THE DEVELOPMENT AND SOME PRACTICAL APPLICATIONS OF A STATISTICAL VALUE DISTRIBUTION THEORY FOR THE WITWATERSRAND AURIFEROUS DEPOSITS.

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CHAPTER I.

INTRODUCTION.

Since the discovery of gold on the Witwatersrand a little more than half a century ago, practical mining methods have improved so rapidly and with such success that South African mining engineers have become internationally famous for their ingenuity and enterprise. In view of this fact it is rather astonishing to find that methods of mine valuation on these fields are still essentially the same today as they were when it first became evident that the Witwatersrand was destined to develop into one of the greatest gold-producing areas in the world.

As soon as systematic sampling of the gold-bearing reefs was started, primarily to establish the location of payable values in the ore, it was realised that a significant difference existed between the expected gold yield as calculated from mine sampling results and that actually obtained in practice from the mill treatment of the ore. Numerous empirical devices, based largely on practical experience, were originated in an attempt to reconcile the gold called for from sampling results with that actually obtained from the mill yield, but it was not until early in the year 1919 that the first really scientific approach to the problem was made by Professor Watermeyer, who used statistical data supplied to him by Hooper. After this initial attempt, a period of about ten years elapsed before the subject was again broached from more or less the same angle by Professor Truscott. Both these authors had as their primary object the evolution of a scientific system of discounting* of mine sampling results to arrive at a true average. In spite of the fact that the method of weighting as originally devised by Professor Watermeyer, and subsequently modified by Professor Truscott, was recently proved to be mathematically unsound by H. S. Sichel, sufficient evidence was furnished to show that the occurrence of gold in the ore of the Witwatersrand is not haphazard, but that the values are distributed throughout a given ore body in accordance with some definite statistical law.

One of the most recent contributions involving the application of the science of statistics to the subject of mine valuation was made by H. S. Sichel, who indicated an ingenious method of eliminating the effect of the errors which are known to exist in mine samples even where the utmost care has been exercised in the actual operation of sampling.

From the foregoing remarks it will be seen that, up to the present, the employment of statistical methods in mine valuation has not only been limited in extent, but confined to one single aspect of the subject only. Summarising, it may be stated that both Watermeyer and Truscott evolved methods of computation of average values by the application of systems of weighting. The system developed was in each case based on the assumption that the values obtained from mine sampling results were correct.

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*There is a distinct difference between the terms "Discounting" and "Cutting".

<u>Discounting</u> refers to the rational adjustment of computed averages of sampling values for some specific purpose.

<u>Cutting</u> is a term which refers to a process devised for an empirical reduction of apparently anomalous sampling values before the average value of any sampled distance is computed. Cutting is being discontinued on an increasing number of mines.

Sichel, on the other hand, proved that mine samples contain unavoidable inherent errors, and originated a method of eliminating the effect of such errors. All three investigators therefor had a common object, namely that of cancelling the well-known Mine Call Factor*. For this purpose a study of the frequency of occurrence of sampling values without any regard to their relative position in the ore body has sufficed. In attempting, however, to determine the statistical nature of the ore <u>in situ</u>, which is the object of this treatise, the actual locality of the individual values in the ore body has to be carefully considered, as the relation between adjoining sampling sections will be shown to be of the utmost importance.

It should be stressed at the outset that the problem has been approached without any regard to the errors inherent in mine sampling. The distribution of sampling values containing these inherent errors will be shown to obey a definite statistical law, and this law will then be used to obtain a reliable estimate of the <u>sampling value</u> of a particular ore body as distinct from its <u>true</u> value.

It is universally recognised that one of the most important problems confronting the mine valuator is undoubtedly the computation and compilation of the ore reserve**. All mining lease agreements are drawn up with the definite stipulation, amongst others, that mining shall be carried out in reasonable accordance with the

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*<u>The Mine Call Factor</u> is the ratio, expressed as a percentage, of the gold accounted for in recovery plus residues (in sands and slimes) and the gold called for at the mill by the measuring methods employed on the mine.

**<u>The Ore Reserve</u> is an estimate of the payable tonnage and mineral content adequately exposed by mining operations up to the date of computation.

average value of the ore reserve, and the fact that the strict observance of this clause constitutes one of the chief concerns of the Inspector of Mining Leases will serve to give some idea of the importance attached to this aspect by the State. The first essential towards satisfying the above requirement is naturally that the value of the ore reserve of a mine shall be assessed with the utmost care and accuracy. As both future policy and current mining methods may be appreciably influenced by the ore reserve value, a reliable assessment of the ore reserve of a mine is also of the greatest importance to the mining engineer, who is unfortunately very frequently misinformed to a greater or lesser degree in this respect as a result of faulty ore reserve valuation.

In support of this contention it may be stated that investigations carried out on a large number of mines have revealed the presence of two distinct and very prevalent types of error, which have been broadly classified as follows:-

A. The Quality or Valuation Error.

The well-known Block Factor*, which shows most ore reserves to be only slightly overvalued when regarded as a whole, is generally accepted as a measure, unfortunately only subsequently available, of the accuracy with which any particular ore reserve has been valued. Block Factors calculated for the different classifications of ore, however, have shown that in the majority of cases this factor is much less for the ore in the upper than for that in the

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*The Block Factor is the ratio, expressed as a percentage, of the total gold content (or average inch-dwt. value) of the ore mined from the ore reserve as indicated by current sampling results, and that content (or average inch-dwt. value) of this ore as computed from the original Block estimates.

lower value groups. This is clearly illustrated in the table below, which shows the Block Factors for the ore in the various value categories of four different mines over a period of a year:-

TABLE I.

Value Group (dwt.)	Block Factors (Per Cent)			
	Mine R	Mine S	Mine C	Mine X
10.0 and over	72	91	79	84
9.0 - 9.9	77	84	91	102
8.0 - 8.9	87	91	97	96
7.0 - 7.9	89	108	102	88
6.0 - 6.9	92	84	108	63
5.0-- 5.9	106	136	113	98
4.0 - 4.9	109	113	118	116
P/L - 3.9	120	122	114	111
Average	99.8	116.2	111.7	101.9

It must be emphasized that both the original block valuation and the current sampling from which the Block Factors have been determined are based on uncut sampling results, and the abnormal Block Factors, assuming these to be fully representative, are therefore the result of faulty ore reserve valuation only. While admitting the limitations - due to certain practical considerations - of such categorised Block Factors for the purpose of checking the accuracy with which the various grades of ore in the ore reserve have been valued, the figures in the above table reveal such a definite trend which has been found to persist in the ore reserve valuation of the majority of mines on which investigations were carried out that its significance cannot be overlooked.

Viewed in this light, two outstanding features emerge from the table. The first is that although the overall Block Factors are in almost all cases reasonably satisfactory and give no cause for undue alarm, the individual factors for the ore in the different value groups show considerable fluctuation. Secondly, there exists in all cases a distinct over-valuation of the ore in the upper as compared with that in the lower value groups.

An extremely important practical consideration arises as a result of this tendency. Assuming again that the Block Factors in the various value groups are fully representative of the errors which could be expected to exist in the original valuation of the ore in these categories, consider what would happen if the grade of the tonnage milled on the mine is to be stepped up. To accomplish this, the tonnage of the ore from the lower value groups will have to be decreased, and that from the upper value groups correspondingly increased. By doing this, a certain increase in yield is expected. Due however to the fact that the ore in the upper value groups has been relatively over-valued, the expected yield will not fully materialise. Thus to achieve the yield originally desired, a disproportionately large amount of low grade ore will have to be replaced by that from the higher grades. Since this must inevitably be reflected in the form of overmining* in the returns, all the complications attendant on the practice of overmining will be experienced.

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*Overmining is the term used when the average value, expressed in dwt. per ton, of the tonnage mined from Ore Reserve Blocks - based on Block Valuation - for any specific period, is greater than that of the Ore Reserve as a whole.

Undermining and Mining to Ore Reserve are the corresponding terms used when this value is less than or equal to that of the Ore Reserve as a whole, respectively.

In order to illustrate the above, the following table has been compiled from the records of one of the leading gold-producing mines on the Witwatersrand. The Block Factors used in the table have been deduced from a comparison of the Ore Reserve valuation with the results obtained from sampling operations subsequently carried out in the respective Ore Reserve Blocks during the ensuing year. While it will therefore not be theoretically permissible to employ these factors for forecasting purposes, they may be used to explain the grade difficulties actually experienced on the mine during the course of the year for which the Ore Reserve was valid.

TABLE II.

(i) The Segregated Ore Reserve.

Value Group (Dwt per Ton)	In Ore Reserve Tons	In Ore Reserve %	Aver. O. R. Value (Dwt / Ton)	Block Factor %
10.0 & over	161,000	4.8	13.3	72
9.0 - 9.9	76,000	2.3	9.4	77
8.0 - 8.9	156,000	4.7	8.4	87
7.0 - 7.9	314,000	9.4	7.5	89
6.0 - 6.9	409,000	12.2	6.5	92
5.0 - 5.9	889,000	26.6	5.4	106
4.0 - 4.9	875,000	26.2	4.5	109
3.4 - 3.9	461,000	13.8	3.6	120
Total	3,341,000	100.0	5.86	99.8

- (Contd.) -

TABLE II - (Contd.)

(ii) Mined from Ore Reserve (First Stage).

Value Group (Dwt./Ton)	Mined from Ore Reserve Tons	%	Aver. O. R. Value (Dwt./Ton)	Block Factor %	Actual Aver. Val. (i.e. O.R. Value corrected by the Block Factor) (Dwt./Ton)
10.0 & over	6,500	5.0	13.3	72	9.6
9.0 - 9.9	4,000	3.1	9.4	77	7.2
8.0 - 8.9	8,300	6.4	8.4	87	7.3
7.0 - 7.9	13,100	10.1	7.5	89	6.7
6.0 - 6.9	16,400	12.6	6.5	92	6.0
5.0 - 5.9	36,700	28.1	5.4	106	5.7
4.0 - 4.9	29,200	22.5	4.5	109	4.9
3.4 - 3.9	15,800	12.2	3.6	120	4.3
Total	130,000	100.0	6.04	99.8	5.83

Expected Yield = 6.04 × 0.7 = 4.23 dwt./ton ------(1)

Actual Yield = 5.83 × 0.7 = 4.08 dwt./ton -----(2)

Approx. Overmining = 6.04/5.86 = 104% --------------(3)

(iii) Mined from Ore Reserve (Second Stage).

Value Group (Dwt./Ton)	Mined from Ore Reserve Tons	%	Aver. O. R. Value (Dwt./Ton)	Block Factor %	Actual Aver. Val. (i.e. O.R. Value corrected by the Block Factor.) (Dwt./Ton)
10.0 & over	7,900	6.1	13.3	72	9.6
9.0 - 9.9	4,900	3.8	9.4	77	7.2
8.0 - 8.9	10,300	7.9	8.4	87	7.3
7.0 - 7.9	16,100	12.4	7.5	89	6.7
6.0 - 6.9	20,200	15.5	6.5	92	6.0
5.0 - 5.9	28,200	21.7	5.4	106	5.7
4.0 - 4.9	27,700	21.3	4.5	109	4.9
3.4 - 3.9	14,700	11.3	3.6	120	4.3
Total	130,000	100.0	6.30	99.8	5.96

Expected Yield = 6.30 × 0.7 = 4.41 dwt./ton --------(1)

Actual Yield = 5.96 × 0.7 = 4.27 dwt./ton --------(2)

Approx. Overmining = 6.30/5.86 = 106% --------------(3)

Table II - (Contd.)

(iv) Mined from Ore Reserves (Third Stage).

Value Group (Dwt./Ton)	Mined from Ore Reserve		Aver. O. R. Value (Dwt./Ton)	Block Factor %	Actual Aver. Val. [i.e. O.R. Value corrected by the Block Factor.] (Dwt./Ton)
	Tons	%			
10.0 & over	9,300	7.2	13.3	72	9.6
9.0 - 9.9	5,900	4.5	9.4	77	7.2
8.0 - 8.9	12,200	9.4	8.4	87	7.3
7.0 - 7.9	19,100	14.7	7.5	89	6.7
6.0 - 6.9	23,900	18.4	6.5	92	6.0
5.0 - 5.9	23,800	18.3	5.4	106	5.7
4.0 - 4.9	23,400	18.0	4.5	109	4.9
3.4 - 3.9	12,400	9.5	3.6	120	4.3
Total	130,000	100.0	6.61	99.8	6.12

Expected Yield = 661×0.7 = 4.63 dwt./ton-----(1)

Actual Yield = 6.12×0.7 = 4.28 dwt./ton-----(2)

Approx. Overmining = $^{6.61}/_{5.86}$ = 113% ------------(3)

Notes on Table II:-

(1) This is the value which would have been obtained had the Ore Reserve been correctly valued.

(2) The value actually obtained, assuming that the categorised Block Factors are a true reflection of the errors existing in the corresponding value groups. The ratio $\frac{\text{Yield}}{\text{O.R. Value}}$ = 0.7

(3) This is an approximation to the true overmining only, since for true overmining purposes, the average value mined from ore reserve is deduced from measured fathomages, block widths* and block values.

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*The Block Width is the average width at which it is estimated that a block of ore will be stoped. It is that width which is used in conjunction with the delineated area of the block to determine the tonnage contained in that block.

In the above illustration the average ore reserve value mined has been based on measured fathomages, current measured widths and block values. Since the block widths, after an allowance for the tonnage discrepancy*, are usually in reasonable agreement with the current measured widths, the error introduced will in most cases be practically negligible. Furthermore, the average value deduced from tonnages based on block widths and block values, as has been done in the above illustration, will be the same as the average value deduced from tonnages based on current measured widths and block values as long as the ratio of the current measured widths to the block widths is constant throughout the entire range of value groups. This has been found almost invariably to be the case.

It will be seen from Table XI that the expected yield based on the average value of the ore reserve is 4.23 dwt. per ton, with the perfectly normal rate of overmining of 3 per cent. The actual yield obtained from this mine will however be only 4.08 dwt. per ton. The grade of 4.23 dwt. per ton having been set as a target, the manager will naturally be desirous of realising this yield. In order to do so, he will find it necessary to push up the apparent overmining to 13 per cent. This can only be done at the risk of incurring the displeasure both of his consulting engineer and the Department of Mines.

B. The Quantity or Area Error.

This error is, if anything, of greater importance than

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*The Tonnage Discrepancy is the difference between the tonnage estimated as milled, based on surveyors' measurements, and that measured as milled by the Reduction Works. The tonnage discrepancy is referred to as an excess when the actual tonnage milled is in excess of the estimated tonnage milled, and as a deficit when it is less than the estimated tonnage milled.

the quality error, due to the fact that its presence is much more difficult to detect. Briefly, it may be stated that errors occur in the valuation of ore reserves as a result of the fact that the average size of the blocks in the different value categories shows considerable variation. This variation is not indiscriminate throughout, investigations having proved that there exists a definite tendency in the ore reserves of many mines towards larger blocks in the upper, and smaller blocks in the lower value groups.

The table below, showing the average size of ore reserve blocks in the various value categories, is the result of investigations carried out on three large Witwatersrand gold mines, and will serve to illustrate the point under discussion.

TABLE III.

Value Group (Dwt / Ton)	Mine "X"		Mine "Y"		Mine "Z"	
	No. of Blocks	Av. Tons per Blocks	No. of Blocks	Av. Tons per Block	No. of Blocks	Av. Tons per Block
10,0 & over	67	29,200	43	5,200	108	9,300
9.0 - 9.9	17	37,700	15	8,300	40	11,500
8.0 - 8.9	21	36,800	24	8,300	47	10,100
7.0 - 7.9	37	30,400	35	5,600	71	9,000
6.0 - 6.9	57	28,[illegible]00	44	7,100	112	8,500
5.0 - 5.9	61	23,300	68	6,700	142	9,500
4.0 - 4.9	67	22,600	100	5,500	184	8,400
3.0 - 3.9	84	20,600	124	5,500	204	7,000
P/L - 2.9	101	17,900	-	-	79	6,900
Totals & Averages	512	24,618	453	6,035	995	8,512

Although there is no scientific justification for this phenomenon, its presence can be simply explained. All ore reserve computers, by the very nature of their task, adopt a more or less conservative attitude towards ore reserve valuation. When dealing with an area or zone of the mine in which higher values occur, valuation apparently presents no difficulty, and the task of blocking out is pursued with full confidence. When areas in which values bordering on the pay limit are encountered, however, a natural conservatism manifests itself, and smaller ore reserve blocks are the result. Indeed, it is not an unusual occurrence for border-line blocks to be entirely excluded from the reserve. The effect of this practice will obviously be to inflate the overall average ore reserve value when the individual blocks are finally summated, due to the fact that the higher grades of ore have been weighted more heavily than the lower grades. The whole economic structure of a mine being built up around the average value of its ore reserve, difficulties will immediately be encountered as a result of this artificial inflation of the value.

Rather less obvious but equally serious complications also arise as a result of this irregularity. Since the average value of the ore reserve will not be representative of the true average value of the mine, mining to the average value of the ore reserve as computed from the summation of individual ore reserve blocks will lead to overmining without this pernicious practice being reflected in the monthly mining returns. Overmining deliberately carried out gives cause for concern, but overmining unknowingly executed in this way can have very serious and far-reaching effects.

From this necessarily brief summary it will be evident that the compilation of ore reserves leaves much to be desired, and it was in an attempt to find a method whereby the accuracy of a completed ore reserve could be tested that the investigations eventually leading to the theories forming the subject of the present treatise were originally embarked upon. During the course of detailed examinations at numerous mines, literally tens of thousands of mine samples were studied with a view to ascertaining not only the statistical distribution of the individual samples themselves, but also the effect on the original distribution of combining adjoining samples into larger and larger groups, as is done when blocks of ore are valued. The succeeding chapters will be devoted mainly to showing how the various theoretical conclusions were deduced from certain practical observations, and to indicating briefly the use that can be made of the theory for the solution of practical mine valuation problems.

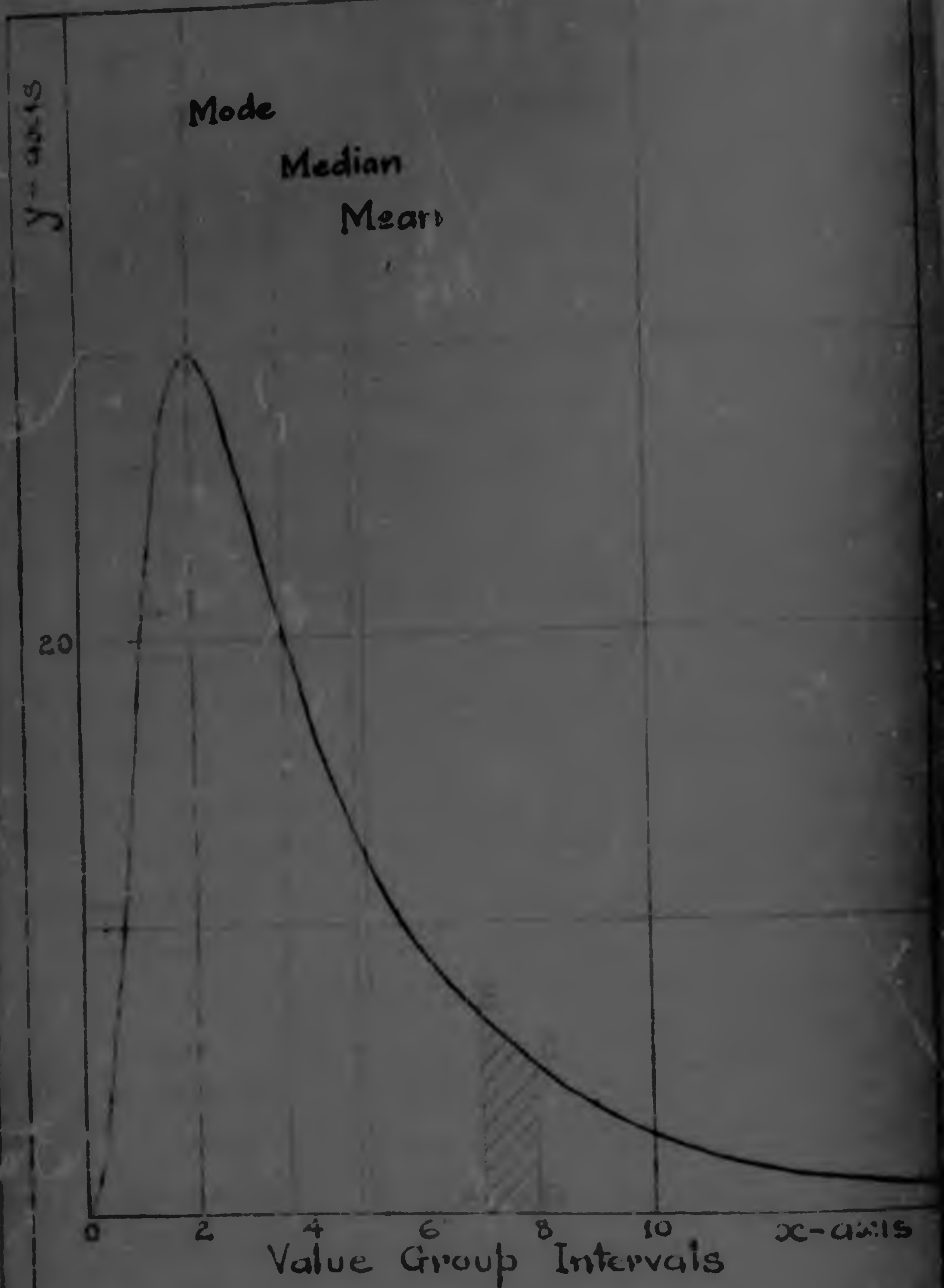

PLATE I

A Typical Value Distribution Curve

CHAPTER II.

PROOF OF THE EXISTENCE OF A STATISTICAL LAW FOR THE DISTRIBUTION OF GOLD VALUES AS DETERMINED FROM UNCUT SAMPLING RESULTS.

Everyone who has had to deal with the subject of Mine Valuation on the Witwatersrand is probably acquainted with the general shape of the frequency curve representing the distribution of gold values. This curve is deduced from practical considerations by plotting the frequencies of occurrence of observed values in the form of a histogram*, and then running a smooth curve through the step-diagram so obtained. The curve is characteristically skewed to the left, rising rapidly from zero to the mode, or the abscissa associated with the maximum ordinate, and then dropping off more gradually in the direction of the higher value groups until it finally approaches the "x"- axis asymptotically.

Plate I illustrates a typical curve of this nature, a characteristic feature of which has been found to be the exceptionally long drawn out tail at the high value end. Since the curve is deduced from practical considerations by passing a smooth curve through the tops of an observed histogram, the percentage frequency of the values falling within a given value group will actually be denoted by the area enclosed between the curve and the "x"- axis. Thus in Plate I the percentage of the total values lying between, say 7 dwt. per ton and 8 dwt. per ton, will be represented by the area marked ABCD. This may also be differently stated

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*A Histogram is a frequency diagram in which the frequencies of occurrence of the values falling within given class intervals are represented, not by ordinates, but by areas based on the given class intervals ~~and enclosed between the x- axis and the curve~~. In this treatise, all histograms, unless otherwise stated, have been plotted on a percentage frequency basis.

by saying that, in taking a sample at random, the probability of striking a value lying between, say 7 dwt. per ton and 8 dwt. per ton will be represented by $\frac{\text{Area ABCD}}{100}$. If the class intervals are all made equal, then the heights of the rectangles based on these class intervals will be proportional to the percentage frequencies, while if the class intervals are chosen as units, these heights will be equal to the percentage frequencies. In all the ensuing work, class intervals have therefore been regarded as units wherever possible.

In order to appreciate fully the theoretical aspects to be dealt with later, the following fundamental definitions and mathematical considerations will be found useful:-

(i) The Mode is the abscissa of the maximum ordinate of the curve, and is in practice that value in a given distribution which occurs most frequently. Thus in Plate I the modal value will be seen to be 2.0 dwt. per ton.

(ii) The Median is the abscissa of the ordinate which bisects the total area between the curve and the x - axis. Because of the skewness of the curve, the median lies to the right of the mode.

(iii) The Mean is the abscissa of the ordinate which passes through the centroid of the distribution. The mean is situated to the right of the median.

If the curve in Plate I can be represented mathematically by the equation $y = g(x)$, then the total frequency of occurrence of all values will be represented by

$$F = \sum_{x=0}^{\infty} y = \int_0^{\infty} g(x)\,dx = 100\% \text{ ---------- [1]}$$

Also, the frequency of occurrence of all values above a given value limit, say "x" dwt per ton, will be

$$y_x = \sum_{x}^{\infty} y = \int_{x}^{\infty} g(x)\,dx \quad \text{------------} \quad (2)$$

Since the mean value "$\bar{x}$" may be defined mathematically as

$$\bar{x} = \frac{\sum_{x=0}^{\infty} x\,y}{\sum_{x=0}^{\infty} y} = \frac{\int_{0}^{\infty} x.g(x)\,dx}{\int_{0}^{\infty} g(x)\,dx}, \quad \text{--------} \quad (3)$$

it follows that the average value "v_x" of all the observations above the value limit of "x" dwt per ton will be

$$v_x = \frac{\sum_{x}^{\infty} x\,y}{\sum_{x}^{\infty} y} = \frac{\int_{x}^{\infty} x.g(x)\,dx}{\int_{x}^{\infty} g(x)\,dx} \quad \text{--------} \quad (4)$$

It will readily be appreciated that in the above analysis a complete knowledge of the equation $y = g(x)$ representing the law of distribution, is essential.

Since the auriferous deposits of the Witwatersrand were all obviously formed in the same manner by the same physical agencies, considered thought has led to the belief that some general statistical law of distribution should exist to describe the peculiarities of any particular Witwatersrand ore deposit, or any portion of such an ore-body.

Before proceeding to give an account of the investigations embarked upon to test the validity of the above contention and the conclusions drawn from the various aspects encountered, it will first be necessary to give a brief mathematical analysis of the curve which has been found to give a very satisfactory "fit" to the observed distribution of gold values in the ore of a representative number of Witwatersrand gold mines. The curve referred to, which therefore represents what has been termed the Fundamental Value Distribution Law, is denoted by the equation

$$y = N e^{-s^2 (\log_e x - b)^2}$$

where "s" and "b" are parameters and "N" a constant, the numerical values of which will depend on the peculiarities of any particular ore-body or portion of an ore-body, as well as on certain practical sampling considerations. In developing the following theory, only such equations and formulae will be derived as will have a direct bearing on the subsequent practical applications and illustrations.

A. <u>Mathematical Analysis of the Equation $y = N e^{-s^2(\log_e x - b)^2}$</u>, <u>denoting the Frequency Curve representing the Fundamental Value Distribution Law</u>.

(a) <u>Determination of the Position of the Mode</u>.

By definition, the mode is the abscissa of the maximum ordinate. Expressed mathematically, this is the value of "x" at the point where $\frac{dy}{dx} = 0$.

Since $y = N e^{-s^2 (\log_e x - b)^2}$

$\therefore \frac{dy}{dx} = N \frac{d}{dx} e^{-s^2 (\log_e x - b)^2}$

Substituting $u = \log_e x - b$, $\frac{dy}{dx} = N(-s^2) e^{-s^2 u^2} .2u$

or $\frac{dy}{dx} = -2N s^2 (\log_e x - b) e^{-s^2(\log_e x - b)^2}$

Hence $\frac{dy}{dx} = 0$ if $\log_e x = b$

or if $x = e^b$

The mode of the curve is therefore defined by the line $x = e^b$ ---------------------- [5]

If $x = e^b$, or $\log_e x = b$,

then $y = N e^{-s^2 . 0}$

or $y = N$ ---------------------- [6]

Thus the maximum ordinate of the curve will be $y = N$, when $x = e^b$.

(b) <u>The Relation between the Constant "N" and the Parameters "a" and "b" to ensure that the Area below the Curve shall be a fixed Quantity, say "A".</u>

The Total area below the curve is represented mathematically by

$$A = \int_0^{\infty} N e^{-a^2 (\log_e x - b)^2} dx$$

Substituting $\log_e x - b = u$,

$$\text{then } \frac{du}{dx} = \frac{1}{x}, \quad \text{or} \quad \frac{dx}{du} = x$$

$$\text{and } x = e^{u+b}$$

Also if $x = 0$, $\quad u = -\infty$

and if $x = +\infty$, $\quad u = +\infty$.

$$A = N \int_{-\infty}^{+\infty} e^{-a^2u^2} e^u e^b du$$

$$= N e^b \int_{-\infty}^{+\infty} e^{-[a^2u^2 - u + (\frac{1}{2a})^2] + (\frac{1}{2a})^2} du$$

$$= N e^b e^{(\frac{1}{2a})^2} \int_{-\infty}^{+\infty} e^{-(au - \frac{1}{2a})^2} du$$

Make the further substitution $t = au - \frac{1}{2a}$

$$\therefore \frac{dt}{du} = a, \quad \text{or} \quad \frac{du}{dt} = \frac{1}{a}.$$

if $u = +\infty$, $\quad t = +\infty$

and if $u = -\infty$, $\quad t = -\infty$.

$$\therefore A = N e^b e^{(\frac{1}{2a})^2} \int_{-\infty}^{\infty} e^{-t^2} \frac{1}{a} dt$$

$$= \frac{N}{a} e^b e^{\frac{1}{4a^2}} \int_{-\infty}^{+\infty} e^{-t^2} dt$$

The numerical value of $\int_0^{\infty} e^{-t^2} dt$ is the constant $\frac{\sqrt{\pi}}{2}$*,

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*This is a well-known statistical relationship, the proof of which can be found in most text books on statistical theory.

and thus $\int_{-\infty}^{+\infty} e^{-t^2} dt = \frac{\sqrt{\pi}}{2} 2 = \sqrt{\pi}$

Hence

$$A = \frac{N}{a} e^{b + \frac{1}{4a^2}} \sqrt{\pi} \quad \text{-----------------------} \quad [7]$$

Since the frequencies are invariably expressed in the form of percentages, the numerical value of "A" will usually be 100.

(c) <u>To Calculate the Mean Value (m) of the Distribution represented by the Equation $y = N e^{-a^2(\log_e x - b)^2}$</u>.

The mean value "m" has been defined as the abscissa of the centroid of the frequency distribution, which may therefore be mathematically written as

$$m = \frac{\int_0^{\infty} x.\ y.\ dx}{\int_0^{\infty} y\ dx}$$

$$m = \frac{\int_0^{\infty} x.\ N.\ e^{-a^2(\log_e x - b)^2}\ dx.}{\int_0^{\infty} N.\ e^{-a^2(\log_e x - b)^2}\ dx}$$

Substituting $\log_e x - b = u$ as before, the above expression may be written as

$$m = \frac{\int_{-\infty}^{+\infty} e^{-a^2u^2}.\ e^{2u}.\ e^{2b}.\ du.}{\int_{-\infty}^{+\infty} e^{-a^2u^2}.\ e^{u}.\ e^{b}.\ du}$$

$$m = \frac{\int_{-\infty}^{+\infty} e^{-a^2u^2 + 2u - (\frac{1}{a})^2 + (\frac{1}{a})^2} du}{\int_{-\infty}^{+\infty} e^{-a^2u^2 + u - (\frac{1}{2a})^2 + (\frac{1}{2a})^2} du}$$

$$= \frac{e^{b + \frac{1}{4a^2}} \int_{-\infty}^{+\infty} e^{-(au - \frac{1}{a})^2} du}{\int_{-\infty}^{+\infty} e^{-(au - \frac{1}{2a})^2} du}$$

$$\text{But} \int_{-\infty}^{+\infty} e^{-\left(au - \frac{1}{a}\right)^2} du = \sqrt{\pi} = \int_{-\infty}^{+\infty} e^{-\left(au - \frac{1}{2a}\right)^2} du$$

$$\therefore \underline{m = e^{b + \frac{3}{4a^2}}} \text{ ---------------------------[8]}$$

It is evident from the above that for a given value of "m" the expression $e^{b + 3/4a^2}$ must necessarily remain constant. The actual <u>shape</u> of the frequency curve however depends on the numerical values of "a" and "b", and since these can vary individually while the mean remains the same, it is obviously possible for a series of curves to obey the same fundamental distribution law with the same mean and yet possess vastly different shapes. This theoretical aspect will be found to be of the utmost importance in certain practical considerations, and should be carefully noted.

(d) <u>Determination of the Locus of the Peak of a Set of Frequency Curves for a given Mean Value</u>.

From equations [6] and [7] it will be seen that at the peak of the curve

$$y = M = \frac{100.\,a}{e^{b + 1/4a^2}\sqrt{\pi}}$$

But from equation [5], $x = e^b$

and from equation [8], $m = e^{b + 3/4a^2}$

The equation of the Peak Locus Curve will be the "a", "b" eliminant of the above equations. This may be determined as follows:-

$$y = \frac{100.\,a}{\sqrt{\pi}.\, e^{1/4a^2}.\, x} \text{ -------------------[9]}$$

$$\text{But } x.e^{3/4a^2} = m$$

$$\therefore e^{3/4a^2} = \frac{m}{x}$$

$$\text{and } e^{1/4a^2} = \left(\frac{m}{x}\right)^{1/3}, \text{ or } e^{1/2a^2} = \left(\frac{m}{x}\right)^{2/3}$$

$$\therefore \frac{1}{2a^2} = \frac{2}{3}\log_e\left(\frac{m}{x}\right)$$

$$s^2 = \frac{3}{4 \log_e \frac{m}{x}}$$

$$\therefore s = + \frac{\sqrt{3}}{2(\log_e \frac{m}{x})^{1/2}} \qquad \text{(since "s" is always +ve)}$$

Substituting this value for "s" in equation [9],

$$y = \frac{100 \sqrt{3}}{2(\log_e \frac{m}{x})^{1/2} x \left(\frac{m}{x}\right)^{1/3} \sqrt{W}}$$

$$y = \frac{100 \sqrt{3}}{2 \sqrt{W}\, m^{1/3} \left[x^{2/3} (\log_e \frac{m}{x})^{1/2} \right]} \quad \text{———— [10]}$$

This locus can now itself be analysed with a view to determining its shape:-

For a given "m": If $x = 0$, $y = \infty$.

Also, if $x = m$, $\log_e \frac{m}{x} = \log_e 1 = 0$, and therefore $y = \infty$.

Thus the curve passes from $+\infty$ to a minimum value, and then again to $+\infty$ as "x" increases from 0 to "m". As will be more fully appreciated when dealing with the practical applications of this frequency distribution law, considerable importance attaches to the position of the minimum point on this peak locus curve. The exact location of the minimum point will therefore be determined mathematically below:-

The lowermost point on the peak locus curve represented by the equation

$$y = \frac{100 \sqrt{3}}{2 \sqrt{W}\, m^{1/3} \left[x^{2/3} (\log_e \frac{m}{x})^{1/2} \right]}$$

occurs where $\frac{dy}{dx} = 0$

As $\frac{100 \sqrt{3}}{2 \sqrt{W}\, m^{1/2}}$ is constant for any given "m", it will be denoted simply by "c".

$$\therefore \frac{dy}{dx} = c\,\frac{d}{dx}[\,x^{-\frac{2}{3}}(\log_e \tfrac{m}{x})^{-\frac{1}{2}}\,]$$

$$= c\,[-\tfrac{2}{3}x^{-\frac{5}{3}}(\log_e \tfrac{m}{x})^{-\frac{1}{2}} + x^{-\frac{2}{3}}\frac{d}{dx}(\log_e \tfrac{m}{x})^{-\frac{1}{2}}\,]$$

$$= c\,[\,-\tfrac{2}{3}x^{-\frac{5}{3}}(\log_e \tfrac{m}{x})^{-\frac{1}{2}}$$

$$+ x^{-\frac{2}{3}}(-\tfrac{1}{2})(\log_e \tfrac{m}{x})^{-\frac{3}{2}}\frac{1}{m/x}(-m)\,x^{-2}\,]$$

$$= c\,[-\tfrac{2}{3}x^{-\frac{5}{3}}(\log_e \tfrac{m}{x})^{-\frac{1}{2}} + x^{-\frac{2}{3}}\frac{x^{-1}}{2}(\log_e \tfrac{m}{x})^{-\frac{3}{2}}\,]$$

$$= c\,[-\tfrac{2}{3}x^{-\frac{5}{3}}(\log_e \tfrac{m}{x})^{-\frac{1}{2}} + \frac{x^{-\frac{5}{3}}}{2}(\log_e \tfrac{m}{x})^{-\frac{3}{2}}\,]$$

$$= c\,x^{-\frac{5}{3}}(\log_e \tfrac{m}{x})\,[\,-\tfrac{2}{3} + \tfrac{1}{2}(\log_e \tfrac{m}{x})^{-1}\,]$$

$$\frac{dy}{dx} = 0 \quad \text{if} \quad \frac{1}{2\log_e \frac{m}{x}} = \frac{2}{3}$$

$$\text{or if} \quad \log_e \tfrac{m}{x} = \tfrac{4}{3}$$

$$\text{i.e. if} \quad \log_e \tfrac{m}{x} - \tfrac{4}{3} = 0$$

$$\therefore \log_e x = \log_e m - \tfrac{4}{3}$$

$$\text{or} \quad x = \frac{e^{\log_e m}}{e^{3/4}}$$

$$\text{or} \quad x = \frac{m}{e^{\frac{3}{4}}}$$

Thus the minimum point in the peak locus occurs

where $x = \dfrac{m}{e^{3/4}} = \dfrac{m}{2.1170}$ ---------------(11)

The corresponding value of "y" is then given by

$$y = \frac{100\sqrt{3}}{2\sqrt{\pi}.\,m^{1/3}}\left[\frac{1}{\left\{\frac{m}{e^{3/4}}\right\}^{2/3}\cdot(\frac{3}{4})^{1/2}}\right]$$

$$\therefore\ y = \frac{100\sqrt{3}\ e^{1/2}\ 2}{2\sqrt{\pi}\ m\ \sqrt{3}}$$

$$= \frac{100\ e^{1/2}}{\sqrt{\pi}\ m}$$

The minimum value of "y" is therefore given by

$$y = \frac{100\ e^{1/2}}{\sqrt{\pi}} \cdot \frac{1}{m} = \frac{93{\cdot}0188}{m} \quad \text{------} \ [12]$$

From equation [11] it will be seen that at the lowest possible position of the peak for a given mean value "m", the value of "x" is given by

$$x = \frac{m}{e^{3/4}}$$

From equation [5], however, the value of "x" at the peak of the curve is given by

$$x = e^{b}$$

Hence, at the minimum point on the peak locus

$$e^{b} = \frac{m}{e^{3/4}}, \quad \text{or } b = \log_e m - \tfrac{3}{4}$$

$$\therefore\ e^{b} = \frac{e^{b + 3/4s^2}}{e^{3/4}}$$

$$\text{or} \quad e^{3/4s^2} = e^{3/4}$$

and therefore $\underline{s = 1}$ ------------------------------ [13]

Hence the frequency curve for the distribution law as defined by the equation

$$y = N\,e^{-s^2(\log_e x - b)^2}$$

has its peak in the lowest possible position for a given mean value "m" (the "height" of the frequency curve then being a minimum, and the "spread" of the corresponding distribution therefore being a maximum) when the parameter "s" is equal to unity. When this condition applies, the Fundamental Distribution Law can consequently be simplified to

$$y = N\,e^{-\left(\log_e \frac{x}{m} + \frac{3}{4}\right)^2} \quad \text{----------} \ [14]$$

(e) <u>Determination of the Total Number of Observations above a certain Given Value, say "x", and the Deduction of an Expression for the Average Value of all the Observations above this Given Value.</u>

The number of observations "T_x" above a given value, say "x", is defined by equation [2] as

$$T_x = N \int_x^{\infty} e^{-a^2(\log_e x - b)^2} \, dx$$

Let $\log_e x - b = u$, as before.

$$\therefore \; T_x = N \int_u^{\infty} e^{-a^2u^2} \, e^{u + b} \, du$$

$$= N e^{b} \int_u^{\infty} e^{-a^2u^2 + u - \left(\frac{1}{2a}\right)^2 + \left(\frac{1}{2a}\right)^2} du$$

$$= N e^{b} \int_u^{\infty} e^{-\left(au - \frac{1}{2a}\right)^2} \, e^{\left(\frac{1}{2a}\right)^2} du$$

$$= N e^{b + \frac{1}{4a^2}} \int_u^{\infty} e^{-\left(au - \frac{1}{2a}\right)^2} du \; .$$

Making the further substitution $v = au - \frac{1}{a}$

$$\text{or} \quad v + \frac{1}{2a} = au - \frac{1}{2a} \, ,$$

$$T_x = N \frac{e^{b + \frac{1}{4a^2}}}{a} \int_v^{\infty} e^{-\left(v + \frac{1}{2a}\right)^2} dv$$

$$\text{or} \quad T_x = N \frac{e^{b + \frac{1}{4a^2}}}{a} \int_{v + \frac{1}{2a}}^{\infty} e^{-v^2} \, dv \quad \text{———————} \; [15]$$

Since $N \dfrac{e^{b + \frac{1}{4a^2}}}{a} \sqrt{\pi} = A$ from equation [7], the above equation may also be written as

$$T_x = \frac{A}{\sqrt{\pi}} \int_{v + \frac{1}{2a}}^{\infty} e^{-v^2} \, dv \, ,$$

where $A = 100$ when dealing with percentage frequencies, as is usually the case.

In the above expressions

$$w = au - \frac{1}{a}$$

$$\text{or } w = a\,(\log_e x - b) - \frac{1}{a}$$

$$\text{and } w + \frac{1}{2a} = a\,(\log_e x - b) - \frac{1}{2a}\ .$$

Since $\frac{2}{\sqrt{\pi}} \int_{w}^{\infty} e^{-w^2}\, dw$ is a form of the well-known "Normal" Integral which is fully tabulated for all limits of integration, the above equation can more conveniently be expressed in the form

$$T_x = \frac{N}{2} \cdot \frac{2}{\sqrt{\pi}} \int_{w + \frac{1}{2a}}^{\infty} e^{-w^2}\, dw \text{ ------------ [16]}$$

The Average value "v_x" of all the observations above the given value "x" has been defined in equation [4] as

$$v_x = \frac{\int_x^{\infty} x\, y\, dx}{\int_x^{\infty} y\, dx}$$

$$\text{or}\quad v_x = \frac{\int_x^{\infty} x\, N\, e^{-a^2 (\log_e x - b)^2}\, dx}{T_x}$$

Substituting $\log_e x - b = u$ as before,

$$v_x . T_x = N\, e^{2b} \int_u^{\infty} e^{-(au - \frac{1}{a})^2}\, e^{(\frac{1}{a})^2}\, du$$

and again making the further substitution

$$w = au - \frac{1}{a}\ ,$$

the expression above finally reduces to

$$v_x . T_x = \frac{N\, e^{2b + \frac{1}{a^2}}}{a} \int_w^{\infty} e^{-w^2}\, dw$$

Substituting for T_x the expression deduced in equation [15] , v_x becomes

$$v_x = \frac{\dfrac{N e^{2b + \frac{1}{a^2}}}{a} \displaystyle\int_w^{\infty} e^{-w^2}\, dw}{\dfrac{N e^{b + \frac{1}{4a^2}}}{a} \displaystyle\int_{w + \frac{1}{2a}}^{\infty} e^{-w^2}\, dw}$$

$$= \frac{e^{b + \frac{1}{4a^2}} \displaystyle\int_w^{\infty} e^{-w^2}\, dw}{\displaystyle\int_{w + \frac{1}{2a}}^{\infty} e^{-w^2}\, dw}$$

Since $e^{b + \frac{1}{4a^2}} = m$, the above equation may be adapted for use with the form of statistical tables already referred to by writing it as

$$v_x = m \cdot \frac{\frac{2}{\sqrt{\pi}} \displaystyle\int_w^{\infty} e^{-w^2}\, dw}{\frac{2}{\sqrt{\pi}} \displaystyle\int_{w + \frac{1}{2a}}^{\infty} e^{-w^2}\, dw} \quad \text{------------} [17].$$

In the above expression,

$$w = a\,(\log_e x - b) - \frac{1}{a}$$

and $w + \frac{1}{2a} = a\,(\log_e x - b) - \frac{1}{2a}$, as before.

(f) <u>Calculation of the Median of the Frequency Distribution.</u>

The <u>total</u> observations in the frequency distribution defined by the fundamental law

$$y = N e^{-a^2 (\log_e x - b)^2}$$

is given by equation [7] as

$$A = \frac{N e^{b + \frac{1}{4a^2}}}{a} \int_{-\infty}^{+\infty} e^{-w^2}\, dw$$

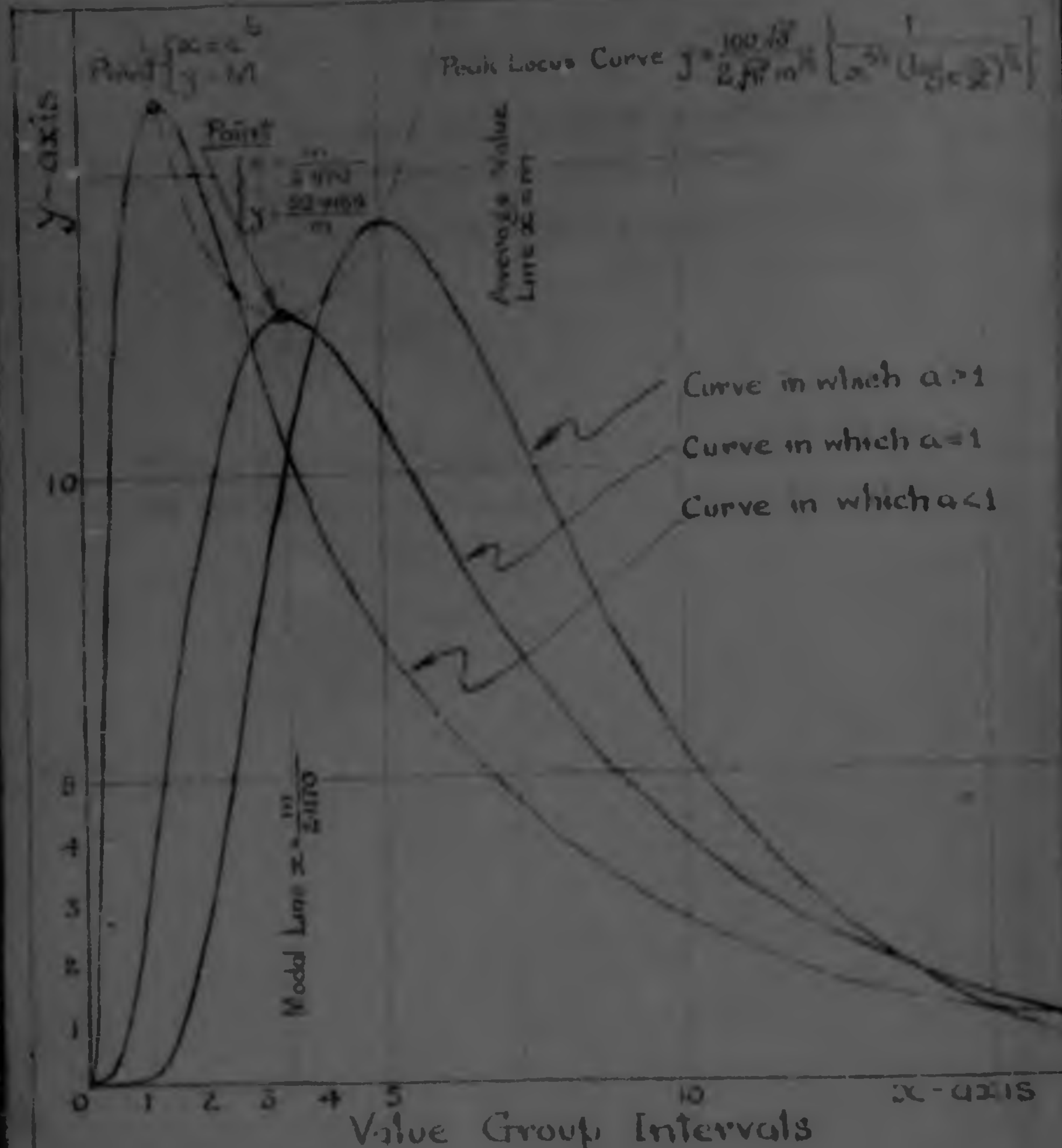

PLATE II

Diagram Illustrating a Set of Frequency Curves of the Type

$$y = Me^{-a^2(\log_e x - b)^2}$$

in which "a" and "b" vary, but the Mean, $m = e^{b + \frac{1}{4a^2}}$, remains constant

Note If $a \to \infty$, $M \to \infty$, and $e^b \to m$, or the Mode of the Curve tends towards the Mean. The Curve will eventually become infinitely elongated and Symmetrical about the Mean.

By definition the
resulting in an o
the frequency curv
readily be approac

$\frac{A}{2}$ = M

But the total numb
value of "x" has b

v_x =

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$\therefore$ x

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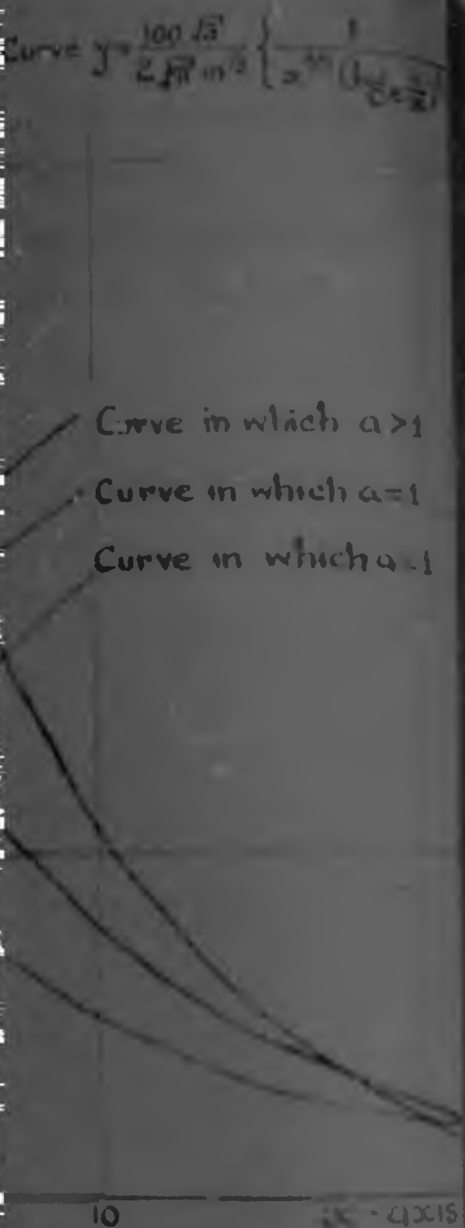


Intervals

...E II—

...ng a Set of
... of the Type
... $x - b)^2$
...b" vary, but the
..., remains constant
...d e^b = m, or the Mo...
... the Mean. The Curv...
...nitely elongated ...
...t the Mean.

By definition the median is that value of "x" resulting in an ordinate which bisects the area below the frequency curve. From equation [V] it will readily be appreciated that

$$\frac{N}{2} = \frac{N e^{b + \frac{1}{4a^2}}}{a} \int_0^{\infty} e^{-w^2}\, dw$$

But the total number of observations above any given value of "x" has been determined in equation [15] as

$$Z_x = \frac{N e^{b + \frac{1}{4a^2}}}{a} \int_{w + \frac{1}{2a}}^{\infty} e^{-w^2}\, dw$$

For the position of the median $Z_x = \frac{N}{2}$

$$\text{or} \int_0^{\infty} e^{-w^2}\, dw = \int_{w + \frac{1}{2a}}^{\infty} e^{-w^2}\, dw$$

i.e. $w + \frac{1}{2a} = 0$

or $a\,(\log_e x - b) = \frac{1}{2a}$

or $\log_e x = b + \frac{1}{4a^2}$

$\therefore\ x = e^{b + \frac{1}{4a^2}}$

and $x = m\,.\,e^{-\frac{1}{2a^2}}$ ———————————— [18]

Hence the position of the median is given by

$$x = \frac{m}{e^{1/2a^2}}\ ,\ \text{and is obviously}$$

different for all the differently shaped curves having the same mean value.

A sketch illustrating graphically most of the above theoretical deductions is shown in Plate II.

B. Practical Investigations into the Frequency of Occurrence of Gold Values carried out on Mine "X".

In the extraction of information from mine sampling records for the purpose of obtaining observed histograms by which the foregoing mathematical analysis could be tested, certain practical considerations had to be carefully observed. These may briefly be summarised as follows:-

(i) A sufficient number of samples to enable the construction of reliable frequency histograms had to be available. This confined the choice to a large well-developed area.

(ii) It was decided that only development sampling results would be used, not only because stope sampling is universally admitted to be less accurate, but also because the interval between sampling sections is usually larger in the case of stope sampling than for development sampling.

(iii) Development headings had to be selected in such a way that long continuous reef exposures were available. Furthermore, the assurance that such development had been carried out in a completely random manner was essential. Although the desirability of extracting sampling data in the form of a development grid is acknowledged, this ideal was found to be unattainable in practice, because raises are put up in selected areas on practically all the mines on the Witwatersrand. In order to randomise the sampling, therefore, the grid conception had to be dispensed with in favour of sampling results obtained solely from development drives spaced at equal intervals.

Bearing those facts i[n]
was carried out on a fully
future work be referred to
imately rectangular area,
a series of reef drives
on the underground plan,
section in the area was
sampling sections, which
care was exercised to
sections for each drive,
preselected area and
delineated is shown on the
reproduced in Plate III.

The required
sampling records of Mine 'E'
below, which represents
data actually collected-

Locality: 3 South

Sect. No.	Inch-Dwt.	Channel Width (Ins.)
1	446	54
	452	60
	500	7[illegible]
	336	65
	169	45
	140	38
	2090	[illegible]
	234	2[illegible]
	551	4[illegible]
10	1035	57
	726	76
	593	76

Bearing these facts in mind, a complete investigation was carried out on a fully developed mine, which will in all future work be referred to as Mine "E". A large and approximately rectangular area, traversed from side to side by a series of reef drives with a random spacing, was demarcated on the underground plan, and each development drive sampling section in the area was recorded. In recording the values of sampling sections, which are spaced at five feet intervals, care was exercised to ensure a complete succession of sampling sections for each drive, starting at one side of the preselected area and finishing off at the other. The area so delineated is shown on the portion of the underground plan reproduced in Plate III.

The required information was extracted from the sampling records of Mine "E" in the form shown in the table below, which represents part of a specimen sheet of the data actually collected:-

TABLE IV.

Locality: 3 South Drive ex Main Incline.

Sect. No.	Inch-Dwt.	Channel Width (Ins.)	Channel Value (Dwt/Ton)	Stoping Width (Ins.)	Stoping Value (Dwt/Ton)	Remarks
1	446	54	8.6	69	6.8	5' from X-cut
	452	60	7.5	75	6.0	
	500	71	7.0	86	5.8	
	336	66	5.1	81	4.1	
	169	46	3.7	61	2.8	
	140	38	3.7	55	2.5	
	2090	10	209.0	50	41.8	Visible gold
	234	28	8.4	52	4.5	
	551	46	12.0	61	9.0	
10	1035	57	18.2	72	14.4	
	726	76	9.6	91	8.0	
	593	76	7.8	91	6.5	

Stoping widths on Mine 'E' are estimated from channel widths in accordance with a sliding scale which was originally compiled for ore reserve purposes from the results of actual mining experience, and which has subsequently been found to conform within reasonable limits to the results obtained in practice. The stoping widths corresponding to various channel widths are shown below:-

TABLE V.

Channel Width (Inches)	Estimated Stoping Width (Inches)
20 and under	50
21 to 25	51
26 to 32	52
33 to 36	53
37	54
38	55
39	56
40	57
41 and 42	58
43	59
44 and 45	60
46 and over	add 15

The manner in which the recorded information was used to deduce certain practical aspects is shown step by step in the following pages.

PLATE A

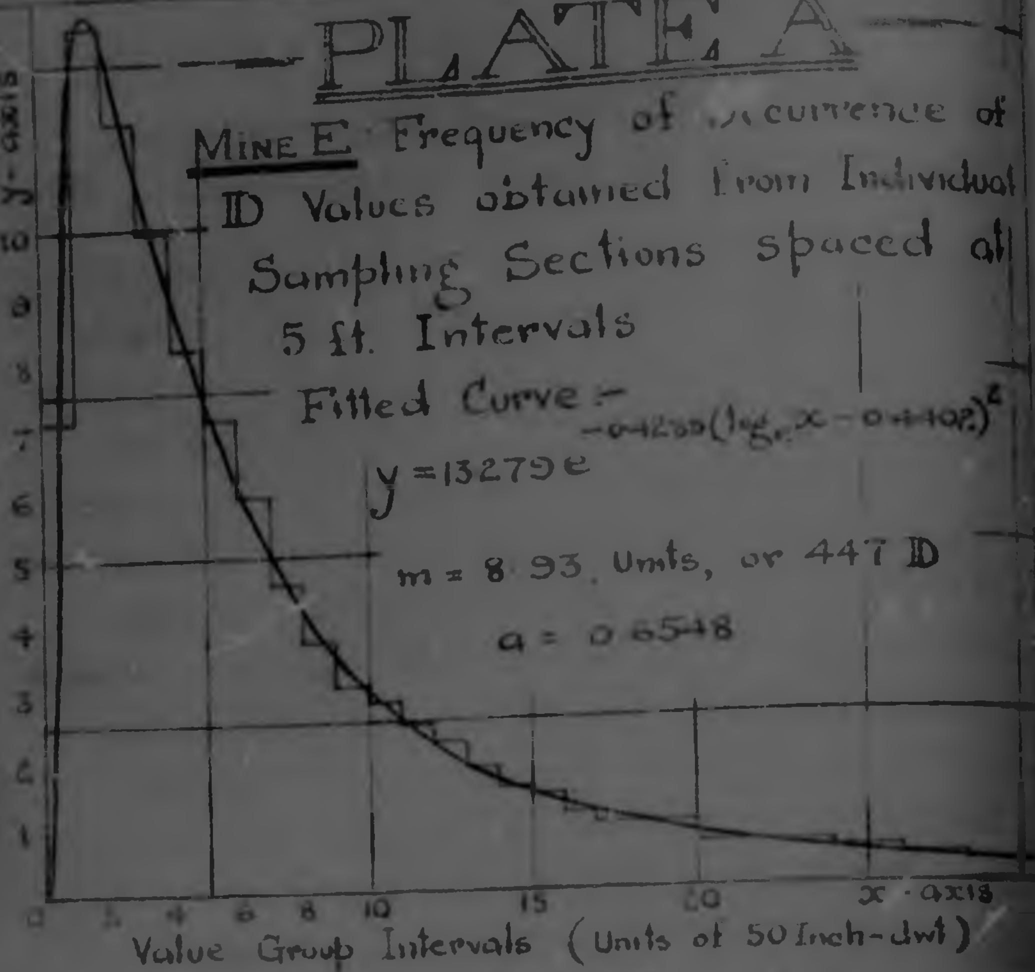


TABLE A

Value Group (Units of 50 ID)	Observed Number	Expected Number	Observed % Freq	Av Observed Value (ID)
0 to 1	348	338	7.06	30
1 " 2	642	639	13.03	75
2 " 3	576	598	11.69	125
3 " 4	489	495	9.92	174
4 " 5	401	399	8.14	225
5 " 6	349	336	7.08	274
6 " 7	288	272	[illegible]	324
7 " 8	225	228	4.57	373
8 " 9	180	189	3.65	424
9 " 10	153	15[illegible]	3.11	471
10 " 11	138	137	2.81	524
11 " 12	121	118	2.46	574
12 " 13	108	102	2.19	626
13 " 14	88	87	1.79	677
14 " 15	74	76	1.50	724
15 " 16	71	66	1.44	773
16 " 17	53	58	1.08	824
17 " 18	45	51	0.91	872
18 " 19	46	47	0.93	921
19 " 20	44	42	[illegible]	973
20 " 22	63	71	1.27	1049
22 " 24	61	61	1.24	[illegible]
24 " 26	57	51	[illegible]	1252
26 " 28	44	46	[illegible]	[illegible]
28 " 30	37	32	[illegible]	[illegible]
30 & over	226	235	[illegible]	[illegible]
Total	4927	4927	100.00	[illegible]

(a) Segregation of ...

Intervals on ...

(i) Distribution of ...

individual sampling ...

The inch-dwt. v...
sections were first ...
this purpose 50 inc...
and the inch-dwt. v...
all, were sorted out...
segregation are sh...
plotted from the ta...
in Plate A, which ...
distribution of ind...
uncut sampling sect...

It was found th...
equation

y = 13.2...

could be ...
histogram, giving ...
seen that this curv...
line in Plate A, ...
where

N = 13.2...

a = 0.6...

and b = 0.4...

The mean valu...

b ...

m = e ...

= 2.77...

= 8.9...

---

*In calculating ...
parameters "a" ...
curve, 50 in...

(a) <u>Segregation of Observations into Selected Value Group Intervals on the Basis of Inch-Dwt. Values.</u>

(i) <u>Distribution of the inch-dwt. values obtained from individual sampling sections spaced at 5 ft. intervals.</u>

The inch-dwt. values obtained from individual sampling sections were first segregated into value groups. For this purpose 50 inch-dwt. class intervals were selected, and the inch-dwt. values of the various sections , 4927 in all, were sorted out accordingly. The results of this segregation are shown in Table A on Plate A. The histogram plotted from the table is denoted by the step-diagram in Plate A, which therefore represents the <u>observed</u> distribution of inch-dwt. values as determined from unewt sampling sections spaced at 5 ft. intervals.

It was found that a curve represented by the equation

$$y = 13.279\, e^{-0.4288\,(\log_e x - 0.4402)^2}$$

could be run through the tops of the observed histogram, giving a very satisfactory "fit". It will be seen that this curve, which is denoted by the full red line in Plate A , is of the type previously referred to, where

$$M = 13.279$$

$$a = 0.6548$$

$$\text{and } b = 0.4402 .$$

The mean value deduced from the "fitted" curve is

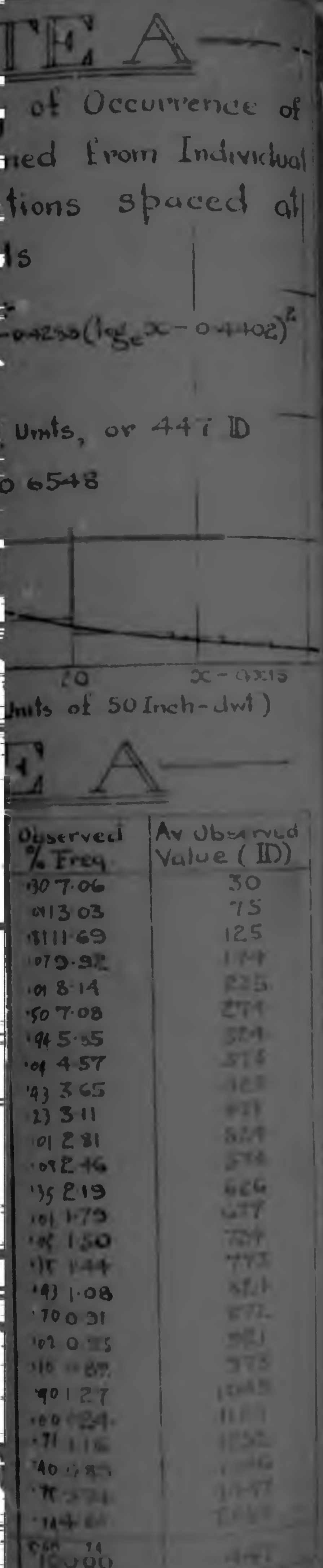
$$m = e^{b + \frac{1}{4a^2}} \qquad \text{[from equation (8)]}$$

$$= 2.7183^{0.4402 + \left(\frac{1}{4 \times 0.4288}\right)}$$

$$= 8.93 \text{ units, or } 447 \text{ inch-dwt.}^{*}$$

---

* In calculating the value of the constant M and the parameters "a" and "b", and in the plotting of the curve, 50 inch-dwt. has been taken as a unit.



The modal value of the distribution is

$$x = e^{b} = 1.553 \text{ units, or } 78 \text{ inch-dwt.}$$

The various aspects of the distribution of uncut inch-dwt. values, as deduced from individual sampling sections spaced at 5 ft. intervals, may now be fully described with the aid of the mathematical expression above.

If successive individual sampling sections be next combined into groups each containing an equal number of sections, a series of averages, each representing the mean of a certain length of development stretch, will be obtained. The averages so calculated will themselves be distributed in a certain manner. This concept of combining individual values into averages over certain stretch lengths is by no means new, and is in fact universally adopted in the compilation of ore reserve plans. For this purpose, the individual sampling sections, which are usually spaced at 5 ft. intervals in development headings on all mines, are combined into groups of ten, the averages so obtained being noted on the assay plans. The reason for averaging out the results over 50 ft. stretch lengths is rather obscure, its almost universal adoption for assay plan work apparently being that this stretch represents the shortest length which can be shown with the requisite degree of clarity on a plan drawn to the scale which has been found to be eminently suited to the purpose of blocking out ore reserves.

In order to obtain a clear conception of the distribution of averages obtained by combining individual sampling sections into groups representing

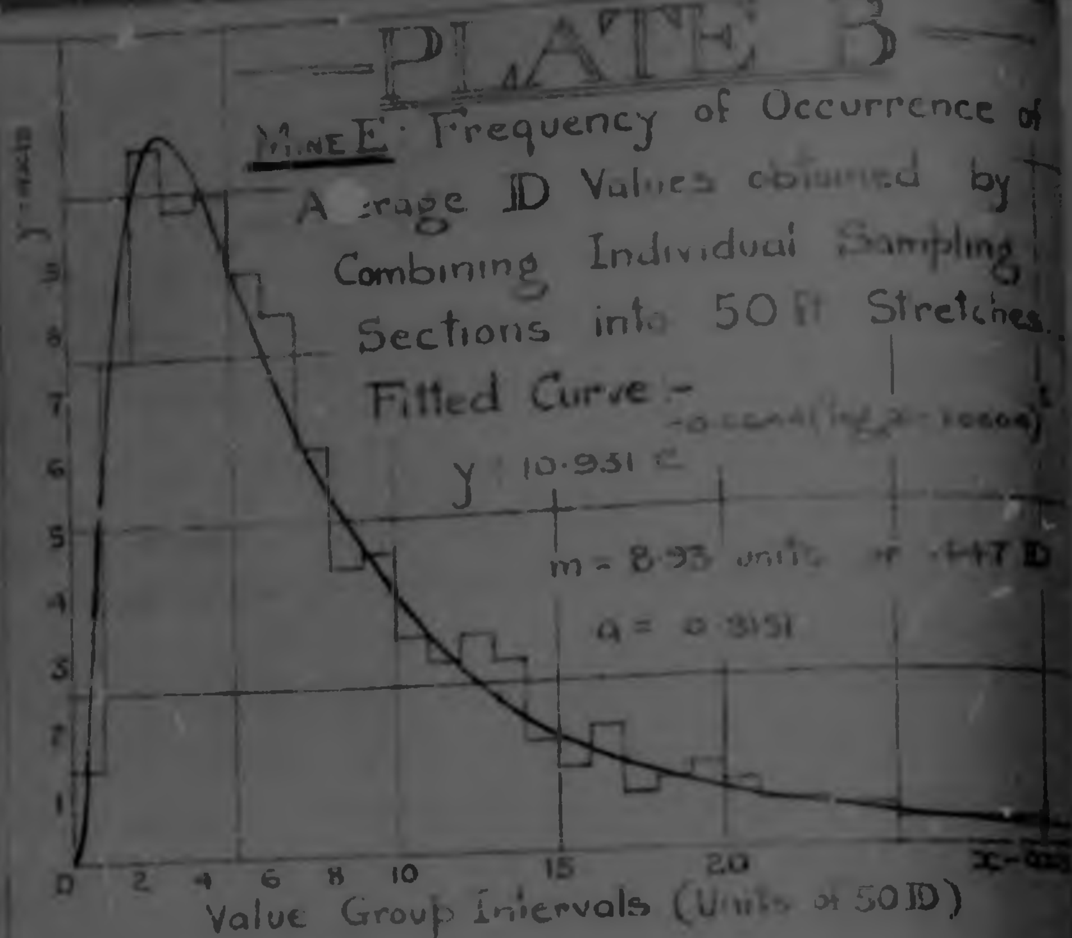


TABLE B

Value Group (Units of 50 ID)	Observed Number	Expected Number	Observed % Freq.	Av. Observed Value (ID)
0 to 1	7	9	1·42	36
1 " 2	37	30	7·50	74
2 " 3	53	52	10·77	125
3 " 4	48	52	9·76	173
4 " 5	49	47	9·96	225
5 " 6	43	41	8·74	275
6 " 7	40	35	8·13	324
7 " 8	30	29	6·10	372
8 " 9	21	25	4·27	425
9 " 10	22	21	4·47	472
10 " 11	16	18	3·25	524
11 " 12	14	15	2·85	573
12 " 13	16	13	3·25	625
13 " 14	14	11	2·85	673
14 " 15	8	9·6	1·63	723
15 " 16	6	8·4	1·22	771
16 " 17	9	7·4	1·83	822
17 " 18	4	6·3	0·81	872
18 " 19	5	5·6	1·02	921
19 " 20	6	4·9	1·22	974
20 " 21	5	4·4	1·02	1022
21 " 23	7	6·9	1·42	1097
23 " 25	6	5·5	1·22	1232
25 " 30	10	8·8	2·03	1373
30 & over	16	14·2	3·25	2211
Total	492	492·0	100·00	[illegible]

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different stretch lengths, the inch-dwt values of the individual sampling sections for Mine "E" were combined into groups of 10, 20, 40 and 60, representing the averages over 50, 100, 200 and 300 feet stretch lengths respectively. Each of these cases will be considered in detail below.

(ii) **Distribution of the Average Inch-dwt values obtained by combining Individual Sampling Sections into 50 ft. Stretch Lengths.**

The results of the segregation of the average values obtained by combining individual sampling sections into groups of ten, each group representing the average value of a 50 ft. stretch length of development, are shown in Table B on Plate B. The observed histogram plotted from the table is represented by the step-diagram in Plate B.

It will be noted that the class intervals have been progressively increased when dealing with the higher values, where the frequencies of occurrence are relatively small. This procedure is perfectly legitimate, and is practised in many other branches of statistical research where relatively low frequencies are associated with certain class intervals.

The continuous red line in Plate B shows the "fitted" curve represented by the distribution law

$$y = 10{\cdot}931\ e^{-0{\cdot}6644\,(\log_e x\ -\ 1{\cdot}0604)^2}$$

This equation is again a form of the fundamental law, in which

$$N = 10{\cdot}931$$
$$a = 0{\cdot}8151$$
$$\text{and} \quad b = 1{\cdot}0604$$

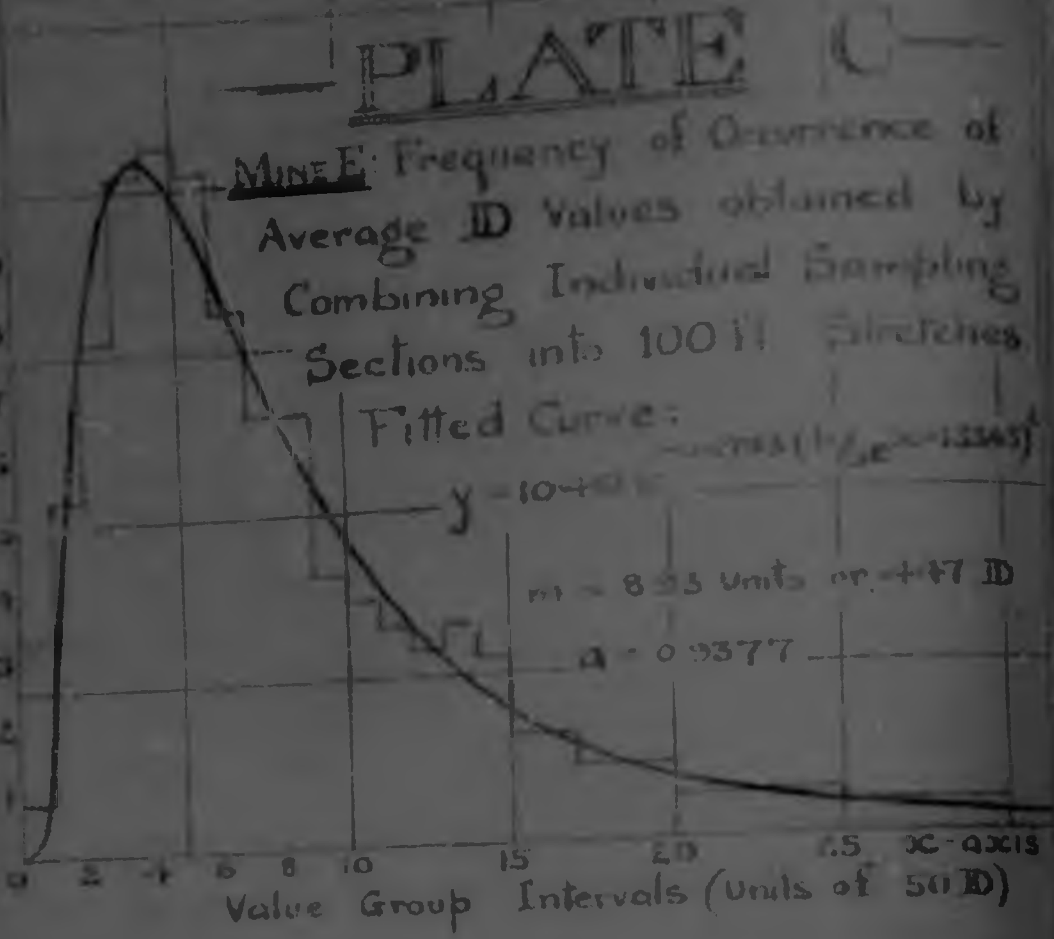

TABLE C

Value Group (Units of 50 D)	Observed Number	Expected Number	Observed % Freq.	Av. Observed Value (D)
0 to 1	2	1·4	0·81	58
1 - 2	13	12·3	5·28	72
2 - 3	19	26·9	7·72	123
3 - 4	25	29·5	10·16	171
4 - 5	26	25·9	10·57	224
5 - 6	25	23·9	10·16	272
6 - 7	20	20·2	8·13	320
7 - 8	16	17·2	6·50	[illegible]
8 - 9	16	14·6	6·50	[illegible]
9 - 10	10	12·5	4·07	[illegible]
10 - 11	9	10·5	3·65	[illegible]
11 - 12	8	8·9	3·25	[illegible]
12 - 13	7	7·5	2·87	[illegible]
13 - 14	3	[illegible]	[illegible]	677
14 - 15	7	[illegible]	2·84	[illegible]
15 - 17	8	8·5	3·26	[illegible]
17 - 20	8	9·0	3·26	[illegible]
20 - 25	6	7·3	[illegible]	[illegible]
25 - 30	6	[illegible]	2·44	[illegible]
30 & over	5	4·2	[illegible]	[illegible]
Total	246	246·0	100·00	[illegible]

The mean value of the distribution is defined by equation [8] as

$$m = e^{b + \frac{1}{4a^2}}$$

$$= 2{\cdot}7183^{1{\cdot}0604 + \frac{1}{4 \times 0{\cdot}6644}}$$

$$= 8{\cdot}93 \text{ units, or } 447 \text{ inch-dwt.}$$

This agrees, as it should, with the mean value of 447 inch-dwt obtained from the distribution of the original individual sampling section values.

The position of the mode is given by equation [5] as

$$x = e^{b} = 2{\cdot}7183^{1{\cdot}0604}$$

$$\text{or } x = 2{\cdot}887 \text{ units, or } 144 \text{ inch-dwt.}$$

(iii) <u>Distribution of the Average Inch-dwt Values obtained by combining Individual Sampling Sections into 100 ft. Stretch Lengths.</u>

The results of the segregation of average values obtained by combining individual sampling sections into groups of 20, each representing the average of a 100 ft. stretch length of development, are shown in Table C on Plate C. The observed histogram plotted from this table is denoted by the step-diagram in Plate C. The continuous red line represents the "fitted" curve plotted from the equation

$$y = 10{\cdot}461\, e^{-0{\cdot}8793\,(\log_e x - 1{\cdot}3365)^2}$$

It will be observed that the fundamental law is again applicable, the constant "N" and the parameters "a" and "b" being

$$N = 10{\cdot}461$$

$$a = 0{\cdot}9377$$

$$\text{and } b = 1{\cdot}3365$$

The mean value of the distribution is, from

PLATE D

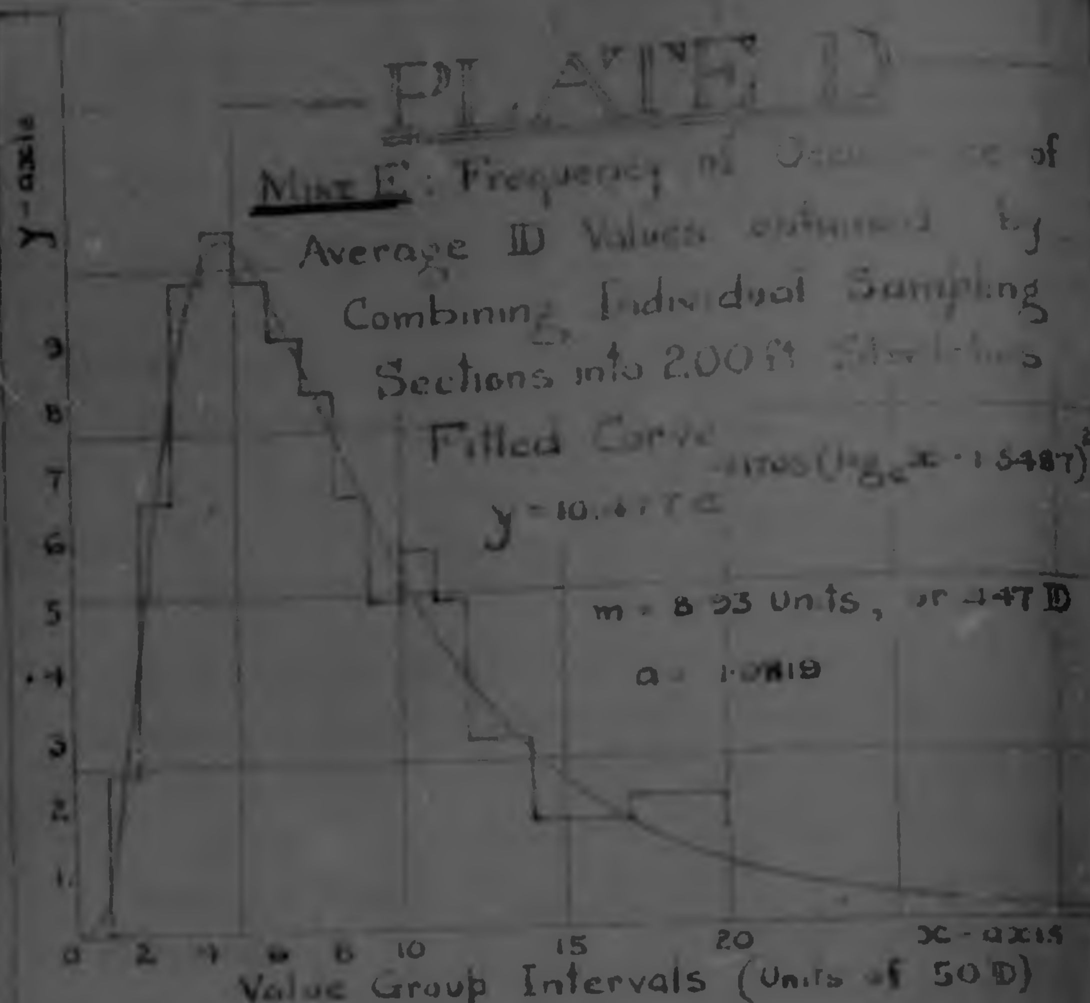

TABLE D

Value Group (Units of 50 D)	Observed Number	Expected Number	Observed % Freq	Av. Observed Value (D)
0 to 1	0	0.2	0.00	—
1 " 2	3	3.0	2.44	78
2 " 3	8	7.9	6.50	135
3 " 4	12	11.6	9.76	174
4 " 5	13	12.7	10.57	222
5 " 6	12	12.5	9.76	270
6 " 7	11	11.4	8.94	321
7 " 8	10	10.0	8.13	371
8 " 9	8	8.5	6.50	421
9 " 10	6	7.3	4.88	472
10 " 11	7	6.0	5.69	520
11 " 12	6	5.1	4.88	574
12 " 14	7	7.7	5.69	642
14 " 17	6	7.4	4.88	772
17 " 20	7	5.4	5.69	923
20 & over	7	6.3	5.69	1242
Total	123	123.0	100.00	447

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$$m = e^{b + \frac{1}{4a^2}}$$

$$= 2{\cdot}7183^{\,1{\cdot}3365 + \frac{1}{4 \times 0{\cdot}8795}}$$

$$= 8{\cdot}93 \text{ units, or 447 inch-dwt, as before.}$$

The modal value of the frequency distribution is given by

$$x = e^{b} = 2{\cdot}7183^{\,1{\cdot}3365}$$

$$= 3{\cdot}805 \text{ units, or 190 inch-dwt.}$$

(iv) <u>Distribution of the Average Inch-dwt Values obtained by combining Individual Sampling Sections into 200 ft. Stretch Lengths.</u>

The results of the segregation of the average values obtained by combining the original individual sampling sections into groups of 40, each representing the average of a 200 ft. development stretch, are shown in Table D on Plate D. The observed histogram has again been denoted by the step-diagram in Plate D, while the continuous red line shows the "fitted" curve represented by the equation

$$y = 10{\cdot}477\, e^{-1{\cdot}1705\,(\log_e x - 1{\cdot}5487)^2}$$

As in all the foregoing cases, the Fundamental Distribution Law is again observed, the constants in this case being

$$M = 10{\cdot}477$$

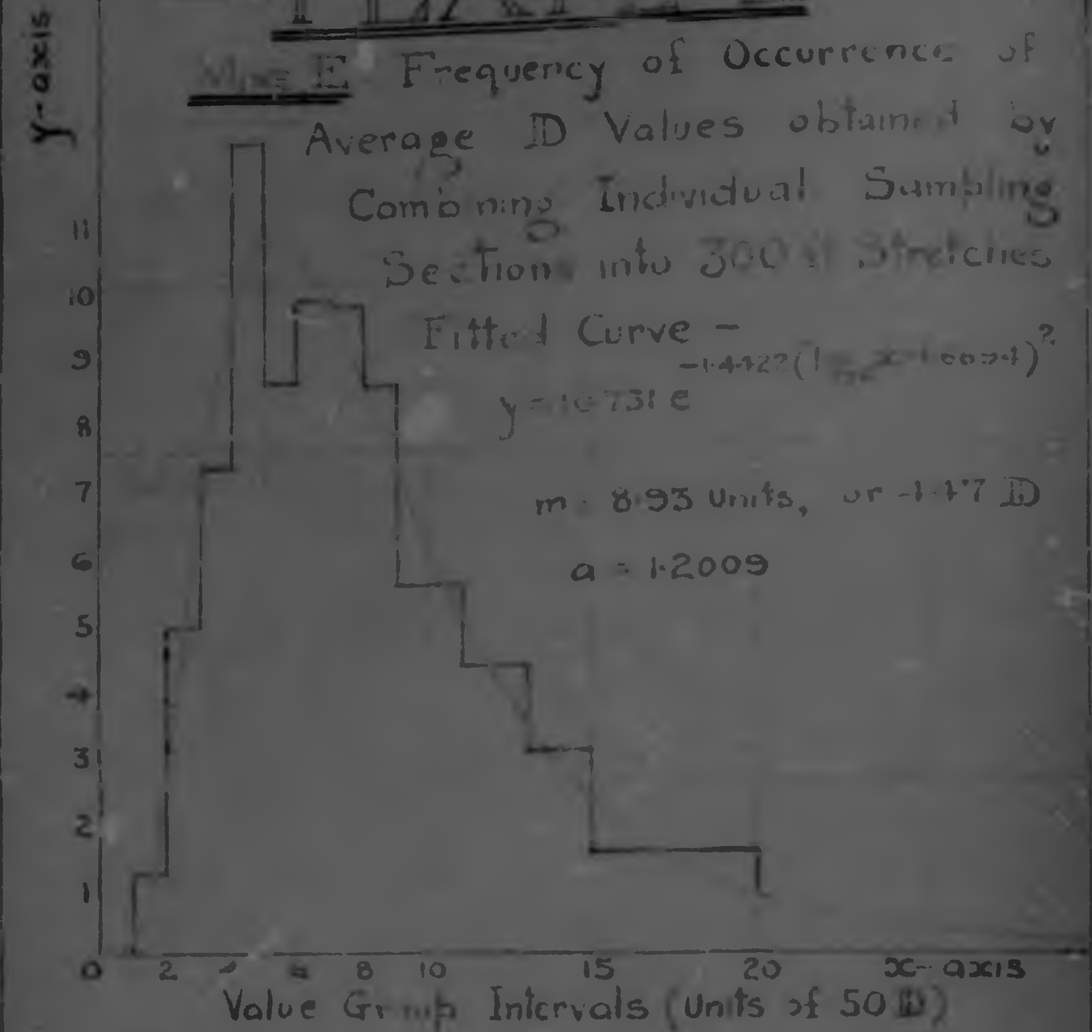
$$a = 1{\cdot}0819$$

$$\text{and } b = 1{\cdot}5487$$

The mean value of the distribution as calculated from equation [8] is

$$m = e^{b + \frac{1}{4a^2}}$$

PLATE E



Map E Frequency of Occurrence of Average D Values obtained by Combining Individual Sampling Sections into 300 ft Stretches

TABLE E

Value Group (Units of 50 D)	Observed Number	Expected Number	Observed % Freq	Av. Observed Value (D)
0 to 1	0	0·1	0·00	--
1 " 2	1	0·9	1·21	62
2 " 3	4	3·8	4·87	13[illegible]
3 " 4	6	6·8	7·32	175
4 " 5	10	[illegible]	12·20	222
5 " 6	7	6·7	8·55	273
6 " 7	8	8·3	9·76	32[illegible]
7 " 8	8	7·5	9·75	375
8 " 9	7	[illegible]	[illegible]	4[illegible]
9 " 11	9	10·1	10·98	494
11 " 13	7	[illegible]	8·54	590
13 " 15	5	[illegible]	6·09	698
15 " 20	6	5·7	7·31	826
20 & over	4	3·6	4·88	1501
Total	82	82·0	100·00	445

or $\quad x = 2{\cdot}7183^{\,1{\cdot}5487 + \frac{3}{4\times1{\cdot}1705}}$

$= 8{\cdot}93$ units, or 447 inch-dwt.

The modal value of the frequency distribution, according to equation [5], will be

$$x = e^{b} = 2{\cdot}7183^{\,1{\cdot}5487}$$

$= 4{\cdot}705$ units, or 235 inch-dwt.

(v) <u>Distribution of the Average Inch-dwt Values obtained by combining Individual Sampling Sections into 300 ft. Stretch Lengths.</u>

The results of the segregation of the average values obtained by combining the original individual sampling sections into groups of 60, each representing the average value of a 300 ft. development stretch length, are shown in Table E on Plate E. The observed histogram plotted from this table is denoted by the step-diagram in Plate E, the continuous red line representing the "fitted" curve being calculated from the equation

$$y = 10{\cdot}731\, e^{-1{\cdot}4422\,(\log_e x \,-\, 1{\cdot}6694)^2}$$

This is again a form of the fundamental distribution law, the constants in this case being

$N = 10{\cdot}731$

$a = 1{\cdot}2069$

and $b = 1{\cdot}6694$.

The mean value of the distribution is

$$x = e^{\,b + \frac{1}{4a^2}}$$

$$= 2{\cdot}7183^{\,1{\cdot}6694 + \frac{3}{4\times1{\cdot}4422}}$$

$= 8{\cdot}93$ units, or 447 inch-dwt.

The modal value of the frequency distribution is given by

$$x = e^{b} = 2{\cdot}7183^{\,1{\cdot}6694}$$

$= 5{\cdot}309$ units, or 265 inch-dwt.

It will be observed that whereas originally 4927 sampling sections were available for segregation, this comparatively large number was successively reduced to 493 for the 50 ft. stretches, 246 for the 100 ft. stretches, 123 for the 200 ft. stretches, and finally to only 82 for the 300 ft. development stretches. Further combination into longer stretches would obviously result in too few averages being available for segregation, and any results so obtained would be too unreliable to merit serious consideration from the value distribution point of view.

(b) Segregation of Observed Values into Selected Value Group Intervals on the Basis of Estimated Stoping Values expressed in Dwt. per Ton.

All the mathematical analysis at the beginning of this chapter was developed on general lines, and no reference was made to the units in which the values are measured. The analysis for Mine "E" has so far been based solely on the segregation of inch-dwt. values. While this is apparently the only basically correct unit to employ, certain practical difficulties are sometimes experienced when dealing with inch-dwt. values. For the purpose of this treatise it has consequently been found far more convenient to base the various segregations on estimated stoping values expressed in dwt. per ton, primarily because it is these values which govern the estimation of the quantity and the quality of the profitable ore which can reasonably be expected to exist within a given mining area. As stoping widths are usually assessed in accordance with a predetermined scale, these can be, and indeed are applied in a perfectly unbiased manner when estimating stoping values.

The table below shows a segregation of the estimated stoping values for Mine "E", which have been deduced from the corresponding inch-dwt values of individual sampling sections by the application of the scale of widths shown in Table V:-

TABLE VI.

Value Group (Dwt/Ton)	Observed Number of Occurrences	Average Stoping Width (Inches)	Average Estimated Stoping Value (Dwt/Ton)
0.0 - 0.9	464	62.0	0.60
1.0 - 1.9	777	61.2	1.45
2.0 - 2.9	[illegible]	62.2	2.43
3.0 - 3.9	506	62.9	3.45
4.0 - 4.9	487	61.7	4.44
5.0 - 5.9	337	[illegible]	5.43
6.0 - 6.9	255	60.5	6.42
7.0 - 7.9	202	60.7	7.42
8.0 - 8.9	165	60.8	8.54
9.0 - 9.9	145	65.4	9.40
10.0 - 10.9	123	59.8	10.45
11.0 - 11.9	99	61.7	11.40
12.0 - 12.9	87	61.0	12.43
13.0 - 13.9	78	59.0	13.49
14.0 - 14.9	58	58.3	14.48
15.0 - 15.9	60	60.0	15.38
16.0 - 16.9	52	64.2	16.46
17.0 - 17.9	37	59.5	17.50
18.0 - 18.9	37	59.7	18.40
19.0 - 19.9	29	63.6	19.48
20.0 - 21.9	59	61.5	21.04
22.0 - 23.9	47	62.2	22.81
24.0 - 25.9	35	59.8	24.86
26.0 - 27.9	27	61.9	26.74
28.0 - 29.9	25	59.8	28.95
30.0 & Over	151	62.5	50.46
Totals & Averages	4987	61.65	7.23

It will be seen from this table that the average stoping widths are reasonably constant throughout the whole range of value groups, and the justifiability of entirely ignoring the variable width factor on the observed frequencies of occurrence in this case will readily be appreciated.

Should the average stoping widths reveal a tendency towards variation throughout the range of value categories, however, an adjustment of the observed frequencies of occurrence will be found necessary. Although comparatively rare, this condition has been found to exist on some mines. The table below illustrates an actually observed case in which the estimated stoping widths show a distinct diminishing tendency in going from the lower to the upper value groups:-

TABLE VIII.

Value Groups (Dwt./Ton)	Observed Number of Occurrences	Average Estimated Stoping Width (inches)	Adjusted Number of Occurrences	Average Estimated Stoping Value (Dwt./Ton)
0.0 - 0.9	403	58.7	402	0.62
1.0 - 1.9	1490	59.2	1498	1.53
2.0 - 2.9	1366	56.8	1391	2.45
3.0 - 3.9	1100	60.2	1125	3.45
4.0 - 4.9	839	59.1	842	4.44
5.0 - 5.9	537	60.0	547	5.42
6.0 - 6.9	415	60.3	425	6.42
7.0 - 7.9	306	58.4	304	7.41
8.0 - 8.9	216	58.0	213	8.40
9.0 - 9.9	151	56.1	144	9.43
10.0 - 10.9	114	55.1	107	10.30
11.0 - 11.9	110	52.9	99	11.44
12.0 - 12.9	65	53.3	59	12.45
13.0 - 13.9	51	52.3	45	13.43
14.0 - 14.9	37	50.7	32	14.38
15.0 & over	182	48.2	149	20.65
Totals & Averages	7382	58.84	7382	4.30

In the above table the "adjusted number of occurrences" has been arrived at by multiplying the corresponding "observed number of occurrences" by the ratio of the average estimated stoping width of all the observations falling within the given value group to the average estimated stoping width of all the observations in the distribution. Justification for this adjustment method is to be found in the following argument:-

In the case of values expressed in inch-dwt, the value of each sampling section in an observed distribution will be representative of a certain area. If the sections are spaced at equal intervals, as is invariably the case, then each section value will be representative of the same area. The fallacy of assigning different weights to sampling section values in accordance with their frequencies of occurrence was pointed out by Sichel*, who stated that since the probability of striking a high value is so much less than that of encountering a low value, the high value, when struck, should have its effect extended over the full area of influence to make up for all the small high value patches which were missed. This argument is perfectly logical, and undoubtedly holds without qualification when dealing with values expressed in inch-dwt. When values are expressed in dwt per ton, however, it must be borne in mind that the observed values obtained from the various sampling sections may occur over different widths. Taking this factor into account, each sampling value may be considered as occurring over a certain <u>volume</u> of influence, made up of a <u>constant</u>

---

*Herbert S. Sichel, B.Sc., in a Paper issued on February 13th, 1947, by the Institution of Mining and Metallurgy, London, entitled "An Experimental and Theoretical Investigation of Bias Error in Mine Sampling with special reference to Narrow Gold Reefs".

area of influence (in the case of sampling sections spaced at equal intervals), multiplied by a variable width. Obviously, if there is no variation in the average width of the sampling sections in the various value groups, then the volume of influence of all sampling sections will be the same, and it will consequently be possible to ignore the effect of the width in arriving at the correct weight to be assigned to any particular sampling section.

If the widths of the different sampling sections vary, however, their effect will have to be allowed for. It has been found more convenient to express the volume of influence in terms of a constant width (the average width of all the sampling sections), multiplied by a variable area of influence. The necessary adjustment can be made to each sampling section by multiplying the original area of influence by the ratio of the width for the particular section under consideration to the average width of all the observations.

Having expressed the weight to be assigned to each individual sampling section in terms of a certain area of influence at a constant width, it is a relatively simple matter to combine these sections into groups. Thus the area of influence of the total observations falling within a certain value group will be represented by the sum of the areas of influence of all the individual sampling sections falling within that value group, at the same average width. It will readily be appreciated that this total area can be expressed in terms of the original constant area of influence at a constant width by suitably adjusting the observed number of values falling within that group. The adjustment is made simply by multiplying the observed number of values falling within a given value group, by the ratio of the average width of the sampling sections contained in that particular group

PLATE F

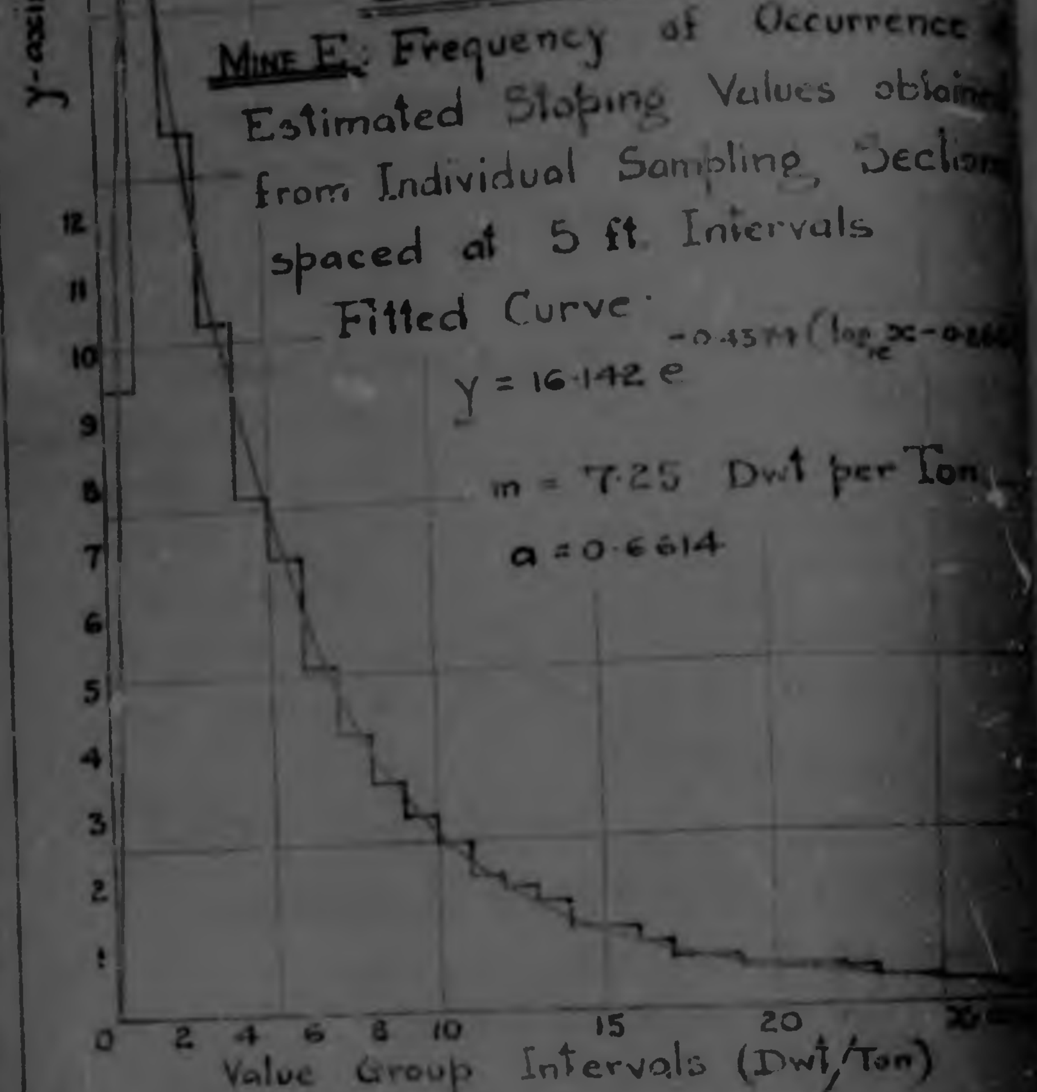


TABLE F

Value Group (Dwt/Ton)	Observed Number	Expected Number	Observed % Freq.	Av. Obs. Value
0 to 1	464	[illegible]48	[illegible]	[illegible]
1 " 2	777	761	[illegible]	[illegible]
2 " 3	650	662	[illegible]	[illegible]
3 " 4	506	524	10·27	[illegible]
4 " 5	427	415	8·67	[illegible]
5 " 6	337	321	6·84	[illegible]
6 " 7	253	262	5·13	[illegible]
7 " 8	202	215	4·10	[illegible]
8 " 9	165	178	3·35	[illegible]
9 " 10	145	140	2·94	[illegible]
10 " 11	123	110	2·50	[illegible]
11 " 12	99	99	2·01	[illegible]
12 " 13	87	84	1·77	[illegible]
13 " 14	78	72	1·58	[illegible]
14 " 15	58	62	1·18	[illegible]
15 " 16	60	54	1·22	[illegible]
16 " 17	51	47	1·04	[illegible]
17 " 18	37	40	[illegible]	[illegible]
18 " 19	37	35	0·75	[illegible]
19 " 20	29	32	0·59	[illegible]
20 " 22	59	55	1·20	[illegible]
22 " 24	47	42	0·96	[illegible]
24 " 26	35	32	0·71	[illegible]
26 " 28	27	26	0·55	[illegible]
28 " 30	23	22	[illegible]	[illegible]
30 & Over	151	15[illegible]	[illegible]	[illegible]
Total	4927	4[illegible]27	100·00	

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This has been done in the table above, and has resulted in an adjusted frequency of occurrence of stoping values which, it may be stated without illustration, is "fitted" very satisfactorily by the fundamental law.

(1) <u>Distribution of Estimated Stoping Values obtained from Individual Sampling Sections spaced at 5 ft. Intervals.</u>

Table F on Plate F shows the results of the segregation of estimated stoping values expressed in dwt per ton, into value groups of 1 dwt per ton. This segregation is graphically represented by the histogram in Plate F. The continuous red line represents the "fitted" curve denoted by the equation

$$y = 16.142\, e^{-0.4374\,(\log_e x - 0.2664)^2}$$

This is once again a form of the fundamental distribution law, in which

$$M = 16.142$$
$$a = 0.6614$$
$$\text{and} \quad b = 0.2634$$

The mean value of the distribution is given by equation [8] as

$$\bar{x} = e^{0.2664 + \frac{1}{4 \times 0.4374}}$$
$$= 7.25 \text{ dwt per ton}$$

The position of the mode as calculated from the equation $x = e^b$ is

$$x = 2.7183^{0.2664}$$
$$= 1.305 \text{ dwt per ton.}$$

PLATE G

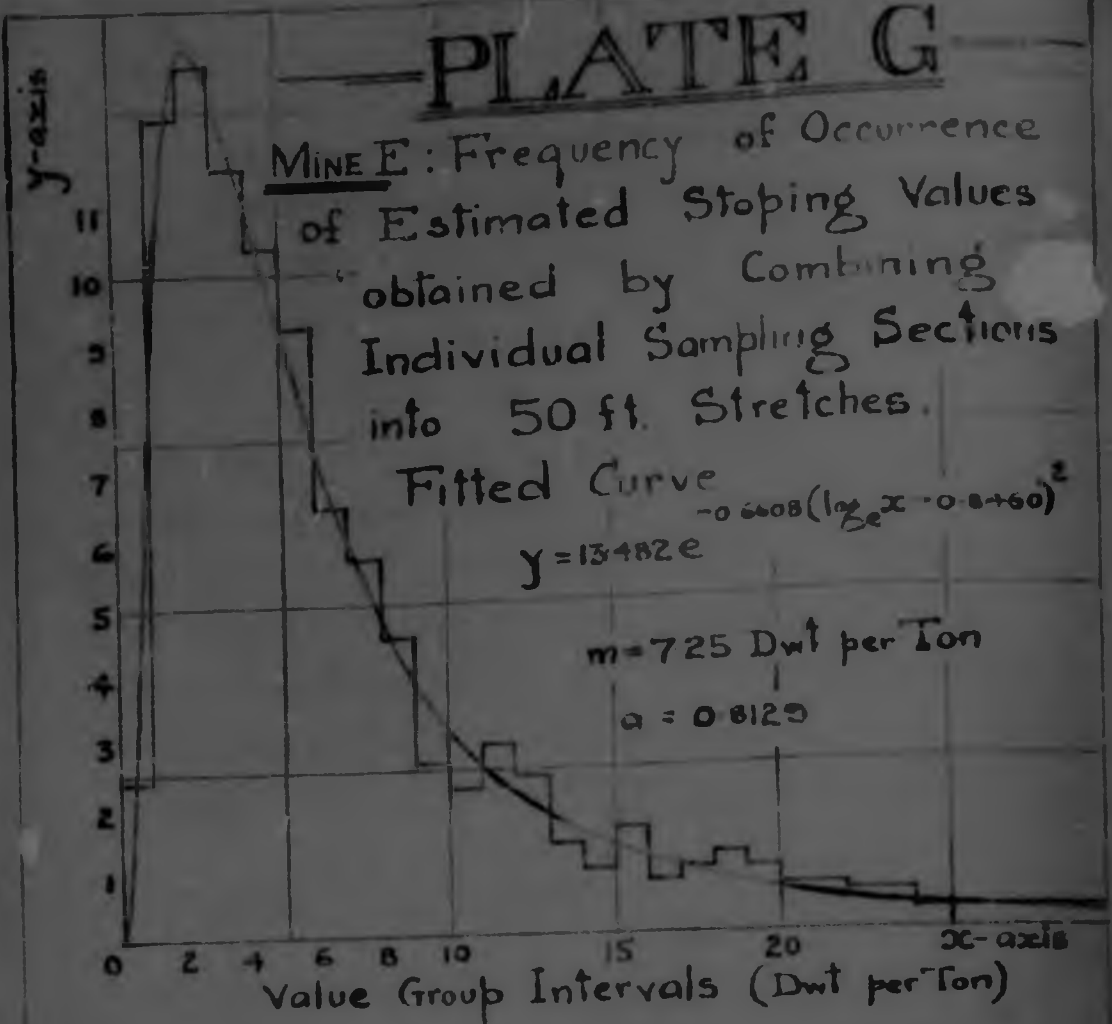


TABLE G

Value Group (Dwt/Ton)	Observed Number	Expected Number	Observed % Freq.	Av. Observed Value (Dwt/Ton)
0 to 1	12	16	2.44	.62
1 " 2	61	57	12.40	1.44
2 " 3	65	65	13.21	2.48
3 " 4	57	60	11.59	3.45
4 " 5	51	51	10.36	4.45
5 " 6	45	41	9.15	5.45
6 " 7	32	33	6.50	6.42
7 " 8	28	27	5.69	7.48
8 " 9	22	21	4.47	8.51
9 " 10	13	17	2.64	9.43
10 " 11	11	14	2.24	[illegible]
11 " 12	14	12	2.85	11.43
12 " 13	12	10	2.44	12.42
13 " 14	7	8.4	1.42	13.50
14 " 15	5	7.2	1.02	[illegible]
15 " 16	8	6.3	1.63	15.55
16 " 17	4	5.5	0.81	16.51
17 " 18	5	4.6	1.02	17.31
18 " 19	6	3.9	1.22	18.48
19 " 20	5	3.4	1.02	19.40
20 " 22	7	5.9	1.42	20.90
22 " 25	6	6.7	1.22	[illegible]
25 " 30	7	6.9	1.42	27.52
30 & Over	9	9.5	1.82	[illegible]
Total	492	492.0	100.00	7.27

(ii) <u>Distribution of the Average Estimated Stoping Values, expressed in Dwt per Ton, obtained by combining Individual Sampling Sections into 50 ft. Stretch Lengths.</u>

The results of the segregation of average estimated stoping values, expressed in dwt per ton, obtained by combining individual sampling sections into groups of ten, each group representing the average value of a 50 ft. stretch length of development, are shown in Table G on Plate G. The observed histogram plotted from the table is represented by the step-diagram in Plate G. The continuous red line shows the "fitted" curve calculated from the equation

$$y = 13{\cdot}482\ e^{-\,0{\cdot}6608\,(\log_e x\ -\ 0{\cdot}8460)^2}$$

This equation is a form of the fundamental law, in which

$$N = 13{\cdot}482$$
$$a = 0{\cdot}8129$$
and $$b = 0{\cdot}8460$$

The mean value of the distribution, as defined by equation [β], is

$$m = 2{\cdot}7183^{\,0{\cdot}8460\ +\ \frac{1}{4 \times 0{\cdot}6608}}$$
$$= 7{\cdot}25 \text{ dwt per ton, as before.}$$

The modal value of the frequency distribution, from equation [β], is

$$x = 2{\cdot}7183^{\,0{\cdot}8460}$$
$$= 2{\cdot}330 \text{ dwt per ton.}$$

PLATE H

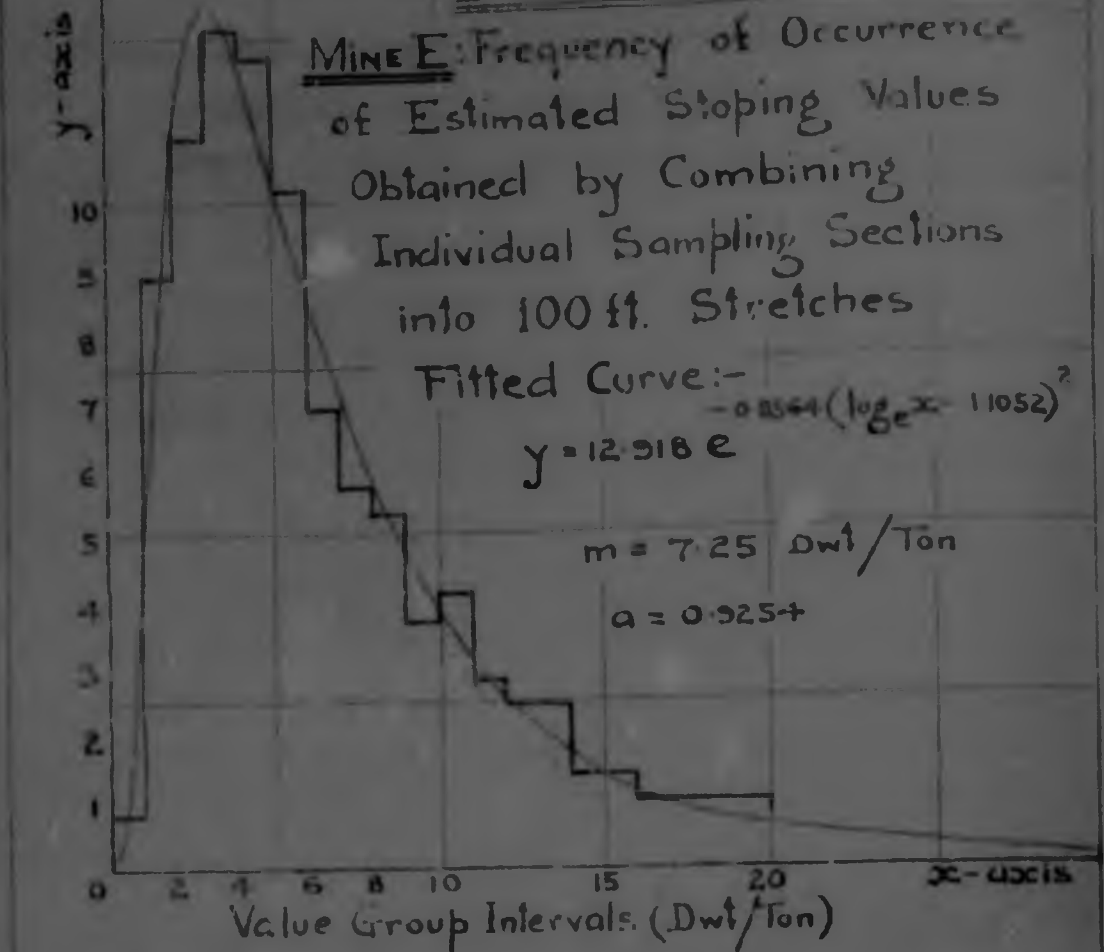


TABLE H

Value Group (Dwt/Ton)	Observed Number	Expected Number	Observed % Freq.	Av. Observed Value (Dwt/Ton)
0 to 1	2	3.3	0.81	0.70
1 " 2	22	20.4	8.94	1.60
2 " 3	27	30.1	10.98	2.62
3 " 4	31	30.6	12.61	3.38
4 " 5	30	27.4	12.19	4.42
5 " 6	25	23.0	10.16	5.46
6 " 7	17	19.1	6.91	6.40
7 " 8	14	15.8	5.69	7.41
8 " 9	13	12.8	5.28	8.45
9 " 10	9	10.4	3.66	9.43
10 " 11	10	8.5	4.07	10.46
11 " 12	7	7.0	2.84	11.40
12 " 13	6	5.7	2.44	12.39
13 " 14	6	4.7	2.44	13.50
14 " 16	7	7.2	2.85	14.97
16 " 20	10	8.5	4.07	17.94
20 & over	10	11.5	4.00	28.53
Total	246	246.0	100.00	7.32

(iii) Distribution of the Average Estimated Stoping Values, expressed in Dwt per Ton, obtained by combining Individual Sampling Sections into 100 ft. Stretch Lengths.

The results of the segregation of the average estimated stoping values, expressed in dwt per ton, obtained by combining individual sampling sections into groups of 20, each group representing the average value of a 100 ft. stretch of development, are shown in Table H in Plate H. The observed histogram plotted from this table is denoted by the step-diagram in Plate H, while the continuous red line represents the "fitted" frequency curve denoted by the equation

$$y = 12{\cdot}918\ e^{-0{\cdot}8564\ (\log_e x\ -\ 1{\cdot}1092)^2}$$

This equation, as in all the foregoing cases, is a form of the Fundamental Distribution Law, in which

$$N = 12{\cdot}918$$
$$a = 0{\cdot}9254$$
$$\text{and}\quad b = 1{\cdot}1092$$

The mean value of the distribution, according to equation [8], is given by

$$m = 2{\cdot}7183^{\,1{\cdot}1092\ +\ \frac{1}{4\times 0{\cdot}8564}}$$
$$= 7{\cdot}25 \text{ dwt per ton, as before.}$$

The modal value of the frequency curve is

$$x = e^{b}$$
$$= 2{\cdot}7183^{\,1{\cdot}1092}$$
$$= 3{\cdot}029 \text{ dwt per ton.}$$

PLATE J

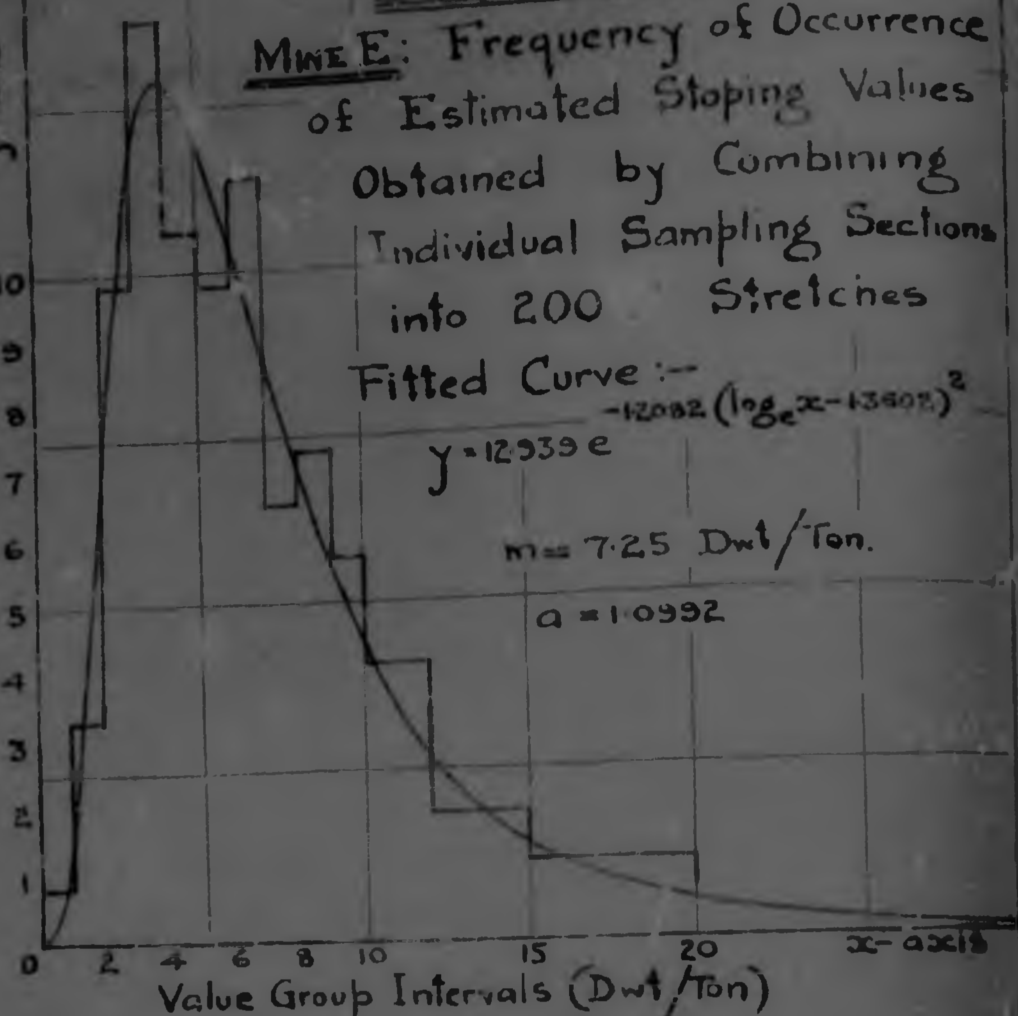
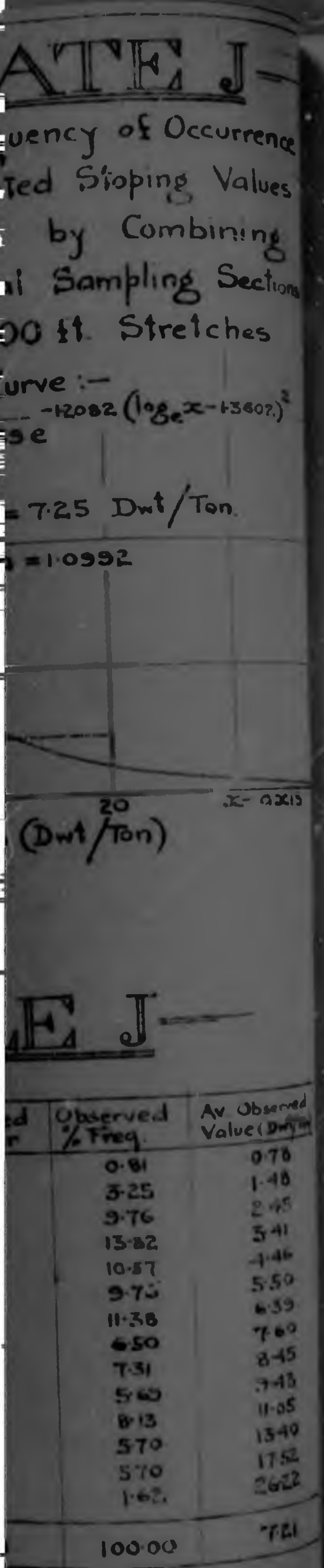


TABLE J

Value Group (Dwt/Ton)	Observed Number	Expected Number	Observed % Freq.	Av. Observed Value (Dwt/Ton)
0 to 1	1	0·4	0·81	0·78
1 " 2	4	4·8	3·25	1·48
2 " 3	12	12·0	9·76	2·45
3 " 4	17	15·5	13·82	3·41
4 " 5	13	15·4	10·57	4·46
5 " 6	12	14·0	9·76	5·50
6 " 7	14	11·6	11·38	6·39
7 " 8	8	9·5	6·50	7·60
8 " 9	9	7·7	7·31	8·45
9 " 10	7	6·2	5·69	9·43
10 " 12	10	8·9	8·13	11·05
12 " 15	7	7·7	5·70	13·40
15 " 20	7	5·6	5·70	17·52
20 & Over	2	3·7	1·63	26·22
Total	123	123·0	100·00	7·21



(iv) <u>Distribution of the Average Estimated Stoping Values, expressed in Dwt per Ton, obtained by combining Individual Sampling Sections into 200 ft. Stretch Lengths</u>.

Table J on Plate J shows the results of the segregation of estimated stoping values, expressed in dwt per ton, obtained by combining individual sampling sections into groups of 40, each group representing the average value of a 200 ft. stretch length of development. The table is represented graphically by the histogram in Plate J, while the "fitted" curve denoted by the equation

$$y = 12.939\, e^{-1.2082\,(\log_e x - 1.3602)^2}$$

is indicated by the continuous red line in Plate J. The above equation will again be observed to be a form of the fundamental law, in which

$$N = 12.939$$
$$a = 1.0992$$
$$\text{and}\quad b = 1.3602$$

The mean value of the distribution, defined by equation [8] as

$$m = e^{\,b + \frac{1}{4a^2}}$$

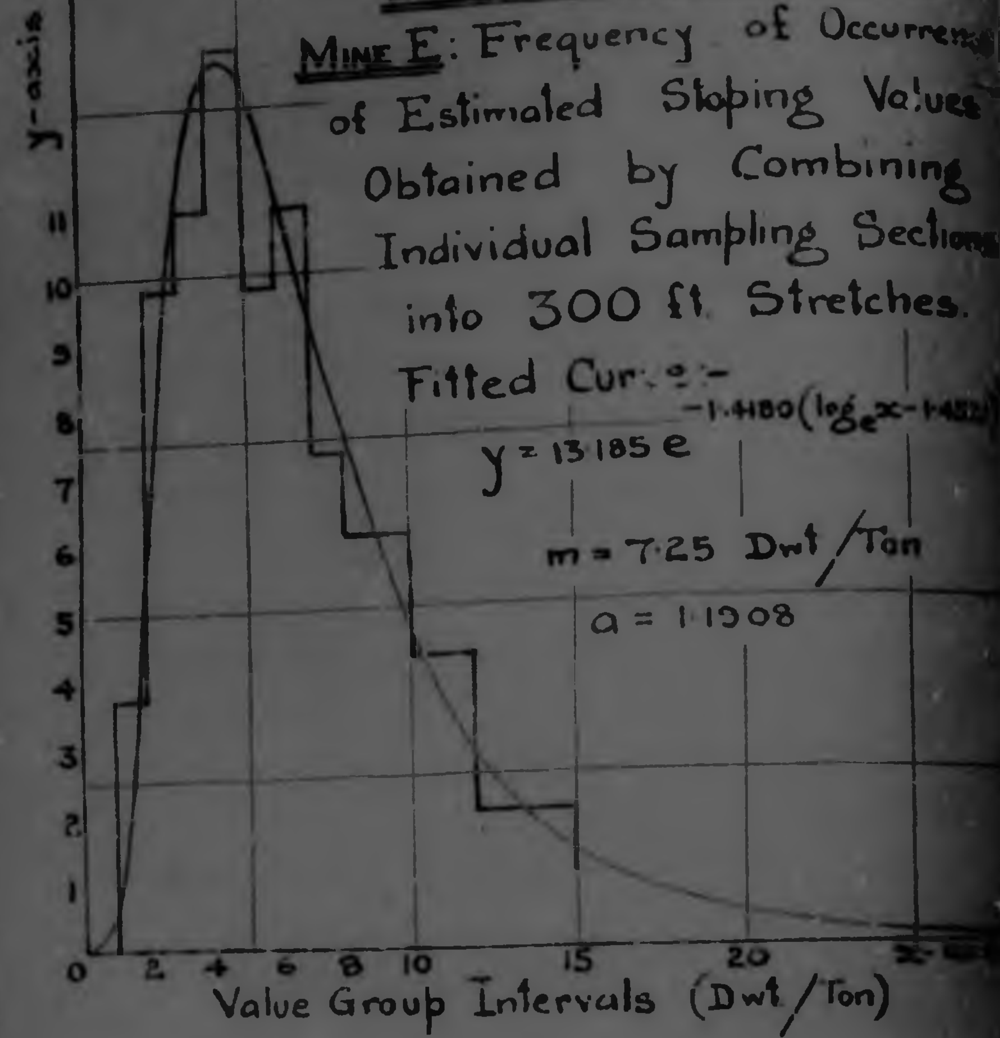
is

$$m = 2.7183^{\,1.3602 + \frac{1}{4 \times 1.0992}}$$
$$= 7.25 \text{ dwt per ton, as before.}$$

The position of the mode of the frequency curve is given by equation [5] as

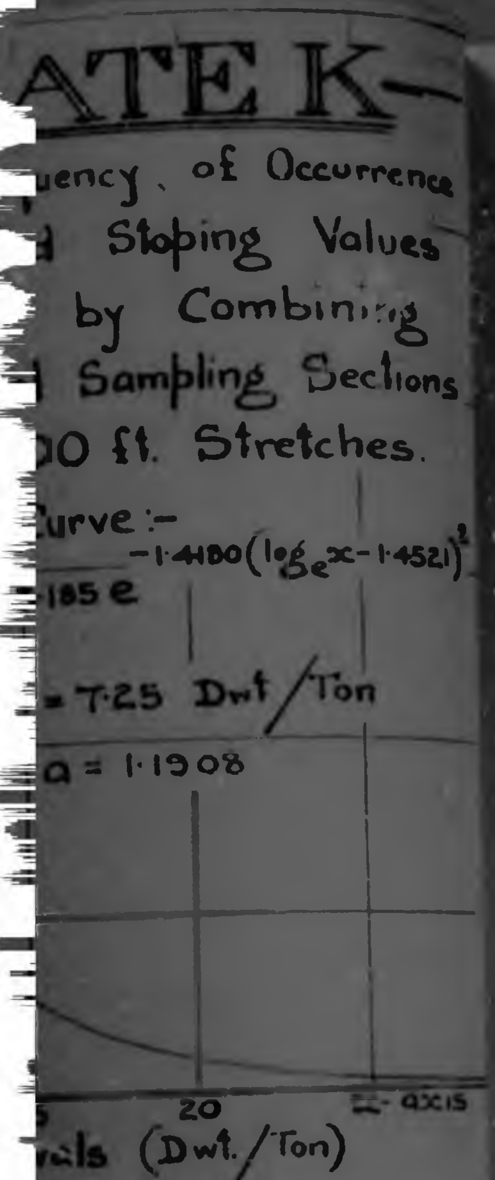
$$x = e^{b}$$
$$= 2.7183^{1.3602}$$
$$= 3.897 \text{ dwt per ton.}$$

—PLATE K—



—TABLE K—

Value Group (Dwt/Ton)	Observed Number	Expected Number	Observed % Freq.	Av. Observed Value Dwt/Ton
0 to 1	0	0·1	—	—
1 " 2	3	2·4	3·66	1·40
2 " 3	8	6·5	9·75	[illegible]
3 " 4	9	10·1	10·98	[illegible]
4 " 5	11	10·6	13·41	[illegible]
5 " 6	8	9·8	9·77	[illegible]
6 " 7	9	8·4	10·97	[illegible]
7 " 8	6	7·0	7·32	7·42
8 " 9	5	5·6	6·10	8·47
9 " 10	5	4·4	6·10	9·32
10 " 12	7	6·3	8·54	[illegible]
12 " 15	5	5·2	6·10	13·41
15 & Over	6	5·2	7·30	19·55
Total	82	82·0	100·00	7·27



Observed % Freq.	Av. Observed Value (Dwt/Ton)
—	—
3·66	1·40
9·75	2·28
10·98	3·44
13·41	4·32
9·77	5·47
0·97	6·30
7·32	7·42
6·10	8·47
6·10	9·32
8·54	10·88
6·10	13·41
7·30	15·55
100·00	7·27

(v) <u>Distribution of the Average Estimated Stoping Values, expressed in Dwt per Ton, obtained by Combining Individual Sampling Sections into 300 ft. Stretch Lengths.</u>

The results of the segregation of average estimated stoping values, expressed in dwt per ton, obtained by combining the original individual sampling sections into groups of 60, each group representing the average value of a 300 ft. stretch length of development, are shown in Table K on Plate K. This table is graphically represented by the histogram in Plate K, while the continuous red line denotes the "fitted" frequency curve represented by the equation

$$y = 13{\cdot}185\ e^{-1{\cdot}4180\ (\log_e x\ -\ 1{\cdot}4521)^2}$$

This is a form of the fundamental distribution law, in which

$$N = 13{\cdot}185$$
$$a = 1{\cdot}1908$$
$$\text{and}\quad b = 1{\cdot}4521$$

The mean value of the distribution is given by equation [8] as

$$m = 2{\cdot}7183^{\,1{\cdot}4521\ +\ \frac{1}{4\times 1{\cdot}4180}}$$
$$= 7{\cdot}25 \text{ dwt per ton, as before.}$$

The position of the mode of the frequency curve is given by equation [5] as

$$x = 2{\cdot}7183^{\,1{\cdot}4521}$$
$$= 4{\cdot}272 \text{ dwt per ton.}$$

(c) A Mathematical Analysis of the Relation between the Frequency Distribution of Gold Values expressed in Inch-Dwt and the Frequency Distribution of the Corresponding Values expressed in Dwt per Ton.

From the various value distributions illustrated for the case of Mine "E" in Plates A to K, it will be appreciated that a certain similarity exists between the various frequency curves representing the distributions of the average inch-dwt values and the corresponding frequency curves representing the distributions of the estimated stoping values, expressed in dwt per ton, derived from these inch-dwt values by the application of an estimated stoping width. This chapter will not be complete without a brief mathematical treatment of this relationship.

Let the unit chosen for the value group interval in the case of a particular frequency distribution of inch-dwt values be represented by "d" inch-dwt, the unit chosen for the value group interval in the case of the frequency distribution of the corresponding estimated stoping values be represented by "p" dwt per ton, and the average estimated stoping width of all the observations, which has either been found to be constant throughout the whole range of value groups, or the variable effect of which has been allowed for by reducing the observed number of values falling within a certain group to a constant width as previously indicated, be represented by "t" inches*.

The unit of "d" inch-dwt employed in the case of the

---

*In all the frequency curves so far deduced for the distribution of values on Mine "E",

$d = 50$ inch-dwt,

$p = 1$ dwt per ton

and $t = 61.65$ inches.

inch-dwt distribution may be expressed in dwt per ton as $\frac{d}{t}$. Hence the unit employed for the frequency curve representing the distribution of estimated stoping values expressed in dwt per ton will be $\frac{1}{p}\cdot\frac{d}{t}$ times as large as that used for the curve representing the corresponding distribution of inch-dwt values. Since the total area under the curve has to be the same and equal to 100 in both cases, the scale of ordinates will have to be adjusted by a factor which is the reciprocal of $\frac{1}{p}\cdot\frac{d}{t}$, i.e. by $\frac{p}{d/t}$.

It will thus be seen that any point (x, y) on the frequency curve representing the distribution of inch-dwt values will be associated with a point (x_1, y_1) on the corresponding curve representing the distribution of estimated stoping values expressed in dwt per ton in such a way that

$$x_1 = x \, \frac{1}{p}\cdot\frac{d}{t}$$

or $$\underline{x = x_1 \, p \, \frac{1}{d/t}}$$

and $$y_1 = y \, \frac{p}{d/t}$$

or $$\underline{y = y_1 \, \frac{1}{p}\cdot\frac{d}{t}}$$

Thus if the frequency curve representing the distribution of inch-dwt values is denoted by the equation

$$y = K \, e^{-a^2 (\log_e x - b)^2}$$

the frequency curve for the distribution of the corresponding estimated stoping values can be derived from this equation by the substitution of the above values for x and y, as follows:-

$$y_1 \frac{d}{t} \cdot \frac{1}{p} = N e^{-a^2 \left[\log_e\left(x_1 \cdot p \cdot \frac{1}{d/t}\right) - b\right]^2}$$

$$\text{or} \quad y_1 = N \frac{p}{d/t} e^{-a^2 \left[\log_e x_1 - \left(b - \log_e \frac{p}{d/t}\right)\right]^2}$$

This equation, representing the distribution of estimated stoping values, may be written in the form of the fundamental law as

$$y = N_1 e^{-a_1^2 (\log_e x - b_1)^2},$$

in which the constants (N_1, a_1, and b_1) can be expressed in terms of those (N, a, and b) for the distribution of the inch-dwt values from which the estimated stoping values were deduced as follows:-

$$N_1 = N \frac{p}{d/t} \text{ --------------------- } [19]$$

$$a_1 = a \text{ ------------------------ } [20]$$

$$\text{and} \quad b_1 = b - \log_e \frac{p}{d/t} \text{ ------------ } [21]$$

It will be observed that although the values of "N" and "b" are changed, that of "a" remains unaltered.

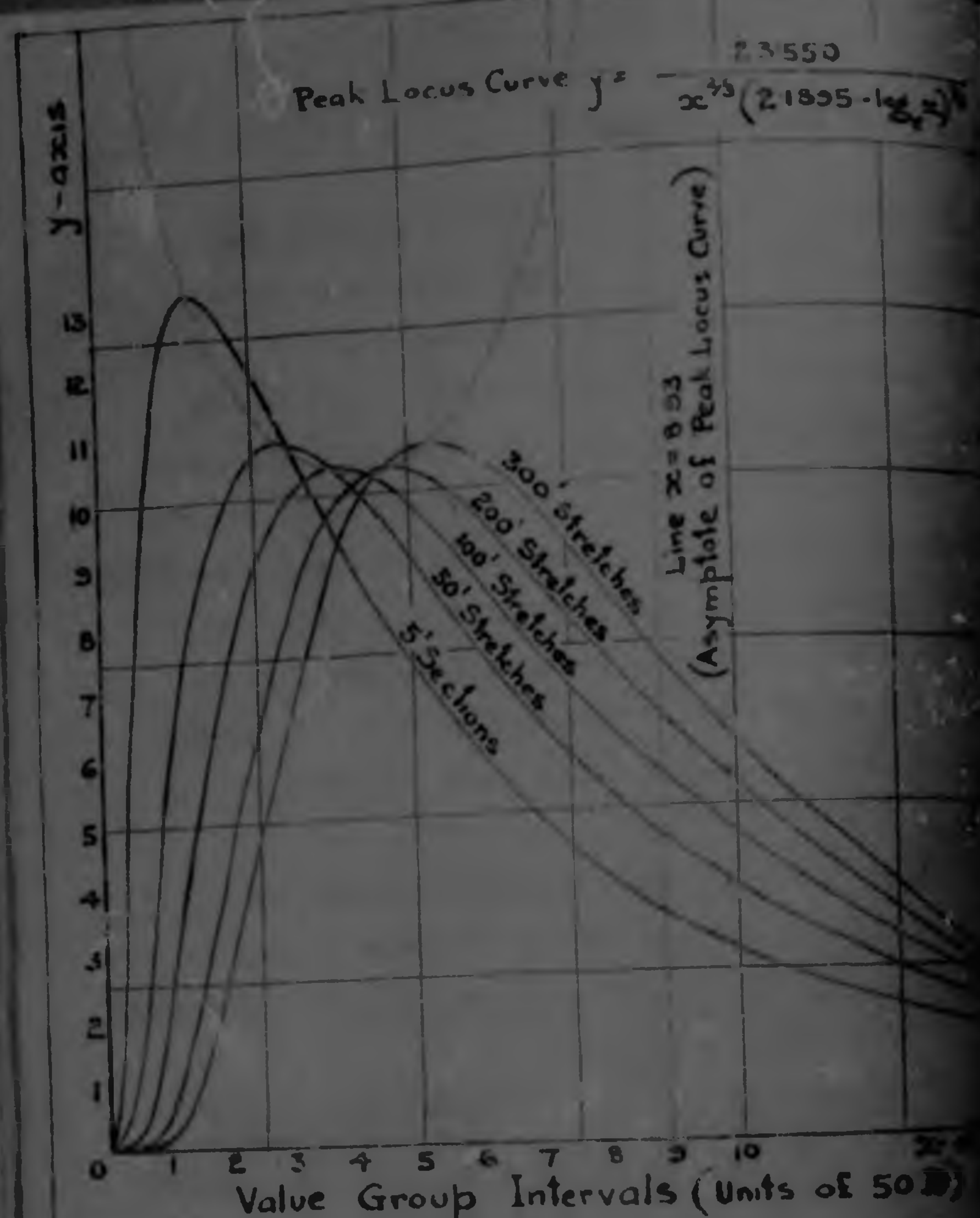
The extent to which the above theoretical conclusions are borne out in practise is illustrated by Table VIII on the following page. This table has been compiled from the "fitted" curves for the various observed inch-dwt value distributions and the corresponding estimated stoping value distributions in the case of Mine "E".

TABLE VIII.

Remarks		Stretch Lengths (in feet)				
		5	50	100	200	300
M	From observed inch-dwt value distribution	13.279	10.931	10.461	10.477	10.731
M_1	From observed estimated stoping value distribution	16.142	13.482	12.918	12.939	13.185
M_1	Deduced theoretically from equation [19]	16.373	13.478	12.898	12.918	13.231
a	From observed inch-dwt value distribution	0.6348	0.8151	0.9577	1.0819	1.2089
a_1	From observed estimated stoping value distribution	0.6614	0.8129	0.9254	1.0992	1.1908
a_1	Deduced theoretically from equation [20]	0.6348	0.8151	0.9577	1.0819	1.2089
b	From the observed inch-dwt value distribution	0.4402	1.0604	1.3365	1.5487	1.6694
b_1	From the observed estimated stoping value distribution	0.2664	0.8460	1.1092	1.3602	1.4921
	Deduced theoretically from equation [21]	0.2306	0.8510	1.1271	1.3395	1.4600

The agreement between practice and theory as shown in the above table is reasonably close, at least to within the limits of certain unavoidable practical errors.

Now that the relationship existing between the frequency curves obtained from the theoretically correct basis of distribution of inch-dwt values, and the practically more convenient basis of distribution of estimated stoping values expressed in dwt per ton has been demonstrated mathematically, and illustrated graphically for the case of Mine "X", it may be claimed that there is sufficient evidence to justify the adoption of the latter in preference to the former.



—PLATE IV—

Mine E: A Set of Frequency Curves for Inch-Dwt Values obtained from Individual Sampling Sections Spaced at 5 ft. Intervals, and by Combining these Individual Sampling Sections into 50, 100, 200 and 300 ft. Stretch Lengths respectively

6. The Optimum Stretch Length

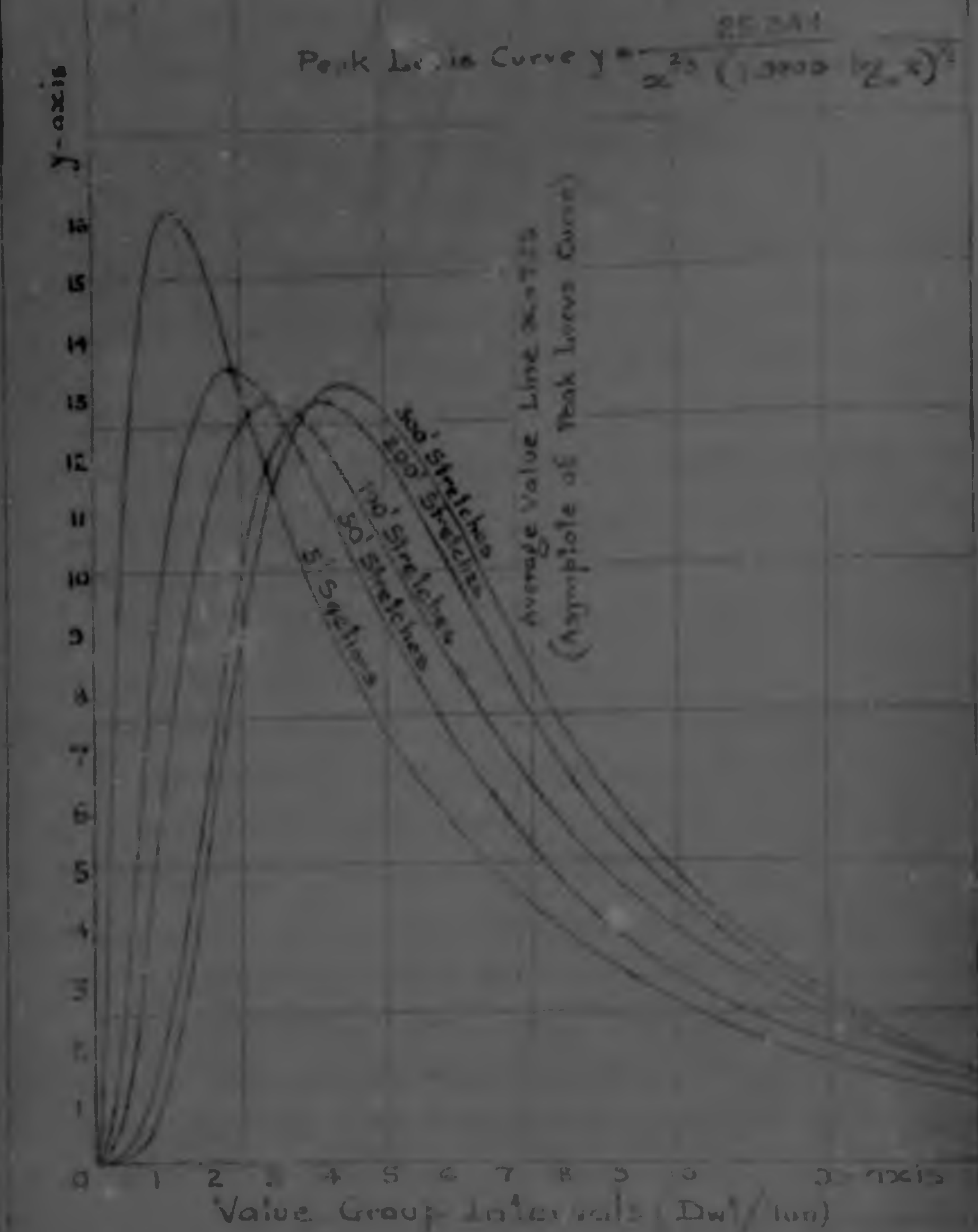
Two notable features emerge from a study of the foregoing examples of value frequency distributions for Mine E, which have without exception been observed to obey the same fundamental distribution law. The first is the essential constancy of the mean (8.93 units or 447 inch-dwt for the various inch-dwt value distributions, and 7.25 dwt per ton in the case of the various distributions representing estimated stoping values expressed in dwt per ton), while the second is the rather less obvious progression in the position of the mode of the frequency curve as the stretch lengths over which averaging takes place are successively increased.

The table below has been compiled to enable the position of the peaks of the various frequency curves to be studied in relation to the corresponding stretch lengths:-

TABLE IX.

Stretch Length (in feet)	Distribution of inch-dwt values (m=8.93 units or 447ID)		Distribution of estimated stoping values (m = 7.25 dwt per ton)	
	$x = e^b$	$y = M$	$x = e^b$	$y = M$
5	1.553	13.279	1.305	16.142
50	2.887	10.931	2.330	13.482
100	3.805	10.461	3.020	12.918
200	4.705	10.477	3.897	12.939
300	5.309	10.731	4.272	13.185

Plate IV shows the five frequency curves for the inch-dwt value distributions plotted with reference to the same co-ordinate axes. The peaks of these curves, shown in Table IX, will all be seen to lie on the peak locus curve



—PLATE V—

MINE F. A Set of Frequency Curves for Estimated Sloping Values obtained from Individual Sampling Sections Spaced at 5ft Intervals, and by Combining these Individual Sampling Sections into 50, 100, 200 and 500 ft Stretch Lengths Respectively

referred to in equation

$$y = \frac{100\sqrt{\ }}{2\sqrt{\ }}$$

where x = 8·95 units, or

Plate V shows the
for the distribution of
plotted with reference
case the peaks, shown
locus curve defined by
being that the value of

As stated before,
observed distribution
enclosed between the "
of the percentage freq
frequency is 100 per c
the frequency curve
will therefore be close
curve is high up on the
the lowermost point
be skew to a marked de
about which the major
clustered. This axis
zero ordinate as the
up along the left-hand
stretch lengths are
argument holds for
curve. As the
sections have been
the peak position
of the locus, and the
and more closely about
which will in this
value. Mathematically

referred to in equation [10], viz.

$$y = \frac{100\sqrt{3}}{2\sqrt{\pi}\, m^{1/3}} \left[\frac{1}{x^{2/3} \left(\log_e \frac{m}{x}\right)^{1/2}} \right]$$

where m = 8·93 units, or 447 inch-dwt.

Plate V shows the five corresponding frequency curves for the distribution of the estimated stoping values plotted with reference to the same co-ordinate axes. In this case the peaks, shown in Table IX, also lie on the peak locus curve defined by equation [10], the only difference being that the value of "m" in this case is 7·23 dwt per ton.

As stated before, in a frequency curve "fitting" an observed distribution represented by a histogram, the area enclosed between the "x"-axis and the curve is representative of the percentage frequency of occurrence. Since the total frequency is 100 per cent, it follows that the area under the frequency curve must be constant and equal to 100. It will therefore be clear that if the peak of the frequency curve is high up on the peak locus curve, and to the left of the lowermost point on this locus, the frequency curve will be skew to a marked degree, and elongated along some axis about which the majority of the observations will tend to be clustered. This axis will approach closer and closer to the zero ordinate as the peak position moves higher and higher up along the left-hand limb of the peak locus, i.e. as the stretch lengths are progressively shortened. A similar argument holds for the right-hand limb of the peak locus curve. As the stretches into which individual sampling sections have been combined are progressively lengthened, the peak position will move up along the right-hand limb of the locus, and the observed values will be clustered more and more closely about a certain ordinate, the position of which will in this case tend to approach that of the mean value. Mathematically stated, this means that as the stretch lengths are progressively increased, the mode will approach the mean, and the frequency curve will tend to become more and more symmetrical. The theoretical limit will occur when the sampling section values have been combined into infinitely large stretch lengths, in which case the mode will coincide with the mean, and a perfectly symmetrical, although infinitely elongated curve will result.

The statistical interpretation of the above conclusions is of the utmost importance. For short stretches, a more or less marked degree of clustering of the values about the mode is exhibited. As the stretch lengths are increased, the extent of this clustering of the observed values about the mode is relatively decreased, until it reaches a minimum when the peak of the frequency curve is at the lowest possible position on the peak locus. As the stretch lengths are further increased, the peak of the frequency curve once more moves upwards, and the clustering of the values about the mode will again become more and more marked. It will therefore be obvious that if the peak of the frequency curve is at the lowest point on the peak locus curve, the values of the corresponding distribution will be spread over the various value groups in the most even possible manner.

From the point of view of the mining engineer, the most suitable length to be employed when grouping individual sampling sections, will be that resulting in a distribution in which the observed values are spread most evenly throughout the entire range of value groups. It is only when the values have been combined in this way that the most selective choice of the stretches to be mined will be rendered possible. The stretch length which will give rise to this particular value distribution has consequently been termed the Optimum Stretch Length.

The conception of combining individual sampling sections into stretch lengths has been further developed to obtain a

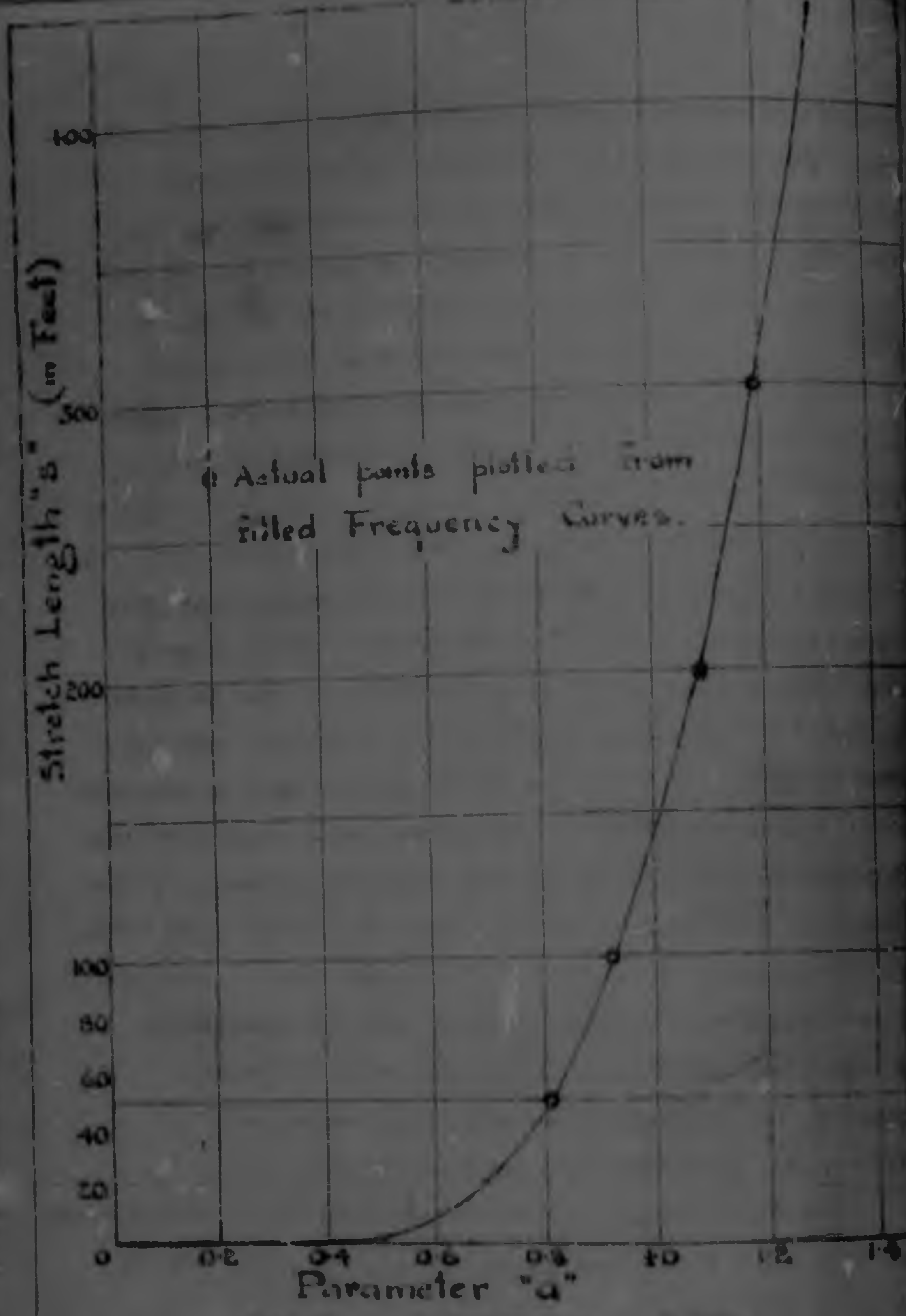


PLATE VI

Mine E: Illustration of the Stretch Length Curve represented by the Equation $s = 300a^3 . e^{-3/4a^2}$, which shows the Relation between the Stretch Length "s" into which Individual Sampling Sections have been Combined, and the Numerical Value of the Parameter "a" for the Corresponding Frequency Distribution relation between the actual stretch length, expressed in feet, into which the individual sampling sections have been combined for averaging , and the corresponding frequency curve representing the distribution of the average values over these stretch lengths. For this purpose the various values of the parameter "a" obtained from the frequency curves representing the distribution of the average values over the different stretch lengths in the case of Mine E, have been plotted as abscissae in Plate VI against ordinates of the corresponding stretch lengths, expressed in feet, into which the individual sampling sections have been combined. The data enabling Plate VI to be plotted will be found in Table VIII on page 50. In this connection it must be borne in mind that the values obtained from individual sampling sections spaced at 5 ft. intervals cannot singly be regarded as being representative of the average which would have been obtained had a number of sampling sections, spaced at closer intervals, been combined to represent the average over 5 ft. stretch lengths. The value of "a" deduced from a "fitting" of the frequency histogram obtained from the segregation of individual sampling sections into value groups can therefore not be regarded as being representative of a point on the Stretch Length Curve shown in Plate VI.

The Stretch Length Curve plotted from actual observations has been found to be accurately represented in the case of Mine "E" by the equation

$$s = 300\, a^3\, e^{-\frac{1}{4a^2}} \text{ ---------------------- } [22]$$

where "s" is the stretch length in feet, and

"a" is the parameter of the corresponding frequency distribution curve.

Theoretically, this curve starts at the origin and sweeps upward at a rapidly increasing rate as the stretch lengths are increased. Although "a" actually approaches infinity only when "s" is made infinitely large, it will be seen from Plate VI that, due to the rapid steepening of the curve, the practical limit to the parameter "a" will be reached even before it has attained the relatively small numerical value of 1·5.

It has been shown in equation [13] that at the minimum point on the Peak Locus Curve, i.e. at the Optimum Stretch Length, the numerical value of "a" is unity. Substituting this value of "a" in equation [22] , a numerical determination of the Optimum Stretch Length for Mine 'E' can be made, as follows:-

$$s_o = 300\ e^{-3/4}$$

$$= \frac{300}{2.1170}$$

$$\text{or} \quad \underline{s_o = 141.7} \qquad [23]$$

The practical significance of the conception of Optimum Stretch Lengths will be more fully appreciated when dealing, in the following chapters, with the Ore Gradation Graph and the application of the Pay Limit to the various frequency distributions.

Subsequent frequency investigations carried out on three other mines of very widely differing characteristics have revealed that the type of curve denoted by equation [22], with different numerical values for the constants, is representative of the relationship existing between the stretch length and the parameter "a" of the corresponding value frequency distribution in all cases. In concluding this chapter, it may be stated that the above particular theoretical aspect of the subject, together with all the various practical implications arising therefrom, provides considerable scope for further investigation.

CHAPTER III.

THE ORE GRADATION GRAPH.

In the foregoing analysis it has already been shown that the mean value "m" of all the ore contained in a given ore-body may be denoted by

$$m = e^{b + \frac{1}{4a^2}},$$

where "a" and "b" are parameters of the Fundamental Frequency Distribution Law. This value of "m" has further been shown to be constant for the particular ore-body under consideration.

If, however, only that portion of the total ore-body having a value equal to or greater than a predetermined value limit, say "x" dwt per ton, is taken into consideration, the average value of all this ore, denoted by "v_x", will not be constant. It will depend not only on the value of "x", but also on the stretch length into which the individual sampling sections have been combined for the purpose of determining the locality and extent of those portions of the ore-body having a value equal to or in excess of "x" dwt per ton. Since the parameter "a" is a function of the stretch length, it follows that "v_x" will be a function of "x" and "a". The properties of this function are not evident off-hand, and all that can be stated with certainty at this stage is that "v_x" will always be greater than "x".

To obtain a clearer conception of the relationship existing between "x" and "v_x", recourse will have to be had to the general mathematical principles referred to in Chapter II. From equation [17] it will be seen that the average value "v_x" of all the ore above a given value "x" may be denoted by

$$v_x = m \frac{\frac{2}{\sqrt{\pi}} \int_{w}^{\infty} e^{-w^2} \, dw}{\frac{2}{\sqrt{\pi}} \int_{w + \frac{1}{2a}}^{\infty} e^{-w^2} \, dw}$$

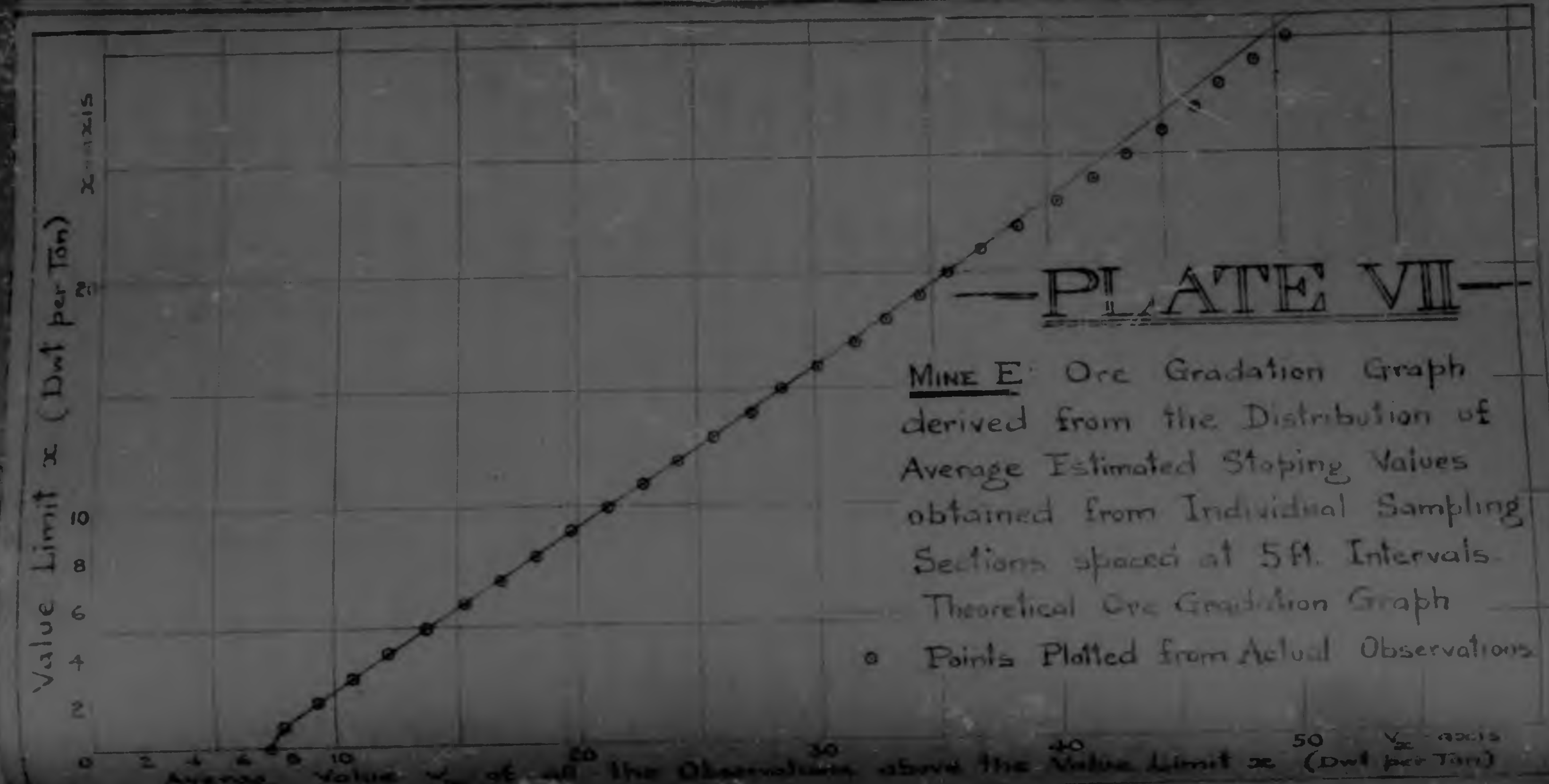
where $w = a\,(\log_e x - b) - \frac{1}{a}$.

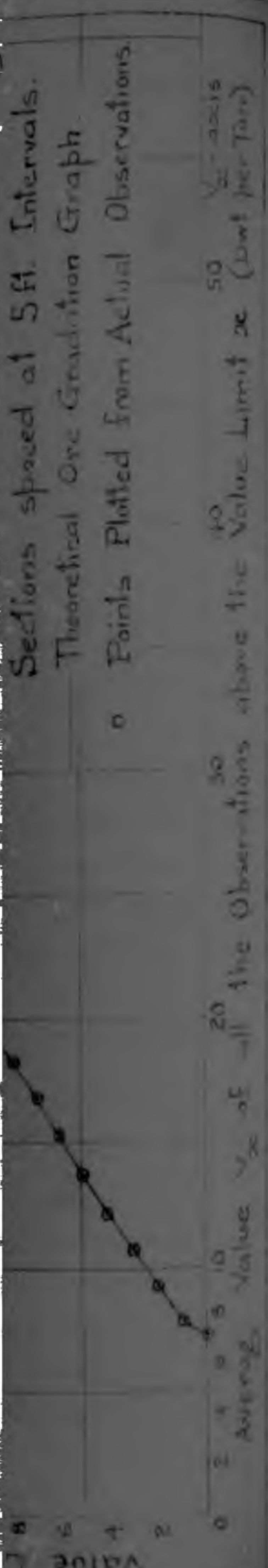
For specific numerical values of "m" and of "a", or, otherwise stated, for a particular ore-body in which valuation has been based on the grouping of individual sampling sections into definite stretch lengths of equal magnitude, "v_x" will be a function of "x" only, and it will therefore be possible to represent "v_x" graphically as a function of "x" for predetermined values of "m" and "a". All the graphs so obtained, by plotting v_x as abscissa against "x" as ordinate for different sets of values for "m" and "a", have revealed such definite characteristics, which have been found to persist without exception for all the gold-bearing reefs of the Witwatersrand, that the name of "Ore Gradation Graph" has been assigned to them. The Ore Gradation Graph may thus be defined as the curve obtained when the average value, say "v_x", of all the ore in a certain ore-body having a value equal to or greater than a predetermined value limit, say "x", is plotted as abscissa against the corresponding value of "x" as ordinate. The remainder of this chapter will be devoted to the practical illustration of various aspects of the Ore Gradation Graph, and to the evolution of certain mathematical formulae which will serve to give useful approximations to the true curves.

A. <u>Numerical Examples of Ore Gradation Graphs deduced from Consideration of the Sampling Data obtained from Mine "X".</u>

(1) <u>The Ore Gradation Graph derived from the Distribution of Estimated Stoping Values, expressed in Dwt per Ton, obtained from Individual Sampling Sections grouped at 5 ft. Intervals.</u>

The theoretical Ore Gradation Graph deduced from


—PLATE VII—
MINE E: Ore Gradation Graph derived from the Distribution of Average Estimated Stoping Values obtained from Individual Sampling Sections spaced at 5 ft. Intervals.
Theoretical Ore Gradation Graph
Points Plotted from Actual Observations
Value Limit x (Dwt per Ton)
x-axis
Average Value V_x of all the Observations above the Value Limit x (Dwt per Ton)
V_x-axis
0
2
4
6
8
10
20
30
40
50



the frequency curve

$$y = 16{\cdot}142\, e^{-0{\cdot}4374\,(\log_e x - 0{\cdot}2664)^2}$$

which has been "fitted" to the observed histogram obtained from the segregation of the values of individual sampling sections , spaced at 5 ft. intervals, has been calculated (from equation [17]) in the table below.

TABLE X.

Value Limit "x" (Dwt per Ton)	$w = a(\log_e x - b) - 1/a$	$\frac{2}{\sqrt{\pi}}\int_w^\infty e^{-w^2}\,dw$	$w + \frac{1}{2a}$	$\frac{2}{\sqrt{\pi}}\int_{w+1/2a}^\infty e^{-w^2}\,dw$	Aver. Freq. Value "v_x" (Dwt/Ton)
30	0·5616	0·4273	1·3175	0·06244	49·615
25	0·4408	0·5331	1·1967	0·09057	42·674
20	0·2933	0·6785	1·0404	0·1379	35·685
15	0·1030	0·8441	0·8589	0·2244	28·564
10	− 0·1658	1·1847	0·5907	0·4055	21·287
8	− 0·3128	1·3418	0·4432	0·5309	18·324
6	− 0·5031	1·5232	0·2528	0·7208	15·321
5	− 0·6236	1·6921	0·1383	0·8516	13·810
4	− 0·7711	1·7245	− 0·0151	1·0169	12·295
3	− 0·9615	1·8261	− 0·2095	1·2286	10·775
2	− 1·2297	1·9179	− 0·4737	1·4970	9·280
1	− 1·6881	1·9831	− 0·9382	1·8125	7·932
0	− ∞	2·0000	− ∞	2·0000	7·290

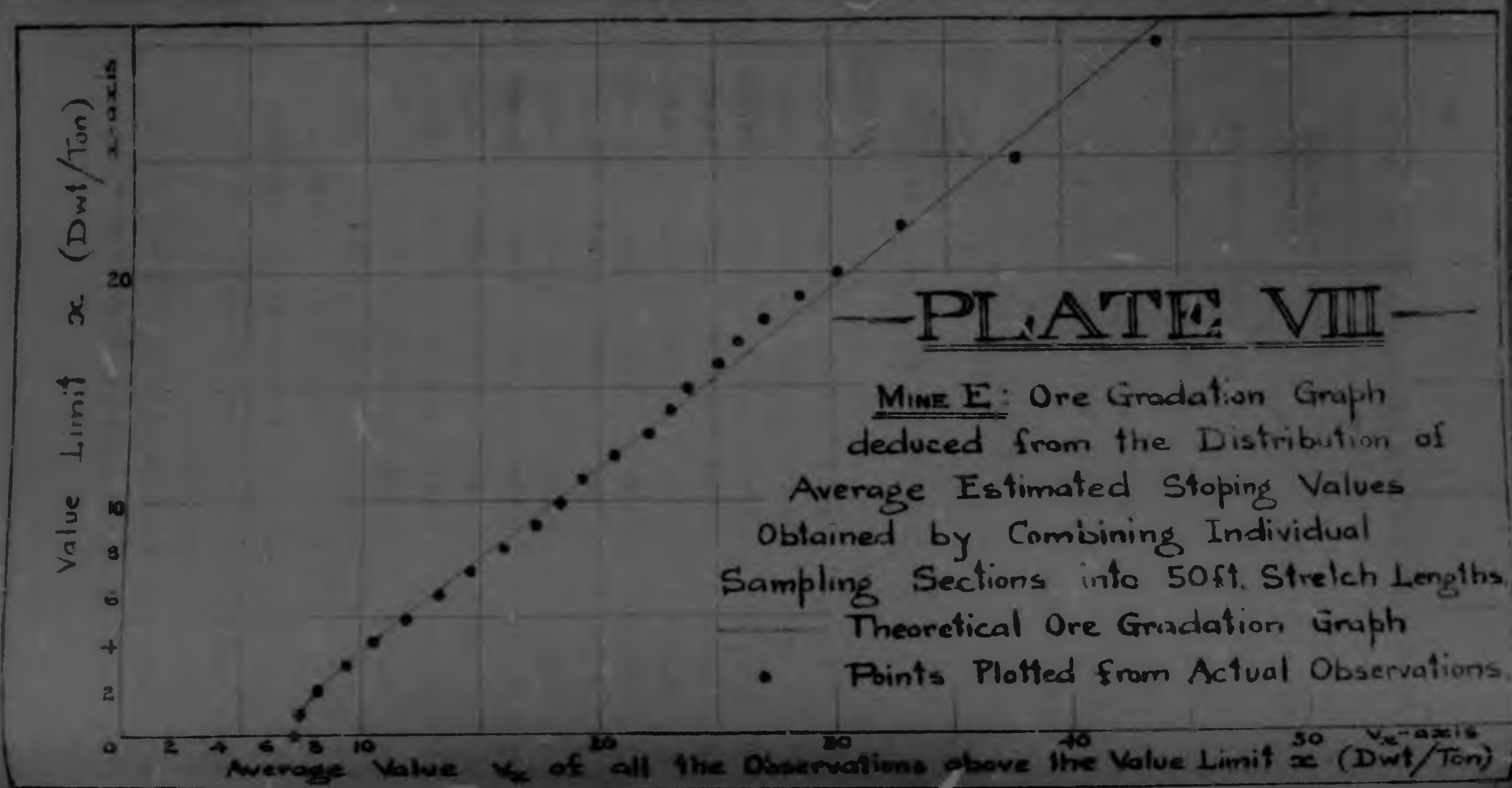
Note: $a = 0{\cdot}6614$; $b = 0{\cdot}2664$; $m = 7{\cdot}290$.

Plate VII has been constructed from this table, the red line illustrating graphically the above results.

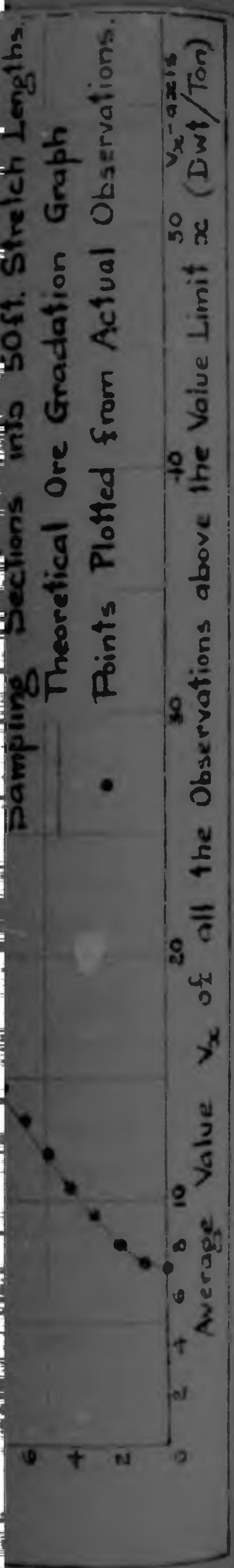
Table XI on the following page shows how the corresponding Ore Gradation Graph may be deduced from the actual observations. The closeness with which the values for "v_x", deduced from practical considerations, approximate to those calculated theoretically will readily be appreciated from a study of Plate VII.

TABLE XI.

Value Group (Dwt per Ton)	Number of Observations	Average Value (Dwt/Ton)	Progressive		
			Value Limit "x" (Dwt/Ton)	Number of Observations.	Average Value "v_x" (Dwt/Ton)
30 & over	151	50.46	30	151	50.46
29 - 30	11	29.43	29	162	49.05
28 - 29	12	28.48	28	174	47.61
27 - 28	10	27.35	27	184	46.51
26 - 27	17	26.38	26	201	45.01
25 - 26	16	25.37	25	217	43.56
24 - 25	19	24.48	24	236	42.05
23 - 24	20	23.34	23	256	40.57
22 - 23	27	22.42	22	283	38.85
21 - 22	30	21.55	21	313	37.18
20 - 21	29	20.52	20	342	35.77
19 - 20	29	19.48	19	371	34.49
18 - 19	37	18.40	18	408	33.05
17 - 18	37	17.50	17	445	31.74
16 - 17	51	16.46	16	496	30.17
15 - 16	60	15.38	15	556	28.58
14 - 15	58	14.48	14	614	27.24
13 - 14	78	13.49	13	692	25.69
12 - 13	87	12.43	12	779	24.21
11 - 12	99	11.40	11	878	22.77
10 - 11	123	10.45	10	1001	21.85
9 - 10	145	9.40	9	1146	19.75
8 - 9	165	8.54	8	1311	18.34
7 - 8	202	7.42	7	1513	16.88
6 - 7	253	6.42	6	1766	15.39
5 - 6	337	5.43	5	2103	13.79
4 - 5	427	4.44	4	2530	12.21
3 - 4	506	3.45	3	3036	10.75
2 - 3	650	2.43	2	3686	9.28
1 - 2	777	1.45	1	4463	7.98
0 - 1	464	0.60	0	4927	7.83



—PLATE VIII—
MINE E: Ore Gradation Graph deduced from the Distribution of Average Estimated Stoping Values Obtained by Combining Individual Sampling Sections into 50ft. Stretch Lengths.
Theoretical Ore Gradation Graph
Points Plotted from Actual Observations.
Value Limit x (Dwt/Ton)
x-axis
0
2
4
6
8
10
20
Average Value v_x of all the Observations above the Value Limit x (Dwt/Ton)
v_x-axis
2
4
6
8
10
20
30
40
50



(31) <u>The Ore Gradation Graph derived from the Distribution of Estimated Stoping Values, expressed in Dwt per Ton, obtained by combining Individual Sampling Sections into 50 ft. Stretch Lengths</u>.

The theoretical Ore Gradation Graph deduced from the equation

$$y = 13.482\ e^{-0.6608\ (\log_e x\ -\ 0.8460)^2}$$

representing the distribution of the average values over 50 ft. stretch lengths, has been calculated in the table below.

<u>TABLE XII.</u>

[illegible]	$w=a(\log_e x-b) - 1/a$	$\frac{2}{\sqrt{\pi}}\int_w^\infty e^{-w^2}\,dw$	$w + \frac{1}{2a}$	$\frac{2}{\sqrt{\pi}}\int_{w+1/2a}^\infty e^{-w^2}\,dw$	Aver. Prog. Value "v_x" (Dwt/Ton)
30	0.8469	0.2311	1.4620	0.03869	43.305
25	0.6986	0.3230	1.3137	0.06320	37.053
20	0.5173	0.4646	1.1324	0.1094	30.[illegible]
15	0.2835	0.6886	0.8986	0.2058	24.496
10	-0.0461	1.0519	0.5690	0.4210	18.115
8	-0.2276	1.2524	0.3875	0.5837	15.[illegible]
6	-0.4614	1.4859	0.1537	0.8276	13.027
5	-0.6096	1.6115	0.0055	0.9938	11.757
4	-0.7909	1.7366	-0.1758	1.1958	10.[illegible]
3	-1.0249	1.8531	-0.4098	1.4377	9.345
2	-1.3545	1.9445	-0.7394	1.7042	8.272
1	-1.9179	1.9953	-1.3028	1.9346	7.470
0	$-\infty$	2.0000	$-\infty$	2.0000	7.250

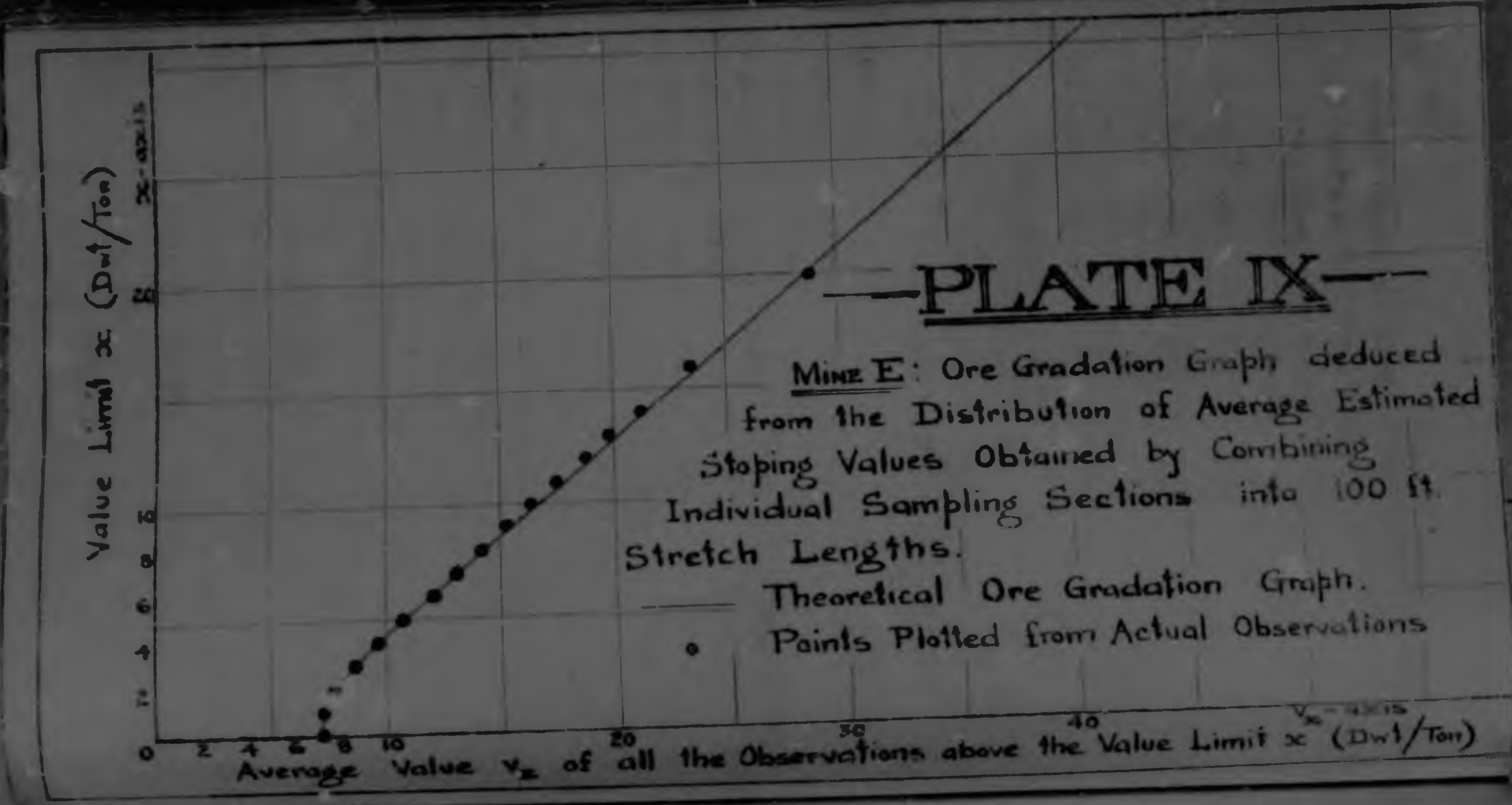
Note: a = 0.8129 ; b = 0.8460 ; m = 7.250 .

The red curve in Plate VIII represents the theoretical Ore Gradation Graph which has been plotted from the results of the above table.

Table XIII on the following page has been compiled to enable the corresponding Ore Gradation Graph deduced from the practical observations to be plotted on Plate VIII for the purpose of comparison with the theoretical Graph.

TABLE XIII.

Value Group (Dwt per Ton)	Number of Observations	Average Value (Dwt per Ton)	Progressive		
			Value Limit "x" (Dwt/Ton)	Number of Observations	Average Value "y_x" Dwt/Ton)
30.0 & over	9	44.26	30	9	44.26
25 - 30	7	27.32	25	16	36.84
2[illegible] 25	6	23.38	22	22	33.16
[illegible] - 22	7	20.90	20	29	30.21
19 - 20	5	19.40	19	34	28.02
18 - 19	6	18.42	18	40	27.09
17 - 18	5	17.31	17	45	26.00
16 - 17	4	16.41	16	49	25.22
15 - 16	8	15.39	15	57	23.84
14 - 15	5	14.46	14	62	23.03
13 - 14	7	13.50	13	69	22.11
12 - 13	12	12.42	12	81	20.68
11 - 12	14	11.43	11	95	19.31
10 - 11	11	10.44	10	106	18.39
9 - 10	13	9.43	9	119	17.41
8 - 9	22	8.51	8	141	16.02
7 - 8	28	7.48	7	16[illegible]	14.61
6 - 7	32	6.42	6	[illegible]	13.30
5 - 6	45	5.45	5	246	11.87
4 - 5	51	4.43	4	297	10.59
3 - 4	57	3.45	3	354	9.[illegible]
2 - 3	65	[illegible].48	2	419	8.36
1 - 2	61	1.44	1	480	7.48
0 - 1	12	0.62	0	492	7.31

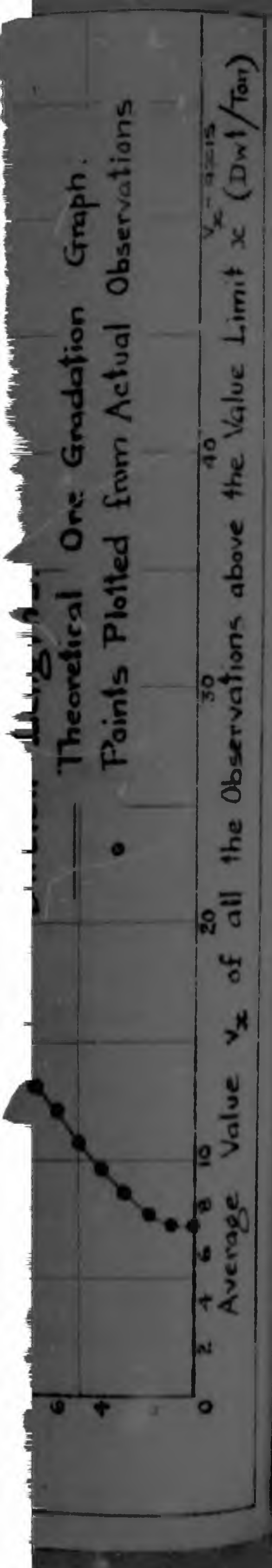


—PLATE IX—

Mine E: Ore Gradation Graph deduced from the Distribution of Average Estimated Stoping Values Obtained by Combining Individual Sampling Sections into 100 ft. Stretch Lengths.

—— Theoretical Ore Gradation Graph.

• Points Plotted from Actual Observations



(iii) The Ore Gradation Graph deduced from the Distribution of Estimated Stoping Values, expressed in Dwt per Ton, obtained by combining Individual Sampling Sections into 100 ft. Stretch Lengths.

The theoretical Ore Gradation Graph deduced from the equation

$$y = 12.918\, e^{-0.8564\,(\log_e x - 1.1052)^2}$$

representing the distribution of average values over 100 ft. stretch lengths, has been calculated in Table XIV below:-

TABLE XIV.

Value Limit x (Dwt per Ton)	$w = a(\log_e x - b) - 1/_a$	$\frac{2}{\sqrt{\pi}}\int_w^\infty e^{-w^2}\,dw$	$w + \frac{1}{2a}$	$\frac{2}{\sqrt{\pi}}\int_{w+1/2a}^\infty e^{-w^2}\,dw$	Aver. Prog. Value v_x (Dwt/Ton)
30	1.0441	0.1399	1.5844	0.02505	40.490
25	0.8753	0.2158	1.4156	0.04530	34.558
20	0.6689	0.3442	1.2092	0.08726	28.598
15	0.4027	0.5690	0.9430	0.1823	22.629
10	0.0275	0.9691	0.5678	0.4220	16.649
8	-0.1791	1.1991	0.3612	0.6095	14.261
6	-0.4453	1.4709	0.0950	0.8951	11.941
5	-0.6139	1.6147	-0.0736	1.0828	10.821
4	-0.8204	1.7540	-0.2801	1.3080	9.722
3	-1.0867	1.8756	-0.5464	1.5605	8.715
2	-1.4620	1.9613	-0.9217	1.8075	7.867
1	-2.1034	1.9970	-1.5632	1.9729	7.338
0	$-\infty$	2.0000	$-\infty$	2.0000	7.250

Note: $a = 0.9254$; $b = 1.1052$; $m = 7.250$.

The theoretical Ore Gradation Graph representing graphically the results shown in the above table is denoted by the red curve in Plate IX.

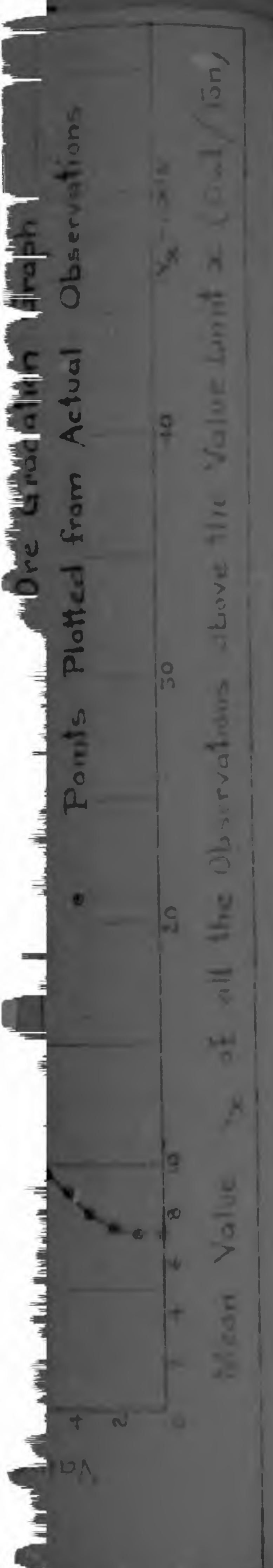
Table XV below has been compiled from the actual observations:-

TABLE XV.

Value Group (Dwt per Ton)	Number of Observations	Average Value (Dwt per Ton)	Progressive		
			Value Limit "x" (Dwt/Ton)	Number of Observations.	Average Value "v_x" (Dwt/Ton)
20 & over	10	28.83	20	10	28.83
16 - 20	10	17.94	16	20	23.39
14 - 16	7	14.97	14	27	21.20
13 - 14	6	13.50	13	33	19.80
12 - 13	6	12.39	12	39	18.66
11 - 12	7	11.40	11	46	17.56
10 - 11	10	10.46	10	56	18.29
9 - 10	9	9.43	9	65	15.34
8 - 9	13	8.45	8	78	14.19
7 - 8	14	7.41	7	92	13.16
6 - 7	17	6.40	6	109	12.11
5 - 6	25	5.46	5	134	10.87
4 - 5	30	4.42	4	164	9.69
3 - 4	31	3.38	3	195	8.68
2 - 3	27	2.62	2	222	7.95
1 - 2	22	1.60	1	244	7.37
0 - 1	2	0.70	0	246	7.32

This table contains data from which the Ore Gradation Graph for 100 ft. stretch lengths, as deduced from practical observations, may be plotted. It is represented graphically in Plate IX for purposes of comparison with the theoretical curve.


— PLATE X —
Mine E: Ore Gradation Graph deduced from the Distribution of Average Estimated Stoping Values obtained by Combining Individual Sampling Sections into 200 ft Stretch Lengths.
Theoretical Ore Gradation Graph
Points Plotted from Actual Observations
Value Limit x (Dwt per Ton)
Mean Value $\bar{v}_x$ of all the Observations above the Value Limit x (Dwt / Ton)



(iv) **The Ore Gradation Graph derived from the Distribution of Estimated Stoping Values, expressed in Dwt per Ton, obtained by combining Individual Sampling Sections into 200 ft. Stretch Lengths.**

The theoretical Ore Gradation Graph, deduced from the equation

$$y = 12.939\, e^{-1.2082\,(\log_e x - 1.3602)^2}$$

representing the distribution of average values over 200 ft. stretch lengths, has been calculated in the table below:-

TABLE XVI.

Value Limit x (Dwt per Ton)	$w = a(\log_e x - b) - 1/a$	$\frac{2m}{\sqrt{\pi}}\int_w^\infty e^{-w^2}\,dw$	$w + \frac{1}{2a}$	$\frac{2m}{\sqrt{\pi}}\int_{w+1/2a}^\infty e^{-w^2}\,dw$	Aver. Prog. Value "v_x" (Dwt/Ton)
30	1.3337	0.05928	1.7886	0.01143	37.682
25	1.1332	0.1090	1.5881	0.02471	31.962
20	0.8879	0.2092	1.3428	0.05757	26.345
15	0.5718	0.4187	1.0267	0.1466	20.707
10	0.1261	0.8586	0.5810	0.4113	15.134
8	-0.1193	1.1339	0.3356	0.6351	12.944
6	-0.4355	1.4620	0.0194	0.9782	10.855
5	-0.6358	1.6314	-0.1809	1.2019	9.841
4	-0.8810	1.7872	-0.4261	1.4531	8.917
3	-1.1974	1.9096	-0.7425	1.7065	8.113
2	-1.6431	1.9800	-1.1882	1.9071	7.527
1	-2.4049	1.9993	-1.9500	1.9942	7.269
0	$-\infty$	2.0000	$-\infty$	2.0000	7.250

Where: $a = 1.0992$; $b = 1.3602$; $m = 7.250$.

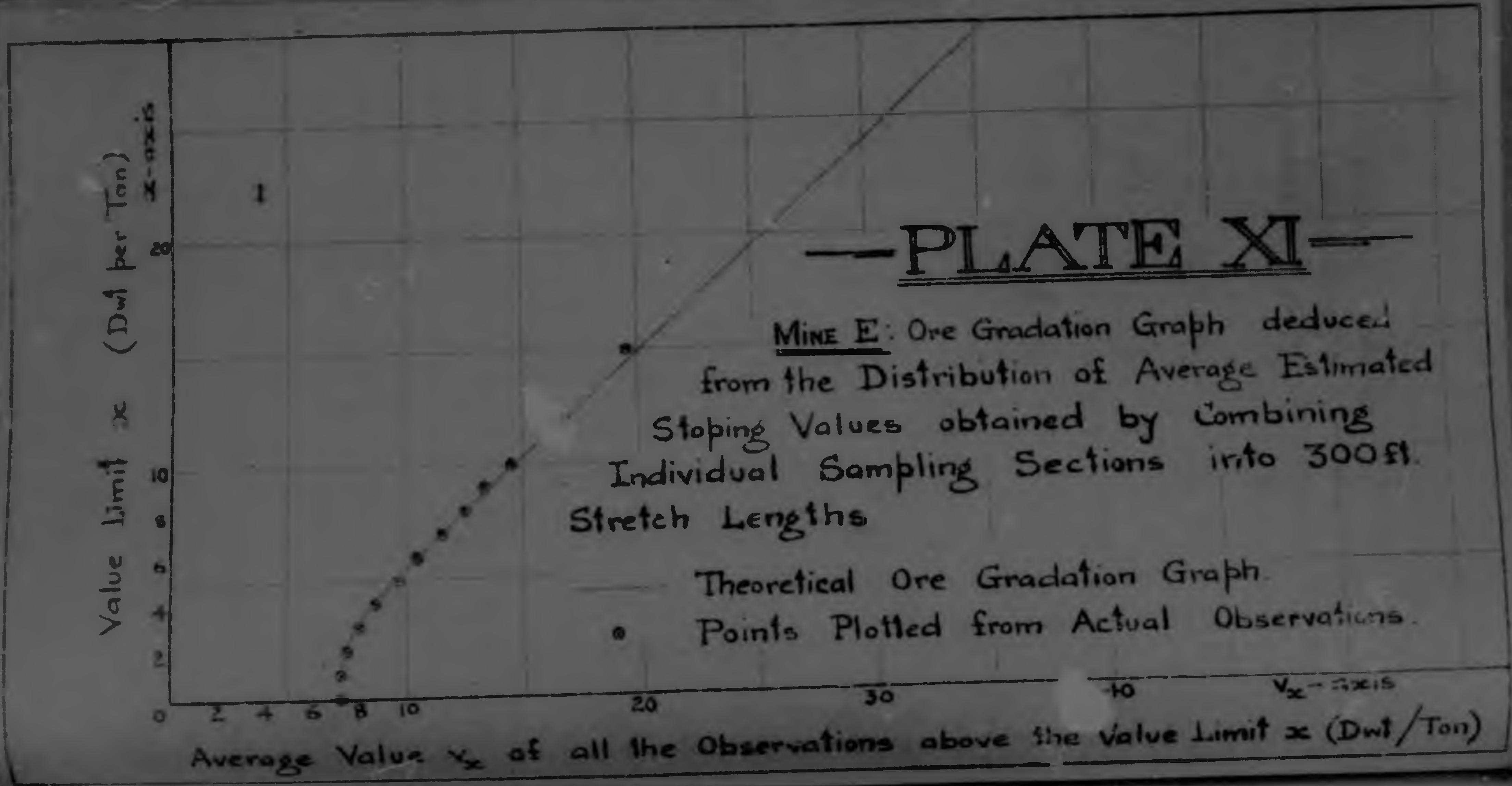
The red curve in Plate X represents graphically the results deduced in Table XVI above.

Table XVII below has been compiled from the actual segregation of the average values for the 200 ft. stretch lengths.

TABLE XVII.

Value Group (Dwt per Ton)	Number of Observations	Average Value (Dwt per Ton)	Progressive		
			Value Limit "x" (Dwt/Ton)	Number of Observations	Average Value V_x = (Dwt/Ton)
20 & over	2	26.72	20	2	26.72
15 - 20	7	17.92	15	9	19.96
12 - 15	7	13.40	12	16	16.86
10 - 12	10	11.05	10	26	14.62
9 - 10	7	9.43	9	33	13.52
8 - 9	9	8.45	8	42	12.44
7 - 8	8	7.60	7	50	11.46
6 - 7	14	6.39	6	64	10.52
5 - 6	12	5.50	5	76	9.72
4 - 5	13	4.46	4	89	8.95
3 - 4	17	3.41	3	106	8.06
2 - 3	12	2.45	2	118	7.49
1 - 2	4	1.48	1	122	7.29
0 - 1	1	0.78	0	123	7.24

The results shown in the above table have been represented graphically in Plate X to show the extent of the agreement between theory and practice.


—PLATE XI—
MINE E: Ore Gradation Graph deduced from the Distribution of Average Estimated Stoping Values obtained by Combining Individual Sampling Sections into 300 ft. Stretch Lengths
Theoretical Ore Gradation Graph.
Points Plotted from Actual Observations.
Value Limit x (Dwt per Ton)
x-axis
v_x-axis
Average Value v_x of all the Observations above the Value Limit x (Dwt/Ton)
0
2
4
6
8
10
20
30
40

(v) The Ore Gradation Graph for the Distribution of Estimated Stoping Values, expressed in Dwt per Ton, obtained by combining Individual Sampling Sections into 300 ft. Stretch Lengths.

Table XVIII below has been calculated from the equation

$$y = 13.185\, e^{-1.4180\,(\log_e x - 1.4581)^2}$$

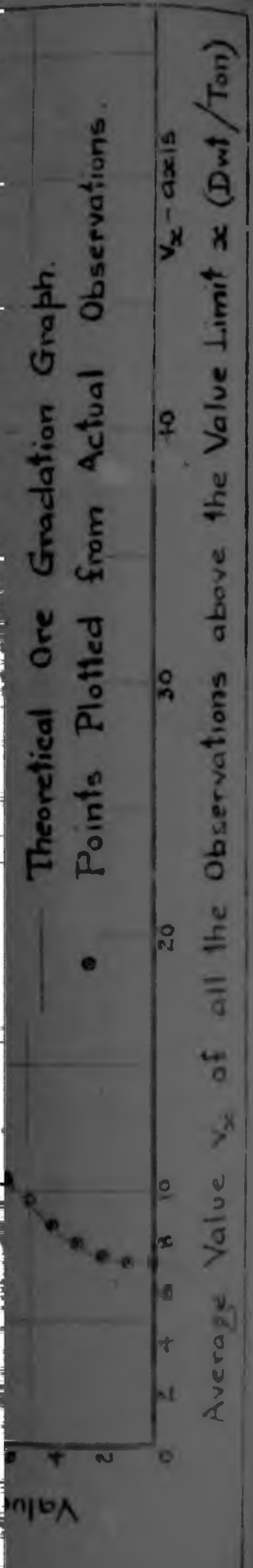
which represents the distribution of the average values obtained by combining sampling sections into groups, each representing a 300 ft. stretch length.

TABLE XVIII.

Value Limit x (Dwt per Ton)	$w = a(\log_e x - b) - 1/a$	$\frac{2}{\sqrt{\pi}}\int_w^\infty e^{-t^2}\,dt$	$w + \frac{1}{2a}$	$\frac{2}{\sqrt{\pi}}\int_{w+\frac{1}{2a}}^\infty e^{-t^2}\,dt$	Aver. Freq. Value v_x (Dwt/Ton)
30	1.4812	0.05620	1.9011	0.007177	36.963
25	1.2640	0.07386	1.6839	0.01725	31.042
20	0.9963	0.1580	1.4182	0.04489	25.518
15	0.6538	0.3538	1.0737	0.1282	20.088
10	0.1730	0.8067	0.5929	0.4018	14.956
8	-0.0928	1.1044	0.3271	0.6437	12.439
6	-0.4354	1.4619	-0.0155	1.0275	10.417
5	-0.6524	1.6438	-0.2325	1.2577	9.476
4	-0.9180	1.8058	-0.4981	1.5188	8.680
3	-1.2604	1.9278	-0.8408	1.7695	7.916
2	-1.7436	1.9863	-1.3237	1.9388	7.496
1	-2.5690	1.9997	-2.1491	1.9976	7.298
0	$-\infty$	2.0000	$-\infty$	2.0000	7.290

Using: $a = 1.1908$; $b = 1.4581$; $m = 7.290$.

The theoretical Ore Gradation Graph, represented graphically by the red curve in Plate XI, has been plotted



from Table XVIII above.

The table below has been compiled from practical considerations.

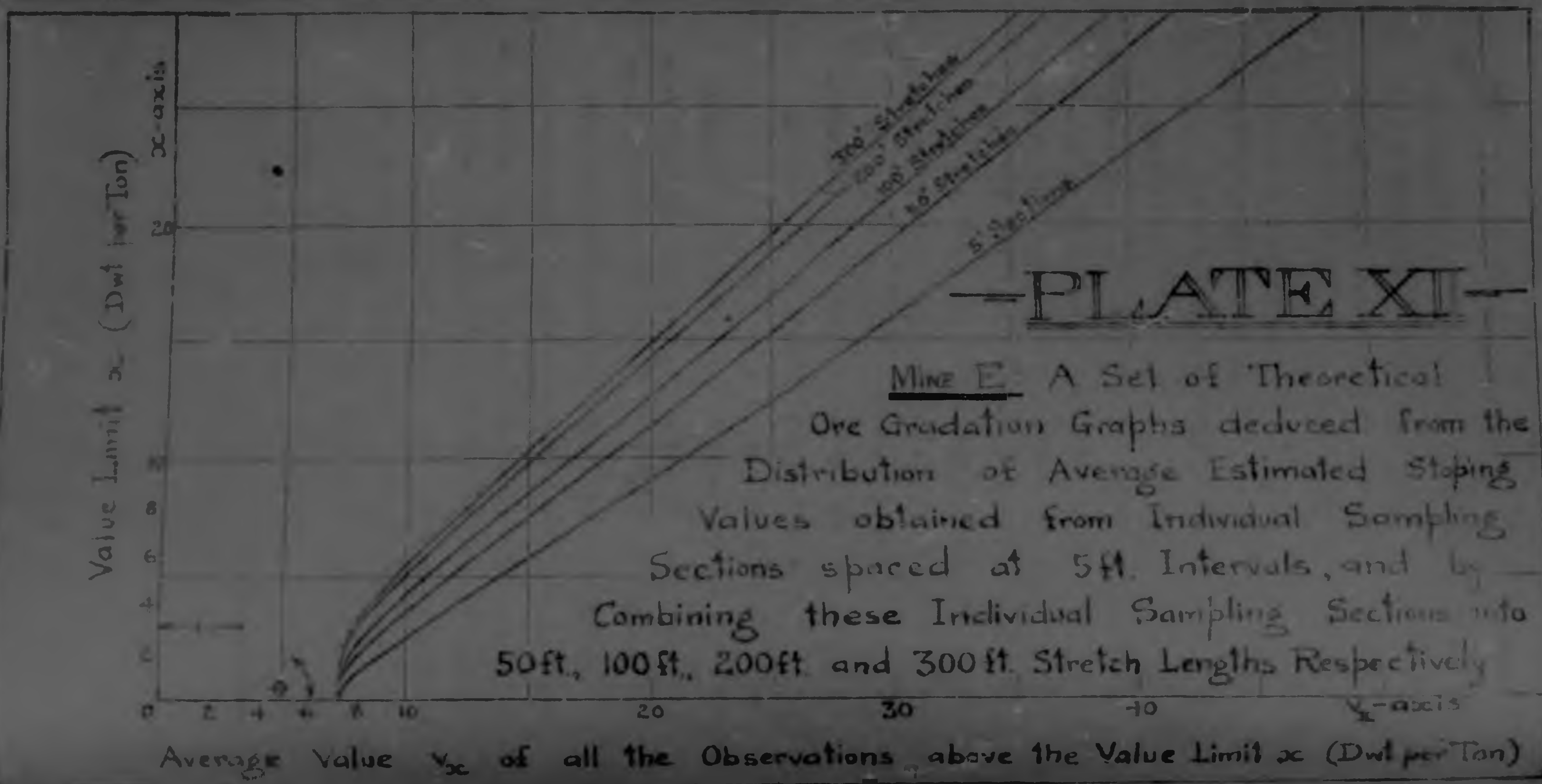
TABLE XIX.

Value Group (Dwt per Ton)	Number of Observations	Average Value (Dwt per Ton)	Progressive: Value Limit "x" (Dwt/Ton)	Progressive: Number of Observations	Progressive: Average Value "v_x" (Dwt/Ton)
15 & over	6	19.95	15	6	19.95
12 - 15	5	13.41	12	11	16.81
10 - 12	7	10.88	10	18	14.51
9 - 10	5	9.32	9	23	13.38
8 - 9	5	8.47	8	28	12.50
7 - 8	6	7.42	7	34	11.61
6 - 7	9	6.30	6	43	10.59
5 - 6	8	5.47	5	51	9.71
4 - 5	11	4.32	4	62	8.75
3 - 4	9	3.44	3	71	8.08
2 - 3	8	2.28	2	79	7.49
1 - 2	3	1.40	1	82	7.27
0 - 1	-	-	0	82	7.27

The Ore Gradation Graph, calculated from the observed practical values for the 300 ft. stretch lengths as shown in the above table, has been plotted in Plate XI for the purpose of comparison with the theoretical curve.

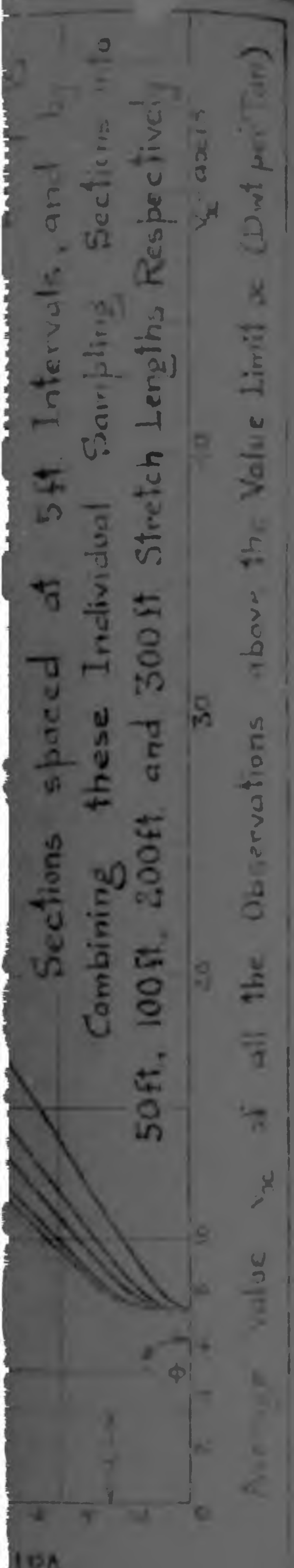
In order to facilitate a closer investigation into the true nature of these Ore Gradation Graphs, all the foregoing theoretically computed curves have been plotted with reference to the same co-ordinate axes in Plate XII, from which the following facts will immediately be evident:-

(1) The mean value "m", represented by the point where the



—PLATE XI—

Mine E. A Set of Theoretical Ore Gradation Graphs deduced from the Distribution of Average Estimated Stoping Values obtained from Individual Sampling Sections spaced at 5 ft. Intervals, and by Combining these Individual Sampling Sections into 50 ft., 100 ft., 200 ft. and 300 ft. Stretch Lengths Respectively



Ore Gradation Graph intersects the "v_x"- axis, is the same in all cases. This is, of course, merely a graphical illustration of the mathematically obvious fact previously referred to.

(2) After a relatively small but varying initial curvature at the low value end, all the curves tend very rapidly towards practically perfect straight lines.

(3) The initial curvature of the Ore Gradation Graphs becomes more and more pronounced as the stretch lengths into which the individual sampling sections have been combined, are increased, i.e. as the numerical value of the parameter "s" in the equation representing the corresponding frequency distribution is increased.

(4) The gradient of the curve is progressively increased as the stretch lengths (or the numerical value of the parameter "s") are increased. Practical considerations seem to indicate that the limiting slope of the Ore Gradation Graph is about 45°.

(5) Although the mean value "$\bar{x}$", represented by the point where the Ore Gradation Graph intersects the "v_x"- axis, is obviously the same in all cases, it is equally clear from a study of the curves plotted in Plate XII that, due both to the variable initial curvature and the variable slope of the different Ore Gradation Graphs, the average value of all the ore above any fixed value will not be constant, but will depend on the stretch length over which averaging has taken place. The longer the stretch length, or in other words the greater the numerical value of the parameter "s", the steeper will be the slope of the corresponding Ore Gradation Graph, and consequently the smaller the value of "v_x".

B. <u>Mathematical Approximation to the true Theoretical Ore Gradation Graph</u>.

The full significance of the observations enumerated above can only be appreciated from a study of the mathematical expression for the Ore Gradation Graph, which is accurately represented by the equation

$$v_x = m \frac{\int_w^\infty e^{-w^2} dw}{\int_{w + 1/2s}^\infty e^{-w^2} dw} .$$

Unfortunately the exact evaluation of this expression is extremely troublesome, as it involves the ratio of two infinite series. A good mathematical approximation to the Ore Gradation Graph as determined from the above equation, for all values of the mean "m" and the parameter "s" likely to be encountered in practice, is given by the general equation

$$v_x = \frac{x}{\tan \theta} + i + r , \text{ ——————————— } (24)$$

in which

$$\tan \theta = \frac{1}{e^{1/4s^2}} \left[\frac{s^{1.2/s}}{30\, s^{0.01}} + 1 \right] \text{ ——————————— } [25]$$

$$i = m\sqrt{1 - (\tan \theta)^3} \text{ ——————————— } [26]$$

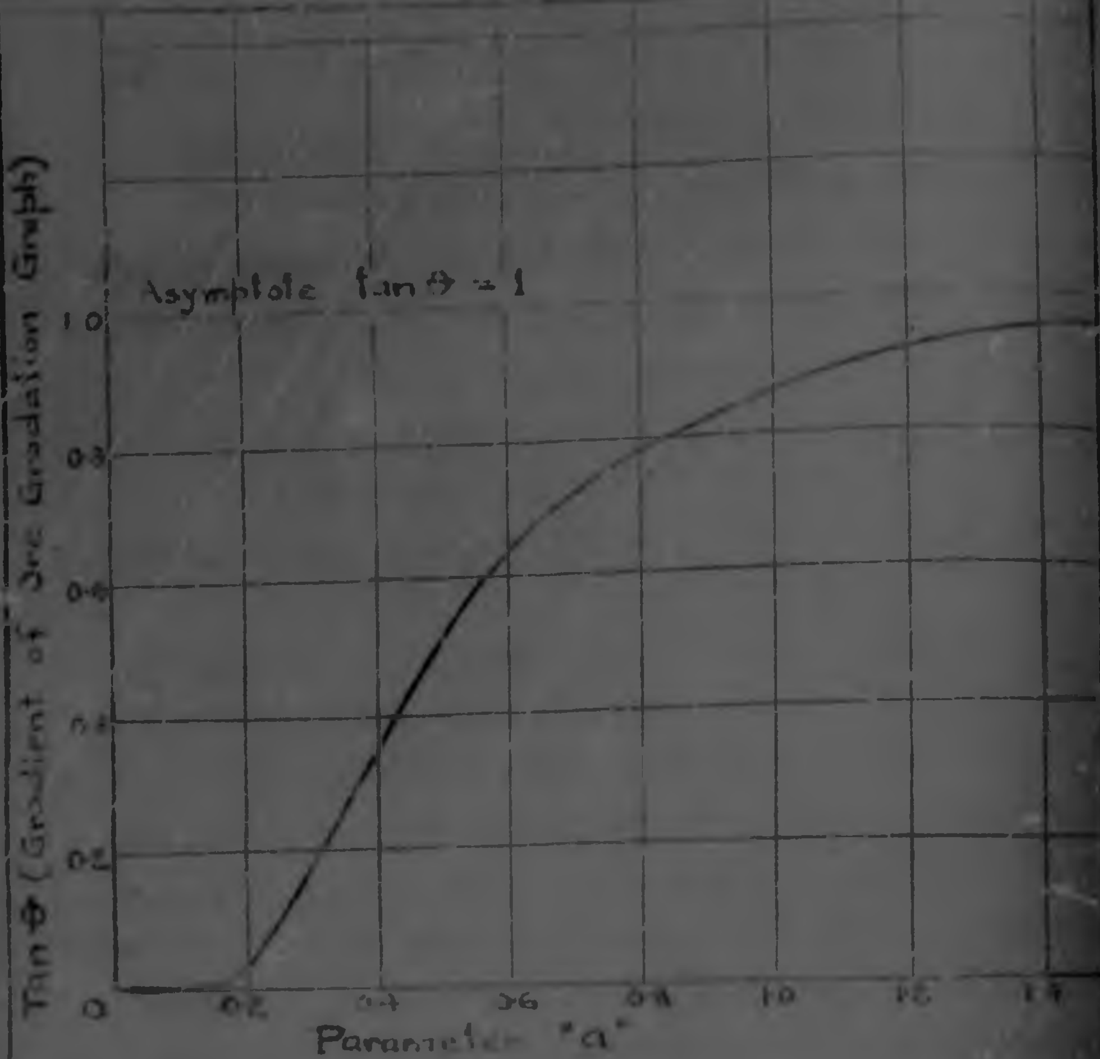
and $$r = (m - i)\, e^{-\frac{4}{s^2}\left[\frac{x}{m}\right]^{1.0835}} \text{ ——————————— } [27]$$

In all the above expressions the following notation has been adopted:-

"v_x" is the average value of all the observations above the given value "x".

"s" is the parameter of the equation representing the corresponding frequency distribution.

"m" ($= e^{b + 3/4s^2}$) is the mean value of all the observations taken for the particular ore-body under consideration.



— PLATE XIII —

Curve of the Equation

$$\operatorname{Tan}\theta = \frac{1}{e^{1/4a^2}}\left\{\frac{e^{1.2/a}}{30\,e^{0.01}} + 1\right\},$$

showing the Relation between the Parameter "a" of the Basic Value Frequency Distribution and the Approximate Gradient (Tan θ) of the Corresponding Ore Gradation Graph.

"g" is the gradient of the straight line portion of the Ore Gradation Graph. (See Plate XII).

"i" is the intercept made on the "v_x"- axis when the straight line portion of the Ore Gradation Graph is produced to cut this axis. (See Plate XII).

"y" is an additive function responsible for the steepening of the Ore Gradation Graph for low values of "x". A discussion on its full significance will be given later.

It may be argued that the theoretical Ore Gradation Graph can much more easily be calculated from statistical tables, in the manner already indicated, than from the comparatively complicated mathematical equation above. This will undoubtedly be the case when dealing with a particular Ore Gradation Graph in which the numerical values of "x" and "s" are known, but equation [24] above provides a general formula giving a good approximation to all Ore Gradation Graphs derived from the Fundamental Distribution Law. It will therefore enable a more detailed investigation to be made into the true nature of these curves.

Consider firstly the expression for the gradient of the Ore Gradation Graph as given in equation [25], viz.

$$\tan \theta = \frac{1}{e^{1/4s^2}} \left[\frac{e^{1.2/s}}{30.s^{0.01}} + 1 \right]$$

A study of this function, a graphical representation of which will be found in Plate XIII, reveals the following important facts:-

(1) The gradient of the Ore Gradation Graph is a function only of the parameter "s" of the corresponding frequency distribution, and is entirely independent of the mean value "x" of the ore-body under consideration. Hence all Ore Gradation Graphs having the same numerical

value for the parameter "s" will have the same gradient.

(ii) Ore Gradation Graphs corresponding to numerically low values of "s" will be relatively flat, while those corresponding to numerically high values will be relatively steeper. Since the curve represented by equation [23] approaches the line $\tan\Theta = 1$ asymptotically, it follows that the slope of the straight line portion of the Ore Gradation Graph will never exceed 45°. Although this slope will theoretically only be attained when "s" is infinitely large, the slope of the graph will actually approximate very closely to the 45° limit for all values of "s" which are in excess of 2.

(iii) It has already been pointed out that all the values of "s" likely to be encountered in practice will probably lie between 0.5 and 1.5 . For numerical values of "s" falling within this range, the factor $s^{0.02}$ approximates very closely to unity, and its effect on the computed value of $\tan\Theta$ may therefore, for all practical purposes, be neglected.

Consider next the expression

$$i = m\sqrt{1 - (\tan\Theta)^2}$$

for the intercept made on the "v_x" - axis by the straight line portion of the Ore Gradation Graph. The following facts are noteworthy:-

(i) Since $\tan\Theta$ has been shown to be a function of "s" only, that factor of the expression under the square root sign will be the same for the Ore Gradation Graphs derived from all frequency distributions having the same numerical value for "s". The intercept "i" for these

Ore Gradation Graphs will therefore depend on the mean values of the different ore-bodies concerned. The slope of all such Ore Gradation Graphs will thus be identical, but their positions will be displaced by varying amounts depending on the relative mean values of the respective ore-bodies.

(ii) For an ore-body having a given mean value "m", the length of the intercept "i" will decrease as the value of $\tan \theta$, or in other words the numerical value of the parameter "a" (or the stretch length), is increased. Since $\tan \theta$ is always less than 1, "i" will always be greater than 0, the length of the intercept actually being equal to zero only when $\tan \theta = 1$ (i.e. when "a", and consequently the stretch length, is infinitely large). The straight portion of the graph will then resolve itself into a line through the origin, inclined at an angle of 45° to the "v_x"- axis.

The complete equation for the Ore Gradation Graph includes yet another additive expression, viz.

$$y = (m - i)\, e^{-\frac{4}{a^2}\left(\frac{x}{m}\right)^{1.0833}}$$

which, for any given "m" and "a", is a function of "x". It is this term in the complete formula for "v_x" (see equation [24]) which is responsible for the steepening of the curve for the lower values of "x".

The following facts emerge from a closer study of this additive expression:-

(i) For any given Ore Gradation Graph (i.e. for a given "m" and "a"), the numerical value of the expression for "y" decreases as "x" is increased until, for sufficiently large values of "x", its effect on the function "v_x" will be negligible. For suitably large "x", therefore, the function "v_x" may completely be represented by the equation

$$v_x = \frac{x}{\tan\theta} + i \quad \text{-----------------} [28]$$

This is a simplification which may frequently be applied in practice. The exact magnitude of "x" for which the expression for "y" becomes negligible can be determined from equation [27] for specific numerical values of "x" and "s".

(ii) For a given value of "x", the numerical value of the above expression for "y" is decreased as the parameter "s" is decreased. Hence the smaller the value of the parameter "s", the more rapidly will the Ore Gradation Graph approximate to the straight line represented by equation [28]. This fact has repeatedly been observed in practice, and is well illustrated in Plate XII.

When individual sampling sections are combined into groups to represent the average values over certain equal stretch lengths, the theoretically ideal grouping to be employed will be that resulting in a series of values, each of which represents the average over the Optimum Stretch Length. Although this actual length will vary from mine to mine depending on the nature and characteristics of the reef body (it has been shown to be approximately 142 feet in the case of Mine "E"), the numerical value of "s" will in all cases be unity when dealing with that particular distribution obtained from values representing averages over the Optimum Stretch Length. The following simplifications in the approximation formula for the Ore Gradation Graph based on the Optimum Stretch Length will therefore be effected:-

(i) Substituting unity in equation [25], the gradient of the straight line portion of any Ore Gradation Graph, obtained from averages over the Optimum Stretch Length, may be calculated as

$$\tan\theta = 0.8649 ,$$

and the slope of the straight line portion of the graph will thus be 40° 51' (or 41°, say) in all cases.

(ii) The value of tan θ having been determined above as 0.8649 , the intercept "i", denoted by equation [26], can therefore be evaluated. This reduces to

$$i = m\sqrt{0.3530}$$

$$\text{or}\quad \underline{i = 0.5942.\ m}$$

(iii) The additive function denoted by equation [27] resolves itself into the comparatively simple equation

$$r = 0.4058.\ m.\ e^{-4\left(\frac{x}{m}\right)^{1.0833}}$$

Combining the above three components, the complete equation for any Ore Gradation Graph, derived from the average values obtained when the individual sampling sections are combined into Optimum Stretch Lengths, will be a function of "m" and "x" only, and may be written as

$$v_x = \frac{x}{0\cdot 8649} + 0.5942\ .\ m + 0.4058.m.e^{-4\left(\frac{x}{m}\right)^{1.0833}}$$

$$\text{or}\quad v_x = \frac{x}{0\cdot 8649} + m\left(0.5942 + 0.4058\ e^{-4\left(\frac{x}{m}\right)^{1.0833}}\right) \text{---[29]}$$

Table XX on the next page has been compiled to show the extent of the agreement between two Ore Gradation Graphs for Mine "X" (mean value "m" = 7.25 dwt per ton), derived respectively from the exact theoretical formula of equation [17] and the approximation formula of equation [24] for two widely differing values of the parameter "m".

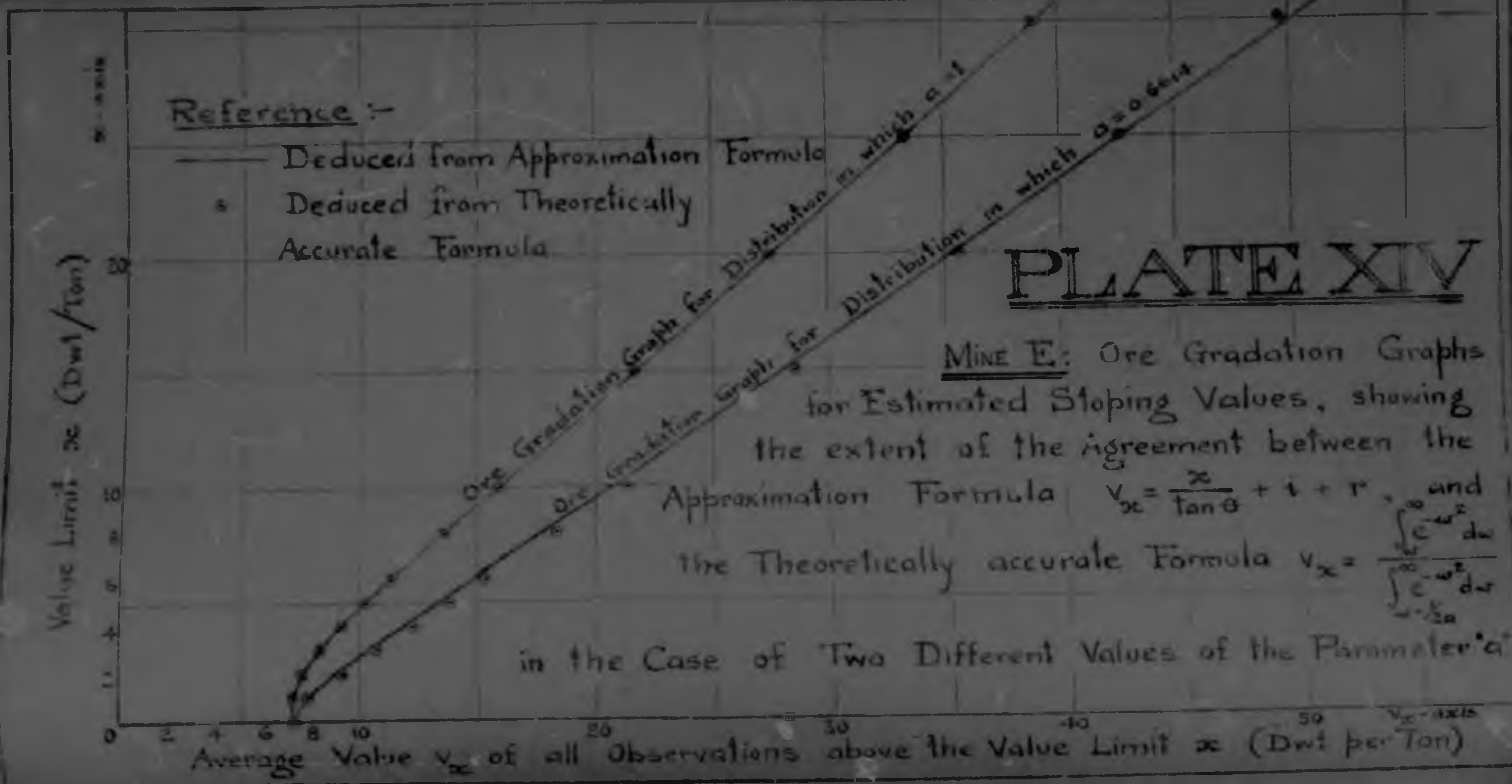


PLATE XIV

MINE E: Ore Gradation Graphs for Estimated Stoping Values, showing the extent of the Agreement between the Approximation Formula $v_x = \frac{x}{\tan\theta} + i + r$, and the Theoretically accurate Formula $v_x = \frac{\int_{w}^{\infty} e^{-w^2} dw}{\int_{-\frac{v}{2a}}^{\infty} e^{-w^2} dw}$ in the Case of Two Different Values of the Parameter "a"

TABLE XX.

Value Limit "x" (Dwt/Ton)	Average Progressive Value "v_x" (Dwt per Ton)			
	As deduced from Equation [17]		As deduced from Equation [24]	
	a = 1.0000	a = 0.6614	a = 1.0000	a = 0.6614
30	39.057	49.615	38.994	50.055
25	33.279	42.674	33.213	42.712
20	27.495	35.663	27.432	35.369
15	21.689	28.564	21.653	28.024
10	15.916	21.287	15.889	20.603
8	13.627	18.324	13.572	17.746
6	11.405	15.321	11.358	14.804
5	10.340	13.810	10.292	13.343
4	9.328	12.295	9.296	11.877
3	8.416	10.775	8.426	10.440
2	7.684	9.289	7.712	9.063
1	7.297	7.932	7.307	7.896
0	7.250	7.250	7.250	7.250

It will readily be seen from Table XX, and its graphical representation in Plate XIV, that whereas the agreement between the theoretically correct Ore Gradation Graph and its approximation is remarkably close in the case for which "a" = 1, it deteriorates appreciably when the numerical value of the parameter "a" is decreased to 0.6614. It should be pointed out, however, that the agreement in the latter, although not nearly as good as in the former case, still falls well within the dictates of practical requirements, the maximum difference being only about $1\frac{1}{3}$ per cent. The agreement will usually be practically acceptable, except possibly for large values of "x" and exceptionally small values of "a", such as may be encountered when dealing with the distribution of values obtained from the segregation of individual sampling sections which have been spaced closely together. As soon as these individual sections are combined into groups, resulting in a distribution of values which have been averaged out over certain stretch lengths, however, the numerical value of the parameter "z" will be increased, and the closeness of the agreement will be improved.

The above feature is primarily due to the fact that the Ore Gradation Graph does not approximate as closely to a straight line for numerical values of "z" which are less than, say 0·6, as will be the case for values greater than 0·6. The theoretical Ore Gradation Graph in the former case develops a distinct steepening tendency for higher values of "x", thereby resulting in a curve which is very slightly concave on the upper side. The effect of this phenomenon may, however, be neglected for all practical purposes.

CHAPTER IV.

PAY LIMITS.

Having proved and illustrated the existence of a fundamental statistical law for the distribution of gold values in the ore of a particular Witwatersrand mine, having explained the conception of stretch length combination and Optimum Stretch Lengths, and having demonstrated the method of deducing the Ore Gradation Graph from this Fundamental Distribution Law, the complications arising from the introduction of the pay limit will now have to be examined.

Before dealing with the conception of pay limit in its relation to the Fundamental Distribution Law, it will first be necessary to define the term, and to dwell briefly on certain aspects of pay limits.

> Definition: A Pay Limit is a computed standard of value which enables differentiation to be made between payable and unpayable ore. Payable ore is ore of which the value is at least equal to that computed as the relevant pay limit.

The basic factor governing the determination of a pay limit is the total cost involved in the extraction and treatment of the ore. For any particular mine, treatment costs are essentially the same once the ore has been delivered to the reduction works. The costs involved in the breaking of the ore and its transport to the mill are, however, subject to considerable variation. Theoretically therefore, pay limits will be different for the ore emanating from different localities in the same mine, and more specifically, from the different individual working places in that mine. This is due to certain variations in the working costs, resulting mainly from

(i) Differences in the stoping width and the cost of support,

(ii) Relative ease of breaking, and

(iii) Differences in the cost of tramming and hoisting, depending on the locality of a particular working place in relation to the available tipping and hoisting facilities.

Although the allocation of different pay limits to individual working places is impracticable, it has nevertheless been found possible to distribute working costs in such a manner as to enable separate average pay limits to be calculated for different reef horizons, and for the various mining operations, such as stoping, development, reclamation, etc. It is not intended here to go into detail regarding the methods of determining these various pay limits. They are all essentially the same, and conform to the general principles governing the determination of pay limits, the only difficulty apparently being the rather practical one of correctly distributing the costs.

Two very distinct and basically different pay limits do exist, however, depending on the specific purpose for which the pay limit is to be used. These may be classified as

(a) <u>The Ore Reserve Pay Limit</u>, which serves as a standard by which the developed ore in the mine may be segregated into payable or unpayable for inclusion in, or exclusion from, the Ore Reserve, and

(b) <u>The Current Pay Limit</u>, which enables an assessment to be made of the payability or otherwise of a current stope face.

Since the essential difference between these two pay limits involves such considerations as the Block Factor and the Tonnage Discrepancy, each will be dealt with separately and in more detail below.

A. <u>The Ore Reserve Pay Limit</u>.

When compiling the Ore Reserve of a mine, the computer, mainly by using the sampling results recorded on the Assay Plans, but also by employing various devices drawn from the rich well of practical experience, estimates the tonnage and the gold content likely to be obtained from each individual ore reserve block, the extent of which is delineated on the ore reserve plan. All these individual ore reserve blocks are then summated and the values averaged to obtain a final estimate of the total tonnage and value of the fully developed ore in the mine. Subsequent mining operations carried out in these ore reserve blocks afford a means of checking the accuracy of the ore reserve estimates in the light of current sampling of stope faces as mining proceeds in each block. By this method, block factors are obtained which show that the average ore reserve values are invariably less than those deduced from current sampling results. While the significance of this phenomenon has been more fully discussed in the introduction to this thesis, it is obvious that the Block Factor, if indicative of the extent to which the ore reserve computer consistently under- or overestimates the value of the ore reserve, must be taken into consideration in attempting to arrive at a true estimate of the Ore Reserve Pay Limit based on undiscounted sampling results.

The Tonnage Discrepancy, usually an excess but rarely also a deficit, has formed the subject of much discussion and detailed investigation by many responsible mining officials on the Witwatersrand. The tonnage of <u>gold-bearing reef</u> broken in any underground excavation is measured by suitably combining an <u>area</u> measurement undertaken by surveyors, a <u>channel width</u> measurement performed by samplers and a <u>specific gravity</u> determination of the reef carried out experimentally in the laboratory. The utmost care is exercised in the measurement of areas, while channel widths, by the very nature of the occurrence of the gold-bearing reefs, can be determined with reasonable accuracy. Bearing in mind that any errors in the measurement of either of these quantities will be compensating in character, it may be assumed that the amount of gold-bearing reef measured as broken and sent to the mill will be reasonably reliable if the laboratory determination of the specific gravity of the reef is accepted. Hence if the total tonnage actually delivered to the mill <u>is less than</u> that measured as broken and sent to the mill, it seems reasonable to assume that the difference may be ascribed to the fact that some of the ore broken has been left underground in stopes and other excavations due to certain practical mining difficulties. In this case the Tonnage Discrepancy may reasonably be assumed to carry a value equal to the average of all the ore broken in stoping operations.

Usually, however, the total tonnage of rock actually delivered at the mill is <u>greater</u> than that measured as being sent to the mill from all underground sources. It has already been suggested that the measured tonnage of gold-bearing reef is probably reasonably accurate, and it may therefore be considered that this tonnage difference has originated entirely from non-auriferous sources. Numerous explanations have been given to account for this difference between the mill measurement and the surveyors' estimate of the tonnage milled, the most important of which probably are:-

(1) Under-estimation of stoping widths, due to the practical difficulties encountered in the measurement of this quantity. The exact extent of this under-estimation of stoping widths is practically impossible to assess, as the settlement of the hanging in present

deep level mining is so rapid that, even with the scaling which invariably occurs, stoping widths obtained from check measurements subsequently carried out will probably prove to be far less than those originally observed. In view of this fact, it is an extremely debatable point whether the common practice of measuring stoping widths ten feet back from the working face, in an attempt to include such waste as may be introduced by possible scaling from the hanging-wall, will actually result in a more reliable estimate of the true stoping width.

(ii) Intentional or unintentional tipping of waste into reef storage bins. Waste control methods have lately been developed to such an extent that the existence of such malpractices has been proved beyond all shadow of doubt. A reliable quantitative determination of the waste introduced into the ore stream in this way is however practically impossible, and the actual extent of such adulteration of the ore is subject to considerable speculation. It is the writer's considered opinion that this waste rock will account for only a small percentage of the usually alarmingly large tonnage discrepancy.

(iii) A feature which has enjoyed increased attention during latter years is that of the specific gravity of Witwatersrand gold ore. The previously determined density of 12 cubic feet to the ton yielded the very satisfactory and simple factor of stoping width ÷ 4 for the conversion of areas measured in fathoms into tonnages. Recent determinations have shown, however, that 11.8 cubic feet of ore to the ton is a closer approximation to the true density of the rock in situ.

This necessitates a factor of stoping width ÷ 3.933 for the conversion referred to above, and the added arithmetical complications introduced probably account for the fact that 11.8 cubic feet to the ton has not been universally adopted for use in surveyors' assessments of tonnage broken. Mill measurements, on the other hand, are invariably based on the latter density. If correctly allowed for, this difference in the basis of assessment in the two cases will have the result of increasing the surveyors' estimate of tonnage by 1.67 per cent. It will be obvious that average value will have to be assigned to that portion of the tonnage discrepancy which is due to this consideration. In view of the small proportion that this tonnage normally forms of the total discrepancy, it is doubtful whether such added refinement is justified.

From the above remarks it will be seen that, except for the known relatively negligible errors which will be introduced, it will be reasonable to assume for all practical purposes that the entire tonnage discrepancy originates from stoping sources. Furthermore, there is strong theoretical evidence to support the contention that if this discrepancy is an excess, it will carry no value, whereas if it is a deficit, it may be credited with the average sampling value of all the ore broken in stoping operations.

In determining an Ore Reserve Pay Limit, the tonnage discrepancy is of considerable importance, and must be taken into consideration. The above assumption, justified on the grounds mentioned, will result in an appreciable simplification of the actual pay limit calculation. This is well illustrated in the example below, in which the results of operations on

Mine "X" for the year 1948 have been taken to form the basis of the computation:-

Price of Gold — 172/6 per ounce.

Working Expenditure (Including Development Costs) — 19/6 per ton milled.

This cost, expressed in pennyweights of gold, will be $\frac{19 \cdot 5}{8 \cdot 625}$, or 2·261 dwt per ton milled.

Gold lost in each ton of Residue is 0·202 dwt.

Thus the required milling Value of the ore delivered for treatment to the reduction works, which will cover the total costs and also allow for the gold discarded in the form of residues, will be 2·261 + 0·202, or 2·463 dwt per ton.

In passing from the mine to the mill, however, the difference between the estimated value of the ore delivered to the mill, deduced from mine sampling results, and the actual value of that ore as calculated from the gold yield, the tonnage milled and the residue value, must be allowed for. This is done by the correct application of the Mine Call Factor.

Hence the required sampling Value of the ore milled in order to cover costs and allow for the gold lost in the residue, the Mine Call Factor being 90 per cent, will be

$$\frac{2 \cdot 463 \times 100}{90}, \text{ or } 2 \cdot 737 \text{ dwt per ton.}$$

Tracing the logical sequence through which the ore is passed, backwards from the mill to its stoping source, and allowing for the ore derived from the various other underground sources in their correct positions, the pay limit for stoping can be deduced from the actual results of past operations, assuming that these will be indicative of future practice, as follows:-

	Tons	Value	Contents
Ore Milled at a <u>Sampling</u> Value which will result in a yield just sufficient to cover the cost of Production	1,800,000	2.737	4,926,600
Plus Waste sorted on Surface	168,000	0.5	84,000
Total Ore hoisted for delivery to the Sorting Plant from all Underground Sources	1,968,000	2.5	5,010,600
Less Ore from Development	52,000	2.9	150,800
Less Ore from all other Underground Sources	115,000	2.1	241,500
Ore sent to Sorting Plant from purely Stoping Sources	1,801,000	2.6	4,618,300
Plus Waste sorted in Stopes	120,000	1.0	120,000
Total Ore actually derived from Stoping Sources before the advent of Underground Sorting	1,921,000	2.466	4,738,300

This value of 2.466 dwt per ton represents the limiting <u>sampling value</u> of a block of ore which will result in the required minimum milling value of 2.465 dwt per ton. As it has been deduced directly from the tonnage milled, it <u>includes</u> the tonnage discrepancy.

Since ore reserve computers usually either under- or over-estimate the gold contained in the blocks by an average amount represented by the Block Factor, the above limiting value will therefore further have to be adjusted by this factor in order to arrive at a pay limit for Ore Reserve blocks. In the case of Mine "X", the Block Factor has proved to be reasonably constant over a number of years at roughly 110 per cent. Hence the Ore Reserve Block Value which will result in a sampling value of 2·466 dwt per ton will be

$$\frac{2{\cdot}466 \times 100}{110}, \text{ or } 2{\cdot}242 \text{ dwt per ton.}$$

Thus, if development costs are included in the total working expenditure, the Ore Reserve Pay Limit for Mine "X" will be 2·242 dwt per ton.

2. The Current Pay Limit.

As stated before, this is the standard by which the payability of current working faces is assessed, and it must therefore be deduced from measured tonnages. In other words, it is an expression of the pay limit in terms of measured stoping widths. Since the measured widths exclude the consideration of tonnage discrepancy, the latter is ignored in the calculation of the Current Pay Limit. The Block Factor also falls away, since individual stope faces, and not blocks, are dealt with.

Thus, pursuing the previous example to its logical conclusion for measured stoping tonnages, the following results are obtained:-

	Tons	Value	Contents
Total Ore actually derived from Stoping Sources before the advent of Underground Sorting	1,921,000	2·466	4,738,300
Less Tonnage Discrepancy, which in this case is an Excess, and is therefore assumed to carry no Value	191,000	Nil	Nil
Ore measured as broken in Stopes before the advent of Underground Sorting	1,730,000	2·739	4,738,300

The Current Pay Limit for stoping, to be used in conjunction with measured stoping widths, is therefore 2·739 dwt per ton. This value for the pay limit has been based on the inclusion of development costs in the total working expenditure.

The tonnage from stoping sources in the above example was obtained from a total area of 128,100 fathoms, for which the average measured width was found to be 54.0 inches. Assuming that the entire Tonnage Discrepancy originated from stoping sources, the average actual width over the area stoped should have been

$$\frac{1,921,000 \times 4}{128,100} = 60.0 \text{ inches},$$

which compares favourably with the estimated block width of 61.65 inches for the portion of Mine "X" selected for these investigations into the frequency of occurrence of values. It will be obvious that the value of 2.242 dwt per ton as calculated above for the Ore Reserve Pay Limit will be applicable only on condition that a scale of widths which takes the tonnage discrepancy into consideration is employed in the estimation of block widths.

6. Working Expenditure and Development Costs.

The pay limits deduced above for Mine "X" are rather below the average for the Industry. This is primarily due to the exceptionally low working expenditure per ton milled for Mine "X", resulting mainly from the following:-

(i) The comparatively high milling rate of approximately 150,000 tons per month. It is a well-known fact that, owing to the high percentage of so-called "standing" or "overhead" expenses, the cost per ton milled can be appreciably reduced by increasing the monthly milling rate to the maximum possible capacity of the mill. In this connection it may be stated that on the average the mines of the Witwatersrand are at present milling at less than 85 per cent of their possible capacity.

(ii) The low development cost. Since Mine "X" is rapidly approaching the fully developed stage, the development has dropped to the rather low total of about 2,500 feet per month. The resultant expenditure on development is therefore approximately 2/6 per ton milled, as compared with the average for the industry of just under 4/- per ton milled.

The conception of development costs raises the much discussed question - should development costs be included in or excluded from the total working expenditure in the calculation of pay limits?

Since the solution to this question is largely a matter of individual opinion, it is not here intended to formulate any fixed policy to be adopted. The following, however, are some of the writer's reasons for considering that development costs should be excluded from the total expenditure for the purpose of determining pay limits:-

(i) In the opening up of a new mine, both shaft sinking and development operations are charged to capital account. Once the mine has embarked upon its production programme, however, the expenditure on development is switched to working costs, while shaft sinking, should this still be in progress or subsequently become necessary for the further exploitation of the reef, remains a capital charge. The logic of such discriminatory treatment of the costs is questionable. The expenditure connected with shaft sinking on producing mines, although a charge against capital account, is eventually appropriated from profits. The same procedure in respect of development costs will in most cases have a material effect on the value of the pay limit.

(ii) In the earlier stages of the life of a mine, development has to proceed at the maximum possible rate in order not only to maintain, but also to augment the ore reserve. The development costs will consequently be comparatively high. As the mine becomes more and more developed, however, the rate of development will naturally decrease, and the costs connected with this operation will therefore also diminish. If development costs are included in the total working expenditure, the latter will be subject to undue fluctuations, which will inevitably be reflected in the value of the pay limit. In this way a certain proportion of low-grade ore, which was considered unprofitable in the early stages of the life of the mine, may subsequently be rendered payable. This would not be serious were it not for the fact that, during the later stages in the life of the mine, a considerable proportion of the ore that has been rendered payable by the decrease in the total expenditure resulting from the reduction in development costs, will either have become altogether unavailable, or will have become entirely uneconomical to extract owing to certain practical mining reasons.

It appears justifiable, therefore, to exclude development costs from the total working expenditure in the calculation of pay limits, and to list the latter as a capital charge, subsequently to be recovered from profits.

The complications introduced into the determination of the pay limit if development costs are charged to capital account may be illustrated by suitably modifying the previous examples for Mine "E" as follows:-

Price of gold = 172/6 per ounce, or	8·625/- per dwt.
Total Working Expenditure	= 19/6 per ton milled.
Total Cost attributable to Development	= 2/6 per ton milled.
Working Expenditure (Excluding Development Costs)	= 17/- per ton milled.

This latter cost, expressed in pennyweights of gold, will be 1·971 dwt per ton.

Gold lost in Residues = 0.202 dwt per ton.

Hence the required value of the ore to be delivered to the mill to cover the cost of production (excluding development) and allow for the gold lost in residues will be 1.971 + 0.202 = 2.173 dwt per ton.

The Mine Call Factor being 90 per cent, the required SAMPLING Value of the ore to be delivered to the mill to cover costs and allow for the gold lost in residues will be $\frac{2.173 \times 100}{90}$, or 2.414 dwt per ton.

	Tons	Value	Contents
Ore Milled at the above required Sampling Value	1,800,000	2.414	4,345,200
Plus Waste Sorted on Surface	168,000	0.5	84,000
Total Ore Hoisted for delivery to the Sorting Plant, from all Underground Sources	1,968,000	2.251	4,429,200
Less Ore from Development	52,000	0.202*	10,500
Less Ore from all other Underground Sources	115,000	2.1	241,500
Ore sent to Sorting Plant from purely Stoping Sources	1,801,000	2.319	4,177,200
Plus Waste Sorted in Stopes	120,000	1.0	120,000
Total actually derived from Stoping Sources before the advent of Underground Sorting	1,921,000	2.237	4,297,200
Less Tonnage Unaccounted for (Tonnage Discrepancy)	191,000	Nil	Nil
Ore Measured as broken before the advent of Underground Sorting	1,730,000	2.484	4,297,200

The Ore Reserve Pay Limit = $\frac{2.237 \times 100}{110}$, or 2.034 dwt/ton.

and the Current Pay Limit = 2.484 dwt per ton.

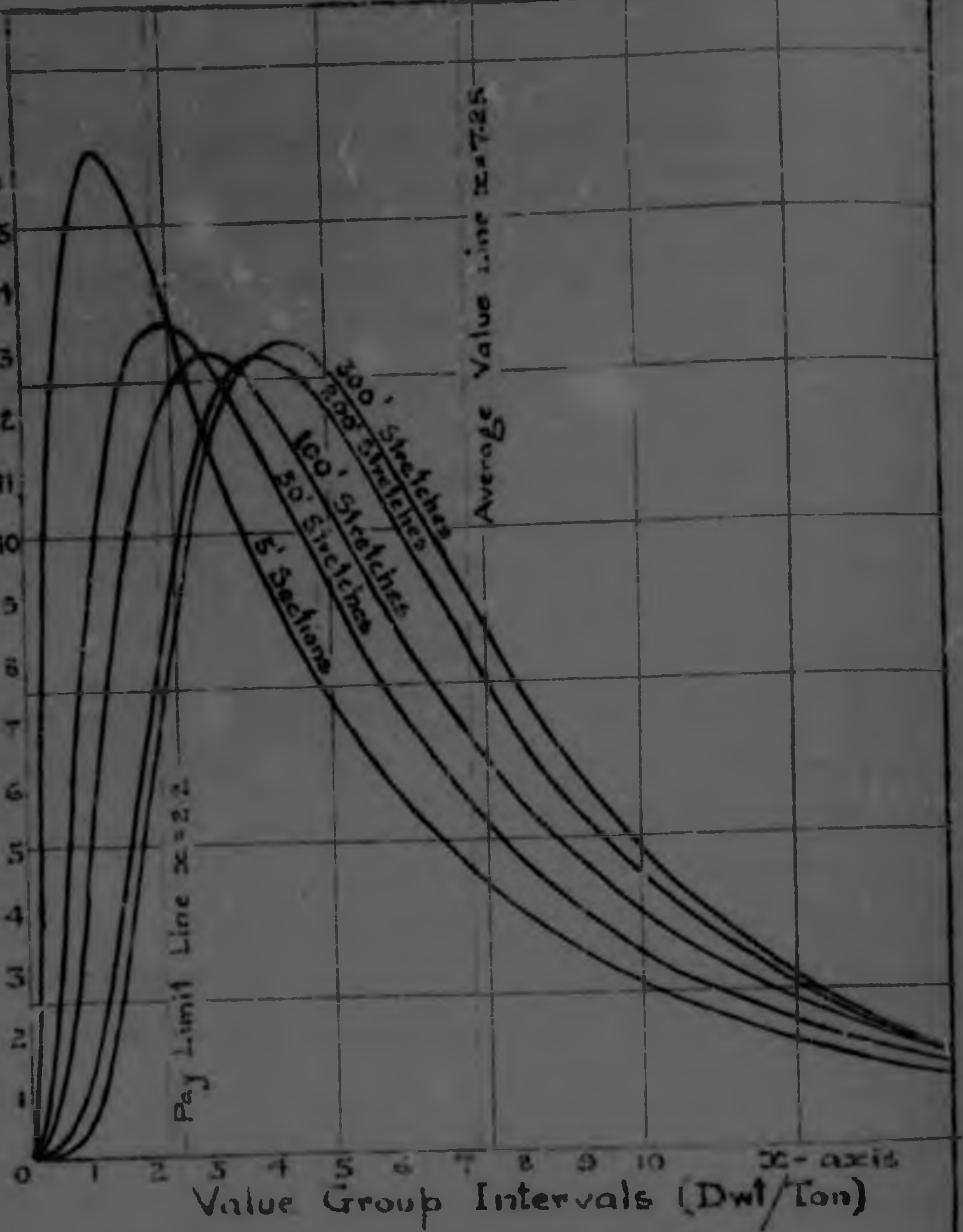
*In the above calculation, effect has been given to the perfectly logical contention that if the cost of development is to be charged to capital account, then the revenue derived from the treatment of development rock will have to be credited to the same account. The revenue derived from the crushing of development rock is represented by the original value of 2.9 dwt per ton, less the residue value of 0.202 dwt per ton. The latter will therefore represent the portion of the total development value which will have to be credited to the current working account, as has been done in the above pay limit determination. In the previous examples, where the cost of development was charged to working expenditure, the full development rock value of 2.9 dwt per ton was credited to the current working account.

It may here be stated that some of the mining groups, in an attempt at striking a middle course, have decided to treat a certain proportion of the total development expenditure as a charge against working costs, the remainder being regarded as a capital charge. Without entering into a discussion on the legitimacy of this cost distribution, it will readily be appreciated that the necessary modification of the pay limit calculation to meet requirements introduced by the adoption of this method is a relatively simple matter.

While it has been found that exclusion of development costs from the total working expenditure results in a reduction of the pay limit from 2.242 dwt per ton to 2.034 dwt per ton in the case of Mine "E", a study of the method employed in the computation will reveal that a decrease in the pay limit need not necessarily result from such a course of procedure in all cases. The magnitude of development costs, as well as the amount and value of the rock broken and sent to the mill from development sources will be the factors which will determine whether the pay limit will be increased or decreased by the inclusion of development costs in, or their exclusion from, the total working expenditure.

In this brief discussion on the determination of pay limits it has been shown that, with the exclusion of development costs from the total working expenditure, the Ore Reserve Pay Limit for Mine "X" will be 2.03 dwt per ton. When estimating the Total Ore Reserve of the mine by the conventional method of blocking, this pay limit of say 2.0 dwt per ton should be used in conjunction with the scale of widths which takes the Tonnage Discrepancy into account.

It should be pointed out, however, that in the determination of average values by the Frequency Method outlined in the preceding chapters, no blocking out of any description has been resorted to, and the conception of Block Factors is therefore entirely eliminated. Hence the pay limit to be applied to the Frequency Distribution Law in order to arrive at an estimate of payability will be the Ore Reserve Pay Limit as determined in the foregoing example, prior to its adjustment by the Block Factor. This will be seen to be 2.237 dwt per ton. The following section of Chapter IV will be devoted to showing how this value of say 2.2 dwt per ton for the Pay Limit may be incorporated in the Basic Value Distribution Theory in the case of Mine "X".



— PLATE XV —

MINE E: The Pay Limit of 2.2 dwt/ton in Relation to a Set of Frequency Curves for Estimated Stoping Values obtained from Individual Sampling Sections spaced at 5ft. Intervals, and by Combining these Individual Sampling Sections into 50, 100, 200 and 300 ft. Stretch Lengths respectively.

B. <u>The Pay Limit in Relation to the Fundamental Value Distribution Law in the Case of Mine "X".</u>

(i) <u>Application of the Pay Limit of 2.2 dwt per ton to the Distribution of Estimated Stoping Values derived from Individual Sampling Sections spaced at 5 ft Intervals.</u>

The Pay Limit of 2.2 dwt per ton in relation to this particular Frequency Distribution, which is represented by the equation

$$y = 16.142\, e^{-0.4374\,(\log_e x - 0.2664)^2}$$

is illustrated in Plate XV.

The percentage payability of Mine "X", deduced from the values of the individual sampling sections spaced at 5 ft. intervals, will be represented by the area under that portion of the frequency curve lying to the right of the line representing the Pay Limit. This area, and thus also the percentage payability, may be calculated from equation [16], as follows:-

$$T_{2.2} = 50\,\frac{2}{\sqrt{\pi}} \int_{w + 1/2a}^{\infty} e^{-w^2}\, dw$$

where $w = 0.6614\,(\log_e 2.2 - 0.2664) - \frac{1}{0.6614}$

$= -1.1666$

and $w + 1/2a = -0.4106$.

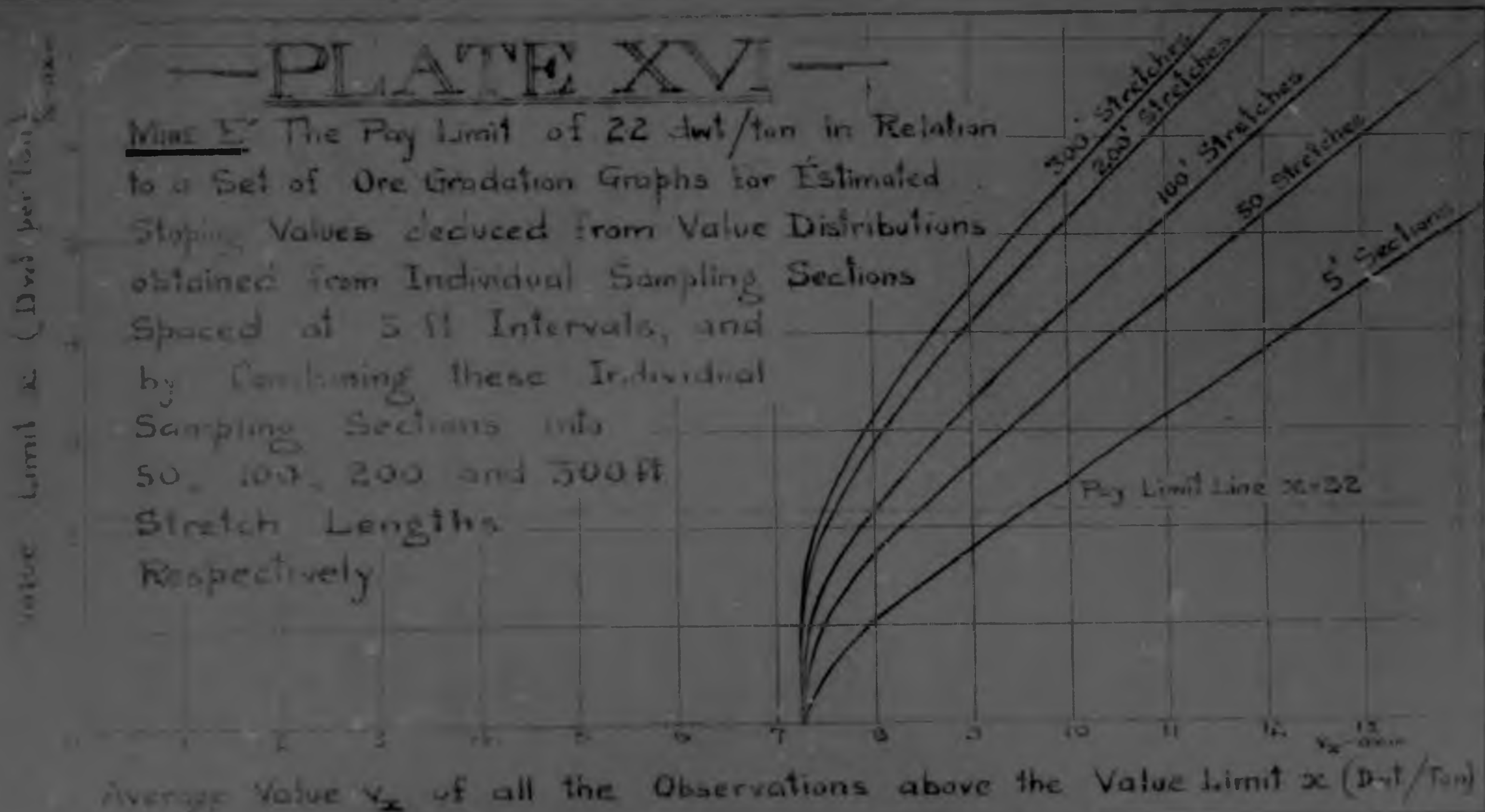
Hence, from statistical tables,

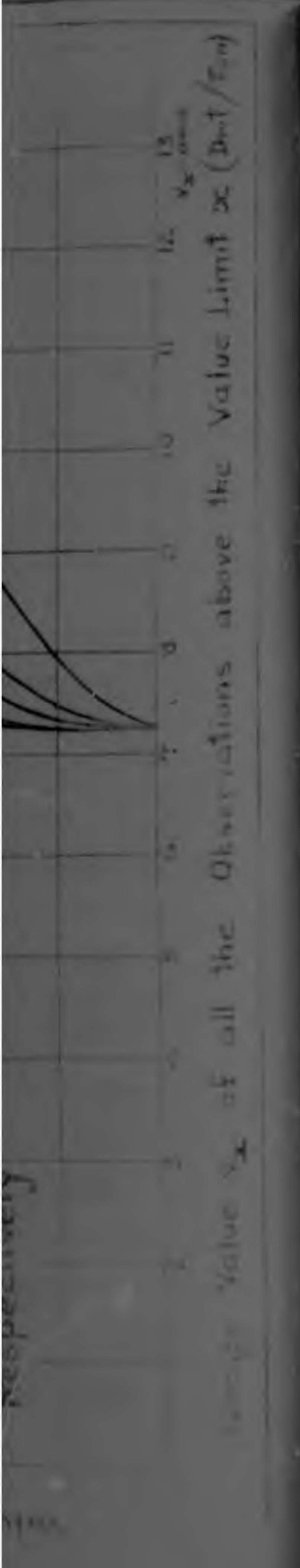
$\underline{T_{2.2} = 71.93 \text{ per cent.}}$

The average value of the ore above the specified pay limit may either be read off from the point where the line representing the pay limit cuts the Ore Gradation Graph for 5 ft. sections (illustrated in Plate XVI), or alternatively it may be calculated from

—PLATE XVI—

Mine "E" The Pay Limit of 22 dwt/ton in Relation to a Set of Ore Gradation Graphs for Estimated Stoping Values deduced from Value Distributions obtained from Individual Sampling Sections Spaced at 5 ft Intervals, and by Combining these Individual Sampling Sections into 50, 100, 200 and 300 ft Stretch Lengths Respectively





equation [17] as shown below:-

$$V_{2.2} = 7{\cdot}250 \frac{\frac{2}{\sqrt{\pi}} \int_{w}^{\infty} e^{-v^2} dv}{\frac{2}{\sqrt{\pi}} \int_{w+1/2a}^{\infty} e^{-v^2} dv}$$

where, as before, $w = -1.1666$

and $w + {}^1/_{2a} = -0.4106$

Thus, from statistical tables,

$$V_{2.2} = 7{\cdot}250 \frac{1.9008}{1.4385}$$

or $\underline{V_{2.2} = 9.58 \text{ dwt per ton.}}$

Hence, for the distribution of estimated stoping values obtained from individual sampling sections, 71.93 per cent of the ore in the selected portion of Mine "X" will be payable at an average value of 9.58 dwt per ton.

(ii) <u>Application of the Pay Limit of 2.2 dwt per ton to the Distribution of Estimated Stoping Values obtained by combining Individual Sampling Sections into 50 ft. Stretch Lengths</u>.

The Pay Limit in relation to the above Frequency Distribution, represented by the equation

$$y = 13.482\, e^{-0.6606 (\log_e x - 0.8460)^2}$$

is illustrated in Plate XV.

The percentage payability is in this case represented by the area under the frequency curve for 50ft. stretches, lying to the right of the line representing the pay limit. This area, and therefore also the percentage payability, may be calculated from equation [16], as follows:-

$$P_{2.2} = 50 \; \frac{2}{\sqrt{\pi}} \int_{v+1/2a}^{\infty} e^{-v^2} \, dv$$

where $v = 0.8129 \, (\log_e 2.2 - 0.8460) - \dfrac{1}{0.8129}$

or $v = -1.2770$

and $v + 1/2a = -0.6619$.

Hence the percentage payability can be deduced from statistical tables as

$$\underline{P_{2.2} = 82.55 \text{ per cent.}}$$

The average value of all the ore above the pay limit is represented graphically in Plate XVI by the point where the line representing the pay limit cuts the relevant Ore Gradation Graph. This value may also be calculated from equation [17] as shown below:-

$$V_{2.2} = 7{\cdot}250 \; \frac{\dfrac{2}{\sqrt{\pi}} \displaystyle\int_{v}^{\infty} e^{-v^2} \, dv}{\dfrac{2}{\sqrt{\pi}} \displaystyle\int_{v+1/2a}^{\infty} e^{-v^2} \, dv}$$

where, as above, $v = -1.2770$

and $v + 1/2a = -0.6619$.

Thus, from statistical tables,

$$V_{2.2} = 7{\cdot}250 \; \frac{1{\cdot}9290}{1{\cdot}6509}$$

or $\underline{V_{2.2} = 8.47 \text{ dwt per ton.}}$

Hence, if values are expressed as the averages obtained by combining individual sampling sections into 50 ft. stretch lengths, 82·55 per cent of the total ore in the selected portion of Mine "Z" will be payable at an average value of 8.47 dwt per ton.

(iii) **Application of the Pay Limit of 2.2 Dwt per Ton to The Distribution of Estimated Stoping Values obtained by Combining Individual Sampling Sections into 100 ft. Stretch Lengths.**

This distribution is represented by the equation

$$y = 12.918\, e^{-0.8564\,(\log_e x - 1.1032)^2}$$

its relation to the Pay Limit being graphically illustrated in Plate XV.

The percentage payability is represented, as before, by the area under that portion of the frequency curve for 100 ft. stretches lying to the right of the line representing the pay limit. This area, and consequently also the percentage payability, may be calculated as follows:-

$$P_{2.2} = 50\,\frac{2}{\sqrt{\pi}} \int_{w + 1/2a}^{\infty} e^{-w^2}\,dw$$

where $\quad w = 0.9254\,(\log_e 2{\cdot}2 - 1.1032) - \frac{1}{0.9254}$

or $\quad w = -1.3738$

and $w + 1/_{2a} = -0.8335$

From statistical tables, therefore,

$$\underline{P_{2.2} = 88.08 \text{ per cent.}}$$

The average value of all the ore above the pay limit is represented graphically in Plate XVI by the point where the pay limit line cuts the Ore Gradation Graph for values grouped to represent averages over 100 ft. stretch lengths. This point may be more accurately calculated from equation [IV] as shown below:-

$$V_{2.2} = 7.250\;\frac{\dfrac{2}{\sqrt{\pi}} \displaystyle\int_{w}^{\infty} e^{-w^2}\,dw}{\dfrac{2}{\sqrt{\pi}} \displaystyle\int_{w+1/2a}^{\infty} e^{-w^2}\,dw}$$

where $w = -1.3738$

and $w + 1/2a = -0.8335$, as above.

From tables, this average value will be found to be

$$V_{2.2} = 7{\cdot}250 \; \frac{1{\cdot}9472}{1{\cdot}7615}$$

or $\underline{V_{2.2} = 8{\cdot}02 \text{ dwt per ton.}}$

Thus, if the original values are so grouped as to represent the averages over 100 ft. stretch lengths, 88·06 per cent of the total ore in the selected area of Mine "E" will be rendered payable at an average value of 8·02 dwt per ton.

(iv) <u>Application of the Pay Limit of 2·2 Dwt per Ton to the Distribution of Estimated Stoping Values obtained by Combining Individual Sampling Sections into 200 ft. Stretch Lengths.</u>

The equation of this particular distribution is

$$y = 12{\cdot}939 \, e^{-1{\cdot}2082 \, (\log_e x - 1.3602)^2}$$

and its relation to the pay limit is illustrated in Plate XV.

The percentage payability, graphically represented by the area under that portion of the corresponding frequency curve lying to the right of the line denoting the pay limit, may be calculated as follows:-

$$P_{2.2} = 50 \, \frac{2}{\sqrt{\pi}} \int_{w+1/2a}^{\infty} e^{-w^2} \, dw \qquad \text{(From eqn. [16])}$$

where $w = 1{\cdot}0992 \, (\log_e 2{\cdot}2 - 1{\cdot}3602) - \frac{1}{1{\cdot}0992}$

$= -1{\cdot}5383$

and $w + 1/2a = -1{\cdot}0834$.

Hence, from statistical tables,

$\underline{P_{2.2} = 93{\cdot}67 \text{ per cent.}}$

The average value of the ore above the pay limit is represented graphically by the point where the line denoting the pay limit cuts the Ore Gradation Graph for 200 ft. stretches in Plate XVI. This value may be calculated from equation [17] as shown below:-

$$v_{2\cdot 2} = 7\cdot 250 \; \frac{\frac{2}{\sqrt{\pi}} \int_{v}^{\infty} e^{-v^2} \, dv}{\frac{2}{\sqrt{\pi}} \int_{v+1/2a}^{\infty} e^{-v^2} \, dv}$$

where, as before, $v = -1\cdot 5383$

and $v + 1/_{2a} = -1\cdot 0834$.

Thus, from tables,

$$\underline{v_{2\cdot 2} = 7\cdot 62 \text{ dwt per ton.}}$$

Hence, if the original observations are combined into groups, each of which represents the average over a 200 ft. stretch length, 95·67 per cent of the total ore in the selected portion of Mine "B" will be returned as payable at an average value of 7·62 dwt per ton.

(v) <u>Application of the Pay Limit of 2·2 Dwt per Ton to the Distribution of Estimated Stoping Values obtained by combining Individual Sampling Sections into 300 ft. Stretch Lengths.</u>

The distribution of average values over 300 ft. stretch lengths is denoted by the equation

$$y = 13\cdot 185 \, e^{-1\cdot 4180 \, (\log_e x - 1\cdot 4921)^2}$$

The relation between the pay limit and this curve is illustrated in Plate XV. The area under that portion of the frequency curve for 300 ft. stretches to the right of the line representing the pay limit, is indicative of the percentage payability. This area, and therefore the percentage payability, may also be calculated as shown below:-

$$z_{2\cdot 2} = 50 \, \frac{2}{\sqrt{\pi}} \int_{v+1/2a}^{\infty} e^{-v^2} \, dv$$

where $\quad w = 1{\cdot}1908\,(\log_e 2{\cdot}2 - 1{\cdot}4281) - \frac{1}{1{\cdot}1908}$

$\qquad = -\,1{\cdot}6301$

and $w + {}^{1}/_{2a} = -\,1{\cdot}2102$

Hence, from statistical tables,

$$\underline{Z_{2\cdot2} = 95{\cdot}65 \text{ per cent.}}$$

The average value of all the ore above the pay limit is graphically illustrated in Plate XVI by the point where the line denoting the pay limit cuts the Ore Gradation Graph for 300 ft. stretch lengths. This average value may be calculated from equation [25], as follows:-

$$V_{2\cdot2} = 7{\cdot}230\,\frac{\frac{2}{\sqrt{\pi}}\int_{w}^{\infty} e^{-w^2}\,dw}{\frac{2}{\sqrt{\pi}}\int_{w+1/2a}^{\infty} e^{-w^2}\,dw}$$

where, as above, $w = -\,1{\cdot}6301$

and $\quad w + {}^{1}/_{2a} = -\,1{\cdot}2102$.

Hence, from tables,

$$\underline{V_{2\cdot2} = 7{\cdot}90 \text{ dwt per ton.}}$$

Thus, judged in the light of values obtained by combining individual sampling sections into 300 ft. stretch lengths, 95·65 per cent of the total ore in the selected portion of Mine "X" will be payable at an average value of 7·90 dwt per ton.

2. Percentage Payability.

The percentage payability and the average value corresponding to the frequency distributions for the various stretch lengths as calculated in the case of Mine "X" above, have been combined into the form of a table, which is reproduced on the following page:-

TABLE XXI.

Pay Limit (Dwt/Ton)	Stretch Length (Feet)	Percentage Payability	Average Value (Dwt/Ton)
2.2	5 (Indiv. Sects.)	71.93	9.58
2.2	50	82.55	8.47
2.2	100	88.08	8.02
2.2	200	93.67	7.62
2.2	300	95.65	7.50

For the conditions existing on Mine "X" it is therefore obvious from the above table, as well as from Plates XV and XVI, that as the stretch lengths into which individual sampling sections have been combined for the purposes of averaging are increased, the percentage payability will also be increased, while the average value of the ore above the pay limit will be decreased. One of the objects of combining sampling sections into stretch lengths should be to obtain as high a percentage payability as possible. As this can only be achieved at the expense of the average value, the desirability of striking the correct balance between a high percentage payability and a high value will at once become evident. The question thus arises as to what stretch length should be employed for the grouping of individual sampling sections in order to obtain this desired balance. The answer is of course to be found in the Optimum Stretch Length, the mathematical significance of which has already been discussed. It will readily be appreciated that the Optimum Stretch Length affords a practical standard by which the percentage payability and the corresponding average value of a mine can be assessed. Its universal adoption will go a long way towards introducing the uniformity which is so desirable when comparing different mines, each of which bases the determination of its percentage payability and average value on methods learned either from practical experience or from old established principles having little or no scientific backing.

The phrase "for the conditions existing on Mine 'X'" at the commencement of the previous paragraph has purposely been underlined, as the trend revealed in Table XXI is not universally applicable to the conditions existing on all mines. In order to explain this statement, recourse will again have to be had to some of the fundamental mathematical deductions of Chapter II.

From equation [14] the total observations having values in excess of a given value limit "x", is denoted by

$$Z_x = 50 \frac{2}{\sqrt{\pi}} \int_{w+1/2a}^{\infty} e^{-w^2} dw$$

where $w = a(\log_e x - b) - \frac{1}{a}$

If the pay limit be now denoted by "p", then the percentage payability "Z_p" will be

$$Z_p = 50 \frac{2}{\sqrt{\pi}} \int_{w+1/2a}^{\infty} e^{-w^2} dw$$

where $w = a(\log_e p - b) - \frac{1}{a}$

and $w + 1/_{2a} = a(\log_e p - b) - \frac{1}{2a}$

From the above expression for "Z_p" it will be seen that if the lower limit of integration $(w + \frac{1}{2a})$ decreases as "s" is increased (i.e. as the stretch lengths are increased), then the percentage payability will actually increase as "s" is increased.

But $w + \frac{1}{2a} = a(\log_e p - b) - \frac{1}{2a}$.

[illegible], from equation [8],

$$m = e^{b + 3/4a^2}$$

$$\text{or} \quad b + \frac{1}{4a^2} = \log_e m$$

and thus

$$b = \log_e m - \frac{1}{4a^2}$$

Substituting this value of "b" in the equation above,

$$w + \frac{1}{2a} = a(\log_e p - \log_e m + \frac{1}{4a^2}) - \frac{1}{2a}$$

$$= a(\log_e \frac{p}{m}) + \frac{1}{4a} - \frac{1}{2a}$$

$$\text{or} \quad \underline{w + \frac{1}{2a} = a \log_e \frac{p}{m} + \frac{1}{4a}} \text{ ---------------------- [30]}$$

It has previously been pointed out that all the values of the parameter "a" likely to be encountered in practice will lie between 0·5 and 1·5. For this range of values, it has been ascertained, from a graphical analysis of equation [30], that if the ratio $\frac{p}{m}$ exceeds $\frac{3}{2}$, then the numerical value of the lower limit of integration $(w + \frac{1}{2a})$ will <u>increase as "a" is increased</u>. Thus, if the ratio of the pay limit to the mean value of an ore-body is greater than $\frac{3}{2}$, the percentage payability will decrease as the stretch lengths into which the individual sampling sections have been combined, are increased. Conversely, if the ratio of the pay limit to the mean value is less than $\frac{3}{2}$, the percentage payability will be increased as the stretch lengths are increased.

Since the average pay limit for the Gold Mining Industry on the Witwatersrand is approximately 3 dwt per ton, the ratio $\frac{p}{m}$ will be equal to $\frac{3}{2}$ if "m" = 2 dwt per ton. Thus it may be stated that, on the average, only in the case of those mines having a mean value of less than 2 dwt per ton, will the percentage payability be <u>increased</u> as the stretch lengths are <u>increased</u>. As an ore-body with a mean value of 2 dwt per ton contains such a low proportion of ore in the upper value groups as to render its economic exploitation an extremely doubtful matter, it may be stated, as a general rule, that increasing the stretch lengths into which individual sampling sections have been combined for valuation purposes, will result in an increase in the percentage payability, with a corresponding decrease in the average value.

The above analysis of the percentage payability completes the theoretical considerations with which it is proposed to deal in this thesis. The following chapter will be devoted entirely to illustrations showing the manner in which the theoretical work of the preceding chapters may be employed in the solution of certain practical problems.

CHAPTER V.

PRACTICAL APPLICATIONS OF THE FUNDAMENTAL STATISTICAL VALUE DISTRIBUTION THEORY.

The manner in which the theory developed in the foregoing chapters may be applied in practice can best be illustrated by suitably chosen examples. Due to lack of space, however, it will be possible to deal with a limited number of illustrations only, and the two examples below have therefore been carefully selected. They both represent practical problems which are of vital interest to the Gold Mining Industry at the present time. The solution of these problems by methods other than those making use of the Fundamental Value Distribution Theory would, if at all capable of a satisfactory scientific solution, at the very least involve a considerable amount of tedious work.

A. Example 1. The Effect of a Change in the Pay Limit on the Ore Reserve of Mine "E".

The average value of all the ore contained in the fully developed portion of Mine "E" (shown in Plate XXI), has been determined from the results of a completely random system of sampling, as 7·250 dwt per ton over an average estimated stoping width of 61·65 inches. The area comprises 748·6 claims, and the reef dips at 20 degrees. For the purposes of this example, it may be assumed that stoping operations have not yet commenced.

It is required to classify the total ore in this selected area of the mine into suitable value groups. Hence, assuming that 5 per cent of the area is not underlain by reef, owing to the presence of faults and dykes, calculate

(a) the total payable ore in the area of the mine shown in Plate XXI, assuming that the pay limit has been determined as 2·2 dwt per ton,

(b) the effect on this Ore Reserve if the pay limit is decreased to 2·0 dwt per ton, and

(c) the effect on the Ore Reserve if the pay limit is increased to 2·5 dwt per ton.

The total ore in the area is first calculated, as follows:-

Total Claim Area	=	748·6
Loss of Reef-bearing Area due to Faults and Dykes (5%)	=	37·4
∴ Reef-bearing Area	=	711·2 claims

$$\text{Total Tonnage} = \frac{711{\cdot}2 \times 64{,}025 \times 61{\cdot}65 \times 1{\cdot}0642}{36 \times 4}$$

$$= 20{,}746{,}000 \text{ tons.}$$

In order to obtain the <u>payable</u> tonnage, this <u>total tonnage</u> will have to be distributed throughout the various value groups in accordance with the Fundamental Value Distribution Law based on the Optimum Stretch Length. Since the mean value "m" is 7·250 dwt per ton, and the parameter "z" is unity for the Value Distribution based on the Optimum Stretch Length, the numerical value of "b" can be determined from equation [8], as follows:-

$$7{\cdot}250 = e^{b + 3/4}$$

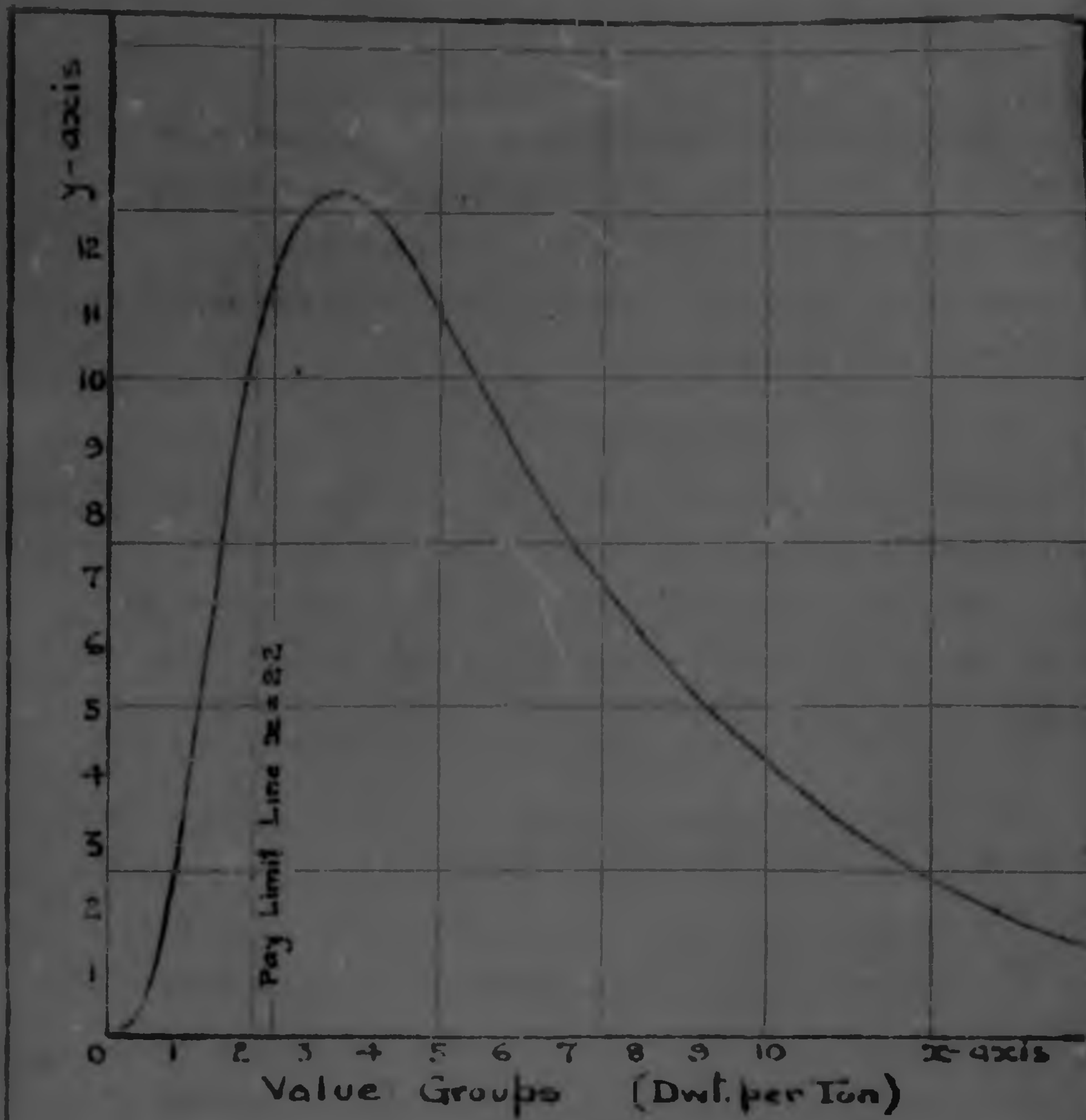
$$\text{or} \quad \underline{b = 1{\cdot}2310}\,.$$

From this value of "b", the corresponding value of the constant "N" for the Distribution can be calculated from equation [7] [illegible] shown below:-

$$N = \frac{100}{\sqrt{\pi}\, e^{b + 1/4}}$$

$$= \frac{93{\cdot}0186}{7{\cdot}250} \qquad \text{(See also equation [12])}$$

$$\text{or} \quad \underline{N = 12{\cdot}830}$$



—PLATE XVII—

MINE E: Frequency Curve representing the Distribution of Estimated Stoping Values which would have been obtained had the Individual Sampling Section Values been Combined into Optimum Stretch Lengths. (ie. Lengths of 14.2 ft.) Equation of Curve is

$$Y = 12.830 . e^{-(\log_e x - 1.2310)^2}$$

The Ideal Distribution for the ore contained in the selected portion of Mine "X", obtainable if the individual sampling sections are combined into Optimum Stretch Lengths (i.e. stretches of 142 ft.), will therefore be represented by the equation

$$y = 12 \cdot 850\, e^{-(\log_e x - 1 \cdot 2310)^2}$$

The frequency curve corresponding to this equation is graphically illustrated in Plate XVII.

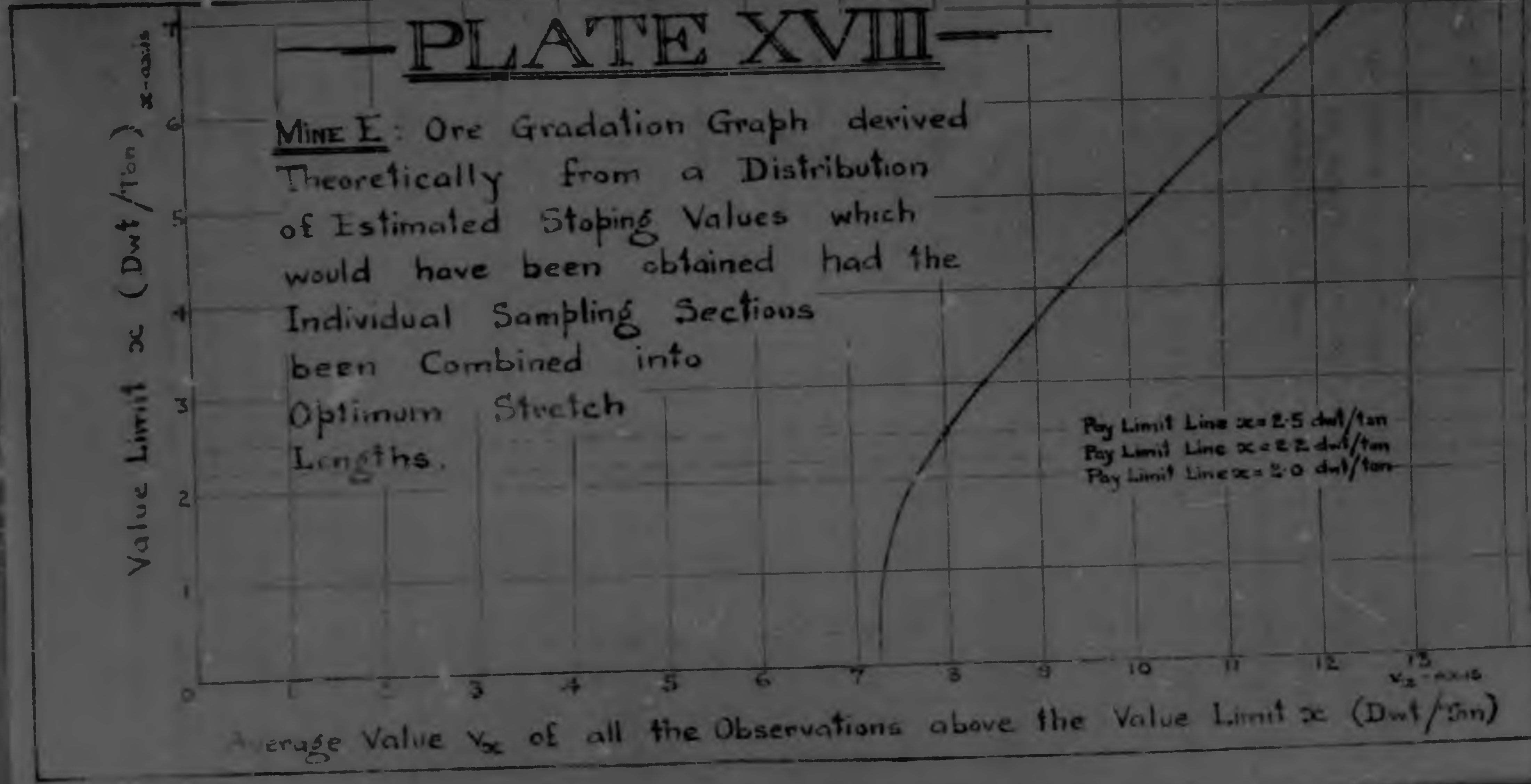
Table XXII below has been compiled to show the total tonnages and the corresponding average values of the ore above certain value limits "x":-

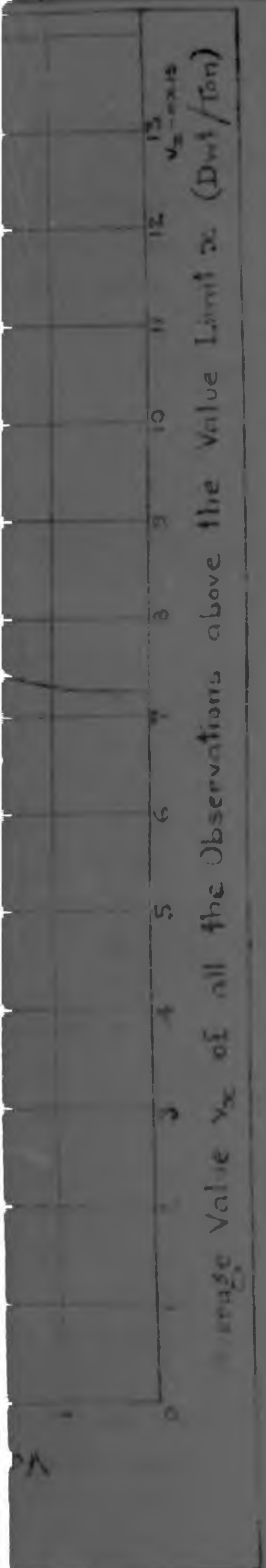
TABLE XXII.

Value Limit "x" (Dwt/Ton)	Total Ore having a Value equal to or greater than the Value Limit "x".		
	Percentage "Z_x" (From eqn. [15])	Tonnage	Average Value "V_x" (From eqn. [17]) Dwt/Ton
30	0.91	189,000	39.057
25	1.77	367,000	33.279
20	3.69	766,000	27.495
15	8.36	1,734,000	21.680
10	20.95	4,346,000	15.916
8	31.12	6,456,000	13.087
6	46.58	9,663,000	11.405
5	56.81	11,786,000	10.340
4	68.69	14,250,000	9.328
3	81.44	16,896,000	8.416
2·5	87.48	18,149,000	8.025
2·2	90.87	18,852,000	7.814
2·0	92.90	19,273,000	7.684
1	99.28	20,597,000	7.297
0	100.00	20,746,000	7.250

—PLATE XVIII—

Mine E: Ore Gradation Graph derived Theoretically from a Distribution of Estimated Stoping Values which would have been obtained had the Individual Sampling Sections been Combined into Optimum Stretch Lengths.





The average values in the last column of this table are represented graphically in Plate XVIII, which shows the Ore Gradation Graph which would have been obtained had the individual sampling sections been combined into Optimum Stretch Lengths on Mine "E".

From Table XXII, in which the total ore in the selected portion of Mine "E" has been classified into suitably chosen value groups, the solutions to the remaining three parts of the problem will immediately be obvious. Thus:-

(a) The Ore Reserve of that portion of Mine "E" shown in Plate III, prior to the commencement of stoping operations, consisted of 18,852,000 tons at an average sampling value of 7·81 dwt per ton over an estimated stoping width of 61·65 inches, the pay limit being 2·2 dwt per ton.

(b) If the pay limit on Mine "E" is reduced to 2·0 dwt per ton, this total Ore Reserve will be increased to 19,273,000 tons at an average sampling value of 7·68 dwt per ton over the estimated stoping width of 61·65 inches.

(c) If the pay limit on Mine "E" be increased to 2·5 dwt per ton, the total Ore Reserve will be decreased to 18,149,000 tons at an average sampling value of 8·03 dwt per ton over the estimated stoping width of 61·65 inches.

It should again be stressed, as has repeatedly been done in this thesis, that the values as determined above are based on uncut and undiscounted sampling results. Since the pay limit has been deduced with due regard to the Mine Call Factor (the Block Factor is automatically eliminated by this Frequency Method of Ore Reserve computation), the Ore Reserve tonnages deduced above will not be subject to any correction. The average values, however, have been derived from sampling results known to contain the normal errors inherent in the operation of sampling. These errors may be compensated for by the application of a globular discount to the Ore Reserve Value in consonance with the Mine Call Factor, <u>subsequent to the determination of the total ore reserve</u>, to obtain the probable true ore reserve value.

B. <u>Example 2</u>: <u>Required Increase in the Rate of Mining from the Higher Grades of Ore to achieve a certain Specified Increase in Yield</u>.

With the fulfilment of the urgent need for increasing the production of gold to assist the country through its present economic crisis rendered practically impossible by the acute shortage of both Native and European labour, the question has arisen as to the practicability, as a temporary expedient, of concentrating all the available labour either in the higher grade mines, or in the high grade areas of the individual mines.

This example is based on actual information obtained from twenty-three producing lease mines on the Witwatersrand, and has been drawn up to illustrate exactly what steps would have to be taken by the Industry to achieve a specified increase in yield while maintaining the tonnage milled at its present level.

Table XXIII on the following page shows the combined Ore Reserves of these twenty-three Witwatersrand producers, segregated into value group intervals, the average pay limit being 2·8 dwt per ton. It is required to investigate the possibility of obtaining an increase in yield of say £ 10 million per annum from this group of mines by the judicious redistribution of the available labour.

—PLATE XIX—

Histogram and Fitted Frequency Curve obtained from the Segregation of the Total Ore Reserve Tonnage of Twenty-three Lease Mines on the Witwatersrand, into Value Group Intervals of 1 Dwt per Ton.

Fitted Curve : $y = 49{\cdot}032 \times 10^{6} \cdot e^{-(\log_e x - 0{\cdot}5288)^2}$

$a = 1$

$m = 5{\cdot}55$ Dwt per Ton.

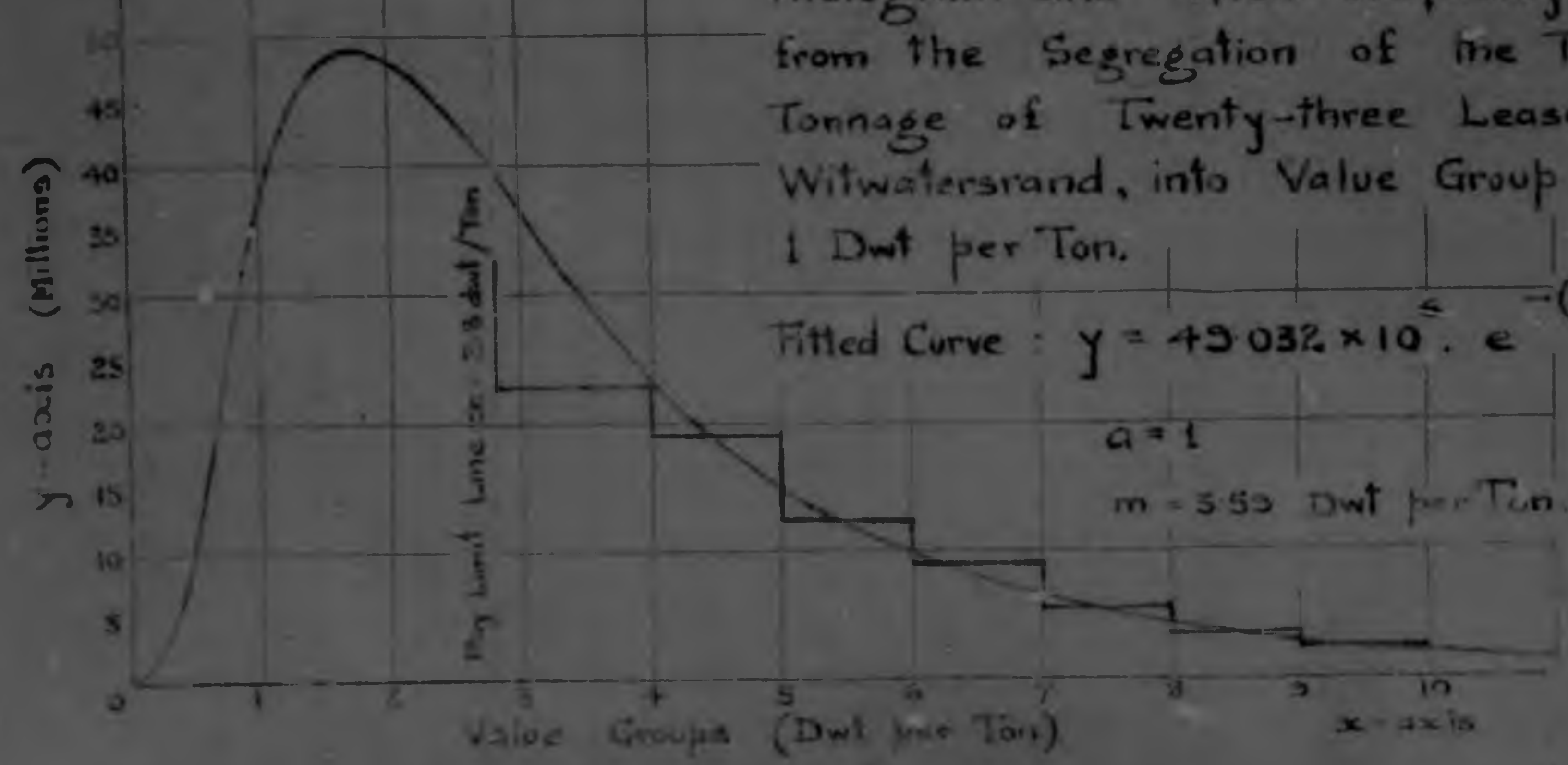
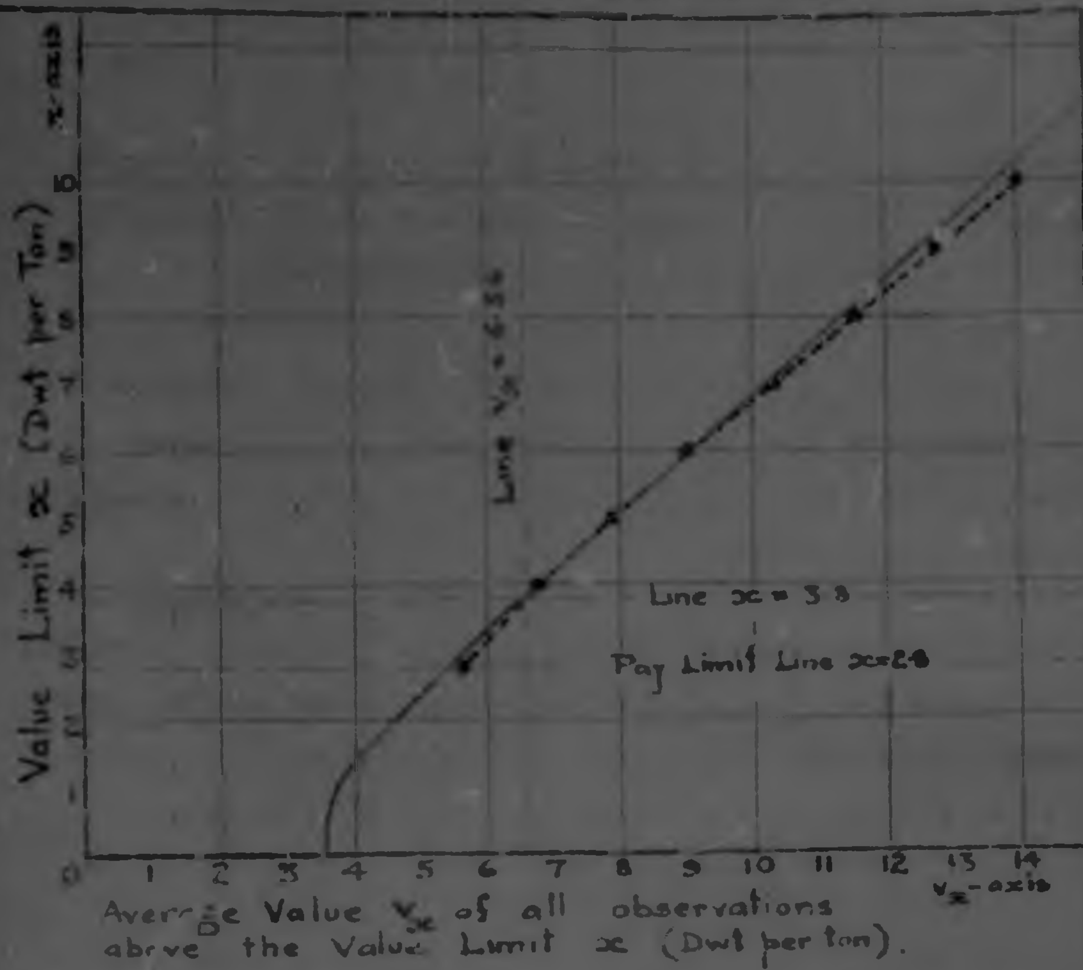


TABLE XXXIII.

Value Group (Dwt/Ton)	Tons	Average Group Value (Dwt/Ton)	Value Limit "x" (Dwt/Ton)	Total Ore above the Value Limit "x"	
				Tons	Average Value "v_x" (Dwt/Ton)
10.0 & over	6,882,000	14.04	10.0	6,882,000	14.04
9.0 - 9.9	2,539,000	9.56	9.0	9,421,000	12.82
8.0 - 8.9	3,564,000	8.44	8.0	12,985,000	11.62
7.0 - 7.9	5,450,000	7.40	7.0	18,435,000	10.37
6.0 - 6.9	9,036,000	6.42	6.0	27,471,000	9.07
5.0 - 5.9	12,281,000	5.42	5.0	39,752,000	7.94
4.0 - 4.9	18,969,000	4.40	4.0	58,721,000	6.80
2.8 - 3.9	27,250,000	3.18	2.8	85,971,000	5.[illegible]6

The Ore Reserve Tonnages in the various value groups are graphically represented by the histogram in Plate XIX. The average values calculated in the last column of the above table will enable the observed Ore Gradation Graph to be plotted. This graph is represented by the dotted line in Plate XX. As blocking out of the Ore Reserves has taken place above the pay limit only, both the observed histogram and the Ore Gradation Graph plotted from Table XXXIII cannot be completed for values below the average pay limit of 2·8 dwt per ton.

It may be pointed out here that the errors in blocking referred to in the Introduction to this thesis are distinctly discernible in the Ore Gradation Graph for the combined Ore Reserves of the twenty-three mines. The flattening of the curve for the higher value limits denotes relative over-valuation of the ore in the upper categories, while the steepening of the curve just above the pay limit apparently indicates exclusion of low-grade ore in the vicinity of the pay limit, which results in an inflation of the average Ore Reserve Value. The effect of these errors in the Ore



—PLATE XX—

Ore Gradation Graph obtained from the Distribution of the Total Ore Reserve Tonnage of Twenty-three Lease Mines on the Witwatersrand

——— Deduced Theoretically from the fitted Curve $y = 49032 \times 10\, e^{c^{2}-(\log_{10} x - 0.5288)^{2}}$

-·-·-·- As Calculated from the Actual Segregation of the Total Ore Reserve Tonnage into Value Group Intervals of 1 dwt per ton.

Reserve valuation is not nearly as marked in the above graph as has been found to be the case for some of the individual mines. This is probably due to the introduction of compensating errors on other mines.

The next step is to graduate the observed histogram in Plate XXX by a smooth curve. The curve which has been found to "fit" the actual observations practically perfectly, is denoted by the equation

$$y = 49\cdot032 \times 10^6 \times e^{-(\log_e x - 0\cdot5288)^2} \qquad \text{---[31]}$$

This equation is represented graphically by the red frequency curve in Plate XXX. It will be noticed that the Fundamental Distribution Law is again observed, the constants being

N = 49,032,000

a = 1·0000

and b = 0·5288

In this case, however, the numerical value of 49,032,000 deduced for the constant "N" will not result in a total frequency of occurrence of 100, since the observed histogram has been plotted on a <u>tonnage</u>, and not on a <u>percentage</u> frequency basis.

The following table has been compiled to show the comparison of the Ore Reserve tonnages obtained from the summation of the Ore Reserve Blocks in the various value categories with the corresponding tonnages deduced theoretically from the frequency curve defined by equation [31] above:-

TABLE XXIV.

Value Limit "x" (Dwt per Ton)	Tonnage deduced from a Segregation of the actual Ore Reserve Blocks	Tonnage deduced Theoretically from the Frequency Distribution of Equation [xi]*
10.0	6,862,000	6,782,000
9.0	9,4[illegible],000	9,323,000
8.0	12,985,000	13,024,000
7.0	18,435,000	18,419,000
6.0	27,471,000	26,568,000
5.0	39,752,000	38,348,000
4.0	58,721,000	58,123,000
2.8	85,971,000	94,599,000
2.0	Not Blocked	129,244,000
1.0	"	175,563,000
0.0	"	189,368,000

*This theoretical tonnage has been calculated from the formula

$$T_x = \frac{N}{a} e^{b + 1/4a^2} \int_{w+1/2a}^{\infty} e^{-w^2} dw$$

in which $a = 1$, and therefore

$$w = (\log_e x - 0{\cdot}5288) - 1 ,$$

$$\text{or } w + 1/_{2a} = (\log_e x - 0{\cdot}5288) - 0{\cdot}5 .$$

The theoretical tonnages calculated in the last column of Table XXIV above are represented graphically by the corresponding areas under the frequency curve in Plate XIX. The exact extent to which ore has been excluded in the 2·8 to 3·9 dwt per ton value group will be seen from Table XXIV to be 8,678,000 tons, or about 10 per cent of the total Ore Reserve Tonnage.

The theoretical Ore Gradation Graph for the distribution denoted by equation [31] may be calculated from the formula

$$v_x = m \frac{\frac{1}{\sqrt{\pi}} \int_{w}^{\infty} e^{-w^2} dw}{\frac{1}{\sqrt{\pi}} \int_{w+1/2a}^{\infty} e^{-w^2} dw}$$

where $a = 1$, and consequently

$$w = \log_e x - 1{\cdot}5288$$

and $$w + 1/_{2a} = \log_e x - 1{\cdot}0288 .$$

The Table below has been compiled in such a way as to afford a direct comparison between the theoretical Ore Gradation Graph deduced from the above formula, and that calculated from the segregation of the actual Ore Reserve Block Tonnages.

TABLE XXV.

Value Limit "x" (Dwt per Ton)	Average Value "v_x" of the Ore above the Value Limit "x"	
	Observed Values (Dwt per Ton)	Theoretical Values (Dwt per Ton)
10.0	14.04	13.73
9.0	12.82	12.56
8.0	11.62	11.38
7.0	10.37	10.24
6.0	9.07	9.08
5.0	7.94	7.93
4.0	6.80	6.78
2.8	9.35	5.46
2.0	Not Blocked	4.64
1.0	"	3.70
0.0	"	3.59

The theoretical Ore Gradation Graph deduced from the "fitted" frequency curve is represented by the red line in

Plate XX. It will be observed from Table XXIV, and its graphical representation in Plate XIX, that once again there is evidence of a remarkable agreement between the Fundamental Value Distribution Law represented by equation [31], and the actual results obtained from practical Ore Reserve considerations. Table XXV and its graphical representation in Plate XX also serve to illustrate the extent of the agreement between the Ore Gradation Graph deduced theoretically, and that obtained from practical observations. These facts provide added and conclusive evidence of the general applicability of this particular Value Distribution Theory to the ore deposits of the Witwatersrand. The fact that it has further been found possible to "fit" the observed frequencies of occurrence of Ore Reserve tonnages with a Fundamental Value Distribution curve in which the parameter "s" is unity, indicates that blocking out of the Ore Reserves has, on the average, taken place by combining sampling results into Optimum Stretch Lengths. The average size of ore reserve blocks may therefore be stated to be approximately in accordance with the Optimum Stretch Length. From the comparatively limited observations which have been made regarding the size of ore reserve blocks, it appears probable that this theoretical conclusion will be substantiated by more detailed practical investigations.

Due to the anomalies which exist in the actual practical Ore Reserve Tonnage Distribution, small though these apparently are in this case, the theoretically deduced tonnage distribution will be used in the further treatment of this problem.

The following additional information regarding the results of operations on the twenty-three producers under consideration during the year 1948 will be required:-

Tonnage Milled	= 27,114,000.
Tonnage drawn from Ore Reserve	= 16,685,000.
Average Ore Reserve Value	= 5·46 dwt per ton.
Total Yield	= £ 50,053,000

The yield is therefore 36·92/- per ton milled.
At the present price of gold of 172/6 per fine ounce, this represents a yield of 4·28 dwt per ton.

The ratio of the yield to the value of the Ore Reserve, which has been found to be reasonably constant over a number of years, is $\frac{4{\cdot}28}{5{\cdot}46}$ or 78·3 per cent.

In order to achieve the specified increase of £ 10,000,000 in the total yield of this group of twenty-three mines, a combined yield of £ 60,053,000 will be required from the <u>same</u> tonnage (no additional labour being available). The required yield will therefore be 5·14 dwt per ton. Making the perfectly logical assumption that the same ratio of yield to Ore Reserve Value will hold in the case of the increased yield, the required average Ore Reserve Value which will result in the specified increase in the total yield will be $\frac{5{\cdot}14}{0{\cdot}783}$, or 6·56 dwt per ton.

From the theoretical Ore Gradation Graph in Plate XX it will be seen that, to obtain an average value of 6·56 dwt per ton from the Ore Reserve, ore will have to be drawn only from sources having a value equal to, or in excess of 3·80 dwt per ton. Satisfying this required increase in yield will therefore have the same effect as raising the Ore Reserve Pay Limit by exactly 1·00 dwt per ton, from 2·8 dwt per ton to 3·8 dwt per ton.

The following table, in which it has been assumed that ore is drawn from the various value groups in the same proportion in which the ore reserve tonnages occur in these different categories, has been compiled to show how the tonnage at present being drawn from the different grades of ore will have to be modified to achieve the desired increase in yield:-

TABLE XXVI.

Value Limit "x" (Dwt/Ton)	In Ore Reserves above the Value Limit "x". (Theoretical) Tons	Per Cent.	Present Rate of Mining from Ore Reserves (Tons)	Revised Rate of Mining from Ore Reserves. (Tons)
10.0	6,782,000	7.17	1,196,000	1,797,000
9.0	9,323,000	9.86	1,645,000	2,472,000
8.0	13,024,000	13.77	2,298,900	3,452,000
7.0	18,419,000	19.47	3,249,000	4,881,000
6.0	26,56[illegible],000	28.08	4,685,000	7,039,000
5.0	38,948,000	41.17	6,869,000	10,320,000
4.0	58,123,000	61.44	10,251,000	15,408,000
3.8	62,965,000	66.56	11,106,000	16,685,000
3.0	87,242,000	92.22	15,387,000	
2.8	94,599,000	100.00	16,685,000	
2.0	129,244,000	136.62		
1.0	175,563,000	185.59		
0.0	189,368,000	200.18		

It will be seen from the table that all the ore below the value of 3·8 dwt per ton will have to be left intact. In order to keep the total tonnage the same, therefore, that tonnage of ore having a value between 2·8 dwt per ton and 3·8 dwt per ton will have to be compensated for by increasing the rate of mining from the ore above 3·8 dwt per ton. This increase in the rate of mining will actually amount to approximately 50 per cent.

Having pointed out exactly what will, on the average, different categories, has been compiled to show how the tonnage at present being drawn from the different grades of ore will have to be modified to achieve the desired increase in yield:-

TABLE XXVI.

Value Limit "x" (Dwt/Ton)	In Ore Reserves above the Value Limit "x". (Theoretical) Tons	Per Cent.	Present Rate of Mining from Ore Reserves (Tons)	Revised Rate of Mining from Ore Reserves. (Tons)
10.0	6,782,000	7.17	1,196,000	1,797,000
9.0	9,323,000	9.86	1,645,000	2,472,000
8.0	13,024,000	13.77	2,298,000	3,452,000
7.0	18,419,000	19.47	3,249,000	4,881,000
6.0	26,568,000	28.08	4,685,000	7,039,000
5.0	38,948,000	41.17	6,869,000	10,320,000
4.0	58,123,000	61.44	10,251,000	15,402,000
3.8	62,965,000	66.56	11,106,000	16,685,000
3.0	87,242,000	92.22	15,387,000	
2.8	94,599,000	100.00	16,685,000	
2.0	129,244,000	136.62		
1.0	175,563,000	185.59		
0.0	189,368,000	200.18		

It will be seen from the table that all the ore below the value of 3·8 dwt per ton will have to be left intact. In order to keep the total tonnage the same, therefore, that tonnage of ore having a value between 2·8 dwt per ton and 3·8 dwt per ton will have to be compensated for by increasing the rate of mining from the ore above 3·8 dwt per ton. This increase in the rate of mining will actually amount to approximately 50 per cent.

Having pointed out exactly what will, on the average, be required from this group of twenty-three mines if the specified increase of £10,000,000 in the yield is to be realised, it is now up to the practical mining engineer to decide whether it will be possible to redistribute the available labour force and to adapt the organisation and the layout of the mines in such a manner as to satisfy the above requirements.

From the above two examples the relative simplicity of calculating the effect of a reduction or an increase in the pay limit on the total ore reserve of a mine, will readily be appreciated. The orthodox methods of ore reserve valuation not only necessitate the preparation of an entirely new set of plans, but practically the whole tedious process of blocking out will also have to be repeated.

C. Conclusion.

In this thesis an attempt has been made to indicate the errors to which the ore reserve of a mine, computed in all good faith by the orthodox methods, is subject. As has been pointed out, rather too much latitude is allowed for the manipulation of both the extent and the value of ore reserve blocks by individual computers. In striving to introduce a more scientific method of evaluating a fully developed reef-bearing area, a single mine only has been chosen and an account given of the various theoretical aspects which were deduced from certain practical observations made on this mine. Similar investigations were subsequently carried out on three other mines of widely differing characteristics both as regards the payability and the mode of occurrence of the ore-bodies. The original intention was to include these examples in the present dissertation, but lack of space forced the abandonment of further practical illustrations in favour of the more detailed development of the theory. Suffice it to say that the investigations referred to have shown that the Fundamental Value Distribution Law illustrated in the preceeding chapters, as well as the various theoretical considerations deduced therefrom, were rigorously substantiated in all cases.

Much additional information, such as for example the method of adapting the Fundamental Distribution Theory to meet the conditions peculiar to the so-called "sheet" mines of the East Rand, has also been acquired. Although falling outside the scope of this treatise, the primary objects of which has been firstly to establish a fundamental value distribution theory for the ore deposits of the Witwatersrand, and secondly to indicate its application to the evaluation of a given ore-body based on sampling results, this interesting development has numerous applications, some of which may possibly form the basis of a further treatise to expand the subject.

In the mathematical treatment of the theory it has been borne in mind that those primarily interested in the subject of Mine Valuation on the Witwatersrand are neither mathematicians nor mathematical statisticians, but intelligent practical mining men. If the development of the statistical theory, and the illustrations of its practical significance, are therefore rather detailed and perhaps a little laborious, this is, in the writer's opinion, justified on the grounds mentioned above. An exhaustive and complete exposition of the basic principles underlying the theories developed has been considered preferable to the covering of a wide field, since experience gained in the subsequent valuation of a number of mines, based on the frequency distribution method outlined in these chapters, has shown that in all cases the fundamental theory could be adapted to overcome problems peculiar to the individual mines.

With the advent of mining activities in the newly discovered goldfields of the Orange Free State, where at least eight mines, each larger than most of the properties on the Witwatersrand, will be going into production within the next few years, the importance of commencing operations according to a predetermined plan formulated on experience gained on the Witwatersrand has been fully appreciated and acted upon by mining engineers. Care should therefore be exercised to ensure that the pitfalls in mine valuation so often encountered in the past are avoided wherever possible in the future. It is in these new fields that statistical valuation methods, of which this thesis is an example, may be employed to full advantage not only to obtain an initially correct valuation of a given area, but also to adjust this initially determined value from time to time in the light of the grade of ore removed by stoping, and the additional sampling information made available as the result of current development operations.

It must be pointed out that this dissertation was not inspired by a spirit of criticism of the present orthodox system of ore reserve valuation, but by the realisation of the shortcomings of this system. An earnest and sincere endeavour has therefore been made to suggest some alternative, based on the application of a scientific value distribution law, the existence of which has long been realised, but the possibilities of which have been only superficially explored.

F.W.J. Ross.

Johannesburg.
August, 1949.