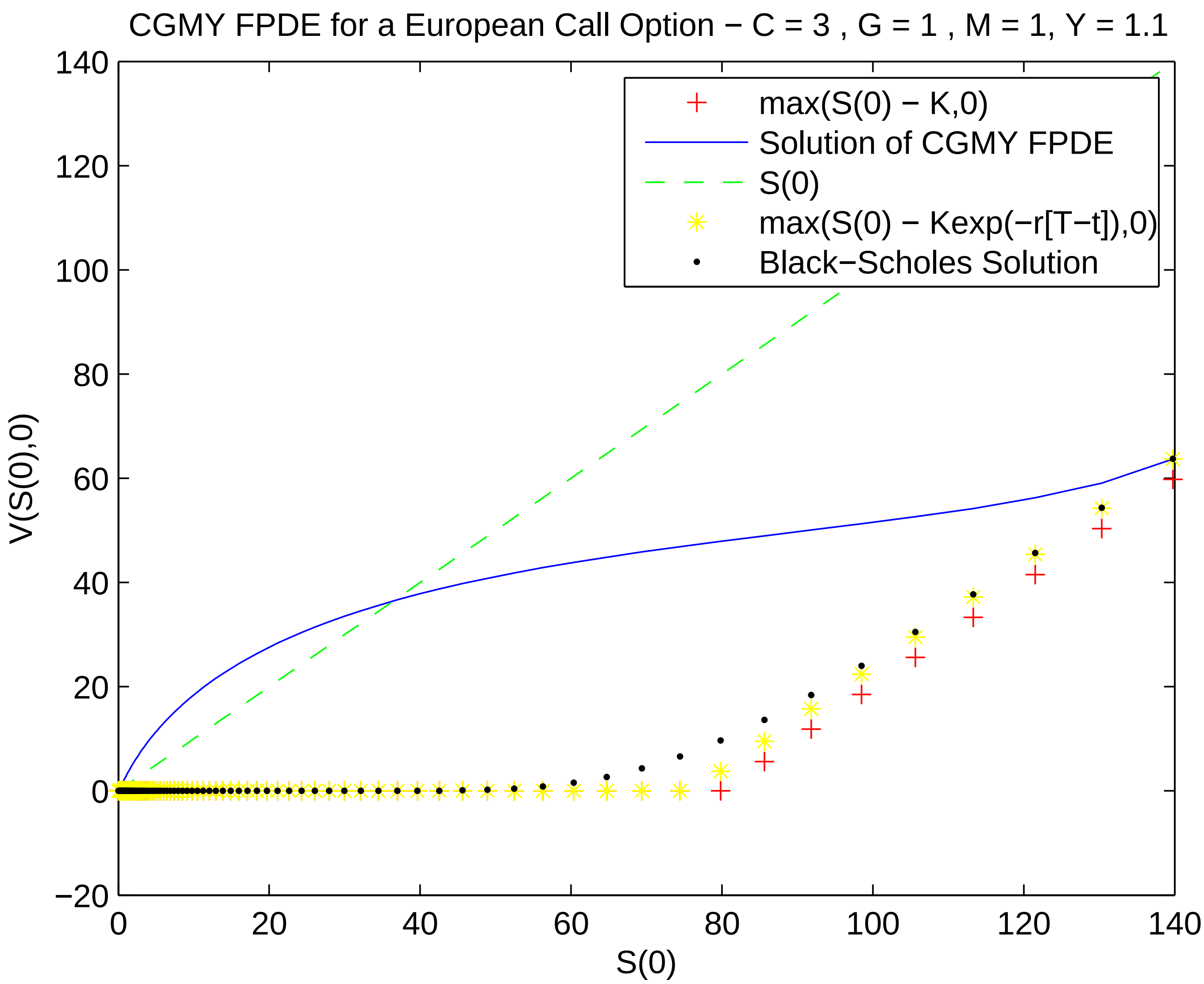

Figure 3.5: Finite Difference, CGMY FPDE, $T = 0.5$, $r = 0.1$, $K = 80$, $C = 3$, $G = 1$, $M = 1$, $Y = 1.1$.

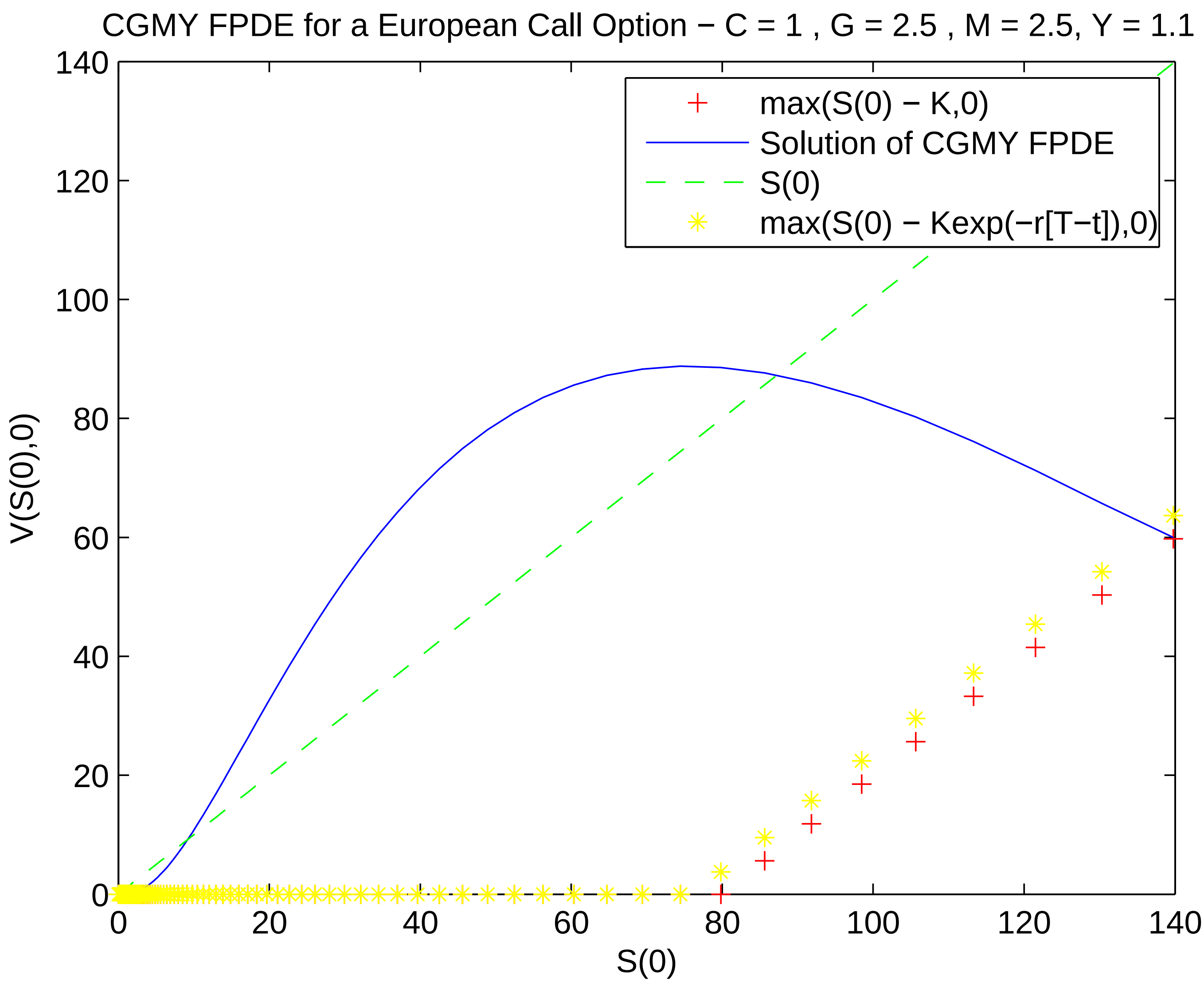

Figure 3.6: Finite Difference, CGMY FPDE, T = 0.5, r = 0.1, K = 80, C = 1, G = 2.5, M = 2.5 , Y = 1.1.

The scheme, when applied to the CGMY FPDE, was extremely unstable and falls way outside the no arbitrage bounds. Empirical evidence suggest that $C \in [0, 300]$, $G, M \in [20, 150]$, $Y \in [-1, 1.8]$ [39], so this model would be impossible to implement under this scheme for any realistic situation. Note that in Figure 3.5 $MAXABSERROR\_CGMY = 1.010$ and in Figure 3.6 $MAXABSERROR\_CGMY = 0.052$. These values are small, indicating that the scheme has converged, however it has obviously converged to the incorrect solution.

As can be seen from the output, these models can produce option premium values which are very different from Black-Merton-Scholes model. Another important feature is that the Black-Merton-Scholes model always produces option premium values which are similar in shape. This is because there is only one parameter in the Black-Merton-Scholes formula, namely σ. However, Lévy models have between two and four parameters. As already stated, this can be viewed as both an advantage and a disadvantage. However, as the output shows, Lévy models have the ability to produce models with differing shapes. This suggests that if good market data is available, these models could be calibrated to produce option price curves that take into account a variety of different market factors, making them more accurate.

This section compared the individual models under different parameter values. Another aspect which practitioners are concerned with is a comparison of the different models under the same conditions to see how different the outputs they produce are. In this case, it is very difficult to compare the models as they have a different parameter structure. We could fix α and σ for the FMLS FPDE, but this leaves us with having to choose p and λ in the Kobol case. However, we do not know which values to pick to make it equivalent to the FMLS FPDE. In the case of the CGMY model the problem is compounded further as there is only one parameter, Y which is similar to α, which we can use. Practitioners are thus advised to calibrate all three models using actual market data, and then compare the model solution to the actual real life data to see which model produces the most accurate output.

3.4 Convergence Under Adomian Decomposition Method

Unfortunately, the use of ADM to price financial derivatives is severely restricted. The reason for this is twofold. Firstly, ADM works in a small interval from the termination date, T - in the case of the CGMY model, $T \geqslant 0.1$ is problematic. This means that derivatives with long maturity cannot be valued using ADM.

Secondly, and far more critically, with the exception of the forward contract which has a linear payoff (and is easily priced without the use of any model) financial derivatives all have terminal payoff functions which are piecewise linear, and are thus not differentiable everywhere. This is because they use either the min or max function (as seen in the previous chapters) which means that there is a point of non-differentiability at the kink of the graph. This is a massive problem when using ADM.

Further, recall that ADM is a Taylor series approximation to the solution, which means that it uses polynomials as its building blocks to approximate the solution. As most financial derivatives have piecewise linear terminal conditions, using polynomials to estimate the solution is inappropriate.

In [43] Bohner and Zheng look at using ADM to solve the simple Black-Merton-Scholes PDE given by (1.2. 3). They assume that the terminal function has derivatives of all orders. This assumption has little practical value.

Further, ADM is extremely computationally intensive for this specific problem. The reason being that the number of summations and differentiations for $V_n(x,t)$ increases exponentially with n. Using $N^* > 2$ with $N1 = 100$ is not viable and requires hours of processing time.

Firstly, to demonstrate that as T increases, the solution becomes unstable, consider using the FMLS model to price a simple forward contract with terminal payoff $S_T - K$. Such a derivative is easily priced and ANY non-arbitrage model should produce $V(x,t) = S_0 - Ke^{-rT}$. Here, $N^* = 1$ and $N1 = 1000$. As the payoff is linear it is not necessary to use $N^* > 1$.

To show this, consider Table 3.1 that looks at the norm $|(S_0 - Ke^{-rT}) - \phi_1(S_0, 0)|$ for $S_0 = 120$. This is the absolute difference between the exact solution and the approximate ADM solution. Notice that as T increases, the difference between the absolute error between the exact solution and the approximate solution increases. This is also illustrated visually in Figures 3.7 and 3.8. Recall that ϕ_{N^*} represents the approximate solution of the solution using ADM, where N^* represents the number of terms in the approximate summation:

ADM to Solve FMLS FPDE for Forward Contract			
T	$S_0 - Ke^{-rT}$	$\phi_1(S_0, 0)$	$\lvert(S_0 - Ke^{-rT}) - \phi_1(S_0, 0)\rvert$
10	90.5696	125.87	35.3
1	40.08	40.0859	0.0059
0.1	40.796	40.8587	0.06289
0.01	47.613	48.587	0.974

Table 3.1: FMLS FPDE

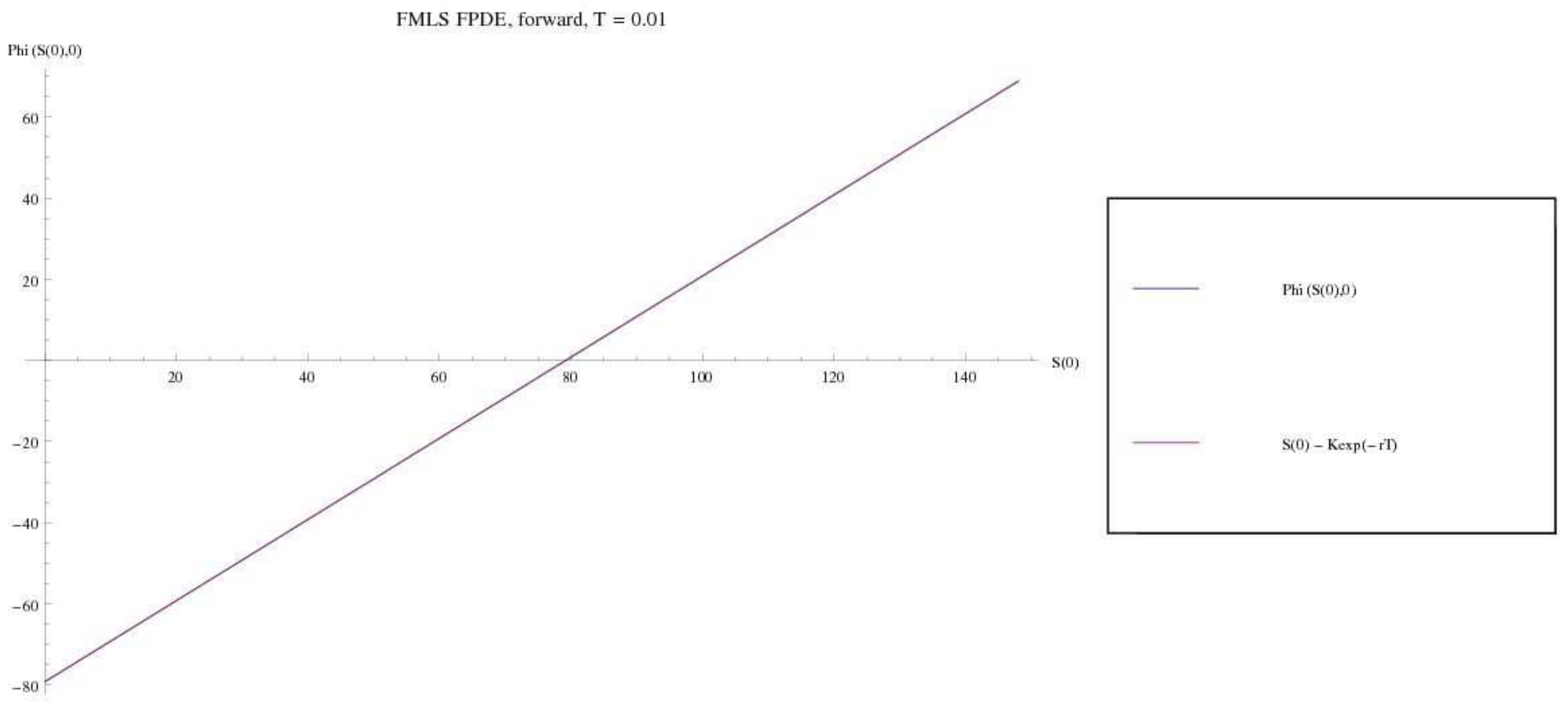

Figure 3.7: ADM, FMLS FPDE, $T = 0.01$, $r = 0.1$, $K = 80$, $\sigma = 0.35$, $\alpha = 1.5$, $L = -6$, $N1 = 100$, $N^* = 1$.

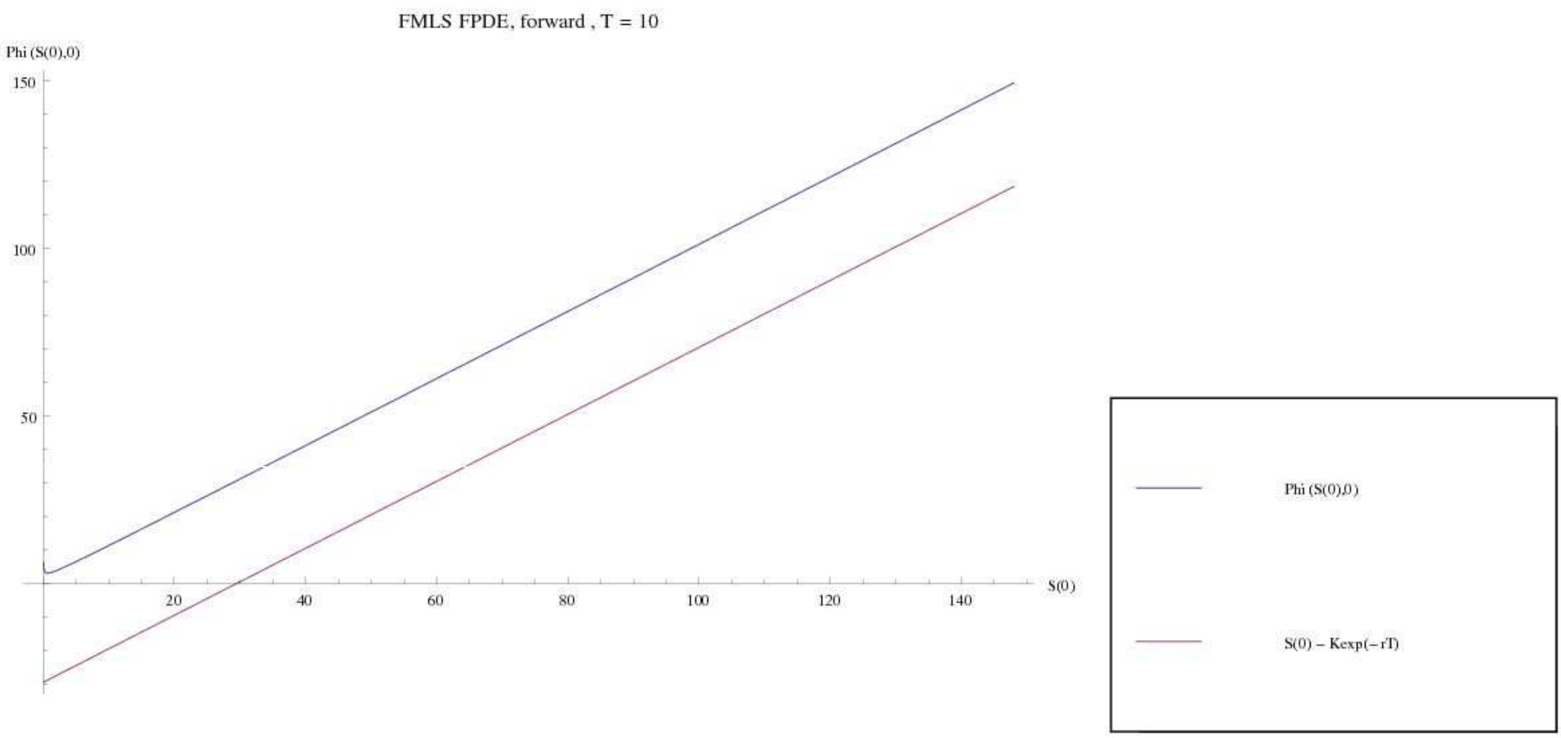

Figure 3.8: ADM, FMLS FPDE, $T = 10$, $r = 0.1$, $K = 80$, $\sigma = 0.35$, $\alpha = 1.5$, $L = -6$, $N1 = 100$, $N^* = 1$.

Similar results hold for the KoBol and CGMY FPDE as seen by Tables 3.2 and 3.3 and Figures 3.9, 3.10, 3.11 and 3.12:

ADM to Solve KoBol FPDE for Forward Contract			
T	$S_0 - Ke^{-rT}$	$\phi_1(S_0, 0)$	$\lvert(S_0 - Ke^{-rT}) - \phi_1(S_0, 0)\rvert$
10	90.5696	473.585	383.0154
1	40.08	40.4336	0.0059
0.1	40.796	44.3359	0.35364
0.01	47.613	83.3585	35.74549

Table 3.2: KoBol FPDE

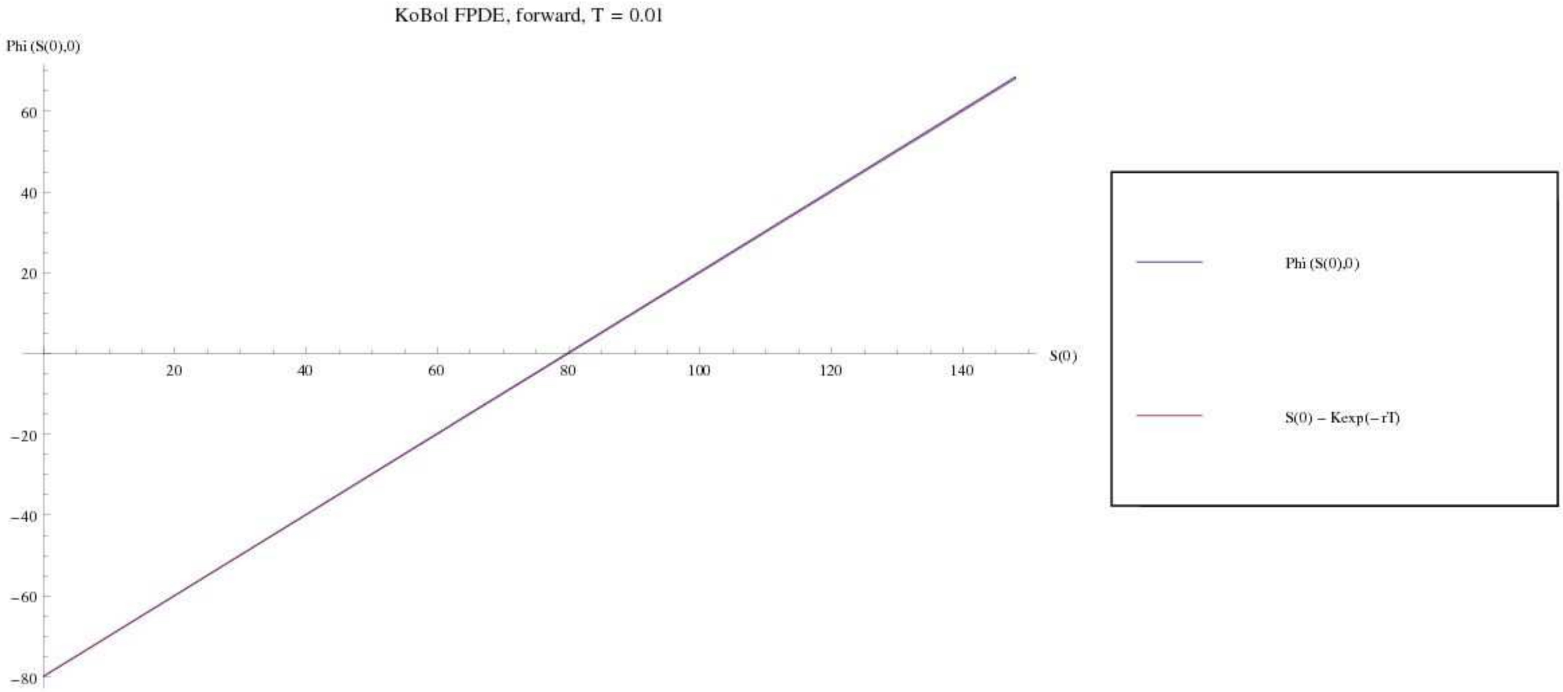

Figure 3.9: ADM, KoBol FPDE, $T = 0.01$, $r = 0.1$, $K = 80$, $\sigma = 0.35$, $\alpha = 1.5$, $L = -6$, $R = 6$, $N1 = 100$, $N^* = 1$, $\lambda = 3$, $p = 0.6$.

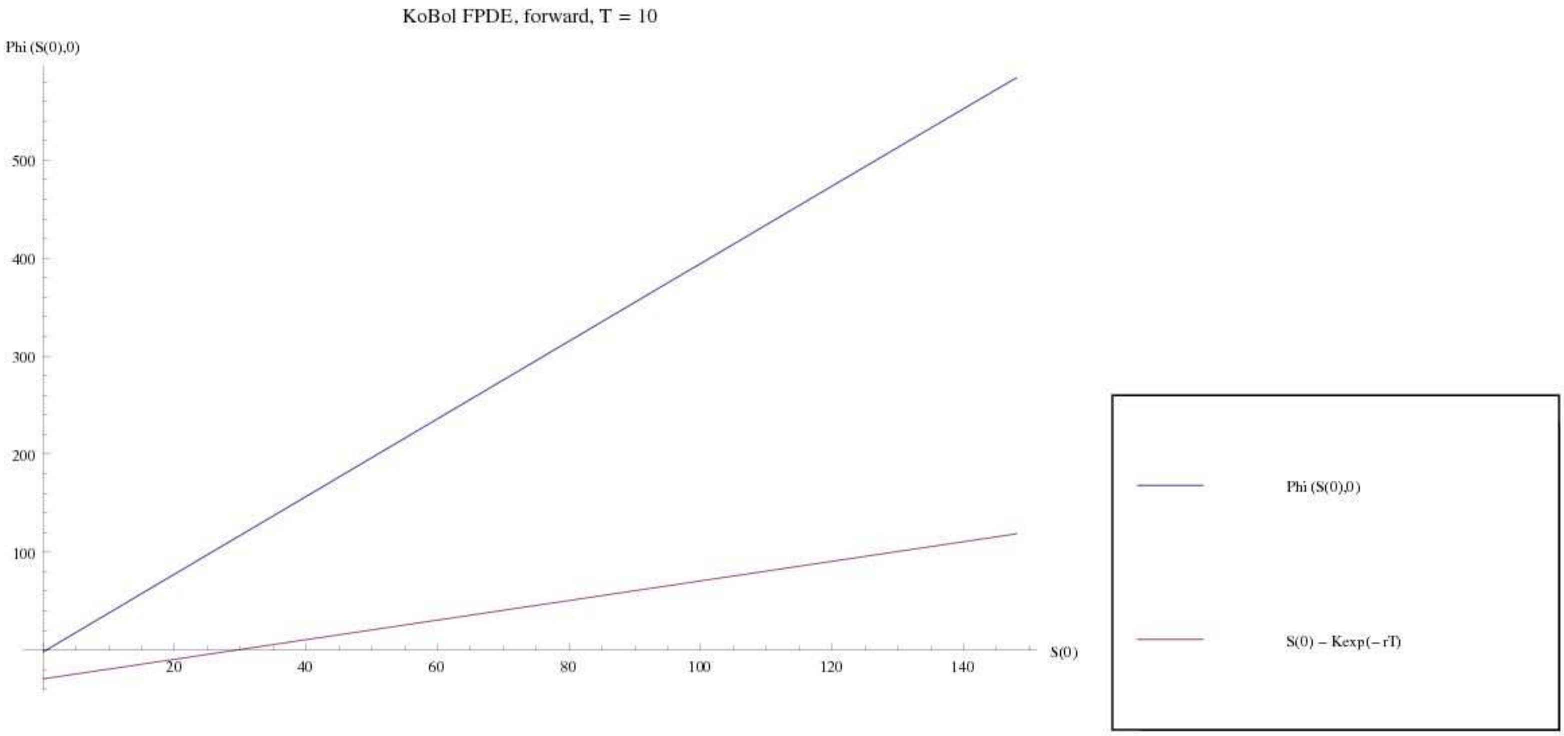

Figure 3.10: ADM, KoBol FPDE, $T = 10$, $r = 0.1$, $K = 80$, $\sigma = 0.35$, $\alpha = 1.5$, $L = -6$, $R = 6$, $N1 = 100$, $N^{*} = 1$, $\lambda = 3$, $p = 0.6$.

ADM to Solve CGMY FPDE for Forward Contract			
T	$S_0 - Ke^{-rT}$	$\phi_1(S_0, 0)$	$\lvert (S_0 - Ke^{-rT}) - \phi_1(S_0, 0) \rvert$
10	90.5696	2012.01	19211.44
1	40.08	41.972	1.89204
0.1	40.796	59.7201	18.9241
0.01	47.613	237.201	189.588

Table 3.3: CGMY FPDE

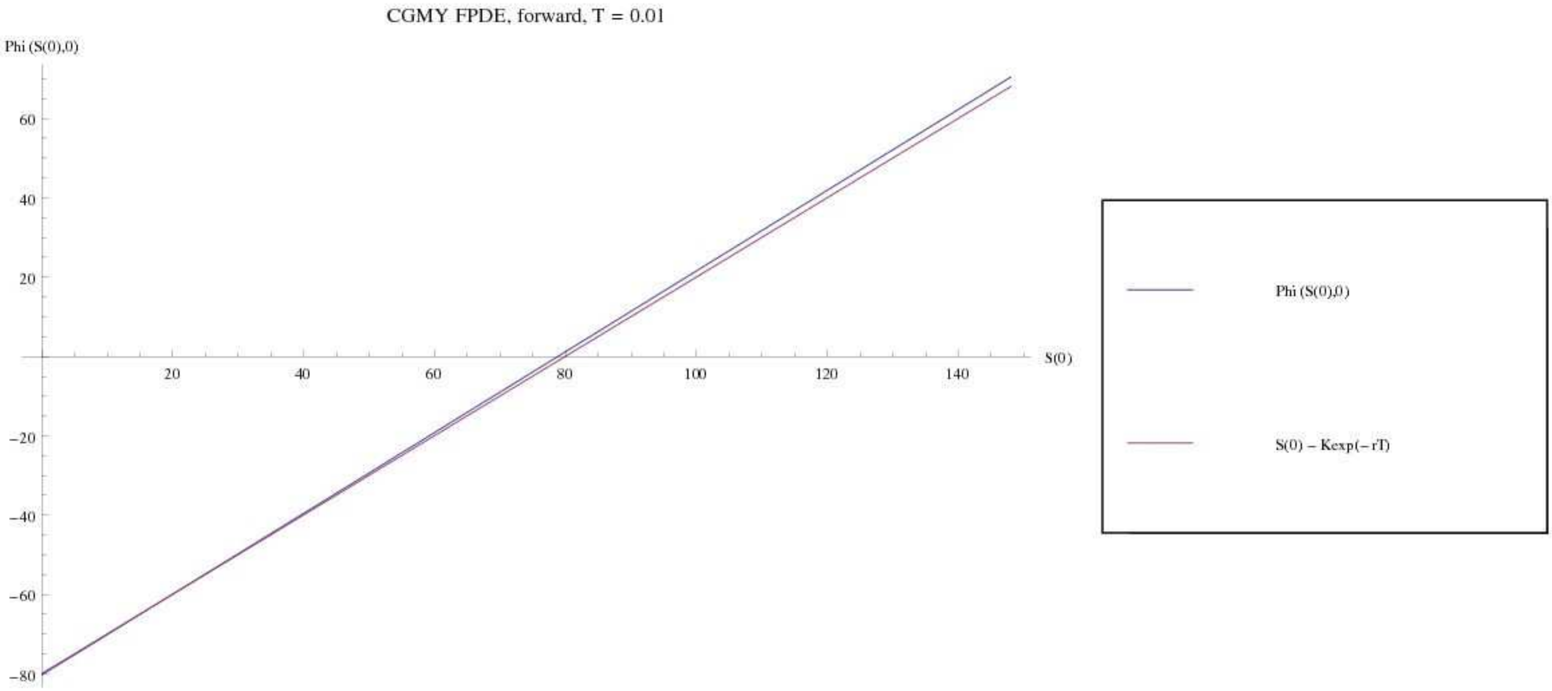

Figure 3.11: ADM, CGMY FPDE, $T = 0.01$, $r = 0.1$, $K = 80$, $C = 1$, $Y = 1.5$, $L = -6$, $R = 6$, $N1 = 100$, $N^* = 1$, $G = 2$, $M = 2$.

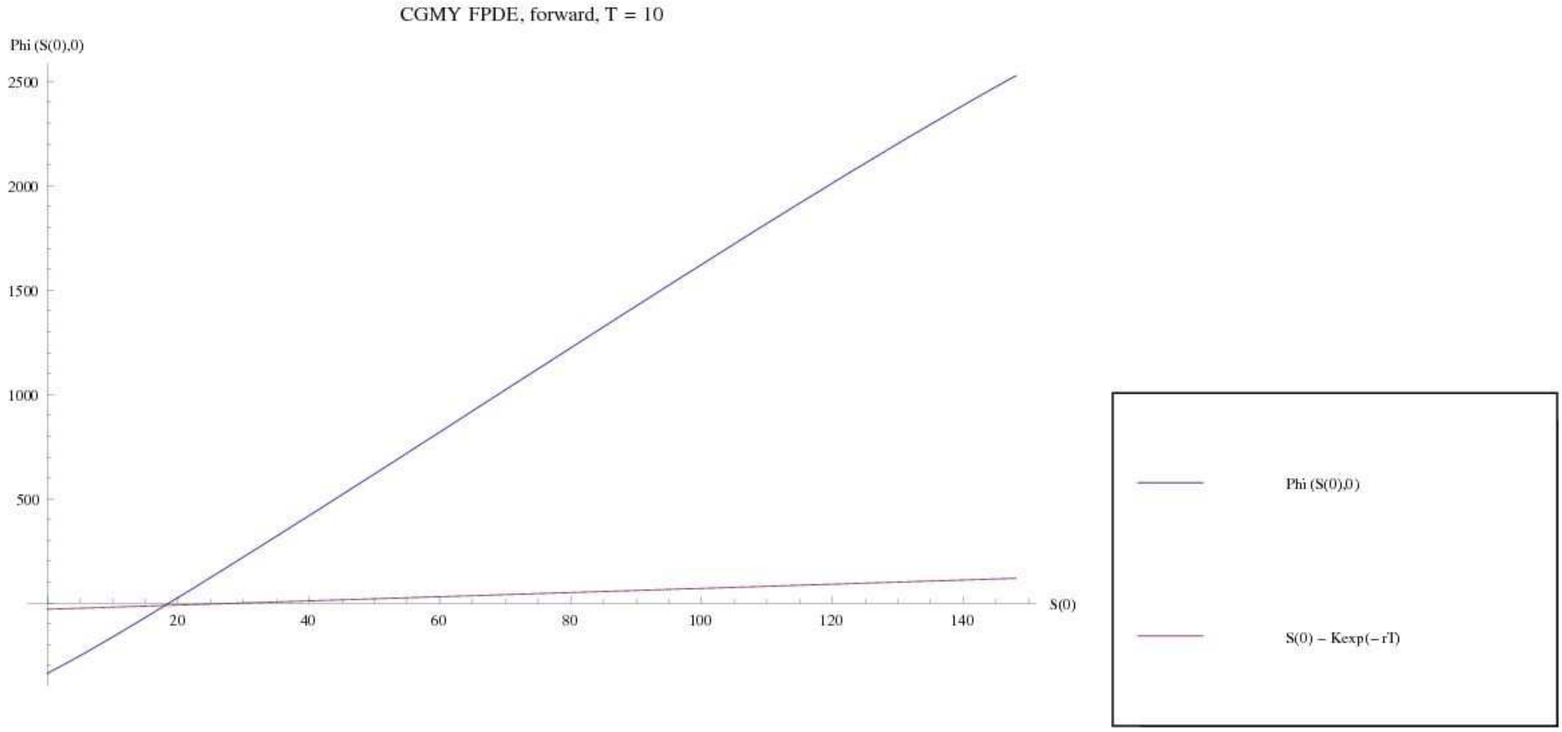

Figure 3.12: ADM, CGMY FPDE, $T = 10$, $r = 0.1$, $K = 80$, $C = 1$, $Y = 1.5$, $L = -6$, $R = 6$, $N1 = 100$, $N^* = 1$, $G = 2$, $M = 2$.

Now, to illustrate the effect that the non-differentiability of the terminal condition has, suppose one tried to price a call option using ADM. The following output was generated with $T = 0.1$ for the FMLS, KoBol and CGMY FPDEs, respectively,

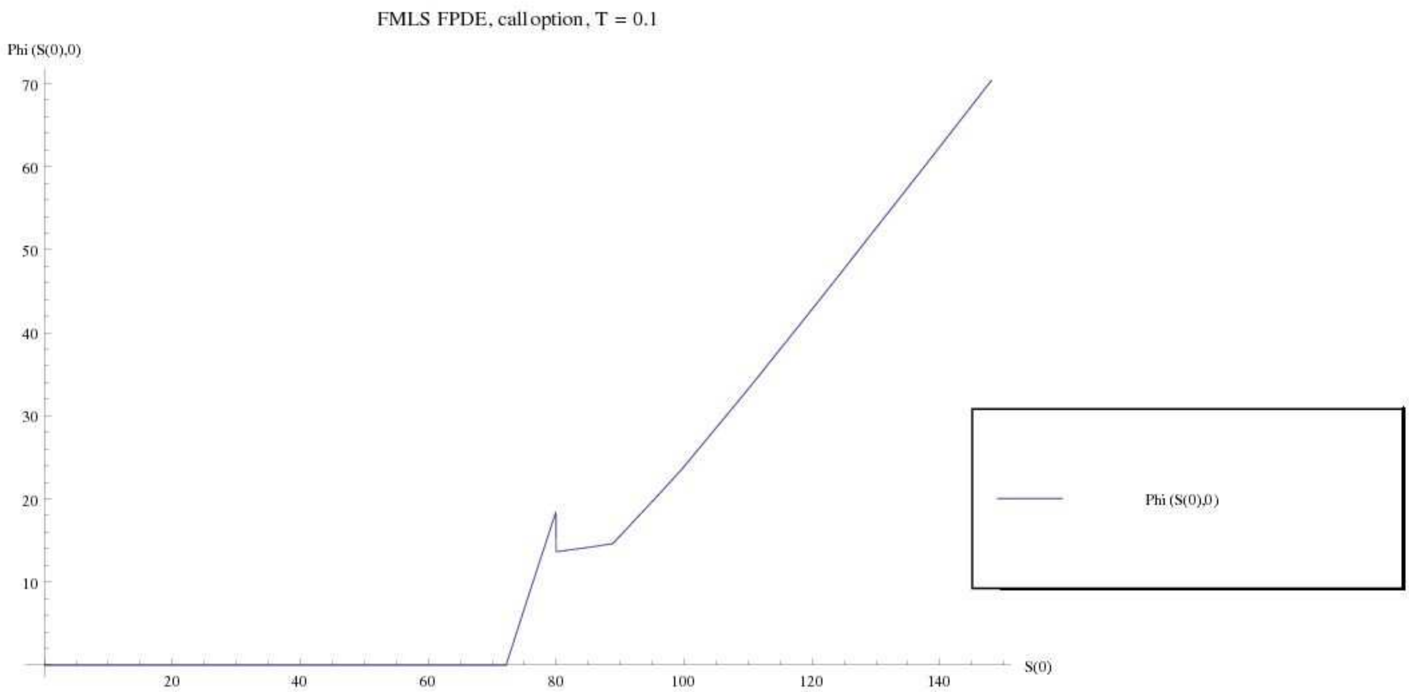

Figure 3.13: ADM, FMLS FPDE, $T = 0.1$, $r = 0.1$, $K = 80$, $\sigma = 1$, $\alpha = 1.5$, $L = -6$, $N1 = 100$, $N^* = 1$.

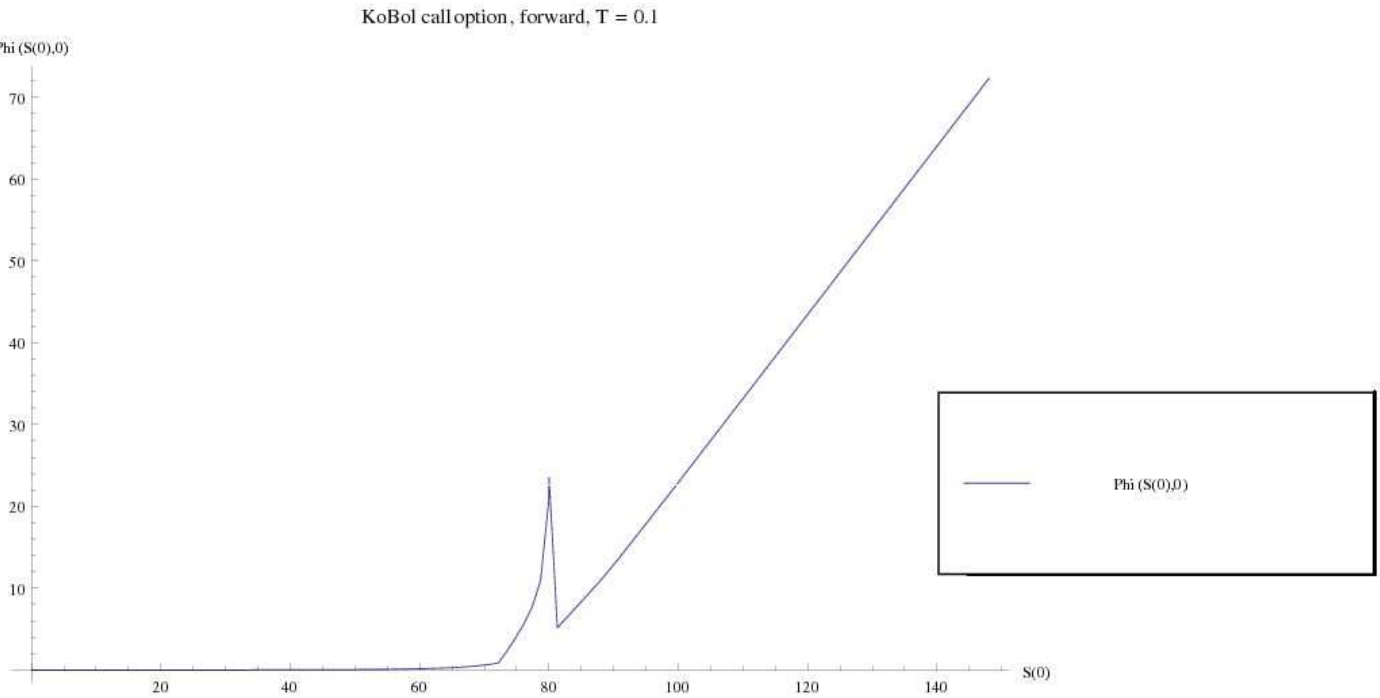

Figure 3.14: ADM, KoBol FPDE, $T = 0.1$, $r = 0.1$, $K = 80$, $\sigma = 1$, $\alpha = 1.5$, $L = -6$, $R = 6$, $N1 = 100$, $N^* = 1$, $\lambda = 3$, $p = 0.6$.

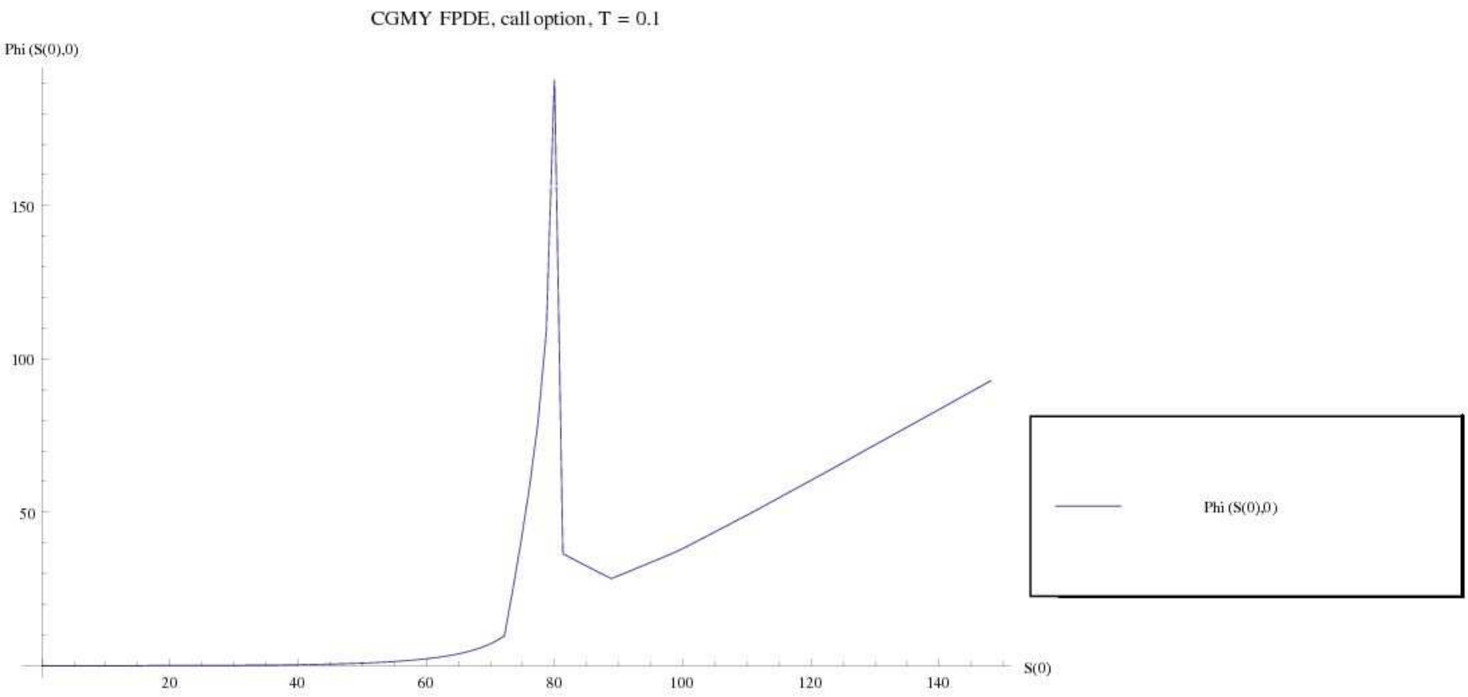

Figure 3.15: ADM, CGMY FPDE, $T = 0.1$, $r = 0.1$, $K = 80$, $C = 1$, $Y = 1.5$, $L = -6$, $R = 6$, $N1 = 100$, $N^{*} = 1$, $G = 2$, $M = 2$.

As can be seen from Figures 3.13, 3.14 and 3.15, there is a kink in the approximate solution at 80. As already stated, this behavior is a result of the non-differentiability at $S_T = 80$ of the terminal function $\max(S_T - K, 0)$.

As can be seen from the output, the finite difference method obtains good convergence for two of the proposed models. However, ADM fails for all three. Reasons for this as well as concluding remarks are discussed in the next chapter.

Chapter 4

Concluding Remarks

This thesis compared three Lévy models, and two methods of approximating their solution. Lévy models tend to be mathematically difficult to work with. They require more parameters than the Black-Merton-Scholes model. This means that more estimation and market calibration is needed. They also produce a problem of intractability and moments that do not exist. Finally, whereas with the Black-Merton-Scholes model one could sometimes obtain closed form solutions to price derivatives, this is not the case with Lévy processes and even the most simple structures need numerical approximations to obtain solutions. Such methods have convergence problems and are computationally intensive. This means that with Lévy models, more time is required to price derivatives and the real time necessity of option pricing is not always viable.

Lévy models become FPDEs when pricing financial derivatives. Two numerical methods were explored to solve FPDEs: the finite difference scheme and ADM.

ADM is simply not suited to deal with the problem of solving financial derivatives. Adomian decomposition method requires terminal or initial functions which are differentiable. Furthermore, these functions need to be able to be built up from polynomials. This is not the case when dealing with practically every traded financial instrument, which tend to have piecewise linear terminal functions.

Finite difference schemes on the other hand work well in comparison. The case of the CGMY model is disappointing. However the FMLS and KoBol models are promising.

The FMLS model in particular had excellent convergence. Convergence was achieved for a wide variety of different values of α and σ. The scheme never failed to converge, even for extreme values. Furthermore, convergence occurred very rapidly - a few seconds at most. This model could easily be implemented in a practical sense.

The KoBol model was also very encouraging, although not to the extent of the FMLS model. This model showed good convergence as long as λ and σ were not chosen to be too extreme. Large values, as seen in the previous chapter, resulted in the solution blowing up and breaking the no arbitrage bounds. The parameter p however, could be chosen to be very large or very small and good convergence was still attained. Recall that p controls the skewness of the distribution. This is advantageous as this model will hold well in markets going through skewed phases, whereas Black-Merton-Scholes fails to deal with such situations. From a time-to-price point of view, this model held up well. Convergence was attained a little slower than the FMLS. When convergence was not attained, the scheme could be adjusted by making Δt smaller. This led to more accurate results but also an increased amount of time for the solution to be obtained. In few cases, this could take up to an hour. This model would also hold up well in practical situations.

In conclusion, ADM is not suited to approximate any of these models. CGMY under the finite difference scheme is also impossible. However, the FMLS and KoBol models are very promising. If sufficient market data is available to calibrate the model then, using these processes is highly recommended.

Chapter 5

Appendix

5.1 Appendix A - Stochastic Calculus

What follows is an outline of results for stochastic calculus required for this thesis. An advanced explanation of stochastic calculus is beyond the scope of this work. The reader is asked to refer to [44, 45] for a detailed explanation of the subject. Stochastic calculus is built from a number of definitions, each relying on the previous one.

Definition: let Ω be an arbitrary set. A set $\mathcal{A}$ of subsets of Ω is called a σ-algebra if the following hold:

(i) $\Omega \in \mathcal{A}$,

(ii) $A \in \mathcal{A} \Rightarrow A^c = \Omega - A \in \mathcal{A}$,

(iii) $A_n \in \mathcal{A} \Rightarrow \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$.

The pair $(\Omega, \mathcal{A})$ for which $\mathcal{A}$ is a σ-algebra is called a *measureable space.*

Definition: for any set $\mathcal{C}$ of subsets of Ω the smallest σ-algebra $\mathcal{A}$ which contains $\mathcal{C}$ can be defined and is denoted by $\sigma(\mathcal{C})$.

Perhaps the most important of all σ-algebras is the Borel σ-algebra defined as follows:

Definition: if $(\mathcal{E}, \mathcal{O})$ is a tautological space, where $\mathcal{O}$ is the set of open sets in $\mathcal{E}$, then $\sigma(\mathcal{O})$ is called the Borel σ-algebra.

Often $\mathcal{E}$ is chosen to be $\mathbb{R}$.

Definition: given a measureable space, $(\Omega, \mathcal{A})$, a function $\mathbb{P} : \mathcal{A} \to \mathbb{R}$ is called a *probability measure* and the triplet $(\Omega, \mathcal{A}, \mathbb{P})$ is called a *probability space* if the following axioms are satisfied:

(i) $\mathbb{P}[A] \geqslant 0$ for all $A \in \mathcal{A}$,

(ii) $\mathbb{P}[\Omega] = 1$,

(iii) $A_n \in \mathcal{A}$ disjoint $\Rightarrow \mathbb{P}[\bigcup_n A_n] = \sum_n \mathbb{P}[A_n]$.

Definition: a map X from a measureable space $(\Omega, \mathcal{A})$ to another measureable space $(\mathcal{D}, \mathcal{B})$ is called $\mathcal{B}$-*measureable* if $X^{-1}(B) \in \mathcal{A}$ for all $B \in \mathcal{B}$

Definition: a function $X : \Omega \to \mathbb{R}$ is called a *random variable* if it is a measureable map from $(\Omega, \mathcal{A})$ to $(\mathbb{R}, \mathcal{B})$ where $\mathcal{B}$ is the Borel σ-algebra of $\mathbb{R}$.

Definition: a set $\{X_t\}_{t \in T}$ of random variables defines a stochastic process.

The index t usually refers to time.

Definition: let $(\Omega, \mathcal{F})$ be a σ-algebra. Then a *filtration* $\mathbb{F}$ is a family $\{\mathcal{F}_t\}$ of increasing σ-algebras on $(\Omega, \mathcal{F})$ with $\mathcal{F}_t \in \mathbb{F}$. If we have a set Ω, a σ-algebra of subsets of Ω, $\mathcal{F}$, and a probability measure $\mathbb{P}$ defined on elements of $\mathcal{F}$ such that

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset ... \subset \mathcal{F}_T = \mathcal{F},$$

then $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is called a *filtered probability space.*

The property that a filtration is increasing is necessary as it implies that information is not forgotten. As one moves through time, one knows the stock price now, as well as what is was at every point in the past.

Definition: a stochastic process, $\{X_t\}_{t\in T}$, is called *adapted* if for all t, X_t is $\mathcal{F}_t$ measureable.

This means that at any time t, $\mathcal{F}_t$ contains all the information about X_t. In reality, the properly that a stochastic process is adapted is too weak and further conditions are required. Often, it is necessary to assume a process is cádlág as well.

Definition: a stochastic process, $\{X_t\}_{t\in T}$, is cádlág if its paths are right continuous with left limits everywhere with a probability of one.

Definition: a filtration is called *right continuous* if $\mathcal{F}_{t+} = \mathcal{F}_t$ where

$$\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s.$$

The intuitive explanation of this definition is that any information known immediately after t is also known at t. This is a standard assumption when dealing with filtrations and referred to as the *usual condition*.

Definition: Brownian motion, $\{B_t\}$ is a stochastic process with the following properties:

(i) $B_0 = 0$

(ii) $B_t - B_s$ is independent of $\mathcal{F}_s$ for $0 \leqslant s < t \leqslant T$,

(iii) $B_t - B_s \sim N(0, t-s)$,

(iv) B_t for $t \geqslant 0$ are continuous functions of t.

The main tool to solve stochastic differential equation is Ito's formula which is a stochastic analog of the fundamental theorem of calculus. This was used implicitly to move from (1.2. 1) to (1.2. 3). Ito's formula is given below:

Let X_t be defined by $dX_t = \mu_t dt + \sigma_t dZ_t$ (X_t is called an *Ito process*). Let $u(t,x)$ be a function that is twice continuously differentiable is x and once continuously differentiable in t. Define $U_t = u(X_t, t)$. The $\{U_t\}$ is an Ito process that satisfies:

$$dU_t = u_x(X_t, t)dX_t + u_t(X_t, t)dt + \frac{1}{2}u_{xx}(X_t, t)\sigma_t^2 dt. \tag{5.1. 1}$$

We conclude this section with the important Girsanov's theorem which allows us to change measure when dealing with stochastic processes:

given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and let $\{B_t\}$ be a Brownian motion on this space. Let θ_t be a process adapted to $\mathcal{F}_t$. Define

$$B_t^{\mathbb{Q}} = \int_0^t \theta_s ds + B_t,$$

and

$$M(t) = e^{-\int_0^t \theta_s dB_s - \frac{1}{2}\int_0^t \theta_s^2 ds},$$

where

$$\mathbb{Q}(F) = \int_F M(T) d\mathbb{P}, \qquad \text{for all } F \in \mathcal{F}.$$

Then under $\mathbb{Q}$, $B_t^{\mathbb{Q}}$ is a Brownian motion.

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