The Geometry of the Hecke Groups Acting on Hyperbolic Plane and their Associated Real Continued Fractions.[1]

Mathematics Master's Research

Lesiba Joseph Maphakela

Under the supervision of Prof. Meira Hockman

School of Mathematics

University of the Witwatersrand

Private Bag 3, Wits 2050, South Africa

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the degree of Master of Science to the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination to any other university.

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Lesiba Joseph Maphakela

_________ day of February 2014, at Johannesburg, South Africa.

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Abstract

Continued fractions have been extensively studied in number theoretic ways. In this text we will consider continued fraction expansions with partial quotients that are in $\lambda\mathbb{Z} = \{\lambda x : x \in \mathbb{Z}\}$ and where $\lambda = 2\cos(\frac{\pi}{q}), q \geq 3$ and with $1 < \lambda < 2$. These continued fractions are expressed as the composition of Möbius maps in $PSL(2, \mathbb{R})$, that act as isometries on $\mathbb{H}^2$, taken at ∞. In particular the subgroups of $PSL(2, \mathbb{R})$ that are studied are the Hecke groups G_λ. The Modular group is the case for $q = 3$ and $\lambda = 1$. In the text we show that the Hecke groups are triangle groups and in this way derive their fundamental domains. From these fundamental domains we produce the v-cell ($\mathbb{P}_0$) that is an ideal q-gon and also tessellate $\mathbb{H}^2$ under G_λ. This tessellation is called the λ-Farey tessellation. We investigate various known λ-continued fractions of a real number. In particular, we consider a geodesic in $\mathbb{H}^2$ cutting across the λ-Farey tessellation that produces a "cutting sequence" or path on a λ-Farey graph. These paths in turn give a rise to a derived λ-continued fraction expansion for the real endpoint of the geodesic. We explore the relationship between the derived λ-continued fraction expansion and the nearest λ-integer continued fraction expansion (reduced λ-continued fraction expansion given by Rosen, [**25**]). The geometric aspect of the derived λ-continued fraction expansion brings clarity and illuminates the algebraic process of the reduced λ-continued fraction expansion.

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CHAPTER 1

Introduction

1. Historical background

Mathematicians have studied general continued fractions of the form,

$$b_0 + \mathbf{K}(a_n|b_n) = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}}$$

where $\{a_i\}_{i\geq 1}$ and $\{b_i\}_{i\geq 0}$ are sequences of non-zero integers or natural numbers [**17**],[**24**]. More generally, continued fractions are given as $b_0 + \mathbf{K}(a_i|b_i)$ with non-zero complex sequences $\{a_i\}_{i\geq 1}$ and $\{b_i\}_{i\geq 0}$.

Continued fractions have been used in mathematics, mainly as a tool for evaluating or approximating real numbers. Mathematicians found approximations for real numbers such as $\sqrt{x}$ where x is non-square integer. Writing $x = a^2 + b$ where a^2 is the largest square integer less than x, we see:

$$\sqrt{x} = \sqrt{a^2 + b} = a + \cfrac{b}{2a + \cfrac{b}{2a + \cfrac{b}{2a + \cdots}}} \qquad e.g \;\; \sqrt{17} = \sqrt{4^2 + 1} = 4 + \cfrac{1}{8 + \cfrac{1}{8 + \cfrac{1}{8 + \cdots}}}$$

where $\sqrt{x} = a + \frac{b}{a + \sqrt{x}}$ is equivalent to $x = a^2 + b$.

The regular continued fraction expansion of e, the base of natural logarithms as a continued fraction was found by Euler [**31**] to be:

$$2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cdots}}}$$

or a sequence of partial quotients given as $[2;1,2,1,1,4,1,1,6,1,1,...,2k,1,1...]$ where $k\in\mathbb{Z}$.

The first recorded study of a general theory of continued fractions appeared in John Wallis' Opera Mathematica in 1695, and introduced the term 'continued fraction'. Many well known mathematicians have added their knowledge to the subject. In particular, the first paper in which continued fractions were properly considered was written by Euler in his exposition from 1737, 'De Fractionibus Continuis dissertatio'.

These mathematicians were primarily interested in the number theory properties of continued fractions. These continued fractions were defined by recurrence relationships: $A_n = b_nA_{n-1} + a_nA_{n-2}$ and $B_n = b_nB_{n-1} + a_nB_{n-2}$ for $n \geq 0$ with initial conditions $A_{-1} = 1, A_0 = b_0, B_{-1} = 0, B_0 = 1$. So $\dfrac{A_n}{B_n} = \dfrac{b_nA_{n-1} + a_nA_{n-2}}{b_nB_{n-1} + a_nB_{n-2}}$. Most number theoretic results on continued fractions have been proven by considering the behavior of the solutions of these recurrence relation equations.

The continued fraction can be represented as a composition of a sequence of complex Möbius maps evaluated at 0 or ∞. Isenkrache (1888), Netto (1892), Schur (1917) and Hamel (1918), used a geometric approach to study continued fraction. This geometrical approach followed the paper by J. F. Paydon and H. S. Wall, "The continued fraction as a sequence of linear transformation" in (1942).

As noted, mathematicians studied continued fractions with $\{a_i\}_{i\geq 1}$ and $\{b_i\}_{i\geq 0}$ being sequences of non-zero integers. If $a_i = 1$ for all i and $b_i \in \mathbb{Z}$ for all i, we call the continued fractions, the integer continued fractions. The integer continued fraction $b_0 + \mathbf{K}(1|b_i)$, when $b_i \in \mathbb{Z}^+$ for $i \geq 1$ with $b_0 \in \mathbb{Z}$ is called a simple integer continued fraction or just simple continued fraction.

DEFINITION 1. *The continued fraction $b_0 + \boldsymbol{K}(a_n|b_n)$ is said to converge classically to a value α if the sequence*

$$b_0\ ,\ b_0 + \frac{a_1}{b_1}\ ,\ b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2}}\ ,\ b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3}}}\ ,\ \cdots$$

of partial quotients converge to α, where $\alpha \in \mathbb{C}$ or $\alpha = \infty$. The partial quotients are called the convergents, or approximates, of the finite or infinite continued fraction.

From [**24**], we examine the recurrence relations and convergents given above more closely, where $a_i = 1$ and $\{b_i\}_{i\geq 0}$ are sequences of real numbers or complex numbers.

Let $A_{-1} = 1$, $B_{-1} = 0$, $A_0 = b_0$ and $B_0 = 1$. Then the sequence of convergents may be given as:

$$\frac{A_0}{B_0} = b_0\ ,\quad \frac{A_1}{B_1} = b_0 + \frac{1}{b_1}\ ,\quad \frac{A_2}{B_2} = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_2}}\ ,\quad ....$$

Since $\dfrac{A_{-1}}{B_{-1}} = \dfrac{1}{0}$, we have

$$\begin{pmatrix} A_{-1} & A_0 \\ B_{-1} & B_0 \end{pmatrix} = \begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix} \text{ and } A_{-1}B_0 - B_{-1}A_0 = 1 - 0 = 1 = det \begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix}.$$

$\begin{pmatrix} A_0 & A_1 \\ B_0 & B_1 \end{pmatrix} = \begin{pmatrix} b_0 & b_0b_1+1 \\ 1 & b_1 \end{pmatrix} = \begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix}$ and $A_0B_1 - B_0A_1 = 0 - 1 = -1 =$ $det \begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix} = det \begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix} det \begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix}$.

$\begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix} = \begin{pmatrix} b_0b_1+1 & b_0b_1b_2+b_0+b_2 \\ b_1 & b_1b_2+1 \end{pmatrix} = \begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & b_2 \end{pmatrix}$ and $A_1B_2 - B_1A_2 =$ $0 + 1 = 1 = det \begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & b_2 \end{pmatrix} = det \begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix} det \begin{pmatrix} 0 & 1 \\ 1 & b_1 \end{pmatrix} det \begin{pmatrix} 0 & 1 \\ 1 & b_2 \end{pmatrix}$

In general,

$\begin{pmatrix} A_{n-1} & A_n \\ B_{n-1} & B_n \end{pmatrix} = \begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix} \cdots\cdots \begin{pmatrix} 0 & 1 \\ 1 & b_n \end{pmatrix}$ and $|A_{n-1}B_n - B_{n-1}A_n| = 1$.

We note from using the association of Möbius maps with matrices as in [**14**], [**3**], the matrix $\begin{pmatrix} 0 & 1 \\ 1 & b_n \end{pmatrix}$ corresponds to the Möbius map, $t_n : \mathbb{C}_\infty \to \mathbb{C}_\infty$ given by $t_n(z) = \dfrac{1}{b_n + z}$ for $n \geq 1$ and $t_0(z) = z + b_0$. So if $T_n(z) = t_0 t_1 \ldots\ldots t_n(z)$ then the matrix of T_n corresponds to $\begin{pmatrix} A_{n-1} & A_n \\ B_{n-1} & B_n \end{pmatrix}$. That is, $T_n(z) = \dfrac{A_{n-1}z + A_n}{B_{n-1}z + B_n}$ with $|A_{n-1}B_n - B_{n-1}A_n| = 1$. Thus, the convergents of the continued fractions could be expressed in terms of the recurrence quotients as above and expressed by fractions of the form $\dfrac{A_n}{B_n}$. These convergents can in turn be expressed in terms of T_n and T_{n+1} with $T_n(0) = T_{n+1}(\infty) = \dfrac{A_n}{B_n}$. If $\dfrac{A_{n-1}}{B_{n-1}}$ and $\dfrac{A_n}{B_n}$ are successive convergents of the continued fraction then $|A_{n-1}B_n - B_{n-1}A_n| = 1$. We may say $\dfrac{A_n}{B_n}$ is adjacent to $\dfrac{A_{n-1}}{B_{n-1}}$ and write $\dfrac{A_n}{B_n} \sim \dfrac{A_{n-1}}{B_{n-1}}$, [**7**].

We note that these convergents can also be expressed in terms of the following Möbius transformation:

$$\text{Let } s_n(z) = b_n + 1/z \quad \text{and } S_n = s_0 s_1 \ldots\ldots s_n \qquad \text{for } n = 0, 1, \ldots.$$

So $b_0 = t_0(0) = s_0(\infty)$; $\quad b_0 + \dfrac{1}{b_1} = t_0 t_1(0) = s_0 s_1(\infty)$

$b_0 + \dfrac{1}{b_1 + \dfrac{1}{b_2}} = t_0 t_1 t_2(0) = s_0 s_1 s_2(\infty)$ etc.

Thus, $S_{n+1}(0) = S_n(\infty) = T_n(0) = T_{n+1}(\infty)$. Hence, we can establish an alternate definition of convergence of continued fraction with the following lemma.

Lemma 1. *The continued fraction* $b_0 + \mathbf{K}(1|b_n)$ *converges classically to* α *if and only if*

$$\lim_{n\to\infty} T_{n+1}(\infty) = \lim_{n\to\infty} T_n(0) = \alpha \quad \text{or} \quad \lim_{n\to\infty} S_{n+1}(0) = \lim_{n\to\infty} S_n(\infty) = \alpha$$

where $T_n(0) = T_n(t_{n+1}(\infty)) = T_{n+1}(\infty)$ *and* $S_n(\infty) = S_n(s_{n+1}(0)) = S_{n+1}(0)$.

In the sequel we will consider $a_i = 1$ for all i and $\{b_i\}_{i\geq 0}$ as a sequence of "λ-integers" in $\lambda\mathbb{Z}$ where $\lambda = 2\cos(\frac{\pi}{q})$, $q \geq 3$ is an integer. These continued fractions are referred to as λ-continued fractions. We note if $q = 3$ then $\lambda = 1$ and the λ-integers, $\lambda\mathbb{Z}$ are just the integers $\mathbb{Z}$. Since $\lambda \in \mathbb{R}$ these λ-continued fractions are real continued fractions.

2. Hyperbolic plane

The hyperbolic plane is the upper half-plane

$$\mathbb{H}^2 = \{z \in \mathbb{C} : Im(z) > 0\}$$

in $\mathbb{C}$ together with the metric induced from the differential, $ds = \frac{|dz|}{y} = \frac{\sqrt{dx^2+dy^2}}{y}$ with $z = x + iy$. Similarly we define the 3-dimensional hyperbolic space as

$$\mathbb{H}^3 = \{z + tj : z \in \mathbb{C}, t > 0\}$$

in $\mathbb{R}^3$ together with the metric $ds = \dfrac{\sqrt{dx^2+dy^2+dt^2}}{t}$. We note that $\mathbb{R}_\infty$ is the boundary of $\mathbb{H}^2$ and $\mathbb{C}_\infty$ is the boundary of $\mathbb{H}^3$.

Definition 2. *Let $E \subseteq \mathbb{H}^2$ and $z \in \mathbb{C}$. The hyperbolic area of E, $\mu(E)$ is given as*

$$\mu(E) = \int\int_E \tfrac{|dz|}{Im(z)}$$

if the integral exists.

Definition 3. *The hyperbolic metric or distance is the map $\zeta : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{R}^+ \cup \{0\}$ between two points z and w in $\mathbb{H}^2$ is defined by the formula: $\zeta(z,w) = inf\ h(\gamma)$ where the infimum is taken over all smooth paths γ joining z to w in $\mathbb{H}^2$.*

$Hyperbolic - lines$ or *geodesics* in $\mathbb{H}^2$ (and $\mathbb{H}^3$) are vertical Euclidean lines or semi$-$circles all in $\mathbb{H}^2$ (or $\mathbb{H}^3$) orthogonal to $\mathbb{R}_\infty$ (or $\mathbb{C}_\infty$). The *geodesic* γ with end points α and β on $\mathbb{R}_\infty$ is denoted by $\gamma_{\alpha,\beta}$ where $\gamma_{\alpha,\beta} = \gamma_{\beta,\alpha}$. The positive imaginary axis is the geodesic $\mathbb{I}_0 = \gamma_{0,\infty}$ and is called the $fundamental$ *geodesic.*

Definition 4. *The horodisc, $\mathcal{H}$ is an open Euclidean disc in $\mathbb{H}^2$ which is tangent to $\mathbb{R}$ at a point w. A horocycle in $\mathbb{H}^2$ is the boundary of a horodisc $\mathcal{H}$ in $\mathbb{H}^2$. A horodisc at ∞ is defined to be $\{(x,y) \in \mathbb{H}^2 : y > k\ \}$, $k > 0$ and its associated horocycle is the line $y = k$.*

Definition 5. *A transformation $\vartheta : \mathbb{H}^2 \to \mathbb{H}^2$ that preserves distance in $\mathbb{H}^2$ is an isometry. That is, $\zeta(\vartheta(z), \vartheta(w)) = \zeta(z,w)$ for $z, w \in \mathbb{H}^2$. The group of isometries of any set E is denoted by $Isom(E)$.*

Definition 6. *Let γ be a geodesic in $\mathbb{H}^2$. A hyperbolic reflection in γ is a hyperbolic isometry other than the identity which fixes each point of γ.*

The geodesic γ is either a Euclidean line or circle. The reflections in these geodesics are either Euclidean reflections in lines or inversions in circles.

DEFINITION 7. [**14**] *If S_1 and S_2 are surfaces in $\mathbb{R}^3$ then we say the differentiable map $f : S_1 \to S_2$ is conformal if it preserves angles. That is, whenever two differentiable curves c_1 and c_2 on S_1 meet at a point w with angle δ, then $f(c_1)$ and $f(c_2)$ meet at $f(w)$ with the same angle δ. This map may be directly conformal (orientation preserving) or indirectly conformal (orientation reversing).*

We note from [**2**], that the following facts are easily established. There is a unique geodesic through any two distinct points of the hyperbolic plane; the distinct geodesics have at most a single point of intersection in $\mathbb{H}^2$ and the reflection in a geodesic is an isometry. Also through any point $p \in \mathbb{H}^2$ not on a geodesic $\gamma \in \mathbb{H}^2$, we can find infinitely many geodesics in $\mathbb{H}^2$ through p not intersecting γ in $\mathbb{H}^2$, [**8**]. We further specifically note the following trigonometric identity which allows us to more easily access $\zeta(z, w)$. This identity has many equivalences found in [**2**].

THEOREM 1. [**2**]

Let ζ be the hyperbolic metric as given above and let $z, w \in \mathbb{H}^2$. Then

$$\sinh \tfrac{1}{2}\zeta(z,w) = \frac{|z-w|}{2(Im[z]Im[w])^{\frac{1}{2}}} .$$

DEFINITION 8. [**14**] *A hyperbolic n-gon is a simply connected open set bounded by n hyperbolic line segments in $\mathbb{H}^2$. A point where two bounding line segment intersect is called a vertex of the n-gon. The vertex may or may not lie on $\mathbb{R}_\infty$. If all the vertices of an n-gon lie on $\mathbb{R}_\infty$ then n-gon is called an ideal n-gon. The angles of an n-gon are the angles between intersecting line segments. The angles of an ideal n-gon are all zero. The n-gon may be referred to by listing its n vertices. A hyperbolic triangle is a hyperbolic 3-gon.*

Theorem 2. *(Gauss-Bonnet),*[**14**].

Let T be a hyperbolic triangle with angles α, β and δ. Then the hyperbolic area of T is given by $\mu(T) = \pi - \alpha - \beta - \delta$.

3. Möbius maps

We recall the set

$$PGL(2,\mathbb{C}) = \left\{ z \longmapsto \frac{az+b}{cz+d} : \quad a,b,c,d \in \mathbb{C}, ad-bc \neq 0 \right\}$$

is the group of Möbius transformations, named in honor of August Ferdinand Möbius (1790-1868), with underlying matrix group $GL(2,\mathbb{C})$. By convention we may associate a map $g \in PGL(2,\mathbb{C})$, the projective linear group, with its underlying matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ or multiple thereof. Let $g(z) = \dfrac{az+b}{cz+d}$, $ad-bc \neq 0$, $a,b,c,d \in \mathbb{C}$ and $\Delta = ad-bc \in \mathbb{C}$. Since $\sqrt{\Delta} \in \mathbb{C}$ for all $\Delta \neq 0$ we can re-write g as $g(z) = \dfrac{\frac{a}{\sqrt{\Delta}}z + \frac{b}{\sqrt{\Delta}}}{\frac{c}{\sqrt{\Delta}}z + \frac{d}{\sqrt{\Delta}}}$ where $\frac{a}{\sqrt{\Delta}}\frac{d}{\sqrt{\Delta}} - \frac{b}{\sqrt{\Delta}}\frac{c}{\sqrt{\Delta}} = \frac{\Delta}{\Delta} = 1$. The matrix associated with g represented in this way is $\begin{pmatrix} \frac{a}{\sqrt{\Delta}} & \frac{b}{\sqrt{\Delta}} \\ \frac{c}{\sqrt{\Delta}} & \frac{d}{\sqrt{\Delta}} \end{pmatrix}$ with trace $\frac{1}{\sqrt{\Delta}}(a+d)$. When g is represented in this way we say it is normalised. Since all g in $PGL(2,\mathbb{C})$, may be normalised, we have $PGL(2,\mathbb{C}) = PSL(2,\mathbb{C}) = \mathcal{M} = \left\{ z \longmapsto \dfrac{az+b}{cz+d} : \quad a,b,c,d \in \mathbb{C}, ad-bc = 1 \right\}$, with the underlying matrix group $SL(2,\mathbb{C})$. Our study of Möbius transformations will be restricted to using only the group of real Möbius transformations, $PSL(2,\mathbb{R})$ where $a,b,c,d \in \mathbb{R}$, $ad-bc = 1$ and $PSL(2,\mathbb{R})$ is a subgroup of $PSL(2,\mathbb{C})$, [**14**]. We recall [**2**], the fact that each Möbius transformation can be represented by composition of even number of reflections of geodesics. We note that the concept of inversions in hyper-spheres exists in $\mathbb{R}^n$ for $n \geq 1$. The use of Möbius transformations to represent continued fractions can be used to address the problem of generalising continued fraction to higher dimensions. We will see that restricting ourself to specific subgroups of the group of Möbius transformations also gives rise to alternate continued fractions expansions.

Definition 9. *Two elements g and h in any group are conjugate if there is some f in the group with $h = fgf^{-1}$.*

The relation of conjugacy on $PSL(2,\mathbb{C})$ is an equivalence relation and the equivalence classes are called the conjugacy classes. Thus, $PSL(2,\mathbb{C})$ is partitioned into distinct, disjoint conjugacy classes. Conjugate Möbius maps have similar properties, both algebraic and geometric, so Möbius maps in the same conjugacy class need not be distinguished when applying such properties. Conjugacy of Möbius maps plays a pivotal role in the classification of Möbius transformations.

Möbius maps can be categorized in the conjugacy classes of parabolic, elliptic and loxodromic elements as follows:

Definition 10. *Let $tr(g)$ be the trace of associated normalised matrix of g in $SL(2,\mathbb{C})$. Let g be a Möbius map other than 1_{map}. Then*

(i) g is parabolic if $tr^2(g) = 4$;

(ii) g is elliptic if $tr^2(g) \in [0,4)$;

(iii) g is loxodromic if $tr^2(g) \notin [0,4]$

The following theorem characterise the different conjugacy classes.

Theorem 3. [**2**] *Let g be a Möbius map other than 1_{map}. Then the following are equivalent:*

(a1) g is parabolic;

(a2) g is conjugate to a translation $z \longmapsto z+1$;

(a3) g has exactly one fixed point w and $g^n \to w$ point-wise on $\mathbb{C}_\infty$.

The following are equivalent:

(b1) g is elliptic;

(b2) g conjugate to a Euclidean rotation $z \longmapsto e^{i\theta} z$, where $e^{i\theta} \neq 1$;

(b3) g has two fixed points, and $g^n(z)$ converges if and only if z is a fixed point of g.

The following are equivalent;

(c1) g is loxodromic;

(c2) g is conjugate to a map $z \longmapsto kz$, where $|k| \neq 0, 1$;

(c3) g has two fixed points u and v which can be chosen so that if $z \neq v$ then $g^n(z) \to u$ as $n \to \infty$.

Note if $g \in PSL(2, \mathbb{R})$ and g is loxodromic then $tr^2(g) > 4$. In this case g is often called hyperbolic with fixed points $u, v \in \mathbb{R}_\infty$. If $z \neq v$ and $g^n(z) \to u$, u is called the attracting fixed point and v is called the repelling fixed point. Further the parabolic and hyperbolic fixed points are in $\mathbb{R}_\infty$. Hence, if $g(v) = v$ and $v \notin \mathbb{R}_\infty$ then v is an elliptic fixed point.

PROPOSITION 1. [**14**]

(i) $PSL(2, \mathbb{R})$ leaves $\mathbb{R}_\infty$ and $\mathbb{H}^2$ invariant.

(ii) $PSL(2, \mathbb{R})$ acts transitively on geodesics in $\mathbb{H}^2$. Thus, the geodesics in $\mathbb{H}^2$ are the orbit of the fundamental geodesic $\mathbb{I}_0$ under $PSL(2, \mathbb{R})$.

THEOREM 4. [**2**]

Let $Isom(\mathbb{H}^2) = G^*$ be a group of isometries of $\mathbb{H}^2$. Then $G^* = PSL(2, \mathbb{R}) \cup PSL(2, \mathbb{R})\omega$ where $\omega(z) = -\overline{z}$. Further, the group of isometries is generated by reflections in hyperbolic lines. $PSL(2, \mathbb{R})$ is the group of orientation preserving conformal isometries of $\mathbb{H}^2$. The group *of orientation preserving conformal isometries is generated by an even number of reflections of hyperbolic lines.*

The Hecke groups that underly the λ-continued fractions are discrete subgroups of the $PSL(2,\mathbb{R})$ and hence are groups of orientation preserving conformal isometries on $\mathbb{H}^2$ (Section 2.2).

We close this introduction with the definition of isometric circles in $\mathbb{C}$. These circles can be used in establishing the fundamental regions of subgroups of $PSL(2,\mathbb{R})$, [**30**].

DEFINITION 11. *Let* $g(z) = \dfrac{az+b}{cz+d}$, $c \neq 0$, *be a real Möbius map that does not fix* ∞. *The isometric circle of* g *is given by* $I_g : |cz+d| = 1$ *or* $|z + \frac{d}{c}| = \frac{1}{|c|}$ *where* $c \neq 0$.

We have that I_g is a Euclidean circle with center $\dfrac{-d}{c} \in \mathbb{R}$ (since $d, c \neq 0 \in \mathbb{R}$) and radius $\dfrac{1}{|c|}$. We note that $|g(z) - g(w)| = |z - w|$ for z and w on I_g. From Ford [**7**], we know g maps I_g onto $I_{g^{-1}}$ and maps the interior of I_g to the exterior of $I_{g^{-1}}$. If z lies within I_g then $|z + \frac{d}{c}| < \frac{1}{|c|}$ and $|g'(z)| > 1$. If z lies outside I_g then $|g'(z)| < 1$. This circle is the locus of points in the neighborhood of which lengths and areas are unchanged by the transformation $g \in PSL(2,\mathbb{R})$.

Finally, it can be noted that $g \in PSL(2,\mathbb{R})$ maps generalised circles to generalised circles in $\mathbb{C}_\infty$, but may map exterior of $\mathcal{C}$ to the interior of $g(\mathcal{C})$ and vice versa where $\mathcal{C}$ is a Euclidean circle in $\mathbb{C}$, [**14**].

CHAPTER 2

The Hecke groups as triangle groups

1. The groups $G_\lambda = \langle \tau_\lambda, \varphi \rangle$

We recall that the group of real Möbius transformations acting on $\mathbb{C}_\infty$ is given by

$$PSL(2,\mathbb{R}) = \left\{ z \longmapsto \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{R}, ad-bc=1 \right\}.$$

We consider the groups G_λ generated by transformations φ and τ_λ. Here $\varphi(z) = -1/z$ is an elliptic generator and $\tau_\lambda(z) = z+\lambda$ is a parabolic generator where $\lambda = 2\cos(\frac{\pi}{q})$, q is an integer ≥ 3 with $\lambda < 2$. We write $G_\lambda = \langle \tau_\lambda, \varphi \rangle = \langle \tau_\lambda, \varphi : (\tau_\lambda \varphi)^q = (\varphi \tau_\lambda^{-1})^q = \varphi^2 = 1_{map} \rangle$. So each element g in G_λ is a word in τ_λ and φ. That is, $g = \tau_\lambda^{r_0} \varphi \tau_\lambda^{r_1} \varphi \cdots \varphi \tau_\lambda^{r_k} \varphi$ where $r_i \in \mathbb{Z}$ for $i \geq 0$ but only r_0 and r_k may be zero . We say g is of length $k+1$ if $r_k \neq 0$ even if $r_0 = 0$. We will show that these groups are the triangle groups known as the Hecke groups. Certainly $G_\lambda \leq PSL(2,\mathbb{R})$.

We see that if $q = 3$ then $\lambda = 1$, if $q = 4$ then $\lambda = \sqrt{2}$ and if $q = 6$ then $\lambda = \sqrt{3}$. Further considering the diagram below and using the cosine rule we see that if $q = 5$ then $\lambda = \frac{1+\sqrt{5}}{2}$.

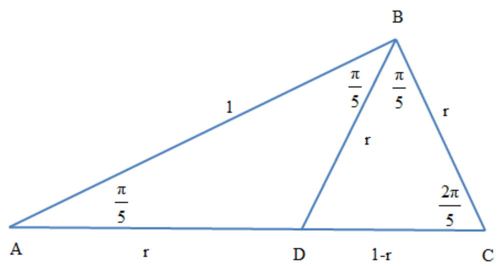

FIGURE 1. Isosceles triangle.

Let $\triangle ABC$ be an isosceles triangle with angle $\pi/5$ at A and sides $AB = AC = 1$. Then $\widehat{B} = \widehat{C} = 2\pi/5$. Bisect $\widehat{B}$ to cut AC at D. Then $B\widehat{D}C = 2\pi/5$ and so $BD = BC = r$. Further since $\triangle DAB$ is isosceles we have that $AD = BD = r$. Since $\triangle ABC|||\triangle BDC$ we have $\dfrac{r}{1} = \dfrac{1-r}{r}$ and so $r^2 + r - 1 = 0$. Therefore $r = \dfrac{-1 \pm \sqrt{5}}{2}$. Since $r > 0$, $r = \dfrac{-1+\sqrt{5}}{2}$. Using the cosine rule in $\triangle ABC$ we see that

$$r^2 = 1 + 1 - 2\cos(\tfrac{\pi}{5}) \text{ or}$$
$$\lambda = 2\cos(\tfrac{\pi}{5}) = 2 - r^2 = 2 - \left(\frac{-1+\sqrt{5}}{2}\right)^2 = \frac{1+\sqrt{5}}{2}.$$

When $\lambda = 1$ $(q = 3)$, G_λ is the modular group. In the sequel we will consider the cases when $q \geq 3$ is an integer and omit cases where $\lambda \geq 2$.

2. Triangle groups

In this section we introduce triangle groups of type $(\frac{\pi}{m_1}, \frac{\pi}{m_2}, \frac{\pi}{m_3})$ with $\frac{\pi}{m_1} + \frac{\pi}{m_2} + \frac{\pi}{m_3} < \pi$. In Section 2.3 we show that the groups $G_\lambda = \langle \tau_\lambda, \varphi \rangle$ are the triangle groups of the type $(\frac{\pi}{2}, \frac{\pi}{q}, 0)$, $\lambda = 2\cos(\frac{\pi}{q})$. Since triangle groups are composed of reflections in the sides of the given triangles, we first establish the following proposition.

Proposition 2. *Let γ be any geodesic in $\mathbb{H}^2$. Then the hyperbolic reflection in γ is an orientation reversing isometry of $\mathbb{H}^2$ of order 2 that fixes each point of γ.*

Proof

Let $\gamma = \mathbb{I}_0$ be the fundamental geodesic, then $R_1 : z \longmapsto -\overline{z}$ is the Euclidean reflection in $\mathbb{I}_0$ and $R_1(ia) = -(-ia) = ia$ for all $a > 0$. So R_1 fixes $\mathbb{I}_0$ point-wise. To establish that R_1 is an isometry of $\mathbb{H}^2$, we use the identity $\sinh^2 \frac{1}{2}\zeta(z,w) = \dfrac{|z-w|^2}{4ImzImw}$, (Theorem 1, page 13). Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be in $\mathbb{H}^2$. Then

$\sinh^2 \frac{1}{2}\zeta(R_1(z_1), R_1(z_2)) = \dfrac{|R_1(z_1) - R_1(z_2)|^2}{4Imz_1Imz_2} = \dfrac{|-\overline{z_1} - (-\overline{z_2})|^2}{4y_1y_2} = \dfrac{|\overline{z_1 - z_2}|^2}{4y_1y_2} = \dfrac{|z_1 - z_2|^2}{4y_1y_2} =$ $\sinh^2 \frac{1}{2}\zeta(z_1, z_2)$, since $Im(R_1(z)) = Imz$.

Therefore $\zeta(R_1(z_1), R_1(z_2)) = \zeta(z_1, z_2)$, so R_1 is an isometry. R_1 is a reflection of order two and thus $R_1 \notin PSL(2, \mathbb{R})$ since it is orientation reversing.

Recall that $PSL(2, \mathbb{R})$ acts transitively on the geodesics of $\mathbb{H}^2$, (Proposition 1, page 16). Let γ be any geodesic in $\mathbb{H}^2$. We can find $g \in PSL(2, \mathbb{R})$ such that $\gamma = g(\mathbb{I}_0)$. Since g is an orientation preserving hyperbolic isometry, gR_1g^{-1} is orientation reversing and fixes each point of γ.

That is, for $z \in \gamma$ we can find $g^{-1} \in PSL(2, \mathbb{R})$ such that $g^{-1}(z) \in \mathbb{I}_0$ and since R_1 fixes $\mathbb{I}_0$, we have $R_1(g^{-1}(z)) = g^{-1}(z)$ Thus, $gR_1g^{-1}(z) = z$. Finally since $R_1^2 = 1_{map}$ we have $(gR_1g^{-1})^2 = 1_{map}$.$\square$

Definition 12. *G^* of isometries of the hyperbolic plane is said to be of type (α, β, δ) if and only if G^* is generated by the reflections across the sides of some hyperbolic triangle with angles α, β and δ.*

Such groups exist if and only if α, β and δ are non-negative and satisfy

$$0 \leq \alpha + \beta + \delta < \pi,$$

because this is the necessary and sufficient conditions for the existence of the triangle with those angles. From [**2**] we know that the group of type (α, β, δ) is discrete if and only if it is also of some type $(\frac{\pi}{m_1}, \frac{\pi}{m_2}, \frac{\pi}{m_3})$, $m_i \in \mathbb{Z}_\infty$. Suppose $m_i \in \mathbb{Z}_\infty$ for $i = 1, 2, 3$ such that $\dfrac{1}{m_1} + \dfrac{1}{m_2} + \dfrac{1}{m_3} < 1$. Then $\dfrac{\pi}{m_1} + \dfrac{\pi}{m_2} + \dfrac{\pi}{m_3} < \pi$. By Gauss-Bonnet Theorem (Theorem 2, page 14), the area of a hyperbolic triangle T with angles $\dfrac{\pi}{m_1}, \dfrac{\pi}{m_2}$ and $\dfrac{\pi}{m_3}$ is $\mu(T) = \pi - \left(\dfrac{\pi}{m_1} + \dfrac{\pi}{m_2} + \dfrac{\pi}{m_3}\right) > 0$. Thus, a hyperbolic triangle T with angles $\dfrac{\pi}{m_i}$ for $i = 1, 2, 3$

exist if $\dfrac{1}{m_1}+\dfrac{1}{m_2}+\dfrac{1}{m_3}<1$. Note if $m_1=2$, $m_2=q$, $m_3=\infty$ then for $q\geq 3$ we have $\dfrac{1}{2}+\dfrac{1}{q}+0<\dfrac{1}{2}+\dfrac{1}{2}=1$. Thus, hyperbolic triangle T having vertices i, v and ∞ with angles $\dfrac{\pi}{2},\dfrac{\pi}{q}$ and 0 respectively exists, with $v=\cos(\frac{\pi}{q})+i\sin(\frac{\pi}{q})$.

In the sequel we restrict our attention to discrete conformal groups and use the terminology from Katok. [**15**].

Definition 13. *$G=G^*\cap PSL(2,\mathbb{R})$ is a (m_1,m_2,m_3)-triangle group and is the subgroup of orientation preserving isometries in G^*.*

Definition 14. *An open region $D\subseteq\mathbb{H}^2$ is fundamental region of group G if*

(i)

$$\bigcup_{g\in G} g(\overline{D})=\mathbb{H}^2$$

and

(ii) $D\cap g(D)=\emptyset$ for all $g\neq 1_{map}$ in G.

The family $\{g(\overline{D}):g\in G\}$ is called a tessellation of $\mathbb{H}^2$ by images of $\overline{D}$, where $\overline{D}$ is the closure of D in $\mathbb{H}^2$.

Definition 15. *A set $\{Ga:a\in A\}$ of subsets of $\mathbb{H}^2$ indexed by elements of a set A is called locally finite if for any compact subset $K\subset\mathbb{H}^2$, $Ma\cap K=\emptyset$ for only finitely many $a\in A$.*

Definition 16. *A group G acts properly discontinuously on $\mathbb{H}^2$ if the G-orbit of any point $z\in\mathbb{H}^2$ is locally finite.*

Katok [**15**], states that a group G acts properly discontinuously on $\mathbb{H}^2$ if and only if each orbit is discrete and the order of the stabilizer of each point is finite. In fact, the discreteness of all orbits already implies the discreteness of the group.

DEFINITION 17. *A subgroup G of $PSL(2,\mathbb{R})$ is called discrete if the induced topology on G is a discrete topology, i.e. if G is a discrete set in the toplogical space $PSL(2,\mathbb{R})$. A Fuchsian group is a discrete subgroup of $PSL(2,\mathbb{R})$.*

THEOREM 5. [**15**] *Let G be a subgroup of $PSL(2,\mathbb{R})$. Then G is a Fuchsian group if and only if G acts properly continues on $\mathbb{H}^2$.*

It has been shown by Hecke [**10**] that G_λ is Fuchsian if and only if $\lambda = 2\cos(\frac{\pi}{q})$ where $q \geq 3$ is an integer, $(1 \leq \lambda < 2)$ and for every real $\lambda > 2$. In our study we will only be interested in the cases where $\lambda = 2\cos(\frac{\pi}{q})$, $1 \leq \lambda < 2$. From [**14**] and [**15**], we explore the proof of the following result.

THEOREM 6. *Let T be a hyperbolic triangle with vertices v_1, v_2 and v_3 where the angles π/m_1, π/m_2 and π/m_3 are at these vertices respectively, with sides M_1, M_2 and M_3 opposite these vertices. Let G^* be the group generated by reflections R_1, R_2 and R_3 in sides M_1, M_2 and M_3 respectively. Then G has representation given as $\langle Z, ZY : Z^{m_3} = Y^{m_1} = (ZY)^{m_2} = 1_{map}\rangle$ where $G = G^* \cap PSL(2,\mathbb{R})$, $Z = R_1R_2$ and $Y = R_2R_3$.*

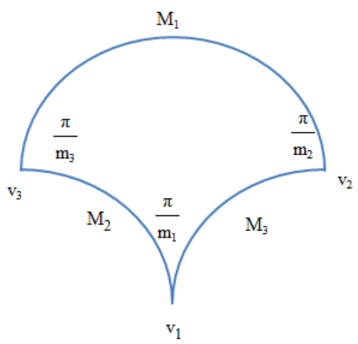

FIGURE 2. Hyperbolic triangle T.

Proof

Let R_i be the hyperbolic reflection in the geodesic containing sides M_i for $i = 1, 2, 3$ respectively and also let G^* be the group generated by R_1, R_2 and R_3. Since $R_i \notin PSL(2, \mathbb{R})$ for $i = 1, 2, 3$, G^* is not a Fuchsian group. Consider $G = G^* \cap PSL(2, \mathbb{R})$. Then G^* is the union of disjoint right cosets of G^* modulo G. That is, $G^*/G = \{G, GR_1\}$ or $G^* = G \cup GR_1$ where $|G^* : G| = 2$. The image of T under R_1 is the hyperbolic triangle with sides $R_1(M_1) = M_1$, $R_1(M_2)$ and $R_1(M_3)$.

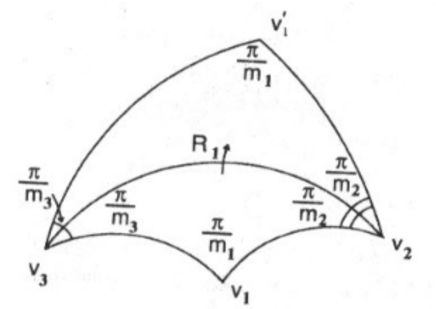

FIGURE 3. Hyperbolic 4-gon $T \cup R_1T$.

We note that $R_1R_2R_1^{-1}(R_1(M_2)) = R_1(M_2)$ so $R_1R_2R_1^{-1} = R_1R_2R_1$ is the reflection in the side $R_1(M_2)$. By this reflection, $R_1(T)$ is transformed to $R_1R_2R_1^{-1}(R_1(T)) = R_1R_2(T)$. Continuing to reflect in the sides of the transformed triangles with vertex v_3, we generated a chain of hyperbolic triangles T, $R_1(T)$, $R_1R_2(T)$, $R_1R_2R_1(T)$,.......$(R_1R_2)^{m_3-1}R_1(T)$. We know that the product of reflections is a rotation so R_1R_2 is a rotation, through $\frac{2\pi}{m_3}$ about the vertex v_3. Thus, $(R_1R_2)^{m_3}$ is a rotation about v_3 through $m_3(\frac{2\pi}{m_3}) = 2\pi$. Hence, $(R_1R_2)^{m_3}(T) = T$ and $(R_1R_2)^{m_3} = 1_{map}$.

Each $g \in G^*$ expressed as a "word" in the reflections R_1, R_2 and R_3. Clearly we see that $R_1^2 = R_2^2 = R_3^2 = 1_{map}$. From above, we also see that $(R_1R_2)^{m_3} = (R_2R_3)^{m_1} = (R_1R_3)^{m_2} = 1_{map}$.

Since $G = G^* \cap PSL(2,\mathbb{R})$, G consist of orientation preserving isometries in G^*. Thus, G is generated by pairs of products of R_1, R_2 and R_3. We note if $Z = R_1R_2$ and $Y = R_2R_3$ then $Z^{m_3} = Y^{m_1} = (ZY)^{m_2} = 1_{map}$ where $ZY = R_1R_2R_2R_3 = R_1R_3$. If we set $P_1 = T \cup R_1(T)$ this polygon has vertices v_1, v_2, $R_1(v_1)$ and v_3 with sides M_3, $R_1(M_2)$, M_2 and $R_1(M_3)$. The presentation of G is $\langle Z, ZY : Z^{m_3} = Y^{m_1} = (ZY)^{m_2} = 1_{map}\rangle$ and G is the triangle group of type $(\frac{\pi}{m_1}, \frac{\pi}{m_2}, \frac{\pi}{m_3})$. $\square$

In [**19**], Magnus shows that $A = \{g(\overline{T}) : g \in G^*\}$ tessellates $\mathbb{H}^2$ in that no two G^* images of T overlap and every point of $\mathbb{H}^2$ belongs to some G^* image of $\overline{T}$. We also note that the sides of $P_1 = T \cup R_1(T)$ are paired by the generators Z and ZY of G. That is, $R_1(M_2) = (R_1R_2R_1)(R_1(M_2)) = R_1R_2(M_2) = Z(M_2)$ and similarly $R_1(M_3) = (R_1R_3R_1)(R_1(M_3)) = R_1R_3(M_3) = ZY(M_3)$. The Poincare Theorem [**2**], then gives G as a discrete group and P_1 is a fundamental polygon for G. Poincare's theorem in fact gives everything, including Theorem 6. We have thus established:

THEOREM 7. *The triangle group G of type $(\frac{\pi}{m_1}, \frac{\pi}{m_2}, \frac{\pi}{m_3})$ given above is generated by $Z = R_1R_2$ and $ZY = R_1R_3$. The group has presentation $\langle Z, ZY : Z^{m_3} = Y^{m_1} = (ZY)^{m_2} = 1_{map}\rangle$. Furthermore $P_i = T \cup R_i(T)$ is a fundamental polygon of G for $i = 1, 2, 3$ and $\mathbb{H}^2$ is tessellated by $\overline{P}_i$ under G.*

3. The Hecke groups

We now consider the triangle groups of type $(\frac{\pi}{2}, \frac{\pi}{q}, 0)$ where $q \geq 3$ is an integer.

DEFINITION 18. *Let T be a hyperbolic triangle having vertices i, $v = \cos(\frac{\pi}{q}) + i\sin(\frac{\pi}{q})$ and ∞ with angles $\frac{\pi}{2}$, $\frac{\pi}{q}$ and 0 respectively. The $(2, q, \infty)$-triangle group is called the Hecke group $H(\lambda)$, where $\lambda = 2\cos(\frac{\pi}{q})$ and $q \geq 3$ is an integer.*

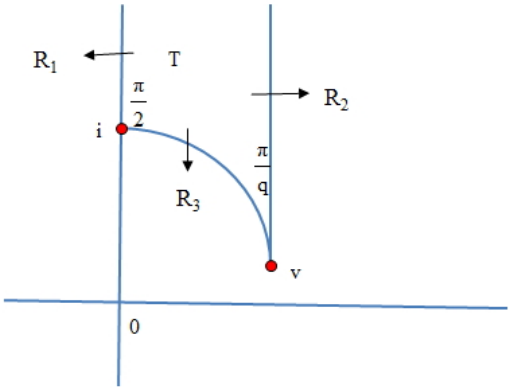

FIGURE 4. Hyperbolic triangle T.

Using the construction of triangle group in Section 2.2, we find H^* to be the group generated by reflections in the sides of triangle T. $H(\lambda) = H^* \cap PSL(2,\mathbb{R})$ is the $(2, q, \infty)$-triangle group. Since $\mathbb{H}^2$ is tessellated by T under H^*, the images under H^* of any point z in $\mathbb{H}^2$ form a discrete set. The following results follow directly from Theorem 7.

THEOREM 8. *A fundamental region of the triangle group $H(\lambda)$ of type $(\frac{\pi}{2}, \frac{\pi}{q}, 0)$ is given by $D = \{z \in \mathbb{C}_\infty : |Re(z)| < \frac{\lambda}{2}$ and $|z| > 1\}$ where $\lambda = 2\cos(\frac{\pi}{q})$, $q \geq 3$ is finite integer. Certainly $D = T \cup R_1 T$ where R_1 is reflection in the side of T opposite the vertex v. Furthermore $H(\lambda)$ is generated by compositions $R_1 R_2$ and $R_1 R_3$ where $R_1(z) = -\overline{z}$, $R_2(z) = -(\overline{z} + \lambda)$ and $R_3(z) = \dfrac{1}{\overline{z}}$.*

We note that $T \cup R_2 T$ or $T \cup R_3 T$ are also fundamental regions of $H(\lambda)$ in $\mathbb{H}^2$.

THEOREM 9. *$G_\lambda = \langle \tau_\lambda, \varphi \rangle = H(\lambda)$.*

Proof

Let R_1, R_2 and R_3 be reflections as in Theorem 8 above then $R_1 R_2 = \tau_\lambda$, $R_1 R_3 = \varphi$ and $R_2 R_3 = \tau_\lambda \varphi$. So the two groups G_λ and $H(\lambda)$ are the same and have the same fundamental region. That is, G_λ is the Hecke group and is the triangle group of type $(\frac{\pi}{2}, \frac{\pi}{q}, 0)$ with $\lambda = 2\cos(\frac{\pi}{q})$ where $q \geq 3$ is an integer. $\square$

In what follows we will represent the Hecke group $H(\lambda)$ by G_λ, $\lambda = 2\cos(\frac{\pi}{q})$ and $q \geq 3$ is an integer.

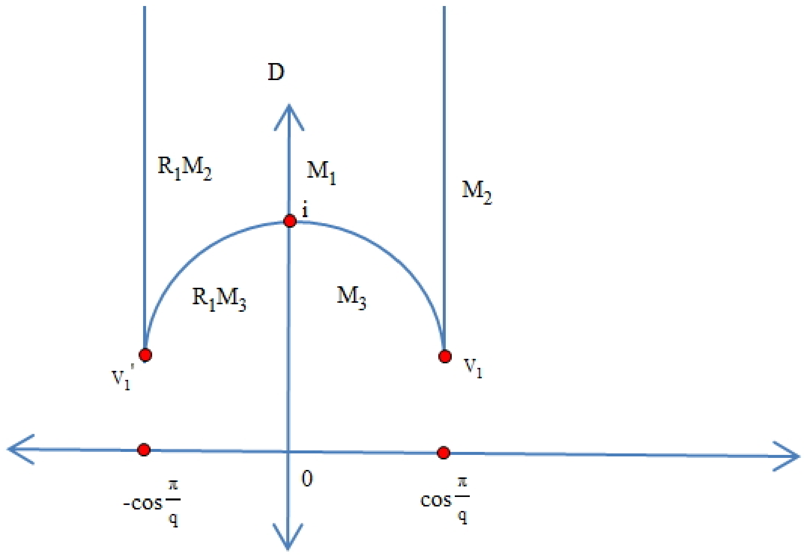

FIGURE 5. $T \cup R_1 T$.

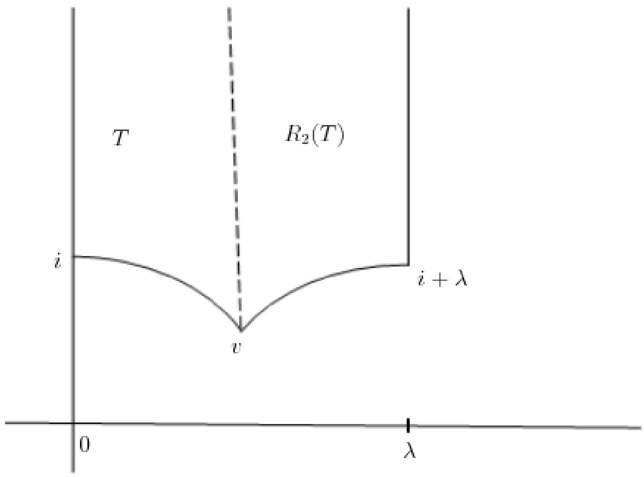

FIGURE 6. $T \cup R_2 T$.

We note if $D = T \cup R_2 T$ where $R_2(z) = -\overline{z} + \lambda$ then the union of the q-images of $\overline{D}$ under $\langle \tau_\lambda \varphi \rangle$ is an ideal q-gon, because this map is a hyperbolic rotation by $2\pi/q$ about the point v. Here the vertex $v = \cos(\frac{\pi}{q}) + i\sin(\frac{\pi}{q})$.

Definition 19. *The union*

$$\overline{D} \cup \tau_\lambda\varphi(\overline{D}) \cup \cdots\cdots \cup (\tau_\lambda\varphi)^{q-1}(\overline{D}) = \bigcup_{i=0}^{q-1} (\tau_\lambda\varphi)^i(\overline{D})$$

is called the v-cell and denoted by $N_q(v)$ where $q \geq 3$ is an integer.

Definition 20. *$Stab(w, G_\lambda) = \{g \in G_\lambda : g(w) = w\} \leq G_\lambda$.*

We note that the fundamental domain $D = T \cup R_2 T$ has exactly one vertex $v = \cos(\frac{\pi}{q}) + i\sin(\frac{\pi}{q})$ that does not lie on a vertical geodesic. Vulakh [**30**] (page 2299) notes that in such a case $N_q(v)$ is a fundamental domain of some subgroup of G_λ with index equal to $|Stab(\infty, G_\lambda)|$.

If $w = \infty$ then $Stab(\infty, G_\lambda) = \langle \tau_\lambda \rangle$. Since $v = \cos(\frac{\pi}{q}) + i\sin(\frac{\pi}{q})$ is in $\mathbb{H}^2$, it is an elliptic fixed point for any $g \in Stab(v, G_\lambda)$. For $g \in Stab(v, G_\lambda)$, $\langle g \rangle$ is a cyclic group that fixes v and is a subgroup of G_λ, a Fuchsian group. Thus, $\langle g \rangle$ is a finite elliptic cyclic group, [**15**].

Following Haas and Series [**9**], we establish that $Stab(v, G_\lambda) = \langle \rho \rangle$, where $\rho = \tau_\lambda \varphi$.

Theorem 10. *Let Γ_λ be the subgroup of G_λ generated by elements $\rho^i \varphi \rho^{-i} = \varphi_i$ for $i = 0, \cdots, q-1$ with $\rho = \tau_\lambda \varphi$. Then $\Gamma_\lambda \lhd G_\lambda$ of index q, $G_\lambda/\Gamma_\lambda \cong \langle \rho \rangle$ and Γ_λ has fundamental region $N_q(v)$. The vertices of $N_q(v)$ are $\rho^i(\infty)$ for $i = 0, 1 \cdots q-1$. Further $Stab(v, G_\lambda) = \langle \rho \rangle$.*

We will prove Theorem 10 by Propositions 3 to 8 that follow. Recall $G_\lambda = \langle \tau_\lambda, \varphi \rangle$, $\rho = \tau_\lambda \varphi$ with $\rho^q = 1_{map}$. We consider the elements $\varphi_i = \rho^i \varphi \rho^{-i}$ for $i = 0, 1, \cdots q-1$.

Set $X = \{\varphi_i : i = 0, \cdots q-1\}$ and let $\Gamma_\lambda = \langle X \rangle$ be the group generated by the elements φ_i.

The group, Γ_λ is the smallest subgroup of G_λ containing X. We note $\varphi_0 = \varphi$ so $\varphi \in \Gamma_\lambda$ and $\varphi_q = \varphi_{nq} = \varphi_0 = \varphi$ for $n \in \mathbb{Z}$.

PROPOSITION 3. $\Gamma_\lambda \lhd G_\lambda$.

Proof

Since $\Gamma_\lambda \le G_\lambda$ by definition, we need only show that Γ_λ is closed under conjugation by elements in G_λ. In particular, we need only show that Γ_λ is closed under conjugation by the generators of G_λ. Specifically, only conjugation of the generators φ_i of Γ_λ by the generators, φ and τ_λ of G_λ need to be considered. Firstly note that

$$\begin{aligned}
\tau_\lambda\varphi_i\tau_\lambda^{-1} &= \tau_\lambda\varphi\varphi(\tau_\lambda\varphi)\{(\tau_\lambda\varphi)^{i-1}\varphi(\tau_\lambda\varphi)^{-i+1}\}(\tau_\lambda\varphi)^{-1}\varphi\varphi\tau_\lambda^{-1} \\
&= (\tau_\lambda\varphi)\varphi(\tau_\lambda\varphi)^{-1}(\tau_\lambda\varphi)^2\{(\tau_\lambda\varphi)^{i-1}\varphi(\tau_\lambda\varphi)^{-i+1}\}(\tau_\lambda\varphi)^{-2}(\tau_\lambda\varphi)\varphi(\tau_\lambda\varphi)^{-1} \\
&= \varphi_1\varphi_{i+1}\varphi_1 \in \Gamma_\lambda.
\end{aligned}$$

Further $\tau_\lambda^{-1}\varphi_i\tau_\lambda = \tau_\lambda^{-1}(\tau_\lambda\varphi)\{(\tau_\lambda\varphi)^{i-1}\varphi(\tau_\lambda\varphi)^{-i+1}\}(\tau_\lambda\varphi)^{-1}\tau_\lambda$

$$\begin{aligned}
&= \varphi\{\rho^{i-1}\varphi\rho^{-(i-1)}\}\varphi \\
&= \varphi\varphi_{i-1}\varphi \in \Gamma_\lambda.
\end{aligned}$$

Thus, since $\Gamma_\lambda = \langle X\rangle$ we have that $\tau_\lambda^k g \tau_\lambda^{-k} \in \Gamma_\lambda$ for all $g \in \Gamma_\lambda$ where $k \in \mathbb{Z}$. So the group, Γ_λ is closed under conjugation by τ_λ^r, $r \in \mathbb{Z}$.

Secondly, $\varphi\varphi_{i-1}\varphi^{-1} \in \Gamma_\lambda$ since $\varphi, \varphi_i \in \Gamma_\lambda$. So Γ_λ is closed under conjugation by φ. Therefore $\Gamma_\lambda \lhd G_\lambda$. $\square$

Since $\Gamma_\lambda \lhd G_\lambda$, G_λ/Γ_λ is a group where elements are cosets of G_λ modulo Γ_λ. Certainly, $\Gamma_\lambda\rho^i \in G_\lambda/\Gamma_\lambda$ for $0 \le i < q$ where $\rho = \tau_\lambda\varphi \in G_\lambda$.

PROPOSITION 4. *$\tau_\lambda^r \in \Gamma_\lambda\rho^i$ for some $0 \le i < q$ where $r \in \mathbb{Z}$ and $\Gamma_\lambda\rho^i = \{g\rho^i : g \in \Gamma_\lambda\}$.*

Proof

If $r = 0$ then $\tau_\lambda^0 = 1_{map} = \rho^0$ and $1_{map} \in \Gamma_\lambda$ since Γ_λ is a group. Assume $0 < r < q$ is an integer. We note that $\varphi_1\varphi_2 \cdots \varphi_r \rho^r = (\rho\varphi)^r = (\tau_\lambda\varphi\varphi)^r = \tau_\lambda^r$. Therefore $\tau_\lambda^r \in \Gamma_\lambda \rho^r$.

If $r = q$ then $\tau_\lambda^q = \varphi_1\varphi_2 \cdots \varphi_q \rho^q = \varphi_1\varphi_2 \cdots \varphi_{q-1}\varphi \in \Gamma_\lambda = \Gamma_\lambda\rho^q$. If $r > q$ then $r = nq + r_0$, $0 \leq r_0 < q$. Now $\varphi_1\varphi_2 \cdots \varphi_q \in \Gamma_\lambda$ and $(\varphi_1\varphi_2 \cdots \varphi_q)^n \in \Gamma_\lambda$. Thus, if $r = nq + r_0$ then $\tau_\lambda^r = \tau_\lambda^{nq}\tau_\lambda^{r_0} = g_1 g_2 \rho^{r_0} \in \Gamma_\lambda \rho^{r_0}$ since $g_1, g_2 \in \Gamma_\lambda$, $g_1 = \tau_\lambda^{nq}$, $g_2 = \varphi_1\varphi_2 \cdots \varphi_{r_0}$ and $0 \leq r_0 < q$. If $r < 0$ then $-r > 0$ so $\tau_\lambda^{-r} \in \Gamma_\lambda\rho^{r_k}$ where $0 \leq k < q$. So $\tau_\lambda^{-r} = (\varphi_1\varphi_2 \cdots \varphi_q)^{nq}(\varphi_1\varphi_2 \cdots \varphi_{r_0})\rho^{r_0}$ by the division algorithm. Thus, $\tau_\lambda^r = \rho^{-r_0}(\varphi_{r_0}\varphi_{r_0-1} \cdots \varphi_1)(\varphi_1\varphi_2 \cdots \varphi_q)^{-nq} \in \rho^{-r_0}\Gamma_\lambda = \Gamma_\lambda\rho^{-r_0}$ since $\Gamma_\lambda \lhd G_\lambda$. Therefore $\tau_\lambda^r \in \Gamma_\lambda\rho^{-r_0} = \Gamma_\lambda\rho^{q-r_0}$ where $0 \leq q - r_0 < q$. $\square$

PROPOSITION 5. *For each $g \in G_\lambda$, $g \in \Gamma_\lambda\rho^i$ for exactly one $0 \leq i < q$.*

Proof

Let $g \in G_\lambda$ with $g = \tau_\lambda^{r_0}\varphi\tau_\lambda^{r_1} \cdots \varphi\tau_\lambda^{r_k}\varphi$ where $r_i \in \mathbb{Z} - \{0\}$, r_0 and r_k may be zero. By Proposition 4, $\tau_\lambda^{r_0} \in \Gamma_\lambda\rho^{k_0}$ for some $0 \leq k_0 < q$. Now $\tau_\lambda^{r_0}\varphi = h\rho^{k_0}\varphi = h\rho^{k_0}\varphi\rho^{-k_0}\rho^{k_0} = h(\rho^{k_0}\varphi\rho^{-k_0})\rho^{k_0} \in \Gamma_\lambda\rho^{k_0}$ where $0 \leq k_0 < q$, $h \in \Gamma_\lambda \lhd G_\lambda$. Also $\tau_\lambda^{r_0}\varphi\tau_\lambda^{r_1} = h_1\rho^{k_0}\varphi h_2\rho^{k_1} = h_1\rho^{k_0}\varphi h_2\rho^{-k_0}\rho^{k_0}\rho^{k_1} = h_1\{\rho^{k_0}\varphi h_2\rho^{-k_0}\}\rho^{k_0+k_1} \in \Gamma_\lambda\rho^k$ where $h_1, h_2 \in \Gamma_\lambda, 0 \leq k \equiv k_0 + k_1 \ mod\ q$ and $\Gamma_\lambda \lhd G_\lambda$. Therefore inductively, $g \in \Gamma_\lambda\rho^i$ for some $0 \leq i < q$.

Suppose $\Gamma_\lambda\rho^i \cap \Gamma_\lambda\rho^j \neq \emptyset$ with $h_1\rho^i = h_2\rho^j$ where $h_1, h_2 \in \Gamma_\lambda, 0 \leq i, j < q$ and $i \neq j$. Then $h_2^{-1}h_1 = \rho^{j-i} \in \Gamma_\lambda$. Let $k = j - i$ then $\Gamma_\lambda\rho^k = \Gamma_\lambda$ where $k \neq 0$. Since $\rho^k \in \Gamma_\lambda$, $(\tau_\lambda\varphi)^k \in \Gamma_\lambda$ and we have $\tau_\lambda^k \in \Gamma_\lambda$, conjugating by τ_λ repeatedly. But $\tau_\lambda^k \in \Gamma_\lambda\rho^t$ for $0 \leq t < q$. So $t = 0$ and $k = nq$, that is $j - i = k = nq$ for $0 \leq t < q$. So $t = 0$ and $k = nq$, that is $j - i = k = nq$ or $j = i \ mod\ q$. But we assumed that $0 \leq i, j < q$ and so $i = j$ which is a contradiction. $\square$

Proposition 6. *G_λ is partitioned into q distinct, disjoint cosets modulo Γ_λ and*

$$G_\lambda = \bigcup_{i=0}^{q-1} \Gamma_\lambda \rho^i$$

with $\Gamma_\lambda \rho^i \cap \Gamma_\lambda \rho^j = \emptyset$ if $i \neq j$ and hence $G_\lambda/\Gamma_\lambda \cong \langle \rho \rangle$.

Proof

$\Gamma_\lambda \lhd G_\lambda$ so G_λ/Γ_λ is a group. Cosets in G_λ/Γ_λ are $\{\Gamma_\lambda g : g \in G_\lambda\}$ and $\Gamma_\lambda g_1 = \Gamma_\lambda g_2$ if and only if $g_1 g_2^{-1} \in \Gamma_\lambda$. We know if $g \in G_\lambda$ then $g \in \Gamma_\lambda \rho^i$ where $0 \leq i < q$. That is, $\Gamma_\lambda g = \Gamma_\lambda h \rho^i = \Gamma_\lambda \rho^i$ where $h \in \Gamma_\lambda$ and $0 \leq i < q$. By above $\Gamma_\lambda \rho^i \cap \Gamma_\lambda \rho^j \neq \emptyset$ for $0 \leq i, j < q$ and $i \neq j$. Thus, $G_\lambda/\Gamma_\lambda = \{\Gamma_\lambda \rho^i : 0 \leq i < q\}$ and

$$G_\lambda = \bigcup_{i=0}^{q-1} \Gamma_\lambda \rho^i.$$

Note that $|G_\lambda : \Gamma_\lambda| = q$. Thus, Γ_λ is a subgroup of index q in Γ_λ. Certainly $G_\lambda/\Gamma_\lambda \cong \langle \rho \rangle$ where $\Gamma_\lambda \rho^i$ is mapped to ρ^i for $0 \leq i < q$. $\Box$

Proposition 7. *A fundamental region for Γ_λ is the open set $N_q(v) - \partial N_q(v)$ where*

$$N_q(v) = \bigcup_{i=0}^{q-1} \rho^i(\overline{D}),$$

$\partial N_q(v)$ is its boundary and D is a fundamental region of G_λ. This is an ideal $q-$gon with vertices $\{\rho^i(\infty) : i = 0, 1 \cdots q-1\}$.

Proof

We have that the v-cell,

$$N_q(v) = \bigcup_{i=0}^{q-1} \rho^i(\overline{D})$$

is the union of images of $\overline{D}$ by powers of ρ. We recall that $\varphi_i = \rho^i \varphi \rho^{-i}$ and note that $tr^2 \varphi_i = tr^2(\rho^i \varphi \rho^{-i}) = tr^2 \varphi = 0$. Each φ_i is of order 2 and is elliptic. From Section 2.3,

$D = T \cup R_2 T$ and is bounded by the curves, $Re(z) = 0$, $Re(z) = \lambda$, $|z| = 1$ and $|z - \lambda| = 1$. The vertex $v = \cos(\frac{\pi}{q}) + i\sin(\frac{\pi}{q})$ is fixed by $\rho = \tau_\lambda \varphi$. We have seen that for $i = 1, 0, \cdots q-1$ the images of D under ρ^i are copies of D with common vertex v and $\rho^0(D) = \rho^q(D) = D$. The union of images of D under ρ^i is a q-gon with vertices $\{\rho^i(\infty) : i = 0, 1 \cdots q-1\}$. Thus, a side of this q-gon is a geodesic with end points $\alpha_i = \rho^i(\infty)$ and $\alpha_{i-1} = \rho^{i-1}(\infty)$ or $\gamma_{\alpha_{i-1},\alpha_i} = s_i$. Then $\varphi_i(\alpha_i) = (\rho^i \varphi \rho^{-i})(\rho^i(\infty)) = \rho^i \varphi(\infty) = \rho^i \varphi(\tau_\lambda^{-1}(\infty)) = \rho^{i-1}(\infty) = \alpha_{i-1}$.
Also $\varphi_i(\alpha_{i-1}) = (\rho^i \varphi \rho^{-i})(\rho^{i-1}(\infty)) = \rho^i(\varphi \rho^{-1}(\infty)) = \rho^i \varphi(\varphi \tau_\lambda^{-1})(\infty)) = \rho^i(\infty) = \alpha_i$. That is, $\varphi_i(s_i) = s_i$. So each side is fixed by a generator of Γ_λ. Further φ_i has a single fixed point corresponding to the fixed point of φ on $\mathbb{I}_0$. So $N_q(v) - \partial N_q(v)$ is a fundamental region of Γ_λ by Theorem 7. $\square$

Proposition 8. *$Stab(v, G_\lambda) = \langle \tau_\lambda \varphi \rangle = \langle \varphi \tau_\lambda^{-1} \rangle$, a cyclic group of order q.*

Proof

Let $\rho = \tau_\lambda \varphi$ and since $\rho(v) = v$ where $v = \cos(\frac{\pi}{q}) + i\sin(\frac{\pi}{q})$ we have $\langle \rho \rangle \subseteq Stab(v, G_\lambda)$. By Proposition 6, G_λ is partitioned into distinct, disjoint cosets modulo Γ_λ. So $G_\lambda = \Gamma_\lambda \cup \Gamma_\lambda \rho \cup \Gamma_\lambda \rho^2 \cup \cdots \cup \Gamma_\lambda \rho^{q-1}$. If $g \in Stab(v, G_\lambda)$ then $g(v) = v$. Suppose $g = \varphi_t$ for some $0 \le t < q$ then $g(v) = \varphi_t(v) = \rho^t \varphi \rho^{-t}(v) = v$. From this equation we see that $\varphi \rho^{-t}(v) = \rho^{-t}(v)$. Thus, $\rho^{-t}(v) = i$, the only fixed point of φ in $\mathbb{H}^2$. But $\rho^{-t}(v) = v$ so $v = i$ which is a contradiction since $q \neq 2$. Hence, if $g \in \Gamma_\lambda$ then $g = 1_{map}$. By Proposition 5, $g \in \Gamma_\lambda \rho^k$ for some $0 \le k < q$. Let $g = h\rho^k$ with $g(v) = v$. Then $h\rho^k(v) = v$ or $h(v) = v$ for $h \in \Gamma_\lambda$. So $h = 1_{map}$ and $g = \rho^k$, $0 \le k < q$. Therefore $Stab(v, G_\lambda) = \langle \tau_\lambda \varphi \rangle = \langle \rho \rangle$, as required. $\square$

It is immediate now that Theorem 10 holds as a result of Proposition 3-8.

Hence, as in [**9**], that $G_\lambda / \Gamma_\lambda \cong \langle \rho \rangle = Stab(v, G_\lambda)$. We explore $Stab(v, G_\lambda)$ and $N_q(v)$ for $q = 3, 4, 5$ and 6 in the following example.

EXAMPLE 1. *(i)* $\lambda = 1$ when $q = 3$. From Theorem 8 (page 26) above, we have fundamental regions $D_1 = \{z \in \mathbb{C}_\infty : |Re(z)| < \frac{1}{2}$ and $|z| > 1\}$ while $D_2 = \{T \cup R_2 T\}$ where $R_2(z) = -\overline{z} + 1$. Then Figure 7 displays $N_3(v)$ as the union of 3 images of $\overline{D_2}$ under $\langle \tau_1 \varphi \rangle$. The triangle group G_λ is the modular group generated by $\tau_1 = z + 1$ and $\varphi(z) = -1/z$. It is the Hecke group of type $(\frac{\pi}{2}, \frac{\pi}{3}, 0)$.

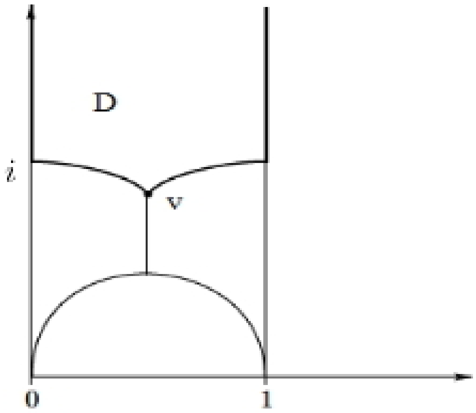

FIGURE 7. $N_3(v)$ for $q = 3$.

(ii) $\lambda = \sqrt{2}$ when $q = 4$. From Theorem 8 above, the fundamental region $D_1 = \{z \in \mathbb{C}_\infty : |Re(z)| < \frac{\sqrt{2}}{2}$ and $|z| > 1\}$. The triangle group G_λ is the Hecke group generated by $\varphi(z) = -1/z$ and $\tau_\lambda = z + \sqrt{2}$ where $\lambda = \sqrt{2}$. It is Hecke group of type $(\frac{\pi}{2}, \frac{\pi}{4}, 0)$. $N_4(v)$ is the union of 4 images of $\overline{D_2}$, $D_2 = T \cup R_2 T$ where $R_2 = -\overline{z} + \sqrt{2}$.

(iii) $\lambda = \frac{\sqrt{5}+1}{2}$ when $q = 5$. From Theorem 8 above, the fundamental region $D_1 = \{z \in \mathbb{C}_\infty : |Re(z)| < \frac{\sqrt{5}+1}{4}$ and $|z| > 1\}$. The triangle group G_λ is the Hecke group generated by $\varphi(z) = \frac{-1}{z}$ and $\tau_\lambda = z + \frac{\sqrt{5}+1}{2}$. It is Hecke group of type $(\frac{\pi}{2}, \frac{\pi}{5}, 0)$. $N_5(v)$ is the union of 5 images of $\overline{D_2}$, $D_2 = T \cup R_2 T$ where $R_2 = -\overline{z} + \frac{\sqrt{5}+1}{2}$.

(iv) $\lambda = \sqrt{3}$ when $q = 6$. From Theorem 8 above, the fundamental regions $D_1 = \{z \in \mathbb{C}_\infty : |Re(z)| < \frac{\sqrt{3}}{2}$ and $|z| > 1\}$ or $D_2 = T \cup R_2 T$ where $R_2(z) = -\overline{z} + \sqrt{3}$. Then

Figure 8 below displays $N_6(v)$ as the union of 6 images of $\overline{D_2}$ under $\langle \tau_\lambda \varphi \rangle$. The triangle group G_λ is the Hecke group generated by $\tau_\lambda = z + \sqrt{3}$ and $\varphi(z) = -1/z$. It is Hecke group of type $(\frac{\pi}{2}, \frac{\pi}{6}, 0)$.

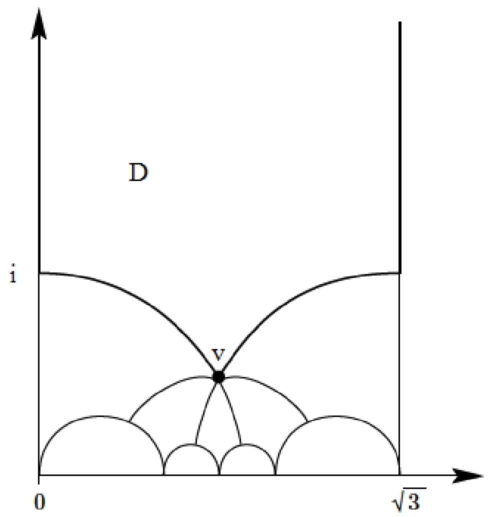

FIGURE 8. $N_6(v)$ for $q = 6$.

CHAPTER 3

λ-fractions

1. λ-continued fractions and elements of G_λ

In what follows we assume $q \geq 3$ is an integer with $\lambda = 2\cos(\frac{\pi}{q})$ and treat λ as an indeterminate. The development follows Rosen [**25**].

Definition 21. *The set $\lambda\mathbb{Z} = \{\lambda x : x \in \mathbb{Z}\}$ is called the set of λ-integers. In fact $\lambda\mathbb{Z}$ is a commutative ring with no zero divisors.*

A finite λ-continued fraction is of the form:

$$r_0\lambda - \cfrac{1}{r_1\lambda - \cfrac{1}{r_2\lambda - \cfrac{1}{\cdots - r_k\lambda}}}$$

where $r_i \in \mathbb{Z} - \{0\}$ for all $1 \leq i < k$, r_0 and r_k may be zero. These λ-continued fractions can be expressed as follows:

Let $\tau_\lambda(z) = z + \lambda$ and $\varphi(z) = -1/z$ then

$$r_0\lambda - \cfrac{1}{r_1\lambda - \cfrac{1}{r_2\lambda - \cfrac{1}{\cdots - r_k\lambda}}} = \tau_\lambda^{r_0} \cdots\cdots \varphi\tau_\lambda^{r_k}\varphi(\infty) = \tau_\lambda^{r_0}\varphi\tau_\lambda^{r_1}\varphi \cdots\ \tau_\lambda^{r_k}(0).$$

Definition 22. *A finite λ-continued fraction (as above) is called a $\lambda-$rational or λ-fraction.*

In particular, if $r_i = 1$ for all i, we have the λ-fraction of the form

$$\lambda - \cfrac{1}{\lambda - \cfrac{1}{\lambda - \cfrac{1}{\cdots - \lambda}}} = \tau_\lambda \varphi \cdots \varphi \tau_\lambda \varphi(\infty) = \rho^r(\infty)$$

where $\rho = \tau_\lambda \varphi(\infty)$ and $r \in \mathbb{Z}^+$.

Theorem 11 and 12 follow from Rosen [**25**] with slight adjustments in the proofs.

Theorem 11. [**25**] *If an element $g(z) = \dfrac{az+b}{cz+d}$ belongs to G_λ then $a/c = g(\infty)$ is a finite λ-fraction. Conversely if a rational a/c is a finite λ-fraction, we can find b/d such that if $g(z) = \dfrac{az+b}{cz+d}$ then $g \in G_\lambda$.*

Proof

Let $g \in G_\lambda = \langle \tau_\lambda, \varphi \rangle$ with $\tau_\lambda(z) = z + \lambda$ and $\varphi(z) = -1/z$. The map g is a word in τ_λ and φ given by $g = \tau_\lambda^{r_0} \varphi \tau_\lambda^{r_1} \varphi \cdots \varphi \tau_\lambda^{r_k} \varphi$ where $r_i \in \mathbb{Z} - \{0\}$ for $1 \le i < k$, r_0 and r_k may be zero. Thus,

$$g(\infty) = a/c = \tau_\lambda^{r_0} \varphi \tau_\lambda^{r_1} \varphi \cdots\ \varphi \tau_\lambda^{r_k} \varphi(\infty) = r_0\lambda - \cfrac{1}{r_1\lambda - \cfrac{1}{r_2\lambda - \cfrac{1}{\cdots - r_k\lambda}}}.$$

That is, a/c is a λ-fraction. Also

$$g(0) = b/d = \tau_\lambda^{r_0} \varphi \tau_\lambda^{r_1} \varphi \cdots\ \varphi \tau_\lambda^{r_{k-1}} \varphi(\infty) = r_0\lambda - \cfrac{1}{r_1\lambda - \cfrac{1}{r_2\lambda - \cfrac{1}{\cdots - r_{k-1}\lambda}}}.$$

Thus, $g(\infty) = a/c$ and $g(0) = b/d$ are consecutive convergents of a λ-continued fraction, with $g(\infty)$ succeeding $g(0)$ and $ad - bc = 1$ as given.

Conversely, if a/c is a λ-fraction then we can find a finite set of λ-integers such that

$$a/c = r_0\lambda - \cfrac{1}{r_1\lambda - \cfrac{1}{r_2\lambda - \cfrac{1}{\cdots - r_k\lambda}}} = \tau_\lambda^{r_0}\varphi\tau_\lambda^{r_1}\varphi\cdots\varphi\tau_\lambda^{r_k}\varphi(\infty) = \tau_\lambda^{r_0}\varphi\tau_\lambda^{r_1}\varphi\cdots\varphi\tau_\lambda^{r_k}(0).$$

Let $b/d = r_0\lambda - \cfrac{1}{r_1\lambda - \cfrac{1}{r_2\lambda - \cfrac{1}{\cdots - r_{k-1}\lambda}}} = \tau_\lambda^{r_0}\varphi\tau_\lambda^{r_1}\varphi\cdots\ \tau_\lambda^{r_{k-1}}\varphi(\infty)$. Then a/c is the k^{th} convergent of a λ-continued fraction where b/d is the $(k-1)^{th}$ convergent of the same λ-continued fraction. We then know that consecutive convergents of any real continued fraction are adjacent and $|ad - bc| = 1$. Let g be the word in τ_λ and φ given by $g(z) = \tau_\lambda^{r_0}\varphi\tau_\lambda^{r_1}\varphi\cdots\ \tau_\lambda^{r_k}\varphi(z)$ where $r_k \neq 0$. Then $g(\infty) = a/c$, $g(0) = b/d$ and $|ad - bc| = 1$. Since $G_\lambda = \langle\tau_\lambda, \varphi\rangle$, $g \in G_\lambda$ as required.□

Corollary 1. *Let $g = \tau_\lambda^{r_0}\varphi\tau_\lambda^{r_1}\varphi\cdots\ \tau_\lambda^{r_k}\varphi \in G_\lambda$, where $g(z) = \dfrac{az+b}{cz+d}$ and the λ-fractions $g(\infty) = \tau_\lambda^{r_0}\varphi\tau_\lambda^{r_1}\varphi\cdots\ \tau_\lambda^{r_k}\varphi(\infty) = a/c$ and $g(0) = \tau_\lambda^{r_0}\varphi\tau_\lambda^{r_1}\varphi\cdots\ \tau_\lambda^{r_k}\varphi(0) = b/d$. Then $\dfrac{a}{c} = \dfrac{\lambda p_1(\lambda^2)}{q_1(\lambda^2)}$ and $\dfrac{b}{d} = \dfrac{p_2(\lambda^2)}{\lambda q_2(\lambda^2)}$ or $\dfrac{a}{c} = \dfrac{p_1(\lambda^2)}{\lambda q_1(\lambda^2)}$ and $\dfrac{b}{d} = \dfrac{\lambda p_2(\lambda^2)}{q_2(\lambda^2)}$ where $p_1(\lambda^2), p_2(\lambda^2), q_1(\lambda^2), q_2(\lambda^2) \in \mathbb{Z}[\lambda^2]$ for k even or odd respectively. Here $\mathbb{Z}[\lambda^2]$ is a ring of polynomials over λ^2 with coefficient in $\mathbb{Z}$. We may write p for $p(\lambda^2)$ for simplicity in the sequel.*

Proof

Let $a/c = \tau_\lambda^{r_0}\varphi\tau_\lambda^{r_1}\varphi\cdots\ \tau_\lambda^{r_k}\varphi(\infty)$ and $b/d = \tau_\lambda^{r_0}\varphi\tau_\lambda^{r_1}\varphi\cdots\ \tau_\lambda^{r_k}\varphi(0)$. We prove the results using induction on $k \geq 0$ where $(k+1)$ is the length of the word in G_λ. If $k = 0$ (even) then $a/c = \tau_\lambda^{r_0}\varphi(\infty) = \lambda r_0 = \dfrac{\lambda p(\lambda^2)}{q(\lambda^2)}$ with $p(\lambda^2) = r_0$ and $q(\lambda^2) = 1$. Also $b/d = \tau_\lambda^{r_o}\varphi(0) = \tau_\lambda^{r_o}(\infty) = \dfrac{p(\lambda^2)}{\lambda q(\lambda^2)}$ where $p(\lambda^2) = 1$ and $q(\lambda^2) = 0$, the zero polynomial. If $k = 1$ (odd) then $a/c = \tau_\lambda^{r_0}\varphi\tau_\lambda^{r_1}\varphi(\infty) = \tau_\lambda^{r_0}\varphi(\lambda r_1) = \lambda r_0 - \dfrac{1}{\lambda r_1} = \dfrac{\lambda^2 r_0 r_1 - 1}{\lambda r_1} = \dfrac{p(\lambda^2)}{\lambda q(\lambda^2)}$ where $p(\lambda^2) = \lambda^2 r_0 r_1 - 1$

and $q(\lambda^2) = r_1$ are in $\mathbb{Z}[\lambda^2]$. Also $b/d = \tau_\lambda^{r_0}\varphi\tau_\lambda^{r_1}\varphi(0) = \tau_\lambda^{r_0}(0) = \lambda r_0 = \dfrac{\lambda p(\lambda^2)}{q(\lambda^2)}$ where $p(\lambda^2) = r_0$ and $q(\lambda^2) = 1$ are in $\mathbb{Z}[\lambda^2]$. Thus, result holds for $k = 0$ (even) and $k = 1$ (odd).

Assume results hold for expansion of length $t \leq k$ where $k \in \mathbb{Z}^+$. Then $a/c = \tau_\lambda^{r_0}\varphi\tau_\lambda^{r_1}\varphi \cdots \tau_\lambda^{r_k}\varphi(\infty) = \tau_\lambda^{r_0}\varphi\{\tau_\lambda^{r_1}\varphi \cdots \tau_\lambda^{r_k}\varphi(\infty)\} = \tau_\lambda^{r_0}\varphi\left(\dfrac{\lambda p(\lambda^2)}{q(\lambda^2)}\right)$ or $\tau_\lambda^{r_0}\varphi\left(\dfrac{p(\lambda^2)}{\lambda q(\lambda^2)}\right)$ for k even or odd respectively, by induction hypothesis. Thus, $a/c = \lambda r_0 - \dfrac{1}{\frac{\lambda p(\lambda^2)}{q(\lambda^2)}} = \lambda r_0 - \dfrac{q(\lambda^2)}{\lambda p(\lambda^2)} = \dfrac{\lambda^2 r_0 p(\lambda^2) - q(\lambda^2)}{\lambda p(\lambda^2)}$ or $\lambda r_0 - \dfrac{1}{\frac{p(\lambda^2)}{\lambda q(\lambda^2)}} = \lambda r_0 - \dfrac{\lambda q(\lambda^2)}{p(\lambda^2)} = \dfrac{\lambda(r_0 p(\lambda^2) - q(\lambda^2))}{p(\lambda^2)}$ as required for $r_0 = 0$ or $r_0 \neq 0$. Thus, the result holds for an expansion of length $k+1$. Hence, result holds for all k.

From above, $g \in G_\lambda$ so $g\varphi \in G_\lambda$ and $g\varphi(\infty) = g(0) = b/d$. So from above $\dfrac{b}{d} = \dfrac{p}{\lambda q}$ or $\dfrac{\lambda p}{q}$. Thus, if $g \in G_\lambda$ then $g(z) = \dfrac{\lambda p_1(\lambda^2)z + p_2(\lambda^2)}{q_1(\lambda^2)z + \lambda q_2(\lambda^2)}$ with $\lambda^2 p_1 q_2 - p_2 q_1 = 1$ or $g(z) = \dfrac{p_1(\lambda^2)z + \lambda p_2(\lambda^2)}{\lambda q_1(\lambda^2)z + q_2(\lambda^2)}$ with $p_1 q_2 - \lambda^2 p_2 q_1 = 1$. $\square$

LEMMA 2. *For $p, q \in \mathbb{Z}[\lambda^2]$, $\dfrac{\lambda p}{q}$ and $\dfrac{p}{\lambda q}$ are λ-fractions.*

Proof

Let $p, q \in \mathbb{Z}[\lambda^2]$ and $\alpha_0 = \dfrac{\lambda p}{q} \in \mathbb{R}$. Since $\mathbb{R}$ is tessellated by the interval $[0, \lambda]$ under $Stab(\infty, G_\lambda) = \langle \tau_\lambda \rangle$, we can find an integer n_0 such that $n_0\lambda \leq \alpha_0 \leq (n_0+1)\lambda$ or $n_0 \leq \alpha_0/\lambda \leq n_0 + 1$. Let $m_0 = [\alpha_0/\lambda]$ be the nearest integer to α_0/λ. Then $\dfrac{\alpha_0}{\lambda} = m_0 + \delta_0$ where $|\delta_0| \leq \dfrac{1}{2}$. Thus, $\alpha_0 = \lambda m_0 + \delta_0\lambda$, where $|\delta_0\lambda| \leq \dfrac{\lambda}{2}$ and $\dfrac{1}{\lambda|\delta_0|} \geq \dfrac{2}{\lambda} > 1$ since $1 \leq \lambda < 2$.

Write $\alpha_1 = \dfrac{1}{\delta_0\lambda} \in \mathbb{R}$. Repeating the above process we can find m_1 and δ_1 such that $\alpha_1 = m_1\lambda + \delta_1\lambda = m_1\lambda + \dfrac{1}{\frac{1}{\delta_1\lambda}} = m_1\lambda + \dfrac{1}{\alpha_2}$ where $m_1 \in \mathbb{Z}$, $|\alpha_2| > 1$. Thus, $\alpha_0 = \lambda m_0 + \dfrac{1}{\lambda m_1 + \dfrac{1}{\alpha_2}}$. Since we know $\alpha_0 = \dfrac{\lambda p}{q}$, a finite λ-fraction, we can repeat this process only a finite number of times. That is, we can find a finite set of integers r_i for $i = 0, ..., k$ such that

$\alpha_0 = \lambda r_0 - \cfrac{1}{\lambda r_1 - \cfrac{1}{\lambda r_2 - \cfrac{1}{\cdots \lambda r_k}}} = \frac{\lambda p}{q}$. Thus, α_0 is a λ-fraction. Similarly if $\alpha = \frac{p}{\lambda q}$ we can show α is λ-fraction. $\square$

We know that $Stab(\infty, G_\lambda) = \langle \tau_\lambda \rangle$ with $\tau_\lambda^n(z) = z + n\lambda$. Then $\tau_\lambda^n(\infty) = \infty = 1/0$ and $\tau_\lambda^n(0) = n\lambda$.

Lemma 3. *If* $\alpha = \tau_\lambda^{r_0}\varphi\tau_\lambda^{r_1}\varphi\cdots\ \tau_\lambda^{r_k}\varphi(\infty) = \frac{\lambda p(\lambda^2)}{1}$ *where* k *is even then* $p(\lambda^2) \in \mathbb{Z}$. *If* $\alpha = \tau_\lambda^{r_0}\varphi\tau_\lambda^{r_1}\varphi\cdots\ \tau_\lambda^{r_k}\varphi(\infty) = \frac{1}{\lambda q(\lambda^2)}$ *where* k *is odd then* $q(\lambda^2) \in \mathbb{Z}$.

Proof

Let α be a λ-fraction so there exist finite integers r_i, $i = 0, \cdots, k$ such that

$$\alpha = r_0\lambda - \cfrac{1}{r_1\lambda - \cfrac{1}{r_2\lambda - \cfrac{1}{\cdots - r_k\lambda}}} = \tau_\lambda^{r_0}\cdots\cdots\varphi\tau_\lambda^{r_k}\varphi(\infty)$$

where r_0 and r_k may be zero with $\lambda = 2\cos(\frac{\pi}{q}) \in \mathbb{R} - \mathbb{Q}$.

If $k = 0$ (k even) then $\alpha = r_0\lambda$ and $p(\lambda^2) = r_0 \in \mathbb{Z}$ as required. If $k = 1$ (k odd) then $\alpha = r_0\lambda - \frac{1}{r_1\lambda} = \frac{r_0r_1\lambda^2 - 1}{r_1\lambda} = \frac{p(\lambda^2)}{\lambda q(\lambda^2)}$. Thus, $q(\lambda^2) = r_1$ and $p(\lambda^2) = r_0r_1\lambda^2 - 1 = 1$, by assumption. Thus, $deg(\lambda, \mathbb{Q}) \leq 2$. So $deg(\lambda, \mathbb{Q}) = 2$ since $\lambda \notin \mathbb{Q}$. Since this is not always true, [**1**] (page 7018), either $r_0 = 0$ or $r_1 = 0$. If $r_0 = 0$ then $\alpha = \frac{1}{-r_1\lambda}$ as required. If $r_1 = 0$ then we revert $k = 0$ case. Thus, the result holds for $k = 0$ and $k = 1$.

Inductively, assume that if $\alpha = \frac{\lambda p}{1}$ or $\frac{1}{\lambda q}$ is a λ-fraction of length $k + 1$ where k is even or odd respectively, the results hold.

Thus, consider

$$\alpha = \tau_\lambda^{r_0}\varphi\tau_\lambda^{r_1}\varphi\cdots\varphi\tau_\lambda^{r_k}\varphi(\infty)$$

with length $k+1$.

Assume k is even, then $\alpha = \dfrac{\lambda p(\lambda^2)}{1}$ (Corollary 1 above). Let $\beta_1 = \varphi\tau_\lambda^{-r_0}(\alpha) = \tau_\lambda^{r_1}\varphi\cdots\varphi\tau_\lambda^{r_k}\varphi(\infty)$. So $\beta_1 = \varphi\tau_\lambda^{-r_0}(\alpha) = \varphi\tau_\lambda^{-r_0}(\lambda p(\lambda^2)) = \dfrac{-1}{\lambda p(\lambda^2) - r_0\lambda} = \dfrac{1}{\lambda(r_0 - p(\lambda^2))}$ is a λ-fraction of length k where $k-1$ is odd. By induction hypothesis $r_0 - p(\lambda^2) \in \mathbb{Z}$, and so $p(\lambda^2) \in \mathbb{Z}$ for k even, as required.

Let k be odd ($k+1$ is even) with $\alpha = \dfrac{1}{\lambda q(\lambda^2)}$ (Corollary 1 above). Then $\beta_2 = \tau_\lambda\varphi(\alpha) = \dfrac{\lambda(1-q(\lambda^2))}{1}$ and it is a λ-fraction of length $k+2$ where $k+1$ is even. By induction hypothesis and the above case, $1 - q(\lambda^2) \in \mathbb{Z}$ and so $q(\lambda^2) \in \mathbb{Z}$. $\square$

Definition 23. *λ-fractions a/c and b/d are said to be λ-Farey neighbors if $|ad - bc| = 1$. If $ad - bc = 1$ then $g(z) = \dfrac{az+b}{cz+d}$ is in G_λ. If $ad - bc = -1$ then $g(z) = \dfrac{-az+b}{-cz+d}$ is in G_λ. We write $\dfrac{a}{c} \sim \dfrac{b}{d}$.*

Definition 24. *If a/c and b/d are λ-Farey neighbors then the geodesics $\gamma_{\frac{a}{c},\frac{b}{d}}$ in $\mathbb{H}^2$ is a λ-Farey geodesic.*

We note from Lemma 3 that all λ-integers (λn) have $1/0 = \infty$ as a λ-Farey neighbor. The λ-Farey geodesic between these λ-Farey neighbors is a Euclidean vertical line $\gamma_{n\lambda,\infty}$. The geodesic $\gamma_{\frac{a}{c},\frac{b}{d}}$ is thus $g(\mathbb{I}_0)$ where $g \in G_\lambda$ and $\mathbb{I}_0 = \gamma_{0,\infty}$.

Lemma 4. *The λ-Farey geodesics in $\mathbb{H}^2$ are the orbit of $\mathbb{I}_0$ under G_λ. Certainly the image under G_λ of any λ-Farey geodesic is again a λ-Farey geodesic. Also if a/c and b/d are λ-Farey neighbors then $h(a/c)$ is a λ-Farey neighbor of $h(b/d)$ for any $h \in G_\lambda$.*

Proof

Let the λ-Farey geodesic $\gamma = \gamma_{\frac{a}{c},\frac{b}{d}}$ and assume $ad - bc = 1$. Then $g(z) = \dfrac{az+b}{cz+d}$ and $g \in G_\lambda$. Certainly $g(\infty) = a/c$ and $g(0) = b/d$ and $\gamma = g(\gamma_{\infty,0})$ as required. If $ad - bc = -1$ then

$g(z) = \dfrac{-az+b}{-cz+d}$ and $g \in G_\lambda$. Once again $\gamma = \gamma_{\frac{a}{c},\frac{b}{d}} = \gamma_{\frac{b}{d},\frac{a}{c}} = g(\gamma_{\infty,0})$ as required. Let $h \in G_\lambda$ and $\dfrac{a}{c} \sim \dfrac{b}{d}$ with $g(z) = \dfrac{-az+b}{-cz+d}$, $g \in G_\lambda$. We know $gh \in G_\lambda$. Then $h(\gamma_{\frac{a}{c},\frac{b}{d}}) = \gamma_{h(\frac{a}{c}),h(\frac{b}{d})} = hg(\gamma_{\infty,0})$ and $h(a/c) = hg(1/0)$ while $h(b/d) = hg(0/1)$. So $h(a/c) \sim h(b/d)$. The result follows similarly if $g(z) = \dfrac{-az+b}{-cz+d}$. $\square$

From Corollary 1, if $\dfrac{a}{c} \sim \dfrac{b}{d}$ and $\dfrac{a}{c} = \dfrac{\lambda p_1}{q_1}$, then $\dfrac{b}{d} = \dfrac{p_2}{\lambda q_2}$. Also if $\dfrac{a}{c} = \dfrac{p_1}{\lambda q_1}$ then $\dfrac{b}{d} = \dfrac{\lambda p_2}{q_2}$ where p_1, p_2, q_1 and q_2 are all in $\mathbb{Z}[\lambda^2]$. We note that if $g(z) = \dfrac{az+b}{cz+d}$, $g \in G_\lambda$ then a/c is in the orbit of ∞ under G_λ. Note, a/c is a λ-fraction of the form $\dfrac{\lambda p}{q}$ or $\dfrac{p}{\lambda q}$ where $p, q \in \mathbb{Z}[\lambda^2]$. The λ-fractions, a/c and b/d are consecutive convergents of a λ-continued fraction. Following Rosen [**25**], we have the following equivalence for λ-rationals.

THEOREM 12. *a/c is in the orbit of ∞ under G_λ if and only if a/c is a parabolic fixed point of some $h \in G_\lambda$.*

Proof

Let $g \in G_\lambda$ be given by $g(z) = \dfrac{az+b}{cz+d}$ and $g(\infty) = a/c$. Assume $\alpha = \dfrac{a}{c} = \dfrac{\lambda p}{q}$ where $p, q \in \mathbb{Z}[\lambda^2]$, and let $s(z) = \dfrac{1}{\alpha - z}$. Certainly $\alpha = \dfrac{\lambda p}{q} \in \mathbb{R}$ since $\lambda, p, q \in \mathbb{R}$. Thus, $s \in PSL(2,\mathbb{R})$ with $s(\alpha) = \infty$ and $s^{-1}(\infty) = \alpha$. Further α is fixed by $s^{-1}\tau_\lambda s$ since $s^{-1}\tau_\lambda s(\alpha) = s^{-1}(\infty) = \alpha$. Since τ_λ is parabolic in $PSL(2,\mathbb{R})$ so too is the conjugate $s^{-1}\tau_\lambda s = h$ parabolic in $PSL(2,\mathbb{R})$. We now show that $s \in G_\lambda$.

$s(z) = \dfrac{1}{\alpha - z} = \dfrac{1}{\frac{\lambda p}{q} - z} = \dfrac{q}{\lambda p - qz} = \dfrac{q}{-qz + \lambda p}$ and $s^{-1}(z) = \dfrac{\lambda p z - q}{qz} = \dfrac{\lambda p}{q} - 1/z$, therefore $s^{-1}(\infty) = \dfrac{\lambda p}{q}$ where $p, q \in \mathbb{Z}[\lambda^2] \subseteq \mathbb{R}$. By Lemma 2, this is a λ-fraction and by Theorem 11, $s^{-1} \in G_\lambda$ so $s \in G_\lambda$. Thus, α is a parabolic fixed point of $h = s^{-1}\tau_\lambda s \in G_\lambda$.

Conversely, assume $\alpha = a/c$ is a parabolic fixed point of some $h \in G_\lambda$. Let $\alpha = 0$ or $\alpha = \infty$. Then since $\tau_\lambda(\infty) = \infty$ and $\varphi(\infty) = 0$ both 0 and ∞ are in the orbit of ∞ under G_λ. Assume

$\alpha \neq 0$ or ∞ with $\alpha = h(\alpha)$ for $h \in G_\lambda$. Then $h(z) = \dfrac{\lambda p_1 z + p_2}{q_1 z + \lambda q_2}$ or $\dfrac{p_1 z + \lambda p_2}{\lambda q_1 z + q_2}$, by Corollary 1. So $(\lambda p_1 \alpha + p_2) = \alpha(q_1\alpha + \lambda q_2)$ or $(p_1\alpha + \lambda p_2) = \alpha(\lambda q_1 \alpha + q_2)$. Then $\alpha^2 q_1 + \alpha(\lambda q_2 - \lambda p_1) - p_2 = 0$ or $\alpha^2(\lambda q_1) + \alpha(q_2 - p_1) - \lambda p_2 = 0$.

Case (i).

Assume $\alpha^2 q_1 + \alpha(\lambda q_2 - \lambda p_1) - p_2 = 0$ and $q_1 = 0$. Then $\alpha = \dfrac{p_2}{\lambda q_2 - \lambda p_1} = \dfrac{p_2}{\lambda(q_2 - p_1)}$ where $p_1, p_2, q_1, q_2, q_2 - p_1 \in \mathbb{Z}[\lambda^2]$. Then by Lemma 2 α is λ-fraction. Then by Theorem 11, $\alpha = a/c = g(\infty)$.

Case (ii).

If $q_1 \neq 0$ then $\alpha = \dfrac{\lambda p_1 - \lambda q_2}{2q_1} = \dfrac{\lambda(p_1 - q_2)}{2q_1}$. So $\alpha = \dfrac{-\lambda(q_2 - p_1)}{2q_1}$ is a λ-fraction (Lemma 2) and hence an orbit of ∞ under G_λ.

Similarly, results hold if $\alpha^2(\lambda q_1) + \alpha(q_2 - p_1) - \lambda p_2 = 0$. $\square$

Hence, we have established the following equivalent statements:

(i) $g \in G_\lambda$, $g(z) = \dfrac{az+b}{cz+d}$ where $ad - bc = 1$ and $a, b, c, d \in \mathbb{R}$.

(ii) a/c and b/d are finite λ-continued fractions (λ-fractions) with $a/c = g(\infty)$ and $b/d = g(0)$ for $g \in G_\lambda$.

(iii) a/c and b/d are of the form $\dfrac{\lambda p_1}{q_2}$ and $\dfrac{p_2}{\lambda q_2}$ or $\dfrac{p_1}{\lambda q_2}$ and $\dfrac{\lambda p_2}{q_2}$.

(iv) a/c is parabolic fixed point.

Hence, λ-fractions are parabolic fixed points of some $h \in G_\lambda$ and are in the orbit of ∞ under G_λ. Further a/c and b/d are λ-Farey neighbors.

Proposition 9. *λ-Farey geodesics cannot intersect in $\mathbb{H}^2$.*

Proof

Let $\gamma_1 = \gamma_{\left[\frac{p_1}{q_1},\frac{p_1'}{q_1'}\right]}$ and $\gamma_2 = \gamma_{\left[\frac{p_2}{q_2},\frac{p_2'}{q_2'}\right]}$ be two distinct λ-Farey geodesics. Assume $\gamma_1 \cap \gamma_2 \neq \emptyset$. Let $g(z) = \dfrac{p_1 z + p_1'}{q_1 z + q_1'}$, $g \in G_\lambda$ and $|p_1 q_1' - p_1' q_1| = 1$. Thus, $g^{-1}(\gamma_1) = \mathbb{I}_0$ (Lemma 4, page 40). Let $g^{-1}(\gamma_2) = \gamma_{\left[\frac{a}{c},\frac{b}{d}\right]} = \gamma_3$ intersects $\mathbb{I}_0$ at a point $t \in \mathbb{I}_0$. Thus, $\dfrac{a}{c} < 0 < \dfrac{b}{d}$.

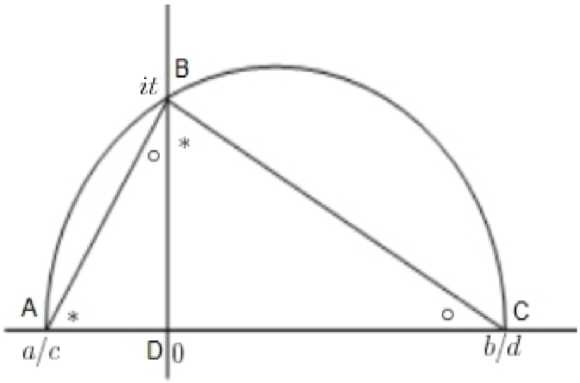

FIGURE 1. Similar triangles.

Since $\Delta DAB|||\Delta BAC|||\Delta DBC$ and by Pythagoras' theorem we have that $t^2 = |AD||DC| = |a/c||b/d|$.

$$
\begin{aligned}
|AC^2| = \left|\frac{b}{d} - \frac{a}{c}\right|^2 &= |AB|^2 + |BC|^2 \\
&= ((a/c-0)^2 + (t-0)^2) + (b/d-0)^2 + (t-0)^2 \\
&= a^2/c^2 + b^2/d^2 + 2t^2
\end{aligned}
$$

Thus, $\left|\dfrac{b}{d} - \dfrac{a}{c}\right|^2 = a^2/c^2 + b^2/d^2 + 2|a/c||b/d| = \left(\dfrac{a}{c} + \dfrac{b}{d}\right)^2$. Therefore $\left(\dfrac{b}{d} - \dfrac{a}{c}\right) = \pm\left(\dfrac{a}{c} + \dfrac{b}{d}\right)$. If $\left(\dfrac{b}{d} - \dfrac{a}{c}\right) = \left(\dfrac{a}{c} + \dfrac{b}{d}\right)$ then $2a/c = 0$, so $a = 0$. This is impossible since $a/c < 0$. If $\left(\dfrac{b}{d} - \dfrac{a}{c}\right) = -\left(\dfrac{a}{c} + \dfrac{b}{d}\right)$ then $2b/d = 0$ and $b = 0$. This is not possible since $b/d > 0$. Thus, the two geodesics γ_1 and γ_2 cannot intersect. $\square$

CHAPTER 4

The λ-Farey tessellation of $\mathbb{H}^2$

1. Tesselation of $\mathbb{H}^2$ by a fundamental region under G_λ

We have seen in Section 2.3 that $D_i = T \cup R_iT$ tessellates $\mathbb{H}^2$ under G_λ for $i = 1, 2, 3$. We have proved the following results, (Theorem 8, page 26).

The triangle group $H(\lambda)$ of type $(\frac{\pi}{2}, \frac{\pi}{q}, 0)$ is generated by $\tau_\lambda(z) = z + \lambda$ and $\varphi(z) = -1/z$. The group has presentation $\langle \tau_\lambda, \varphi : \varphi^2 = (\tau_\lambda\varphi)^q = 1_{map}\rangle$. Further, $D_i = T \cup R_iT$ is the fundamental polygon of the group $H(\lambda)$ for $i = 1, 2, 3$ where $R_1(z) = -\overline{z}$, $R_2(z) = -(\overline{z} + \lambda)$ and $R_3(z) = \dfrac{1}{\overline{z}}$. We also established that $H(\lambda) = G_\lambda$.

From Vulukh [**30**], we note that the fundamental regions could alternately be determined using isometric circles of elements in G_λ as follows.

We know ∞ is the parabolic fixed point of τ_λ. We have seen that $Stab(\infty, G_\lambda) = \langle\tau_\lambda\rangle$. Let $B = [\frac{-\lambda}{2}, \frac{\lambda}{2}]$ be a Dirichlet interval of $Stab(\infty, G_\lambda)$ and let $B_\infty = \{(x, t) \in \mathbb{H}^2 : x \in B\}$. We recall that $g \in G_\lambda$ with $g(z) = \dfrac{az + b}{cz + d}$ and $ad - bc = 1$ has isometric circle $|cz + d| = 1$ if $c \neq 0$. So $g'(z) = \dfrac{a(cz + d) - c(az + b)}{(cz + d)^2} = \dfrac{1}{(cz + d)^2}$ and $|g'(z)| = \dfrac{1}{|cz + d|^2}$. If $|g'(z)| < 1$ then $|cz + d| > 1$ and z lies outside the isometric circle. Note that the isometric circle of $\varphi(z) = \dfrac{-1}{z}$ is $|z| = 1$.

Thus, $D = \{z \in \mathbb{C}_\infty : |Re(z)| < \frac{\lambda}{2}$ and $|z| > 1\} = T \cup R_1T$, can be written as the intersection of B_∞ and the exterior of all isometric circles of G_λ. That is, $D = B_\infty \cap \{x \in \mathbb{H}^2 : |g'(x)| < 1,$ $g \in G_\lambda\}$. We note the closure of D is $\overline{D} = \{z \in \mathbb{C}_\infty : |Re(z)| \leq \frac{\lambda}{2}$ and $|z| \geq 1\}$.

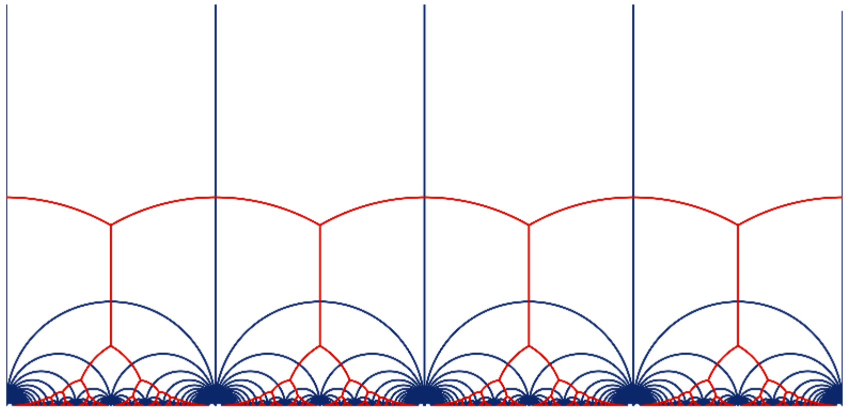

FIGURE 1. Tessellation of $\mathbb{H}^2$ by $\overline{D}$ under modular group, $\lambda = 1$.

2. λ-Farey tessellation and q-gons

In this section we show that the v-cell $N_q(v)$ introduced in Definition 19 (page 28), tessellates $\mathbb{H}^2$ under G_λ.

DEFINITION 25. *Let $K = K(\infty) = \langle \tau_\lambda \rangle(\overline{D})$, the union of the orbit of $\overline{D}$ under $Stab(\infty, G_\lambda)$.*

We note K has a vertex at ∞. We may note that if $K(g(\infty)) := g(K(\infty))$ then $K(g(\infty))$ has a vertex at $g(\infty)$. Since $g(\infty)$ is a λ-fraction, $K(g(\infty))$ has a vertex at a λ-fraction.

Let ∂K be the boundary of K. We say that $\partial K \cap \overline{D}$ is the floor of D where

$$\partial K \cap \overline{D} = \{z \in \mathbb{C} : |z| = 1 \text{ and } |Re(z)| \leq \tfrac{\lambda}{2}\}.$$

Recall that $v = \cos(\frac{\pi}{q}) + i\sin(\frac{\pi}{q})$ and $Stab(v, G_\lambda) = \langle \tau_\lambda \varphi \rangle$, (Proposition 8, page 32).

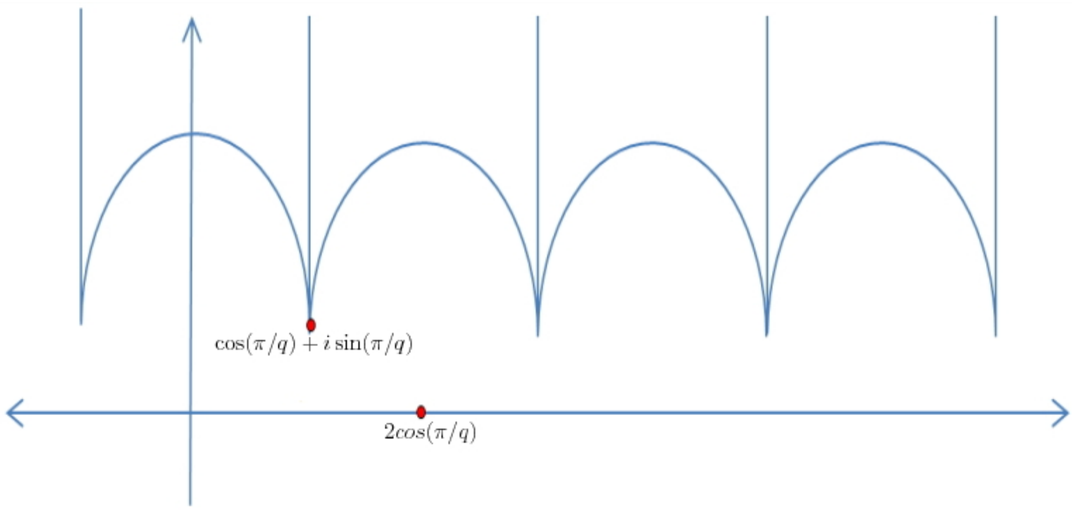

FIGURE 2. K(∞).

In what follows we will consider $D = T \cup R_2T$ as in Section 2.3, Figure 6 (page 27).

We have seen for $q \geq 3$, v-cell $N_q(v)$ is defined as

$$N_q(v) = \overline{D} \cup \tau_\lambda\varphi(\overline{D}) \cup (\tau_\lambda\varphi)^2(\overline{D}) \ldots \cup (\tau_\lambda\varphi)^{q-1}(\overline{D}) = \cup_{r=0}^{q-1}(\tau_\lambda\varphi)^r\overline{D}.$$

Further, the v-cell $N_q(v)$ is a q-gon with vertices on $\mathbb{R}_\infty$. From Proposition 7 (page 31), the vertices of $N_q(v)$ are the orbit of ∞ under $\langle\tau_\lambda\varphi\rangle$ being

$$\{\infty, \rho(\infty), \rho^2(\infty), \cdots, \rho^{q-1}(\infty)\} = \{\infty, \sigma^{q-1}(\infty), \sigma^{q-2}(\infty), \cdots, \sigma^1(\infty)\}$$

where $\rho = \tau_\lambda\varphi = \sigma^{q-1}$, $\rho^q = 1_{map}$.

LEMMA 5. *The vertices of a v-cell $N_q(v)$, can be written as*

$$\{\infty, \lambda, \lambda - \frac{1}{\lambda}, \lambda - \frac{1}{\lambda - \frac{1}{\lambda}}, \cdots, \frac{1}{\lambda}, 0\}$$

and are all parabolic fixed points of G_λ and hence they are λ-fractions. Consecutive vertices are λ-Farey neighbors. Every edge or side of $N_q(v)$ is a λ-Farey geodesic.

Proof

The set of vertices of the v-cell $N_q(v)$ is given by $\{(\tau_\lambda\varphi)^r(\infty) : r = 0, \cdots, q-1\}$ or alternately as $\{(\varphi\tau_\lambda^{-1})^r(\infty) : r = 0, \cdots, q-1\}$ where $\varphi(z) = -1/z$ and $\tau_\lambda(z) = z + \lambda$. Evaluating the vertices for $r = 0, 1, \cdots, q-1$, we get the vertices $\varphi\tau_\lambda^{-1}(\infty) = 0$; $(\varphi\tau_\lambda^{-1})^2(\infty) = (\varphi\tau_\lambda^{-1})(0) = \varphi(-\lambda) = \dfrac{1}{\lambda}$; $(\varphi\tau_\lambda^{-1})^3(\infty) = (\varphi\tau_\lambda^{-1})(1/\lambda) = \varphi(-\lambda + 1/\lambda) = \dfrac{1}{\lambda - 1/\lambda}$; $\cdots$; $(\varphi\tau_\lambda^{-1})^{q-1}(\infty) = (\tau_\lambda\varphi)^{-1}(\tau_\lambda\varphi)^q(\infty) = (\tau_\lambda\varphi)(\infty) = \lambda$.

Thus, the vertices are all λ-fractions and are consecutive convergents of a λ-continued fraction expansion of $\sigma^l(\infty)$ where $q = 2l(l \geq 2)$ or $q = 2l - 1(l \geq 3)$. If $\sigma^r(\infty)$ and $\sigma^{r-1}(\infty)$ are consecutive vertices of $N_q(v)$ then $\sigma^r(\infty) = \sigma^{r-1}(\varphi\tau_\lambda^{-1})(\infty) = \sigma^{r-1}(\varphi(\infty)) = \sigma^{r-1}(0)$, $\sigma^{r-1} \in G_\lambda$ where $1 \leq r \leq l$ for both $q = 2l$ and $q = 2l - 1$. Thus, consecutive vertices of the q-gon are λ-Farey neighbors and the geodesics between them are the λ-Farey geodesics given by $g(\mathbb{I}_0)$, where $g = \sigma^{r-1} \in G_\lambda$. $\square$

Note that the v-cell tessellates $\mathbb{H}^2$ under G_λ, since D is a fundamental region of G_λ and each v-cell is the union of q-images of $\overline{D}$ under $\langle\tau_\lambda\varphi\rangle = Stab(v, G_\lambda)$. Further $B = [\frac{-\lambda}{2}; \frac{\lambda}{2}]$ tessellates $\mathbb{R}_\infty$ under $\langle\tau_\lambda\rangle = Stab(\infty, G_\lambda)$.

We have noted that $N_q(g(v))$ is defined as $g(N_q(v))$. The vertices of the $g(v)$-cell $N_q(g(v))$ are just $\{g\sigma^r(\infty) : \sigma = \rho^{-1}, r = 0, \cdots, q-1\}$ since $\{\sigma^r(\infty) : \sigma = \rho^{-1}, r = 0, \cdots, q-1\}$ are the vertices of $\mathbb{P}_0 = N_q(v)$. We may write $N_q(g(v)) = g(\mathbb{P}_0)$, $g \in G_\lambda$. From now we will denote the v-cell $N_q(v)$ by $\mathbb{P}_0$ and use them interchangeably.

DEFINITION 26. *The tessellation of* $\mathbb{H}^2$ *by the* v*-cells will be called the* λ*-Farey tessellation of* $\mathbb{H}^2$ *associated with* G_λ. *The ordered set* $[0,\ 1/\lambda,\ \dfrac{1}{\lambda - 1/\lambda}, \cdots, \lambda, \infty]$ *is called the* λ*-Farey subdivision of the interval* $[0,\ \infty]$.

DEFINITION 27. *We call the* v*-cell* $\mathbb{P}_0$, *the fundamental* λ*-Farey* q*-gon. That is,* $\mathbb{P}_0$, *the closed convex hull given by the vertices in the set* $\{\infty, \lambda, \lambda - 1/\lambda, \lambda - \dfrac{1}{\lambda - 1/\lambda}, \cdots, 1/\lambda, 0\}$, *is the fundamental* λ*-Farey* q*-gon for* $\lambda = 2\cos(\frac{\pi}{q})$ *where* $q \geq 3$ *is an integer.*

In the following example we will use the triangle $T = \{i, v, \infty\}$ (Definition 18, page 25), to explore $\mathbb{P}_0$ for $q = 3, 4, 5$ and 6.

EXAMPLE 2. (i) $q = 3$ then $\tau_\lambda(z) = z + \lambda = z + 1$ where $\lambda = 2\cos(\frac{\pi}{3}) = 1$ and $(\tau_\lambda\varphi)^3 = 1_{map}$. The 3-gon with vertices $\{\infty, 0, 1\}$ is the fundamental λ-Farey 3-gon or Farey triangle with $\lambda = 1$. Then $v = \dfrac{1+\sqrt{3}i}{2}$.

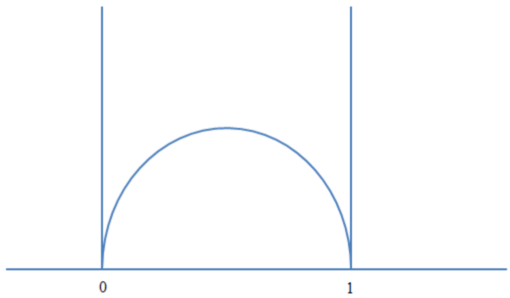

FIGURE 3. The v-cell : $N_3(v)$.

(ii) $q = 4$ then $\tau_\lambda(z) = z + \lambda = z + \sqrt{2}$, $\lambda = 2\cos(\frac{\pi}{4})$ and $(\tau_\lambda\varphi)^4 = 1_{map}$. The 4-gon with vertices $\{\infty, 0, \frac{1}{\sqrt{2}}, \sqrt{2}\}$ is the fundamental λ-Farey 4-gon with $\lambda = \sqrt{2}$. Then $v = \frac{1+i}{\sqrt{2}}$.

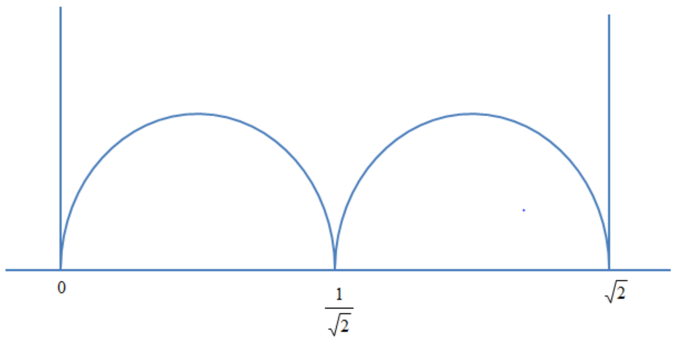

FIGURE 4. The v-cell : $N_4(v)$.

(iii) $q = 5$ then $\tau_\lambda(z) = z + \lambda = z + \frac{1+\sqrt{5}}{2}$ where $\lambda = 2\cos(\frac{\pi}{5})$ and $(\tau_\lambda\varphi)^5 = 1_{map}$. Note that $\lambda^2 - \lambda - 1 = 0$. The 5-gon with vertices $\{\infty, 0, \frac{\sqrt{5}-1}{2}, 1, \frac{\sqrt{5}+1}{2}\}$ is the fundamental λ-Farey 5-gon with $\lambda = \frac{1+\sqrt{5}}{2}$. Here we note $\lambda - 1/\lambda = \frac{\lambda^2-1}{\lambda} = \frac{\lambda}{\lambda} = 1$ and $\frac{1}{\lambda} = \frac{1}{\frac{1+\sqrt{5}}{2}} = \frac{2}{\sqrt{5}+1} = \frac{2(\sqrt{5}-1)}{5-1} = \frac{\sqrt{5}-1}{2}$. Then $v = \cos(\frac{\pi}{5}) + i\sin(\frac{\pi}{5})$.

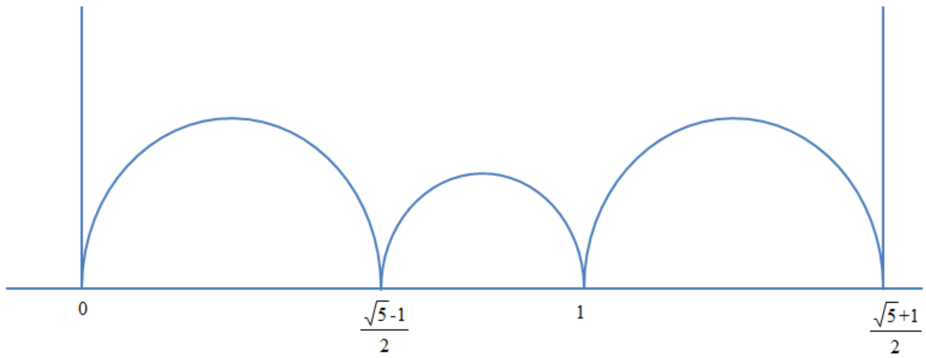

FIGURE 5. The v-cell : $N_5(v)$.

(iv) $q = 6$ then $\tau_\lambda(z) = z + \lambda = z + \sqrt{3}$ where $\lambda = 2\cos(\frac{\pi}{6})$ and $(\tau_\lambda\varphi)^6 = 1_{map}$. The 6-gon with vertices $\{\infty, 0, \frac{1}{\sqrt{3}}, \frac{\sqrt{3}}{2}, \frac{2}{\sqrt{3}}, \sqrt{3}\}$ is the fundamental λ-Farey 6-gon with $\lambda = \sqrt{3}$. Here we note $\lambda - \dfrac{1}{\lambda} = \dfrac{\lambda^2 - 1}{\lambda} = \dfrac{3-1}{\sqrt{3}} = \dfrac{2}{\sqrt{3}}$ and $\lambda - \dfrac{1}{\lambda - \frac{1}{\lambda}} = \sqrt{3} - \dfrac{1}{\frac{2}{\sqrt{3}}} = \dfrac{\sqrt{3}}{2}$. Then $v = \dfrac{\sqrt{3}+i}{2}$.

LEMMA 6. *Let* $\varphi(z) = -1/z$, $\phi(z) = \dfrac{1}{z}$ *and* $\mu(z) = -z$. *Then* $\{1, \varphi, \phi, \mu\}$ *is Klein 4-group in* $PSL(2, \mathbb{C})$ *and* $\phi(\rho^k) = \rho^{-k}\phi = \sigma^k\phi$ *where* $\rho = \tau_\lambda\varphi = \sigma^{-1}$.

Proof

It is easy to see that $\{1, \varphi, \phi, \mu\}$ is a Klein 4-group, since $\varphi\phi = \phi\varphi = \mu$ and $\varphi^2 = \phi^2 = \mu^2 = 1_{map}$. Note that $\phi = \varphi\mu = \mu\varphi$ and $\mu\tau_\lambda = \tau_\lambda^{-1}\mu$. Thus, $\phi\rho^k = \phi\rho\rho^{k-1} = \phi(\tau_\lambda\varphi)\rho^{k-1} = \varphi(\mu\tau_\lambda)\varphi\rho^{k-1} = \varphi\tau_\lambda^{-1}\varphi\mu\rho^{k-1} = \cdots = (\varphi\tau_\lambda^{-1})^k\phi = \sigma^k\phi = \rho^{-k}\phi$. $\square$

LEMMA 7. (a) *If q is even with $q = 2l$, then the q vertices of the fundamental λ-Farey q-gon that form the v-cell can be written as* $\{\rho^{\pm t}(\infty) : t = 0, \cdots, l\}$. *That is, each vertex has inverse in the set. Further* $\sigma^l(\infty) = \rho^l(\infty) = \lambda/2$ *and* $\sigma^{l+1}(\infty) = \rho^{l-1}(\infty) = 2/\lambda$ *where* $\lambda/2 < 1 < 2/\lambda$.

(b) *If q is odd with $q = 2l-1$, then the q vertices of the fundamental λ-Farey q-gon can be written as* $\{\rho^{\pm t}(\infty) : t = 0, \cdots, l-1\}$. *That is, each vertex has inverse in the set. Further* $\rho^{l-1}(\infty) = 1 = \sigma^l(\infty)$.

In both cases we see that $\sigma^r(\infty)$ and $\rho^r(\infty)$ have the same denominators in their representation as λ-fractions for $1 \le r \le l-1$.

Proof

The q vertices of $\mathbb{P}_0$ are the set $\{\infty, \rho(\infty), \cdots, \rho^{q-1}(\infty)\}$ or $\{\infty, \sigma(\infty), \cdots, \sigma^{q-1}(\infty)\}$ where $\sigma = \rho^{-1}$.

(a) Let $q = 2l$, $\sigma^{2l} = 1_{map}$ and $\sigma^{-l} = \rho^l$. Then $\phi(\rho^t(\infty)) = \sigma^t\phi(\infty) = \sigma^t(0) = \sigma^t(\sigma(\infty)) = \sigma^{t+1}(\infty)$ where $t = 0, \cdots, l$. Therefore $\phi(\rho^l(\infty)) = \phi(\sigma^l(\infty)) = 1/\sigma^l(\infty) = \sigma^{l+1}(\infty) = \sigma(\sigma^l(\infty)) = \varphi(\sigma^l(\infty) - \lambda) = \dfrac{1}{\lambda - \sigma^l(\infty)}$. So $\lambda - \sigma^l(\infty) = \sigma^l(\infty) = \lambda/2$.

(b) Let $q = 2l-1$, $\sigma^{2l-1} = 1_{map}$ and again $\phi(\rho^t(\infty)) = \sigma^t\phi(\infty) = \sigma^t(0) = \sigma^t(\sigma(\infty)) =$

$\sigma^{t+1}(\infty)$. Thus, $\phi(\rho^{l-1}(\infty)) = 1/\rho^{l-1}(\infty) = \sigma^l(\infty)$. Therefore $\{\sigma^l(\infty)\}^2 = 1$ and hence $\sigma^l(\infty) = \rho^{l-1}(\infty) = 1$.

We see that $\sigma^r(\infty) = \cfrac{1}{\lambda - \cfrac{1}{\lambda - \cfrac{1}{\dots - \cfrac{1}{\lambda}}}}$ with $r-1$ appearance of λ while

$\rho^r(\infty) = \lambda - \cfrac{1}{\lambda - \cfrac{1}{\lambda - \cfrac{1}{\dots - \cfrac{1}{\lambda}}}}$ with r appearance of λ. So $\rho^r(\infty) = \lambda - \sigma^r(\infty)$. If $\sigma^r(\infty) = \dfrac{p_1}{q_1}$

then $\rho^r(\infty) = \dfrac{\lambda q_1 - p_1}{q_1}$. So $\rho^r(\infty)$ and $\sigma^r(\infty)$ have the same denominators. $\square$

3. λ-Ford circles and their tessellation of $\mathbb{H}^2$

In this section, following Vulakh [**30**] , we introduce the λ-Ford discs and explore these λ-Ford discs together with the associated mesh polygons that tessellate $\mathbb{H}^2$ under G_λ for $q = 4$ and 5. For $\lambda = 1$ ($q = 3$), we know that the closed Ford discs together with the mesh triangles "tessellate" $\mathbb{H}^2$ under modular group, [**6**]. We build an analogous description of this type of tessellation for $q \geq 4$ using [**30**].

Recall for $g(z) = \dfrac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{R}$, $ad - bc = 1$, $c \neq 0$, the isometric circle of g is $I_g : |cz + d| = 1$ with radius $r = 1/|c|$. From Vulakh [**30**] (page 2296), we define k_{G_λ} to be the largest value of k such that the connected parts of D lying below the line $y = k/2$ are pyramidal regions bounded by the edges of D that meet vertex v. We know that the non-vertical edges of D are segments of the isometric circles $|z| = 1$ and $|z - \lambda| = 1$ where $v = \cos(\frac{\pi}{q}) + i\sin(\frac{\pi}{q})$ or $\dfrac{\lambda}{2} + \dfrac{i\sqrt{4-\lambda^2}}{2}$. We see that $k_{G_\lambda} = 2$ and $y = 1$ in all our cases.

From Vulakh [**30**] (page 2300), consider any $g \in G_\lambda$. For any $k > 0$, let $\mathcal{R}(g, k)$ be a horodisc in $\mathbb{H}^2$ tangent to $\mathbb{R}_\infty$ at $g(\infty)$ having radius r^2/k where r is the radius of an isometric circle,

I_g (see Definition 4, page 12). That is, $r = 1/|c|$. When $k = 2 = k_{G_\lambda}$, the radius of $\mathcal{R}(g, 2)$ is $\dfrac{1}{2|c|^2}$ and the horodisc $\mathcal{R}(g, k) = \mathcal{R}(g, 2)$ is called a λ-Ford disk and it's boundary (horocycle) is the λ-Ford circle, denoted by $\mathcal{C}_{g(\infty)}$. We may write $\mathcal{R}(g, 2) = \mathcal{R}_{g(\infty)}$ or just $\mathcal{R}_g$. If $c = 0$, the isometric circle is undefined for g. We define $\mathcal{R}_\infty$ to be line $y = 1$ or $z = x + i$, $x \in \mathbb{R}$.

Definition 28. *The horocycles $\mathcal{C}_{g(\infty)}$ for $g \in G_\lambda$, are called λ-Ford circles at $g(\infty)$. The λ-Ford circle at ∞, $\mathcal{C}_\infty$ is the line $y = 1$ and is called the fundamental λ-Ford circle.*

In the following example the λ-Ford circles are at the vertices of v-cell ($\mathbb{P}_0$).

Example 3. Assume $g \in G_\lambda$, $g(z) = \dfrac{az + b}{cz + d}$ where $a, b, c, d \in \mathbb{R}$, $c \neq 0$ and $ad - bc = 1$.

(i) Consider $q = 4$. The fundamental λ-Farey 4-gon is given by the vertices $\{\infty, \varphi\tau_\lambda^{-1}(\infty), (\varphi\tau_\lambda^{-1})^2(\infty), (\varphi\tau_\lambda^{-1})^3(\infty)\} = \{\infty, \sigma(\infty), \sigma^2(\infty), \sigma^3(\infty)\}$ or $\{\infty, 0, \frac{1}{\lambda}, \lambda\}$ with $\sigma(z) = \varphi\tau_\lambda^{-1}(z) = \dfrac{-1}{z - \lambda}$. We consider the λ-Ford circles $\mathcal{C}_\infty, \mathcal{C}_{\sigma(\infty)}, \mathcal{C}_{\sigma^2(\infty)}$ and $\mathcal{C}_{\sigma^3(\infty)}$ of each vertex. The λ-Ford circle, $\mathcal{C}_\infty$ is the line $y = 1$; $\mathcal{C}_{\sigma(\infty)} = \mathcal{C}_0$ is the horocycle center $(0, \frac{1}{2})$; $\mathcal{C}_{\sigma^2(\infty)} = \mathcal{C}_{\frac{1}{\lambda}}$ is horocycle with center $(\frac{1}{\lambda}, \frac{1}{2\lambda^2})$; $\mathcal{C}_{\sigma^3(\infty)} = \mathcal{C}_\lambda$ is horocycle with center $(\lambda, \frac{1}{2})$. The $\sqrt{2}$-Ford circle at $1/\sqrt{2}$ has radius $1/4$ and the horocycles at 0 and $\sqrt{2}$ has radius $1/2$. See figure below.

(ii) Let $q = 5$ and $\lambda = \frac{1+\sqrt{5}}{2}$. The fundamental λ-Farey 5-gon is given by the vertices $\{\infty, \sigma(\infty), \sigma^2(\infty), \sigma^3(\infty), \sigma^4(\infty)\} = \{\infty, 0, \frac{1}{\lambda}, \lambda, 1\}$ or $\{\infty, 0, \frac{2}{1+\sqrt{5}}, \frac{1+\sqrt{5}}{2}, 1\}$ with $\sigma(z) = \dfrac{-1}{z - \lambda}$. We consider the λ-Ford circles at each of these vertices i.e $\mathcal{C}_\infty, \mathcal{C}_{\sigma(\infty)}, \mathcal{C}_{\sigma^2(\infty)}, \mathcal{C}_{\sigma^3(\infty)}$ and $\mathcal{C}_{\sigma^4(\infty)}$. $\mathcal{C}_\infty$ is the line $y = 1$; $\mathcal{C}_{\sigma(\infty)} = \mathcal{C}_0$ is the horocycle with center $(0, \frac{1}{2})$; $\mathcal{C}_{\sigma^2(\infty)} = \mathcal{C}_{\frac{1}{\lambda}}$ is the horocycle with center $(\frac{1}{\lambda}, \frac{1}{3+\sqrt{5}})$; $\mathcal{C}_{\sigma^3(\infty)} = \mathcal{C}_1$ is the horocycle with center $(1, \frac{1}{3+\sqrt{5}})$; $\mathcal{C}_{\sigma^4(\infty)} = \mathcal{C}_\lambda$ is the horocycle with center $(\lambda, \frac{1}{2})$. Thus, for $q = 5$ and $\lambda = \frac{1+\sqrt{5}}{2}$, the λ-Ford circles at $\{0, \frac{1}{\lambda}, 1, \lambda\}$ have radii

$$\left\{\frac{1}{2}, \frac{1}{2\lambda^2}, \frac{1}{2(\lambda^2-1)^2}, \frac{1}{2\lambda^2(2-\lambda^2)^2}\right\} \text{ or } \left\{\frac{1}{2}, \frac{1}{3+\sqrt{5}}, \frac{1}{3+\sqrt{5}}, \frac{1}{2}\right\} \text{ respectively.}$$

See figure below.

DEFINITION 29. *The parts of* $\mathbb{H}^2$ *exterior to all the* λ*-Ford circles consist of an infinite number of circular arc polygons to which the name of "mesh polygons" is given. Any two sides of a mesh polygon that share a vertex lie on* λ*-Ford circles at* λ*-Farey neighbors.*

The following proposition is an adaption of a similar proposition for the modular group, [**5**].

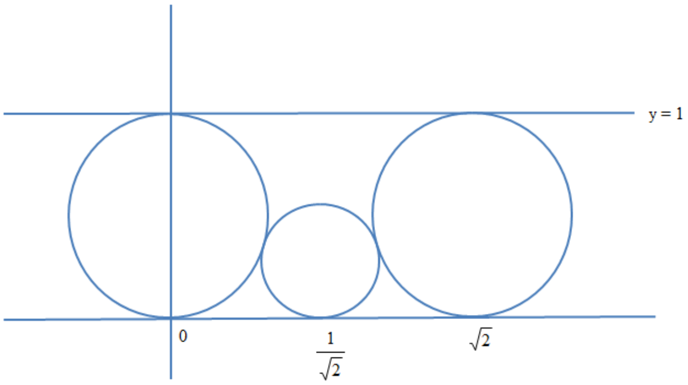

FIGURE 6. λ-Ford circles for $q = 4$.

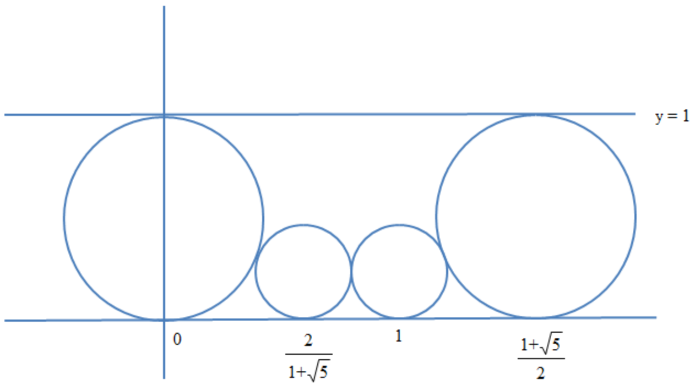

FIGURE 7. λ-Ford circle for $q = 5$.

PROPOSITION 10. *For $g \in G_\lambda$, $\mathcal{C}_{g(\infty)} = g(\mathcal{C}_\infty)$. That is, the λ-Ford circles are the orbit of $\mathcal{C}_\infty$ under G_λ.*

Proof

We may assume that if $g(\infty) = a/c$ then $c > 0$. The λ-Ford circle $\mathcal{C}_{\frac{1}{0}} = \mathcal{C}_\infty$ is the line $y = 1$. This is a generalised circle in $\mathbb{C}_\infty$ that passes through ∞. In fact it is the boundary of a horodisc tangent to $\mathbb{R}_\infty$ at ∞. Let $g \in G_\lambda$ with $g(\infty) = a/c$. Since $g \in \mathcal{M}$, $g(\mathcal{C}_\infty)$ is a circle or generalised circle in $\mathbb{C}_\infty$ that passes through a/c and preserves $\mathbb{H}^2$, [**14**]. Thus, $g(\mathcal{C}_\infty)$ is a circle tangent to $\mathbb{R}_\infty$ at a/c in $\mathbb{H}^2$. Certainly $g(\mathcal{C}_\infty) = \{g(x+i) : x \in \mathbb{R}_\infty\}$. The Euclidean diameter is thus the supremum of $Im\{g(x+i) : x \in \mathbb{R}_\infty\}$. Further $Img(x+i) =$
$Im\left\{\dfrac{ax+ai+b}{(cx+d)+ci}\right\} = Im\left\{\dfrac{((ax+b)+ai)((cx+d)-ci)}{(cx+d)^2+c^2}\right\} = \dfrac{1}{(cx+d)^2+c^2}$.
Therefore diameter of $g(\mathcal{C}_\infty)$ is $Sup\left\{\dfrac{1}{(cx+d)^2+c^2}\right\} = \dfrac{1}{c^2}$. Thus, radius of $g(\mathcal{C}_\infty)$ is $\dfrac{1}{2c^2}$. So $g(\mathcal{C}_\infty) = \mathcal{C}_{g(\infty)}$. $\square$

PROPOSITION 11. *λ-Ford circles $\mathcal{C}_{\frac{p_1}{q_1}}$ and $\mathcal{C}_{\frac{p_2}{q_2}}$ are externally tangential if and only if $|p_1q_2 - p_2q_1| = 1$ or $\dfrac{p_1}{q_1} \sim \dfrac{p_2}{q_2}$.*

Proof

Let $\mathcal{C}_{\frac{p_1}{q_1}}$ be the λ-Ford circle tangent to $\mathbb{R}$ at A given by $\left|z - \left(\dfrac{p_1}{q_1} + i\dfrac{1}{2q_1^2}\right)\right| = \dfrac{1}{2q_1^2}$. Let $\mathcal{C}_{\frac{p_2}{q_2}}$ be the λ-Ford circle tangent to $\mathbb{R}$ at B given by $\left|z - \left(\dfrac{p_2}{q_2} + i\dfrac{1}{2q_2^2}\right)\right| = \dfrac{1}{2q_2^2}$. Assume $\mathcal{C}_{\frac{p_1}{q_1}}$ and $\mathcal{C}_{\frac{p_2}{q_2}}$ are externally tangential and meet at $a + ib$ where $a, b \in \mathbb{R}$. Then

$$|CD|^2 = \left(\frac{1}{2q_1^2} + \frac{1}{2q_2^2}\right)^2 = \left(\frac{2q_1^2 + 2q_2^2}{4q_1^2q_2^2}\right)^2 \qquad \cdots (1)$$

and by Pythagoras' theorem

$$|CD|^2 = \left(\frac{1}{2q_1^2} + \frac{1}{2q_2^2}\right)^2 + \left(\frac{p_1}{q_1} - \frac{p_2}{q_2}\right)^2 = \left(\frac{2q_1^2 - 2q_2^2}{4q_1^2q_{2^2}}\right)^2 + \left(\frac{p_1q_2 - p_2q_1}{q_1q_2}\right)^2 \qquad \cdots (2)$$

Therefore
$\left(\frac{2q_1^2 - 2q_2^2}{4q_1^2q_2^2}\right)^2 = \frac{4q_1^2q_2^2}{(2q_1^2q_2^2)^2} = \frac{1}{q_1^2q_2^2}$. That is, $(p_1q_2 - p_2q_1)^2 = 1$, thus $|p_1q_2 - p_2q_1| = 1$.

Conversely, assume $|p_1q_2 - p_2q_1| = 1$. Then $(p_1q_2 - p_2q_1)^2 = 1$. From (2)
$|CD|^2 = \frac{(q_1^2 - q_2^2)^2}{(2q_1^2q_2^2)^2} + \frac{1}{q_1^2q_1^2} = \left(\frac{1}{2q_1^2} + \frac{1}{2q_2^2}\right)^2$, as required. Therefore $\mathcal{C}_{\frac{p_1}{q_1}} \cap \mathcal{C}_{\frac{p_2}{q_2}} = g(i)$ where $g(z) = \frac{p_1z + p_2}{q_1z + q_2}$ since $\mathcal{C}_{0/1} \cap \mathcal{C}_{1/0} = \{i\}$. Since $\mathcal{C}_{\frac{p_1}{q_1}}$ and $\mathcal{C}_{\frac{p_2}{q_2}}$ are distinct λ-Ford circles, they cannot be internally tangential. $\square$

Note that since $n\lambda \sim \frac{1}{0}$ with $n \in \mathbb{Z}$, the λ-Ford circles $\mathbb{C}_\infty$ and $\mathcal{C}_{n\lambda}$ are tangent for all λ-integers.

PROPOSITION 12. *If* $\mathcal{C}_{\frac{p_1}{q_1}}$ *is externally tangential to* $\mathcal{C}_{\frac{p_2}{q_2}}$ *and* $\mathcal{C}_{\frac{p_3}{q_3}}$ *is externally tangential to* $\mathcal{C}_{\frac{p_1}{q_1}}$ *then* $\mathcal{C}_{\frac{p_2}{q_2}}$ *cannot overlap* $\mathcal{C}_{\frac{p_3}{p_3}}$.

Proof

Assume $\frac{p_1}{q_1} \sim \frac{p_2}{q_2}$ with $\mathcal{C}_{\frac{p_1}{q_1}}$ externally tangent to $\mathcal{C}_{\frac{p_2}{q_2}}$. Assume we have a λ-rational p_3/q_3 with $\mathcal{C}_{\frac{p_3}{q_3}}$ externally tangent to $\mathcal{C}_{\frac{p_1}{q_1}}$ but overlapping $\mathcal{C}_{\frac{p_2}{q_2}}$. Let $g(z) = \frac{p_1z + p_2}{q_1z + q_2}$, then $g^{-1}(\mathcal{C}_{\frac{p_1}{q_1}}) = \mathcal{C}_\infty$ and $g^{-1}(\mathcal{C}_{\frac{p_2}{q_2}}) = \mathcal{C}_0$. Since $\mathcal{C}_{\frac{p_3}{q_3}}$ is tangent to $\mathcal{C}_\infty$ so too must $g^{-1}(\mathcal{C}_{\frac{p_3}{q_3}})$ be tangent to $\mathcal{C}_\infty$. So

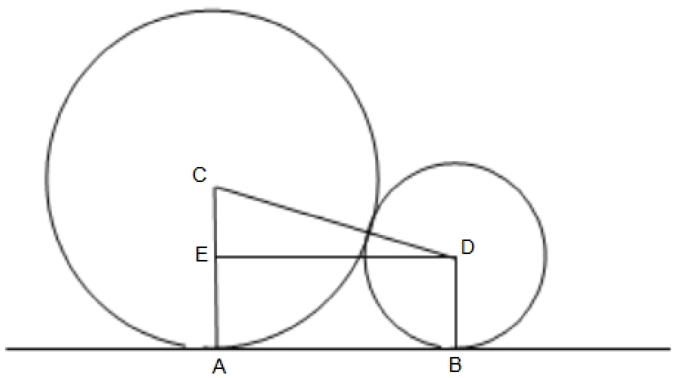

FIGURE 8. Tangent λ-Ford circles.

$g^{-1}(p_3/q_3)$ must be a λ-integer. If $g^{-1}(\mathcal{C}_{\frac{p_3}{q_3}}) = \lambda$ the λ-Ford circle $\mathcal{C}_\lambda$ cannot meet $\mathcal{C}_0$ since $1 < \lambda < 2$ and radius of $\mathcal{C}_\lambda$ is $1/2$ as is the radius of $\mathcal{C}_0$. Similarly if $g^{-1}(p_3/q_3) = -\lambda$. Hence, there cannot be any overlapping of λ-Ford circles. $\square$

We have seen (Lemma 7) that $\sigma^r(\infty)$ and $\rho^r(\infty)$ have the same denominators for $1 \leq r \leq l-1$. Hence, the λ-Ford circles at this points have the same radius.

Proposition 13. *Distinct λ-Ford circles cannot overlap.*

Proof

Let $\mathcal{C}_{\frac{a}{c}}$ and $\mathcal{C}_{\frac{b}{d}}$ be two distinct λ-Ford circles with $|ad - bc| \neq 1$. We can find $g \in G_\lambda$ such that $g^{-1}(\mathcal{C}_{\frac{a}{c}}) = \mathcal{C}_\infty$. Assume that $g^{-1}(\mathcal{C}_{\frac{b}{d}}) \cap g^{-1}(\mathcal{C}_{\frac{a}{c}}) \neq \emptyset$. That is, $\mathcal{C}_{g^{-1}(\frac{b}{d})} \cap \mathcal{C}_\infty \neq \emptyset$. Thus, the denominator q of $g^{-1}(b/d) = p/q$ is greater that 1, since the radius of $\mathcal{C}_{g^{-1}(\frac{b}{d})}$ is $\dfrac{1}{2q^2}$ and $\dfrac{1}{2q^2} > \dfrac{1}{2}$. Now we can find $n \in \mathbb{Z}$ such that $n\lambda \leq \dfrac{p}{q} \leq (n+1)\lambda$. We shall show in Theorem 15 and 16 that the denominators of the cusps of $\mathbb{P}_0$ are non-decreasing as they tend to $\sigma^l(\infty)$, for $q = 2l$ or $q = 2l - 1$. So $q \geq 1$. Thus, $\dfrac{1}{2q^2} \leq \dfrac{1}{2}$. Hence, $\mathcal{C}_{g^{-1}(\frac{b}{d})}$ can at most touch $\mathcal{C}_\infty$ but cannot intersect $\mathcal{C}_\infty$. Therefore $\mathcal{C}_{\frac{a}{c}}$ cannot intersect $\mathcal{C}_{\frac{b}{d}}$ if $\dfrac{b}{d} \not\sim \dfrac{a}{c}$. $\square$

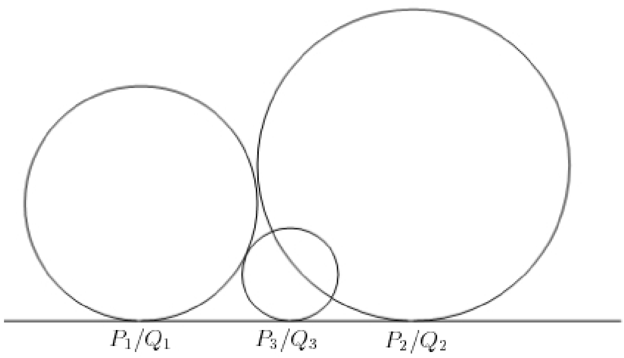

Figure 9. The λ-Ford circles.

The λ-Ford circles and the λ-Farey geodesics have a dual relationship. The λ-Ford circles corresponds to parabolic fixed points (λ-rationals) and the λ-Farey geodesic corresponding to tangency points between adjacent λ-Ford circles. The parabolic fixed points are the orbit of ∞ under G_λ while the tangency points of the λ-Ford circles are the orbit of i under G_λ. Further more, the λ-Ford circles are the orbit of $\mathcal{C}_\infty$ under G_λ and the λ-Farey geodesics are the orbit of $\mathbb{I}_0$ under G_λ. Consequently, there is a duality between results about λ-Ford circles and about λ-Farey geodesics.

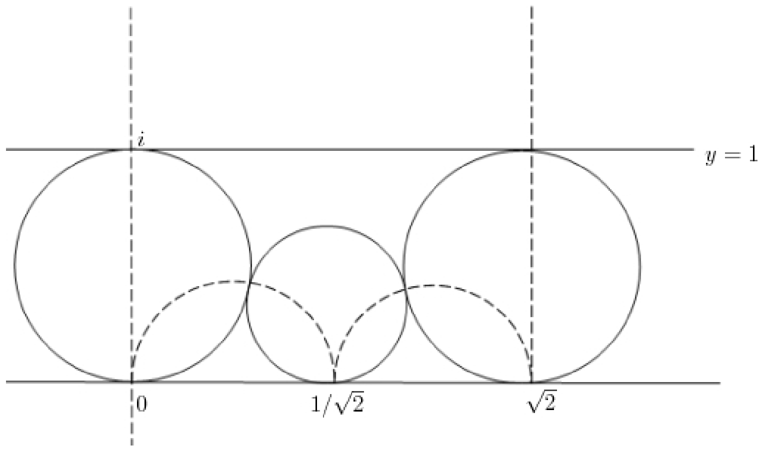

FIGURE 10. The dual relationship of λ-Ford circles and λ-Farey geodesics for $q = 4$.

CHAPTER 5

λ-continued fractions

In this chapter we introduce special λ-continued fraction expansions. Firstly, we consider "minus" or backward λ-continued fractions as introduced by Ressler, [**23**], following Schmidt and Sheingorn, [**27**]. In this case we show that every real number α can be expressed as an infinite λ-continued fraction. Further, every λ-continued fraction of this form, called an admissible λ-continued fraction, will converge. In the case of α being a λ-rational the admissible λ-continued fraction is periodic and $\lim_{k\to\infty} g^k(z) = \lambda$ where g is the generator of the period.

Secondly, we consider the "nearest λ-integer" continued fraction expansion of any real number, α. Following Rosen [**25**], we will choose an option that creates a unique expansion for each real number. Thirdly, we consider the "λ-integer part" (the floor) continued fraction where each $\alpha \in \mathbb{R}$ can be expressed as $\tau_\lambda^{r_0}\varphi\tau_\lambda^{r_1}\varphi\cdots\tau_\lambda^{r_k}\varphi(\alpha_k)$ where $r_i \in \mathbb{Z}^-$ for $i \geq 1$.

Finally we introduce Rosen's reduced λ-continued fractions. We note that Rosen, [**25**] has shown that the expansion of $\alpha \in \mathbb{R}$ using the nearest λ-integer algorithm, satisfies the condition of being a reduced λ-continued fraction and that every reduced λ-continued fraction converges.

1. Minus or backward λ-continued fractions

We consider $G_\lambda = \langle \tau_\lambda, \varphi \rangle$ with generators $\tau_\lambda(z) = z + \lambda$ and $\varphi(z) = -1/z$ where $\lambda = 2\cos(\frac{\pi}{q})$ and $q \geq 3$ is a finite integer. Since the minus λ-continued fractions are known for the case $\lambda = 1$ ($q = 3$), [**16**], we now consider $q \geq 4$. We have also seen that the fundamental q-gon is given with vertices $\{\infty, \sigma(\infty), \sigma^2(\infty), \cdots, \sigma^{q-1}(\infty)\}$ where $\sigma = \varphi\tau_\lambda^{-1}$, $\sigma^q = 1_{map}$ and

$\rho = \sigma^{-1} = \tau_\lambda \varphi = \sigma^{q-1}$. We recall that a finite λ-continued fraction (λ-fraction) can be expressed as $T_k(\infty) = \tau_\lambda^{r_0}\varphi\tau_\lambda^{r_1}\varphi \cdots \tau_\lambda^{r_k}\varphi(\infty)$ where $r_i \in \mathbb{Z}, i \geq 0$ and only r_0 and r_k may be zero, while an infinite λ-continued fraction has terms $T_k(\infty)$ as its convergents.

From Ressler [**23**], we expand any finite real number α as a unique λ-continued fraction according to the next-multiple-of-λ algorithm. Let $\alpha_0 = \alpha$. For $j \geq 0$ define $r_j = \lfloor \alpha_j/\lambda \rfloor + 1$ where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. We call $\alpha_{j+1} = \varphi\tau_\lambda^{-r_j}(\alpha_j) = \varphi(\alpha_j - \lambda r_j) = \dfrac{1}{\lambda r_j - \alpha_j}$ the $j+1^{th}$ complete quotient or $j+1^{th}$ tail of $\alpha_0 = \alpha$. Since $\lfloor \alpha_j/\lambda \rfloor \leq \alpha_j/\lambda$, we know that $\lfloor \alpha_j/\lambda \rfloor \lambda \leq \alpha_j$ and so $\lambda r_j - \alpha_j = \lambda(\lfloor \alpha_j/\lambda \rfloor + 1) - \alpha_j = \lambda\lfloor \alpha_j/\lambda \rfloor + \lambda - \alpha_j \leq \lambda$. Therefore $\alpha_{j+1} = \dfrac{1}{\lambda r_j - \alpha_j} \geq \dfrac{1}{\lambda} > 0$ for all $j \geq 0$ and $r_j \geq 1$ for all j. Then $\alpha_j = r_j\lambda - \dfrac{1}{\alpha_{j+1}}$ for $j \geq 0$ where $\alpha = \alpha_0 = \tau_\lambda^{r_0}\varphi\tau_\lambda^{r_1}\varphi \cdots \tau_\lambda^{r_k}\varphi(\alpha_{k+1})$.

Following Ressler [**23**], we define an admissible λ-continued fraction of a finite real number.

Definition 30. *The λ-continued fraction expansion of $\alpha \in \mathbb{R}$ is called admissible λ-continued fraction if $\alpha = \tau_\lambda^{r_0}\varphi\tau_\lambda^{r_1}\varphi \cdots \tau_\lambda^{r_k}\varphi(\alpha_{k+1})$ and $\alpha_{j+1} \geq \dfrac{1}{\lambda}$ with $r_j \geq 1$ for all $j \geq 1$ and α_{j+1} are the complete quotients or tails of the λ-continued fraction expansion.*

Lemma 8. [**23**]

Fix $q \geq 4$ with $\lambda = 2\cos(\frac{\pi}{q})$. Then every admissible λ-continued fraction converges.

Proof

Consider an admissible λ-continued fraction given as $\tau_\lambda^{r_0}\varphi\tau_\lambda^{r_1}\varphi \cdots \varphi\tau_\lambda^{r_k}\varphi(\alpha_{k+1})$ where $r_j \geq 1$ for all $j \geq 1$ while $r_0 \in \mathbb{Z}$ and $\alpha_{k+1} = \dfrac{1}{\lambda r_k - \alpha_k} \geq \dfrac{1}{\lambda} > 0$. Thus, $\lambda r_k - \alpha_k > 0$ with $\lambda r_k > \alpha_k$. Let $T_k = \tau_\lambda^{r_0}\varphi\tau_\lambda^{r_1}\varphi \cdots \tau_\lambda^{r_k}\varphi(\infty)$ be the k^{th} convergent of the λ-continued fraction. We show that $\{T_k\}_{k=0}^{\infty}$ is a decreasing sequence that is bounded below by some $\alpha \in \mathbb{R}$. Let $T_{m,k} = \tau_\lambda^{r_m}\varphi\tau_\lambda^{r_{m+1}}\varphi \cdots \tau_\lambda^{r_k}\varphi(\infty)$ where $0 \leq m \leq k$. We note $T_{k,k} = \tau_\lambda^{r_k}\varphi(\infty) = r_k\lambda > \alpha_k$.

For all $k, \alpha_k \geq 1/\lambda > 0$ and $T_{k-1,k} = \tau_\lambda^{r_{k-1}}\varphi\tau_\lambda^{r_k}\varphi(\infty) = \tau_\lambda^{r_{k-1}}\varphi(T_{k,k}) > \tau_\lambda^{r_{k-1}}\varphi(\alpha_k) = \alpha_{k-1}$ since $\tau_\lambda^{r_j}\varphi(x) = r_j\lambda - 1/x$ preserves order on $(0, +\infty)$. Continuing, we have that $T_{m,k} > \alpha_m$ for all $0 \leq m \leq k$. In particular $T_k = T_{0,k} > \alpha_0 = \alpha$ for all $k \geq 0$. In order to show that $\{T_n\}_{n=0}^{\infty}$ is decreasing, we fix $n \geq 0$ and note that $T_{n,n} = r_n\lambda > r_n\lambda - \dfrac{1}{r_{n+1}\lambda} = T_{n,n+1}$, $r_{n+1} \geq 1$. So $T_{n,n} > T_{n,n+1}$. Then for all $n > 0$, $T_{n-1,n} = \tau_\lambda^{r_{n-1}}\varphi(T_{n,n}) > \tau_\lambda^{r_{n-1}}\varphi(T_{n,n+1}) = T_{n-1,n+1}$. Continuing we have $T_{m,n} > T_{m,n+1}$ for all m, $0 \leq m \leq n$. In particular $T_n > T_{n+1}$ for all $n \geq 0$. $\square$

Definition 31. *A λ-continued fraction expansion of $\alpha \in \mathbb{R}$ where*

$$\alpha = \lambda b_0 - \cfrac{1}{\lambda b_1 - \cfrac{1}{\lambda b_2 - \cdots}} \quad \textit{or following Rosen } [\mathbf{25}],\ \alpha = \left[\lambda b_0, \frac{-1}{\lambda b_1}, \frac{-1}{\lambda b_2}, \cdots\right]$$

is periodic if there is a positive integer k such that $b_n = b_{n+k}$ for $n = 0, 1, \cdots$ and is pre-periodic if there is some positive integer k such that $b_n = b_{n+k}$ for all sufficiently large n.

That is, a λ-continued fraction for α is pre-periodic if the sequence b_0, b_1, $b_2, \cdots$ is periodic after a finite number of initial terms have been deleted. The period k of the λ-continued fraction is the smallest positive integer that is the period of the sequence b_0, $b_1, \cdots$.

Suppose α has a pre-periodic λ-continued fraction expansion with period k. Then we can find $h, g \in G_\lambda$ with $h = \tau_\lambda^{b_0}\varphi \cdots \tau_\lambda^{b_n}\varphi$ and $g = \tau_\lambda^{b_{n+1}}\varphi \cdots \varphi\tau_\lambda^{b_{n+k}}$ so that

$$\alpha = \lim_{m\to\infty} hg^m(\infty).$$

We call g the generator of the λ-continued fraction. From Ressler [**23**] we note that, T. A. Schmidt and M. Sheingorn [**27**] (Lemma 1, 2, 3) have established a close link between pre-periodic admissible λ-continued fraction expansions, parabolic and loxodromic fixed points of elements in G_λ as follows.

LEMMA 9. [**27**] *A real number* α *is a fixed point of* G_λ *if and only if* α *has a periodic admissible* λ*-continued fraction expansion. Moreover, such a number* α *is a parabolic fixed point (*λ*-rational) if and only if its admissible* λ*-continued fraction expansion has generator* $g = \tau_\lambda^2\varphi(\tau_\lambda\varphi)^{q-3} = \tau_\lambda\sigma^2$ *where* $\sigma = \varphi\tau_\lambda^{-1}$*. Further* α *is hyperbolic if and only if its admissible* λ*-continued fraction expansion has a generator other than* $g = \tau_\lambda\sigma^2$ *since* $\rho^q = 1_{map}$*.*

We recall that the λ-rationals have the form, $\dfrac{\lambda p(\lambda^2)}{q(\lambda^2)}$ or $\dfrac{p(\lambda^2)}{\lambda q(\lambda^2)}$ (Theorem 8 and Corollary 1), and thus will have the generator $\tau_\lambda\sigma^2$ in their admissible λ-fraction expression.

LEMMA 10. $g = \tau_\lambda\sigma^2$ *is a parabolic Möbius map and has exactly one fixed point* λ *in* $\mathbb{R}$*. If* α *is a* λ*-rational (parabolic fixed point in* G_λ*) with periodic admissible* λ*-continued fraction expansion and generator* g*, then we must have* $\alpha_k = \lambda$ *for some* k *in the admissible continued fraction expansion of* α*. Here* α_k *is the* k^{th} *complete quotient or tail of the* λ*-continued fraction.*

Proof

Let $g = \tau_\lambda\sigma^2$ with associated matrix $A = \begin{pmatrix} -1-\lambda^2 & \lambda^3 \\ -\lambda & -1+\lambda^2 \end{pmatrix}$ and $tr^2(A) = (-2)^2 = 4$. Thus, g is a parabolic Möbius map. We note that $g(\infty) = \tau_\lambda\sigma^2(\infty) = \tau_\lambda(1/\lambda) = \lambda + 1/\lambda$. The admissible λ-continued fraction for λ is as follows:

Let $\lambda = \alpha_0$ with $1 < \lambda < 2$. Consider $\varphi\tau_\lambda^{-2}(\lambda) = \alpha_1$ or $\lambda = \tau_\lambda^2\varphi(\alpha_1)$. Therefore $\alpha_1 = \varphi\tau_\lambda^{-2}(\lambda) = 1/\lambda = \sigma^2(\infty)$, $1/2 < 1/\lambda < 1$. Thus, $\alpha_2 = \varphi\tau_\lambda^{-1}(\alpha_1) = \varphi\tau_\lambda^{-1}(\varphi\tau_\lambda^{-2}(\lambda))$. Therefore $\lambda = \tau_\lambda^2\varphi\tau_\lambda\varphi(\alpha_2) = \tau_\lambda\{\tau_\lambda\varphi\tau_\lambda\varphi\}(\alpha_2) = \tau_\lambda(\sigma^{-2}(\alpha_2))$ where $\alpha_2 = \dfrac{-1}{-\lambda+1/\lambda} = \dfrac{1}{\lambda-1/\lambda} = \sigma^3(\infty)$ and $\dfrac{1}{\lambda-1/\lambda} > \dfrac{1}{\lambda}$. Continuing, this way we have that $\alpha_k = \sigma^{k+1}(\infty)$ and $\alpha_k > \dfrac{1}{\lambda}$ for $k \geq 1$. When $k = q-1$, $\alpha_{q-1} = \sigma^q(\infty) = \infty$ and when $k = q-2$, $\alpha_{q-2} = \sigma^{q-1}(\infty) = \sigma^{-1}(\infty) = \lambda$. Therefore $\lambda = \tau_\lambda^2\varphi\tau_\lambda\cdots\tau_\lambda\varphi(\alpha_k) = \tau_\lambda(\sigma^{-k}(\alpha_k))$. Thus,

$$\tau_\lambda\tau_\lambda\varphi\tau_\lambda\cdots\tau_\lambda\varphi(\alpha_{q-2}) = \tau_\lambda\sigma^{-q}\sigma^2(\lambda) = \tau_\lambda\sigma^2(\lambda) = g(\lambda) = \tau_\lambda\varphi(\infty) = \lambda. \ \square$$

Note that $\alpha = \lim_{k\to\infty} hg^k(\infty) = h\lim_{k\to\infty} g^k(\infty) = h(\lambda)$ since $\lim_{k\to\infty} g^k(\infty) = \lambda$ for all $z \in \mathbb{R}_\infty$. Thus, the admissible λ-continued fractions have $\alpha_k = \lambda$ for some k and the admissible λ-continued fraction expansion may be terminated with $\lambda = \tau_\lambda\varphi(\infty)$. That is, we replace $\lim_{k\to\infty}(\tau_\lambda^2\varphi(\tau_\lambda\varphi)^{q-3})^k(\lambda) = \lim_{k\to\infty} g^k(\lambda)$ with $\lambda = \tau_\lambda\varphi(\infty)$ and stop the expansion. Therefore $\alpha_0 = \tau_\lambda^{r_0}\varphi\tau_\lambda^{r_1}\varphi\cdots\tau_\lambda^{r_k}\varphi(\tau_\lambda\varphi(\infty))$ and is a finite expansion.

2. Nearest λ-integer continued fractions

We consider $\lambda = 2\cos(\frac{\pi}{q})$, $q \geq 4$ so $\sqrt{2} \leq \lambda < 2$. By $[u]$- we mean the nearest integer to u, as the unique integer such that $-1/2 < [u] - u \leq 1/2$, $u \in \mathbb{R}$. Thus, if $u = x + 1/2$ then $[u] = x + 1$, $x \in \mathbb{Z}$.

Consider $\alpha = \alpha_0 \in \mathbb{R}$ and we can find $n_0 \in \mathbb{Z}$ such that $n_0 \leq \alpha_0/\lambda \leq (n_0 + 1)$. Choose $r_0 = [\alpha_0/\lambda]$. Then $\alpha = \alpha_0 = r_0\lambda + \epsilon_1 a_1$ where $\epsilon_1 = \pm 1$ and $0 \leq a_1 \leq \lambda/2$. If $a_1 = \lambda/2$ then $\epsilon_1 = -1$. If $a_1 = 0$, the expansion terminates and $\alpha = \alpha_0 = r_0\lambda = \tau_\lambda^{r_0}\varphi(\infty)$. If $a_1 \neq 0$, $\alpha_0 = r_0\lambda + \epsilon_1 a_1 = r_0\lambda + \dfrac{\epsilon_1}{1/a_1} = r_0\lambda + \dfrac{\epsilon_1}{\alpha_1}$, $\alpha_1 = \dfrac{1}{a_1} \geq \dfrac{2}{\lambda}$. Consider $n_1 < \alpha_1/\lambda \leq n_1 + 1$ and let $r_1 = [\alpha_1/\lambda] = [1/a_1\lambda]$ and $\alpha_0 = r_0\lambda + \dfrac{\epsilon_1}{r_1\lambda + \epsilon_2 a_2'}$ with $\dfrac{1}{a_1} = r_1\lambda + \epsilon_2 a_2'$, $\epsilon_2 = \pm 1$ and $0 \leq a_2' \leq \lambda/2$. Thus, $1 = a_1 r_1\lambda + \epsilon_2 a_1 a_2' = a_1 r_1 \lambda + \epsilon_2 a_2$ and $0 \leq a_2 = a_1 a_2' \leq \dfrac{a_1\lambda}{2} \leq \dfrac{\lambda^2}{4}$. Continuing $\alpha_0 = r_0\lambda + \dfrac{\epsilon_1}{r_1\lambda + \epsilon_2 a_2'} = r_0\lambda + \dfrac{\epsilon_1}{r_1\lambda + \dfrac{\epsilon_2 a_2}{a_1}} = r_0\lambda + \dfrac{\epsilon_1}{r_1\lambda + \dfrac{\epsilon_2}{a_1/a_2}}$. Write $\alpha_2 = \dfrac{a_1}{a_2} = r_2\lambda + \epsilon_3 a_3'$ and $a_1 = a_2 r_2\lambda + \epsilon_3 a_2 a_3' = a_2 r_2 \lambda + \epsilon_3 a_3$ where $a_3 = a_3' a_2 \leq \dfrac{\lambda a_2}{2} \leq \dfrac{\lambda^2 a_1}{4} \leq \dfrac{\lambda^3}{8}$ and $0 \leq a_3' \leq \lambda/2$. Continuing this way we find $a_i = a_{i+1}r_{i+1}\lambda + \epsilon_{i+2}a_{i+2}$ where $r_i = [a_{i+1}/\lambda a_{i+2}]$, $\epsilon_{i+2} = \pm 1$ and $0 \leq a_{i+2} \leq \dfrac{\lambda a_{i+1}}{2} \leq \dfrac{\lambda^{i+2}}{2^{i+2}}$. If $a_i = 0$ for some i, the expansion terminates and $\alpha = \alpha_0 = \tau_\lambda^{r_0}\varsigma_1\tau_\lambda^{r_1}\varsigma_2\cdots\tau_\lambda^{r_k}\varsigma_{k+1}(\infty)$ where $\varsigma_i = \pm 1/z$ for $i \geq 1$. If $a_i \neq 0$ then $\alpha = \tau_\lambda^{r_0}\varsigma_1\tau_\lambda^{r_1}\varsigma_2\cdots\tau_\lambda^{r_k}\varsigma_{k+1}(a_{i-1}/a_i)$ where $a_{i-1}/a_i > 2/\lambda$. The $\alpha_i = a_{i-1}/a_i$ are the tails of the λ-continued fraction.

3. λ-Integer part continued fractions

We may expand any real number α as a unique λ-continued fraction according to the λ-integer part algorithm. Let $\alpha_0 = \alpha \in \mathbb{R}$. For $j \geq 0$ define $r_j = \lfloor \alpha_j/\lambda \rfloor$ where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. Let $\alpha_{j+1} = \varphi\tau_\lambda^{-r_j}(\alpha_j) = \varphi(\alpha_j - \lambda r_j) = \dfrac{1}{\lambda r_j - \alpha_j} < 0$ where $r_j \leq -1$ for $j \geq 1$, $r_0 \in \mathbb{Z}$. That is, $\alpha_0 = \tau_\lambda^{r_0}\varphi\tau_\lambda^{r_1}\varphi \cdots \tau_\lambda^{r_k}\varphi(\alpha_k)$. Since $r_j = \lfloor \alpha_j/\lambda \rfloor$, we will terminate when $\tau_\lambda^{-r_j}(\alpha_j) = \alpha_j - \lambda r_j = 0 = \varphi(\infty)$ or $\varphi\tau_\lambda^{-r_j}(\alpha_j) = \infty$.

4. Reduced λ-continued fractions

From Rosen [**25**], we have the following definition of a reduced λ-continued fraction for $q = 2l-1$, $l \geq 3$ or $q = 2l$, $l \geq 2$ where $B(l-2) = \rho^{l-2}(\infty) = [\lambda, -1/\lambda, -1/\lambda, \cdots, -1/\lambda]$.

Definition 32. *If $\lambda = 2\cos(\frac{\pi}{q})$, $q \geq 4$, the λ-continued fraction $[r_0\lambda, \epsilon_1/r_1\lambda, \cdots]$, where $\epsilon_i = \pm 1$, $r_i \in \mathbb{Z}^+$ for $i \geq 1$, r_0 may be zero, is a reduced λ-continued fraction if and only if the following properties are satisfied:*

(i) The inequality $r_i\lambda + \epsilon_{i+1} < 1$ is satisfied for no more than $l-2$ consecutive values of i, $i = j,\ j+1,\ j+2, \cdots, j-l+1,\ j \geq 1$.

(ii) If $q = 2l-1$, and if $r_i\lambda + \epsilon_{i+1} < 1$ is satisfied for $l-2$ consecutive values of $i = j,\ j+1,\ j+2, \cdots, j-l+1$, then $r_{j+l-2} \geq 2$.

(iii) If $q = 2l-1$, and if $[B(l-2), -1/2\lambda, -1/B(l-2)]$ occurs, the succeeding sign is plus.

(iv) If $q = 2l-1$, the λ-continued fraction terminates with $\epsilon/B(l-1)$, then $\epsilon = 1$.

(v) If some tail of a finite λ-continued fraction has the value $2/\lambda$, then $r\lambda + \dfrac{1}{2/\lambda} = (r+1)\lambda - \dfrac{1}{2/\lambda}$, and $r\lambda - \dfrac{2}{\lambda} = (r-1)\lambda + \dfrac{1}{2/\lambda}$. We shall choose the plus sign.

We note that Rosen [**25**] establishes that every reduced λ-continued fraction converges and that the nearest λ-continued fraction is reduced. We omit this proof but establish a variation of it in Chapter 8 when we introduce cutting sequences across the λ-Farey tessellation. Rosen also establishes that in a reduced λ-continued fraction with convergents P_n/Q_n we have, $Q_n \geq 1$ and Q_n are non-decreasing with $Q_n \to \infty$ as $n \to \infty$.

The following theorem is from Lehner, [**18**].

THEOREM 13. *Let P/Q be a λ-rational and $\alpha \in \mathbb{R} - G_\lambda(\infty)$. Suppose that $|\alpha - P/Q| < \dfrac{1}{2Q^2}$. Then P/Q is a convergent of the nearest λ-integer continued fraction of α and hence of a reduced λ-continued fraction of α.*

Proof

Write $\alpha - P/Q = \dfrac{\epsilon\theta}{Q^2}$, $0 < \theta < 1/2$, $\epsilon = \pm 1$. That is, $\alpha - P/Q = \dfrac{\alpha Q - P}{Q} = \dfrac{(\alpha Q - P)Q}{Q^2}$, and $\epsilon\theta = (\alpha Q^2 - PQ)$ or $\theta = |\alpha Q^2 - PQ| = |Q^2||\alpha - P/Q| < \dfrac{|Q^2|}{2Q^2} < \dfrac{1}{2}$, therefore $0 < \theta < 1/2$. Expand P/Q as a nearest λ-integer continued fraction which is reduced. Thus, $P/Q = [r_0\lambda, \epsilon_1/r_1\lambda, \epsilon_2/r_2\lambda, \cdots, \epsilon_{n-1}/r_{n-1}\lambda]$ where $\epsilon_i = \pm 1$. Since P/Q is a λ-rational, the λ-continued fraction is finite. Call the convergents of P/Q, P_i/Q_i so that $\dfrac{P}{Q} = \dfrac{P_{n-1}}{Q_{n-1}}$. We define w by $\alpha = \dfrac{(P_{n-1}w + \epsilon_n P_{n-2})}{(Q_{n-1}w + \epsilon_n Q_{n-2})}$ where the ϵ_n are defined recursively by $\epsilon = \epsilon_1\epsilon_2\cdots\epsilon_n(-1)^{n-1}$. That is, when $n = 1$, $\epsilon = \epsilon_1$. When $n = 2$, $\epsilon = \epsilon_1\epsilon_2(-1)$, so $\epsilon_2 = -1$. When $n = 3$, $\epsilon = \epsilon_1\epsilon_2\epsilon_3(-1)^2$, so $\epsilon_2\epsilon_3 = 1$ and hence $\epsilon_3 = -1$. Continuing this way we note that $\epsilon_n = -1$ for all $n \geq 2$ where $\epsilon_1 = \epsilon$. Since $P/Q = [r_0\lambda, \epsilon_1/r_1\lambda, \epsilon_2/r_2\lambda, \cdots, \epsilon_{n-1}/r_{n-1}\lambda] = P_{n-1}/Q_{n-1}$ is a λ-continued fraction we have $P_{n-1}Q_{n-2} - Q_{n-1}P_{n-2} = (-1)^n\epsilon_1\epsilon_2\cdots\epsilon_{n-1}$.

Thus, $\dfrac{P}{Q} = \dfrac{P_{n-1}}{Q_{n-1}}$, $\dfrac{\epsilon\theta}{Q_{n-1}^2} = \alpha - P/Q$

$$= \frac{(P_{n-1}w + \epsilon_n P_{n-2})}{(Q_{n-1}w + \epsilon_n Q_{n-2})} - \frac{P_{n-1}}{Q_{n-1}}$$

$$= \frac{Q_{n-1}P_{n-1}w + \epsilon_n P_{n-2}Q_{n-1} - P_{n-1}Q_{n-1}w - \epsilon_n P_{n-1}Q_{n-2}}{Q_{n-1}(Q_{n-1}w + \epsilon_n Q_{n-2})}$$

$$= \frac{\epsilon_n(P_{n-2}Q_{n-1} - P_{n-1}Q_{n-2})}{Q_{n-1}(Q_{n-1}w + \epsilon_n Q_{n-2})}$$
$$= \frac{(-1)\epsilon_n\epsilon_1\epsilon_2\cdots\epsilon_{n-1}(-1)^n}{Q_{n-1}(Q_{n-1}w + \epsilon_n Q_{n-2})}.$$

Therefore, $\frac{\epsilon\theta}{Q_{n-1}^2} = \frac{(-1)\epsilon_n\epsilon_1\epsilon_2\cdots\epsilon_{n-1}(-1)^n}{Q_{n-1}(Q_{n-1}w+\epsilon_n Q_{n-2})}$ and so $\theta = \frac{Q_{n-1}}{Q_{n-1}w+\epsilon_n Q_{n-2}}$. Hence, $w = \frac{Q_{n-1}-\theta\epsilon_n Q_{n-2}}{\theta Q_{n-1}} = \frac{1}{\theta} - \frac{\epsilon_n Q_{n-2}}{Q_{n-1}} > 0$, since the denominators of a reduced λ-continued fraction are non-decreasing and Thus, $Q_{n-1} \geq Q_{n-2}$. We have assumed that P_{n-1}/Q_{n-1} has a nearest λ-integer continued fraction expansion. We have also seen that, $0 < \theta < 1/2$ and $\epsilon_n = \pm 1$. Now expand w in a nearest λ-integer continued fraction algorithm. Then $w = [r_n\lambda, \epsilon_{n+1}/r_{n+1}\lambda, \epsilon_{n+2}/r_{n+2}\lambda, \cdots]$. That is, $r_n = [w/\lambda]$ is the nearest integer to w/λ. Since $w = \frac{1}{\theta} - \frac{\epsilon Q_{n-2}}{Q_{n-1}} > 2-1 = 1 > \frac{\lambda}{2}$, it follows that $r_n \geq 1$. We have $\alpha = \frac{P_{n-1}w+\epsilon_n P_{n-2}}{Q_{n-1}w+\epsilon_n Q_{n-2}}$ where $[r_0\lambda, \epsilon_1/r_1\lambda, \epsilon_2/r_2\lambda, \cdots, \epsilon_{n-1}/r_{n-1}\lambda]$ and $w = [r_n\lambda, \epsilon_{n+1}/r_{n+1}\lambda, \epsilon_{n+2}/r_{n+2}\lambda, \cdots]$. So $\alpha = [r_0\lambda, \epsilon_1/r_1\lambda, \epsilon_2/r_2\lambda, \cdots, \epsilon_{n-1}/r_{n-1}\lambda, \epsilon_n/r_n\lambda, \epsilon_{n+1}/r_{n+1}\lambda, \cdots]$ and w is the n^{th} complete quotient or n^{th} tail of α. Thus, P/Q is a convergent to α, a nearest λ-integer continued fraction and hence is a convergent of a reduced λ-continued fraction of α. □

Example 4. Let $q = 6$ and $\lambda = \sqrt{3}$ with $\alpha = \frac{12\sqrt{3}}{13}$. We find the following λ-continued fractions for α:

(i) Admissible λ-continued fraction,

(ii) Nearest λ-integer continued fraction,

(iii) Integer part λ-continued fraction.

(i) The admissible λ-continued fraction expansion:
$\frac{12\sqrt{3}}{13} = \lim_{k\to\infty} \tau_\lambda\varphi\tau_\lambda^5\varphi\tau_\lambda\varphi\tau_\lambda\varphi g^k(\infty)$ where $g = \tau_\lambda^2\varphi(\tau_\lambda\varphi)^3$ is the generator of the periodic λ-continued fraction. We know from Lemma 10 (page 62), that g fixes $\sqrt{3}$. It is of infinite length but it can be made finite by considering $\lim_{k\to\infty} g^k(\infty) = \tau_\lambda\varphi(\infty)$.

The convergents are:
$\tau_\lambda\varphi(\infty) = \sqrt{3}$; $\tau_\lambda\varphi\tau_\lambda^5\varphi(\infty) = \frac{14\sqrt{3}}{15}$; $\tau_\lambda\varphi\tau_\lambda^5\varphi\tau_\lambda\varphi(\infty) = \frac{13\sqrt{3}}{12}$; $\tau_\lambda\varphi\tau_\lambda^5\varphi\tau_\lambda\varphi\tau_\lambda\varphi(\infty) = \frac{25\sqrt{3}}{27}$; $\tau_\lambda\varphi\tau_\lambda^5\varphi\tau_\lambda\varphi\tau_\lambda\varphi\tau_\lambda^2\varphi(\infty) = \frac{37\sqrt{3}}{40}; \cdots$.

(ii) The nearest λ-integer algorithm:
$\frac{12\sqrt{3}}{13} = \tau_\lambda\varphi\tau_\lambda^4\varphi\tau_\lambda^{-1}\varphi(\infty)$. It is of length 3.

The convergents are:
$\tau_\lambda\varphi(\infty) = \sqrt{3}$; $\tau_\lambda\varphi\tau_\lambda^4\varphi(\infty) = \sqrt{3} - \frac{1}{4\sqrt{3}} = \frac{11\sqrt{3}}{12}$; $\tau_\lambda\varphi\tau_\lambda^4\varphi\tau_\lambda^{-1}\varphi(\infty) = \frac{12\sqrt{3}}{13}$.

(iii) The λ-integer part continued fraction:
$\frac{12\sqrt{3}}{13} = (\varphi\tau_\lambda^{-1}\varphi\tau_\lambda^{-1}\varphi\tau_\lambda^{-1}\varphi\tau_\lambda^{-2})^4\varphi(\infty)$. It is of length 17.

The convergents are:
$\varphi(\infty) = 0$; $\varphi\tau_\lambda^{-1}\varphi(\infty) = \frac{1}{\sqrt{3}}$; $\varphi\tau_\lambda^{-1}\varphi\tau_\lambda^{-1}\varphi(\infty) = \frac{\sqrt{3}}{2}$; $\varphi\tau_\lambda^{-1}\varphi\tau_\lambda^{-1}\varphi\tau_\lambda^{-1}\varphi(\infty) = \frac{2\sqrt{3}}{3}$; $\varphi\tau_\lambda^{-1}\varphi\tau_\lambda^{-1}\varphi\tau_\lambda^{-1}\varphi\tau_\lambda^{-2}\varphi(\infty) = \frac{3\sqrt{3}}{4}; \cdots ; \frac{12\sqrt{3}}{13}$.

We note that given a λ-rational α, the expansion for admissible λ-continued fraction is periodic and infinite, while the λ-continued fraction expansion with the nearest λ-integer algorithm yield an expansion of finite length less than or equal to the expansion with respect to the λ-integer part algorithm.

CHAPTER 6

Geometry of λ-continued fraction

1. Cutting sequence across the λ-Farey tesselation of $\mathbb{H}^2$ by G_λ

We consider an orientated geodesic γ in $\mathbb{H}^2$ that passes through $\mathbb{I}_0$, the fundamental polygon $\mathbb{P}_0 = N_q(v) = \{\infty, \sigma(\infty), \cdots, \sigma^{q-1}(\infty)\}$ and ends at a point $\alpha \in \mathbb{R}$. As γ moves through $\mathbb{I}_0$ and $\mathbb{P}_0$ to α it cuts across copies of $\mathbb{P}_0$ under G_λ. Each of these polygons can be labeled as $g(\mathbb{P}_0)$ for $g \in G_\lambda$. Following [**11**], [**20**], [**21**],[**22**] and [**26**] this chain of polygons is called the cutting sequence of γ ending at α (see Theorem 17, page 88). This chain of polygons and their spanning intervals will be shown to converge to α, yielding a λ-continued fraction for α.

Following [**12**], we established the following preliminary results. It is obvious that $\mathbb{I}_0$ divides $\mathbb{H}^2$ into two halves. The first, called the inner half, is bounded by $\mathbb{R}^+_\infty$ while the other, the outer half, is bounded by $\mathbb{R}^-_\infty$. The inner half contains the λ-Farey subdivision of $[0,\infty]$ on its boundary. The λ-Farey subdivision is the set $\{\infty, \sigma(\infty), \cdots, \sigma^{q-1}(\infty)\} \subseteq \mathbb{R}^+_\infty$ and is the vertex set of $\mathbb{P}_0$. The outer half contains the image of the λ-Farey subdivision under reflection in $\mathbb{I}_0$ on its boundary. This set of vertices is the vertex set of $\varphi(\mathbb{P}_0)$.

Definition 33. (*i*) *Let γ be a geodesic with end points $\dfrac{b}{d} < \dfrac{a}{c}$. Then $\left[\dfrac{b}{d}, \dfrac{a}{c}\right]$ is called its spanning interval of γ.*

(*ii*) *If $\mathbb{P}_k = g(\mathbb{P}_0)$ is a polygon in the λ-Farey tessellation of $\mathbb{H}^2$ then the spanning interval of $\mathbb{P}_k$ is $[g(0),\, g(\infty)] = \mathbf{I}_k$ where $g \in G_\lambda$. In particular the span of $\mathbb{P}_0$ is given by $\mathbf{I}_0 = [0,\, \infty]$.*

Since each λ-Farey geodesic γ can be written as $g(\mathbb{I}_0)$ for $g \in G_\lambda$ we immediately have the following result.

LEMMA 11. *Every λ-Farey geodesic γ with the spanning interval $\left[\frac{b}{d}, \frac{a}{c}\right]$ with $\frac{b}{d} < \frac{a}{c}$ divides $\mathbb{H}^2$ into two sections. The inner section is bounded by the open interval $\left(\frac{b}{d}, \frac{a}{c}\right)$ while the outer section is bounded by $\mathbb{R} - \left(\frac{b}{d}, \frac{a}{c}\right)$. Further the λ-Farey sub-division of γ lies in $\left[\frac{b}{d}, \frac{a}{c}\right]$ while its image with respect to reflection in γ lies on $\mathbb{R} - \left(\frac{b}{d}, \frac{a}{c}\right)$.*

Proof

Assume γ is a λ-Farey geodesic that spans the interval $\left[\frac{b}{d}, \frac{a}{c}\right]$ on $\mathbb{R}$. Thus, $\frac{b}{d} \sim \frac{a}{c}$ and $\frac{b}{d} < \frac{a}{c}$. Let $g : z \longmapsto \frac{az+b}{cz+d}$ in G_λ. Thus, $g^{-1}(\gamma) = \mathbb{I}_0$ and g is a Möbius map (Lemma 4, page 40). So $g(\mathbb{I}_0) = \gamma$ divides $\mathbb{H}^2$ into 2 sections. The λ-Farey subdivision of γ is given by $\{g(\infty) = a/c$, $g(\sigma(\infty)) = g(0) = b/d$, $g(\sigma^2(\infty)), \cdots, g(\sigma^{q-1}(\infty))\}$.

Since $\frac{b}{d} < \frac{a}{c}$ and $g \in G_\lambda$ preserves order, we have that this subdivision lies on the boundary of the inner subdivision of $\mathbb{H}^2$ by γ. Since $\varphi\{\infty, 0, 1/\lambda \cdots, \lambda\}$ is the vertex set of $\varphi(\mathbb{P}_0)$ we have that $g\varphi\{\infty, 0, 1/\lambda \cdots, \lambda\} = g\varphi\{\infty, \sigma(\infty), \sigma(\infty), \sigma^2(\infty), \cdots, g(\sigma^{q-1}(\infty))\}$ lies on the boundary of the outer subdivision of $\mathbb{H}^2$ by γ. $\square$

LEMMA 12. *Every λ-Farey geodesic γ is an edge to exactly two (adjacent) q-gons in the λ-Farey tessellation. These two q-gons are inverse of each other with respect to γ. We may refer to these q-gons as an inner subdivision and an outer subdivision of the geodesic γ, determined by the images of the inner and outer subdivision of $\mathbb{I}_0$ in $\mathbb{H}^2$.*

Proof

Let $\dfrac{b}{d} \sim \dfrac{a}{c}$, and $[b/d, a/c]$ span the λ-Farey geodesic γ, with $\dfrac{b}{d} < \dfrac{a}{c}$. Then $ad - bc = 1$, $g(z) = \dfrac{az+b}{cz+d}$ with $g(\infty) = a/c$, $g(0) = b/d$ and $g \in G_\lambda$. Thus, $g^{-1}(\gamma) = \mathbb{I}_0$.

The imaginary geodesic $\mathbb{I}_0$ is a common geodesic between $\mathbb{P}_0$ and $\varphi(\mathbb{P}_0)$. Since we have assumed $\dfrac{b}{d} < \dfrac{a}{c}$ and $g \in G_\lambda$ preserves order, we know g maps the inner side of $\mathbb{I}_0$ to the inner side of γ. Also $g(\varphi(\mathbb{P}_0))$ and $g(\mathbb{P}_0)$ are inverses with respect to γ. Finally, it is noted that the interval with end points $g(\sigma^r(\infty))$ and $g(\sigma^{r-1}(\infty))$ lies inside the interval $[g(0), g(\infty)] = [b/d, a/c]$, and thus $|g(\sigma^r(\infty)) - g(\sigma^{r-1}(\infty))| < |g(\sigma(\infty)) - g(\sigma^{-1}(\infty))| = |g(0) - g(\lambda)| = \left|\dfrac{b}{d} - \dfrac{\lambda a + b}{c\lambda + d}\right| = \dfrac{|(bc-ad)\lambda|}{|d(c\lambda+d)|} = \dfrac{\lambda}{|d(c\lambda+d)|} \leq \lambda$, $2 \leq r \leq q-1$. $\square$

We note that these results are analogous to the results by A. L. Schmidt on the Farey triangles [**28**] and Farey simplices [**29**], where every Farey triangle and Farey simplex has an inner and outer subdivision.

2. The λ-Farey graph

In this section we define a graph whose vertex set is the orbit of the fundamental q-gon $\mathbb{P}_0$, under G_λ and whose edges are the λ-Farey geodesic. Ultimately we wish to establish that paths on this graph converge to points on $\mathbb{R}$.

We know that the vertices of $\mathbb{P}_0$ are given by what we call the λ-Farey subdivision of $\mathbb{R}^+_\infty = [0, \infty]$. That is, $\{\sigma^r(\infty) : r = 0, \cdots, q-1\}$. Equivalently this subdivision could be expressed as $\{\rho^r(\infty) : r = 0, \cdots, q-1\}$ where $\rho = \sigma^{-1}$. Using the representations given above for the λ-Farey subdivision of $[0, \infty]$, the span of $\mathbb{P}_0$ can also be given as:

$$\{\sigma^r(\infty) : -l \leq r \leq l-1,\ l \geq 2\} \text{ for } q = 2l \text{ (q is even) and}$$
$$\{\sigma^r(\infty) : -l+1 \leq r \leq l-1,\ l \geq 3\} \text{ for } q = 2l-1 \text{ (q is odd).}$$

We specifically note that $\sigma^0(\infty) = \infty$, $\sigma(\infty) = 0 = \rho^{-1}(\infty)$, $\sigma^2(\infty) = \dfrac{1}{\lambda} = \rho^{-2}(\infty)$, $\sigma^3(\infty) = \dfrac{1}{\lambda - 1/\lambda} = \dfrac{\lambda}{\lambda^2 - 1} = \rho^{-3}(\infty), \cdots, \sigma^{q-2}(\infty) = \lambda - 1/\lambda = \dfrac{\lambda^2 - 1}{\lambda} = \rho^2(\infty)$ and $\sigma^{-1}(\infty) = \lambda = \rho(\infty)$.

Let $g(z) = \dfrac{az + b}{cz + d}$, $g \in G_\lambda$ and $ad - bc = 1$. The vertices of $g(\mathbb{P}_0)$ can be seen to be the set $\left\{\dfrac{a}{c}; \dfrac{b}{d}; \dfrac{a + \lambda b}{c + \lambda d}; \cdots; \dfrac{a\lambda + b}{c\lambda + d}\right\}$. On the other hand, the vertices of $g(\varphi(\mathbb{P}_0))$ are given as $\left\{\dfrac{b}{d}; \dfrac{a}{c}; \dfrac{-a\lambda + b}{-c\lambda + d}; \cdots; \dfrac{-a + b\lambda}{-c + d\lambda}\right\}$.

Definition 34. *Let $\mathcal{G}$ be a graph whose vertices are the q-gons of the λ-Farey tessellation of $\mathbb{H}^2$ by $\mathbb{P}_0$ under G_λ. The edges of $\mathcal{G}$ are to be determined by the pairs of adjacent q-gons in the tessellations. We call this graph $\mathcal{G}$, the λ-Farey graph. By Lemma 12, we may understand the edges of $\mathcal{G}$ to be determined by the λ-Farey geodesics.*

In the sequel we will call the q-gons the vertices of $\mathcal{G}$ while the vertices of the individual q-gons will be referred to as cusps.

Theorem 14. *$\mathcal{G}$ is a regular connected q-graph with no cycles and hence $\mathcal{G}$ is a regular tree. Specifically each vertex of $\mathcal{G}$ can be represented in the following ways:*

Since $\sigma^q = 1_{map} = \rho^q$ and $\rho = \sigma^{-1}$,

(i) $\sigma^{r_0}\varphi\sigma^{r_1}\varphi\cdots\sigma^{r_k}\varphi(\mathbb{P}_0)$ for $0 \leq r_i \leq q-1$, $\sigma^0 = \sigma^q = 1_{map}$.

(ii) $\rho^{r_0}\varphi\rho^{r_1}\varphi\cdots\rho^{r_k}\varphi(\mathbb{P}_0)$ for $0 \leq r_i \leq q-1$, $\rho^0 = \rho^q = 1_{map}$.

(iii) $\sigma^{r_0}\varphi\sigma^{r_1}\varphi\cdots\sigma^{r_k}\varphi(\mathbb{P}_0)$ for $-l \leq r_i \leq l-1$ if $q = 2l$ (q is even) or for $-l+1 \leq r_i \leq l-1$ if $q = 2l-1$ (q is odd).

Proof

We know that $\mathbb{I}_0$ is adjacent to exactly two q-gons namely $\mathbb{P}_0$ and $\varphi(\mathbb{P}_0)$, and hence each $g(\mathbb{I}_0)$ for $g \in G_\lambda$ is adjacent to exactly two q-gons. The λ-Farey tessellation is disconnected by the removal of $\mathbb{P}_0$ and hence with the removal of any $g(\mathbb{P}_0)$, $g \in G_\lambda$. Thus, $\mathcal{G}$ is a regular tree with each vertex having exactly q adjacent vertices. The adjacent vertices to $\mathbb{P}_0$ can be represented in the following ways:

(i) $\sigma^0\varphi(\mathbb{P}_0)$, $\sigma^1\varphi(\mathbb{P}_0)$,$\cdots$, $\sigma^{q-1}\varphi(\mathbb{P}_0)$

(ii) $\rho^0\varphi(\mathbb{P}_0)$, $\rho^1\varphi(\mathbb{P}_0)$,$\cdots$, $\rho^{q-1}\varphi(\mathbb{P}_0)$

(iii) $\{\sigma^r\varphi(\mathbb{P}_0) : -l \leq r \leq l-1\}$ for $q = 2l$ and $\{\sigma^r\varphi(\mathbb{P}_0) : -l+1 \leq r \leq l-1\}$ for $q = 2l-1$.

Equivalently representations exist in terms of $\rho = \sigma^{-1}$ where $\rho^q = \sigma^q = 1_{map}$.

We note that $\sigma^{q-1}\varphi(\mathbb{P}_0) = \sigma^{-1}\varphi(\mathbb{P}_0) = \rho\varphi(\mathbb{P}_0) = \tau_\lambda\varphi\varphi(\mathbb{P}_0) = \tau_\lambda(\mathbb{P}_0)$. Thus, $(\rho\varphi)^t(\mathbb{P}) = \tau_\lambda^t(\mathbb{P}_0)$ and $(\varphi\sigma)\varphi(\mathbb{P}_0)) = (\varphi\varphi\tau_\lambda^{-1})\varphi(\mathbb{P}_0)) = \tau_\lambda^{-1}\varphi(\mathbb{P}_0)$. Thus, $(\varphi\sigma)^t\varphi(\mathbb{P}_0) = \tau_\lambda^{-t}\varphi(\mathbb{P}_0)$, $t \in \mathbb{Z}^+$. We also note that $\rho(\mathbb{P}_0) = \sigma(\mathbb{P}_0) = \mathbb{P}_0$ and thus $\rho^t(\mathbb{P}_0) = \sigma^t(\mathbb{P}_0) = \mathbb{P}_0$ for any t.

Inductively, let $\mathbb{P}_k = g_k(\mathbb{P}_0)$ be any vertex in $\mathcal{G}$ with $g_k = \sigma^{r_0}\varphi\sigma^{r_1}\varphi\cdots\sigma^{r_k}\varphi$ where $0 \leq r_i \leq q-1$ for any i. Then $\mathbb{P}_0 = g_k^{-1}(\mathbb{P}_k)$. The vertices adjacent to $\mathbb{P}_0$ are given above and thus the vertices adjacent to $\mathbb{P}_k$ are given as

$$\{g_k(\sigma^0\varphi(\mathbb{P}_0)),\ g_k(\sigma^1\varphi(\mathbb{P}_0)),\cdots, g_k(\sigma^{q-1}\varphi(\mathbb{P}_0))\}$$

or in an equivalent form.

We note that $g_k\varphi = \sigma^{r_0}\varphi\sigma^{r_1}\varphi\cdots\varphi\sigma^{r_k} = g_k\sigma^0\varphi$ and that $\sigma(\mathbb{P}_0) = \varphi\tau_\lambda^{-1}(\mathbb{P}_0) = \mathbb{P}_0$ and $\sigma^{r_k}(\mathbb{P}_0) = \mathbb{P}_0$ for all $r_k \in \mathbb{Z}$. So

$$g_k\varphi(\mathbb{P}_0) = \sigma^{r_0}\varphi\sigma^{r_1}\varphi\cdots\sigma^{r_{k-1}}\varphi(\mathbb{P}_0) = g_{k-1}(\mathbb{P}_0) = \mathbb{P}_{k-1}.$$

The adjacent vertices to $\mathbb{P}_k$ are given as $\sigma^{r_0}\varphi\sigma^{r_1}\varphi\cdots\sigma^{r_k}\varphi\sigma^t\varphi(\mathbb{P}_0)$ where $t = 0$,1, 2, $\cdots, q-1$. By above, when $t = 0$, $\mathbb{P}_{k-1}$ is adjacent to $\mathbb{P}_k$. Letting $t = r_{k+1}$ results is proved.

Hence, each vertex in $\mathcal{G}$ (any q-gon in $\mathbb{H}^2$) can be written as $\sigma^{r_0}\varphi\sigma^{r_1}\varphi\cdots\sigma^{r_k}\varphi(\mathbb{P}_0)$ where $1 \leq r_i \leq q-1$ for $i \geq 1$ since if $r_i = 0$ then the expansion will collapse. Re-writing in terms of

$\rho = \sigma^{-1}$ we can see that any vertex can be written as $\rho^{r_0}\varphi\rho^{r_1}\varphi\cdots\rho^{r_k}\varphi(\mathbb{P}_0)$ where $r_0 \in \mathbb{Z}$ and where $1 \le r_i \le q-1$, $i \ge 1$. Treating the odd and the even separately and writing $\sigma^t = \sigma^{t-q}$, we have for $q = 2l$ (even) each vertex can be written as $\mathbb{P}_k = \sigma^{r_0}\varphi\sigma^{r_1}\varphi\cdots\varphi\sigma^{r_k}(\mathbb{P}_0)$ where $-l \le r_i \le l-1$, $r_i \ne 0$ for $i \ne 0, k$. For $q = 2l-1$ (odd) each vertex can be written as $\mathbb{P}_k = \sigma^{r_0}\varphi\sigma^{r_1}\varphi\cdots\varphi\sigma^{r_k}\varphi(\mathbb{P}_0)$ where $-l+1 \le r_i \le l-1$, $r_i \ne 0$ for $i \ne 0, k$. $\square$

Figures 1 and 2 show the initial parts of the graph of $\mathcal{G}$ for q even and odd.

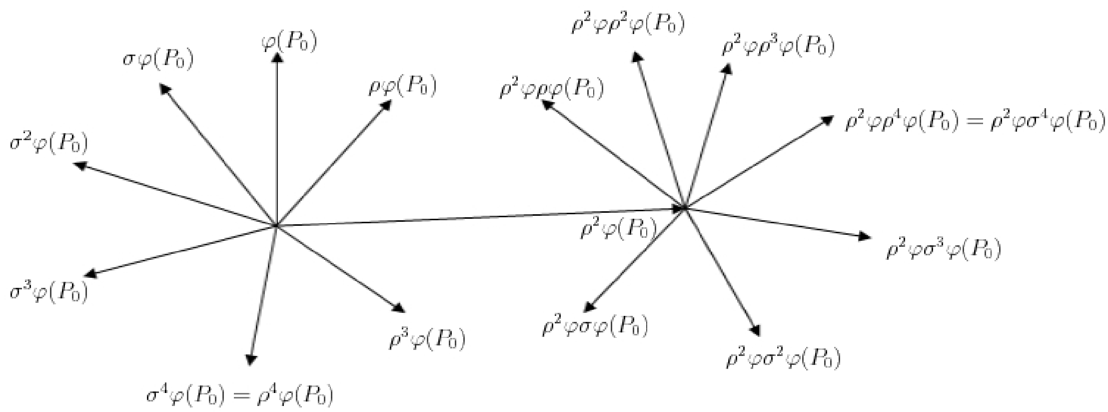

FIGURE 1. λ-Farey graph for $q = 8$, $l = 4$.

We note that each λ-fraction or cusp of a vertex in $\mathcal{G}$ can be expressed as $\sigma^{r_0}\varphi\sigma^{r_1}\varphi\cdots\varphi\sigma^{r_k}\varphi(\infty)$ where $-l \le r_i \le l-1$, $r_i \ne 0$ if $q = 2l$ and $-l+1 \le r_i \le l-1$, if $q = 2l-1$. Using $\rho = \sigma^{-1}$, $\sigma^q = \rho^q = 1_{map}$.

Further, we note the following results that follow from the above representations:

$$\sigma\varphi\sigma^{-1}\varphi = (\varphi\tau_\lambda^{-1})\varphi(\tau_\lambda\varphi)\varphi = \varphi\tau_\lambda^{-1}\varphi\tau_\lambda \text{ and } \sigma^{-1}\varphi\sigma\varphi = (\tau_\lambda\varphi)\varphi(\varphi\tau_\lambda^{-1})\varphi = \tau_\lambda\varphi\tau_\lambda^{-1}\varphi. \text{ But}$$
$$(\sigma\varphi)(\sigma\varphi) = (\varphi\tau_\lambda^{-1})\varphi(\varphi\tau_\lambda^{-1})\varphi = \varphi\tau_\lambda^{-2}\varphi \text{ and similarly } (\sigma^{-1}\varphi)(\sigma^{-1}\varphi) = (\tau_\lambda\varphi)\varphi(\tau_\lambda\varphi)\varphi = \tau_\lambda^2\varphi.$$

Hence, words of the form $\sigma^{r_0}\varphi\sigma^{r_1}\varphi\cdots\varphi\sigma^{r_k}\varphi$ will reduce if r_i and r_{i+1} are of the same sign, in that the powers of τ_λ or τ_λ^{-1} will increase and the length of the word will decrease.

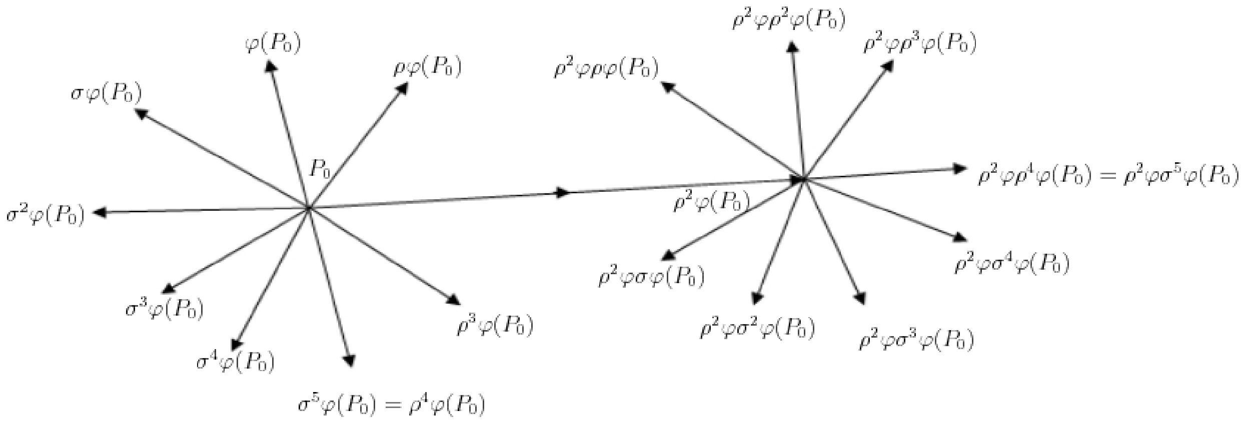

FIGURE 2. λ-Farey graph for $q = 9$, $l = 5$.

CHAPTER 7

Convergence of λ-Farey intervals

In this chapter we will show that every non λ-rational, α in $\mathbb{R}$ is contained in a nested chain of λ-intervals, $\mathbf{I}_1 \supset \mathbf{I}_2 \supset \cdots$ where $\mathbf{I}_k = [p_k/q_k, P_k/Q_k]$, $\dfrac{p_k}{q_k} \sim \dfrac{P_k}{Q_k}$ are adjacent λ-rationals and such that $\lim\limits_{k\to\infty} p_k/q_k = \lim\limits_{k\to\infty} P_k/Q_k = \alpha$, [**12**]. We note that $|\mathbf{I}_k| = \dfrac{1}{q_k Q_k}$ and we will see that $\lim\limits_{k\to\infty} |\mathbf{I}_k| = 0$. This result can be compared to Rosen's [**25**] (page 559) result that every infinite reduced λ-continued fraction converges. To prove these theorems we first establish some preliminary results, [**12**]. We recall from Lemma 7 (page 51) that when q is even ($q = 2l$ with $l \geq 2$) $\sigma^l(\infty) = \rho^l(\infty) = \lambda/2$ and $\sigma^{l+1}(\infty) = \rho^{l-1}(\infty) = 2/\lambda$ with $\lambda/2 < 1 < 2/\lambda$. When q is odd with $q = 2l-1$, $l \geq 3$, we have $\rho^{l-1}(\infty) = \sigma^l(\infty) = 1$ where $\lambda/2 < 1$. In both cases $\sigma^q = \rho^q = 1_{map}$. Finally we note from Lemma 7 (page 51) that $\sigma^{r+1}(\infty) = 1/\rho^r(\infty) = \{\rho^r(\infty)\}^{-1}$ for $r = 0, \cdots, q-1$.

The results about the converging λ-Farey intervals, leads to the interpretation of a λ-continued fraction as a path on the λ-Farey graph. Equivalently, the results can be interpreted as a λ-continued fraction derived from the cutting sequence of a geodesic ending on a non-rational α, across the λ-Farey tessellation of $\mathbb{H}^2$ under G_λ. We first consider $\mathbb{P}_0$, when q is both even and odd, and show that the denominators of cusps of the polygon $\mathbb{P}_0$ on either side of $\sigma^l(\infty)$ ($q = 2l$ or $q = 2l-1$) are non-decreasing. We follow these results with a generalisation to a general polygon $\mathbb{P}_k = g(\mathbb{P}_0), g \in G_\lambda$.

1. The fundamental q-gon $\mathbb{P}_0$ and the length of its spanning intervals

LEMMA 13. [**12**] *If $q = 2l$, $l \geq 2$ then the cusps of $\mathbb{P}_0$, $\sigma(\infty)$, $\sigma^2(\infty), \cdots, \sigma^{l-1}(\infty)$ are convergents to $\sigma^{l-1}(\infty) = \rho(\lambda/2) = \dfrac{\lambda^2-2}{\lambda}$ with respect to the nearest λ-integer continued fraction while $\rho(\infty)$, $\rho^2(\infty), \cdots, \rho^{l-1}(\infty)$ are convergents to $\rho^l(\infty) = \lambda/2$ with respect to the nearest λ-integer continued fraction. Further we see that $|\sigma(\infty) - \sigma^2(\infty)| \leq \lambda/2$ and $|\rho(\infty) - \rho^2(\infty)| \leq \lambda/2$. In general, $|\sigma^r(\infty) - \sigma^{r+1}(\infty)| \leq |\sigma(\infty) - \sigma^2(\infty)|$ and $|\rho^r(\infty) - \rho^{r+1}(\infty)| \leq |\rho(\infty) - \rho^2(\infty)|$ for $1 \leq r \leq l-1$. The denominators of the cusps of $\mathbb{P}_0$ on either side of $\sigma^l(\infty) = \rho^l(\infty) = \lambda/2$ are non-decreasing.*

Proof

For $q = 2l$, $l \geq 2$ we have $1 < \sqrt{2} \leq \lambda < 2$ with the λ-Farey subdivision of $\mathbb{P}_0$ given by the set of cusps $\{\sigma^r(\infty) : 1 \leq r \leq q-1\}$ where $\rho = \sigma^{-1}$. Since (Lemma 7, page 51) $1/\rho^r(\infty) = \sigma^{r+1}(\infty) = \sigma(\sigma^r(\infty)) = \varphi\tau_\lambda^{-1}(\sigma^r(\infty)) = \dfrac{1}{\lambda - \sigma^r(\infty)}$, the expansions of $\sigma^r(\infty)$ satisfy the nearest λ-integer continued fraction algorithm.

We see that $|\sigma(\infty) - \sigma^2(\infty)| = |0 - 1/\lambda| = 1/\lambda \leq \lambda/2$ since for $l \geq 2$, $1/2 < 1/\lambda \leq 1/\sqrt{2} = \sqrt{2}/2 \leq \lambda/2 < 1$. Similarly $|\rho(\infty) - \rho^2(\infty)| = |\lambda - (\lambda - 1/\lambda)| = 1/\lambda \leq \lambda/2$. Since each of the cusps has a nearest λ-integer continued fraction representation, we have that $\sigma(\infty)$, $\sigma^2(\infty), \cdots, \sigma^{l-1}(\infty)$ are nearest λ-integer convergents of $\sigma^{l-1}(\infty) = \dfrac{\lambda^2-2}{\lambda}$ and hence $\sigma^r(\infty) = \rho^r(\infty) = \lambda/2$ and similarly $\rho(\infty)$, $\rho^2(\infty), \cdots, \rho^{l-1}(\infty)$ are nearest λ-integer convergents to $\rho^l(\infty) = \lambda/2$. Thus, $\dfrac{p_1}{q_1} = \dfrac{0}{1}, \dfrac{p_2}{q_2} = \dfrac{1}{\lambda}, \dfrac{p_3}{q_3} = \dfrac{\lambda}{\lambda^2-1}, \dfrac{p_4}{q_4} = \dfrac{\lambda^2-1}{\lambda^3}, \cdots, \dfrac{p_{l-2}}{q_{l-2}}$ are convergents to $\sigma^{l-1}(\infty)$. We see that $p_{i+1} = q_i$ for $i = 1, \cdots, l-2$ and $\dfrac{p_1}{q_1} = \dfrac{0}{1} < \dfrac{p_2}{q_2} = \dfrac{1}{\lambda} < \dfrac{p_3}{q_3} = \dfrac{\lambda}{\lambda^2-1} < \dfrac{p_4}{q_4} = \dfrac{\lambda^2-1}{\lambda^3} < \cdots < \dfrac{p_{l-2}}{q_{l-2}} < \dfrac{\lambda}{2} < 1$. Thus, $q_i = p_{i+1} < q_{i+1}$ for $i = 1, \cdots, l-2$. Thus, the denominators of the convergents $\dfrac{p_i}{q_i}$, of the cusps of $\mathbb{P}_0$ are non-decreasing. We have

$q_{l-1} > q_{l-2} > \cdots > q_2 > q_1 = 1$. Further if $\dfrac{m_1}{n_1} = \dfrac{\lambda}{1} > \dfrac{m_2}{n_2} = \lambda - 1/\lambda = \dfrac{\lambda^2-1}{\lambda} > \dfrac{m_3}{n_3} = \lambda - \dfrac{1}{\lambda - 1/\lambda} = \dfrac{\lambda^3 - 2\lambda}{\lambda^2-1} > \dfrac{m_4}{n_4} = \dfrac{\lambda^4 - 3\lambda^2 - 1}{\lambda^3 - 2\lambda} > \cdots > \dfrac{m_{l-1}}{n_{l-1}} > 1 > \lambda/2$, then $n_{i+1} = m_i > n_i$ for $i = 1, \cdots, l-1$. So the denominators of the convergents $\dfrac{m_i}{n_i}$, of the λ-continued fraction expansion of the cusps of $\mathbb{P}_0$ are non-decreasing for $i = 1, \cdots, l-1$. Thus,

$|\sigma^r(\infty) - \sigma^{r+1}(\infty)| = \left|\dfrac{p_r}{q_r} - \dfrac{p_{r+1}}{q_{r+1}}\right| = \dfrac{1}{|q_r q_{r+1}|} \leq \dfrac{1}{q_1 q_2} = |\sigma(\infty) - \sigma^2(\infty)| \leq \lambda/2$ where $r = 1, \cdots, l-2$ while $|\sigma^r(\infty) - \sigma^{r+1}(\infty)| = \left|\dfrac{m_r}{n_r} - \dfrac{m_{r+1}}{n_{r+1}}\right| \leq \dfrac{1}{n_1 n_2} = |\sigma(\infty) - \sigma^2(\infty)| < \lambda/2$ where $-l+1 \leq r \leq -1$, or $|\rho^r(\infty) - \rho^{r+1}(\infty)| \leq |\rho(\infty) - \rho^2(\infty)| \leq \lambda/2$ where $1 \leq r \leq l-1$. $\square$

Lemma 14. [**12**] *Let $q = 2l-1$ for $l \geq 3$. The cusps of $\mathbb{P}_0$, given as $\sigma(\infty)$, $\sigma^2(\infty), \cdots, \sigma^{l-1}(\infty)$ are convergents to $\sigma^{l-1}(\infty)$ with respect to nearest λ-integer continued fraction while the cusps $\rho(\infty)$, $\rho^2(\infty), \cdots, \rho^{l-1}(\infty)$ are convergents to $\rho^{l-1}(\infty)$ with respect to nearest λ-integer continued fraction. Further $|\sigma(\infty) - \sigma^2(\infty)| \leq \lambda/2$ and $|\rho(\infty) - \rho^2(\infty)| \leq \lambda/2$ while $|\sigma^r(\infty) - \sigma^{r+1}(\infty)| \leq |\sigma(\infty) - \sigma^2(\infty)| \leq \lambda/2$ for $1 \leq r \leq l-2$ and $|\rho^r(\infty) - \rho^{r+1}(\infty)| \leq |\rho(\infty) - \rho^2(\infty)| \leq \lambda/2$ for $1 \leq r \leq l-2$. The denominators of the cusps of $\mathbb{P}_0$ on either side of $\sigma^l(\infty) = \rho^{l-1}(\infty)$ are non-decreasing.*

Proof

Let $q = 2l-1$, $l \geq 3$ then $\sqrt{2} < \lambda < 2$. As in Lemma 13, we have that the λ-Farey subdivision for $\mathbb{P}_0$ is given by the set of cusps $\{\sigma^r(\infty) : 1 \leq r \leq q-1\}$. The representation of the cusps is in terms of the nearest λ-integer continued fraction and we have that $\sigma(\infty)$, $\sigma^2(\infty), \cdots, \sigma^{l-1}(\infty)$ are convergents to $\sigma^{l-1}(\infty)$ (and hance of $\sigma^l(\infty) = \rho^{l-1}(\infty) = 1$) while $\rho(\infty)$, $\rho^2(\infty), \cdots, \rho^{l-2}(\infty)$ are convergents to $\rho^{l-1}(\infty)$.

Thus, if $0/1 = p_1/q_1,\ p_2/q_2, \cdots, p_{l-2}/q_{l-2}$ are convergents to $\sigma^{l-1}(\infty)$ then as in Lemma 13, $q_{l-2} > q_{l-3} > \cdots > 1$. If $m_1/n_1 = \lambda/1,\ m_2/n_2, \cdots, m_{l-2}/n_{l-2}$ are the consecutive cusps to $\rho^{l-1}(\infty)$ then $m_{l-2} > m_{l-3} > \cdots n_1 = 1$. Thus, $|\sigma^r(\infty) - \sigma^{r+1}(\infty)| = \left|\dfrac{p_r}{q_r} - \dfrac{p_{r+1}}{q_{r+1}}\right| = \dfrac{1}{|q_r q_{r+1}|} \leq \dfrac{1}{q_1 q_2} = |\sigma(\infty) - \sigma^2(\infty)| \leq \lambda/2$ where $r = 1, \cdots, l-2$ and $|\rho^r(\infty) - \rho^{r+1}(\infty)| = \left|\dfrac{m_r}{n_r} - \dfrac{m_{r+1}}{n_{r+1}}\right| \leq \dfrac{1}{n_1 n_2} = |\rho(\infty) - \rho^2(\infty)| < \lambda/2$ where $1 \leq r \leq l-1$. $\square$

In the next section we will generalise these two lemmas to any q-gon $\mathbb{P}_k = g(\mathbb{P}_0), g \in G_\lambda$ as given in Theorem 15 and 16, [**12**].

2. The q-gon $\mathbb{P}_k$ and the length of its spanning intervals

THEOREM 15. [**12**] *Let $q = 2l$, $l \geq 2$ with $1 < \lambda < 2$. Let $\mathbb{P}_k = g(\mathbb{P}_0)$ with $g = \sigma^{r_0}\varphi\sigma^{r_1}\varphi \cdots \varphi\sigma^{r_k}\varphi$ and where $-l \leq r_i \leq l-1$, $r_i \neq 0$ for all $i = 1, \cdots, k$ except r_0 which may be zero. If consecutive cusps $a/c = g\sigma^t(\infty)$ and $b/d = g\sigma^{t+1}(\infty) = g\sigma^t(0)$ of $g(\mathbb{P}_0)$ are given then $g_1 = g\sigma^t \in G_\lambda$ and $g_1^{-1}(\infty) = d/c$. Further $\dfrac{d}{c} > 1$ if $1 \leq t \leq l-1$ and $\dfrac{d}{c} < 1$ if $-l < t \leq -1$.*

THEOREM 16. [**12**] *Let $q = 2l-1$, $l \geq 3$ with $1 < \lambda < 2$. Let $\mathbb{P}_k = g(\mathbb{P}_0)$ with $g = \sigma^{r_0}\varphi \cdots \varphi\sigma^{r_k}\varphi$ where $-l+1 \leq r_i \leq l-1$ $r_i \neq 0$ for all $i = 1, \cdots, k$ except r_0 which may be zero. If consecutive cusps $a/c = g\sigma^t(\infty)$ and $b/d = g\sigma^{t+1}(\infty) = g\sigma^t(0)$ are given then $g_1 = g\sigma^t \in G_\lambda$ and $g_1^{-1}(\infty) = d/c$. Further $\dfrac{d}{c} > 1$ if $1 \leq t \leq l-1$ and $\dfrac{d}{c} < 1$ if $-l+1 < t \leq -1$.*

Lemmas 15-19 will be used in these generalisations.

LEMMA 15. [**12**] *Let $\mathbb{P}_k = g(\mathbb{P}_0)$ where $g = \sigma^{r_0}\varphi\sigma^{r_1}\varphi \cdots \varphi\sigma^{r_k}\varphi$, $0 \leq r_i \leq q-1$, $r_i \neq 0$, for $i = 1, \cdots, k$ and r_0 may be zero. If the cusps of $\mathbb{P}_k$ are given by the λ-Farey subdivision*

$\{g\sigma^t(\infty) : 0 \leq t \leq q-1\}$ *with* $g_1 = g\sigma^t$*, then* $g_1(z) = \dfrac{-az+b}{-cz+d}$ *where* $g_1(\infty) = a/c$ *and* $g_1(0) = b/d$*, and* $g_1 \in G_\lambda$*. Further* $g_1^{-1}(\infty) = d/c$*.*

Proof

Let $\mathbb{P}_k = g(\mathbb{P}_0)$ where $g = \sigma^{r_0}\varphi\sigma^{r_1}\varphi\cdots\varphi\sigma^{r_k}\varphi$, $0 \leq r_i \leq q-1$ and where $r_i \neq 0$, for $i = 1, \cdots, k$ and r_0 may be zero. The λ-Farey subdivision of $\mathbb{P}_k$ is given as $\{g\sigma^t(\infty) : t = 0, \cdots, q-1\}$. We know that for $0 \leq t \leq q-1$ that $\sigma^t(\infty) < \sigma^{t+1}(\infty)$. Since $g \in G_\lambda$ and g preserves order, we have $g\sigma^t(\infty) < g\sigma^{t+1}(\infty) = g\sigma^t(0)$. We note that when $t = 0$, $g(\infty)$ and $g(0)$ are bounds of the interval spanned by $\mathbb{P}_k = g(\mathbb{P}_0)$. Let $a/c = g\sigma^t(\infty)$ and $g\sigma^{t+1}(\infty) = g\sigma^t(0) = b/d$, with $a/c < b/d$, be consecutive cusps of $\mathbb{P}_k$ and let $g_1 = g\sigma^t \in G_\lambda$. Then $g_1(z) = \dfrac{-az+b}{-cz+d}$, $ad - bc = -1$ so $g_1^{-1}(z) = \dfrac{dz-b}{cz-a}$ and $g_1^{-1}(\infty) = d/c$. $\square$

Lemma 16. [**12**] *Let* $g = \sigma^{r_0}\varphi\sigma^{r_1}\varphi\cdots\varphi\sigma^{r_k}\varphi$ *and* $\phi(z) = 1/z$ *where* $0 \leq r_i \leq q-1$*,* $r_i \neq 0$*, for* $i = 1, \cdots, k$ *and* r_0 *may be zero. Then* $\phi g = \phi\sigma^{r_0}\varphi\sigma^{r_1}\varphi\cdots\varphi\sigma^{r_k}\varphi = \rho^{r_0}\varphi\rho^{r_1}\varphi\cdots\varphi\rho^{r_k}\varphi\phi$*.*

Proof

Recall in Lemma 6 (page 51) we have the results that $\phi\rho^k = \rho^{-k}\phi$. Therefore

$\phi g = \rho^{r_0}\varphi\rho^{r_1}\varphi\cdots\varphi\rho^{r_k}\varphi\phi$ since $\phi\sigma = \rho\phi$. $\square$

Lemma 17. [**12**] *Let* $q = 2l$*,* $l \geq 2$ *and* $g = \tau_\lambda^2\varphi\rho^{l-2}$ *where* $\rho = \tau_\lambda\varphi$*. Then* g *is a loxodromic map and* $\lambda + 1$ *is its attracting fixed point. Thus,* $\lim\limits_{k\to\infty} g^k(\infty) = \lambda + 1$*.*

Proof

Let $\tau_\lambda(z) = z + \lambda$, $\varphi(z) = -1/z$ and $\rho = \tau_\lambda\varphi$ for $q = 2l$, $l \geq 2$. Thus, $g = \tau_\lambda^2\varphi\rho^{l-2} = \tau_\lambda^2\varphi\rho^{-2}\rho^l = (\tau_\lambda\varphi\tau_\lambda^{-1})\rho^l$. Now $\rho^l(\infty) = \lambda/2$ and $\rho^{l-1}(\infty) = 2/\lambda$. Written in λ-fraction notation by Corollary 1 (page 37), we have either $\rho^{l-1}(\infty) = \rho^l(0) = \lambda p_1/q_1$ or $p_1/\lambda q_1$ where p_1, q_1 are polynomials in λ^2. Thus, $\rho^l(\infty) = \rho(\rho^{l-1}(\infty)) = \tau_\lambda\varphi(\lambda p_1/q_1)$ or $\tau_\lambda\varphi(p_1/\lambda q_1)$. That is, $\rho^l(\infty) = \dfrac{\lambda^2 p_1 - q_1}{\lambda p_1}$ or $\dfrac{\lambda(p_1 - q_1)}{p_1}$. Say $\rho^l(0) = \dfrac{\lambda p_1}{q_1}$. Then since $\rho^l \in G_\lambda$ we may write $\rho^l(z) = \dfrac{-(\lambda^2 p_1 - q_1)z + \lambda p_1}{-(\lambda p_1)z + q_1}$ where $\rho^l(\infty) < \rho^{l-1}(\infty) = \rho^l(0)$ and $-q_1(\lambda^2 p_1 - q_1) - (-\lambda p_1)(\lambda p_1) = 1$. Thus, $q_1^2 + \lambda^2 p_1^2 = 1 + \lambda^2 p_1 q_1$ where $\lambda p_1/q_1 = \rho^{l-1}(\infty) = 2/\lambda$ or $\lambda p_1 = 2q_1/\lambda$. Thus, $\rho^l(z) = \dfrac{\lambda z - 2}{2z - \lambda}$.

Since $g = \tau_\lambda^2\varphi\rho^{l-2}$ we have $g(z) = \dfrac{(-\lambda^2 - 2)z + \lambda(\lambda^2 - 1)}{-\lambda z - 2 + \lambda^2}$ and $tr^2 g = ((-\lambda^2 - 2) + (\lambda^2 - 2))^2 = 16$. So g is loxodromic in G_λ. The fixed points of g are established by solving $g(z) = z$. That is, $\lambda z^2 - z(2\lambda^2) + \lambda(\lambda^2 - 1) = 0$. Thus, $z = \dfrac{(2\lambda^2 \pm 2\lambda)}{2\lambda} = \lambda \pm 1$. Let $\alpha = \lambda + 1$ and $\beta = \lambda - 1$ be the fixed points. Consider $s(z) = \dfrac{z - \alpha}{z - \beta}$ with $s(\alpha) = 0$ and $s(\beta) = \infty$ then $sgs^{-1}(z) = uz$ where $sgs^{-1}(\infty) = \infty$, $sgs^{-1}(0) = 0$ and $u = \dfrac{2\lambda - 4}{-(2\lambda + 4)} = \dfrac{4 - 2\lambda}{2\lambda + 4}$. Thus, $|u| = \dfrac{|4 - 2\lambda|}{2\lambda + 4} = \dfrac{4 - 2\lambda}{2\lambda + 4} < \dfrac{2}{6} = \dfrac{1}{3}$. Therefore α is the attracting fixed point of g since 0 is the attracting fixed point of sgs^{-1}, [**14**]. If $\rho^l(0) = p_1/\lambda q_1$ the same results can be established. $\square$

LEMMA 18. [**12**]

$\tau_\lambda^2\varphi\rho^r(\infty) \geq \lambda + 1$ for all $-1 \leq r \leq l - 2$ where $q = 2l$ or $q = 2l - 1$.

Proof

(i) Let $q = 2l$ and $\sqrt{2} \leq \lambda < 2$. Then $\rho^r(\infty) \geq \rho^{l-2}(\infty)$ for $1 \leq r \leq l - 2$. Thus,

$\tau_\lambda^2\varphi\rho^r(\infty) \geq \tau_\lambda^2\varphi\rho^{l-2}(\infty) = \tau_\lambda^2\varphi\sigma^2\rho^l(\infty) = \tau_\lambda^2\varphi\sigma^2(\lambda/2) = \tau_\lambda\varphi(-\lambda + \lambda/2) = \tau_\lambda\varphi(-\lambda/2) = \lambda + 2/\lambda > \lambda + 1$ since $2/\lambda > 1$ for $\sqrt{2} \leq \lambda < 2$.

If $r = -1$, $\tau_\lambda^2\varphi\rho^{-1}(\infty) = \tau_\lambda^2(\infty) = \infty > \lambda + 1$.

If $r = 0$, $\tau_\lambda^2\varphi(\infty) = 2\lambda > \lambda + 1$ since $\lambda > 1$.

(ii) Let $q = 2l - 1$ then $\sqrt{2} < \lambda < 2$. Since $\rho^{l-1}(\infty) = 1$, $\tau_\lambda^2\varphi\rho^r(\infty) = \tau_\lambda(\tau_\lambda\varphi\rho^r(\infty)) = \tau_\lambda(\rho^{r+1}(\infty)) \geq \tau_\lambda\rho^{l-1}(\infty) = \tau_\lambda(1) = 1 + \lambda$, $0 \leq r + 1 \leq l - 1$. We note that if $r + 1 = 0$ or $r = -1$ we have $\tau_\lambda^2\varphi\rho^{-1}(\infty) = \tau_\lambda^2\varphi(0) = \tau_\lambda^2(\infty) = \infty > \lambda + 1$.

LEMMA 19. [**12**] *$\tau_\lambda^2\varphi\rho^{r-2}(\lambda + 1) \geq \lambda + 1$ for $1 \leq r \leq l - 1$ where $q = 2l$ or $q = 2l - 1$.*

Proof

Let $1 \leq r \leq l-1$. Then $-1 \leq r-2 \leq l-3 < l-2$. We know that $1 < \sqrt{2} \leq \lambda < 2$ implies that $0 < 2 < \lambda+1 < 3$. That is, $1/3 < 1/(\lambda+1) < 1/2$. We note that $0 < \lambda - 1/2 < \lambda - \dfrac{1}{\lambda+1} < \lambda - 1/3 < \lambda + 1$, with $\lambda - \dfrac{1}{\lambda+1} = \tau_\lambda\varphi(\lambda + 1)$. Thus, $\lambda - \dfrac{1}{\lambda+1} = \tau_\lambda\varphi(\lambda + 1) < \lambda + 1$ or $\rho(\lambda + 1) < \lambda + 1$ and so $\rho^2(\lambda + 1) < \rho(\lambda + 1) < \lambda + 1$, since $\rho \in G_\lambda$ preserves order. In fact $\rho^k(\lambda+1) \leq (\lambda+1)$ for all $k \geq 0$. Thus, since $l-r \geq l-(l-1) = 1$, we have $\rho^{l-r}(\lambda+1) \leq \lambda+1$ and hence $\rho^r(\lambda+1) \geq \rho^l(\lambda+1)$ where $1 \leq r \leq l-1$. Finally $\tau_\lambda^2\varphi\rho^{r-2}(\lambda+1) \geq \tau_\lambda^2\varphi\rho^{l-2}(\lambda+1) = \lambda+1$ for all $1 \leq r \leq l - 1$. $\square$

The proofs of Theorems 15 and 17 can now be completed.

Proof of Theorem 15.

We do not consider the case where $t = 0$ since $g(\infty)$ and $g(0)$ are bounds of the interval spanned by $\mathbb{P}_k$. The consecutive cusps of $\mathbb{P}_k = g(\mathbb{P}_0)$ are given as $a/c = g\sigma^t(\infty)$ and $b/d = g\sigma^{t+1}(\infty) = g\sigma^t(0)$ for $-l \leq t \leq l - 1$. We have $\sigma^l(\infty) = \rho^l(\infty) = \rho^{-l}(\infty) = \lambda/2$. The cusps $\sigma^l(\infty)$ is a consecutive cusp to $\rho^{l-1}(\infty)$ and cusp $\sigma^{l-1}(\infty)$. So we consider the cases for $1 \leq t \leq l - 1$ and $-l + 1 \leq t \leq -1$ only. From Lemma 15 (page 80) we know $g_1 = g\sigma^t$ can be expressed as $g_1(z) = \dfrac{-az + b}{-cz + d}$, $ad - bc = -1$ with $g_1^{-1}(z) = \dfrac{dz - b}{cz - a}$ and $g_1^{-1}(\infty) = d/c$. Further $g_1^{-1}(\infty) = (g\sigma^t)^{-1}(\infty) = \rho^t\varphi\rho^{r_k}\varphi \cdots \varphi\rho^{r_0}(\infty)$.

We consider two cases.

Case A: $1 \leq t \leq l-1$ and Case B: $-l+1 \leq t \leq -1$.

Case A: $1 \leq t \leq l-1$. We show that $d/c > 1$.

Consider (i) $0 \leq r_0 \leq l-1$ and (ii) $-l \leq r_0 \leq -1$.

(i) Assume that $0 \leq r_0 \leq l-1$. Let r_j be the first index in the expansion of g_1^{-1}, checking from the right, such that $r_j \leq -1$. Thus, $r_i \geq 1$ for all $i = 1, 2, \cdots, j-1$ and where r_0 may be zero.

$$
\begin{aligned}
g_1^{-1}(\infty) &= \rho^t \varphi \rho^{r_k} \varphi \cdots \varphi \rho^{r_j} \varphi \cdots \varphi \rho^{r_0}(\infty) \\
&= \rho^t \varphi \rho^{r_k} \varphi \cdots \varphi \rho^{r_j}(\rho^{2l}) \varphi \cdots \rho^{r_1} \varphi \rho^{r_0}(\infty),\ \rho^{2l} = \rho^q = 1_{map} \\
&= \rho^t \varphi \rho^{r_k} \varphi \cdots \varphi \rho^{r_j+l}(\rho \rho^{l-2} \rho) \varphi (\rho \rho^{r_{j-1}-2} \rho) \cdots (\rho \rho^{r_1-2} \rho) \varphi (\rho \rho^{r_0-1}(\infty)) \\
&= \rho^t \varphi \rho^{r_k} \varphi \cdots \varphi \rho^{r_j+l}(\rho \rho^{l-2})(\rho \varphi \rho) \rho^{r_{j-1}-2}(\rho \varphi \rho) \cdots (\rho \varphi \rho) \rho^{r_1-2}(\rho \varphi \rho) \rho^{r_0-1}(\infty) \\
&= \rho^t \varphi \rho^{r_k} \varphi \cdots \varphi \rho^{r_j+l}(\tau_\lambda \varphi \rho^{l-2}(T)), \text{ where}
\end{aligned}
$$

$T = (\rho\varphi\rho)\rho^{r_{j-1}-2}(\rho\varphi\rho) \cdots (\rho\varphi\rho)\rho^{r_1-2}(\rho\varphi\rho)\rho^{r_0-1}(\infty) \geq \tau_\lambda^2 \varphi \rho^{r_{j-1}-2} \tau_\lambda^2 \varphi \cdots \tau_\lambda^2 \varphi \rho^{r_1-2}(\lambda+1)$, by Lemma 18 (page 82) since $-1 \leq r_0 - 1 \leq l-2$. By Lemma 19 (page 83) we see that $T \geq \lambda + 1$. Thus,

$$
\begin{aligned}
g_1^{-1}(\infty) &\geq \rho^t \varphi \rho^{r_k} \varphi \cdots \varphi \rho^{r_j+l}(\tau_\lambda \varphi \rho^{l-2}(\lambda+1)) \\
&= \rho^t \varphi \rho^{r_k} \varphi \cdots \varphi \rho^{r_j+l}(\tau_\lambda^{-1}(\tau_\lambda^2 \varphi \rho^{l-2}(\lambda+1))) \\
&= \rho^t \varphi \rho^{r_k} \varphi \cdots \varphi \rho^{r_j+l} \tau_\lambda^{-1}(\lambda+1) \text{ since } \tau_\lambda^2 \varphi \rho^{l-2}(\lambda+1) = \lambda+1, \text{ Lemma 17 (page 81)} \\
&= \rho^t \varphi \rho^{r_k} \varphi \cdots \varphi \rho^{r_j+l}(1) > \rho^t \varphi \rho^{r_k} \varphi \cdots \varphi \rho^{r_j+l}(\lambda/2) > \rho^t \varphi \cdots \rho^{r_j+l}(\rho^l(\infty)) \\
&= \rho^t \varphi \cdots \rho^{r_j}(\infty) \text{ since } \rho^{2l} = 1_{map} \text{ and } -l \leq r_j \leq -1.
\end{aligned}
$$

This process will continue in a way analogous to Case $A(ii)$ (See below).

(ii) Assume that $-l \leq r_0 \leq -1$.

$$
\begin{aligned}
g_1^{-1}(\infty) &= \rho^t\varphi\rho^{r_k}\varphi\cdots\varphi\rho^{r_1}\varphi\rho^{r_0}(\infty),\ -l \le r_0 \le -1\\
&= \rho^t\varphi\rho^{r_k}\varphi\cdots\varphi\rho^{r_1}\varphi\rho^{r_0}\phi(0)\\
&= \rho^t\varphi\rho^{r_k}\varphi\cdots\varphi\rho^{r_1}\varphi\phi\rho^{-r_0}(0),\ 1 \le -r_0 \le l\\
&= \rho^t\varphi\rho^{r_k}\varphi\cdots\varphi\rho^{r_1}\varphi\phi\rho^{-r_0-1}(\infty) \text{ where } 0 = \rho^{-1}(\infty) \text{ and } 0 \le -r_0 - 1 \le l-1.\\
&\ge \rho^t\varphi\rho^{r_k}\varphi\cdots\varphi\rho^{r_1}\varphi\phi\rho^{l-1}(\infty),\ \rho^l(\infty) = \lambda/2 < 1 < \rho^{l-1}(\infty) = 2/\lambda\\
&> \rho^t\varphi\rho^{r_k}\varphi\cdots\varphi\rho^{r_1}\varphi\phi(1)\\
&= \rho^t\varphi\rho^{r_k}\varphi\cdots\varphi\rho^{r_1-1}(\tau_\lambda(1))\\
&= \rho^t\varphi\rho^{r_k}\varphi\cdots\varphi\rho^{r_1-1}(\lambda+1)\\
&> \rho^t\varphi\rho^{r_k}\varphi\cdots\varphi\rho^{r_1-1}(\lambda) \text{ since } \lambda+1 > \lambda\\
&= \rho^t\varphi\rho^{r_k}\varphi\cdots\varphi\rho^{r_1}(\infty) \text{ since } \lambda = \rho(\infty)
\end{aligned}
$$

If $1 \le r_1 \le l-1$, continue in a way analogous to Case $A(i)$, If $-l \le r_1 \le -1$, continue in a way analogous to Case $A(ii)$.

Continuing the processes through repetitions of Case A (i) or (ii) as needed, we reach one of the following:

1. $g_1^{-1}(\infty) > \rho^t\varphi\rho^{r_k}(\infty)$, $r_k \ge 1$ or
2. $g_1^{-1}(\infty) > \rho^t\varphi\rho^{r_k}(\infty)$, $r_k \le -1$.

For (1)
$$
\begin{aligned}
g_1^{-1}(\infty) &> \rho^{t-1}(\rho\varphi\rho)\rho^{r_k-1}(\infty)\\
&= \rho^{t-1}(\tau_\lambda^2\varphi)\rho^{r_k-1}(\infty),\ 0 \le r_k - 1 \le l-2\\
&\ge \rho^{t-1}(\tau_\lambda^2\varphi)\rho^{l-2}(\infty)\\
&= \rho^{t-1}(\lambda+1) \text{ by Lemma 17}\\
&> \rho^{t-1}(\lambda) = \rho^t(\infty),\ \rho(\infty) = \lambda\\
&> 1, \text{ since } 1 \le t \le l-1 \text{ and } \rho^t(\infty) \ge \rho^{l-1}(\infty) = 2/\lambda > 1.
\end{aligned}
$$

Therefore $g_1^{-1}(\infty) = d/c > 1$ or $d > c$.

For (2): $g_1^{-1}(\infty) > \rho^t\varphi\rho^{r_k}(\infty) = \rho^t\varphi\rho^{r_k}\phi(0)$

$$= \rho^t\varphi\phi\rho^{-r_k-1}(\infty) > \rho^t\varphi\phi(1) \text{ where } 0 \le -r_k - 1 \le l-1 \text{ and } \rho^{l-1}(\infty) = 2/\lambda > 1$$

$$= \rho^{t-1}\tau_\lambda(1) = \rho^{t-1}(\lambda+1) > \rho^{t-1}(\lambda) = \rho^t(\infty) > 1 \text{ since } 1 \le t \le l-1.$$

Therefore $g_1^{-1}(\infty) = d/c > 1$ or $d > c$ as required.

Case B. $-l+1 \le t \le -1$.

Let $g_1^{-1}(\infty) = \rho^t\varphi\rho^{r_k}\varphi\cdots\varphi\rho^{r_1}\varphi\rho^{r_0}(\infty)$. We use Lemma 16 (page 81) to transform Case B to Case A as follows:

$$\begin{aligned}\phi g_1^{-1}(\infty) &= \phi\{\rho^t\varphi\rho^{r_k}\varphi\cdots\varphi\rho^{r_1}\varphi\rho^{r_0}(\infty)\}\\ &= \rho^{-t}\varphi\rho^{-r_k}\varphi\cdots\varphi\rho^{-r_1}\varphi\rho^{-r_0}\phi(\infty)\\ &= \rho^{-t}\varphi\rho^{-r_k}\varphi\cdots\varphi\rho^{-r_1}\varphi\rho^{-r_0-1}(\infty),\ 0 = \rho^{-1}(\infty) \text{ and } 1 \le -t \le l-1.\end{aligned}$$

By Case A, $\phi g_1^{-1}(\infty) > 1$ and so $g_1^{-1}(\infty) < 1$ so $d/c < 1$ and $d < c$. $\square$

In a very similar manner we now complete Theorem 16.

Proof of Theorem 16

From Lemma 15 (page 80), $g_1(z) = \dfrac{-az+b}{-cz+d}$, $ad - bc = -1$ and $g_1^{-1}(\infty) = d/c$. We note for $q = 2l-1$, $l \ge 3$ we have $\rho^{l-1}(\infty) = 1 = \sigma^l(\infty)$ and $g_1^{-1}(\infty) = \rho^t\varphi\rho^{r_k}\varphi\cdots\varphi\rho^{r_0}(\infty)$. Again consider the two cases: Case A: $1 \le t \le l-1$ and Case B: $-l+1 \le t \le -1$.

Case A: $1 \le t \le l-1$. We show that $d/c > 1$.

Consider (i) $0 \le r_0 \le l-1$ and (ii) $-l \le r_0 \le -1$.

Case $A(i)$: We proceed exactly as in Case A(i) of Theorem 15 (page 8) to the stage where we insert $\rho^{2l-1} = 1_{map}$ instead of $\rho^{2l} = 1_{map}$. Then $g_1^{-1}(\infty) = \rho^t\varphi\rho^{r_k}\varphi\cdots\varphi\rho^{r_j+l-1}(\tau_\lambda\varphi\rho^{l-2}(T))$. Again using Lemmas 18 and 19 (page 82) we have:

$$
\begin{aligned}
g_1^{-1}(\infty) &\geq \rho^t\varphi\rho^{r_k}\varphi\cdots\varphi\rho^{r_j+l-1}(1)\\
&= \rho^t\varphi\rho^{r_k}\varphi\cdots\varphi\rho^{r_j+l-1}(\rho^{l-1}(\infty)),\ 1=\rho^{l-1}(\infty)\\
&= \rho^t\varphi\rho^{r_k}\varphi\cdots\varphi\rho^{r_j+2l-1-1}(\infty)\\
&= \rho^t\varphi\rho^{r_k}\varphi\cdots\varphi\rho^{r_j-1}(\infty) \text{ since } \rho^{2l-1}=1_{map},\ -l\leq r_j-1<-1
\end{aligned}
$$

We note that if $r_j-1=-l$ then $g_1^{-1}(\infty)\geq \rho^t\varphi\rho^{r_k}\varphi\cdots\varphi\rho^{l-1}(\infty)$ where $\rho^{-l}=\rho^{2l-1-l}=\rho^{l-1}$. The process can be continued repeating Case $A(i)$. If $r_{j-1}\neq -l$, we continue in a way analogous to Case $A(ii)$ below.

Case A(ii): Assume that $-l+1\leq r_0\leq -1$. Again we proceed as in Case A(ii) of Theorem 15 (page 84) to the stage:

$$
\begin{aligned}
g_1^{-1}(\infty) &= \rho^t\varphi\cdots\varphi\rho^{r_1}\varphi\phi\rho^{-r_0}(0),\ 1\leq -r_0\leq l-1\\
&= \rho^t\varphi\cdots\varphi\rho^{r_1}\varphi\phi\rho^{-r_0-1}(\infty) \text{ where } 0=\rho^{-1}(\infty) \text{ and } 0\leq -r_0-1\leq l-2<l-1\\
&> \rho^t\varphi\cdots\varphi\rho^{r_1}\varphi\phi\rho^{l-1}(\infty),\ 1=\rho^{l-1}(\infty)\\
&= \rho^t\varphi\cdots\varphi\rho^{r_1-1}(\tau_\lambda(1))\\
&> \rho^t\varphi\cdots\varphi\rho^{r_1}(\infty) \text{ as before.}
\end{aligned}
$$

Once again, continuing to repeat Case A(i) and (ii) as required, we reach the two possibilities.

1. $g_1^{-1}(\infty)>\rho^t\varphi\rho^{r_k}(\infty),\ 1\leq r_k\leq l-1$ or
2. $g_1^{-1}(\infty)>\rho^t\varphi\rho^{r_k}(\infty),\ -l+1\leq r_k\leq -1$

For (1), $g_1^{-1}(\infty)>\rho^{t-1}(\tau_\lambda\rho^{r_k}(\infty))>\rho^{t-1}\tau_\lambda\rho^{l-1}(\infty)$

$$= \rho^{t-1}(\lambda+1)>\rho^{t-1}(\lambda)=\rho^t(\infty)\geq 1 \text{ where } \rho(\infty)=\lambda \text{ and } 1\leq t\leq l-1.$$

For (2), $g_1^{-1}(\infty)>\rho^t(\varphi\phi\rho^{-r_k}(0))=\rho^t(\varphi\phi\rho^{-r_k-1}(\infty))$ where $0\leq -r_k-1\leq l-2<l-1$.

$$> \rho^{t-1}\tau_\lambda(1) \text{ where } \rho^{-r_k-1}(\infty)>\rho^{l-1}(\infty)=1 \text{ and}$$

$$= \rho^{t-1}(\lambda + 1) > \rho^{t-1}(\lambda)$$
$$= \rho^{t}(\infty) \geq 1 \text{ for } 1 \leq t \leq l-1,\ \rho(\infty) = \lambda$$

Hence, in Case A(i) and (ii), we have $g_1^{-1}(\infty) > 1$ and thus $d/c > 1$ or $d > c$.

Case B: Assume $-l+1 \leq t \leq -1$ and $g_1^{-1}(\infty) = \rho^t\varphi\cdots\varphi\rho^{r_0}(\infty)$. Then as in Case B of Theorem 15, consider

$$\begin{aligned}\phi g_1^{-1}(\infty) &= \phi\{\rho^t\varphi\cdots\varphi\rho^{r_0}(\infty)\}\\ &= \rho^{-t}\varphi\cdots\varphi\rho^{-r_0}(0)\\ &= \rho^{-t}\varphi\cdots\varphi\rho^{-r_0-1}(\infty) \text{ where } \rho^{-1}(\infty) = 0 \text{ and } 1 \leq -t \leq l-1.\end{aligned}$$

Thus, from Case A(i) and (ii) above, we have that $\phi g_1^{-1}(\infty) > 1$ or $g_1^{-1}(\infty) < 1$. So $d/c < 1$ and $d < c$. $\square$

From Theorem 15 and 16 we have established that the denominators of cusps of $\mathbb{P}_k$ are non-decreasing. In the following theorem we show that the spanning interval of these cusps converge to a point on $\mathbb{R}$.

THEOREM 17. [**12**] *The cutting sequence of a geodesic γ in $\mathbb{H}^2$ ending at a non λ-rational real number $\alpha \in \mathbb{R}$ is an infinite path on the λ-Farey graph. The sequence of vertices on this path is a sequence of λ-Farey q-gons $\{\mathbb{P}_k\}$ where each $\mathbb{P}_k = g_k(\mathbb{P}_0)$ spans on interval $\mathbf{I}_k = [p_k/q_k; P_k/Q_k]$ containing α and where $g_k = \sigma^{r_0}\varphi\cdots\varphi\sigma^{r_k}\varphi$. The sequence of spanning intervals $\{\mathbf{I}_k\}$ form a nested chain $\mathbf{I}_k \subseteq \cdots \subseteq \mathbf{I}_1$, converging to α with $\lim\limits_{k\to\infty} p_k/q_k = \lim\limits_{k\to\infty} P_k/Q_k = \alpha$.*

Proof

By Theorem 17 and 18 above, we have seen that the spanning intervals of a vertex $\mathbb{P}_k$ on the path has end points p_k/q_k and P_k/Q_k where $g\sigma^t(\infty) = p_k/q_k$ and $g\sigma^t(0) = P_k/Q_k$. We have seen that if $1 \leq t \leq l-1$ then $Q_k > q_k$ while if $-l+1 \leq t \leq -1$ $(q = 2l-1)$ or $-l \leq t \leq -1$ $(q = 2l)$ then $Q_k < q_k$. Certainly $\mathbf{I}_k \subseteq \cdots \subseteq \mathbf{I}_1$ and $\alpha \in \mathbf{I}_k$ for all k. Further $|\mathbf{I}_k| = \left|\frac{p_k}{q_k} - \frac{P_k}{Q_k}\right| = \frac{1}{|q_k Q_k|}$ since $|p_k Q_k - q_k P_k| = 1$, the end points of the intervals are λ-Farey neighbors. Then $|\mathbf{I}_k| < 1/q_k^2$, where $Q_k > q_k$ for $1 \leq t \leq l-1$. Also $|\mathbf{I}_k| < 1/Q_k^2$, where $Q_k < q_k$ for $-l+1 \leq t \leq -1$ $(q = 2l-1)$ or $-l \leq t \leq -1$ $(q = 2l)$.

Thus, the denominators of the endpoints of the spanning intervals of the vertices on the path are non-decreasing for $1 \leq t \leq l-1$ and for $-l+1 \leq t \leq -1$ $(q = 2l-1)$ or $-l \leq t \leq -1$ $(q = 2l)$. In fact, for all $P_k = g(\mathbb{P}_0)$, the denominators of the cusp are strictly increasing on each side of $g\sigma^l(\infty)$. Thus, $\lim\limits_{k\to\infty} q_k = \infty$ and $\lim\limits_{k\to\infty} Q_k = \infty$. Hence, we have $\lim\limits_{k\to\infty} |\mathbf{I}_k| = 0$ with $\lim\limits_{k\to\infty} p_k/q_k = \lim\limits_{k\to\infty} P_k/Q_k = \alpha$. $\square$

Example 5. Let $q = 6$, $\lambda = \sqrt{3}$ and $\alpha = \frac{12\sqrt{3}}{13}$. We examine the cutting sequence of a geodesic ending at α across the λ-Farey tessellation (path on a λ-Farey graph) in the light of the following λ-continued fractions of α.

(i) Admissible λ-continued fraction,

(ii) Nearest λ-integer continued fraction,

(iii) Integer part λ-continued fraction.

In each case we introduce ρ and φ into the expansions. In the following examples assume $\mathbb{P}_0$ has a λ-Farey subdivision $\{\infty, 0, 1/\sqrt{3}, \frac{1}{\sqrt{3}-1/\sqrt{3}}, \sqrt{3}-1/\sqrt{3}, \sqrt{3}\}$ and $\mathbf{I}_0 = [0,\infty]$ spans $\mathbb{P}_0$ on $\mathbb{R}_\infty$.

(i) The admissible λ-continued fraction of $\frac{12\sqrt{3}}{13}$ is given as $\lim\limits_{k\to\infty} \tau_\lambda\varphi\tau_\lambda^5\varphi\tau_\lambda\varphi\tau_\lambda\varphi g^k(\infty)$ where $g = \tau_\lambda^2\varphi(\tau_\lambda\varphi)^3$ is the generator. Re-writing the λ-continued fraction in terms of ρ and φ we have, $\lim\limits_{k\to\infty} \rho^2\varphi\rho\varphi\rho\varphi\rho\varphi\rho^4\varphi h^k(\infty) = \rho^2\varphi\rho\varphi\rho\varphi\rho\varphi\rho^4\varphi \lim\limits_{k\to\infty} h^k(\infty)$ where $h = \rho^5\varphi = \sigma\varphi = \rho^{-1}\varphi$.

We note that $h \in G_\lambda$ is a parabolic map with attracting fixed point 0. Thus, $\rho^4\varphi \lim_{k\to\infty} h^k(\infty) = \sigma^2\varphi \lim_{k\to\infty} h^k(\infty) = \sigma^2(\infty) = \sigma\varphi(\infty)$. So $\dfrac{12\sqrt{3}}{13}$ can be expressed as $\rho^2\varphi\rho\varphi\rho\varphi\rho\varphi\rho^{-1}\varphi(\infty)$.

(ii) The nearest λ-integer continued fraction of $\dfrac{12\sqrt{3}}{13}$ is given as $\tau_\lambda\varphi\tau_\lambda^4\varphi\tau_\lambda^{-1}\varphi(\infty)$. Introducing ρ and φ we see that this λ-integer continued fraction can be written as, $\rho^2\varphi\rho\varphi\rho\varphi\rho\varphi\sigma\rho^{-1}(\infty)$ as above.

(iii) The integer part λ-continued fraction of $\dfrac{12\sqrt{3}}{13}$ is given as $(\varphi\tau_\lambda^{-1}\varphi\tau_\lambda^{-1}\varphi\tau_\lambda^{-1}\varphi\tau_\lambda^{-2})^4\varphi(\infty)$. Again introducing ρ and φ we find that the expansion can be re-written as $\sigma^4\varphi\sigma^5\varphi\sigma^5\varphi\sigma^5\varphi\sigma\varphi(\infty) = \rho^2\varphi\rho\varphi\rho\varphi\rho\varphi\rho^{-1}\varphi(\infty)$.

Thus, the λ-continued fraction expansion for α derived from the cutting sequence of a geodesic ending at α is given as $\rho^2\varphi\rho\varphi\rho\varphi\rho\varphi\rho^{-1}\varphi(\infty)$. We note that when substituting ρ and φ into the λ-continued fraction above, (ii) and (iii) immediately yield the same finite expansion. In the case of (i), when we replace $\lim_{k\to\infty} h^k(\infty)$ with 0, the finite expansion for α which is the same as case of (ii) and (iii). We now write down the chain of nested intervals converging to α.

Let $\mathbf{I}_1 = \rho^2\varphi(\mathbf{I}_0) = [\rho^2\varphi(0); \rho^2\varphi(\infty)] = [\tau_\lambda\varphi\tau_\lambda(0); \tau_\lambda\varphi\tau_\lambda(\infty)] = \left[\sqrt{3} - \dfrac{1}{\sqrt{3}}; \sqrt{3}\right] = \left[\dfrac{2}{\sqrt{3}}; \sqrt{3}\right]$,

$\mathbf{I}_2 = \rho^2\varphi\rho\varphi(\mathbf{I}_0) = [\rho^2\varphi\rho\varphi(0); \rho^2\varphi\rho\varphi(\infty)] = \left[\sqrt{3} - \dfrac{1}{\sqrt{3}}; \sqrt{3}\right] = \left[\dfrac{5\sqrt{3}}{6}; \sqrt{3}\right]$,

$\mathbf{I}_3 = \rho^2\varphi\rho\varphi\rho\varphi(\mathbf{I}_0) = [\rho^2\varphi\rho\varphi\rho\varphi(0); \rho^2\varphi\rho\varphi\rho\varphi(\infty)] = \left[\sqrt{3} - \dfrac{1}{3\sqrt{3}}; \sqrt{3}\right] = \left[\dfrac{8\sqrt{3}}{9}; \sqrt{3}\right]$,

$\mathbf{I}_4 = \rho^2\varphi\rho\varphi\rho\varphi\rho\varphi(\mathbf{I}_0) = [\rho^2\varphi\rho\varphi\rho\varphi\rho\varphi(0); \rho^2\varphi\rho\varphi\rho\varphi\rho\varphi(\infty)] = \left[\sqrt{3} - \dfrac{1}{4\sqrt{3}}; \sqrt{3}\right] = \left[\dfrac{11\sqrt{3}}{12}; \sqrt{3}\right]$,

$\mathbf{I}_5 = \rho^2\varphi\rho\varphi\rho\varphi\rho\varphi\sigma\varphi(\mathbf{I}_0) = [\rho^2\varphi\rho\varphi\rho\varphi\rho\varphi\sigma\varphi(0); \rho^2\varphi\rho\varphi\rho\varphi\rho\varphi\sigma\varphi(\infty)] = \left[\sqrt{3} - \dfrac{1}{4\sqrt{3} - \dfrac{1}{-\sqrt{3}}}; \sqrt{3}\right] =$
$\left[\dfrac{11\sqrt{3}}{12}; \dfrac{12\sqrt{3}}{13}\right]$. Thus, $\mathbf{I}_0 \supset \mathbf{I}_1 \supset \mathbf{I}_2 \supset \mathbf{I}_3 \supset \mathbf{I}_4 \supset \mathbf{I}_5$ since
$[0, \infty] \supset \left[\dfrac{2\sqrt{3}}{3}; \sqrt{3}\right] \supset \left[\dfrac{5\sqrt{3}}{6}; \sqrt{3}\right] \supset \left[\dfrac{8\sqrt{3}}{9}; \sqrt{3}\right] \supset \left[\dfrac{11\sqrt{3}}{12}; \sqrt{3}\right] \supset \left[\dfrac{11\sqrt{3}}{12}; \dfrac{12\sqrt{3}}{13}\right]$.

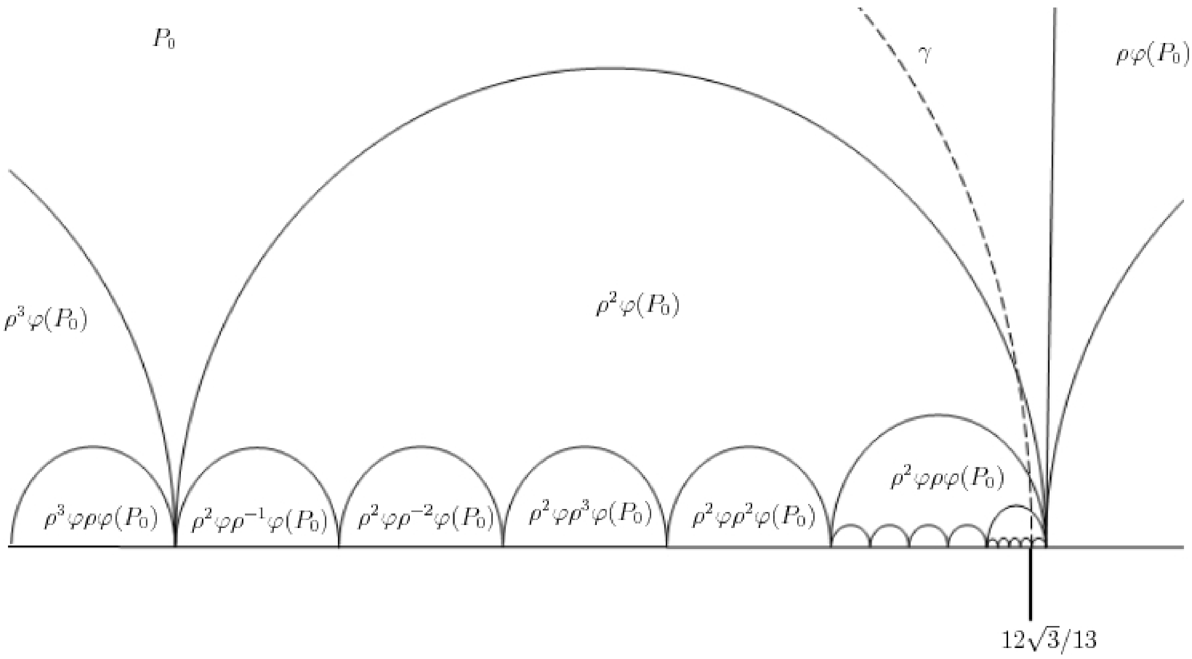

FIGURE 1. Geodesic cutting across the λ-Farey tessellation and ending at $\dfrac{12\sqrt{3}}{13}$.

CHAPTER 8

The equivalence of reduced and derived λ-continued fractions

In this chapter, following [**12**], we show that the reduced λ-continued fraction expansion for any $\alpha \in \mathbb{R}$ is equivalent to the λ-continued fraction for α derived from the cutting sequence. We note that the derived λ-continued fraction expansion has a strong geometric flavor, while the reduced λ-continued fraction expansion is heavily dependent on an algebraic definition. Thus, using the derived λ-continued fraction has greater geometric advantages. From Rosen [**25**], we see that a reduced λ-continued fraction converges. By Theorem 17, we have shown that a λ-continued fraction derived from the cutting sequence of a geodesic will converge to the end point of the geodesic.

We recall again, the definition of a reduced λ-continued fraction (Definition 31, page 60) given as:

Let $B(l-2) = \rho^{l-2}(\infty) = [\lambda, -1/\lambda, -1/\lambda, \cdots, -1/\lambda]$ *where* $q = 2l-1$, $l \geq 3$ *or* $q = 2l$, $l \geq 2$. *If* $\lambda = 2\cos(\frac{\pi}{q})$, $q \geq 4$, *the* λ*-continued fraction* $[r_0\lambda, \epsilon_1/r_1\lambda, \cdots,]$, *where* $\epsilon_i = \pm 1$, $r_i \in \mathbb{Z}^+$ *for* $i \geq 1$, r_0 *may be zero, is a reduced* λ*-continued fraction if and only if the following properties are satisfied:*

(i) The inequality $r_i\lambda + \epsilon_{i+1} < 1$ *is satisfied for no more than* $l-2$ *consecutive values of* i, $i = j, j+1, j+2, \cdots, j-l+1$, $j \geq 1$.

(ii) If $q = 2l-1$, *and if* $r_i\lambda + \epsilon_{i+1} < 1$ *is satisfied for* $l-2$ *consecutive values of* $i = j, j+1, j+2, \cdots, j+l+1$, *then* $r_{j+l-2} \geq 2$.

(iii) If $q = 2l-1$, *and if* $[B(l-2), -1/2\lambda, -1/B(l-2)]$ *occurs, the succeeding sign is plus.*

(iv) If $q = 2l - 1$, the λ-continued fraction terminates with $\epsilon/B(l-1)$, then $\epsilon = 1$.

(v) If some tail of a finite λ-continued fraction has the value $2/\lambda$, then $r\lambda + \dfrac{1}{2/\lambda} = (r+1)\lambda - \dfrac{1}{2/\lambda}$, and $r\lambda - \dfrac{2}{\lambda} = (r-1)\lambda + \dfrac{1}{2/\lambda}$. We shall choose the plus sign.

We note that since $\epsilon_i = \pm 1$ this definition involves the maps $\varphi(z) = -1/z$, $\phi(z) = 1/z$ and $\tau_\lambda^r(z) = z + \lambda r$ for $r \geq 1$.

We also recall that if we let γ be a geodesic in $\mathbb{H}^2$ cutting through $\mathbb{P}_0$ and ending at $\beta \in \mathbb{R}$. The geodesic γ yields a path in $\mathcal{G}$. Each vertex $\mathbb{P}_k$ on the path is given as $g(\mathbb{P}_0) = \rho^{r_0}\varphi\rho^{r_1}\varphi\cdots\varphi\rho^{r_k}\varphi(\mathbb{P}_0)$, $-l+1 \leq r_i \leq l-1$ $(q = 2l-1)$ and $-l \leq r_i \leq l-1$ $(q = 2l)$ for $i \geq 1$ and where only r_0 may be zero. If β is λ-rational then it is a cusp of a vertex on the graph, so there exist k such that $\beta = \rho^{r_0}\varphi\rho^{r_1}\varphi\cdots\varphi\rho^{r_k}\varphi(\infty)$. If β is non-λ-rational then $\beta = \lim_{k\to\infty} \rho^{r_0}\varphi\rho^{r_1}\varphi\cdots\varphi\rho^{r_k}\varphi(\infty)$. In either case the expansions are λ-continued fractions and can be written in terms of φ and τ_λ where $\rho = \tau_\lambda\varphi$.

DEFINITION 35. *The λ-continued fraction expansion for $\beta \in \mathbb{R}$ derived from the cutting sequence of γ, as described above, is called the derived λ-continued fraction expansion for β.*

We note that this derived expression is given in terms of $\varphi(z) = -1/z$ and $\rho^r(z)$ with $\rho(z) = \tau_\lambda\varphi(z)$ and $-l \leq r \leq l-1$ $(q = 2l)$ or $-l+1 \leq r \leq l-1$ $(q = 2l-1)$.

The following theorem involves converting the reduced λ-reduced continued fraction expansion to a derived λ-continued fraction expansion. In this conversion the following equalities will be used:

$\phi\rho^r = \sigma^r\phi$ where $\rho = \tau_\lambda\varphi$ and $\rho^{-1} = \sigma = \varphi\tau_\lambda^{-1}$

$\phi\tau_\lambda^r\phi = \varphi\tau_\lambda^{-r}\varphi$ and $\varphi\tau_\lambda^r\varphi = \phi\tau_\lambda^{-r}\phi$.

We note that Rosen [**25**], uses the term $B(t) = \rho^t(\infty)$ in his development of reduced λ-continued fractions. We have also noted previously (page 74) that a sequence $\varphi\rho^{r_1}\varphi\rho^{r_2}\varphi$ will collapse or reduce if and only if r_1 and r_2 are of the same sign.

THEOREM 18. [**12**] *Let $\alpha \in \mathbb{R}$. The derived λ-continued fraction expansion for α satisfies the conditions for a reduced λ-continued fraction of α. Similarly the reduced λ-continued fraction expansion of α can be converted to a derived λ-continued fraction expansion.*

Proof

Assume $\alpha \in \mathbb{R}$ is a non-rational real number with reduced λ-continued fraction expansion given in terms of partial quotients as $\alpha = \left[r_0\lambda, \frac{\epsilon_1}{r_1\lambda}, \frac{\epsilon_2}{r_2\lambda}, \cdots\right]$, [**25**], where $\epsilon_i = \pm 1$ and $r_i \in \mathbb{Z}^+$ for all $i \geq 1$ and $r_0 \in \mathbb{Z}$.

Property (i) for reduced λ-continued fraction states that the inequality $r_i\lambda + \epsilon_{i+1} < 1$ is satisfied for at most $l-2$ consecutive values of i where $q = 2l, l \geq 2$ or $q = 2l-1, l \geq 3$. It is easily seen that if $\epsilon_{i+1} = 1$ we have no solutions to the inequality since $r_i \in \mathbb{Z}^+$ and $1 < \lambda < 2$. Hence, the inequality states that $r_i = 1$ and $\epsilon_{i+1} = -1$ for at most $l-2$ consecutive values of i. Thus, in the reduced λ-continued fraction expansion of α we may have a sequence $B(t) = \rho^t(\infty) = \left[\lambda, \frac{-1}{\lambda}, \frac{-1}{\lambda}, \cdots, \frac{-1}{\lambda}\right]$ with $t \leq l-2$ partial quotients. Thus, in the expansion of α we may have a sequence $\tau_\lambda^{r_{j-1}}\varsigma_1\tau_\lambda\varphi\tau_\lambda\varphi\cdots\tau_\lambda\varphi\tau_\lambda^{r_{j+t}}\varsigma_2$ where $t \leq l-2$, $\varsigma_m(z) = \pm 1/z$ for $m = 1, 2$ and r_{j-1} and $r_{j+t} \in \mathbb{Z}^+ \cup \{0\}$. Replacing $\tau_\lambda\varphi$ with ρ we can rewrite the expansion as $\tau_\lambda^{r_{j-1}}\varsigma_1\rho^t\tau_\lambda^{r_{j+t}}\varsigma_2$. Since $\varsigma_m(z) = \pm 1/z$ for $m = 1, 2$, this expansion needs to be considered in the cases where $\varsigma_1 = \varphi$ and $\varsigma_1 = \phi$.

Case 1. Say $\varsigma_1 = \varphi$, then

$$
\begin{aligned}
\tau_\lambda^{r_{j-1}}\varsigma_1\rho^t\tau_\lambda^{r_{j+t}}\varsigma_2 &= \tau_\lambda^{r_{j-1}}\varphi\rho^t\tau_\lambda^{r_{j+t}}\varsigma_2 \\
&= \tau_\lambda^{r_{j-1}-1}(\tau_\lambda\varphi)\rho^t(\tau_\lambda\varphi)\varphi\tau_\lambda^{r_{j+t}-1}\varsigma_2 \text{ where } r_{j-1}-1 \geq 0 \text{ and } r_{j+t}-1 \geq 0 \\
&= \tau_\lambda^{r_{j-1}-1}\rho^{t+2}\varphi\tau_\lambda^{r_{j+t}-1}\varsigma_2 \text{ where } t+2 \leq l.
\end{aligned}
$$

Hence, we see in converting a reduced λ-continued fraction to an expansion in terms of ρ and φ, that if ρ^{t_0} occurs then $t_0 \leq l$. If $t_0 = l$ then $\rho^{t_0} = \rho^l = \rho^{-l}$ for $q = 2l$ and $\rho^l = \rho^{-l+1}$ for $q = 2l-1$. So $-l \leq t_0 \leq l-1$ for $q = 2l$ and $-l+1 \leq t_0 \leq l-1$ for $q = 2l-1$.

Case 2. Say $\varsigma_1 = \phi$, then

$$
\begin{aligned}
\tau_\lambda^{r_{j-1}}\varsigma_1\rho^t\tau_\lambda^{r_{j+t}}\varsigma_2 &= \tau_\lambda^{r_{j-1}}\phi\rho^t(\tau_\lambda\varphi)\varphi\tau_\lambda^{r_{j+t}-1}\varsigma_2 \\
&= \tau_\lambda^{r_{j-1}}\rho^{-(t+1)}\phi\varphi\tau_\lambda^{r_{j+t}-1}\varsigma_2 \\
&= \tau_\lambda^{r_{j-1}+1}\varphi\varphi\tau_\lambda^{-1}\rho^{-(t+1)}\phi\varphi\tau_\lambda^{r_{j+t}-1}\varsigma_2,\ r_{j+t}-1 \geq 0,\ r_{j-1}+1 \geq 2 \\
&= \tau_\lambda^{r_{j-1}+1}\varphi\rho^{-(t+2)}(\varphi\tau_\lambda^{-(r_{j+t}-1)}\varphi)\phi\varsigma_2 \\
&= \tau_\lambda^{r_{j-1}+1}\varphi\rho^{-(t+1)}\varphi\tau_\lambda^{-(r_{j+t})}\varphi\phi\varsigma_2 \text{ where } -(t+1) \geq -(l-1) = -l+1.
\end{aligned}
$$

Hence, in converting a reduced λ-continued fraction expansion to a derived λ-continued fraction expansion we may have terms ρ^{t_0} where $-l \leq t_0 \leq l-1$ for $q = 2l$ and $-l+1 \leq t_0 \leq l-1$ for $q = 2l-1$. We note that if $r_i\lambda + \epsilon_{i+1} < 1$ is satisfied for exactly $l-2$ consecutive values of i then t_0 may be $l-1$ and must be greater or equal to $-l+1$ and $-l$ for q-odd and even respectively. Thus, each reduced λ-continued fraction expansion α may be converted to a derived λ-continued fraction expansion satisfying the restriction of $-l \leq r_i \leq l-1$ for $q = 2l$ and $-l+1 \leq r_i \leq l-1$ for $q = 2l-1$.

Conversely, let α be a non rational real with a sequence $\cdots\varphi\rho^{r_1}\varphi\rho^{r_2}\varphi\rho^{r_3}\varphi\cdots$ in its derived λ-continued fraction expansion where $r_i \neq 0$, $-l \leq r_i \leq l-1$ for $q = 2l$ and $-l+1 \leq r_i \leq l-1$

for $q = 2l - 1$. We show that α will have a reduced λ-continued fraction expansion when we make the conversion from φ and ρ to $\tau_\lambda, \varphi, \phi$ and $r_i \geq 1$.

Let $r_2 = l - 1$ and consider the following cases;

(i) $r_1 \geq 1$ and $r_3 \geq 1$

(ii) $r_1 \leq -1$ and $r_3 \geq 1$

(iii) $r_1 \leq -1$ and $r_3 \leq -1$

(iv) $r_1 \geq 1$ and $r_3 \leq -1$

Case (i): The sequence $\cdots \varphi\rho^{r_1}\varphi\rho^{r_2}\varphi\rho^{r_3}\varphi\cdots$ will be written as

$$\cdots \varphi\rho^{r_1}\varphi\rho^{l-1}\varphi\rho^{r_3}\varphi\cdots = \cdots\varphi\rho^{r_1}\varphi\rho^{l-2}(\tau_\lambda\varphi)\varphi(\tau_\lambda\varphi)\rho^{r_3-1}\varphi\cdots = \cdots\varphi\rho^{r_1}\varphi\rho^{l-2}\tau_\lambda^2\varphi\rho^{r_3-1}\varphi\cdots,$$

$r_3 - 1 \geq 0$ and $r_1 \geq 1$.

Case (ii): Let $r_1 \leq -1$ and $r_3 \geq 1$. Then the sequence may be written as;

$$\begin{aligned}
\cdots \varphi\rho^{r_1}\varphi\rho^{l-1}\varphi\rho^{r_3}\varphi\cdots &= \cdots\varphi\rho^{r_1+1}(\varphi\tau_\lambda^{-1})\varphi\rho^{l-2}(\tau_\lambda\varphi)\varphi(\tau_\lambda\varphi)\rho^{r_3-1}\varphi\cdots \text{ where } r_3 \geq 1, r_1 \leq -1\\
&= \cdots\varphi\rho^{r_1+1}(\varphi\tau_\lambda^{-1})\varphi\rho^{l-2}\tau_\lambda^2\varphi\rho^{r_3-1}\varphi\cdots\\
&= \cdots\varphi\rho^{r_1+1}\phi\tau_\lambda\phi\rho^{l-2}\tau_\lambda^2\varphi\rho^{r_3-1}\varphi\cdots\\
&= \cdots\phi\varphi\rho^{-(r_1+1)}\tau_\lambda\phi\rho^{l-2}\tau_\lambda^2\varphi\rho^{r_3-1}\varphi\cdots \text{ where } r_3-1 \geq 0 \text{ and } -(r_1+1) \geq 0.
\end{aligned}$$

Case (iii): Let $r_1 \leq -1$ and $r_3 \leq -1$. If $q = 2l - 1$, the sequence may be rewritten as

$$\begin{aligned}
\cdots \varphi\rho^{r_1}\varphi\rho^{l-1}\varphi\rho^{r_3}\varphi\cdots &= \cdots\varphi\rho^{r_1}\varphi\rho^{l-1-(2l-1)}\varphi\rho^{r_3}\varphi\cdots\\
&= \cdots\varphi\rho^{r_1+1}(\varphi\tau_\lambda^{-1})\varphi\rho^{-l}\varphi(\varphi\tau_\lambda^{-1})\rho^{r_2+1}\varphi\cdots, \text{ where } r_1+1 \leq 0 \text{ and } r_2+1 \leq 0.\\
&= \cdots\varphi\rho^{r_1+1}(\varphi\tau_\lambda^{-1}\varphi\varphi\tau_\lambda^{-1})\sigma^{l-2}(\varphi\tau_\lambda^{-1}\varphi\varphi\tau_\lambda^{-1})\rho^{r_3+1}\varphi\cdots\\
&= \cdots\varphi\rho^{r_1+1}(\varphi\tau_\lambda^{-2}\sigma^{l-2}\varphi\tau_\lambda^{-2})\rho^{r_3+1}\varphi\cdots\\
&= \cdots\varphi\rho^{r_1+1}\varphi\tau_\lambda^{-2}\sigma^{l-2}\phi\tau_\lambda^2\phi\varphi\rho^{r_3+1}\varphi\cdots,\ \varphi\tau_\lambda^{-2}\varphi = \phi\tau_\lambda^2\phi\\
&= \cdots\varphi\rho^{r_1+1}\varphi\tau_\lambda^{-2}\varphi\phi\rho^{l-2}\tau_\lambda^{-2}\varphi\rho^{-(r_3+1)}\varphi\phi\cdots
\end{aligned}$$

$$= \cdots \varphi\rho^{r_1+1}\varphi\varphi\phi\tau_\lambda^2\varphi\rho^{l-2}\tau_\lambda^2\varphi\rho^{-(r_3+1)}\varphi\phi\cdots$$
$$= \cdots \phi\varphi\rho^{-(r_1+1)}\tau_\lambda^2\varphi\rho^{l-2}\tau_\lambda^2\varphi\rho^{-(r_3+1)}\varphi\phi\cdots$$

where $-(r_1+1), -(r_3+1) \geq 0$.

We note that in the Cases (i), (ii) and (iii) the appearance of ρ^{l-2} occurs and is followed by τ_λ^2 when $q = 2l-1$, thus meeting Property (i) and (ii) of Rosen's reduced λ-continued fraction expansion definition.

Case (iv). This case results in a situation where ρ^{l-2} does not appear and hence if the following term is not τ_λ^2 then there is no contradiction to Property (ii) of Rosen's reduced λ-continued fraction expansion definition.

The expansion $\cdots\varphi\rho^{r_1}\varphi\rho^{l-1}\varphi\rho^{r_3}\varphi\cdots$ can be re-written as:

$$\cdots\varphi\rho^{r_1}\varphi\rho^{l-1}\varphi\rho^{r_3}\varphi\cdots = \cdots\varphi\rho^{r_1-1}\rho\varphi\rho^{l-1}\varphi\rho^{-1}\rho^{r_3+1}\varphi\cdots \text{ where } r_3+1 \leq 0 \text{ and } r_1-1 \geq 0,$$
$$= \cdots\varphi\rho^{r_1-1}(\tau_\lambda\varphi)\varphi(\tau_\lambda\varphi)\rho^{l-2}\varphi(\varphi\tau_\lambda^{-1})\rho^{r_3+1}\varphi\cdots$$
$$= \cdots\varphi\rho^{r_1-1}\tau_\lambda^2\varphi\rho^{l-2}\tau_\lambda^{-1}\varphi\varphi\rho^{r_3+1}\varphi\cdots$$
$$= \cdots\varphi\rho^{r_1-1}\tau_\lambda^2\varphi\rho^{l-3}(\tau_\lambda\varphi\tau_\lambda^{-1}\varphi)\varphi\rho^{r_3+1}\varphi\cdots$$
$$= \cdots\varphi\rho^{r_1-1}\tau_\lambda^2\varphi\rho^{l-3}\tau_\lambda\phi\tau_\lambda\phi(\varphi\rho^{r_3+1}\varphi)\cdots$$
$$= \cdots\varphi\rho^{r_1-1}\tau_\lambda^2\varphi\rho^{l-3}\tau_\lambda\phi\tau_\lambda\varphi\rho^{-(r_3+1)}\varphi\phi\cdots \text{ where } r_1-1 \geq 0,\ -(r_3+1) \geq 0$$

Thus, sequence $\rho\varphi\rho^{l-1}\varphi\rho^{-1}$ leads to a sequence $\rho^{l-3}\tau_\lambda\phi\tau_\lambda\phi$ and does not contradict Property (ii) of the reduced λ-continued fraction expansion definition.

If we now assume that $r_2 = -l+1$ with $q = 2l-1$ then a sequence $\cdots\varphi\rho^{r_1}\varphi\rho^{r_2}\varphi\rho^{r_3}\varphi\cdots$ can be rewritten as:

$$\cdots\varphi\rho^{r_1}\varphi\rho^{-l+1}\varphi\rho^{r_3}\varphi\cdots = \cdots\varphi\rho^{r_1}\varphi\sigma^{l-1}\varphi\rho^{r_3}\varphi\cdots$$
$$= \cdots\varphi\rho^{r_1}\varphi(\phi\rho\phi)^{l-1}\varphi\rho^{r_3}\varphi\cdots$$

$$= \cdots \varphi\rho^{r_1}\varphi(\phi\rho^{l-1}\phi)\varphi\rho^{r_3}\varphi\cdots$$

$$= \cdots \phi\{\varphi\rho^{s_1}\varphi\rho^{l-1}\varphi\rho^{s_2}\varphi\}\phi\cdots$$

where $-l+1 \leq s_1, s_2 \leq l-1$, $-r_i = s_i \neq 0$.

Using the process for $r_2 = l-1$ (above), we establish that Property (i) and (ii) of reduced λ-continued fraction holds, when $q = 2l-1$ (odd).

Property (iii) for reduced λ-continued fraction states that if $q = 2l-1$ and if $[B(l-2), -1/2\lambda, -1/B(l-2)]$, is in the sequence then the succeeding sign is plus. That is, if we have a sequence $(\rho^{l-2}\varphi)\varphi\tau_\lambda^2\varphi(\rho^{l-2}) = \rho^{l-2}\tau_\lambda\varphi\varphi\tau_\lambda\varphi\rho^{l-2}\varphi = (\rho^{l-1}\varphi\rho^{l-1}\varphi)$ in the expansion, then the subsequent map in the expansion must be ϕ.

Let the following term of the expansion be $\rho^r\varphi$ with $r \geq 1$ or $r \leq -1$.

If $r = -1$ then,

$$\begin{aligned}
\rho^{l-1}\varphi\rho^{l-1}\varphi\rho^{-1}\varphi &= (\rho^{l-1}\varphi\rho^{l-1}\varphi)(\varphi\tau_\lambda^{-1})\varphi \\
&= (\rho^{l-2}(\tau_\lambda\varphi)\varphi(\tau_\lambda\varphi)\rho^{l-2}\varphi)(\varphi\tau_\lambda^{-1})\varphi \\
&= \rho^{l-2}\tau_\lambda^2\varphi\rho^{l-2}\varphi\phi\tau_\lambda\phi \\
&= (\rho^{l-2}\varphi)(\varphi\tau_\lambda^2)(\varphi\rho^{l-2}\varphi)\phi\tau_\lambda\phi
\end{aligned}$$

so $[B(l-2), -1/2\lambda, -1/B(l-2)]$ is followed by a plus sign.

If $r = 1$ then,

$$\begin{aligned}
(\rho^{l-1}\varphi\rho^{l-1}\varphi)\rho\varphi &= (\rho^{l-2}(\tau_\lambda\varphi)\varphi(\tau_\lambda\varphi)\rho^{l-2}\varphi)(\tau_\lambda\varphi)\varphi \\
&= \rho^{l-2}\tau_\lambda^2\varphi\rho^{l-2}\varphi\tau_\lambda \\
&= \rho^{l-2}\tau_\lambda^2\varphi\rho^{l-3}(\tau_\lambda\varphi)\varphi\tau_\lambda \\
&= \rho^{l-2}\tau_\lambda^2\varphi\rho^{l-3}\tau_\lambda^2
\end{aligned}$$

$$= (\rho^{l-2}\varphi)(\varphi\tau_\lambda^2)(\varphi\rho^{l-3}\varphi)\varphi\tau_\lambda^2$$

where $(\rho^{l-2}\varphi)(\varphi\tau_\lambda^2)(\varphi\rho^{l-3}\varphi)$ is equivalent to $[B(l-2), -1/2\lambda, -1/B(l-3)]$ in the sequence. So this case does not satisfy the given requirements.

We notice if we have $[B(l-2), -1/2\lambda, -1/B(l-2)]$ in the sequence and it is followed by a plus sign then we have:

$$\begin{aligned}
(\rho^{l-2}\varphi)(\varphi\tau_\lambda^2)(\varphi\rho^{l-2}\varphi)\phi\tau_\lambda^r &= \rho^{l-2}(\tau_\lambda\varphi\varphi\tau_\lambda)\varphi\rho^{l-2}\varphi\phi\tau_\lambda^r,\ r \geq 1 \\
&= \rho^{l-1}\varphi\rho^{l-1}\varphi\phi\tau_\lambda^r \\
&= \rho^{l-1}\varphi\rho^{l-1}\varphi(\varphi\tau_\lambda^{-r}\varphi\phi) \\
&= \rho^{l-1}\varphi\rho^{l-1}(\varphi\sigma)^r\varphi\phi,\ \sigma = \varphi\tau_\lambda^{-1} \text{ therefore } \varphi\sigma = \tau_\lambda^{-1} \\
&= \rho^{l-1}\varphi\rho^{l-1}\varphi\sigma\varphi\sigma(\varphi\sigma)^{r-2}\varphi\phi \\
&= \rho^{l-1}\varphi\rho^{l-1}\varphi\rho^{-1}\varphi\rho^{-1}(\varphi\sigma)^{r-1}\varphi\phi,\ \rho^{-1} = \sigma.
\end{aligned}$$

So we see that if $[B(l-2), -1/2\lambda, -1/B(l-2)]$ in the sequence is followed by a plus sign then the equivalently $(\rho^{l-1}\varphi\rho^{l-1}\varphi)$ must be followed by ρ^{-1}. Thus, Property (iii) of reduced λ-continued fraction expansion holds.

Property (iv) of reduced λ-continued fraction states that the expansion terminates with at most a block $B(l-2)$ or $\rho^{l-2}(\infty)$. Consider a cusp α on the λ-Farey graph with $\alpha = \rho^{r_0}\varphi\cdots\varphi\rho^{r_k}\varphi(\infty)$.

Let $r_k = l-1$. Then $\alpha = \rho^{r_0}\varphi\cdots\varphi\rho^{l-1}\varphi(\infty) = \rho^{r_0}\varphi\cdots\varphi\rho^{l-2}(\tau_\lambda\varphi\varphi(\infty)) = \rho^{r_0}\varphi\cdots\varphi\rho^{l-2}(\infty)$, as required.

Let $r_k = -l+1$, then

$$\begin{aligned}
\alpha = \rho^{r_0}\varphi\cdots\varphi\rho^{-l+1}\varphi(\infty) &= \rho^{r_0}\varphi\cdots\varphi\rho^{-l+1}\phi(\infty), \varphi(\infty) = \phi(\infty) \\
&= \rho^{r_0}\varphi\cdots\varphi\phi\rho^{l-1}(\infty)
\end{aligned}$$

$$= \rho^{r_0}\varphi \cdots \phi\varphi\rho\rho^{l-2}(\infty)$$
$$= \rho^{r_0}\varphi \cdots \rho^{r_{k-1}}\phi\varphi(\tau_\lambda\varphi)\rho^{l-2}(\infty)$$

If $r_k = -l+1$ and $r_{k-1} \leq -1$, then

$$\begin{aligned}\alpha &= \rho^{r_0}\varphi \cdots \phi\rho^{-(r_{k-1})}\varphi(\tau_\lambda\varphi)\rho^{l-2}(\infty)\\ &= \rho^{r_0}\varphi \cdots \phi\rho^{-(r_{k-1}-1)}(\tau_\lambda\varphi)\varphi(\tau_\lambda\varphi)\rho^{l-2}(\infty),\ -r_{k-1}-1 \geq 0\\ &= \rho^{r_0}\varphi \cdots \phi\rho^{-(r_{k-1}-1)}\tau_\lambda^2\varphi\rho^{l-2}(\infty), \text{ as required.}\end{aligned}$$

If $r_k = -l+1$ and $r_{k-1} \geq 1$, then.

$$\begin{aligned}\alpha &= \rho^{r_0}\varphi \cdots \rho^{r_{k-1}}\varphi\rho^{-l+1}\varphi(\infty)\\ &= \rho^{r_0}\varphi \cdots \rho^{r_{k-1}}\varphi\rho^{-l+1+2l-1}\varphi(\infty),\ \rho^{-l+1+2l-1} = \rho^l\\ &= \rho^{r_0}\varphi \cdots \rho^{r_{k-1}}\varphi\rho\rho^{l-2}(\rho\varphi(\infty))\\ &= \rho^{r_0}\varphi \cdots \rho^{r_{k-1}-1}(\rho\varphi\rho)\rho^{l-2}(\infty)\\ &= \rho^{r_0}\varphi \cdots \rho^{r_{k-1}-1}\tau_\lambda^2\varphi\rho^{l-2}(\infty),\ r_{k-1}-1 \geq 0, \text{ as required.}\end{aligned}$$

Thus, Property (iv) of a reduced λ-continued fraction is satisfied by a derived λ-continued fraction expansion.

Property (v) of a reduced λ-continued fraction states that if the tail of the λ-continued fraction is $2/\lambda$ then $r\lambda + \dfrac{1}{2/\lambda} = (r+1)\lambda - \dfrac{1}{2/\lambda}$ and $r\lambda - \dfrac{1}{2/\lambda} = (r-1)\lambda + \dfrac{1}{2/\lambda}$. That is, the expansion $\tau_\lambda^r\phi(2/\lambda) = \tau_\lambda^{r+1}\varphi(2/\lambda)$ and $\tau_\lambda^r\varphi(2/\lambda) = \tau_\lambda^{r-1}\phi(2/\lambda)$ or $\tau_\lambda^r(\lambda/2) = \tau_\lambda^{r+1}\varphi(2/\lambda)$ and $\tau_\lambda^r\varphi(2/\lambda) = \tau_\lambda^{r-1}(\lambda/2)$. That is, the ϕ and φ can easily be interchanged or the tail can be replaced with $\lambda/2$.

Assume $\rho^{r_0}\varphi\rho^{r_1}\varphi\cdots\varphi\rho^{r_{k-1}}\varphi(2/\lambda)$. Then since $2/\lambda = \varphi\tau_\lambda^{-1}\phi(2/\lambda)$, we have $\varphi(2/\lambda) = \tau_\lambda^{-1}\phi(2/\lambda)$. Thus,

$$\begin{aligned}
\rho^{r_{k-1}}\varphi(2/\lambda) &= \rho^{r_{k-1}}\varphi\varphi\tau_{\lambda}^{-1}\phi(2/\lambda), \varphi\varphi = 1_{map} \\
&= \rho^{r_{k-1}}\varphi\sigma\phi(2/\lambda). \\
&= \rho^{r_{k-1}}\varphi\rho^{-1}\phi(2/\lambda). \\
&= \rho^{r_{k-1}}\varphi\rho^{-1}(\lambda/2).
\end{aligned}$$

That is, $2/\lambda$ is replaced by $\rho^{-1}(\lambda/2)$. Thus, the tail of $2/\lambda$ can be replaced with a tail of $\lambda/2$. Property (v) of a reduced λ-continued fraction is also satisfied by a derived λ-continued fraction. Thus, in general we have an equivalence of reduced λ-continued fraction and a derived λ-continued fraction. $\square$

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