The Adomian Decomposition Method Applied to Blood Flow through Arteries in the presence of a Magnetic Field

Tendani Patrick Ungani
Student Number: **688768**

Supervisor: Professor Shirley Abelman

School of Computational and Applied Mathematics
University of Witwatersrand.
Johannesburg, South Africa

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Declaration

While I may have benefited from discussions with other people, I declare that this dissertation is my own, unaided work, except where appropriately documented acknowledgments and references to the literature are included. It is being submitted as fulfillment for the Degree for Master of Science to the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination to any other University.

Tendani Patrick Ungani

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Abstract

The Adomian decomposition method is an effective procedure for the analytical solution of a wide class of dynamical systems without linearization or weak nonlinearity assumptions, closure approximations, perturbation theory, or restrictive assumptions on stochasticity. Our aim here is to apply the Adomian decomposition method to steady two-dimensional blood flow through a constricted artery in the presence of a uniform transverse magnetic field. Blood flow is the study of measuring blood pressure and determining flow through arteries. Blood flow is assumed to be Newtonian and is governed by the equation of continuity and the momentum balanced equation (which are known as the Navier-Stokes equations). This model is consistent with the principles of ferro-hydrodynamics and magnetohydrodynamics and takes into account both magnetization and electrical conductivity of blood. We apply the Adomian decomposition method to the equations governing blood flow through arteries in the presence of an external transverse magnetic field. The results show that the effect of a uniform external transverse magnetic field applied to blood flow through arteries favors the physiological condition of blood. The motion of blood in stenosed arteries can be regulated by applying a magnetic field externally and increasing/decreasing the intensity of the applied field.

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To my younger brother Phindulo, rest in peace. I hope you have now found peace at last.

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Chapter 1

Introductory Remarks

1.1 Research Objectives

The objectives of this study are to:

1. demonstrate the application of the Adomian decomposition method (ADM) and the modified ADM with illustrative examples

2. analyze blood flow mathematically by using equation of continuity and the momentum balance equation (these equations are known as the Navier-Stokes equations)

3. approximate blood flow through arteries in the presence of a uniform transverse magnetic field using the ADM

4. investigate the effects of homogeneous magnetic field on the blood flow characteristics by using the Frobenius method

1.2 Outline of Dissertation

Chapter 1, provides the basic definitions and introductory concepts. We present the origin and the advantages of the ADM based on the literature. The ADM is a significant, powerful method, which provides an efficient means for the analytical and numerical solution of differential equations, which model real-world physical applications. A considerable amount of research work has been invested recently in applying this method to a wide class of linear and nonlinear ordinary differential equations, partial differential equations and integral differential equations as well. We outline applications of both the ADM and the modified ADM. We present a background introduction of blood flow through an artery in the presence of a uniform external transverse magnetic field.

In chapter 2, we present mathematical background of the ADM and modified ADM. We demonstrate the application of both the ADM and modified ADM by illustrative examples. The ADM and the improvements made by the noise terms phenomenon and the modified ADM are reliable and effective techniques with promising results. Adomian polynomials are key in solving nonlinear equations and hence we present both the Adomian and alternative algorithms for calculating Adomian polynomials. We also present a useful tool which accelerates the convergence of the ADM, namely the noise terms phenomenon. We present the convergence analysis of both the ADM and modified ADM. In the literature, the effects of different types of magnetic field on flow characteristics in tubes of uniform circular cross-section have been studied; but the corresponding problem in the presence of a constriction is more important from the physiological point of view.

In chapter 3, blood flow like any other fluids (such as air, water, oil, etc.) may be analyzed mathematically by the use of two equations. The first, often referred to as the continuity equation, requires that the mass of fluid entering a fixed control volume either leaves that volume or accumulates within it. It is thus a mass balance requirement posed in mathematical form, and is a scalar equation. The second, is the momentum equation, and may be thought of as a momentum balance. The Navier-Stokes equations are the fluid dynamics equivalent of Newton's second law, force equals mass times acceleration and they are of crucial importance in fluid dynamics. We consider blood as an incompressible Newtonian fluid with uniform viscosity and present the Navier-Stokes equations in vector form and in cylindrical polar coordinates to be the equations governing blood flow through arteries in the presence of a magnetic field.

The novelty in chapter 4 is the extension of the work of ([21], [31]) to include a three-term approximation to the solution of the stream function. The expressions for a two-term and a three-term approximation to the solution of the stream function, axial velocity component and wall shear stress are obtained. We apply the ADM to the equation governing blood flow through a constricted artery in the presence of an external transverse magnetic field which is applied uniformly. Blood flowing through the tube is assumed to be Newtonian in character. The numerical solutions of the wall shear stress for different values of the Reynolds number R_e and the Hartmann number M are shown graphically.

In chapter 5, we investigate the effect of an externally applied homogeneous magnetic field on the flow characteristics in a single constricted blood vessel using the Frobenius method.

Chapter 6 states concluding remarks and recommendations about possible future research work.

1.3 Introduction

At the beginning of the 1980's, an American Applied Mathematician named George Adomian (1922-1996) presented a powerful decomposition methodology for practical solution of linear or nonlinear and deterministic or stochastic operator equations, including ordinary differential equations, partial differential equations, integral equations, etc. Since then, the method has been known as the Adomian decomposition method or in short the ADM. The ADM is a significant, powerful method, which provides an efficient means for the analytical and numerical solution of differential equations, which model real-world physical applications. A considerable amount of research work has been invested recently in applying this method to a wide class of linear and nonlinear ordinary differential equations, partial differential equations and integral differential equations as well [4], [17]-[29].

Many phenomena in mathematics, engineering, physics, chemistry, etc., can be described very successfully by models using mathematical tools from fractional calculus, i.e. the theory of derivatives and integrals of fractional (non-integer) order. Most fractional differential equations do not have exact analytical solutions, so that approximation methods and numerical techniques must be used. Most of these models depict problems which are nonlinear and nonlinear phenomena play a crucial role in applied mathematics, physics, mechanics, biology, etc., and, therefore, explicit solutions to the nonlinear equations are of fundamental importance to preserve the actual physical character of the problem and to understand deeply the described process. It is very important to apply efficient methods such as the ADM to solve these nonlinear problems.

Adomian ([1]-[4]) and others have successfully applied the ADM to algebraic equations, ordinary, partial, delay, and non-integer order or fractional differential equations ([5], [6], [7]) for a wide class of nonlinearities, including polynomial, exponential, trigonometric, hyperbolic, composite, negative power, radical and even decimal power nonlinearities. The ADM solves nonlinear differential equations for any analytic nonlinearity. The ADM allows one to solve nonlinear differential equations without having to appeal to the decidedly questionable practices of perturbation or linearization.

The solution algorithm yields a rapidly convergent sequence of analytic approximants, which are readily computable, without recourse to linearization, perturbation, discretization, etc. The solution is viewed by Adomian as a decomposition of the pre-existent, unique, analytic function, which identically satisfies the mathematical statement under consideration into components to be determined or resolved by recursion, rather than as a formal series, i.e. some suspect expansion series of uncertain convergence, where the convergence question is held in suspense until after the series coefficients or such other parameters are actually computed. Of course, the notion of decomposition presupposes the existence and uniqueness of a mathematical construct, since decomposition is the partitioning of the solution into basic components or parts, which represents a specific performance measure for some aspect of a natural phenomenon or man-made device.

Decomposition then may be viewed as a rearrangement, depending upon the particular recursive algorithm, of the terms of the uniformly convergent series of an analytic function. It is definitely not an expansion of an arbitrary or pathological function. Decomposition is the rearrangement of the analytic function, which is the solution of a mathematical statement modeling a physical system. It is a rapidly convergent series. Furthermore, Adomian et al. ([14], [15]) developed a new approach to numerical integration algorithms based upon the ADM [16]. Also the ADM does not require very large computations inherent in discretization methods such as finite differences.

Since the ADM solves nonlinear problems rather than linearizing them, the resulting solutions are physically more realistic. The objective of the decomposition method is to make possible physically realistic solutions of complex systems without the usual modeling and solution compromises to achieve tractability. A bonus is that it essentially combines the fields of ordinary and partial differential equations. The prior art in mathematical analysis as seen in the literature necessarily relies on such limiting procedures. Thus it may well be said that physics is usually perturbative theory and mathematics is essentially linear operator theory. Of course there are some methods of solving nonlinear equations, but not general methods. For example, clever transformation of variables sometimes results in a linear equation: however, this rarely works.

The ADM provides an analytical solution in the form of an infinite power series. From a practical perspective, it is necessary to evaluate this analytical solution, and to obtain numerical values from the infinite power series. An advantage of the decomposition method is that it can provide analytical approximation to a rather wide class of nonlinear (and stochastic) equations without linearization, perturbation, closure approximations, or discretization methods which can result in very large numerical computation. Common analytical procedures, to solve nonlinear differential equations, linearize the system or assume that the nonlinearities are relatively small, transforming the physical problem into a purely mathematical one with an available solution. This procedure may change the real solution of the mathematical model which represents the physical reality. Usually, the numerical methods are based on discretization techniques, and permit only to calculate the approximate solutions for some values of time and space variables, which have the disadvantage of causing overlooking for some important phenomena occurring in very small time and space intervals, such as chaos and bifurcations. Closed-form analytical solutions are considered ideal when possible. However, they may necessitate changing the actual or real-life problem to a more tractable mathematical problem. Except for a small class of equations in which clever transformations can result in linear equations, it becomes necessary to resort to linearization or statistical linearization techniques, or assumptions of "weak nonlinearity," etc. What we obtain then is a solution of the simpler mathematical problem. The resulting solution can deviate significantly from the solution of the actual problem; nonlinear systems can be extremely sensitive to small changes. These small changes can occur because of inherent stochastic effects or computer errors; the resulting solutions (especially in strongly nonlinear equations) can show violent, erratic (or "chaotic") behavior. Of course, it is clear that considerable progress has been made with the generally used procedures, and, in many problems, these methods remain adequate. Thus, in problems which are close to linear, or where perturbation theory is adequate, excellent solutions are obtained.

In recent years, the decomposition method has emerged as an alternative method for solving a wide range of problems whose mathematical models involve algebraic, differential, integral, integro-differential, higher-order ordinary differential equations (ODEs), and partial differential equations (PDEs) and systems, arising from Physics, Chemistry, Biology, Engineering, etc, subject to boundary or initial conditions. For nonlinear models, the ADM has shown reliable results in analytical approximation that converges very rapidly.

1.4 Advantages of the Adomian Decomposition Method

The ADM provides an analytical approximation to a rather wide class of nonlinear (and stochastic) equations without linearization, perturbation, closure approximations, or discretization methods which can result in very large numerical computation.

An advantage, other than the fact that problems are considered more realistically than by customary constraints, is that solutions are not obtained here by discretized methods: solutions are continuous and computationally much more efficient.

The ADM can be applied directly for all types of differential and integral equations, linear or nonlinear, homogeneous or inhomogeneous, with constant coefficients or with variable coefficients.

The ADM does not require discretization of the variables. Hence, the solution is not affected by computation round off errors and the necessity of large computer memory.

ADM is capable of greatly reducing the size of computation work while still maintaining high accuracy of the numerical solution.

The advantage of this method is the avoidance of simplifications and restrictions, which change the nonlinear problem to a mathematically tractable one, whose solution is not consistent with the physical solution.

The decomposition method has a significant advantage in that it provides the solution in a rapid convergent series with elegantly computable components.

The ADM uses the linear composite operator coefficients of the Adomian integral equation to construct the solution. The ADM is a constructive method.

Furthermore, the ADM is preferred for solving stochastic differential equations (Adomian, [8]-[13]) for the statistical measures, e.g. the expectation, correlation and so forth, of solution processes as pioneered by Adomian, when dealing with physically realistic applications without a priori assumptions of stationarity, ergodicity, white noise, Gaussian processes, etc.

The decomposition solution is also an approximation, but one which does not change the problem. All modeling is an approximation and this methodology approximates (accurately and in an easily computable manner) the solution of the real nonlinear and possibly stochastic problem rather than a grossly simplified linearized or averaged problem. It provides analytical expressions, explicitly displaying the expected nonlinear dependence versus any of the physical parameters of the problem.

The ADM rigorously solves practical problems, and goes beyond vicious circles appearing in the study of different schemes.

The ADM can be carried out easily and the non-numerical results obtained are very useful.

Of paramount interest is that the ADM provides a universal approach to real-world problems at the frontiers of science, including both initial and boundary value problems. Thus, Adomian's breakthrough providing a unified theory of differential equations in mathematics is without precedent! Although his work has been regarded as controversial due to a sometimes-evident disregard of rigorous terminology, Adomian was the first to state necessary conditions for convergence of the ADM. For example, in Adomian [8] he discusses the notion of mean-square convergence for analytic correlation functions as solution of stochastic differential equations [16].

1.5 Application of the Adomian Decomposition Method

Applications of the ADM and its modifications have caught the attention of several researchers, so it could be possible to solve a great diversity of both ordinary as well as partial linear and nonlinear differential equations, deterministic and also stochastic [4], [17]-[29]. The ADM has extensive applications in fields such as physics, biology, chemistry, and engineering.

Adomian [1] applied the ADM to the Navier-Stokes equations in a Cartesian coordinate system by studying the flow of viscous incompressible continuous fluids. Chiu and Chen [30] applied the ADM in solving convective longitudinal fin problems with variable thermal conductivity, in which the nonlinear problems were treated in a manner similar to linear problems. Haldar [31] applied the ADM to approximate the analytical solution of the Navier-Stokes equations in cylindrical coordinates by studying steady two-dimensional irrotational fluid flow problems in tubes of nonuniform circular cross-sections. ADM was applied to time-fractional Navier-Stokes equations for unsteady flow of viscous fluid in a tube by Momani and Odibat [32]. The time-fractional Navier-Stokes equations are nonlinear and as such there is no known general method to solve these equations and there are very few cases where an exact solution can be obtained. Momani [17] applied the ADM to find the exact and approximate the solutions of convection-diffusion problems. The approximate solution was calculated in the form of a convergent series with easily computable components. The calculations were accelerated by using the noise terms phenomenon for non-homogeneous problems. Mamaloukas et al [23] applied double decomposition to pulsatile flow of incompressible viscous fluid through a circular rigid tube provided with constriction and concluded that pulsatile blood flow through an artery with stenosis depicts flow characteristics and the phenomena of flow separation very well. Ngarhasta et al [33] proved convergence of the ADM applied to linear or nonlinear diffusion equations. Their results showed that convergence of the ADM is not influenced by the choice of the linear invertible operator L in the equation to be solved. Also other researchers [18], [34]-[41] have since more rigorously investigated the necessary conditions for convergence of the ADM.

1.6 Blood Flow through Arteries

Modeling and analysis of a number of biological problems involving interactions of physiological systems, such as those between the respiratory system and the cardiovascular system, can benefit significantly from new advances in mathematical methodology, which allow solution of dynamical systems involving coupled systems, anharmonic oscillators, nonlinear ordinary or partial differential equations, and delay equations [42].

Blood flow is a study of measuring blood pressure and determining flow through the blood vessels. This study is important for human health. Most researchers study blood flow in the arteries and veins. Blood flow under normal physiological conditions is an important field of study, as is blood flow under disease conditions. The majority of deaths in developed countries result from cardiovascular diseases, most of which are associated with some form of abnormal blood flow in arteries. One of the motivations to study blood flow is to understand conditions that may contribute to high blood pressure. Past studies indicated that one of the reasons a person has hypertension is when the blood vessels become narrow. The flow of blood through an artery depends upon the pumping action of the heart which gives rise to a pressure gradient which produces an oscillatory flow in the blood vessel. The blood vessels distribute blood to different organs and supply themselves with nutrition. Blood flow problems through arteries are of increasing interest due to physiological and clinical importance. A number of diseases of blood vessels are being covered by a general term called "stenosis" which is one of the present health hazards. Stenosis refers to the occlusion of the arterial lumen partly or fully due to the deposition of fatty substance. It is the most wide spread arterial disease of blood vessels.

Localized narrowing in an artery, commonly referred to as a stenosis, is a frequent result of arterial disease. Such constrictions disturb normal blood flow through the artery and there is considerable evidence that hydrodynamic factors can play a significant role in the development and progression of this disease. Several flow characteristics, such as wall shearing stress, pressure, and turbulence, may have potential medical significance. In addition, higher resistance to flow can become increasingly important as the stenosis becomes more severe.

Biomagnetic fluid dynamics (BFD) is a relatively new area in fluid mechanics investigating the fluid dynamics of biological fluids in the presence of magnetic field. Biomagnetic fluid is a fluid that exists in a living creature and its flow is influenced by the presence of a magnetic field. The most characteristic biomagnetic fluid is blood, which behaves like a magnetic fluid, due to the complex interaction of the intercellular protein, cell membrane and the hemoglobin, a form of iron oxide, which is present at a uniquely high concentration in the mature red blood cells, while its magnetic property is affected by factors such as the state of oxygenation.

The human body experiences magnetic fields of moderate to high intensity in many situations of day to day life. In recent times, many medical diagnostic devices especially those used in diagnosing cardiovascular disease make use of magnetic fields. It is known from magnetohydrodynamics that when a stationary, transverse magnetic field is applied externally to a moving electrically conducting fluid, electrical currents are induced in the fluid. The interaction between these induced currents and the applied magnetic field produces a body force (known as the Lorentz force) which tends to retard the movement of blood. i.e. the Lorentz force (electromagnetic force) acts on the blood and this force opposes the motion of blood and thereby flow of blood is impeded, so that the external magnetic field can be used in treatment of some kinds of diseases like cardiovascular disease and in the disease with accelerated blood circulation such as hemorrhages and hypertension. When the body is subjected to a magnetic field, the positively and negatively charged blood particles, flowing transversally to the field, are deflected by the Lorentz force in opposite directions. This induces electrical currents and voltages across the vessel walls and in the surrounding tissues, strong enough to be detected at the surface of the thorax in the electrocardiogram.

In general, biological systems are affected by application of an external magnetic field on blood flow through the human arterial system. Many mathematical models have already been investigated by several research workers to explore the nature of blood flow under the influence of an external magnetic field. The biological effects of magnetic fields have often been linked to nitric oxide, which is responsible for the changes in vessel diameter following magnetic field exposure. Recently magnetic fields have been shown to have positive effects on numerous human systems. For instance, it is documented that magnetic field exposure can provide analgesia, decrease healing time for fractures, increase the speed of nerve regeneration, act as a treatment for depression and provide other medical benefits.

Magnetic force therapy could be useful for the reperfusion of ischemic tissue or during sepsis. When blood flow to a tissue becomes blocked or reduced, necrosis will eventually occur. Local exposure of a magnetic field could potentially result in blood vessel relaxation and increased blood flow. The effects of magnetism on blood vessels and the cardiovascular system are very interesting.

Chapter 2

Adomian Decomposition Method

2.1 Introduction

In this section we present the mathematical background of the ADM.

The ADM has received much attention in recent years in Applied Mathematics in general, and in the area of series solutions in particular. The ADM proved to be powerful, effective and can easily handle a wide class of linear or nonlinear, ordinary or partial differential equations, and integral equations.

The ADM demonstrates fast convergence of the solution and therefore provides several significant advantages. We will use the method to handle most types of partial differential equations that appear in several physical models and scientific applications. The ADM attacks the problem in a direct way and in a straightforward fashion without linearization, perturbation or any restrictive assumption that may change the physical behavior of the model under discussion. A brief description of the ADM follows along with a list of the necessary Adomian Polynomials, an essential component of the method.

2.2 Analysis of Adomian Decomposition Method

Definition 1 (Decomposition Scheme). *Let $\sum C_k(X_0, \ldots, X_k)$ be a strongly convergent decomposition series. The decomposition scheme associated with $\sum C_k$ is the recurrent scheme*

$u_0 = 0, \qquad u_{n+1} = C_n(u_0, \ldots, u_n)$

which constructs a series $\sum C_n$ in a Banach space E

Definition 2 (Decomposition Method). *The decomposition method is the method consisting of constructing the solution of an equation with a decomposition scheme*

The ADM consists of decomposing the unknown function $u(x,t)$ of any equation into sum of infinite number of components defined by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t),$$

where the components $u_n(x,t), n \geq 0$ are to be determined in a recursive manner.

The ADM concerns itself by finding the components $u_0, u_1, u_2, \ldots$ individually. The determination of these components can be achieved in an easy way through a recursive relation that involves simple integrals. This technique is very simple in an abstract formulation but the difficulty arises in calculating the Adomian polynomials and proving convergence of the series of the function.

The ADM consists of splitting the given equation into linear and nonlinear parts, inverting the highest-order derivative operator contained in the linear operator on both sides, identifying the initial and/or boundary conditions and the terms involving the independent variable alone as initial approximation, decomposing the unknown function into a series whose components are to be determined, decomposing the nonlinear function in terms of special polynomials called Adomian polynomials and finding the successive terms of the series solution by recurrent relation using Adomian polynomials. The solution is found as an infinite series in which each term can be easily determined and that converges quickly towards an accurate solution. The

ADM is quantitative rather than qualitative, analytic, requiring neither linearization nor perturbation and continuous with no resort to discretization and consequent computer-intensive calculations.

For a clear overview of the ADM, we consider a differential equation

$$Fu(t) = g(t),$$

where F represents a general nonlinear ordinary or partial differential operator comprising of both linear and nonlinear terms. Linear terms are decomposed into $L + R$, where L is invertible and is taken as the highest order derivative to avoid difficult integrations involving complicated Green's functions, and R is the remainder of the linear operator. Thus the equation may be written as

$$Lu + Nu + Ru = g \tag{2.1}$$

where Nu represents nonlinear terms. Solving for Lu, we obtain

$$Lu = g - Ru - Nu. \tag{2.2}$$

L is invertible and L^{-1} is a twofold integration operator and is defined as a definite integration from 0 to t. i.e.

$$L^{-1} = \int_0^t \int_0^t (.)\mathrm{d}t\mathrm{d}t. \tag{2.3}$$

For the operator $L = \dfrac{d^2}{dt^2}$, we have

$$L^{-1}Lu = u(x,t) - u(x,0) - tu_t(x,0). \tag{2.4}$$

Operating on both sides of eq. (2.2) with L^{-1} we have,

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu. \tag{2.5}$$

Combining eqs. (2.4) and (2.5) yields,

$$u(x,t) = u(x,0) + tu_t(x,0) + L^{-1}g - L^{-1}Ru - L^{-1}Nu. \tag{2.6}$$

The decomposition method represents the solution $u(x,t)$ as a series of this form,

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{2.7}$$

The nonlinear term Nu is decomposed as

$$Nu = \sum_{n=0}^{\infty} A_n \tag{2.8}$$

Substitute eqs. (2.7) and (2.8) into eq. (2.6),

$$\sum_{n=0}^{\infty} u_n(x,t) = u_0 - L^{-1}R\sum_{n=0}^{\infty} u_n - L^{-1}\sum_{n=0}^{\infty} A_n, \tag{2.9}$$

where

$$u_0 = f, \tag{2.10}$$

and f represents the terms arising from integrating the source term g and from the given conditions all are assumed to be prescribed.

Consequently, we can write

$$\begin{aligned}
u_1 &= -L^{-1}Ru_0 - L^{-1}A_0,\\
u_2 &= -L^{-1}Ru_1 - L^{-1}A_1,\\
&\vdots\\
u_{n+1} &= -L^{-1}Ru_n - L^{-1}A_n, \quad n \geq 0,
\end{aligned} \tag{2.11}$$

where A_n are the Adomian polynomials generated for each nonlinearity so that A_0 depends only on u_0, A_1 depends only on u_0 and u_1, A_2 depends on u_0, u_1, u_2, and etc. The Adomian polynomials are obtained from the formula

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{n=0}^{\infty} \lambda^n u_n\right)\right]_{\lambda=0} \quad , n = 0, 1, 2, \ldots \tag{2.12}$$

We write the first five Adomian polynomials

$$\begin{aligned}
A_0 &= F(u_0),\\
A_1 &= u_1F'(u_0),\\
A_2 &= u_2F'(u_0) + \frac{1}{2!}u_1^2F''(u_0),\\
A_3 &= u_3F'(u_0) + u_1u_2F''(u_0) + \frac{1}{3!}u_1^3F'''(u_0),\\
A_4 &= u_4F'(u_0) + \left[\frac{1}{2!}u_2^2 + u_1u_3\right]F''(u_0) + \frac{1}{2!}u_1^2u_2F'''(u_0) + \frac{1}{4!}u_1^4F''''(u_0),\\
&\vdots
\end{aligned} \tag{2.13}$$

So, the practical solution for the $n - term$ approximation is,

$$\phi_n = \sum_{i=0}^{n-1} u_i \tag{2.14}$$

where

$$u(x,t) = \lim_{n\to\infty} \phi_n(x,t) = \sum_{i=0}^{\infty} u_i(x,t) \tag{2.15}$$

We now demonstrate the ADM on the following illustrative examples.

Example 2.2.1. Consider a nonlinear ordinary differential equation

$$\begin{aligned} \frac{du}{dt} &= u^2 \\ u(0) &= 1. \end{aligned} \tag{2.16}$$

The differential equation represented by eq. (2.16) has an exact solution given by

$$u(t) = \frac{1}{1-t}. \tag{2.17}$$

Decomposing the differential equation in eq. (2.16) we obtain

$$\begin{aligned} Lu &= Nu \\ \Rightarrow \quad L^{-1}Lu &= L^{-1}(Nu), \end{aligned} \tag{2.18}$$

where $Nu = u^2$ and N a nonlinear operator.
Define an invertible linear operator $L = \dfrac{d}{dt}$ such that

$$L^{-1} = \int_0^t (.)\mathrm{d}t, \tag{2.19}$$

then

$$\begin{aligned} L^{-1}Lu &= u(t) - u(0) \\ \Rightarrow \quad L^{-1}Lu &= u(t) - 1. \end{aligned} \tag{2.20}$$

Combining eqs. (2.18) and (2.20) we have

$$u(t) = 1 + L^{-1}(Nu). \tag{2.21}$$

Applying eqs. (2.7) and (2.8) we have

$$\sum_{n=0}^{\infty} u_n(t) = 1 + L^{-1} \sum_{n=0}^{\infty} A_n. \tag{2.22}$$

Recursively we determine $u_0, u_1, u_2, \ldots$ to obtain

$$\begin{aligned} u_0 &= 1 \\ u_{n+1} &= L^{-1}(A_n) \quad n \geq 0. \end{aligned} \tag{2.23}$$

The Adomian polynomials are given as follows

$$\begin{aligned} A_0 &= u_0^2 \\ A_1 &= 2u_0u_1 \\ A_2 &= u_1^2 + 2u_0u_2 \\ A_3 &= 2u_1u_2 + 2u_0u_3 \\ A_4 &= u_2^2 + 2u_1u_3 + 2u_0u_4 \\ &\vdots \end{aligned}$$

Solving eq. (2.23) yields

$$\begin{aligned} u_0 &= 1 \\ u_1 &= \int_0^t A_0 \mathrm{d}t = t \\ u_2 &= \int_0^t A_1 \mathrm{d}t = t^2 \\ u_3 &= \int_0^t A_2 \mathrm{d}t = t^3 \\ u_4 &= \int_0^t A_3 \mathrm{d}t = t^4 \\ &\vdots \end{aligned} \tag{2.24}$$

It follows that the solution series generated by the ADM in eq. (2.24) is represented by

$$u(t) = 1 + t + t^2 + t^3 + t^4 + t^5 + \ldots \tag{2.25}$$

which we recognize as a Taylor series expansion of the function

$$u(t) = \frac{1}{1-t}. \tag{2.26}$$

Hence the differential equation in eq. (2.16) is solved.

Example 2.2.2. Consider the following inhomogeneous partial differential equation (PDE)

$$u_x + u_y = x + y, \quad u(0,y) = 0, \quad u(x,0) = 0. \tag{2.27}$$

Solution

In operator form eq. (2.27) can be written as

$$L_x u = x + y - L_y u, \tag{2.28}$$

where

$$L_x = \frac{\partial}{\partial x}, \quad L_y = \frac{\partial}{\partial x}. \tag{2.29}$$

It is clear that L_x is invertible, hence L_x^{-1} exists and is given by

$$L_x^{-1} = \int_0^x (.)\mathrm{d}x. \tag{2.30}$$

Applying L_x^{-1} on both sides of eq. (2.28) we have

$$L_x^{-1} L_x u = L_x^{-1}(x+y) - L_x^{-1}(L_y u), \tag{2.31}$$

or equivalently

$$\begin{aligned} u(x,y) &= u(0,y) + \frac{x^2}{2} + xy - L_x^{-1}(L_y u(x,y)) \\ &= \frac{x^2}{2} + xy - L_x^{-1}(L_y u(x,y)) \end{aligned} \tag{2.32}$$

obtained using the given condition $u(0,y) = 0$ and by integrating $f(x,y) = x + y$ with respect to x. The ADM identifies the unknown function $u(x,y)$ as an infinite number of components $u_n(x,y), \quad n \geq 0$ given by

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y). \tag{2.33}$$

Substituting eq. (2.33) into eq. (2.32) we have

$$\sum_{n=0}^{\infty} u_n(x,y) = \frac{x^2}{2} + xy - L_x^{-1}\left(L_y\left(\sum_{n=0}^{\infty} u_n(x,y)\right)\right), \tag{2.34}$$

or equivalently

$$u_0 + u_1 + u_2 + \ldots = \frac{x^2}{2} + xy - L_x^{-1}(L_y(u_0 + u_1 + u_2 + \ldots)). \tag{2.35}$$

The ADM identifies the zeroth component u_0 by all terms arising from the given condition and from integrating $f(x, y) = x + y$, therefore

$$u_0(x, y) = \frac{x^2}{2} + xy. \tag{2.36}$$

Consequently, the recursive scheme that will enable us to completely determine the successive components is constructed by

$$\begin{aligned} u_0(x, y) &= \frac{x^2}{2} + xy \\ u_{n+1}(x, y) &= -L_x^{-1}(L_y(u_n)), \quad n \geq 0. \end{aligned} \tag{2.37}$$

This in turn gives

$$\begin{aligned} u_1(x, y) &= -L_x^{-1}(L_y(u_0)) = -L_x^{-1}(L_y(\frac{x^2}{2} + xy)) = -\frac{x^2}{2}, \quad (2.38) \\ u_2(x, y) &= -L_x^{-1}(L_y(u_1)) = -L_x^{-1}(L_y(-\frac{x^2}{2})) = 0. \end{aligned}$$

Accordingly, $u_n = 0, \quad n \geq 2$. Having determined the components of $u(x, y)$, we find

$$u(x, y) = u_0(x, y) + u_1(x, y) + u_2(x, y) + \ldots = \frac{x^2}{2} + xy - \frac{x^2}{2} = xy \tag{2.39}$$

Hence the exact solution of the inhomogeneous PDE in eq. (2.27) is given by

$$u(x, y) = xy. \tag{2.40}$$

2.3 Adomian Algorithm for Calculating Adomian Polynomials

Adomian polynomials, a notion due to Adomian [43], are key in solving nonlinear equations, and which notion was named the Adomian polynomials by Rach [44] in obvious recognition of Adomian's breakthrough in mathematics. Recall that the Adomian decomposition technique suggests that the unknown solution $u(x, t)$ can be represented by the following decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{2.41}$$

with u_n being computed recursively in an elegant way. However, the nonlinear term $F(u)$, such as $u^2, u^3, u^4, \sin u, e^u, uu_x$, etc. can be expressed by an infinite series of the Adomian polynomials A_n given on the form

$$F(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \ldots, u_n) \tag{2.42}$$

where the Adomian polynomials A_n can be evaluated for all forms of nonlinearity. Several schemes have been introduced in the literature by researchers to calculate the Adomian polynomials. Adomian introduced a scheme for calculation of the Adomian polynomials that was formally justified and considered by many as simple and practical. In the following, we present the general Adomian algorithm for the calculation of the Adomian polynomials and a summary of the necessary steps to calculate the first few Adomian polynomials.

Definition 3 (Adomian Polynomials)**.** *Let F be an analytical function and $\sum u_n$ a convergent series in a Banach space E. Then the Adomian polynomials A_n for the nonlinear term $F(u)$ can be evaluated by the following expression*

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[f\left(\sum_{n=0}^{\infty} \lambda^n u_n\right)\right]_{\lambda=0} \quad , n = 0, 1, 2, \ldots \tag{2.43}$$

The general formula of the Adomian polynomials in eq. (2.43) can be simplified as follows. Assuming that the nonlinear function is $F(u)$. Therefore by using eq. (2.43), the Adomian polynomials are given by

$$\begin{aligned}
A_0 &= F(u_0), \\
A_1 &= u_1 F'(u_0), \\
A_2 &= u_2 F'(u_0) + \frac{1}{n!}u_1^2 F''(u_0), \\
A_3 &= u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{1}{3!}u_1^3 F'''(u_0),
\end{aligned} \tag{2.44}$$

$$
\begin{aligned}
A_4 &= u_4F'(u_0) + \left(\frac{1}{2!}u_2^2 + u_1u_3\right)F''(u_0) + \frac{1}{2!}u_1^2u_2F'''(u_0) + \frac{1}{4!}u_1^4F''''(u_0), \\
A_5 &= u_5F'(u_0) + (u_2u_3 + u_1u_4)F''(u_0) + \left(\frac{1}{2!}u_1u_2^2 + \frac{1}{2!}u_1^2u_3\right)F'''(u_0) \\
&+ \frac{1}{3!}u_1^3u_2F''''(u_0) + \frac{1}{5!}u_1^5F'''''(u_0) \\
&\vdots
\end{aligned}
$$

Other polynomials can be generated in the same manner.

Two important observations can be made here. First, A_0 depends only on u_0, A_1 depends only on u_0 and u_1, A_2 depends on u_0, u_1, u_2, and so on.

Secondly, substituting eq. (2.44) into eq. (2.42) we obtain

$$
\begin{aligned}
F(u) &= A_0 + A_1 + A_2 + A_3 + A_4 + \ldots \\
&= (u_1 + u_2 + u_3)F'(u_0) \\
&\quad + \frac{1}{2!}(u_1^2 + 2u_1u_2 + 2u_1u_3 + u_2^2 + \ldots)F''(u_0)) \\
&\quad + \frac{1}{3!}(u_1^3 + 3u_1^2u_2 + 3u_1^2u_3 + 6u_1u_2u_3 + \ldots)F'''(u_0) \\
&= F(u_0) + (u - u_0)F'(u_0) + \frac{1}{2!}(u-u_0)^2F''(u_0) + \frac{1}{3!}(u-u_0)^3F'''(u_0) + \ldots \\
&= \sum_{n=0}^{\infty} \frac{F^{(n)}(u_0)}{n!}(u-u_0)^n. \qquad (2.45)
\end{aligned}
$$

The last expansion confirms that the series of A_n polynomials is a Taylor series expansion about a function u_0 and not about a point as is usually used. As mentioned before, it is clear that A_0 depends only on u_0, A_1 depends only on u_0 and u_1, A_2 depends only on u_0, u_1 and u_2. The same conclusion holds for other polynomials.

In the following, we will calculate Adomian polynomials for two forms of nonlinearity that may arise in nonlinear ordinary or partial differential equations.

Example 2.3.1. $F(u) = u^2$

The Adomian polynomials are

$$
\begin{aligned}
A_0 &= u_0^2 \\
A_1 &= 2u_0u_1
\end{aligned}
$$

$$\begin{aligned}
A_2 &= u_1^2 + 2u_0u_2 \\
A_3 &= 2u_1u_2 + 2u_0u_3 \\
A_4 &= u_2^2 + 2u_1u_3 + 2u_0u_4 \\
A_5 &= 2u_2u_3 + 2u_1u_4 + 2u_0u_5 \\
&\vdots
\end{aligned}$$

Example 2.3.2. $F(u) = \sin u$
The Adomian polynomials are

$$\begin{aligned}
A_0 &= \sin u_0 \\
A_1 &= u_1 \cos u_0 \\
A_2 &= u_2 \cos u_0 - \frac{1}{2!} u_1^2 \sin u_0 \\
A_3 &= u_3 \cos u_0 - u_1 u_2 \sin u_0 - \frac{1}{3!} u_1^3 \cos u_0 \\
&\vdots
\end{aligned} \tag{2.46}$$

2.4 Alternative Algorithm for Calculating Adomian Polynomials

A considerable amount of research work has been invested to develop an alternative method to the Adomian algorithm for calculating Adomian polynomials A_n. The objective was to construct a practical technique that would calculate Adomian polynomials in a practical way without any need to the formulae introduced before. However, the methods developed are identical to those used by Adomian. We believe that a simple and reliable technique can be established to make the calculations less dependable on the formula given in eq. (2.43). We will introduce an alternative algorithm that can be used to calculate Adomian polynomials for nonlinear terms in an easy way. The newly developed technique depends mainly on algebraic and trigonometric identities, and on Taylor series expansions as well. Moreover, we should use the fact that the sum of subscripts of the components of u in each term of the polynomials A_n equals n. The alternative algorithm suggests that we substitute u as a sum of components of $u_n, n \geq 0$ as defined by the Adomian decomposition method. It is clear that A_0 is always determined independently of the other polynomials $A_n, n \geq 1$, where A_0 is defined by

$$A_0 = F(u_0). \tag{2.47}$$

The alternative algorithm assumes that we first separate $A_0 = F(u_0)$ for every nonlinear term $F(u)$. With the separation done, the remaining components of $F(u)$ can be expanded by using algebraic operations, trigonometric identities, and Taylor series as well. Next we collect all the terms of the expansion obtained such that the sum of the subscripts of the components of u in each term is the same and thus complete the calculation of the Adomian polynomials. Several examples have been tested, and the obtained results have shown that the Adomian polynomials can be elegantly computed without the formula established by Adomian. We explain the alternative algorithm by means of worked examples.

Case 1. $F(u) = u^2$

We first set

$$u = \sum_{n=0}^{\infty} u_n. \tag{2.48}$$

Substitute eq. (2.48) into $F(u) = u^2$ gives

$$\begin{aligned}
F(u) &= (u_0 + u_1 + u_2 + u_3 + u_4 + u_5 + \ldots)^2 \\
&= u_0^2 + 2u_0u_1 + 2u_0u_2 + u_1^2 + 2u_0u_3 + 2u_1u_2 + \ldots \qquad (2.49)\\
&= \underbrace{u_0}_{A_0} + \underbrace{2u_0u_1}_{A_1} + \underbrace{2u_0u_2 + u_1^2}_{A_2} + \underbrace{2u_0u_3 + 2u_1u_2}_{A_3} \\
&+ \underbrace{2u_0u_4 + 2u_1u_3 + u_2^2}_{A_4} + \underbrace{2u_0u_5 + 2u_1u_4 + 2u_2u_3}_{A_5} + \ldots
\end{aligned}$$

This is consistent with the results obtained before using Adomian's algorithm.

Case 2. $F(u) = \sin u$

Note that it is impossible to perform algebraic operations here. Therefore, our main aim is to separate $A_0 = F(u_0)$ from other terms. To achieve this goal, we first substitute

$$u = \sum_{n=0}^{\infty} u_n \tag{2.50}$$

into $F(u) = \sin u$ to obtain

$$\begin{aligned}
F(u) &= \sin(u_0 + u_1 + u_2 + \ldots) \\
&= \sin(u_0 + [u_1 + u_2 + \ldots]). \qquad (2.51)
\end{aligned}$$

Recall that $\sin(A+B) = \sin A\cos B + \cos A\sin B$.
Thus

$$\sin(u_0 + [u_1 + u_2 + \ldots]) = \sin u_0 \cos(u_1 + u_2 + \ldots) + \cos u_0 \sin(u_1 + u_2 + \ldots). \tag{2.52}$$

Applying the Taylor expansion for $\sin(u_1 + u_2 + \ldots)$ and $\cos(u_1 + u_2 + \ldots)$.

$$\begin{aligned}
F(u) &= \sin u_0 \left[1 - \frac{(u_1 + u_2 + \ldots)^2}{2!} + \frac{u_1 + u_2 + \ldots)^4}{4!} - \ldots\right] \\
&+ \cos u_0 \left[(u_1 + u_2 + \ldots) - \frac{(u_1 + u_2 + \ldots)^3}{3!} + \ldots\right] \\
&= \sin u_0 \left[1 - \frac{1}{2!}(u_1^2 + 2u_1u_2 + \ldots) + \ldots\right] \\
&+ \cos u_0 \left[(u_1 + u_2 + \ldots) - \frac{1}{3!}(u_1^3 + 3u_1^2u_2 + 3u_1^2u_3 + \ldots) + \ldots\right] \\
&= \underbrace{\sin u_0}_{A_0} + \underbrace{u_1 \cos u_0}_{A_1} + \underbrace{u_2 \cos u_0 - \frac{1}{2!}u_1^2 \sin u_0}_{A_2} \\
&+ \underbrace{u_3 \cos u_0 - u_1u_2 \sin u_0 - \frac{1}{3!}u_1^3 \cos u_0}_{A_3} + \ldots
\end{aligned} \tag{2.53}$$

When we compare the Adomian polynomials found in eq. (2.53) with the ones found in eq. (2.46) we see that we have the same Adomian polynomials computed using two different methods.

2.5 The Noise Terms Phenomenon

In this section, we present a useful tool that will accelerate the convergence of the ADM. In problems solved by the decomposition method, the appearance of **noise terms** sometimes makes it necessary to compute more terms to observe the self-cancellations and separate solution terms from the terms whose sum vanishes in the limit. In the study of inhomogeneous equations, the noise term phenomenon is rather useful because of the role it plays in the rapid convergence of solutions obtained by the ADM. The noise terms phenomenon provides a major advantage in that it demonstrates a fast convergence of the solution. It is of great importance to note that the noise terms phenomenon may appear only for inhomogeneous partial differential equations of any order. The noise terms, if they exist in components u_0 and u_1, will provide the solution in a closed form with only two successive iterations.

The **noise terms** are defined as the identical terms with opposite signs that arise in the components u_0 and u_1. By cancelling out the noise terms between u_0 and u_1, even though u_1 contains further terms, the remaining non-cancelled of u_0 may give the exact solution of the partial differential equation. Therefore, it is necessary to verify that the non-cancelled terms of u_0 satisfy the partial differential equation under discussion [28, 29]. On the other hand, if the non-cancelled terms of u_0 did not justify the given partial differential equation, or the noise terms did not appear between u_0 and u_1, then it is necessary to determine more components of u to determine the solution in a series form. It was formally proved [29] that the noise terms appear for specific cases for inhomogeneous PDE's, whereas homogeneous PDE's do not show noise terms. The conclusion about the self-cancelling noise terms was based on observations drawn from solving specific models where no proof was presented. It was formally proved by researchers that a necessary condition for the appearance of the noise terms is required. The conclusion made in this regard is that the zeroth component u_0 must contain the exact solution u among other terms. The phenomenon of the useful noise terms will be explained by the following illustrative examples.

Example 2.5.1. In this example we use the ADM and the noise terms phenomenon to solve the following inhomogeneous partial differential equation

$$u_x + u_y = (1+x)e^y, \quad u(0,y) = 0, \quad u(x,0) = x \tag{2.54}$$

Equation (2.54) can be written in an operator form as follows

$$L_x u = (1+x)e^y - L_y u \tag{2.55}$$

Clearly L_x is invertible and therefore the inverse operator L_x^{-1} exists and given by

$$L_x^{-1} = \int_0^x (.)dx \tag{2.56}$$

Applying L_x^{-1} on both sides of eq. (2.55) and using the given condition leads to

$$u(x,y) = (x + \frac{x^2}{2!})e^y - L_x^{-1}(L_y u(x,y)) \tag{2.57}$$

Using the decomposition series $u(x,y) = \sum_{n=0}^{\infty} u_n(x,y)$ into eq. (2.57) yields

$$\sum_{n=0}^{\infty} u(x,y) = (x + \frac{x^2}{2!})e^y - L_x^{-1}\left(L_y\left(\sum_{n=0}^{\infty} u_n(x,y)\right)\right) \tag{2.58}$$

or equivalently

$$u_0 + u_1 + u_2 + \ldots = (x + \frac{x^2}{2!})e^y - L_x^{-1}(L_y(u_0 + u_1 + u_2 + \ldots)) \tag{2.59}$$

The components of $u_0 + u_1 + u_2 + \ldots$ are determined in a recursive manner by

$$\begin{aligned}
u_0(x,y) &= \left(x + \frac{x^2}{2!}\right)e^y \\
u_1(x,y) &= -L_x^{-1}(L_y u_0) = -\left(\frac{x^2}{2!} + \frac{x^3}{3!}\right)e^y, \\
u_2(x,y) &= -L_x^{-1}(L_y u_2) = \left(\frac{x^4}{4!} + \frac{x^5}{5!}\right)e^y, \\
&\vdots
\end{aligned} \tag{2.60}$$

Considering the first two components u_0 and u_1 in eq. (2.60), it is easily observed that the noise terms $\frac{x^2}{2!}e^y$ and $-\frac{x^2}{2!}e^y$ appear in u_0 and u_1 respectively. By cancelling the noise term $\frac{x^2}{2!}e^y$ in u_0, and by verifying that the remaining non cancelled terms of u_0 satisfy eq. (2.54), we find that the exact solution is given by

$$u(x,y) = xe^y. \tag{2.61}$$

Notice that the exact solution is verified through substitution in eq. (2.54) and not only upon the appearance of the noise terms. In addition, the other noise terms that appear between other components will vanish in the limit.

2.6 Modified Adomian Decomposition Method

In this section we present a reliable modification of the ADM presented by Wazwaz [45, 46] which gives us a variation of the recursive relation in eq. (2.11), which in turn leads to a faster and easier way of determining the components of u.
The modelling of physical problems can lead to ordinary or partial differential equations which are quite generally nonlinear, such as the Navier-Stokes equations in fluid mechanics, the Lane-Emden equation for stellar structure, nonlinear Schrödinger equations in quantum theory, soliton equations, etc. We present a variation of the decomposition method which can also be applied to such equations to obtain accurate quantitative solutions. A mathematical advantage of the various adaptations of decomposition is that linear equations are an easily solved special case and ordinary differential equations are a special case of the theory for partial differential equations, so we have a single unified field. This alternative formulation will be referred to as " **modified Adomian decomposition method**."

A modification of the ADM was proposed by Wazwaz [45] to overcome computational difficulties arising when obtaining the solution of differential equations containing radicals while inverting the operator, particularly when the initial approximation is not a constant. Wazwaz [45] proposed a modification of the ADM in order to simplify the calculations and accelerate rapid convergence of the series solution, and validated the method through several examples. The modified decomposition further accelerates the convergence of the series solution. Note that, the modified ADM will be applied wherever it is appropriate, to all partial differential equations of any order. The modified ADM has been shown to be computationally efficient in several examples that are important to researchers in applied fields. In addition, the modified ADM may give the exact solution after just two iterations only and without using the Adomian polynomials.

To give a clear description of the modified ADM we consider the partial differential equation in an operator form

$$Lu + Nu + Ru = g \tag{2.62}$$

where L is the highest order derivative which is assumed to be easily

invertible, R is a linear differential operator of less or equal order to L, Nu represents the nonlinear terms.
Applying the operator L^{-1} to both sides of eq. (2.62) yields

$$u = f - L^{-1}Ru - L^{-1}Nu, \tag{2.63}$$

where f represents the terms arising from the given conditions and from integrating the source term g.
The standard ADM defines a solution $u(x,t)$ by the series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t). \tag{2.64}$$

The aim of the ADM is to determine the components $u_n, n \geq 0$ recursively and elegantly. To achieve this goal, the ADM use the following recursive relation

$$\begin{aligned}
u_0 &= f, \\
u_1 &= -L^{-1}Ru_0 - L^{-1}A_0, \\
&\vdots \\
u_{n+1} &= -L^{-1}Ru_n - L^{-1}A_n, \quad n \geq 1.
\end{aligned} \tag{2.65}$$

The modified decomposition method introduces a slight variation to the recursive relation in eq. (2.65) that leads to the determination of u in a faster and easier way. For specific cases, the function f can be set as the sum of two partial functions, namely f_1 and f_2. i.e.

$$f = f_1 + f_2. \tag{2.66}$$

Using eq. (2.66), we introduce a qualitative change in the formation of the recursive relation in eq. (2.65). To reduce the size of calculations, we identify the zeroth component u_0 by one part of f, namely f_1 or f_2. The other part can be added to the component u_1 among other terms. In other words, the modified recursive relation can be identified by

$$u_0 = f_1,$$

$$\begin{aligned} u_1 &= f_2 - L^{-1}u_0 - L^{-1}A_0, \\ &\vdots \\ u_{n+1} &= -L^{-1}Ru_n - L^{-1}A_n, \quad n \geq 1. \end{aligned} \tag{2.67}$$

Comparing the recursive scheme in eq. (2.11) of the ADM with the recursive scheme in eq. (2.67) of the modified ADM leads to the conclusion that in eq. (2.65) the zeroth component was defined by the function f, whereas in eq. (2.67) the zeroth component u_0 is defined by f_1 of f. The remaining part f_2 is added to the definition of the component u_1 in eq. (2.65). Two important remarks related to the modified method can be made here.

First, by proper selection of the functions f_1 and f_2, the exact solution u may be obtained by using very few iterations. The success of this modification depends only on the choice of f_1 and f_2, and this can be made through trials. Second, if f consists of one term only, the standard decomposition method should be employed. It is worth mentioning that the modified decomposition method will be used for linear and nonlinear equations of any order.

Example 2.6.1. Consider the ordinary differential equation

$$u'(x) - u(x) = x\cos x - x\sin x + \sin x, \quad u(0) = 0. \tag{2.68}$$

The **ADM** Applying the inverse operator L^{-1} to both sides of eq. (2.68) we obtain

$$u(x) = x + x\cos x - \sin x + L^{-1}(u(x)). \tag{2.69}$$

Using the recursive scheme in eq. (2.11) yields

$$\begin{aligned} u_0 &= x\sin x + x\cos x - x\sin x, \\ u_1 &= \int_0^x u_0(t)\mathrm{d}t \\ &= -x\cos x + \sin x + x\sin x + 2\cos x - 2, \\ u_2 &= \int_0^x u_1(t)\mathrm{d}t \\ &= -x\sin x - 2\cos x - x\cos x + 3\sin x - 2x + 2, \\ &\vdots \end{aligned} \tag{2.70}$$

We need more components to obtain an insight through the series of functions obtained. i.e. the ADM takes long to converge to the desired solution. We notice that the noise terms $x\cos x$ and $-\sin x$ appear with opposite signs in the components u_0 and u_1.
By cancelling these terms from u_0 and justifying that the remaining non-cancelled term of u_0 given by

$$u_0 = x\sin x \tag{2.71}$$

satisfies the equation. This means that the exact solution is given by

$$u(x) = x\sin x. \tag{2.72}$$

Modified ADM We split f into two parts defined by

$$f_1 = x\sin x \tag{2.73}$$

$$f_2 = x\cos x - \sin x \tag{2.74}$$

Applying the recursive scheme in eq. (2.67) we have the following relation

$$\begin{aligned} u_0 &= x\sin x \\ u_1 &= x\cos x - \sin x + \int_0^x u_0(t)\mathrm{d}t = 0 \\ &\vdots \\ u_{n+1} &= 0, \quad n \geq 0. \end{aligned} \tag{2.75}$$

Looking at eq. (2.75) we see that the exact solution is given by

$$u(x) = x\sin x. \tag{2.76}$$

Note that the power of the modified ADM depends mainly on the proper selection of f_1 and f_2 which in turn accelerates the convergence of the solution by employing two iterations only. An important observation that can be made here is that the success of this method depends mainly on the choice of f_1 and f_2. We have been unable to establish any theory for the selection of f_1 and f_2. It appears that trials are the only criteria that can be applied.

2.7 Convergence Analysis

The convergence concept of the decomposition series was thoroughly investigated by many researchers [18], [34]-[41] to confirm the rapid convergence of the resulting series. They obtained a number of important results allowing the easy use of the decomposition method in concrete situations.

2.7.1 Convergence of the Adomian Decomposition Method

Consider the general functional equation

$$y - N(y) = f(t) \tag{2.77}$$

where N is a nonlinear operator from a Hilbert space H to H. f is a given function in H and we are seeking a $y \in H$ satisfying eq. (2.77). We assume that eq. (2.77) has a unique solution for $f \in H$.

Theorem 2.7.1. *Consider an entire series*

$$\sum_{n=0}^{\infty} a_n x^n, \tag{2.78}$$

with a convergence radius R. Suppose that

$$x = \sum_{n=0}^{\infty} b_n \lambda^n. \tag{2.79}$$

If we replace x in eq. (2.78) by the expression in eq. (2.79), we have an entire series

$$\sum_{n=0}^{\infty} c_n \lambda^n, \tag{2.80}$$

where the c_n are given by

$$\begin{aligned} c_0 &= a_0 + a_1 b_0 + a_2 b_0^2 + \ldots + a_n b_0^n + \ldots \\ c_1 &= a_1 b_1 + 2a_2 b_1 b_0 + \ldots + n a_n b_0^{n-1} + \ldots \\ c_2 &= a_1 b_2 + a_2 (b_1^2 + 2b_0 b_2) + \ldots + \ldots \end{aligned} \tag{2.81}$$

If we have

$$\begin{cases} |b_n| & \leq \dfrac{M}{(1+\epsilon)^n}(n \geq 0), \quad \epsilon > 0 \\ M & < R \\ \epsilon & \geq \dfrac{M}{R-M}, \end{cases} \tag{2.82}$$

then the series in eq. (2.81) has a radius of convergence $(R \geq 1)$.

Proof. It is sufficient to prove that

$$\sum_{n=0}^{\infty} |b_n||\lambda|^n < R \quad for \quad |\lambda| < 1. \tag{2.83}$$

From eq. (2.79) we have

$$\sum_{n=0}^{\infty} |b_n||\lambda|^n \leq \sum_{n=0}^{\infty} M \left(\frac{|\lambda|}{1+\epsilon}\right)^n. \tag{2.84}$$

Suppose we let $|\lambda| < 1+\epsilon$; then from eq. (2.84) it follows that

$$\sum_{n=0}^{\infty} |b_n||\lambda|^n \leq \frac{M}{1 - \dfrac{|\lambda|}{1+\epsilon}} \tag{2.85}$$

$$\frac{1}{1 - \dfrac{|\lambda|}{1+\epsilon}} < R \Leftrightarrow |\lambda| < (1+\epsilon)\left(1 - \frac{M}{R}\right). \tag{2.86}$$

From eq. (2.85) we have

$$(1+\epsilon)\left(1 - \frac{M}{R}\right) \geq \left(1 + \frac{M}{R-M}\right)\left(1 - \frac{M}{R}\right) = 1 \tag{2.87}$$

so that the result is proved. □

Lemma 2.7.1. *If*

$$\sum_{|i|=n} (i)^0 = \sum_{|i|=n} (1). \tag{2.88}$$

then

$$\sum_{|i|=n} (1) = \frac{(m+n)!}{m!n!}, \tag{2.89}$$

where $i = (i_1, i_2, \ldots, i_{m+1}) \in IN^m$.

Lemma 2.7.2. *Suppose that N is an analytic function in* $:]x_0 - R; x_0 + R[:$, *and furthermore*

$$\| N^n(x_0) \| \leq n! M \alpha^n. \tag{2.90}$$

Then the Adomian polynomials satisfy the following expression

$$\| A_n \| \leq \frac{(2n!)}{(n+1)!n!} M^{n+1} \alpha^n \quad (n \geq 0). \tag{2.91}$$

The above lemmas lead to the following theorem.

Theorem 2.7.2. *Suppose that N satisfies the following condition*

$$\| N^n x_0 \| \leq n! M \alpha^n. \tag{2.92}$$

Then the sufficient conditions for convergence of the method are

1. $4M\alpha \leq 1$ *if R is infinite.*

2. $5M\alpha \leq 1$ *if R is finite.*

Proof. **Case 1**

It is sufficient to prove that

$$\sum_{n=0}^{\infty} \| A_n \| < \infty$$

$$\sum_{n=0}^{\infty} \| A_n \| \leq \sum_{n=0}^{\infty} \frac{(2n)!}{(n+1)!n!} M^{n+1} \alpha^n.$$

Using the Stirling [41] formula we obtain

$$\frac{(2n)!}{(n+1)!n!} M^{n+1} \alpha^n \sim \frac{(4M\alpha)^n M}{\sqrt{\pi}(n+1)^{\frac{3}{2}}} (n \to \infty).$$

Case 2

$$\| A_n \| \leq \frac{(2n)!}{(n+1)!n!} M^{n+1} \alpha^n = \frac{(2n)!M}{4^n (n+1)!n!} 4^n M^n \alpha^n.$$

If

$$X_n = \frac{(2n)!}{4^n (n+1)!n!}, \tag{2.93}$$

then we have

$$\frac{X_{n+1}}{X_n} = \frac{2n+1}{2n+4} < 1 \Rightarrow X_n < X_0 = 1. \tag{2.94}$$

Consequently,

$$\| A_n \| \leq \frac{(2n)!}{(n+1)!n!} M^{n+1}\alpha^n = M4^n M^n \alpha^n = \frac{M}{(1+\epsilon)^n}, \tag{2.95}$$

where

$$\epsilon = \frac{1}{4M\alpha - 1} \quad (4M\alpha < 1). \tag{2.96}$$

It follows that

$$\epsilon \geq \frac{M}{R-M} \Leftrightarrow 5M\alpha \leq 1, \tag{2.97}$$

where R is a convergence radius. □

2.7.2 Convergence of the Modified Adomian Decomposition Method

Case 1

Consider an ordinary differential equation

Suppose that $H = \Re$,

$$\begin{cases} u'(t) & = f(u(t)) + g(t) \\ u(0) & = c, t \in [a,b] \end{cases} \tag{2.98}$$

If we integrate between 0 to t we have

$$u(t) = \int_0^t f(u(s))\mathrm{d}s + c + \int_0^t g(s)\mathrm{d}s.$$

Suppose that

$$g(t) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} t^n,$$

then

$$u(t) = \int_0^t f(u(s))\mathrm{d}s + c + \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{(n+1)!} t^{n+1}.$$

By setting

$$b_n = \frac{g^{(n)}(0)}{(n+1)!} t^{n+1},$$

we obtain

$$|f^{(n)}(c)| \leq M\alpha^n,$$

and

$$|g^{(n)}(0)| \leq n! M_1 \alpha_1^n.$$

Then the sufficient conditions for convergence of the modified Adomian decomposition are

$$\begin{aligned} \alpha_1 &< \alpha M|t| \\ M_1 &< M \\ |t|\alpha M &\leq \frac{1}{e}, \end{aligned}$$

where

$$N(u)(t) = \int_0^t f(u(s))ds.$$

Then

$$\| N^{(n)}(c) \| \leq M\alpha^n |t|,$$

and we have

$$\begin{aligned} |b_n| &\le M_1\alpha_1^n|t|^{n+1} \\ &= (M_1|t|).(\alpha_1|t|)^n. \end{aligned}$$

Case 2

We begin with the following preliminary results.

Lemma 2.7.3. *Suppose that*

$$|N^{(n)}(a)| \le n!M\alpha^n,$$

then

$$\left|(N^{[n]}(a))^{(m)}\right| \le \frac{(n+m-1)!)!}{(n-1)!m!}M^n\alpha^m.$$

Proof. This can be proved by using the general Leibnitz formulae and **lemma 2.7.1**.

□

Theorem 2.7.3. *Suppose that*

$$|N(n)(a)| \le n!M\alpha^n \tag{2.99}$$

and

$$|b_n| \le M_1\alpha_1^n \tag{2.100}$$

with

$$\begin{cases} M_1 & < M \\ \alpha_1 & < M, \alpha \end{cases} \tag{2.101}$$

then the sufficient conditions for ensuring convergence towards the solution of eq. (2.78) are

(a) If R is finite

$$\begin{cases} \beta & \le \dfrac{1}{\alpha} \\ \dfrac{1}{4M\alpha} - 1 & \ge \dfrac{\alpha\beta}{1-\alpha\beta} \end{cases} \tag{2.102}$$

(b) If R is infinite

$$4M\alpha \leq 1. \tag{2.103}$$

Proof. See [41] page 537 □

2.7.3 The Fundamental Convergence Theorem

Theorem 2.7.4 (Convergence of the decomposition schemes)**.** *Every decomposition scheme associated with a strongly convergent decomposition series the degeneration sum of which is S, gives a convergent series the sum of which U verifies $U = S(U)$, where f is contracting and f is an analytic function in Banach space E*

Proof. Let $\sum C_k$ be a strongly convergent decomposition series, let S be its degenerated sum and let $\bar{S}$ be its sum (i.e for all convergent series $\sum v_i$, $\bar{S}(\sum v_i) = S(\sum_0^\infty v_i))$.

Note, $\sum u_i$ is the series given by the decomposition scheme associated with $\sum C_k$.

Suppose that S is a contraction, then there exists u such that $u = S(u)$. $\bar{u} = \sum \bar{u}_i$ the series defined by $\bar{u}_0 = u$, and $\forall i \geq n$, $\bar{u}_i = 0$.
Using the following scheme

$$\bar{u}^{(0)} = \bar{u}, \quad \bar{u}^{(n+1)} = \bar{S}\left(0 + \bar{u}^{(n)}\right)$$

we build out a sequence of series $(u^{(n)})$.

It can also be verified that the first N terms of $u^{(N)}$ are equal to the first N terms of $\sum u_i$.

If, for every convergent series $\bar{w}$ and $\bar{v}$, we note $\bar{R}_n(\bar{w}) = \sum_{n+1}^\infty C_k(\bar{w})$ and $\bar{R}_n(\bar{v}) = \bar{R}_n(0 + \bar{v})$, then we have

$$\begin{aligned} \|u - \sum_{i=0}^\infty u_i\| &= \|\sum_{i=0}^\infty \bar{u}^{(N)} - \sum_{i=0}^\infty u_i\| \\ &= \|\bar{R}_N\left(\bar{u}^{(N)}\right)\| \\ &= \|\tilde{R}_N \tilde{R}_{N-1} \ldots \tilde{R}_0(\bar{u})\|. \end{aligned} \tag{2.104}$$

As $\sum C_k$ is a strongly convergent decomposition series, $\bar{R}_n$ and $\tilde{R}_n$ converge towards 0. So $\sum_0^N u_i$ converges towards u. The decomposition scheme leads to a series $\sum u_i$ the sum of which is the solution u of the equation $u = S(u)$.

Note. We cannot obtain the convergence of the scheme by adding the equalities defining u_n

$$\begin{aligned} u_0 &= 0 \\ u_1 &= C_0(u_0) \\ u_2 &= C_0(u_0, u_1) \\ &\vdots \\ u_{n+1} &= C_n(u_0, \ldots, u_n). \end{aligned} \tag{2.105}$$

We do not know if the series of elements of $E \sum C_k(u_0, \ldots, u_k)$ converges. We only know that the decomposition scheme $\sum C_k$ strongly converges, i.e. $\sum C_k(u_0, \ldots, u_k)$ converges if $\sum u_i$ converges and that is what we want to prove. It is dangerous to use the hypothesis $\lim \|S_n - S\| = 0$ with $S_n = \sum_0^n C_k$ because $S_n - S = \sum_{n+1}^{\infty} C_k$ converges towards 0 if $\sum C_k$ converges, and that is what we want to prove. This hypothesis can be verified only if the expression of $\sum_0^N C_k$ is known and this case rarely occurs. Finally, the hypothesis $\sum C_k$, is a strongly convergent decomposition series and is more easily verified without computation.

Note, the terms of a convergent decomposition series always verify the Cherruault equalities [47] $u_{n+1} = S_n(U_n) - S_{n-1}(U_{n-1})$. Those relations can be considered as an extension to the definition of the B_k to the C_k (S_n replacing S).

Conclusion. The practical problem is to solve equation $u = G(u)$, where G is a given operator. Using the convergence theorem, it is possible to do this if we have a strongly convergent decomposition series, the degeneration sum of which is G. Then the solution U is obtained by applying the decomposition scheme associated with this series.

Note, usually, all the terms of the series $\sum u_i$ cannot be computed. So the exact solution $\sum_{i=0}^{\infty} u_i$ cannot be obtained but an approximation as close as needed can be worked out using only the first terms of the series $\varphi_n = \sum_{i=0}^{n-1} u_i$ [47]. $\square$

Chapter 3

Mathematical Analysis

3.1 Introduction

Fluid dynamics deals with the motion of liquids and gases, which when studied macroscopically, appear to be continuous in structure. All the variables are considered to be continuous functions of the spatial coordinates and time. Blood flow like any other fluids (such as air, water, oil, etc.) may be analyzed mathematically by the use of two equations. The first, often referred to as the *continuity equation*, requires that the mass of fluid entering a fixed control volume either leaves that volume or accumulates within it. It is thus a *"mass balance"* requirement posed in mathematical form, and is a scalar equation. The second governing equation, is the *momentum equation*, and may be thought of as a *"momentum balance"*. The Navier-Stokes equations are the fluid dynamics equivalent of Newton's second law, force equals mass times acceleration and they are of crucial importance in fluid dynamics. The Navier-Stokes equations are vector equations, meaning that there is a separate equation for each of the coordinate directions (usually three). The equations were derived independently by G.G. Stokes, in England, and M. Navier, in France, in the early 1800's [48]. The equations are extensions of the Euler equations and include the effects of viscosity on the flow. The Navier-Stokes equations are a set of nonlinear partial differential equations (relating first and second derivatives of fluid velocity) that describe the flow of fluids under quite general conditions, and they appear in the study of many important phenomena, either alone or coupled with other equations. They model weather, the movement of air in the atmosphere, ocean currents, water flow in a pipe, as well as many other fluid flow phenomena.

For instance, they are used in theoretical studies in aeronautical sciences, in meteorology, in thermo-hydraulics, in the petroleum industry, in plasma physics, etc. From the point of view of continuum mechanics the Navier-Stokes equations are essentially the simplest equations describing the motion of a fluid, and they are derived under a quite simple physical assumption, namely, the existence of a linear local relation between stresses and strain rates. The Navier-Stokes equations are nonlinear partial differential equations in every real situation. The nonlinear term $(\mathbf{u}.\nabla)\mathbf{u}$ contained in the equations comes from kinematical considerations (i.e., it is the result of an elementary mathematical operation) and does not result from assumptions about the nature of the physical model; consequently this term cannot be avoided by changing the physical model.

While the physical model leading to the Navier-Stokes equations is simple, the situation is quite different from the mathematical point of view. In particular, because of their nonlinearity, the mathematical study of these equations is difficult and requires the full power of modern functional analysis. Despite all the important work carried out on these equations, our understanding of them remains fundamentally incomplete [49].

The Navier-Stokes equations may be used to analyze the flow of most common fluids in internal (pipes) or external (wings) flow situations. Mathematically speaking, these equations are extremely difficult to solve in their raw form. The Navier-Stokes equations are second order, non-homogeneous, nonlinear partial differential equations that require at least two boundary conditions for solution. Most solutions that exist are for highly simplified flow situations where certain terms in the equations have been eliminated through some rational process.

3.2 Navier-Stokes Equations in Cylindrical Polar Coordinates

An ultimate objective in fluid dynamics is to obtain the general solution for the Navier-Stokes equations describing viscous compressible fluid flow. The incompressible case occurs where pressure depends only on the velocity field and vice-versa [1, 4]. A complete solution of the compressible case

requires determination of the three components of the velocity vector and the state of the fluid described by the pressure P, density ρ, and temperature T as a function of space and time. The objective of this research is an analytical, non-perturbative, non-linearized solution avoiding discretization and resulting intensive numerical computation. The procedure is the ADM which has been shown to offer accurate solutions of general mathematical models involving systems of nonlinear partial differential equations.
Cylindrical polar coordinates (r, θ, z) are such that
$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$ as shown in fig. 3.1

$$\nabla^2 = \frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{3.1}$$

and

$$\bar{u}.\nabla = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z}, \tag{3.2}$$

where u_r, u_θ, and u_z are the physical components of the fluid velocity in cylindrical polar coordinates (r, θ, z) respectively.

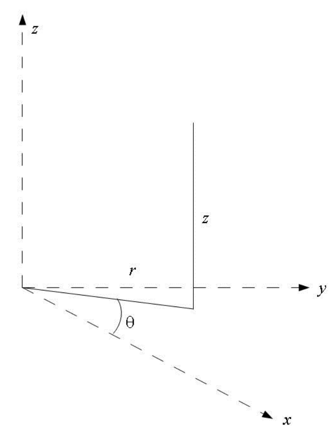

Figure 3.1: Cylindrical polar coordinates.

We consider an incompressible, Newtonian fluid with uniform viscosity and present the Navier-Stokes equations in vector form and in cylindrical polar coordinates.
In vector form the equation of motion of an incompressible Newtonian fluid is given by

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p + \mu\nabla^2\mathbf{u} + \rho\bar{g}, \tag{3.3}$$

$$\bar{\nabla}.\mathbf{u} = 0. \tag{3.4}$$

Mathematically the condition of incompressibility (which is the equation of continuity) is simply

$$\nabla.\mathbf{u} = 0.$$

In cylindrical coordinates (r, θ, z) the **continuity equation** for an incompressible fluid is given by

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0. \tag{3.5}$$

In cylindrical coordinates, the Navier-Stokes equations of motion for an incompressible fluid of constant dynamic viscosity ν, and density ρ are

r-component

$$\begin{gathered}\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} + u_z\frac{\partial u_r}{\partial z}\right) = \\ -\frac{\partial p}{\partial r} + \nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial^2 u_r}{\partial z^2}\right] + F_r \quad ,\end{gathered} \tag{3.6}$$

θ-component

$$\begin{gathered}\rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} - \frac{u_r u_\theta}{r} + u_z\frac{\partial u_\theta}{\partial z}\right) = \\ -\frac{1}{r}\frac{\partial p}{\partial \theta} + \nu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} + \frac{\partial^2 u_\theta}{\partial z^2}\right] + F_\theta \quad ,\end{gathered} \tag{3.7}$$

z-component

$$\begin{gathered}\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta} + u_z\frac{\partial u_z}{\partial z}\right) = \\ \frac{\partial p}{\partial z} + \nu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}\right] + F_z \quad ,\end{gathered} \tag{3.8}$$

where u_r, u_θ, u_z are the velocities in the r, θ, z cylindrical coordinates direction respectively, ρ is the density of the fluid, ν is the kinematic viscosity, p is the pressure, F_r, F_θ, F_z are the body force components in the r, θ, z directions and the operators $\dfrac{D}{Dt}$ and ∇^2 are defined as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z},$$

$$\nabla^2 = \frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.$$

Equations (3.6)-(3.8) are often called the momentum equations.

3.3 Mathematical Model

During the last decades extensive research work has been carried out on fluid dynamics of biological fluids in the presence of a magnetic field. In general, biological systems are affected by the application of an external magnetic field on blood flow through the human arterial system. Many mathematical models have already been investigated by several researchers [50]-[60] to explore the nature of blood flow under the influence of a magnetic field.

Blood flow is the study of measuring blood pressure and finding flow through blood vessels. This area of study is of great importance to our health. Most researchers conduct studies in blood flow in order to understand conditions which contribute to high blood pressure and other blood related diseases. An electromagnetic force acts on the blood and this force opposes the motion of blood and thus flow of blood is impeded, so that the external magnetic force can be used in the treatment of some kinds of diseases like cardiovascular diseases and in the disease with accelerated blood circulation such as hemorrhages and hypertension. Heat transfer in a biological system is relevant in many diagnostic and therapeutic applications that involve changes in temperature. The cardiovascular system is sensitive to changes in the environment, and flow characteristics of blood are modified to satisfy the changing demands of an organism. In addition to transporting oxygen, metabolites and other dissolved substances to and from tissues, blood flow alters heat transfer within the human body.

The simulation of complex, dynamic processes that appear in nature or in industrial applications poses challenging mathematical problems, opening a long road from the basic problem, to the mathematical modelling, the numerical simulation, and finally to the interpretation of results. In order to achieve these goals, interdisciplinarity research between applied mathematicians and experts in other fields becomes of increasing importance, since mathematical knowledge alone does not suffice in order to obtain a solution, but understanding the physics of the process is required as well.

The following steps should be followed in order to construct a mathematical model that will preserve the actual physical character of our problem

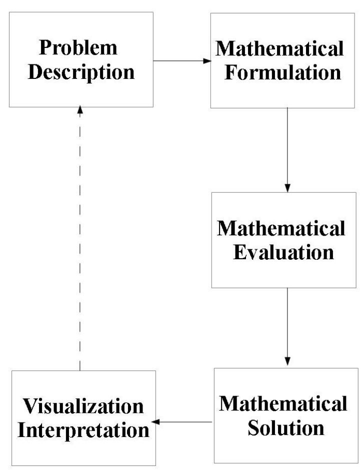

Figure 3.2: The process of mathematical modelling.

3.4 Modelling Blood Flow

Adequate modelling of blood flow in the human cardiovascular system and ability to solve the resulting very complex nonlinear equations would contribute materially to better understanding of pathogenesis of arterial disease and the design of artificial hearts. Such a solution would depend on solving the general Navier-Stokes equations without resorting to linearization and assumptions of "smallness."

Despite this, however, the solutions indeed leave much to be desired. For example, the ocean dealt with in hydrodynamics studies is a mathematized ocean, not the real ocean in which pressure, density, and velocity are fluctuating or stochastic variables. Nonlinearities are linearized and complex terms are dropped in order to obtain an approximate solution and the flow is assumed to be laminar since turbulence has not been tractable to mathematical analysis. Turbulence is a strongly nonlinear and stochastic phenomenon and mathematics needed is more complicated. The blood flow problem is modeled by these same equations as well as by boundary conditions that may be even more complicated. We are dealing with flow of a complex fluid through an elastic or viscoelastic vessel with branches, organs, prosthetic devices or natural heart valves, and tubes of varying diameter with possible motion of the walls under pulsatile flow because of elasticity. Detailed analysis of arterial flow should consider unsteady viscous flow and retain all the nonlinear terms. The assumptions of smallness-e.g., in wall motions-are unrealistic since such motions may have a significant effect on flow-as in the aorta.

The objective is to determine the velocity, knowing quantities such as pressure. Then it is possible to estimate stresses on the walls and possible wall damage or filtration, thrombus growth, or the behavior in an artificial heart. To be able to describe mathematically the flow of blood through prosthetic or natural heart valves or the flow about any obstacle in a vessel, or the analytical description of flow under conditions where it becomes non-laminar, could help diagnoses and design of prosthetic devices. Finally, we must be able to handle another hitherto unsolved problem-that of nonlinear boundary conditions as a result of nonlinear changes of area with pressure changes [42].

Consider steady, laminar and axially symmetric flow of blood through a locally constricted straight artery of infinite length under the influence of an external transverse magnetic field which is applied uniformly and perpendicular to the flow of blood. Blood exhibits remarkable non-Newtonian properties but in this case we assume that blood flowing through arteries to be a Newtonian, incompressible, homogeneous, and viscous fluid. The assumption of Newtonian behaviour of blood is acceptable for high shear rate flow, i.e blood flow in large arteries behaves almost like a Newtonian fluid. Also the viscosity is assumed to be constant. The flow of blood through an artery depends upon the pumping action of the heart which gives rise to a pressure gradient which produces an oscillatory flow in the blood vessel. The effect of an applied magnetic field is perpendicular to the flow of blood as shown in fig. 3.3. The appropriate equations governing the flow field in the tube are the momentum equations and these equations, after introducing the electro-magnetic force, are

$$\mathbf{u}.\nabla\mathbf{u} = -\frac{1}{\rho}.\nabla\mathbf{p} + \nu\nabla^2\mathbf{u} + \frac{1}{\rho}(\mathbf{I}\times\mathbf{B}) \tag{3.9}$$

where $\mathbf{u}$ is the velocity vector of the field, $\mathbf{p}$ is the pressure, $\nu = \dfrac{\mu}{\rho}$ is the kinematic viscosity, ρ is the density of the fluid, $\mathbf{I}$ is the current density, $\mathbf{B}$ is the magnetic field and the operator is the same as defined in eq. (3.1)

The current density and the magnetic field are expressed by Maxwell's equations and Ohm's Law, given by

$$\mathbf{I} = \sigma_e[\mathbf{E} + \mu_e(\mathbf{u}\times\mathbf{B})] \tag{3.10}$$

$$\nabla.\mathbf{B} = 0 \tag{3.11}$$

$$\nabla\times\mathbf{B} = 0 \tag{3.12}$$

where $\mathbf{E}$ is the electric field, σ_e is the conductivity of the field and μ_e is the magnetic permeability [31].

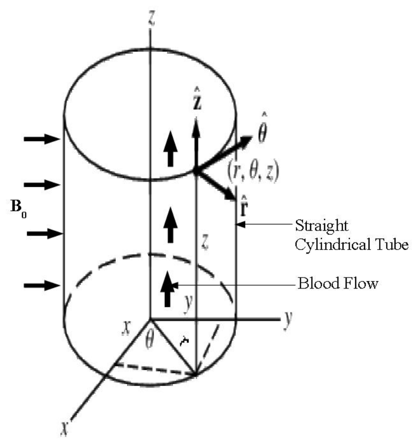

Figure 3.3: The geometry of the problem.

The electric fluid produced due to the motion of an electrically conducting fluid is very small. We assume that due to the effects of the induced magnetic field there is no external force applied. With these assumptions and the assumption of axially symmetric flow of fluid, the governing equations of motion of the fluid are the Navier-Stokes equations in cylindrical polar coordinates

z-component

$$\begin{aligned}
\frac{\partial \bar{u}_{\bar{z}}}{\partial \bar{t}} + (\mathbf{u}.\bar{\nabla})\bar{u}_{\bar{z}} &= -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{z}} + \nu \bar{\nabla}^2 \bar{u}_{\bar{z}} \\
\left(\bar{u}_{\bar{r}}\frac{\partial}{\partial \bar{r}} + \frac{\bar{u}_{\bar{\theta}}}{\bar{r}}\frac{\partial}{\partial \bar{\theta}} + \bar{u}_{\bar{z}}\frac{\partial}{\partial \bar{z}}\right)\bar{u}_{\bar{z}} &= -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{z}} + \nu\left[\frac{\partial^2}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}} + \frac{1}{\bar{r}^2}\frac{\partial^2}{\partial \bar{\theta}^2} + \frac{\partial^2}{\partial \bar{z}^2}\right]\bar{u}_{\bar{z}} + \bar{F}_{\bar{z}} \\
\bar{u}_{\bar{r}}\frac{\partial \bar{u}_{\bar{z}}}{\partial \bar{r}} + \bar{u}_{\bar{z}}\frac{\partial \bar{u}_{\bar{z}}}{\partial \bar{z}} &= -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{z}} + \nu\left(\frac{\partial^2 \bar{u}_{\bar{z}}}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial \bar{u}_{\bar{z}}}{\partial \bar{r}} + \frac{\partial^2 \bar{u}_{\bar{z}}}{\partial \bar{z}^2}\right) - \frac{B_0^2}{\rho}\bar{u}_{\bar{z}}.
\end{aligned}$$

Rearranging the terms on the left hand side we have

$$\bar{u}_{\bar{z}}\frac{\partial \bar{u}_{\bar{z}}}{\partial \bar{z}} + \bar{u}_{\bar{r}}\frac{\partial \bar{u}_{\bar{z}}}{\partial \bar{r}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{z}} + \nu\left(\frac{\partial^2 \bar{u}_{\bar{z}}}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial \bar{u}_{\bar{z}}}{\partial \bar{r}} + \frac{\partial^2 \bar{u}_{\bar{z}}}{\partial \bar{z}^2}\right) - \frac{B_0^2}{\rho}\bar{u}_{\bar{z}}. \tag{3.13}$$

r-component

$$\frac{\partial \bar{u}_{\bar{r}}}{\partial \bar{t}} + (\mathbf{u}.\bar{\nabla})\bar{u}_{\bar{r}} - \frac{\bar{u}_{\bar{\theta}}^2}{\bar{r}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{r}} + \nu\left(\bar{\nabla}^2\bar{u}_{\bar{r}} - \frac{\bar{u}_{\bar{r}}}{\partial \bar{r}} - \frac{2}{\bar{r}^2}\frac{\partial \bar{u}_{\bar{\theta}}}{\partial \bar{\theta}}\right) + \bar{F}_{\bar{r}}$$

$$\left(\bar{u}_{\bar{r}}\frac{\partial}{\partial \bar{r}} + \frac{\bar{u}_{\bar{\theta}}}{\bar{r}}\frac{\partial}{\partial \bar{\theta}} + \bar{u}_{\bar{z}}\frac{\partial}{\partial \bar{z}}\right)\bar{u}_{\bar{r}} - \frac{\bar{u}_{\bar{\theta}}}{\bar{r}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{r}} + \nu\left(\frac{\partial^2}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}} + \frac{1}{\bar{r}^2}\frac{\partial}{\partial \bar{r}} + \frac{\partial^2}{\partial \bar{z}^2} - \frac{1}{\bar{r}^2}\right)\bar{u}_{\bar{r}}$$

$$\bar{u}_{\bar{z}}\frac{\partial \bar{u}_{\bar{r}}}{\partial \bar{z}} + \bar{u}_{\bar{r}}\frac{\partial \bar{u}_{\bar{r}}}{\partial \bar{r}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{r}} + \nu\left(\frac{\partial^2 \bar{u}_r}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial u_r}{\partial \bar{r}} - \frac{\bar{u}_{\bar{r}}}{\bar{r}^2} + \frac{\partial^2 \bar{u}_r}{\partial \bar{z}^2}\right). \tag{3.14}$$

Since we are considering a steady state and if we let $\bar{u}_{\bar{z}} = \bar{u}$ and $\bar{u}_{\bar{r}} = \bar{v}$, then eqs. (3.13) and (3.14) become

$$\bar{u}\frac{\partial \bar{u}}{\partial \bar{z}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{r}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{z}} + \nu\left(\frac{\partial^2 \bar{u}}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial \bar{u}}{\partial \bar{r}} + \frac{\partial^2 \bar{u}}{\partial \bar{z}^2}\right) - \frac{B_0^2}{\rho}\bar{u}, \tag{3.15}$$

$$\bar{u}\frac{\partial \bar{v}}{\partial \bar{z}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{r}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial \bar{r}} + \nu\left(\frac{\partial^2 \bar{v}}{\partial \bar{r}^2} + \frac{1}{\bar{r}}\frac{\partial \bar{v}}{\partial \bar{r}} - \frac{\bar{v}}{\bar{r}^2} + \frac{\partial^2 \bar{v}}{\partial \bar{z}^2}\right), \tag{3.16}$$

and the continuity equation is

$$\frac{\partial}{\partial \bar{z}}(\bar{r}\bar{u}) + \frac{\partial}{\partial \bar{r}}(\bar{r}\bar{v}) = 0, \tag{3.17}$$

where $\bar{u}$ and $\bar{v}$ are the components of the fluid velocity in the axial and radial directions respectively, $B_0 = \mu_e H_0$ is the electromagnetic induction and H_0 is the transverse component of the magnetic field.
The geometry of the constriction is described by

$$\frac{\bar{R}(\bar{z})}{R_0} = 1 - \frac{\bar{\sum}}{R_0}\bar{f}(\bar{z}) \tag{3.18}$$

where $\bar{R}_0$ is the radius of the normal tube, $\bar{R}(\bar{z})$ is the radius of the tube in the stenotic region and $\bar{\sum}$ is the maximum height stenosis [21].
The boundary conditions are

$$\bar{u} = \bar{v} = 0 \quad \text{at} \quad \bar{r} = \bar{R}(\bar{z}), \tag{3.19}$$

$$\frac{\partial \bar{u}}{\partial \bar{r}} = \bar{r} \quad \text{at} \quad \bar{r} = 0, \tag{3.20}$$

$$\int_0^{\bar{R}(\bar{z})} \bar{r}\bar{u}\,\mathrm{d}\bar{r} = \frac{\bar{Q}}{2\pi}, \tag{3.21}$$

where $\bar{Q}$ is the constant volumetric flux across any cross-section of the tube. The equations that govern flow, under the assumed conditions, are the Navier-Stokes equations with continuity equation [21]. For convenience we write the system of equations from eq. (3.15) to eq. (3.21) in non-dimensional form using the following transformation variables

$$\begin{aligned} u &= \frac{\bar{u}}{U_0}, \quad v = \frac{\bar{v}}{U_0} \\ r &= \frac{\bar{r}}{R_0}, \quad z = \frac{\bar{z}}{R_0}, \quad p = \frac{\bar{p}}{\rho U_0^2} \end{aligned} \tag{3.22}$$

where (u, v) are the dimensionless velocity components, U_0 is the characteristic velocity, and p is the dimensionless fluid pressure.
Applying the above transformations to the dimensionless momentum equations in eqs. (3.15) and (3.16) yields

$$u\frac{\partial u}{\partial z} + v\frac{\partial u}{\partial r} = -\frac{\partial p}{\partial z} + \frac{1}{R_e}\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2}\right) - M^2 u \tag{3.23}$$

in the axial direction and

$$u\frac{\partial v}{\partial z} + v\frac{\partial v}{\partial r} = -\frac{\partial p}{\partial z} + \frac{1}{R_e}\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{3.24}$$

in the radial direction where R_e and M are the Reynolds number and the Hartmann number respectively, defined by

$$R_e = \frac{U_0 R_0}{\nu} \tag{3.25}$$

$$M = B_0 R_0 \sqrt{\frac{\sigma_e}{\mu}}. \tag{3.26}$$

Similarly, applying the transformation variables from eqs. (3.22) on the dimensionless continuity equation from eq. (3.17), the continuity equation can thus be written as

$$\frac{\partial}{\partial z}(ru) + \frac{\partial}{\partial r}(rv) = 0, \tag{3.27}$$

and the geometry of constrictions from eq. (3.18) takes the form

$$\eta(z) = 1 - \sum f(z), \tag{3.28}$$

where

$$\begin{aligned}
\eta(z) &= \frac{\bar{R}_{\bar{z}}}{R_0} \\
f(z) &= \frac{\bar{f}(\bar{z})}{R_0} \\
\sum &= \frac{\bar{\sum}}{R_0}.
\end{aligned} \tag{3.29}$$

Applying transformations from eq. (3.22) to the dimensionless boundary conditions in eqs. (3.19) to (3.21) we have

$$u = v = 0 \quad \text{at} \quad r = \eta \tag{3.30}$$

$$\frac{\partial u}{\partial r} = 0 \quad \text{at} \quad r = 0 \tag{3.31}$$

$$\int_0^{\eta(z)} r u \mathrm{d}r = -\frac{1}{2}. \tag{3.32}$$

Let us now introduce the stream function ψ defined by

$$u = -\frac{1}{r}\frac{\partial \psi}{\partial r}, \tag{3.33}$$

$$v = \frac{1}{r}\frac{\partial \psi}{\partial z}. \tag{3.34}$$

The continuity equation in eq. (3.27) is satisfied identically and by using eqs. (3.33) and (3.34) we eliminate p between eqs. (3.23) and (3.24). Hence we have the following governing equation

$$R_e\left[\frac{1}{r}J - \frac{2}{r^2}\nabla^2\psi\frac{\partial \psi}{\partial z}\right] = \nabla^4\psi - M^2 r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right), \tag{3.35}$$

where J is the Jacobian defined as

$$\begin{aligned}
J &= \frac{\partial(\nabla^2\psi, \psi)}{\partial(r, z)} \\
&= \begin{vmatrix} \dfrac{\partial(\nabla^2\psi)}{\partial r} & \dfrac{\partial \psi}{\partial r} \\ \dfrac{\partial(\nabla^2\psi)}{\partial z} & \dfrac{\partial \psi}{\partial z} \end{vmatrix} \\
&= \frac{\partial(\nabla^2\psi)}{\partial r}.\frac{\partial \psi}{\partial z} - \frac{\partial(\nabla^2\psi)}{\partial z}.\frac{\partial \psi}{\partial r};
\end{aligned} \tag{3.36}$$

and the operator ∇^2 is given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{3.37}$$

The boundary conditions relating to eq. (3.35) in terms of ψ are as follows

$$-\frac{1}{r}\frac{\partial \psi}{\partial r} = 0, \quad \psi = -\frac{1}{2} \quad \text{at} \quad r = \eta \tag{3.38}$$

$$-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) = \psi = 0 \quad \text{at} \quad r = 0. \tag{3.39}$$

Equation (3.35) governing the flow of blood through arteries in the presence of a magnetic field is a nonlinear partial differential equation subject to the boundary conditions eqs. (3.38) and (3.39) and the exact solution of this equation is not always available. This equation can be solved using traditional numerical techniques which will result in very large numerical computations. We will use the ADM to solve this nonlinear partial differential equation in the next chapter.

Chapter 4

Mathematical Evaluation

4.1 Introduction

As ambitious as it may appear, it is fortuitous that the ADM appears to be capable of solutions in a fairly wide class of problems. The superficial resemblance of this method to some other methods can be misleading; the proof is in the fact that it solves problems not always solvable by other methods, or only solvable with much more difficulty or computation. The method is an "approximation" method, not a "closed form" solution. The usual significance of these terms is that in the one case, we have an exact answer and in the other an approximate one. Clearly, however, a method of solution which changes the problem to a different, easier mathematical problem and then solves it exactly is not to be preferred to one in which the actual nonlinear and/or stochastic model is treated with an "approximate" method which provides accurate, rapidly convergent, and computable series of terms.

4.2 The Adomian Decomposition Method Applied to the Equation Governing Blood Flow in the Presence of a Magnetic Field

From eq. (3.35) the equation governing flow of blood through arteries in the presence of a magnetic field as a nonlinear partial differential equation given by the following expression

$$R_e\left[\frac{1}{r}J-\frac{2}{r^2}\nabla^2\psi\frac{\partial\psi}{\partial z}\right]=\nabla^4\psi-M^2r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right), \tag{4.1}$$

where R_e and M are the Reynolds number and the Hartmann number respectively defined by

$$R_e = \frac{U_0R_0}{\nu}, \tag{4.2}$$

$$M = B_0R_0\sqrt{\frac{\sigma_e}{\mu}}; \tag{4.3}$$

J is the Jacobian defined as

$$\begin{aligned} J &= \frac{\partial(\nabla^2\psi,\psi)}{\partial(r,z)} \\ &= \begin{vmatrix} \dfrac{\partial(\nabla^2\psi)}{\partial r} & \dfrac{\partial\psi}{\partial r} \\ \dfrac{\partial(\nabla^2\psi)}{\partial z} & \dfrac{\partial\psi}{\partial z} \end{vmatrix} \\ &= \frac{\partial(\nabla^2\psi)}{\partial r}\cdot\frac{\partial\psi}{\partial z}-\frac{\partial(\nabla^2\psi)}{\partial z}\cdot\frac{\partial\psi}{\partial r}; \end{aligned} \tag{4.4}$$

and the operator ∇^2 is given by

$$\nabla^2=\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}. \tag{4.5}$$

The boundary conditions are

$$-\frac{1}{r}\frac{\partial\psi}{\partial r}=0, \quad \psi=-\frac{1}{2} \quad \text{at} \quad r=\eta, \tag{4.6}$$

$$-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) = \psi = 0 \quad \text{at} \quad r = 0. \tag{4.7}$$

We now recall basic principles of the ADM for solving differential equations subject to boundary conditions.

In order for us to apply the ADM we need an invertible operator L to decompose eq. (4.1). Consider the operator ∇^2 in eq. (4.5)

$$\nabla^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{4.8}$$

Let $L = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}$ be an operator
i.e

$$\begin{aligned}
\nabla^2 &= L + \frac{\partial^2}{\partial z^2} \\
\nabla^4 &= L^2 + 2L\frac{\partial^2}{\partial z^2} + \frac{\partial^4}{\partial z^4} \\
\nabla^4\psi &= L^2\psi + 2L\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^4\psi}{\partial z^4}.
\end{aligned} \tag{4.9}$$

Substituting eq. (4.9) and the operator L into eq. (4.1), we obtain

$$L^2\psi = R_e N\psi - \frac{\partial^4\psi}{\partial z^4} - 2\frac{\partial^2}{\partial z^2}(L\psi) + M^2 L\psi \tag{4.10}$$

where

$$N\psi = \frac{1}{r}J - \frac{2}{r^2}\nabla^2\psi\frac{\partial\psi}{\partial z} \tag{4.11}$$

represents the nonlinear term. Also note that

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) = L\psi. \tag{4.12}$$

(see Appendix B).

Operating on both sides of eq. (4.10) with the inverse operator L^{-2}, we obtain

$$\psi = \psi_0 + L^{-2}\left[R_e N\psi - \frac{\partial^4\psi}{\partial z^4} - 2\frac{\partial^2}{\partial z^2}(L\psi) + M^2 L\psi\right]. \tag{4.13}$$

In this case ψ_0 is a solution to the homogeneous equation

$$L^2\psi_0 = 0 \tag{4.14}$$

and it is given by

$$\psi_0 = \frac{1}{4}r^4 B(z) + \left[\frac{1}{2}\log r - \frac{1}{4}\right] r^2 C(z) + \frac{1}{2}r^2 E(z) + F(z). \tag{4.15}$$

The functions B, C, E and F in eq. (4.15) are to be determined using the given boundary conditions in eqs. (4.6) and (4.7). The ADM decomposes the solution ψ and the nonlinear term $N\psi$ into the following parametrized forms

$$\psi = \sum_{n=0}^{\infty} \lambda^n \psi_n \tag{4.16}$$

$$N\psi = \sum_{n=0}^{\infty} \lambda^n A_n \tag{4.17}$$

where A_n are the Adomian polynomials and the parameter λ used in eqs. (4.16) and (4.17) is not a perturbation parameter; it is only used for grouping the terms of different orders. Then the parameterized form of eq. (4.13) is given by

$$\psi = \psi_0 + \lambda^n L^{-2}\left[R_e N\psi - \frac{\partial^4\psi}{\partial z^4} - 2\frac{\partial^2}{\partial z^2}(L\psi) + M^2 L\psi\right]. \tag{4.18}$$

Substituting eqs. (4.16) and (4.17) into eq. (4.13) we obtain

$$\sum_{n=0}^{\infty} \lambda^n \psi_n = \psi_0 + \lambda^n L^{-2}\left[R_e \sum_{n=0}^{\infty} \lambda^n A_n - \frac{\partial^4\psi}{\partial z^4} - 2\frac{\partial^2}{\partial z^2}(L\psi) + M^2 L\psi\right]. \tag{4.19}$$

Comparing the like-power terms of λ on both sides of eq. (4.19) we obtain the following expression

$$\psi_{n+1} = L^{-2}\left[R_e A_n - \frac{\partial^4 \psi_n}{\partial z^4} - 2\frac{\partial^2 (L\psi_n)}{\partial z^2} + M^2 L\psi_n\right] \tag{4.20}$$

where $n = 0, 1, 2, \ldots$ and A_n are the Adomian polynomials. Once the component ψ_0 is determined, the other components of ψ such as, ψ_1, ψ_2, ψ_3 can be easily determined from eq. (4.20). We have applied the regular ADM on the above expression.
If we further take the parameterized decomposition of ψ_0 given by

$$\psi_0 = \sum_{n=0}^{\infty} \lambda^n \psi_{0,n} \tag{4.21}$$

i.e we have applied the Adomian double decomposition.
By substituting eqs. (4.16), (4.17), and (4.21) into eq. (4.13) we obtain

$$\sum_{n=0}^{\infty} \lambda^n \psi_n = \sum_{n=0}^{\infty} \lambda^n \psi_{0,n} + \lambda^n L^{-2}\left[R_e \sum_{n=0}^{\infty} \lambda^n A_n - \frac{\partial^4 \psi}{\partial z^4} - 2\frac{\partial^2}{\partial z^2}(L\psi) + M^2 L\psi\right]. \tag{4.22}$$

This is the double decomposition of ψ given by

$$\psi_{n+1} = \psi_{0,n+1} + L^{-2}\left[R_e A_n - \frac{\partial^4 \psi_n}{\partial z^4} - 2\frac{\partial^2 (L\psi_n)}{\partial z^2} + M^2 L\psi_n\right]. \tag{4.23}$$

Since the expression ψ_0 contains the constants B, C, E, and F then the parameterized decomposition forms of all these constants are as follows

$$\begin{aligned} B &= \sum_{n=0}^{\infty} \lambda^n B_n, \\ C &= \sum_{n=0}^{\infty} \lambda^n C_n, \\ E &= \sum_{n=0}^{\infty} \lambda^n E_n, \\ F &= \sum_{n=0}^{\infty} \lambda^n F_n. \end{aligned} \tag{4.24}$$

Substitute eqs. (4.21) and (4.24) into eq. (4.15) to obtain the following expression

$$\sum_{n=0}^{\infty} \lambda^n \psi_{0,n} = \frac{1}{4} r^4 \sum_{n=0}^{\infty} \lambda^n B_n + \left[\frac{1}{2}\log r - \frac{1}{4}\right] r^2 \sum_{n=0}^{\infty} \lambda^n C_n + \frac{1}{2} r^2 \sum_{n=0}^{\infty} \lambda^n E_n + \sum_{n=0}^{\infty} \lambda^n F_n. \tag{4.25}$$

Comparing the like-power terms of λ on both sides of eq. (4.25) we obtain

$$\psi_{0,n+1} = \frac{1}{4} r^4 B_{n+1}(z) + \left[\frac{1}{2}\log r - \frac{1}{4}\right] r^2 C_{n+1}(z) + \frac{1}{2} r^2 E_{n+1}(z) + F_{n+1}(z), \tag{4.26}$$

Using the relations (4.23) and (4.26) together we will obtain the components of ψ. The constants involved in each ψ_n will be determined by their respective boundary conditions.
The approximations of ψ from eq. (4.23) are given as follows.
When $n = 0$, we have

$$\psi_1 = \psi_{0,1} + L^{-2}\left[R_e A_0 - \frac{\partial^4 \psi_0}{\partial z^4} - 2\frac{\partial^2 (L\psi_0)}{\partial z^2} + M^2 L\psi_0\right], \tag{4.27}$$

where

$$\psi_{0,1} = \frac{1}{4} r^4 B_1(z) + \left[\frac{1}{2}\log r - \frac{1}{4}\right] r^2 C_1(z) + \frac{1}{2} r^2 E_1(z) + F_1(z). \tag{4.28}$$

Likewise, when $n = 1$, we have

$$\psi_2 = \psi_{0,2} + L^{-2}\left[R_e A_1 - \frac{\partial^4 \psi_1}{\partial z^4} - 2\frac{\partial^2 (L\psi_1)}{\partial z^2} + M^2 L\psi_1\right], \tag{4.29}$$

where

$$\psi_{0,2} = \frac{1}{4} r^4 B_2(z) + \left[\frac{1}{2}\log r - \frac{1}{4}\right] r^2 C_2(z) + \frac{1}{2} r^2 E_2(z) + F_2(z), \tag{4.30}$$

and so on.
The polynomials $A_0, A_1, \ldots, A_n$ are the Adomian polynomials. They are

defined in such a way that $A_0 \equiv A_0(\psi_0)$, $A_1 \equiv A_1(\psi_0, \psi_1)$, $A_2 \equiv A_2(\psi_0, \psi_1, \psi_2)$, ..., $A_n \equiv A_n(\psi_0, \psi_1, \ldots, \psi_n)$.
In order to determine these polynomials, we substitute eqs. (4.16) and (4.17) into eq. (4.11)

$$\sum_{n=0}^{\infty} \lambda^n A_n = \frac{1}{r} \cdot \frac{\partial(\nabla^2 \sum_{n=0}^{\infty} \lambda^n \psi_n, \sum_{n=0}^{\infty} \lambda^n \psi_n)}{\partial(r, z)} - \frac{2}{r^2} \cdot \frac{\partial}{\partial z} \left(\sum_{n=0}^{\infty} \lambda^n \psi_n \right) \cdot \nabla^2 \sum_{n=0}^{\infty} \lambda^n \psi_n. \tag{4.31}$$

Comparing the terms of the like powers of λ on both sides of eq. (4.31) yields the following expression

$$A_n = \frac{1}{r} \cdot \frac{\partial(\nabla^2 \psi_n, \psi_n)}{\partial(r, z)} - \frac{2}{r^2} \cdot \frac{\partial \psi_n}{\partial z} \cdot \nabla^2 \psi_n. \tag{4.32}$$

Equation (4.32) gives the following set of Adomian polynomials

$$\begin{aligned} A_0 &= \frac{1}{r} \cdot \frac{\partial(\nabla^2 \psi_0, \psi_0)}{\partial(r, z)} - \frac{2}{r^2} \cdot \frac{\partial \psi_0}{\partial z} \cdot \nabla^2 \psi_0 \\ A_1 &= \frac{1}{r} \left[\frac{\partial(\nabla^2 \psi_1, \psi_0)}{\partial(r, z)} + \frac{\partial(\nabla^2 \psi_0, \psi_1)}{\partial(r, z)} \right] \\ &- \frac{2}{r^2} \left[\frac{\partial \psi_0}{\partial z} \cdot \nabla^2 \psi_1 + \frac{\partial \psi_1}{\partial z} \cdot \nabla^2 \psi_0 \right] \\ A_2 &= \frac{1}{r} \left[\frac{\partial(\nabla^2 \psi_2, \psi_0)}{\partial(r, x)} + \frac{\partial(\nabla^2 \psi_1, \psi_1)}{\partial(r, z)} + \frac{\partial(\nabla^2 \psi_0, \psi_2)}{\partial(r, z)} \right] \\ &- \frac{2}{r^2} \left[\frac{\partial \psi_0}{\partial z} . \nabla^2 \psi_2 + \frac{\partial \psi_1}{\partial z} . \nabla^2 \psi_1 + \frac{\partial \psi_2}{\partial z} . \nabla^2 \psi_0 \right] \\ &\vdots \end{aligned} \tag{4.33}$$

Now we need to substitute eq. (4.16) into the boundary conditions (4.6) and (4.7). This gives the boundary conditions for the respective components of ψ_0, ψ_1, ψ_2, etc. as follows

$$\left. \begin{aligned} -\frac{1}{r} \cdot \frac{\partial \psi_0}{\partial r} = 0, \quad \psi_0 = -\frac{1}{2} \quad &\text{at} \quad r = \eta, \\ -\frac{\partial}{\partial r} \left(\frac{1}{r} \cdot \frac{\partial \psi_0}{\partial r} \right) = \psi_0 = 0 \quad &\text{at} \quad r = 0, \end{aligned} \right\} \tag{4.34}$$

and

$$\left.\begin{aligned} -\frac{1}{r}\cdot\frac{\partial\psi_n}{\partial r} = \psi_n = 0 \quad &\text{at} \quad r = \eta, \\ -\frac{\partial}{\partial r}\left(\frac{1}{r}\cdot\frac{\partial\psi_n}{\partial r}\right) = \psi_n = 0 \quad &\text{at} \quad r = 0, \end{aligned}\right\} \tag{4.35}$$

for any positive integer.

4.3 Two-Term Approximation of ψ

Before we proceed to the solutions we need to determine the inverse operator L^{-1} and to do that we consider the following equation for ψ

$$L\psi = F, \tag{4.36}$$

which on solving, gives

$$\psi = [L_1^{-1}r(L_1^{-1}r^{-1})]F, \tag{4.37}$$

remembering that the boundary condition terms vanish and L_1^{-1} is a one-fold indefinite integral. It is obvious from the relation (4.37) that the inverse L_1^{-1} is identified as

$$L^{-1} = [L_1^{-1}r(L_1^{-1}r^{-1})]. \tag{4.38}$$

Hence we obtain

$$L^{-2} = L_1^{-1}[rL_1^{-1}\{r^{-1}L_1^{-1}(rL_1^{-1}r^{-1})\}]. \tag{4.39}$$

Using the boundary conditions in eqs. (4.34) into eq. (4.15) we obtain the expression for ψ_0 (see Appendix C) as

$$\psi_0 = \frac{1}{2\eta^4}(r^4 - 2\eta^2 r^2). \tag{4.40}$$

To obtain the expression for ψ_1 we use eqs. (4.27) and (4.28) and this expression involves the operator L^{-2} given by eq. (4.39). Performing the operation of the inverse operator (see eq. (D.22)) , we obtain

$$\begin{aligned}\psi_1 &= \frac{\alpha(z)r^{10}}{5040} + \frac{\beta(z)r^8}{1680} + \frac{\gamma(z)r^6}{360} \\ &+ \frac{1}{4}r^4 B_1 + \left[\int r\log r \mathrm{d}r\right] C_1 + \frac{1}{2}r^2 E_1 + F_1, \end{aligned} \tag{4.41}$$

where B_1, C_1, D_1 and F_1 are integration constants to be obtained from the boundary conditions (F.14).

We can rewrite eq. (4.41) as

$$\begin{aligned}\psi_1 &= \alpha r^{10} + \beta r^8 + \gamma r^6 \\ &+ \frac{1}{4}r^4 B_1 + \left[\int r\log r \mathrm{d}r\right] C_1 + \frac{1}{2}r^2 E_1 + F_1 \end{aligned} \tag{4.42}$$

where

$$\alpha = \frac{R_e}{315\eta^{11}}\left(20\eta_1^3 - 13\eta\eta_1\eta_2 + \eta^2\eta_3\right), \tag{4.43}$$

$$\begin{aligned}\beta &= \frac{R_e}{105\eta^9}\left(8\eta_1 - 16\eta_1^3 + 11\eta\eta_1\eta_2 - \eta^2\eta_3\right) \\ &+ \frac{1}{420\eta^8}\left(180\eta\eta_1^2\eta_2 - 210\eta_1^4 - 15\eta^2\eta_2^2 - 20\eta^2\eta_1\eta_3 + \eta^3\eta_4\right), \end{aligned} \tag{4.44}$$

$$\begin{aligned}\gamma &= \frac{1}{180\eta^6}\left(60\eta_1^4 - 160\eta_1^2 + 32\eta\eta_2 - 72\eta\eta_1^2\eta_2 + 9\eta^2\eta_2^2 + 12\eta^2\eta_1\eta_3 - \eta^3\eta_4\right) \\ &+ \frac{R_e}{90\eta^7}\left(12\eta_1^3 - 9\eta\eta_1\eta_2 + \eta^2\eta_3 - 16\eta_1\right) + \frac{M^2}{45\eta^4}. \end{aligned} \tag{4.45}$$

The resulting expression of ψ_1 as in eq. (E.9) is

$$\begin{aligned}\psi_1 &= \alpha r^{10} + \beta r^8 + \gamma r^6 - \eta^2(4\alpha\eta^4 + 3\beta\eta^2 + 2\gamma)r^4 \\ &+ \eta^4(3\alpha\eta^4 + 2\beta\eta^2 + \gamma)r^2, \end{aligned} \tag{4.46}$$

where α, β, γ are defined in eqs. (4.43), (4.44) and (4.45) respectively, η_1, η_2, η_3 and η_4 are the derivatives of η with respect to z indicating the orders according to their subscripts.

The two-term approximation of the solution of ψ from eq. (4.16), is

$$\begin{aligned}\psi &= \psi_0 + \psi_1 \\ &= \frac{1}{2\eta^4}(r^4 - 2\eta^2 r^2) + \alpha r^{10} + \beta r^8 + \gamma r^6 - \eta^2(4\alpha\eta^4 + 3\beta\eta^2 + 2\gamma)r^4 \\ &+ \eta^4(3\alpha\eta^4 + 2\beta\eta^2 + \gamma)r^2, \end{aligned} \tag{4.47}$$

where α, β, γ are defined in eqs. (4.43), (4.44) and (4.45) respectively and remembering that the convergence of the solution ψ is $\lambda = 1$.

The axial velocity component is given by the expression

$$\begin{aligned} u &= -\left[\frac{1}{r} \cdot \frac{\partial \psi_0}{\partial r} + \frac{1}{r} \cdot \frac{\partial \psi_1}{\partial r}\right], \\ &= \frac{2}{\eta^2} - \frac{4r^2}{\eta^4} - 2(5\alpha r^8 + 4\beta r^6 + 3\gamma r^4 - 8r^2\alpha\eta^6 \\ &- 6r^2\beta\eta^4 - 4r^2\gamma\eta^2 3\alpha\eta^8 + 2\beta\eta^6 + \gamma\eta^4) \end{aligned} \tag{4.48}$$

The wall shear stress is defined by

$$\tau = -\frac{1}{4}\left(\frac{\partial u}{\partial r}\right)_{r=\eta} (1 + \eta_1^2), \tag{4.49}$$

$$= \frac{1}{\eta^3}\left[2 + 2\eta^6\left(6\alpha\eta^4 + 3\beta\eta^2 + \gamma\right)\right](1 + \eta_1^2). \tag{4.50}$$

The wall shear stress in eq. (4.50) is a result of substituting u from eq. (4.48) into eq. (4.49).

To assess the validity of the assumptions made in section 3.4 a series of experiments was performed. The following stenosis geometry(see fig. 4.1) was selected and was described by the expression

$$f(z) = \frac{1}{2}\left(1 + \cos\frac{\pi z}{L_0}\right), \quad -L_0 \leq z \leq L_0 \tag{4.51}$$

The geometry of constriction from eq. (3.28) takes the following form

$$\eta(z) = 1 - \frac{1}{2}\Sigma\left(1 + \cos\frac{\pi z}{L_0}\right), \quad -L_0 \leq z \leq L_0 \tag{4.52}$$

The wall shear stress of the two-term approximation solution ψ in eq. (4.50) is a result of substituting the axial velocity u from eq. (4.48) into eq. (4.49).

Figures. 4.2 and 4.3 show variations of the wall shear stress (4.58) along the length of the constricted artery for different values of the Reynolds number and the Hartmann number for a two-term solution of ψ.

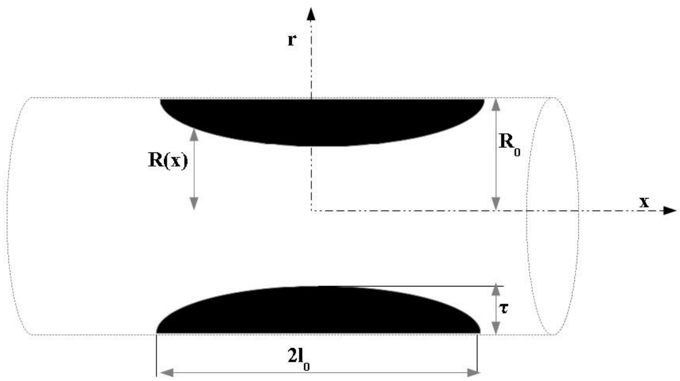

Figure 4.1: Schematic diagram of the model geometry.

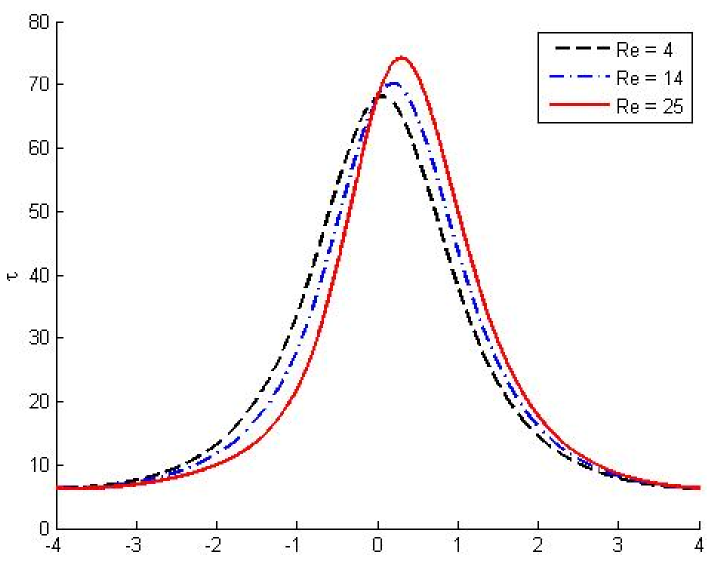

Figure 4.2: Distribution of wall shear stress for $\Sigma = \frac{2}{3}$ and $M = 10$.

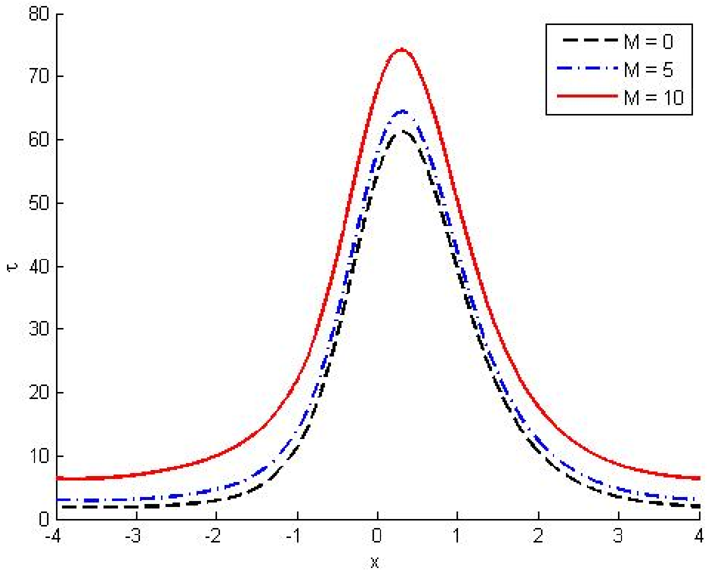

Figure 4.3: Distribution of wall shear stress for $R_e = 25$ and $\Sigma = \frac{2}{3}$.

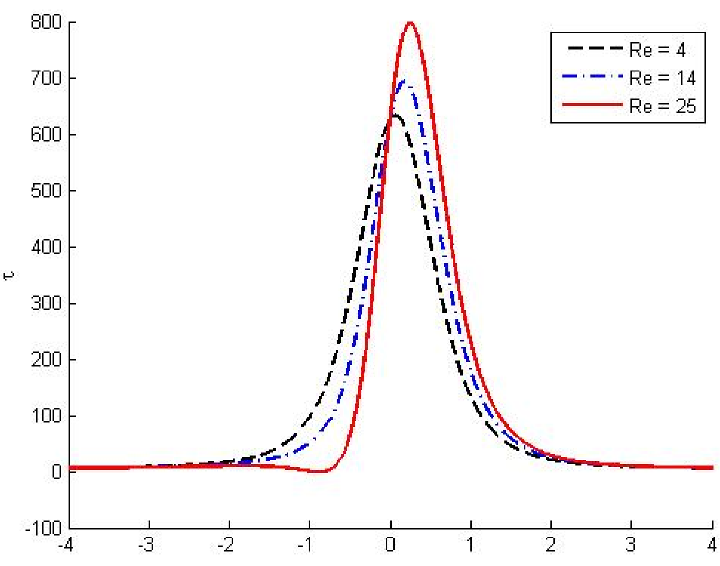

Figure 4.4: Distribution of wall shear stress for $\Sigma = 0.85$ and $M = 10$.

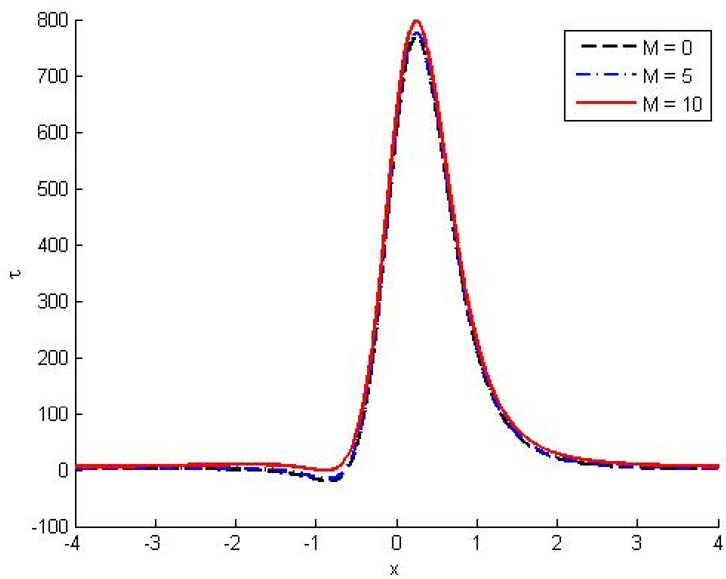

Figure 4.5: Distribution of wall shear stress for $R_e = 25$ and $\Sigma = 0.85$.

Figures 4.2, 4.3, 4.4, and 4.5 show that the maximum value of the solution occurs just ahead of the throat of stenosis for different values of the Reynolds number R_e, Hartmann number M and Σ for a two-term approximation to the solution. In figs. 4.4 and 4.5, the negative distribution of the solution is observed over some length in the diverging section as we increase the value of Σ. This negative behavior of the wall shear stress indicates separation which involves circulation with back flow near the wall. As a result of this back flow, a low shear exists at the wall and a high velocity core surrounded by the separated region is formed. Figure 4.5 shows the variation of wall shear stress with x for different values of Hartmann number M. It is seen that the negative behavior of the solution observed in the diverging section of the tube decreases with increasing Hartmann number. As a result the circulation diminishes indicating the favorable physiological condition of blood flow.

4.4 Three-Term Approximation of ψ

In this section we seek the expression of ψ_2 such that the solution of ψ is given by

$$\psi = \psi_0 + \psi_1 + \psi_2 \tag{4.53}$$

that is, to obtain the expression of ψ_2 we use eqs. (4.29), and (4.30). To do this we also require the operator L^{-2} from eq.(4.39). Performing the inverse operator(see eq. (F.11)) from eq. (4.39), we obtain

$$\begin{aligned}\psi_2 &= \zeta r^{14} + \vartheta r^{12} + \xi r^{10} + \varsigma r^8 + \chi r^6 - r^4\eta^2\left(6\zeta\eta^8 + 5\vartheta\eta^6 + 4\xi\eta^4 + 3\varsigma\eta^2 + 2\chi\right) \\ &+ r^2\left(5\zeta\eta^{12} + 4\vartheta\eta^{10} + 3\xi\eta^8 + 2\varsigma\eta^6 + \chi\eta^4\right)\end{aligned} \tag{4.54}$$

where $\zeta, \vartheta, \xi, \varsigma, \chi$ are defined in eq. (F.13). Hence, considering the three-term solution of ψ from eq. (4.16)is given by

$$\begin{aligned}\psi &= \psi_0 + \psi_1 + \psi_2 \\ &= \frac{1}{2\eta^4}(r^4 - 2\eta^2 r^2) + \alpha r^{10} + \beta r^8 + \gamma r^6 - \eta^2 r^4(4\alpha\eta^4 + 3\beta\eta^2 + 2\gamma) \\ &+ \eta^4 r^2(3\alpha\eta^4 + 2\beta\eta^2 + \gamma) + \zeta r^{14} + \vartheta r^{12} + \xi r^{10} + \varsigma r^8 + \chi r^6 \\ &- r^4\eta^2\left(6\zeta\eta^8 + 5\vartheta\eta^6 + 4\xi\eta^4 + 3\varsigma\eta^2 + 2\chi\right) + r^2\left(5\zeta\eta^{12} + 4\vartheta\eta^{10} + 3\xi\eta^8 + 2\varsigma\eta^6 + \chi\eta^4\right)\end{aligned} \tag{4.55}$$

where α, β, γ are defined in eqs. (4.43), (4.44) and (4.45) respectively; $\zeta, \vartheta, \xi, \varsigma, \chi$ are defined in eq. (F.13) and remembering that the convergence of the solution ψ is $\lambda = 1$.

The axial velocity component of the three-term approximation of the solution ψ is given by

$$\begin{aligned}
u &= -\left[\frac{1}{r}\cdot\frac{\partial\psi_0}{\partial r} + \frac{1}{r}\cdot\frac{\partial\psi_1}{\partial r} + \frac{1}{r}\cdot\frac{\partial\psi_2}{\partial r}\right], \\
&= \frac{2}{\eta^2} - \frac{4r^2}{\eta^4} - 2(5\alpha r^8 + 4\beta r^6 + 3\gamma r^4) - 2(3\alpha\eta^8 + 2\beta\eta^6 + \gamma\eta^4) \\
&- 14\zeta r^{12} - 12\vartheta r^{10} - 10\xi r^8 + 8\varsigma r^6 - 6\chi r^4 + 4r^2\eta^2\left(6\zeta\eta^8 + 5\vartheta\eta^6 + 4\xi\eta^4 + 3\varsigma\eta^2 + 2\chi\right) \\
&- 2\left(5\zeta\eta^{12} + 4\vartheta\eta^{10} + 3\xi\eta^8 + 2\varsigma\eta^6 + \chi\eta^4\right). \qquad (4.56)
\end{aligned}$$

The wall shear stress of the three-term approximation of the solution ψ is defined by

$$\tau = -\frac{1}{4}\left(\frac{\partial u}{\partial r}\right)_{r=\eta}(1+\eta_1^2) \qquad (4.57)$$

$$= \frac{1}{\eta^3}\left[2 + 2\eta^3(6\alpha\eta^7 + 3\beta\eta^5 + \gamma\eta^3 + 15\zeta\eta^{11} + 10\vartheta\eta^9 + 6\xi\eta^7 + 3\varsigma\eta^5 + \chi\eta^3)\right](1+\eta_1^2). \qquad (4.58)$$

The wall shear stress of the three-term approximation solution ψ in eq. (4.58) is a result of substituting the axial velocity u from eq. (4.56) into eq. (4.57).

Figures 4.6 and 4.7 show the variations of the wall shear stress eq. (4.58) along the length of the constricted artery for different values of Reynolds number R_e and the Hartmann number M for a three-term solution of ψ.

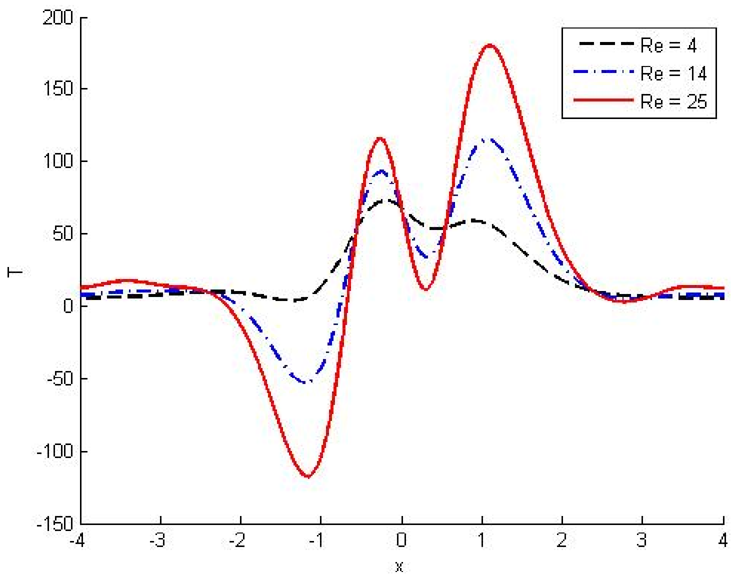

Figure 4.6: Distribution of wall shear stress for $\Sigma = \frac{2}{3}$ and $M = 10$.

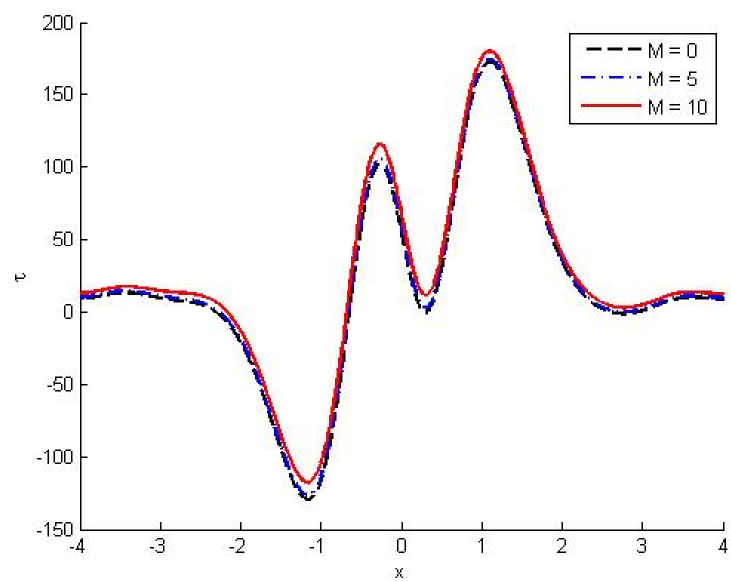

Figure 4.7: Distribution of wall shear stress for $R_e = 25$ and $\Sigma = \frac{2}{3}$.

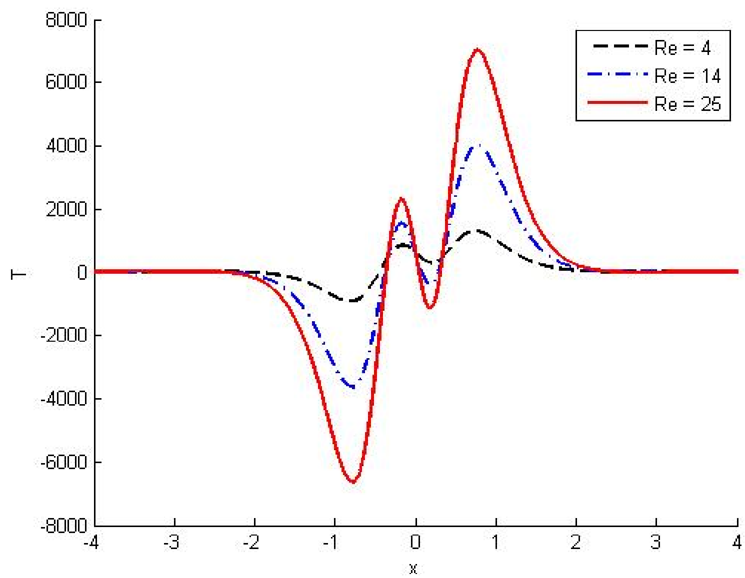

Figure 4.8: Distribution of wall shear stress for $\Sigma = 0.85$ and $M = 10$.

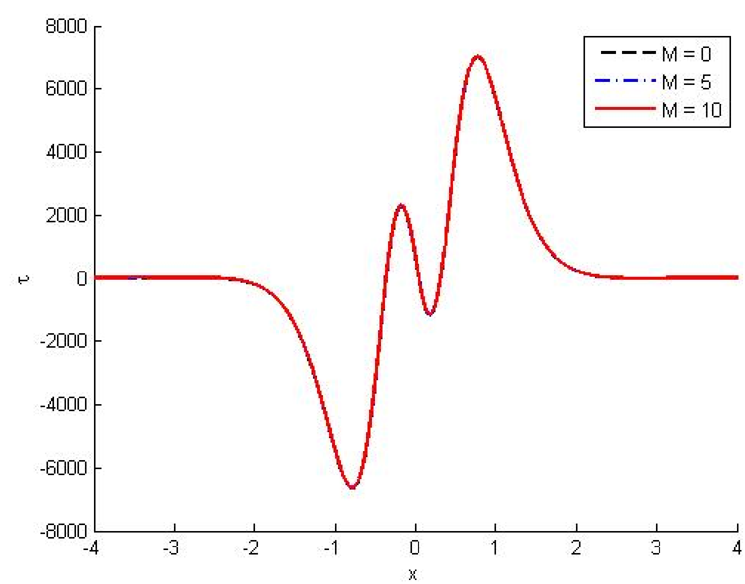

Figure 4.9: Distribution of wall shear stress for $R_e = 25$ and $\Sigma = 0.85$.

Figures 4.6, 4.7, 4.8, and 4.9 show that the maximum value of the solution occurs just ahead of the throat of stenosis for different values of the Reynolds number R_e, Hartmann number M and Σ for a three-term approximation to the solution. With the increase of the Reynolds number the negative behavior of the solution increases showing the enlargement of circulation which is physiologically unfavorable. Figure 4.8 shows the variation of wall shear stress with x for different values of the Reynolds number R_e. It is seen that the negative behavior of the solution observed in the diverging section of the tube decreases with increasing Reynolds number. As we increase the values of Hartmann number M in fig. 4.9 the negative behavior is constant in the diverging section of the tube. As a result the circulation diminishes indicating the favorable physiological condition of blood flow.

The application of an external magnetic field B_0 to blood flow generates electromagnetic inductions in the fluid that reduce the flow rate and flatten the velocity profile while stretching it more prominently in the same direction as the applied field. These effects heighten when B_0 increases; the induced magnetic fields however remain very weak. Approximating the results by neglecting these induced fields, while assuming velocity profile symmetry, overestimates flow reductions. Even though the induced magnetic fields are insignificant, solving the problem while neglecting them is inexact. Therefore, it can be concluded that the effect of an external transverse magnetic field applied uniformly favors the physiological condition of blood flow.

Chapter 5

Effect of a Magnetic Field on Blood Flow

5.1 Introduction

Many cardiovascular diseases, particularly atherosclerosis (medically called stenosis), found to be responsible for deaths in developed and developing countries, are closely related to the nature of blood movement and the dynamic behaviour of blood vessels. From medical surveys, it is well known that more than 80% of the total deaths are due to the diseases of blood vessel walls. Among them stenosis is a dangerous disease that is caused due to deposition of cholesterol and some other substances on the endothelium and by the proliferation of connective tissues in the arterial wall. The reason for formation of stenosis in the lumen of an artery is not known but its effect over the flow characteristics has been studied by many researchers [50]-[66]. Stenosis may develop at more than one location of the cardiovascular system. Stenosis means the abnormal and unnatural growth in the lumen of an artery that develops at various locations in the cardiovascular system under unfavorable conditions.

The study of blood rheology and blood flow has several objectives such as not only understanding health and disease but also in essence, what kind of fluid it is. Blood is assumed to be Newtonian in nature in large blood vessels such as the aorta and in medium and small vessels blood is a non-Newtonian fluid. Blood is a suspension of plasma and the plasma which is a solution of proteins, electrolytes and other substances, is considered to be a

Newtonian fluid. From a biochemical point of view, blood is considered an intelligent fluid, probably the most one in nature capable of adapting itself to a great extent in order to provide nutrients to the organs. Blood is regarded as a magnetic fluid, in which red blood cells are magnetic in nature. Liquid carriers in the blood contain the magnetic suspension of the particles [66].

Human body experiences magnetic fields of moderate to high intensity in many situations of day to day life. In recent times, many medical diagnostics especially those used in diagnosing cardiovascular disease make use of magnetic fields. In this section we present the effect of an externally applied magnetic field over the flow characteristics of blood in a single stenosed artery.

5.2 Mathematical Model

Consider steady, laminar and axially symmetric flow of blood through an artery with a mild stenosis under the influence of an externally applied homogeneous magnetic field. The blood flowing in the tube is assumed to be a suspension of red blood cells in plasma. We also assume that the density of the fluid is constant but the viscosity varies radially and that the electromagnetic force produced is very small. Under this assumption of small electrical conductivity, the one dimensional equation of motion is given by

$$\frac{\partial p}{\partial x} + \frac{1}{r}\frac{\partial (r\tau_{rx})}{\partial r} + \beta_0^2 \sigma_e u = 0, \tag{5.1}$$

where p is the fluid pressure, u is the axial velocity component, $\beta_0 = \mu_e H_0$ is the electromagnetic induction, μ_e is the magnetic permeability, H_0 is the intensity of the magnetic field, and σ_e is the conductivity of the fluid.

The shear stress τ_{rx} is given by

$$\tau_{rx} = -\mu(r)\frac{du}{dr} \tag{5.2}$$

where $\mu(r)$ is the coefficient of viscosity of blood proposed by Einstein defined as

$$\mu(r) = \mu_0[1 + \beta h(r)]. \tag{5.3}$$

where μ_0 is the coefficient of viscosity of plasma, $\beta = 2.5$ is a constant and

$h(r)$ is the hematocrit described by an imperial formula

$$h(r) = h_m \left[1 - \left(\frac{r}{R_0}\right)^n\right]. \tag{5.4}$$

R_0 is the radius of the normal tube, h_m is the maximum hematocrit at the center of the tube and $n(\geq 2)$ is a parameter determining the exact shape of the profile. The relation (5.4) is valid only for a very dilute suspension of red cells which are supposed to be spherical in shape.

The stenosis develops symmetrically about the tube axis but it is non-symmetric with respect to radial coordinates and its geometry is described by

$$\frac{R(x)}{R_0} = 1 - A[l_0^{s-1}(x-d) - (x-d)^s], \quad d \leq x \leq d + l_0, \tag{5.5}$$

where $R(x)$ is the radius of the stenosed artery, l_0 is the length of stenosis, d indicates its location, and A is given by

$$A = \frac{\varepsilon}{R_0 l_0^s}, \tag{5.6}$$

where $s(\geq 2)$ is a parameter determining the shape of stenosis and ε denotes the maximum height of stenosis at

$$x = d + \frac{l_0}{S^{\frac{1}{(s-1)}}} \tag{5.7}$$

such that $\frac{\varepsilon}{R_0} < 1$ see fig. 5.1

The boundary conditions are

$$u = 0 \quad \text{at} \quad r = R(x) \tag{5.8}$$

$$\frac{du}{dr} = 0 \quad \text{at} \quad r = 0. \tag{5.9}$$

Consider eq. (5.4). We introduce the following transformations

$$\begin{aligned} y &= \frac{r}{R_0} \\ r &= yR_0 \\ dr &= R_0 dy \end{aligned} \tag{5.10}$$

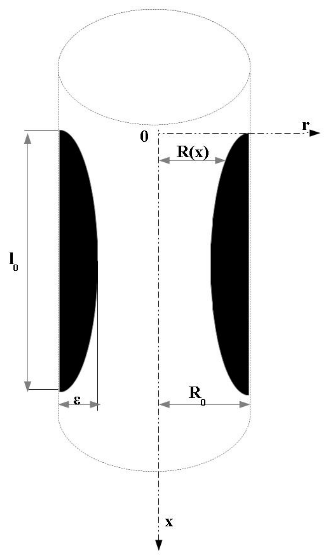

Figure 5.1: Geometry of Constrictions.

then substituting eqs. (5.10), (5.4), (5.3), and (5.2) into eq. (5.1), we obtain

$$\begin{aligned}
\frac{dp}{dx} + \frac{1}{r}\frac{d}{dr}\left(r\tau_{rx}\right) + \beta_0^2\sigma_e u &= 0 \\
\frac{dp}{dx} + \frac{1}{yR_0}\frac{d}{R_0 dy}\left[-yR_0\mu_0\left(1+\beta h_m\left(1-y^n\right)\frac{du}{R_0 dy}\right)\right] + \beta_0^2\sigma_e u &= 0 \\
\frac{-R_0^2}{\mu_0}\frac{dp}{dx} + \frac{1}{y}\frac{d}{dy}\left[-y\left(1+\beta h_m\left(1-y^n\right)\frac{du}{dy}\right)\right] - \frac{\beta_0^2 R_0^2\sigma_e}{\mu_0}u &= 0
\end{aligned}$$

$$\frac{1}{y}\frac{d}{dy}\left[y(a-ky^n)\frac{du}{dy}\right] - M^2 u = \frac{R_0^2}{\mu_0}\cdot\frac{dp}{dx} \tag{5.11}$$

where

$$\begin{aligned}
k &= \beta h_m. \\
a &= 1+k
\end{aligned} \tag{5.12}$$

and M is the Hartmann number defined as

$$M = B_0 R_0\sqrt{\frac{\sigma_e}{\mu_0}}. \tag{5.13}$$

The corresponding boundary conditions in eqs. (5.8) and (5.9) take the form

$$u = 0 \quad \text{at} \quad y = \frac{R(x)}{R_0} \tag{5.14}$$

$$\frac{du}{dy} = 0 \quad \text{at} \quad y = 0. \tag{5.15}$$

5.3 Effect of a Magnetic Field on Blood Flow using the Frobenius Method

In mathematics, the method of Frobenius, named after Ferdinand Georg Frobenius, is a way of determining an infinite series solution for a second-order ordinary differential equation in the vicinity of a regular singular point. Equation (5.11) subject to boundary conditions in eqs. (5.14) and (5.15) can be solved using the Frobenius method. It is required that u is bounded at $y = 0$ and that the only admissible series solution of the governing equation eq. (5.11) is

$$u = D\sum_{m=0}^{\infty} C_m y^m + \frac{R_0^2}{4a\mu_0} \cdot \frac{dp}{dx} \cdot \sum_{m=0}^{\infty} \bar{C}_m y^{m+2} \tag{5.16}$$

where D is an arbitrary constant to be determined by the boundary conditions in eqs. (5.14) and (5.15). C_m and $\bar{C_m}$ are given by the following expressions

$$C_{m+1} = \frac{k(m+1)(m-n+1)C_{m-n+1} + M^2 C_{m-1}}{a(m+1)^2} \tag{5.17}$$

and

$$\bar{C}_{m+1} = \frac{k(m+3)(m-n+3)\bar{C}_{m-n+1} + M^2 \bar{C}_{m-1}}{a(m+3)^2} \tag{5.18}$$

remembering that C_0 and $\bar{C}_0$ are to be taken equal to unity and

$$C_{-m} = \bar{C}_{-m} = 0. \tag{5.19}$$

Applying the boundary condition in eq. (5.14) into eq. (5.16) we obtain

$$D = -\frac{\dfrac{R_0^2}{4a\mu_0}\left[\displaystyle\sum_{m=0}^{\infty} \bar{C}_m \left(\frac{R}{R_0}\right)^{m+2}\right]}{\displaystyle\sum_{m=0}^{\infty} C_m \left(\frac{R}{R_0}\right)^m} \tag{5.20}$$

and the resulting expression of u is given by

$$\begin{aligned} u &= -\frac{R_0^2}{4a\mu_0}\cdot\frac{dp}{dx}\left[\sum_{m=0}^{\infty}\bar{C}_m\left(\frac{R}{R_0}\right)^{m+2}\cdot\sum_{m=0}^{\infty}\bar{C}_m y^m\right. \\ &- \left.\sum_{m=0}^{\infty}C_m\left(\frac{R}{R_0}\right)^{m}\cdot\sum_{m=0}^{\infty}\bar{C}_m y^{m+2}\right]/\sum_{m=0}^{\infty}C_m\left(\frac{R}{R_0}\right)^{m}. \end{aligned} \tag{5.21}$$

If u_0 is the average velocity given by

$$u_0 = -\frac{R_0^2}{8\mu_0}\left(\frac{dp}{dx}\right)_0 \tag{5.22}$$

where $\left(\frac{dp}{dx}\right)_0$ is the pressure gradient of blood flow in the normal tube in the absence of a magnetic field and hematocrit, then the non-dimensional form of u with respect to u_0 is given by the following expression

$$\begin{aligned} \frac{u}{u_0} &= \frac{2}{a}\cdot\frac{\left(\frac{dp}{dx}\right)}{\left(\frac{dp}{dx}\right)_0}\left[\sum_{m=0}^{\infty}\bar{C}_m\left(\frac{R}{R_0}\right)^{m+2}\cdot\sum_{m=0}^{\infty}C_m y^m\right. \\ &- \left.\sum_{m=0}^{\infty}C_m\left(\frac{R}{R_0}\right)^{m}\cdot\sum_{m=0}^{\infty}\bar{C}_m y^{m+2}\right]/\sum_{m=0}^{\infty}C_m\left(\frac{R}{R_0}\right)^{m}. \end{aligned} \tag{5.23}$$

The volumetric flow rate Q of fluid in the stenotic region is given by the expression

$$Q = 2\pi R_0\int_0^{R(x)/R_0} u(y)\cdot y\mathrm{d}y. \tag{5.24}$$

Substituting u from eq. (5.21) into eq. (5.24) and then integrating with respect to y we obtain the following expression

$$\begin{aligned} Q &= -\frac{\pi R_0^3}{2a\mu_0}\cdot\frac{dp}{dx}\left[\sum_{m=0}^{\infty}\bar{C}_m\left(\frac{R}{R_0}\right)^{m+2}\cdot\sum_{m=0}^{\infty}\frac{C_m}{m+2}\left(\frac{R}{R_0}\right)^{m+2}\right. \\ &- \left.\sum_{m=0}^{\infty}C_m\left(\frac{R}{R_0}\right)^{m}\cdot\sum_{m=0}^{\infty}\frac{\bar{C}_m}{m+4}\left(\frac{R}{R_0}\right)^{m+4}\right]/\sum_{m=0}^{\infty}C_m\left(\frac{R}{R_0}\right)^{m} \end{aligned} \tag{5.25}$$

If Q_0 is the flow rate of plasma fluid in the unconstricted tube in the absence of magnetic field and hematocrit, then

$$Q_0 = \frac{\pi R_0^3}{8\mu_0}\cdot\left(\frac{dp}{dx}\right)_0 \tag{5.26}$$

where $\left(\dfrac{dp}{dx}\right)_0$ is the pressure gradient of fluid. If the flow is steady and the system is closed, then $\left(\dfrac{Q}{Q_0}\right) = 1$ and we have the relative pressure gradient from eqs. (5.25) and (5.26) as

$$\left(\frac{Q}{Q_0}\right) = 1$$

$$\Rightarrow \frac{\left(\frac{dp}{dx}\right)}{\left(\frac{dp}{dx}\right)_0} =$$

$$\frac{-a\sum_{m=0}^{\infty} C_m\left(\frac{R}{R_0}\right)^m}{4\sum_{m=0}^{\infty}\bar{C}_m\left(\frac{R}{R_0}\right)^{m+2}\cdot\sum_{m=0}^{\infty}\frac{C_m}{m+2}\left(\frac{R}{R_0}\right)^{m+2} - 4\sum_{m=0}^{\infty}C_m\left(\frac{R}{R_0}\right)^m\cdot\sum_{m=0}^{\infty}\frac{\bar{C}_m}{m+4}\left(\frac{R}{R_0}\right)^{m+4}}. \tag{5.27}$$

The wall shear stress is defined as

$$\tau_R = -\left[-\mu(r)\frac{du}{dr}\right]_{r=R(x)}. \tag{5.28}$$

Note that $\left(\dfrac{du}{dr}\right)_{r=R(x)}$ from eq. (5.28) can be obtained by differentiating eq. (5.21) where $y = \frac{r}{R_0}$, that is

$$\begin{aligned}
\left(\frac{du}{dr}\right)_{r=R(x)} &= -\frac{R_0^2}{4a\mu_0}\cdot\frac{dp}{dx}\frac{1}{R_0}\left[\sum_{m=0}^{\infty}\bar{C}_m\left(\frac{R}{R_0}\right)^{m+2}\cdot\sum_{m=0}^{\infty}m\bar{C}_m\left(\frac{r}{R_0}\right)^{m-1}\right.\\
&\quad - \left.\sum_{m=0}^{\infty}C_m\left(\frac{R}{R_0}\right)^m\cdot\sum_{m=0}^{\infty}(m+2)\bar{C}_m\left(\frac{r}{R_0}\right)^{m+1}\right]\Big/\sum_{m=0}^{\infty}C_m\left(\frac{R}{R_0}\right)^m\\
&= -\frac{R_0}{4a\mu_0}\cdot\frac{dp}{dx}\left[\sum_{m=0}^{\infty}\bar{C}_m\left(\frac{R}{R_0}\right)^{m+2}\cdot\sum_{m=0}^{\infty}m\bar{C}_m\left(\frac{R}{R_0}\right)^{m-1}\right.\\
&\quad - \left.\sum_{m=0}^{\infty}C_m\left(\frac{R}{R_0}\right)^m\cdot\sum_{m=0}^{\infty}(m+2)\bar{C}_m\left(\frac{R}{R_0}\right)^{m+1}\right]\Big/\sum_{m=0}^{\infty}C_m\left(\frac{R}{R_0}\right)^m
\end{aligned} \tag{5.29}$$

and

$$-\mu(r)\mid_{r=R(x)} = -\mu_0\left[1+\beta h_m\left(1-\left(\frac{R}{R_0}\right)^n\right)\right]. \tag{5.30}$$

Hence, the wall shear stress in the absence of magnetic field and hematocrit is given by

$$\tau_R = \frac{R_0}{4a}\cdot\frac{dp}{dx}\left[\sum_{m=0}^{\infty}\bar{C}_m\left(\frac{R}{R_0}\right)^{m+2}\cdot\sum_{m=0}^{\infty}m\bar{C}_m\left(\frac{R}{R_0}\right)^{m-1}\right. \tag{5.31}$$

$$- \sum_{m=0}^{\infty} C_m \left(\frac{R}{R_0}\right)^m \cdot \sum_{m=0}^{\infty} (m+2)\bar{C}_m \left(\frac{R}{R_0}\right)^{m+1} \Bigg] \Bigg/ \sum_{m=0}^{\infty} C_m \left(\frac{R}{R_0}\right)^m .$$

If $\tau_N = -\frac{R_0}{2}\left(\frac{dp}{dx}\right)_0$ is the shear stress of the plasma fluid at the normal tube wall in the absence of a magnetic field, then the non-dimensional form of eq. (5.31) is given by

$$\tau = \frac{1}{2a} \cdot \frac{\left(\frac{dp}{dx}\right)}{\left(\frac{dp}{dx}\right)_0} \left[\sum_{m=0}^{\infty} \bar{C}_m \left(\frac{R}{R_0}\right)^{m+2} \cdot \sum_{m=0}^{\infty} m\bar{C}_m \left(\frac{R}{R_0}\right)^{m-1} \right. \tag{5.32}$$

$$\left. - \sum_{m=0}^{\infty} C_m \left(\frac{R}{R_0}\right)^m \cdot \sum_{m=0}^{\infty} (m+2)\bar{C}_m \left(\frac{R}{R_0}\right)^{m+1} \right] \Bigg/ \sum_{m=0}^{\infty} C_m \left(\frac{R}{R_0}\right)^m$$

where $\frac{\left(\frac{dp}{dx}\right)}{\left(\frac{dp}{dx}\right)_0}$ is given by the expression in eq. (5.27).

Since the wall shear stress in stenosed arteries increases with a rise in the intensity of the magnetic field, there is a possibility that if the magnetic field strength is quite high, the wall shear stress may become so amplified that the stenosis may rupture and the concerned portion of the body may become paralyzed. The motion of blood in stenosed arteries can be regulated by applying a magnetic field externally and increasing/decreasing the intensity of the applied field.

Chapter 6

Conclusions

It is well known that a general method for determining analytical solutions for partial differential equations has not yet been found among traditional methods [28]. However, we believe that the ADM, the noise terms phenomenon, and the modified ADM provide an effective, reliable, and powerful tool for handling nonlinear differential equations.

We point out that the ADM works effectively for nonlinear ordinary and partial differential equations. The ADM and the improvements made by the noise terms phenomenon and the modified ADM are reliable and effective techniques with promising results. This was demonstrated by applying the ADM and the modified ADM to problems of ordinary and partial differential equations. The ADM can be used generally to all types of differential and integral equations. The ADM can be applied in a straightforward manner and it provides a rapidly convergent series solution. The speed of convergence depends upon the choice of operator which may be a highest-ordered differential operator or a combination of linear operators or a multidimensional operator. The ADM essentially unifies the subjects of linear and nonlinear, deterministic and stochastic, ordinary and partial differential equations, initial value and boundary value problems as well as systems of coupled equations into a single fundamental method [16].

We successfully applied the ADM to the equation governing blood flow through a constricted artery in the presence of an external transverse magnetic field which was applied uniformly. We assumed that blood flowing through the tube was a Newtonian in character. The flow of blood through an artery depends upon the pumping action of the heart which gives rise to a pressure gradient which produces an oscillatory flow in the blood vessel. The electric fluid produced due to the motion of an electrically conducting fluid was very small. We assumed that due to the effects of the induced magnetic fields no external applied force exists and the governing equations of motion of the fluid are the Navier-Stokes equations in cylindrical polar coordinates. We applied the ADM to obtain a two-term approximation and a three-term approximation to the solution of the stream function. The axial velocity component and wall shear stress were obtained. The numerical solutions of the wall shear stress for different values of the Reynolds number R_e and the Hartmann number M indicated that the maximum value of the solution occurred just ahead of the throat of stenosis for different values of the Reynolds number R_e, Hartmann number M and Σ for a three-term approximation to the solution. With the increase of the Reynolds number the negative behavior of the solution increases showing the enlargement of circulation which is physiologically unfavorable. It was observed that the negative behavior of the solution in the diverging section of the tube decreased with increasing Reynolds number. As we increased the values of the Hartmann number M the negative behavior was constant in the diverging section of the tube. As a result the circulation diminished indicating the favorable physiological condition of blood flow.

We investigated the effect of a externally applied homogeneous magnetic field on the flow characteristics in a single constricted blood vessel analytically by using the Frobenius method and concluded that the motion of blood in stenosed arteries can be regulated by applying a magnetic field externally and increasing/decreasing the intensity of the applied field.

The application of an external magnetic field B_0 to blood flow generates electromagnetic inductions in the fluid that reduce the flow rate and flatten the velocity profile while stretching it more prominently in the same direction as the applied field. These effects heighten when B_0 increases; the induced magnetic fields however remain very weak. Approximating the results by neglecting these induced fields, while assuming velocity profile symmetry, overestimates flow reductions. Even though the induced magnetic fields are insignificant, solving the problem while neglecting them is inexact.

Therefore, it can be concluded that the effect of an external transverse magnetic field applied uniformly favors the physiological condition of blood flow. This will encourage medical researchers/biomedical engineers to control the flow of blood in human cardiovascular and neural circulation systems artificially by applying a uniform magnetic field perpendicular to direction of blood flow. It will be of great importance in the treatment of cardiovascular lesions such as atherosclerosis plaques, intimal cushions and aneurysms etc, which tend to occur near the apex of bifurcation and the disease related with accelerated circulation like hypertension and brain hemorrhages etc.

The results can be further improved by considering more terms of the stream function ψ in the approximation to the solution. We extended the work of [31] to include a three-term approximation to the solution of the stream function. Future work could consider applying the Differential Transform Method (DTM) to the problem and comparing results obtained with ADM.

Appendix A

The Foundation of the Adomian Decomposition Method

A.1 The Basic Concepts Of The Decomposition Theory

The goal of the ADM is to solve an equation $u = G(u)$, in a Banach space E, where G is an operator which can be nonlinear. The Banach space E is not necessarily a finite-dimensional space, it can be a functional space. The ADM is an original approach to this kind of problem.

Definition 4 (Decomposition series of finite-order p)**.** *A decomposition series of finite-order p is a series $\sum C_k$, where each C_k is an E-valued function of the $p(k+1)$ variables $X_0^{(1)}, \ldots, X_k^{(1)}, \ldots, X_0^{(p)}, \ldots, X_k^{(p)}$.*

The decomposition series of first order is simply called the decomposition series.

Definition 5 (Weak convergence of the decomposition series of finite-order p)**.** *A decomposition series of finite-order p is weakly convergent if for each collection of p convergent series in E $\left(\sum u_n^{(1)}, \ldots, {}_n^{(p)}\right)$, the series*

$$\sum C_k \left(u_0^{(1)}, \ldots, u_k^{(1)}, \ldots, u_0^{(p)}, \ldots, u_k^{(p)}\right)$$

in E converge.

Definition 6 (Sum of a convergent decomposition series of finite-order p)**.** *The sum of a decomposition series of finite-order p is a function of p variables mapping the set of convergent series (in E) into E.*

$$S\left(\sum u_n^{(1)}, \dots, \sum u_n^{(p)}\right) = \sum_{k=0}^{\infty} C_k\left(u_0^{(1)}, \dots, u_k^{(1)}, \dots, u_0^{(p)}, \dots, u_k^{(p)}\right).$$

Definition 7 (Strong convergence of the decomposition of finite-order p)**.** *A decomposition series of finite-order p is strongly convergent if it is weakly convergent and if its sum is depends only on the sum of the series in E, i.e.*

$$\sum_{n=0}^{\infty} u_n^{(i)} = \sum_{n=0}^{\infty} v_n^{(i)}$$

$$\Rightarrow \quad S\left(\sum u_n^{(1)}, \dots, \sum u_n^{(p)}\right) = S\left(\sum v_n^{(1)}, \dots, \sum v_n^{(p)}\right), \quad \forall i \epsilon [1,p].$$

Definition 8 (Degenerated sum of strongly convergent decomposition series of finite-order p)**.** *Using the previously defined sum S of a convergent decomposition series of finite-order p, a new operator S^*, mapping E^p into E can be created when the convergence is strong. S and S^* can be identified.*

Let S be the sum of a strongly convergent decomposition series of finite-order p. Then for each collection $(u^{(1)}, \dots, u^{(p)})$ of elements of E, $S^*(u^{(1)}, \dots, u^{(p)})$ can be defined (because of the strong convergence) by $S(\sum u^{(1)}, \dots, \sum u^{(p)})$, where each $\sum u_n^{(i)}$ is any convergent series in E the sum of which is $u^{(i)}$. As a series of this kind, the the one which is reduced to one term equal to $u^{(i)}$ can be chosen. So, it can be written

$$S^*(u^{(1)}, \dots, u^{(p)}) = S(u^{(1)}, \dots, u^{(p)}).$$

Definition 9 (Decomposition scheme)**.** *Let $\sum C_k(X_0, \dots, X_k)$ be a strongly convergent decomposition series. The decomposition scheme associated with $\sum C_k$ is the recurrent scheme*

$$\begin{aligned} u_0 &= 0 \\ u_{n+1} &= C_n(u_0, \dots, u_n) \end{aligned} \tag{A.1}$$

which constructs a series $\sum u_n$ in E.

Such a series can be constructed because each C_n is a function of $u_0, \dots, u_n$ but not of the following terms.

A.2 The Basic Decomposition Series

Definition 10 (Basic decomposition series). *The basic decomposition series associated with the operator G is the series $\sum B_n$, where*

$$\begin{aligned} B_0 &= 0, \\ B_n &= G\left(\sum_{i=0}^{n} X_i\right) - G\left(\sum_{i=0}^{n-1} X_i\right). \end{aligned} \tag{A.2}$$

Each B_n is mapping E^{n+1} into E.

Theorem A.2.1 (Convergence of the basic decomposition series). *The basic decomposition series $\sum B_n$ associated with a continuous operator G is a decomposition series (of first order) which strongly converges and the degenerated sum of which is G.*

Proof. If $\sum u_n$ converges, then the series in E, $\sum B_n(u_0, \ldots, u_n)$ converges and its sum $\sum B_n G(\sum_{n=0}^{\infty} u_n)$ only depends on the sum of $\sum u_n$ $\square$

If G is a nonlinear operator, the basic decomposition series is useless because the Adomian decomposition method needs much more calculus that the successive approximations method to solve the equation $u = G(u)$. However, if G is a linear operator, the Adomian decomposition scheme becomes simpler as shown below.

Definition 11 (Basic decomposition series associated with a linear operator). *The basic decomposition series $\sum B_k$ associated with the linear operator L is*

$$\begin{aligned} B_0 &= L(X_0) \\ B_n &= L(X_n). \end{aligned} \tag{A.3}$$

A.3 The Adomian Decomposition Series

We apply the Adomian decomposition theorem to decompose these analytic functions into components to be computed by recursion as originally proposed by Adomian. A key notion is that we thus effectively rearrange the Banach-space analog of the Taylor expansion series about the solution's initial component function, or Taylor-Adomian expansion series, into an Adomian decomposition series with corresponding degenerate sums. This effective rearrangement via decomposition is justified because the Banach-space analog of the Taylor expansion series uniformly converges and thus the sum will not be altered by rearrangement of the terms in such a series. Thus, we say that the Adomian decomposition series for the solution and the Adomian decomposition series for nonlinear functions of the solution are designed to converge since we are clearly rearranging a uniformly convergent series of an analytic function into another uniformly convergent series with identical degenerate sums. We decompose the solution and nonlinear functions of the solution in order to solve the Adomian integral equation for the solution by recursion, hence thereby solving the associated nonlinear partial differential equation. As Adomian often stated, the decomposition of a function is non-unique, and hence permits the exercise of the creative art of computation and algorithm synthesis. Of course, the degenerate sum is unique. It is important to observe that we only bring in the Banach contraction theorem in the setting of the Adomian integral equation for the solution, because of how the ADM uses the linear composite operator coefficients of the Adomian integral equation to construct the solution. The ADMis a constructive method. Others, e.g. Hazewinkel in the Series Editor's Preface [13], observed that the Adomian decomposition series is distinct from the Neumann series and indeed is very much unlike Picard's method of successive approximations. For solving nonlinear differential equations ([4], [11]-[13]), the ADM is distinctly different from other techniques such as ad hoc transformations, perturbation, successive approximations, finite differences, or sinc-Galerkin methods [16]. Furthermore, the ADM is preferred for solving stochastic differential equations [8]-[11] for the statistical measures, e.g. the expectation, correlation and so forth, of solution processes as pioneered by Adomian, when dealing with physically realistic applications without a priori assumptions of stationarity, ergodicity, white noise, Gaussian processes, etc.

Definition 12 (Adomian's polynomials)**.** *Let G be an analytical function and $\sum u_n$ a convergent series in a Banach space E. The Adomian polynomials are defined by*

$$A_k = \frac{1}{k!}\left[\frac{d^k}{d\lambda^k} G\left(\sum_{n=0}^{\infty} u_n \lambda^n\right)\right]_{\lambda=0}. \tag{A.4}$$

Define $U = \sum_{n=0}^{\infty} u_n$ and $u^+ = \sum_{n=0}^{\infty} u_n \lambda^n$. This power series converges when $\lambda = 1$. But it is known that the sum of a power series, whose converge radius is ρ, is analytical over $OD(O, \rho)$ (open disc whose center is O and whose radius is ρ), then u^+ is analytical over $DO(O, \rho)$, i.e. there are A_k so that $G \circ u^+(\lambda) = \sum_{k=0}^{\infty} A_k \lambda^k$ and these A_k verify

$$A_k = \frac{1}{k!} \left[\frac{d^k}{d\lambda^k} G \left(\sum_{n=0}^{\infty} u_n \lambda^n \right) \right]_{\lambda=0} . \tag{A.5}$$

Note. We do not need to assume that the convergence radius is greater than 1. If $\rho = 1$, as u^+ converges and its sum is U, then the Abel's theorem leads to $\lim_{\lambda \to 1^-} u^+(\lambda) = U$ (λ being a real number) and so $\lim_{\lambda \to 1^-} G \circ u^+(\lambda) = G(U)$.

Theorem A.3.1 (Convergence of the Adomian decomposition series)**.** *The Adomian decomposition series $\sum C_k$ associated with the analytical function G define a decomposition series (of the first order) which strongly converges and the degeneration sum of which is G.*

Proof. To verify that each A_k depends only on $u_0, \ldots, u_k$, we express A_k as a function of the coefficients of the two series that are composed using a classical theorem of power series composition [38]. We note that the expression obtained is only used to prove this dependence. We have just prove that the $\sum A_k$ is a decomposition series. If $\sum u_n$ is a convergent series, we have seen that $\sum A_k(u_0, \ldots, u_k)$ converges and that its sum is $G \circ u^+(1) = G(U)$, that is to say that the decomposition series $\sum A_k$ weakly converges, and that its sum is G. If $\sum u_n$ and $\sum v_n$ are two series having the same sum U, and if their Adomian's polynomials are A_k^u and A_k^v respectively, then we write

$$\begin{aligned} \sum_{k=0}^{\infty} A_k^u &= G \circ u^+(\lambda = 1) \\ &= G \circ v^+(\lambda = 1) \\ &= \sum_{k=0}^{\infty} A_k^v. \end{aligned} \tag{A.6}$$

So, the sum of the Adomian decomposition series depends only on the sum of the considered series the convergence is strong. □

Appendix B

Equation Governing Blood Flow

From eq. (4.1) for the last term we have $r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right)$ and comparing with the operator $L = \dfrac{\partial^2}{\partial r^2} - \dfrac{1}{r}\dfrac{\partial}{\partial r}$ in chapter 4, we have

$$\begin{aligned} r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) &= r\left[-\frac{1}{r^2}\frac{\partial \psi}{\partial r} + \frac{1}{r}\frac{\partial \psi^2}{\partial r^2}\right] \\ &= -\frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial \psi^2}{\partial r^2} \\ &= L\psi \end{aligned} \tag{B.1}$$

Appendix C

Applying boundary conditions (4.34) to obtain an expression for ψ_0

Using boundary conditions (4.34) to solve the functions $B(z), C(z), E(z), F(z)$ from eq. (4.15) and thus to obtain the expression of ψ_0.

From eq. (4.15)

$$\psi_0 = \frac{1}{4}r^4 B(z) + \left[\int r \log r dr\right] C(z) + \frac{1}{2}r^2 E(z) + F(z) \tag{C.1}$$

and the boundary conditions from eq. (4.34) are

$$\left.\begin{array}{l} -\dfrac{1}{r} \cdot \dfrac{\partial \psi_0}{\partial r} = 0, \quad \psi_0 = -\dfrac{1}{2} \quad \text{at} \quad r = \eta, \\ -\dfrac{\partial}{\partial r}\left(\dfrac{1}{r} \cdot \dfrac{\partial \psi_0}{\partial r}\right) = \psi_0 = 0 \quad \text{at} \quad r = 0. \end{array}\right\} \tag{C.2}$$

The first equation for solving for constants is given by the following expression

$$\begin{aligned} & -\frac{1}{r} \cdot \frac{\partial \psi_0}{\partial r}|_{r=\eta} = 0 \\ \Rightarrow \quad & -\frac{1}{r}\left[\frac{\partial}{\partial r}\left(\frac{1}{4}r^4 B(z) + \left[\int r \log r dr\right] C(z) + \frac{1}{2}r^2 E(z) + F(z)\right)\right]_{r=\eta} = 0 \\ \Rightarrow \quad & -\frac{1}{r}\left[r^3 B(z) + r \log r C(z) + r E(z)\right]_{r=\eta} = 0 \end{aligned}$$

$$\Rightarrow \quad -\eta^2 B(z) - \log \eta C(z) - E(z) = 0 \tag{C.3}$$

and for the second equation we use the boundary condition

$$\psi_0|_{r=\eta} = -\frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{4}\eta^4 B(z) + \left[\frac{\eta^2}{2}\log\eta - \frac{\eta^2}{4}\right] C(z) + \frac{1}{2}\eta^2 E(z) + F(z) = -\frac{1}{2}. \tag{C.4}$$

From the boundary conditions in eq. (4.34) we have

$$-\frac{\partial}{\partial r}\left[\frac{1}{r}\cdot\left(r^3 B(z) + r\log r C(z) + rE(z)\right)\right]_{r=0} = 0 \tag{C.5}$$

$$\begin{aligned}
&\Rightarrow \quad \left[-\frac{3r^2 B(z) + (1+\log r)C(z) + E(z)}{r} + \frac{r^3 B(z) + r\log r C(z) + rE(z)}{r^2}\right]_{r=0} = 0 \\
&\Rightarrow \quad \left[\frac{-3r^3 B(z) - r\log r C(x) - rC(z) - rE(z) + r^3 B(z) + r\log r C(z) + rE(z)}{r^2}\right]_{r=0} = 0 \\
&\Rightarrow \quad \left[-2r^2 B(z) - C(z)\right]_{r=0} = 0 \\
&\Rightarrow \quad C(z) = 0.
\end{aligned} \tag{C.6}$$

To obtain the fourth and final equation we use the boundary condition

$$\psi_0|_{r=0} = 0 \tag{C.7}$$

$$\begin{aligned}
&\Rightarrow \quad \frac{1}{4}r^4 B(z) + \left[\frac{1}{2}\log r - \frac{1}{4}\right] r^2 C(z) + \frac{1}{2}r^2 E(z) + F(z)|_{r=0} = 0 \\
&\Rightarrow \quad F(z) = 0.
\end{aligned} \tag{C.8}$$

We have obtained the following equations

$$\begin{aligned}
-\eta^2 B(z) - \log\eta C(z) - E(z) &= 0 \\
\frac{1}{4}\eta^4 B(z) + \left[\frac{\eta^2}{2}\log\eta - \frac{\eta^2}{4}\right] C(z) + \frac{1}{2}\eta^2 E(z) + F(z) &= -\frac{1}{2} \\
C(z) &= 0 \\
F(z) &= 0
\end{aligned}$$

After solving the simultaneous equations we find $B(z) = \frac{2}{\eta^4}$ and $E(z) = -\frac{2}{\eta^2}$

Substituting $B(z)$ and $E(z)$ into eq. (4.15) we have the following expression for ψ_0

$$\begin{aligned}\psi_0 &= \frac{1}{4}r^4\left(\frac{2}{\eta^4}\right) - \frac{1}{2}r^2\left(\frac{2}{\eta^2}\right) \\ &= \frac{r^4}{\eta^4} - \frac{r^2}{\eta^2} \\ &= \frac{1}{2\eta^4}(r^4 - 2\eta^2 r^2). \end{aligned} \tag{C.9}$$

Appendix D

Evaluating the second term of the expression on eq. (4.23) when n=0

Setting $n = 0$ in eq.(4.23), we have the following expression

$$L^{-2}\left[R_e A_0 - \frac{\partial^4 \psi_0}{\partial z^4} - 2\frac{\partial^2 (L\psi_0)}{\partial z^2} + M^2 L\psi_0\right], \tag{D.1}$$

where

$$\begin{aligned}
L^{-2} &= L_1^{-1}[rL_1^{-1}\{r^{-1}L_1^{-1}(rL_1^{-1}r^{-1})\}];\\
\psi_0 &= \frac{r^4}{\eta^4} - \frac{r^2}{\eta^2};\\
A_0 &= \frac{1}{r}\cdot\frac{\partial(\nabla^2\psi_0, \psi_0)}{\partial(r,z)} - \frac{2}{r^2}\cdot\frac{\partial\psi_0}{\partial z}\cdot\nabla^2\psi_0;\\
\frac{\partial(\nabla^2\psi_0, \psi_0)}{\partial(r,z)} &= \frac{\partial(\nabla^2\psi_0)}{\partial r}\cdot\frac{\partial\psi_0}{\partial z} - \frac{\partial(\nabla^2\psi_0)}{\partial z}\cdot\frac{\partial\psi_0}{\partial r};\\
\nabla^2 &= \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2};\\
L &= \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}.
\end{aligned} \tag{D.2}$$

Note that $\eta = \eta(z)$ and

$$\frac{\partial\psi_0}{\partial r} = \frac{4r^3}{\eta^4} - \frac{2r}{\eta^2}, \tag{D.3}$$

$$\frac{\partial^2\psi_0}{\partial r^2} = \frac{12r^2}{\eta^4} - \frac{2}{\eta^2}. \tag{D.4}$$

From eq.(D.1), we evaluate term by term starting with $M^2L\psi_0$

$$\begin{aligned} M^2L\psi_0 &= M^2\left(\frac{\partial^2\psi_0}{\partial r^2} - \frac{1}{r}\cdot\frac{\partial\psi_0}{\partial r}\right) \\ &= M^2\left[\frac{12r^2}{\eta^4} - \frac{2}{\eta^2} - \frac{1}{r}\left(\frac{4r^3}{\eta^4} - \frac{2r}{\eta^2}\right)\right] \\ &= M^2\left[\frac{12r^2}{\eta^4} - \frac{2}{\eta^2} - \frac{4r^2}{\eta^4} + \frac{2}{\eta^2}\right] \\ &= M^2\cdot\frac{8r^2}{\eta^4}. \end{aligned} \tag{D.5}$$

$$\begin{aligned} \frac{\partial^2(L\psi_0)}{\partial z^2} &= \frac{\partial^2}{\partial z^2}\left[\frac{\partial^2\psi_0}{\partial r^2} - \frac{1}{r}\cdot\frac{\partial\psi_0}{\partial r}\right] \\ &= \frac{\partial^2}{\partial z^2}\left[\frac{8r^2}{\eta^4}\right] \\ &= 8r^2\left(\frac{20\eta'(z)^2}{\eta(z)^6} - \frac{4\eta''(z)}{\eta(z)^5}\right) \end{aligned} \tag{D.6}$$

$$\frac{\partial\psi_0}{\partial z} = \frac{2r^2\eta'(z)}{\eta(z)^3} - \frac{4r^4\eta'(z)}{\eta(z)^5}. \tag{D.7}$$

$$\begin{aligned} \frac{\partial^4\psi_0}{\partial z^4} &= r^4\left(\frac{840\eta'(z)^4}{\eta(z)^8} - \frac{720\eta'(z)^2\eta''(z)}{\eta(z)^7} + \frac{60\eta''(z)^2}{\eta(z)^6} + \frac{80\eta'(z)\eta'''(z)}{\eta(z)^6} - \frac{4\eta''''(z)}{\eta(z)^5}\right) \\ &- r^2\left(\frac{120\eta'(z)^4}{\eta(z)^6} - \frac{144\eta'(z)^2\eta''(z)}{\eta(z)^5} + \frac{18\eta''(z)^2}{\eta(z)^4} + \frac{24\eta'(z)\eta'''(z)}{\eta(z)^4} - \frac{2\eta''''(z)}{\eta(z)^3}\right). \end{aligned} \tag{D.8}$$

$$\begin{aligned} \nabla^2\psi_0 &= \frac{\partial^2\psi_0}{\partial r^2} - \frac{1}{r}\frac{\partial\psi_0}{\partial r} + \frac{\partial^2\psi_0}{\partial z^2} \\ &= \frac{8r^2}{\eta(z)^4} + r^4\left(\frac{20\eta'(z)^2}{\eta(z)^6} - \frac{4\eta''(z)}{\eta(z)^5}\right) - r^2\left(\frac{6\eta'(z)^2}{\eta(z)^4} - \frac{2\eta''(z)}{\eta(z)^3}\right). \end{aligned} \tag{D.9}$$

$$\frac{\partial(\nabla^2\psi_0)}{\partial r} = \frac{16r}{\eta(z)^4} + 4r^3\left(\frac{20\eta'(z)^2}{\eta(z)^6} - \frac{4\eta''(z)}{\eta(z)^5}\right) - 2r\left(\frac{6\eta'(z)^2}{\eta(z)^4} - \frac{2\eta''(z)}{\eta(z)^3}\right). \tag{D.10}$$

$$\frac{\partial(\nabla^2\psi_0)}{\partial z} = r^4\left(\frac{60\eta'(z)\eta''(z)}{\eta(z)^6} - \frac{120\eta'(z)^3}{\eta(z)^7} - \frac{4\eta'''(z)}{\eta(z)^5}\right) - r^2\left(\frac{18\eta'(z)\eta''(z)}{\eta(z)^5} - \frac{24\eta'(z)^3}{\eta(z)^4} - \frac{2\eta'''(z)}{\eta(z)^3} - \frac{32\eta'(z)}{\eta(z)^5}\right). \tag{D.11}$$

Now we multiply eqs. (D.10) with (D.7) to obtain the first part of the determinant of A_0, i.e.

$$\begin{aligned}\frac{\partial(\nabla^2\psi_0)}{\partial r}\cdot\frac{\partial\psi_0}{\partial z} &= \frac{32r^3\eta'(z)}{\eta(z)^7} - \frac{64r^5\eta'(z)}{\eta(z)^9} - \frac{320r^7\eta'(z)^3}{\eta(z)^{11}} + \frac{208r^5\eta'(z)^3}{\eta(z)^9} \\ &- \frac{24r^3\eta'(z)^3}{\eta(z)^7} + \frac{64r^7\eta'(z)\eta''(z)}{\eta(z)^{10}} - \frac{48r^5\eta'(z)\eta''(z)}{\eta(z)^8} \\ &+ \frac{8r^3\eta'(z)\eta''(z)}{\eta(z)^6}.\end{aligned} \tag{D.12}$$

Similarly, we multiply eqs. (D.11) with (D.3) to obtain the second part of the determinant of A_0

$$\begin{aligned}\frac{\partial(\nabla^2\psi_0)}{\partial z}\cdot\frac{\partial\psi_0}{\partial r} &= \frac{64r^3\eta'(z)}{\eta(z)^7} - \frac{128r^5\eta'(z)}{\eta(z)^9} - \frac{480r^7\eta'(z)^3}{\eta(z)^{11}} + \frac{336r^5\eta'(z)^3}{\eta(z)^9} \\ &- \frac{48r^3\eta'(z)^3}{\eta(z)^7} + \frac{240r^7\eta'(z)\eta''(z)}{\eta(z)^{10}} - \frac{192r^5\eta'(z)\eta''(z)}{\eta(z)^8} \\ &+ \frac{36r^3\eta'(z)\eta''(z)}{\eta(z)^6} - \frac{16r^7\eta'''(z)}{\eta(z)^9} + \frac{16r^5\eta'''(z)}{\eta(z)^7} \\ &- \frac{4r^3\eta'''(z)}{\eta(z)^5}.\end{aligned} \tag{D.13}$$

Since

$$\frac{\partial(\nabla^2\psi_0,\psi_0)}{\partial(r,z)} = \frac{\partial(\nabla^2\psi_0)}{\partial r}\cdot\frac{\partial\psi_0}{\partial z} - \frac{\partial(\nabla^2\psi_0)}{\partial z}\cdot\frac{\partial\psi_0}{\partial r}, \tag{D.14}$$

then

$$\frac{1}{r}\cdot\frac{\partial(\nabla^2\psi_0,\psi_0)}{\partial(r,z)} = \frac{1}{r}\left(\frac{\partial(\nabla^2\psi_0)}{\partial r}\cdot\frac{\partial\psi_0}{\partial z} - \frac{\partial(\nabla^2\psi_0)}{\partial z}\cdot\frac{\partial\psi_0}{\partial r}\right)$$

$$
\begin{aligned}
&= \frac{64r^4\eta'(z)}{\eta(z)^9} - \frac{32r^2\eta'(z)}{\eta(z)^7} + \frac{160r^6\eta'(z)^z}{\eta(z)^{11}} - \frac{128r^4\eta'(z)^3}{\eta(z)^9} \\
&+ \frac{24r^2\eta'(z)^3}{\eta(z)^7} - \frac{176r^6\eta'(z)\eta''(z)}{\eta(z)^{10}} + \frac{144r^4\eta'(z)\eta''(z)}{\eta(z)^8} - \frac{16r4\eta'''(z)}{\eta(z)^7} \\
&- \frac{28r^2\eta'(z)\eta''(z)}{\eta(z)^6} + \frac{16r^6\eta'''(z)}{\eta(z)^9} + \frac{4r^2\eta'''(z)}{\eta(z)^5} \qquad \text{(D.15)}
\end{aligned}
$$

To obtain the full expression for A_0 we multiply $\frac{2}{r^2}$ with the product of eqs. (D.9) and (D.7)

$$
\begin{aligned}
\frac{2}{r^2}\cdot\frac{\partial\psi_0}{\partial z}\cdot\nabla^2\psi_0 &= \frac{32r^2\eta'(z)}{\eta(z)^7} - \frac{64r^4\eta'(z)}{\eta(z)^9} - \frac{160r^6\eta'(z)^3}{\eta(z)^{11}} + \frac{128r^4\eta'(z)^3}{\eta(z)^9} \\
&- \frac{24r^2\eta'(z)^3}{\eta(z)^7} + \frac{32r^6\eta'(z)\eta''(z)}{\eta(z)^{10}} - \frac{32r^4\eta'(z)\eta''(z)}{\eta(z)^8} \\
&+ \frac{8r^2\eta'(z)\eta''(z)}{\eta(z)^6}. \qquad \text{(D.16)}
\end{aligned}
$$

Hence R_eA_0 will be eqs. (D.15) − (D.16)

$$
\begin{aligned}
R_eA_0 &= \frac{R_er^6}{\eta^{11}}\left(320\eta_1^3 - 208\eta\eta_1\eta_2 + 16\eta^2\eta_3\right) \\
&+ \frac{R_er^4}{\eta^9}\left(128\eta_1 - 256\eta_1^3 + 176\eta\eta_1\eta_2 - 16\eta^2\eta_3\right) \\
&+ \frac{R_er^2}{\eta^7}\left(48\eta_1^3 - 64\eta_1 - 36\eta\eta_1\eta_2 + 4\eta^2\eta_3\right). \qquad \text{(D.17)}
\end{aligned}
$$

Thus

$$
L^{-2}\left[R_eA_0 - \frac{\partial^4\psi_0}{\partial z^4} - 2\frac{\partial^2(L\psi_0)}{\partial z^2} + M^2L\psi_0\right] = L^{-2}\left[\alpha(z)r^6 + \beta(z)r^4 + \gamma(z)r^2\right], \qquad \text{(D.18)}
$$

where

$$
\begin{aligned}
\alpha(z) &= \frac{R_e}{\eta^{11}}\left(320\eta_1^3 - 208\eta\eta_1\eta_2 + 16\eta^2\eta_3\right), \qquad \text{(D.19)} \\
\beta(z) &= \frac{R_e}{\eta^9}\left(128\eta_1 - 256\eta_1^3 + 176\eta\eta_1\eta_2 - 16\eta^2\eta_3\right) \\
&+ \frac{1}{\eta^8}\left(720\eta\eta_1^2\eta_2 - 840\eta_1^4 - 60\eta^2\eta_2^2 - 80\eta^2\eta_1\eta_3 + 4\eta^3\eta_4\right), \qquad \text{(D.20)} \\
\gamma(z) &= \frac{R_e}{\eta^7}\left(48\eta_1^3 - 64\eta_1 - 36\eta\eta_1\eta_2 + 4\eta^2\eta_3\right) \\
&+ \frac{1}{\eta^6}\left(120\eta_1^4 - 320\eta_1^2 + 64\eta\eta_2 - 144\eta\eta_1^2\eta_2 + 18\eta^2\eta_2^2 + 24\eta^2\eta_1\eta_3 - 2\eta^3\eta_4\right)
\end{aligned}
$$

$$+ \quad \frac{8M^2}{\eta^4}. \tag{D.21}$$

where η_1, η_2, η_3 and η_4 are the derivatives of η with respect to z indicating the orders according to their suffies.
Using eq. (4.39) to integrate eq. (D.18) we have

$$\begin{aligned}
L^{-2}\left[\alpha(z)r^6 + \beta(z)r^4 + \gamma(z)r^2\right] &= L_1^{-1}[rL_1^{-1}\{r^{-1}L_1^{-1}(rL_1^{-1}r^{-1})\}] \\
&= \int r \int r^{-1} \int r \int r^{-1}\left(\alpha(z)r^6 + \beta(z)r^4 + \gamma(z)r^2\right) \mathrm{d}r\mathrm{d}r\mathrm{d}r\mathrm{d}r \\
&= \int\int\int\int \left(\alpha(z)r^6 + \beta(z)r^4 + \gamma(z)r^2\right) \mathrm{d}r\mathrm{d}r\mathrm{d}r\mathrm{d}r \\
&= \frac{\alpha(z)r^{10}}{5040} + \frac{\beta(z)r^8}{1680} + \frac{\gamma(z)r^6}{360}.
\end{aligned} \tag{D.22}$$

Appendix E

Applying boundary conditions (F.14) to obtain an expression for ψ_1

Applying boundary conditions (F.14) (set $n = 1$) to solve for constants B_1, C_1, E_1, F_1 in eq. (4.42) to obtain an expression for ψ_1.

Recall that eq. (4.42) is given by

$$\begin{aligned} \psi_1 &= \alpha r^{10} + \beta r^8 + \gamma r^6 \\ &+ \frac{1}{4} r^4 B_1 + \left[\int r \log r \mathrm{d}r \right] C_1 + \frac{1}{2} r^2 E_1 + F_1 \end{aligned} \tag{E.1}$$

and the boundary conditions (F.14) when $n = 1$ are as follows

$$\left. \begin{array}{l} -\dfrac{1}{r} \cdot \dfrac{\partial \psi_n}{\partial r} = \psi_n = 0 \qquad \text{at} \quad r = \eta, \\ -\dfrac{\partial}{\partial r} \left(\dfrac{1}{r} \cdot \dfrac{\partial \psi_n}{\partial r} \right) = \psi_n = 0 \quad \text{at} \quad r = 0, \end{array} \right\} \tag{E.2}$$

That is

$$\begin{aligned} & -\frac{1}{r} \cdot \frac{\partial \psi_1}{\partial r} \Big|_{r=\eta} = 0 \\ \Rightarrow \quad & -\frac{1}{r} \left[\frac{d}{dr} \left(\alpha r^{10} + \beta r^8 + \gamma r^6 + \frac{1}{4} r^4 B_1 + \left[\int r \log r \mathrm{d}r \right] C_1 + \frac{1}{2} r^2 E_1 + F_1 \right) \right]_{r=\eta} = 0 \\ \Rightarrow \quad & -10\alpha r^8 - 8\beta r^6 - 6\gamma r^4 - B_1 r^2 - E_1 - C_1 \log r \Big|_{r=\eta} = 0 \end{aligned}$$

$$\Rightarrow \quad -10\alpha\eta^8 - 8\beta\eta^8 - 6\gamma\eta^4 - B_1\eta^2 - E_1 - C_1 \log\eta = 0$$
$$\Rightarrow \quad 10\alpha\eta^8 + 8\beta\eta^8 + 6\gamma\eta^4 + B_1\eta^2 + E_1 + C_1 \log\eta = 0 \tag{E.3}$$

and

$$\psi_1 \mid_{r=\eta} = 0$$
$$\Rightarrow \quad \alpha r^{10} + \beta r^8 + \gamma r^6 + \frac{1}{4} r^4 B_1 + \left(\frac{r^2}{2}\log r - \frac{r^2}{4}\right) C_1 + \frac{1}{2} r^2 E_1 + F_1 \mid_{r=\eta} = 0$$
$$\Rightarrow \quad \alpha\eta^{10} + \beta\eta^8 + \gamma\eta^6 + \frac{\eta^4 B_1}{4} + \left(\frac{\eta^2}{2}\log\eta - \frac{\eta^2}{4}\right) C_1 + \frac{\eta^2 E_1}{2} + F_1 = 0. \tag{E.4}$$

We now use the second boundary condition from the boundary conditions (F.14) when $n = 1$.

$$-\frac{\partial}{\partial r}\left(\frac{1}{r}\cdot\frac{\partial\psi_1}{\partial r}\right) \mid_{r=0} = 0$$
$$\Rightarrow \quad -\frac{d}{dr}\left[\frac{1}{r}\left(10\alpha r^9 + 8\beta r^7 + 6\gamma r^5 + B_1 r^3 + rE_1 + C_1 \log r\right)\right] \mid_{r=0} = 0$$
$$\Rightarrow \quad -\frac{(1+\log r)C_1 + E_1 + 3B_1 r^2 + 90\alpha r^8 + 56\beta r^6 + 30\gamma r^4}{r}$$
$$+\frac{10\alpha r^9 + 8\beta r^7 + 6\gamma r^5 + B_1 r^3 + rE_1 + C_1 r\log r}{r^2} \mid_{r=0} = 0$$
$$\Rightarrow \quad rC_1 - 2B_1 r^3 - 80\alpha r^9 - 48\beta r^7 - 24\gamma r^5 \mid_{r=0} = 0$$
$$\Rightarrow \quad C_1 - 2B_1 r^2 - 80\alpha r^8 - 48\beta r^6 - 24\gamma r^4 \mid_{r=0} = 0$$
$$\Rightarrow \quad C_1 = 0 \tag{E.5}$$

and

$$\psi_1 \mid_{r=0} = 0$$
$$\Rightarrow \quad \alpha r^{10} + \beta r^8 + \gamma r^6 + \left(\frac{r^2}{2}\log r - \frac{r^2}{4}\right) C_1 + \frac{1}{2} r^2 E_1 + F_1 \mid_{r=0} = 0$$
$$\Rightarrow \quad F_1 = 0. \tag{E.6}$$

Solving eqs. (E.3) and (E.4) where $C_1 = 0$ and $F_1 = 0$ yields the following results

$$B_1 = -4\eta^2(4\alpha\eta^4 + 3\beta\eta^2 + 2\gamma), \tag{E.7}$$
$$C_1 = 2\eta^4(3\alpha\eta^4 + 2\beta\eta^2 + \gamma). \tag{E.8}$$

Thus

$$\begin{aligned} \psi_1 &= \alpha r^{10} + \beta r^8 + \gamma r^6 - \eta^2(4\alpha\eta^4 + 3\beta\eta^2 + 2\gamma)r^4 \\ &+ \eta^4(3\alpha\eta^4 + 2\beta\eta^2 + \gamma)r^2. \end{aligned} \tag{E.9}$$

Appendix F

Applying boundary conditions (F.14) to obtain an expression for ψ_2

After a rigorous evaluation we obtain the following expression for ψ_2 as

$$\begin{aligned}
\psi_2 &= L^{-2}\left[\Gamma(z)r^{10}+\Upsilon(z)r^8+\Phi(z)r^6+\Psi(z)r^4+\Omega(z)r^2\right] \\
&+ \frac{1}{4}r^4B_2+\left[\int r\log r\mathrm{d}r\right]C_2+\frac{1}{2}r^2E_2+F_2 \qquad \text{(F.1)}
\end{aligned}$$

where

$$\begin{aligned}
\Gamma &= \frac{102400R_e^2\eta_1^6}{21\eta^{18}}+\frac{4576R_e\eta_1^7}{3\eta^{15}}-\frac{1177R_e^2\eta_1^4\eta_2}{21\eta^{17}}-\frac{8008R_e\eta_1^5\eta_2}{3\eta^{14}}+\frac{33280R_e^2\eta_1^2\eta_2^2}{21\eta^{16}} \\
&+ \frac{3520Re\eta_1^3\eta_2^2}{3\eta^{13}}-\frac{110R_e\eta_1\eta_2^3}{\eta^{12}}+\frac{25600R_e^2\eta_1^3\eta_3}{63\eta^{16}}+\frac{440R_e\eta_1^4\eta_3}{\eta^{13}}-\frac{2048R_e^2\eta_1\eta_2\eta_3}{9\eta^{15}} \\
&- \frac{2200R_e\eta_1^2\eta_2\eta_3}{9\eta^{12}}+\frac{110R_e\eta_2^2\eta_3}{9\eta^{11}}+\frac{512R_e^2\eta_3^2}{63\eta^{14}}+\frac{80R_e\eta_1\eta_3^2}{9\eta^{11}}-\frac{440R_e\eta_1^3\eta_4}{9\eta^{12}} \\
&+ \frac{130R_e\eta_1\eta_2\eta_4}{9\eta^{11}}-\frac{5R_e\eta_3\eta_4}{9\eta^{10}}+\frac{32R_e\eta_1^2\eta_5}{9\eta^{11}}-\frac{17R_e\eta_2\eta_5}{45\eta^{10}}-\frac{7R_e\eta_1\eta_6}{45\eta^{10}} \\
&+ \frac{R_e\eta_7}{315\eta^9}; \qquad \text{(F.2)}
\end{aligned}$$

$$
\begin{aligned}
\Upsilon \;=\; & 80M^2\alpha + \frac{2048R_e^2\eta_1^4}{63\eta^{16}} - \frac{47168R_e\eta_1^5}{21\eta^{13}} - \frac{20480R_e^2\eta_1^6}{21\eta^{16}} - \frac{17280R_e\eta_1^7}{\eta^{15}} + \frac{3960\eta_1^8}{\eta^{12}} \\
+ & \frac{23552R_e^2\eta_1^2\eta_2}{63\eta^{15}} + \frac{15840R_e\eta_1^3\eta_2}{7\eta^{12}} + \frac{104704R_e^2\eta_1^4\eta_2}{21\eta^{15}} + \frac{164160R_e\eta_1^5\eta_2}{7\eta^{14}} - \frac{5808R_e\eta_1^5\eta_2}{7\eta^{12}} \\
- & \frac{10080\eta_1^6\eta_2}{\eta^{11}} - \frac{7600R_e\eta_1\eta_2^2}{21\eta^{11}} - \frac{16000R_e^2\eta_1^2\eta_2^2}{7\eta^{14}} - \frac{8640R_e\eta_1^3\eta_2^2}{\eta^{13}} + \frac{5280R_e\eta_1^3\eta_2^2}{7\eta^{11}} \\
+ & \frac{7560\eta_1^4\eta_2^2}{\eta^{10}} + \frac{4320R_e\eta_1\eta_2^3}{7\eta^{12}} - \frac{792R_e\eta_1\eta_2^3}{7\eta^{10}} - \frac{1680\eta_1^2\eta_2^3}{\eta^9} + \frac{105\eta_2^4}{2\eta^8} - \frac{13312R_e^2\eta_1\eta_3}{315\eta^{14}} \\
- & \frac{17440R_e\eta_1^2\eta_3}{63\eta^{11}} - \frac{36864R_e^2\eta_1^3\eta_3}{7\eta^{16}} - \frac{28928R_e^2\eta_1^3\eta_3}{63\eta^{14}} - \frac{2880R_e\eta_1^4\eta_3}{7\eta^{13}} + \frac{2448R_e\eta_1^4\eta_3}{7\eta^{11}} \\
+ & \frac{2016\eta_1^5\eta_3}{\eta^{10}} + \frac{656R_e\eta_2\eta_3}{21\eta^{10}} + \frac{18432R_e^2\eta_1\eta_2\eta_3}{7\eta^{15}} + \frac{118016R_e^2\eta_1\eta_2\eta_3}{315\eta^{13}} + \frac{2880R_e\eta_1^2\eta_2\eta_3}{7\eta^{12}} \\
- & \frac{9948R_e\eta_1^2\eta_2\eta_3}{35\eta^{10}} - \frac{2240\eta_1^3\eta_2\eta_3}{\eta^9} + \frac{2104R_e\eta_2^2\eta_3}{105\eta^9} + \frac{420\eta_1\eta_2^2\eta_3}{\eta^8} - \frac{6144R_e^2\eta_3^2}{35\eta^{14}} \\
- & \frac{4736R_e^2\eta_3^2}{315\eta^{12}} + \frac{544R_e\eta_1\eta_3^2}{35\eta^9} + \frac{140\eta_1^2\eta_3^2}{\eta^8} - \frac{20\eta_2\eta_3^2}{\eta^7} + \frac{1208R_e\eta_1\eta_4}{63\eta^{10}} \\
- & \frac{3648R_e\eta_1^3\eta_4}{7\eta^{12}} - \frac{2256R_e\eta_1^3\eta_4}{35\eta^{10}} - \frac{280\eta_1^4\eta_4}{\eta^9} + \frac{864R_e\eta_1\eta_2\eta_4}{7\eta^{11}} + \frac{2696R_e\eta_1\eta_2\eta_4}{105\eta^9} \\
+ & \frac{210\eta_1^2\eta_2\eta_4}{\eta^8} - \frac{15\eta_2^2\eta_4}{\eta^7} - \frac{144R_e\eta_3\eta_4}{35\eta^{10}} - \frac{29R_e\eta_3\eta_4}{21\eta^8} - \frac{20\eta_1\eta_3\eta_4}{\eta^7} + \frac{5\eta_4^2}{12\eta^6} \\
- & \frac{184R_e\eta_5}{315\eta^9} + \frac{576R_e\eta_1^2\eta_5}{7\eta^{11}} + \frac{688R_e\eta_1^2\eta_5}{105\eta^9} + \frac{28\eta_1^3\eta_5}{\eta^8} - \frac{288R_e\eta_2\eta_5}{35\eta^{10}} - \frac{97R_e\eta_2\eta_5}{105\eta^8} \\
- & \frac{13\eta_1\eta_2\eta_5}{\eta^7} + \frac{2\eta_3\eta_5}{3\eta^6} - \frac{192R_e\eta_1\eta_6}{35\eta^{10}} - \frac{13R_e\eta_1\eta_6}{35\eta^8} - \frac{2\eta_1^2\eta_6}{\eta^7} + \frac{\eta_2\eta_6}{3\eta^6} + \frac{16R_e\eta_7}{105\eta^9} \\
+ & \frac{R_e\eta_7}{105\eta^7} + \frac{2\eta_1\eta_7}{21\eta^6} - \frac{\eta_8}{420\eta^5};
\end{aligned}
\tag{F.3}
$$

$$
\begin{aligned}
\Phi \;=\;& 48M^2\beta - \frac{4608R_e\eta_1^3}{7\eta^{11}} + \frac{149504R_e^2\eta_1^4}{105\eta^{14}} + \frac{26624R_e\eta_1^5}{\eta^{13}} + \frac{28288R_e^2\eta_1^6}{5\eta^{14}} + \frac{6144\eta_1^6}{\eta^{10}} \\
+& \frac{13440R_e\eta_1^7}{\eta^{13}} - \frac{672R_e\eta_1^7}{\eta^{11}} - \frac{1008\eta_1^8}{\eta^{10}} + \frac{6912R_e\eta_1\eta_2}{35\eta^{10}} - \frac{91648R_e^2\eta_1^2\eta_2}{105\eta^{13}} + \frac{33280R_e\eta_1^3\eta_2}{\eta^{12}} \\
-& \frac{25344R_e\eta_1^3\eta_2}{35\eta^{10}} - \frac{476032R_e^2\eta_1^4\eta_2}{35\eta^{13}} - \frac{10240\eta_1^4\eta_2}{\eta^9} - \frac{216736R_e\eta_1^5\eta_2}{7\eta^{12}} + \frac{8736R_e\eta_1^5\eta_2}{5\eta^{10}} \\
+& \frac{3136\eta_1^6\eta_2}{\eta^9} - \frac{7680R_e\eta_1\eta_2^2}{\eta^{11}} + \frac{8448R_e\eta_1\eta_2^2}{35\eta^9} + \frac{742016R_e^2\eta_1^2\eta_2^2}{105\eta^{12}} + \frac{3840\eta_1^2\eta_2^2}{\eta^8} + \frac{147200R_e\eta_1^3\eta_2^2}{7\eta^{11}} \\
-& \frac{3472R_e\eta_1^3\eta_2^2}{3\eta^9} - \frac{2940\eta_1^4\eta_2^2}{\eta^8} - \frac{5248R_e^2\eta_2^3}{15\eta^{11}} - \frac{1280\eta_2^3}{7\eta^7} - \frac{24144R_e\eta_1\eta_2^3}{7\eta^{10}} + \frac{168R_e\eta_1\eta_2^3}{\eta^8} \\
+& \frac{840\eta_1^2\eta_2^3}{\eta^7} - \frac{35\eta_2^4}{\eta^6} - \frac{256R_e\eta_3}{35\eta^9} + \frac{5632R_e^2\eta_1\eta_3}{105\eta^{12}} - \frac{23040R_e\eta_1^2\eta_3}{7\eta^{11}} + \frac{7424R_e\eta_1^2\eta_3}{35\eta^9} + \frac{36864R_e^2\eta_1^3\eta_3}{35\eta^{14}} \\
+& \frac{95936R_e^2\eta_1^3\eta_3}{35\eta^{12}} + \frac{5120\eta_1^3\eta_3}{3\eta^8} + \frac{44192R_e\eta_1^4\eta_3}{7\eta^{11}} - \frac{6328R_e\eta_1^4\eta_3}{15\eta^9} - \frac{784\eta_1^5\eta_3}{\eta^8} + \frac{15872R_e\eta_2\eta_3}{21\eta^{10}} \\
-& \frac{256R_e\eta_2\eta_3}{7\eta^8} - \frac{27648R_e^2\eta_1\eta_2\eta_3}{35\eta^{13}} - \frac{12192R_e^2\eta_1\eta_2\eta_3}{7\eta^{11}} - \frac{5120\eta_1\eta_2\eta_3}{7\eta^7} - \frac{234336R_e\eta_1^2\eta_2\eta_3}{35\eta^{10}} \\
+& \frac{5446R_e\eta_1^2\eta_2\eta_3}{15\eta^8} + \frac{1120\eta_1^3\eta_2\eta_3}{\eta^7} + \frac{23792R_e\eta_2^2\eta_3}{35\eta^9} - \frac{439R_e\eta_2^2\eta_3}{15\eta^7} - \frac{280\eta_1\eta_2^2\eta_3}{\eta^6} + \frac{3072R_e^2\eta_3^2}{35\eta^{12}} \\
+& \frac{464R_e^2\eta_3^2}{7\eta^{10}} + \frac{1280\eta_3^2}{63\eta^6} + \frac{16416R_e\eta_1\eta_3^2}{35\eta^9} - \frac{104R_e\eta_1\eta_3^2}{5\eta^7} - \frac{280\eta_1^2\eta_3^2}{3\eta^6} + \frac{56\eta_2\eta_3^2}{3\eta^5} + \frac{13568R_e\eta_1\eta_4}{105\eta^{10}} \\
-& \frac{160R_e\eta_1\eta_4}{7\eta^8} - \frac{5632R_e^2\eta_1^2\eta_4}{15\eta^{11}} - \frac{1280\eta_1^2\eta_4}{7\eta^7} - \frac{32496R_e\eta_1^3\eta_4}{35\eta^{10}} + \frac{994R_e\eta_1^3\eta_4}{15\eta^8} + \frac{140\eta_1^4\eta_4}{\eta^7} \\
+& \frac{1232R_e\eta_2\eta_4}{15\eta^{10}} + \frac{640\eta_2\eta_4}{21\eta^6} + \frac{22832R_e\eta_1\eta_2\eta_4}{35\eta^9} - \frac{476R_e\eta_1\eta_2\eta_4}{15\eta^7} - \frac{140\eta_1^2\eta_2\eta_4}{\eta^6} + \frac{14\eta_2^2\eta_4}{\eta^5} \\
-& \frac{6248R_e\eta_3\eta_4}{105\eta^8} + \frac{37R_e\eta_3\eta_4}{18\eta^6} + \frac{56\eta_1\eta_3\eta_4}{3\eta^5} - \frac{7\eta_4^2}{12\eta^4} - \frac{256R_e\eta_5}{105\eta^9} + \frac{32R_e\eta_5}{35\eta^7} + \frac{464R_e^2\eta_1\eta_5}{15\eta^{10}} \\
+& \frac{256\eta_1\eta_5}{21\eta^6} + \frac{1648R_e\eta_1^2\eta_5}{15\eta^9} - \frac{20R_e\eta_1^2\eta_5}{3\eta^7} - \frac{56\eta_1^3\eta_5}{3\eta^6} - \frac{488R_e\eta_2\eta_5}{15\eta^8} + \frac{11R_e\eta_2\eta_5}{9\eta^6} + \frac{56\eta_1\eta_2\eta_5}{5\eta^5} \\
-& \frac{14\eta_3\eta_5}{15\eta^4} - \frac{16R_e^2\eta_6}{15\eta^9} - \frac{128\eta_6}{315\eta^5} - \frac{152R_e\eta_1\eta_6}{15\eta^8} + \frac{2R_e\eta_1\eta_6}{5\eta^6} + \frac{28\eta_1^2\eta_6}{15\eta^5} - \frac{7\eta_2\eta_6}{15\eta^4} + \frac{8R_e\eta_7}{15\eta^7} \\
-& \frac{R_e\eta_7}{90\eta^5} - \frac{2\eta_1\eta_7}{15\eta^4} + \frac{\eta_8}{180\eta^3}; \qquad \text{(F.4)}
\end{aligned}
$$

$$
\begin{aligned}
\Psi \;=\;& 24M^2\gamma + \frac{12288R_e^2\eta_1^2}{35\eta^{12}} - \frac{20480R_e\eta_1^3}{3\eta^{11}} + \frac{19968R_e^2\eta_1^4}{35\eta^{12}} + \frac{1792\eta_1^4}{\eta^8} + \frac{189184R_e\eta_1^5}{9\eta^{11}} \\
+\;& \frac{128R_e\eta_1^5}{5\eta^9} - \frac{14080R_e^2\eta_1^6}{3\eta^{12}} - \frac{6496\eta_1^6}{3\eta^8} - \frac{12416R_e\eta_1^7}{3\eta^{11}} + \frac{2624R_e\eta_1^7}{3\eta^9} - \frac{700\eta_1^8}{\eta^8} + \frac{38912R_e\eta_1\eta_2}{15\eta^{10}} \\
-\;& \frac{9728R_e^2\eta_1^2\eta_2}{15\eta^{11}} - \frac{1536\eta_1^2\eta_2}{\eta^7} - \frac{9445376\eta_1^3\eta_2}{315\eta^{10}} + \frac{192\eta_1^3\eta_2}{\eta^8} + \frac{3575872R_e^2\eta_1^4\eta_2}{315\eta^{11}} + \frac{13664\eta_1^4\eta_2}{3\eta^7} \\
+\;& \frac{502976R_e\eta_1^5\eta_2}{35\eta^{10}} - \frac{6424R_e\eta_1^5\eta_2}{3\eta^8} + \frac{16624\eta_1^6\eta_2}{7\eta^7} + \frac{2048R_e^2\eta_2^2}{15\eta^{10}} + \frac{128\eta_2^2}{\eta^6} + \frac{2467712R_e\eta_1\eta_2^2}{315\eta^9} \\
-\;& \frac{624\eta_1\eta_2^2}{7\eta^7} - \frac{87584R_e^2\eta_1^2\eta_2^2}{15\eta^{10}} - \frac{6704\eta_1^2\eta_2^2}{3\eta^6} - \frac{1379552R_e\eta_1^3\eta_2^2}{105\eta^9} + \frac{29800R_e\eta_1^3\eta_2^2}{21\eta^7} - \frac{17212\eta_1^4\eta_2^2}{7\eta^6} \\
+\;& \frac{16064R_e^2\eta_2^3}{45\eta^9} + \frac{2192\eta_2^3}{15\eta^5} + \frac{101104R_e\eta_1\eta_2^3}{35\eta^8} - \frac{4580R_e\eta_1\eta_2^3}{21\eta^6} + \frac{27592\eta_1^2\eta_2^3}{35\eta^5} - \frac{2633\eta_2^4}{70\eta^4} \\
-\;& \frac{2048R_e\eta_3}{15\eta^9} - \frac{24576R_e^2\eta_1\eta_3}{35\eta^{12}} + \frac{19328R_e^2\eta_1\eta_3}{105\eta^{10}} + \frac{512\eta_1\eta_3}{3\eta^6} + \frac{1540096R_e\eta_1^2\eta_3}{315\eta^9} - \frac{2416R_e\eta_1^2\eta_3}{35\eta^7} \\
+\;& \frac{38912R_e^2\eta_1^3\eta_3}{35\eta^{12}} - \frac{83504R_e^2\eta_1^3\eta_3}{35\eta^{10}} - \frac{8768\eta_1^3\eta_3}{9\eta^6} - \frac{582208R_e\eta_1^4\eta_3}{105\eta^9} + \frac{9344R_e\eta_1^4\eta_3}{21\eta^7} - \frac{12272\eta_1^5\eta_3}{21\eta^6} \\
-\;& \frac{3200R_e\eta_2\eta_3}{3\eta^8} + \frac{1768R_e\eta_2\eta_3}{105\eta^6} - \frac{37888R_e^2\eta_1\eta_2\eta_3}{35\eta^{11}} + \frac{33568R_e^2\eta_1\eta_2\eta_3}{21\eta^9} + \frac{25792\eta_1\eta_2\eta_3}{45\eta^5} \\
+\;& \frac{18656R_e\eta_1^2\eta_2\eta_3}{3\eta^8} - \frac{25712R_e\eta_1^2\eta_2\eta_3}{63\eta^6} + \frac{96608\eta_1^3\eta_2\eta_3}{105\eta^5} - \frac{11344R_e\eta_2^2\eta_3}{15\eta^7} + \frac{11696R_e\eta_2^2\eta_3}{315\eta^5} \\
-\;& \frac{8964\eta_1\eta_2^2\eta_3}{35\eta^4} - \frac{1024R_e^2\eta_3^2}{35\eta^{10}} - \frac{21512R_e^2\eta_3^2}{315\eta^8} - \frac{344\eta_3^2}{15\eta^4} - \frac{7792R_e\eta_1\eta_3^2}{15\eta^7} + \frac{7208R_e\eta_1\eta_3^2}{315\eta^5} \\
-\;& \frac{2516\eta_1^2\eta_3^2}{35\eta^4} + \frac{316\eta_2\eta_3^2}{21\eta^3} - \frac{128R_e^2\eta_4}{15\eta^9} - \frac{128\eta_4}{15\eta^5} - \frac{30400R_e\eta_1\eta_4}{63\eta^8} + \frac{856R_e\eta_1\eta_4}{105\eta^6} + \frac{16816R_e^2\eta_1^2\eta_4}{45\eta^9} \\
+\;& \frac{688\eta_1^2\eta_4}{5\eta^5} + \frac{76432R_e\eta_1^3\eta_4}{63\eta^8} - \frac{3788R_e\eta_1^3\eta_4}{63\eta^6} + \frac{3236\eta_1^4\eta_4}{35\eta^5} - \frac{4168R_e^2\eta_2\eta_4}{45\eta^8} - \frac{1516\eta_2\eta_4}{45\eta^4} \\
-\;& \frac{249736R_e\eta_1\eta_2\eta_4}{315\eta^7} + \frac{10204R_e\eta_1\eta_2\eta_4}{315\eta^5} - \frac{690\eta_1^2\eta_2\eta_4}{7\eta^4} + \frac{1061\eta_2^2\eta_4}{105\eta^3} + \frac{548R_e\eta_3\eta_4}{7\eta^6} - \frac{136R_e\eta_3\eta_4}{63\eta^4} \\
+\;& \frac{3284\eta_1\eta_3\eta_4}{315\eta^3} - \frac{97\eta_4^2}{1260\eta^2} + \frac{1088R_e\eta_5}{45\eta^7} - \frac{32R_e\eta_5}{105\eta^5} - \frac{1496R_e^2\eta_1\eta_5}{45\eta^8} - \frac{568\eta_1\eta_5}{45\eta^4} - \frac{59744R_e\eta_1^2\eta_5}{315\eta^7} \\
+\;& \frac{1676R_e\eta_1^2\eta_5}{315\eta^5} - \frac{988\eta_1^3\eta_5}{105\eta^4} + \frac{15772R_e\eta_2\eta_5}{315\eta^6} - \frac{361R_e\eta_2\eta_5}{315\eta^4} + \frac{548\eta_1\eta_2\eta_5}{105\eta^3} - \frac{34\eta_3\eta_5}{315\eta^2} \\
+\;& \frac{56R_e^2\eta_6}{45\eta^7} + \frac{28\eta_6}{45\eta^3} + \frac{1276R_e\eta_1\eta_6}{63\eta^6} - \frac{89R_e\eta_1\eta_6}{315\eta^4} + \frac{18\eta_1\eta_6}{35\eta^3} - \frac{\eta_2\eta_6}{35\eta^2} - \frac{68R_e\eta_7}{63\eta^5} \\
+\;& \frac{2R_e\eta_7}{315\eta^3} + \frac{2\eta_1\eta_7}{315\eta^2} - \frac{\eta_8}{252\eta};
\end{aligned}
\tag{F.5}
$$

$$
\begin{aligned}
\Omega \;=\;& \frac{26624R_e\eta_1^3}{9\eta^9} - 8M^2\eta^2(2\gamma+3\beta\eta^2+4\alpha\eta^4) - \frac{4096R_e^2\eta_1^2}{35\eta^{10}} + \frac{768R_e\eta_1^3}{7\eta^7} - \frac{196864R_e^2\eta_1^4}{315\eta^{10}} \\
&- \frac{5120\eta_1^4}{9\eta^6} - \frac{13312R_e\eta_1^5}{3\eta^9} + \frac{4096R_e\eta_1^5}{21\eta^7} + \frac{63872R_e^2\eta_1^6}{105\eta^{10}} - \frac{160\eta_1^6}{\eta^6} + \frac{1536R_e\eta_1^7}{\eta^9} - \frac{816R_e\eta_1^7}{7\eta^7} \\
&+ \frac{80\eta_1^8}{\eta^6} - \frac{54272R_e\eta_1\eta_2}{45\eta^8} - \frac{384R_e\eta_1\eta_2}{7\eta^6} + \frac{29696R_e^2\eta_1^2\eta_2}{63\eta^9} + \frac{28672\eta_1^2\eta_2}{45\eta^5} + \frac{749824R_e\eta_1^3\eta_2}{105\eta^8} \\
&- \frac{16768R_e\eta_1^3\eta_2}{63\eta^6} - \frac{274816R_e^2\eta_1^4\eta_2}{105\eta^9} + \frac{28384\eta_1^4\eta_2}{105\eta^5} - \frac{15856R_e\eta_1^5\eta_2}{5\eta^8} + \frac{7576R_e\eta_1^5\eta_2}{21\eta^6} - \frac{12704\eta_1^6\eta_2}{35\eta^5} \\
&- \frac{1024R_e^2\eta_2^2}{15\eta^8} - \frac{3328\eta_2^2}{45\eta^4} - \frac{215552R_e\eta_1\eta_2^2}{105\eta^7} + \frac{2176R_e\eta_1\eta_2^2}{35\eta^5} + \frac{144352R_e^2\eta_1^2\eta_2^2}{105\eta^8} - \frac{440\eta_1^2\eta_2^2}{7\eta^4} \\
&+ \frac{91936R_e\eta_1^3\eta_2^2}{35\eta^7} - \frac{6584R_e\eta_1^3\eta_2^2}{21\eta^5} + \frac{2652\eta_1^4\eta_2^2}{5\eta^4} - \frac{1408R_e^2\eta_2^3}{15\eta^7} - \frac{232\eta_2^3}{35\eta^3} - \frac{23864R_e\eta_1\eta_2^3}{35\eta^6} \\
&+ \frac{466R_e\eta_1\eta_2^3}{7\eta^4} - \frac{9088\eta_1^2\eta_2^3}{35\eta^3} + \frac{752\eta_2^4}{35\eta^2} + \frac{1024R_e\eta_3}{15\eta^7} + \frac{128R_e\eta_3}{35\eta^5} + \frac{8192R_e^2\eta_1\eta_3}{35\eta^{10}} - \frac{31552R_e^2\eta_1\eta_3}{315\eta^8} \\
&- \frac{4096\eta_1\eta_3}{45\eta^4} - \frac{53504R_e\eta_1^2\eta_3}{35\eta^7} + \frac{10816R_e\eta_1^2\eta_3}{315\eta^5} + \frac{175552R_e^2\eta_1^3\eta_3}{315\eta^8} - \frac{1184\eta_1^3\eta_3}{105\eta^4} + \frac{45392R_e\eta_1^4\eta_3}{35\eta^7} \\
&- \frac{9908R_e\eta_1^4\eta_3}{105\eta^5} + \frac{4304\eta_1^5\eta_3}{35\eta^4} + \frac{110528R_e\eta_2\eta_3}{315\eta^6} - \frac{592R_e\eta_2\eta_3}{105\eta^4} + \frac{13824R_e^2\eta_1\eta_2\eta_3}{35\eta^9} \\
&- \frac{125984R_e^2\eta_1\eta_2\eta_3}{315\eta^7} - \frac{11168\eta_1\eta_2\eta_3}{315\eta^3} - \frac{50912R_e\eta_1^2\eta_2\eta_3}{35\eta^6} + \frac{12382R_e\eta_1^2\eta_2\eta_3}{105\eta^4} - \frac{30752\eta_1^3\eta_2\eta_3}{105\eta^3} \\
&+ \frac{1032R_e\eta_2^2\eta_3}{5\eta^5} - \frac{4951R_e\eta_2^2\eta_3}{315\eta^3} + \frac{4832\eta_1\eta_2^2\eta_3}{35\eta^2} + \frac{512R_e^2\eta_3^2}{35\eta^8} + \frac{5968R_e^2\eta_3^2}{315\eta^6} + \frac{1784\eta_3^2}{315\eta^2} \\
&+ \frac{2048R_e\eta_1\eta_3^2}{15\eta^5} - \frac{2768R_e\eta_1\eta_3^2}{315\eta^3} + \frac{184\eta_1^2\eta_3^2}{5\eta^2} - \frac{96\eta_2\eta_3^2}{7\eta} + \frac{64R_e^2\eta_4}{15\eta^7} + \frac{256\eta_4}{45\eta^3} + \frac{67808R_e\eta_1\eta_4}{315\eta^6} \\
&- \frac{512R_e\eta_1\eta_4}{315\eta^4} - \frac{1408R_e^2\eta_1^2\eta_4}{15\eta^7} - \frac{3112\eta_1^2\eta_4}{315\eta^3} - \frac{9616R_e\eta_1^3\eta_4}{35\eta^6} + \frac{1678R_e\eta_1^3\eta_4}{105\eta^4} - \frac{2956\eta_1^4\eta_4}{105\eta^3} \\
&+ \frac{1168R_e^2\eta_2\eta_4}{45\eta^6} + \frac{276\eta_2\eta_4}{35\eta^2} + \frac{21208R_e\eta_1\eta_2\eta_4}{105\eta^5} - \frac{542R_e\eta_1\eta_2\eta_4}{45\eta^3} + \frac{1744\eta_1^2\eta_2\eta_4}{35\eta^2} - \frac{64\eta_2^2\eta_4}{7\eta} \\
&- \frac{352R_e\eta_3\eta_4}{15\eta^4} + \frac{13R_e\eta_3\eta_4}{14\eta^2} - \frac{64\eta_1\eta_3\eta_4}{7\eta} + \frac{307\eta_4^2}{1260} - \frac{4768R_e\eta_5}{315\eta^5} - \frac{16R_e\eta_5}{315\eta^3} + \frac{80R_e^2\eta_1\eta_5}{9\eta^6} \\
&+ \frac{776\eta_1\eta_5}{315\eta^2} + \frac{5104R_e\eta_1^2\eta_5}{105\eta^5} - \frac{544R_e\eta_1^2\eta_5}{315\eta^3} + \frac{472\eta_1^3\eta_5}{105\eta^2} - \frac{328R_e\eta_2\eta_5}{21\eta^4} + \frac{148R_e\eta_2\eta_5}{315\eta^2} - \frac{32\eta_1\eta_2\eta_5}{7\eta} \\
&+ \frac{118\eta_3\eta_5}{315} - \frac{16R_e^2\eta_6}{45\eta^5} - \frac{76\eta_6}{315\eta} - \frac{2104R_e\eta_1\eta_6}{315\eta^4} + \frac{31R_e\eta_1\eta_6}{315\eta^2} - \frac{16\eta_1^2\eta_6}{35\eta} + \frac{17\eta_2\eta_6}{105} \\
&+ \frac{152R_e\eta_7}{315\eta^3} - \frac{R_e\eta_7}{630\eta} + \frac{2\eta_1\eta_7}{63} + \frac{\eta\eta_8}{1260};
\end{aligned} \tag{F.6}
$$

$$\alpha = \frac{R_e}{315\eta^{11}}\left(20\eta_1^3 - 13\eta\eta_1\eta_2 + \eta^2\eta_3\right); \tag{F.7}$$

$$\begin{aligned}\beta &= \frac{R_e}{105\eta^9}\left(8\eta_1 - 16\eta_1^3 + 11\eta\eta_1\eta_2 - \eta^2\eta_3\right) \\ &+ \frac{1}{420\eta^8}\left(180\eta\eta_1^2\eta_2 - 210\eta_1^4 - 15\eta^2\eta_2^2 - 20\eta^2\eta_1\eta_3 + \eta^3\eta_4\right);\end{aligned} \tag{F.8}$$

$$\begin{aligned}\gamma &= \frac{1}{180\eta^6}\left(60\eta_1^4 - 160\eta_1^2 + 32\eta\eta_2 - 72\eta\eta_1^2\eta_2 + 9\eta^2\eta_2^2 + 12\eta^2\eta_1\eta_3 - \eta^3\eta_4\right) \\ &+ \frac{R_e}{90\eta^7}\left(12\eta_1^3 - 9\eta\eta_1\eta_2 + \eta^2\eta_3 - 16\eta_1\right) + \frac{M^2}{45\eta^4};\end{aligned} \tag{F.9}$$

where $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, \eta_7, \eta_8$ are the derivatives of η with respect to z indicating the orders according to their suffixes, and B_2, C_2, E_2, F_4 are constants to be determined by the boundary conditions.
Thus

$$\begin{aligned}L^{-2}\left[R_eA_1 - \frac{\partial^4\psi_1}{\partial z^4} - 2\frac{\partial^2(L\psi_1)}{\partial z^2} + M^2L\psi_1\right] &= L^{-2}\left[\Gamma(z)r^{10} + \Upsilon(z)r^8\right. \\ &+ \left.\Phi(z)r^6 + \Psi(z)r^4 + \Omega(z)r^2\right],\end{aligned} \tag{F.10}$$

Using eq. (4.39) to integrate eq. (F.10) we have

$$\begin{aligned}&L^{-2}\left[\Gamma(z)r^{10} + \Upsilon(z)r^8 + \Phi(z)r^6 + \Psi(z)r^4 + \Omega(z)r^2\right] \\ &= L_1^{-1}[rL_1^{-1}\{r^{-1}L_1^{-1}(rL_1^{-1}r^{-1})\}] \\ &= \int r\int r^{-1}\int r\int r^{-1}\left(\Gamma(z)r^{10} + \Upsilon(z)r^8 + \Phi(z)r^6 + \Psi(z)r^4 + \Omega(z)r^2\right)\mathrm{d}r\mathrm{d}r\mathrm{d}r\mathrm{d}r, \\ &= \frac{\Gamma(z)r^{14}}{24024} + \frac{\Upsilon(z)r^{12}}{11880} + \frac{\Phi(z)r^{10}}{5040} + \frac{\Psi(z)r^8}{1680} + \frac{\Omega(z)r^6}{360}.\end{aligned} \tag{F.11}$$

Hence

$$\begin{aligned}\psi_2 &= \frac{\Gamma(z)r^{14}}{24024} + \frac{\Upsilon(z)r^{12}}{11880} + \frac{\Phi(z)r^{10}}{5040} + \frac{\Psi(z)r^{10}}{1680} + \frac{\Omega(z)r^6}{360} \\ &+ \frac{r^44}{B\ _2} + \left[\int r\log r\mathrm{d}r\right]C_2 + \frac{r^2}{2}E_2 + F_2 \\ &= \zeta r^{14} + \vartheta r^{12} + \xi r^{10} + \varsigma r^8 + \chi r^6 \\ &+ \frac{r^4}{4}B_2 + \left[\int r\log r\mathrm{d}r\right]C_2 + \frac{r^2}{2}E_2 + F_2,\end{aligned} \tag{F.12}$$

where

$$\begin{aligned}
\zeta &= \frac{\Gamma(z)}{24024}, \\
\vartheta &= \frac{\Upsilon(z)}{11880}, \\
\xi &= \frac{\Phi(z)}{5040}, \\
\varsigma &= \frac{\Psi(z)}{1680}, \\
\chi &= \frac{\Omega(z)}{360}.
\end{aligned} \tag{F.13}$$

We now apply the boundary conditions (F.14) setting $n = 2$ to solve for the constants B_2, C_2, D_2, E_2 in eq. (F.12). Hence we obtain the expression for ψ_2

Recall that the boundary conditions (F.14) when $n = 2$ is given by

$$\left.\begin{array}{l}
-\dfrac{1}{r}\cdot\dfrac{\partial\psi_2}{\partial r} = \psi_2 = 0 \qquad \text{at} \quad r = \eta, \\
-\dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\cdot\dfrac{\partial\psi_2}{\partial r}\right) = \psi_2 = 0 \quad \text{at} \quad r = 0,
\end{array}\right\} \tag{F.14}$$

that is

$$\begin{aligned}
& -\frac{1}{r}\cdot\frac{\partial\psi_2}{\partial r}\,|_{r=\eta} = 0, \\
\Rightarrow\ & 14\zeta r^{12} + 12\vartheta r^{10} + 10\xi r^8 + 8\varsigma r^6 + 6\chi r^4 + B_2 r^2 + C_2\log r + E_2\,|_{r=\eta} = 0, \\
\Rightarrow\ & 14\zeta\eta^{12} + 12\vartheta\eta^{10} + 10\xi\eta^8 + 8\varsigma\eta^6 + 6\chi\eta^4 + B_2\eta^2 + C_2\log\eta + E_2 = 0.
\end{aligned} \tag{F.15}$$

To obtain the second equation we apply

$$\begin{aligned}
& \psi_2\,|_{r=\eta} = 0, \\
\Rightarrow\ & \zeta r^{14} + \vartheta r^{12} + \xi r^{10} + \varsigma r^8 + \chi r^6 + \frac{r^4}{4}B_2 + \left[\frac{r^2}{2}\log r - \frac{r^2}{4}\right]C_2 + \frac{1}{2}r^2 E_2 + F_2\,|_{r=\eta} = 0, \\
\Rightarrow\ & \zeta\eta^{14} + \vartheta\eta^{12} + \xi\eta^{10} + \varsigma\eta^8 + \chi\eta^6 + \frac{\eta^4}{4}B_2 + \left[\frac{r^2}{2}\log\eta - \frac{\eta^2}{4}\right]C_2 + \frac{1}{2}\eta^2 E_2 + F_2 = 0.
\end{aligned} \tag{F.16}$$

For the third equation we apply

$$-\frac{\partial}{\partial r}\left(\frac{1}{r}\cdot\frac{\partial\psi_2}{\partial r}\right)|_{r=0}=0,$$
$$\Rightarrow \quad C_2=0. \tag{F.17}$$

To obtain the fourth equation we apply

$$\psi_2\,|_{r=0}=0$$
$$\Rightarrow \quad F_2=0. \tag{F.18}$$

We now have the simultaneous equations given by

$$\zeta\eta^{12}+12\vartheta\eta^{10}+10\xi\eta^{8}+8\varsigma\eta^{6}+6\chi\eta^{4}+B_2\eta^2+E_2 \quad = \quad 0, \tag{F.19}$$
$$\zeta\eta^{14}+\vartheta\eta^{12}+\xi\eta^{10}+\varsigma\eta^{8}+\chi\eta^{6}+\frac{\eta^4}{4}B_2++\frac{1}{2}\eta^2E_2+ \quad = \quad 0. \tag{F.20}$$

Solving the above simultaneous equations we obtain the following solutions

$$B_2=-4\eta^2\left(6\zeta\eta^8+5\vartheta\eta^6+4\xi\eta^4+3\varsigma\eta^2+2\chi\right), \tag{F.21}$$

and

$$E_2=2\left(5\zeta\eta^{12}+4\vartheta\eta^{10}+3\xi\eta^8+2\varsigma\eta^6+\chi\eta^4\right). \tag{F.22}$$

Substituting eq. (F.21) and eq. (F.22) into eq. (F.12), the expression for ψ_2 is given by

$$\begin{aligned}\psi_2 \quad &= \quad \zeta r^{14}+\vartheta r^{12}+\xi r^{10}+\varsigma r^8+\chi r^6-r^4\eta^2\left(6\zeta\eta^8+5\vartheta\eta^6+4\xi\eta^4+3\varsigma\eta^2+2\chi\right)\\ &+ \quad r^2\left(5\zeta\eta^{12}+4\vartheta\eta^{10}+3\xi\eta^8+2\varsigma\eta^6+\chi\eta^4\right).\end{aligned} \tag{F.23}$$

More terms like $\psi_3, \psi_4, \ldots, \psi_n$ were not evaluated because of their very large differentiation.

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