ADS/CFT CORRESPONDENCE IN A NON-SUPERSYMMETRIC γ_i-DEFORMED BACKGROUND

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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Abstract

A non-supersymmetric γ_i-deformed AdS/CFT correspondence has recently been conjectured by Frolov. A detailed description of both sides of this proposed gauge/string duality is presented. The analogy that exists between single trace gauge theory operators in the $SU(3)$ sector and γ_i-deformed $SU(3)$ integrable spin chains is also discussed. Frolov, Roiban and Tseytlin's leading order comparison between the γ_i-deformed spin chain coherent state action and γ_i-deformed string worldsheet action in the semiclassical limit is reviewed. A particular Lax pair representation for the first order semiclassical γ_i-deformed spin chain/string action is then constructed.

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Chapter 1

Introduction

The modern picture of physics involves a quantum field theoretical description of the three non-gravitational forces (electromagnetism, the strong and the weak interactions), with the gravitational interaction separately described by general relativity. A quantum field theory is a framework consistent with both quantum mechanics and special relativity in which point-like particles are the excitations of local quantized operator fields. Attempts to quantize general relativity in the usual way and so unify gravity with the other three interactions are beset with difficulties - one of the most important is that the resulting quantum field theory is non-renormalizable. Other suggestions for a unified theory including gravity have been made and, at present, the most likely candidate appears to be string theory. Here the fundamental objects are not point-like, but are rather one dimensional strings and multi-dimensional branes. These strings are allowed to oscillate and the different modes correspond to various particles with different masses.

An interesting suggestion [1] of t'Hooft, reviewed in [2], is the notion of gauge/string duality. He examined a quantum field theory with $SU(N)$ gauge invariance in the t'Hooft limit

$$N \longrightarrow \infty, \qquad \text{with} \quad \lambda = g^2 N = \text{fixed}, \tag{1.1}$$

where the t'Hooft coupling λ is defined in terms of the gauge theory coupling constant g and the order N of the $SU(N)$ gauge group. Perturbative expansions in terms of Feynman diagrams were written as expansions of two dimensional surfaces with genus counting parameter $\frac{1}{N}$ and loop parameter λ, and were hence identified with string expansions in terms of the string coupling constant $g_s = \frac{1}{N}$. This suggests that, despite their very different appearance, an $SU(N)$ gauge theory and a string theory may describe the same underlying physics. This gauge/string duality is of great significance, not only because it may allow us to solve hitherto intractable problems in string theory, but also because it may yield a string theory dual to Quantum Chromodynamics (QCD). This is the quantum field theory with $SU(3)$ gauge invariance describing strong interactions, which remains, as yet, imperfectly understood.

1.1 AdS/CFT Correspondence

In 1997, Maldacena proposed the first concrete example of a gauge/string duality, which has become known as AdS/CFT correspondence [3]. This states that $\mathcal{N} = 4$ Super Yang-Mills (SYM) conformal field theory with $SU(N)$ gauge invariance in four dimensional Minkowski spacetime is dual to type IIB string theory in an $AdS_5 \times S^5$ background. We shall now briefly describe the arguments that led to the Maldacena conjecture based on [2, 3, 4, 5]:

Consider a system of N evenly spaced parallel D3 branes, each of which forms a 3+1 dimensional hypervolume in 9+1 dimensional flat spacetime. We can describe this system in two different ways in terms of a type IIB string theory in the low energy limit

$$\alpha' \longrightarrow 0 \qquad \text{with} \quad \frac{\alpha'}{r} = \text{fixed}, \tag{1.2}$$

in which the string tension $\sim \frac{1}{\alpha'}$ becomes large and the spacing r between two consecutive D3 branes shrinks to zero. Firstly, we can view the D3 branes as the end-points of open strings. Closed strings propagate in the empty space surrounding the D3 branes, which is known as the bulk. In the low energy limit (1.2), only massless modes survive, and the open and closed string theories decouple. The closed strings in the bulk become free, while the open strings are described by a 3+1 dimensional $U(N)$ $\mathcal{N} = 4$ SYM gauge theory on the D3 branes. Note that the $U(N)$ gauge group can be split into an $SU(N)$ gauge group plus some extra $U(1)$ degrees of freedom. Secondly, the D3 branes can be viewed as massive objects, which are stacked together and warp the spacetime around them. An observer at infinity will see two main types of low energy string modes - massless free closed string modes far from the D3 branes and all the string modes in the near horizon region close to the D3 branes (which are red-shifted to low energies). The near horizon geometry is that of an $AdS_5 \times S^5$

spacetime. Thus, comparing these two descriptions, we obtain the Maldacena conjecture that an $SU(N)$ invariant $\mathcal{N} = 4$ SYM gauge theory is dual to type IIB string theory in an $AdS_5 \times S^5$ background. The extra $U(1)$ degrees of freedom from the original $U(N)$ gauge group correspond to modes in the space separating the near horizon region from the region far from the D3 branes. These modes appear on the boundary of the $AdS_5 \times S^5$ spacetime and shall be omitted from our string theory description.

Now AdS/CFT correspondence is a strong/weak coupling duality with respect to the t'Hooft coupling, since the Maldacena conjecture identifies

$$\sqrt{\lambda} = \sqrt{g^2 N} = \frac{R^2}{\alpha'}, \tag{1.3}$$

where R is the radius of the AdS_5 and S^5 spaces (which is the same) [3]. Performing perturbative gauge theory calculations is easiest when the t'Hooft coupling is small, but string theory problems can most easily be solved when the strings are nearly point-like in comparison to the background space, so that $\frac{R^2}{\alpha'}$ is large. Although this makes testing the proposed gauge/string duality difficult, it also means that, if established, AdS/CFT correspondence will be exceedingly useful in allowing us to perform strong coupling gauge theory calculations in the dual string theory where the coupling is weak and vice versa.

There has recently been great interest in finding string theories dual to less supersymmetric deformations of $\mathcal{N} = 4$ SYM theory. Leigh and Strassler were able to construct $\mathcal{N} = 1$ supersymmetric marginal deformations of $\mathcal{N} = 4$ SYM theory [6], which include the so-called β-deformations. The string theory dual to this β-deformed SYM theory was described by Lunin and Maldacena [7]. Frolov showed [8] that, in the case of a real deformation parameter $\beta = \gamma$, the classical string worldsheet action in the Lunin-Maldacena background can be derived using a TsT-transformation, with shift parameter $\hat{\gamma} = \sqrt{\lambda}\gamma$. This insight allowed him to demonstrate the existence of a Lax pair for strings moving on a γ-deformed five-sphere. Furthermore, Frolov also constructed a γ_i-deformed string theory by performing a series of three TsT-transformations, with shift parameters $\hat{\gamma}_i = \sqrt{\lambda}\gamma_i$, on the original classical string worldsheet action and showed that strings moving on a γ_i-deformed five-sphere also admit a Lax pair representation. He conjectured a duality between this γ_i-deformed string theory and a non-supersymmetric γ_i-deformed Yang-Mills (YM) theory, which has been studied in more detail by Frolov, Roiban and Tseytlin [9]. We are especially interested in this proposed non-supersymmetric γ_i-deformed gauge/string duality because any agreement found cannot be the result of matching supersymmetric structures on either side.

After Maldacena's initial conjecture, further details of AdS/CFT correspondence were described in [10, 11]. Specifically, it was established that string energies should be dual to the conformal dimensions of the corresponding gauge theory operators. Agreement was found between the energies of point-like strings and the conformal dimensions of chiral primary (half-BPS) operators, which are preserved from quantum corrections by supersymmetry. Berenstein, Maldacena and Nastase (BMN) extended this result by matching the conformal dimensions of long 'nearly BPS' operators to the energies of nearly point-like strings in a pp-wave background [12]. The R-charge J of these BMN operators, which is dual to the total angular momentum of the corresponding strings, was assumed to be large. More explicitly, they considered these quantities in the BMN or semiclassical limit

$$J \longrightarrow \infty \qquad \text{with} \quad \tilde{\lambda} \equiv \frac{\lambda}{J^2} = \text{fixed} \ll 1, \tag{1.4}$$

in which it is possible to perform perturbative expansions in terms of the small parameter $\tilde{\lambda}$. This semiclassical limit was further discussed in [13, 14, 15] and is remarkably useful in allowing calculations to be performed in both the gauge and string theories despite the difficulties associated with the strong/weak coupling nature of the duality.

An interesting development on the gauge theory side was the realization of Minahan and Zarembo [16] that single trace operators in the scalar sector of $\mathcal{N} = 4$ SYM theory are analogous to $SO(6)$ spin chain states. They showed that the planar one-loop matrix of anomalous dimensions in the scalar sector, the eigenvalues of which should correspond to string excitation energies, is simply the Hamiltonian of an $SO(6)$ spin chain. Similar results apply to other sectors of $\mathcal{N} = 4$ SYM theory as well as to various sectors of the β-deformed SYM and γ_i-deformed YM theories. A semiclassical limit of the relevant gauge theory operator corresponds to a continuum limit of the analogous spin chain. It is thus possible to compare the coherent state effective action of a spin chain in the continuum limit with the corresponding string worldsheet action in the fast motion limit [9, 17, 18, 19, 20, 21, 22]. Special mention should be made of Frolov, Roiban and Tseytlin's leading order semiclassical comparison [9] between γ_i-deformed $SU(3)$ spin chains and strings in a γ_i-deformed $\mathbb{R} \times S^5$ background at the level of the action.

1.2 Aim and Structure of Thesis

The aim of this thesis is to study the non-supersymmetric γ_i-deformed gauge/string duality and to ultimately construct a Lax pair representation for the first order semiclassical γ_i-deformed $SU(3)$ spin chain/string action.

This thesis is arranged into six chapters. Chapter 2 contains a review of $\mathcal{N} = 4$ SYM conformal field theory. We discuss the derivation of the $\mathcal{N} = 4$ SYM Lagrangian by dimensional reduction, together with its supersymmetric and conformal nature. Marginal deformations of $\mathcal{N} = 4$ SYM theory are then described and special mention is made of the $\mathcal{N} = 1$ supersymmetric β-deformed and non-supersymmetric γ_i-deformed YM theories. In chapter 3, we discuss the representation of the matrix of anomalous dimensions corresponding to single trace operators in the $SU(3)$ sector of our γ_i-deformed YM theory as the Hamiltonian of a γ_i-deformed $SU(3)$ spin chain. We also explain how this Hamiltonian can be diagonalized using an algebraic Bethe ansatz. Chapter 4 involves a description of the γ_i-deformed string theory. We construct the classical string worldsheet action for strings moving in an $\mathbb{R} \times S^5$ background and, by performing various TsT-transformations, derive the γ-deformed and γ_i-deformed string worldsheet actions. The Lax pair representations of these string theories are also discussed. Chapter 5 contains a review of the first order semiclassical comparison between the γ_i-deformed $SU(3)$ spin chain coherent state action in the continuum limit and the γ_i-deformed string worldsheet action in the fast motion limit. Furthermore, we extend the calculation of the semiclassical γ_i-deformed string worldsheet action to second order for the purpose of constructing the conserved $U(1)$ charge and current densities. Finally, we demonstrate that the γ_i-deformed semiclassical spin chain/string action to leading order admits a Lax pair representation. In other words, the γ_i-deformed spin chain and string systems remain integrable in the semiclassical limit. This new result has been published [23]. A few concluding remarks are presented in chapter 6.

Chapter 2

Conformal Field Theory

2.1 $\mathcal{N} = 4$ Super Yang-Mills Theory

2.1.1 Yang-Mills theory with $SU(N)$ gauge invariance

The original YM theory [24] was developed in 1954 in an attempt to explain the strong interaction in terms of pion exchange. Unlike the analogous theory of electrodynamics, which contains $U(1)$ gauge invariance, this YM theory was invariant under $SU(2)$ isospin rotations. More recently, with the advent of the quark model, a more correct description of the strong interaction in terms of gluon exchange was developed using a YM theory with an $SU(3)$ colour gauge group. The main difference between these YM theories and electrodynamics is their non-abelian nature - the components of the gauge field do not commute. This results in the self-interaction of the gauge bosons and is responsible for much of the extra complexity inherent in the theory.

We are interested in a more general YM theory, which contains $SU(N)$ gauge invariance (with $N \geq 2$ an arbitrary integer, often taken to be large). We shall now explain how to construct such a theory based on discussions in [25, 26]:

Let us start by considering some free field theory containing a complex scalar field $\Phi(x) = (\Phi_1(x), \Phi_2(x), \ldots, \Phi_N(x))$ with N components and a Dirac spinor field $\Psi(x)$, the four components of which are themselves N-component fields, so that $\Psi_i(x) = (\Psi_{i1}(x), \Psi_{i2}(x), \ldots, \Psi_{iN}(x))$. This representation of the fields $\Phi(x)$ and $\Psi(x)$ in terms of N-component vectors is called the *fundamental representation.* The free field Lagrangian is then given by

$$\mathcal{L} = \partial^\mu \Phi^\dagger \partial_\mu \Phi - m_b^2 \Phi^\dagger \Phi + \bar{\Psi} \left(i\gamma^\mu \partial_\mu - m_f \right) \Psi, \tag{2.1}$$

with $\bar{\Psi} \equiv \Psi^\dagger \gamma^0$. Here m_b and m_f are the masses of the scalar (boson) and spinor (fermion) fields respectively, and γ^μ are the usual 4×4 gamma matrices (A.11).

We would now like to include a gauge field in this description so as to obtain an $SU(N)$ invariant field theory. In other words, we would like our Lagrangian to remain unchanged when

$$\Phi(x) \longrightarrow U(x)\Phi(x) \qquad \text{and} \qquad \Psi_i(x) \longrightarrow U(x)\Psi_i(x), \tag{2.2}$$

where $U(x)$ is an arbitrary element of $SU(N)$. This is clearly not the case for the Lagrangian (2.1) due to the extra derivative terms that arise as a result of the local nature of the transformation $U(x)$. We shall therefore introduce the covariant derivative

$$D_\mu \equiv \partial_\mu - igA_\mu(x), \tag{2.3}$$

where $A_\mu(x)$ are real $N \times N$ gauge field matrices, chosen to be traceless and hermitean, and g is the YM coupling constant. The Lagrangian then becomes

$$\mathcal{L} = \left(D^\mu \Phi \right)^\dagger D_\mu \Phi - \mu^2 \Phi^\dagger \Phi + \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi, \tag{2.4}$$

which remains invariant under any local $SU(N)$ transformation $U(x)$, if we insist that the gauge field $A_\mu(x)$ must transform as follows:

$$A_\mu(x) \longrightarrow U(x)A_\mu(x)U^\dagger(x) + \frac{i}{g}U(x)\partial_\mu U^\dagger(x). \tag{2.5}$$

Finally, we would like to determine the field strength contribution to the YM Lagrangian, which contains the kinetic terms associated with the gauge field A_μ. Let us first define the field strength as

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\mu A_\nu - ig \left[A_\mu, A_\nu \right]. \tag{2.6}$$

Note that the last term in this expression, which is zero in electrodynamics, is now present because the components A_μ and A_ν of the YM gauge field do not commute. Since the gauge field A_μ is a traceless hermitean $N \times N$ matrix, we can expand

$A_\mu(x) = \sum_a A_\mu^a(x) T^a$ in terms of the $N^2 - 1$ generators T^a of $SU(N)$, where $A_\mu^a(x)$ are real field coefficients. Hence the field strength can be written as

$$F_{\mu\nu}(x) = \sum_a F_{\mu\nu}^a(x) T^a \qquad \text{with} \quad F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \sum_{b,c} f^{bca} A_\mu^b A_\nu^c, \tag{2.7}$$

where the structure constants f^{abc} are defined such that $\left[T^a, T^b\right] = i \sum_c f^{abc} T^c$ and are a property of the Lie algebra of $SU(N)$. This field strength transforms under a local $SU(N)$ transformation $U(x)$ as

$$F_{\mu\nu}(x) \longrightarrow U(x) F_{\mu\nu}(x) U^\dagger(x), \tag{2.8}$$

so that $\mathrm{Tr}\{F_{\mu\nu} F^{\mu\nu}\}$ is $SU(N)$ invariant. Thus we shall define the field strength or pure YM Lagrangian as follows:

$$\mathcal{L}_{YM} \equiv -\frac{1}{4} \mathrm{Tr}\{F_{\mu\nu} F^{\mu\nu}\} = -\frac{1}{8} \sum_a F_{\mu\nu}^a F^{\mu\nu\ a}, \tag{2.9}$$

where we have made use of the conventional normalization $\mathrm{Tr}\left\{T^a T^b\right\} = \frac{1}{2}\delta^{ab}$.

A full YM Lagrangian can, of course, contain terms other than just (2.4) and (2.9) - further $SU(N)$ invariant interaction terms can also be included. One possibility is a scalar potential of the form $V(\Phi^\dagger \Phi)$.

Now there is another possible representation for the fields $\Phi(x)$ and $\Psi(x)$ in terms of traceless hermitean $N \times N$ matrices rather than N-component vectors. This is called the *adjoint representation* and is spanned by the generators of $SU(N)$. The Lagrangian (2.4) in this representation is given by

$$\mathcal{L} = \frac{1}{2} \mathrm{Tr}\left\{(D^\mu \Phi)^\dagger D_\mu \Phi - \mu^2 \Phi^\dagger \Phi + \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi\right\}, \tag{2.10}$$

where the covariant derivatives of the fields Φ and Ψ are

$$D_\mu \Phi \equiv \partial_\mu \Phi - ig\left[A_\mu, \Phi\right] \qquad \text{and} \qquad D_\mu \Psi \equiv \partial_\mu \Psi - ig\left[A_\mu, \Psi\right]. \tag{2.11}$$

Notice that one must introduce a trace into this Lagrangian, due to the fact that the fields are now matrices, and a commutator between the gauge field A_μ and the field into the covariant derivative.

Lastly, we make a few general observations about this YM theory: The gauge field matrix A_μ really consists of $N^2 - 1$ different real fields A_μ^a. This is the reason that the YM theory with $SU(3)$ gauge invariance contains eight distinct gauge bosons called gluons. Furthermore, the field strength term $-\frac{1}{8} \sum_a F_{\mu\nu}^a F^{\mu\nu\ a}$ in the YM Lagrangian contains cubic and quartic terms as well as the usual quadratic ones. This results in the self-interaction of the gauge bosons.

2.1.2 $\mathcal{N}=4$ SYM theory by dimensional reduction

$\mathcal{N}=4$ SYM theory was originally constructed [27] by dimensionally reducing a ten dimensional $\mathcal{N}=1$ SYM theory to four dimensions. YM theories and dimensional reductions thereof were further discussed in [28]. We shall now derive the $\mathcal{N}=4$ SYM Lagrangian by dimensional reduction following [27].

The relevant ten dimensional $\mathcal{N}=1$ SYM theory contains the gauge field $B_M\left(x^N\right)$ and the massless Weyl-Majorana spinor field $\lambda\left(x^N\right)$, chosen to be in the adjoint representation of $SU(N)$. Here the capital roman letters, which index the coordinates in our ten dimensional Minkowski spacetime, run from 0 to 9. Hence the ten dimensional SYM Lagrangian is given by [27]

$$\mathcal{L}=\mathrm{Tr}\left\{-\frac{1}{4}G_{MN}G^{MN}-\frac{i}{2}\bar{\lambda}\Gamma^M D_M\lambda\right\}, \tag{2.12}$$

where $G_{MN}\equiv\partial_M B_N-\partial_N B_M-ig\left[B_M,B_N\right]$ is the ten dimensional field strength and Γ^M are gamma matrices satisfying the Clifford algebra in ten dimensions.

In order to reduce this SYM Lagrangian to four spacetime dimensions, we shall split up our ten coordinates x^M into four reduced coordinates x^μ and six extra coordinates x^{3+m}. Here the greek and small roman indices run from 0 to 3 and from 1 to 6 respectively. We shall assume that the extra six dimensions are very small and compact, and take a zero slope limit, so that all dependence on these dimensions and all derivatives with respect to the extra six spacetime coordinates vanish. We shall also separate the ten gauge field matrices B_M into four gauge field matrices A_μ and six real scalar field matrices ϕ_m. Thus

$$B_\mu\left(x^N\right)\longrightarrow A_\mu\left(x^\nu\right)\qquad\text{and}\qquad B_m\left(x^N\right)\longrightarrow\phi_m\left(x^\nu\right), \tag{2.13}$$

so that the components of the ten dimensional field strength can be written as

$$\begin{aligned}
G_{\mu\nu}\left(x^R\right)&\longrightarrow F_{\mu\nu}\left(x^\rho\right)=\partial_\mu A_\nu\left(x^\rho\right)-\partial_\nu A_\mu\left(x^\rho\right)-ig\left[A_\mu\left(x^\rho\right),A_\nu\left(x^\rho\right)\right],\\
G_{\mu\ 3+m}\left(x^R\right)&\longrightarrow D_\mu\phi_m\left(x^\rho\right)=\partial_\mu\phi_m\left(x^\rho\right)-ig\left[A_\mu\left(x^\rho\right),\phi_m\left(x^\rho\right)\right],\\
G_{3+m\ \mu}\left(x^R\right)&\longrightarrow -D_\mu\phi_m\left(x^\rho\right)=-\partial_\mu\phi_m\left(x^\rho\right)+ig\left[A_\mu\left(x^\rho\right),\phi_m\left(x^\rho\right)\right],\\
G_{3+m\ 3+n}(x^R)&\longrightarrow -ig\left[\phi_m\left(x^\rho\right),\phi_n\left(x^\rho\right)\right].
\end{aligned} \tag{2.14}$$

The SYM Lagrangian (2.12) therefore reduces to

$$\begin{aligned}
\mathcal{L}=&-\frac{1}{4}\mathrm{Tr}\left\{F_{\mu\nu}F^{\mu\nu}\right\}-\frac{1}{2}\mathrm{Tr}\left\{D_\mu\phi_m D^\mu\phi^m\right\}+\frac{1}{4}g^2\mathrm{Tr}\left\{\left[\phi_m,\phi_n\right]\left[\phi^m,\phi^n\right]\right\}\\
&-\frac{i}{2}\mathrm{Tr}\left\{\bar{\lambda}\Gamma^\mu D_\mu\lambda\right\}-\frac{1}{2}g\mathrm{Tr}\left\{\bar{\lambda}\Gamma^{3+m}\left[\phi_m,\lambda\right]\right\}.
\end{aligned} \tag{2.15}$$

Further simplification of (2.15) requires us to choose an explicit representation for the gamma matrices Γ^M. This representation must be at least 32 dimensional[1]. Now, as in [29], one can choose an off-diagonal block representation in terms of the 16×16 matrices Σ^M and $\bar{\Sigma}^M$ in analogy to (A.25). However, for the purposes of dimensional reduction, we shall rather make use of the representation of [27], which involves a tensor product of 8×8 and 4×4 matrices, as follows:

$$\Gamma^\mu = \begin{pmatrix} 1_4 & 0 \\ 0 & -1_4 \end{pmatrix} \otimes \gamma^\mu, \quad \Gamma^{3+j} = \begin{pmatrix} \rho^j & 0 \\ 0 & \rho'^j \end{pmatrix} \otimes (-i\gamma^5), \quad \Gamma^{6+j} = \begin{pmatrix} 0 & \gamma^j \\ \gamma^j & 0 \end{pmatrix} \otimes 1_4, \tag{2.16}$$

where μ runs from 0 to 3, as usual, and j runs from 1 to 3. Here γ^μ are the usual 4×4 gamma matrices (A.11) in four spacetime dimensions, and $\rho^1 \equiv \rho'^1 \equiv \gamma^0$, $\rho^2 \equiv \rho'^2 \equiv \gamma^5$ and $\rho^3 \equiv -\rho'^3 \equiv -i\gamma^0\gamma^5$. We can easily verify that this collection of matrices satisfy the Clifford algebra in ten dimensions. Hence the chirality matrix is given by

$$-i\Gamma^{11} = \Gamma^0\Gamma^1 \dots \Gamma^9 = -i \begin{pmatrix} 0 & \rho^3 \\ -\rho^3 & 0 \end{pmatrix} \otimes 1_4. \tag{2.17}$$

Now λ is a 32-component Weyl-Majorana spinor satisfying both the Weyl or chirality condition $-i\Gamma^{11}\lambda = \lambda$ and the Majorana condition $\lambda = \lambda^C$ that the spinor must be the same as its charge conjugate[2]. Let us define

$$\lambda \equiv \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \tag{2.18}$$

where λ_1 and λ_2 each consist of four 4-component Dirac spinors. The Weyl condition then implies that $\lambda_2 = i(\rho_3 \otimes 1_4)\lambda_1$, which yields

$$\lambda = \begin{pmatrix} \lambda_1 \\ i(\rho_3 \otimes 1_4)\lambda_1 \end{pmatrix} \qquad \text{with} \quad \lambda_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{pmatrix}. \tag{2.19}$$

Furthermore, for λ to be Majorana, it was shown in [27] that the four Dirac spinors χ_a must also be Majorana, so that

$$\chi_a = \begin{pmatrix} \psi_{a\alpha} \\ \bar{\psi}_a{}^{\dot{\alpha}} \end{pmatrix}, \tag{2.20}$$

[1] The gamma matrices in D spacetime dimensions (with D even), which satisfy the Clifford algebra, have a minimal representation of dimension $2^{D/2}$ [27].

[2] It turns out that ten dimensional spacetime is the lowest dimensional spacetime (aside from the rather trivial $D = 2$ case) in which it is possible for a spinor to satisfy both the Weyl and Majorana conditions [27].

where $\psi_{a\alpha}$ are four 2-component Weyl spinors. (A more detailed discussion of Weyl spinors, and dotted and undotted notation is available in appendix A.)

Therefore, using the expressions (2.16) and (2.19) for the gamma matrices Γ^M and the Weyl-Majorana spinor λ in our 32 dimensional representation, we can calculate

$$-\frac{i}{2}\mathrm{Tr}\left\{\bar{\lambda}\Gamma^{\mu}D_{\mu}\lambda\right\} = -\frac{1}{2}\sum_{a=1}^{4}\mathrm{Tr}\left\{\bar{\chi}_a\gamma^{\mu}D_{\mu}\chi_a\right\}, \tag{2.21}$$

$$-\frac{1}{2}g\mathrm{Tr}\left\{\bar{\lambda}\Gamma^{3+j}\left[\phi_j,\lambda\right]\right\} = \frac{i}{2}g\sum_{a,b=1}^{4}\mathrm{Tr}\left\{\bar{\chi}_a(\beta^j)_{ab}\gamma^5\left[\phi_j,\chi_b\right]\right\}, \tag{2.22}$$

$$-\frac{1}{2}g\mathrm{Tr}\left\{\bar{\lambda}\Gamma^{6+j}\left[\phi_{3+j},\lambda\right]\right\} = \frac{i}{2}g\sum_{a,b=1}^{4}\mathrm{Tr}\left\{\bar{\chi}_a(\alpha^j)_{ab}\left[\phi_{3+j},\chi_b\right]\right\}, \tag{2.23}$$

where $\beta^j \equiv \rho^j$ and $\alpha^j \equiv -\rho^3\gamma^j$, which are explicitly given by

$$\beta^1 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \qquad \beta^2 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \qquad \beta^3 = \begin{pmatrix} 0 & i1_2 \\ -i1_2 & 0 \end{pmatrix}, \tag{2.24}$$

$$\alpha^1 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & i\sigma^1 \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}, \quad \alpha^3 = \begin{pmatrix} -i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}. \tag{2.25}$$

The $\mathcal{N}=4$ SYM Lagrangian in our reduced four dimensional Minkowski spacetime is thus [27]

$$\begin{aligned}\mathcal{L}_{\mathrm{SYM}} = &-\frac{1}{4}\mathrm{Tr}\left\{F_{\mu\nu}F^{\mu\nu}\right\} - \frac{1}{2}\mathrm{Tr}\left\{D_{\mu}\phi_m D^{\mu}\phi^m\right\} + \frac{1}{4}g^2\mathrm{Tr}\left\{\left[\phi_m,\phi_n\right]\left[\phi^m,\phi^n\right]\right\} \\ &- \frac{1}{2}\sum_{a=1}^{4}\mathrm{Tr}\left\{\bar{\chi}_a\gamma^{\mu}D_{\mu}\chi_a\right\} + \frac{1}{2}ig\sum_{a,b=1}^{4}\mathrm{Tr}\left\{\bar{\chi}_a(\beta^j)_{ab}\gamma^5\left[\phi_j,\chi_b\right]\right\} \\ &+ \frac{1}{2}ig\sum_{a,b=1}^{4}\mathrm{Tr}\left\{\bar{\chi}_a(\alpha^j)_{ab}\left[\phi_{3+j},\chi_b\right]\right\}.\end{aligned} \tag{2.26}$$

The six massless real scalar fields ϕ_m, four gauge fields A_μ and components of the four massless Majorana spinor fields χ_a are all $N\times N$ matrices in the adjoint representation of $SU(N)$. The scalar fields ϕ_m are invariant under $SO(6)$ rotations and this internal symmetry is locally isomorphic to the internal $SU(4)$ symmetry of the spinor fields χ_a [29].

Finally, let us check that the number of bosonic and fermionic degrees of freedom match (as one would expect for a supersymmetric theory). There are six degrees of freedom in the real scalar fields and two in the gauge boson fields (A_μ has two polarization states). This yields a total of eight bosonic degrees of freedom. One would expect each of the four Majorana spinors to contain two complex (four real) degrees of freedom. These spinors must, however, satisfy the Dirac equation and this complex constraint limits the number of real degrees of freedom associated with each spinor to two. Thus there are also eight fermionic degrees of freedom.

2.1.3 The scalar potential

We shall now consider the scalar interaction term in the SYM Lagrangian (2.26) in more detail. This scalar potential is given by

$$V = \frac{1}{4}g^2 \mathrm{Tr}\left\{[\phi^m, \phi^n][\phi_m, \phi_n]\right\} = \frac{1}{4}g^2 \mathrm{Tr}\left\{[\phi^m, \phi^n]^2\right\}, \tag{2.27}$$

where we note that $\phi^m = \eta^{3+m\ 3+n}\phi_n$, with $\eta^{3+m\ 3+n} = -\delta^{mn}$ the ten dimensional Minkowski metric confined to the six compact dimensions, so that $\phi^m = -\phi_m$.

It is now possible [22] to rewrite this scalar potential in terms of three complex scalar fields $\Phi_j \equiv \phi_j + i\phi_{3+j}$, with complex conjugates $\Phi_j^* = \phi_j - i\phi_{3+j}$, as follows:

$$\begin{aligned} V = -\frac{1}{4}g^2 \Big\{ & \mathrm{Tr}\left[|\Phi_1\Phi_2 - \Phi_2\Phi_1|^2 + |\Phi_2\Phi_3 - \Phi_3\Phi_2|^2 + |\Phi_3\Phi_1 - \Phi_1\Phi_3|^2\right] \\ & - \frac{1}{4}\mathrm{Tr}\left[\left([\Phi_1, \Phi_1^*] + [\Phi_2, \Phi_2^*] + [\Phi_3, \Phi_3^*]\right)^2\right] \Big\}. \end{aligned} \tag{2.28}$$

The first term is known as the F-term and the second as the D-term. (The reason for this will become apparent when we discuss supersymmetry). It is the F-term that will be modified when we introduce the β-deformed $\mathcal{N} = 1$ supersymmetric and γ_i-deformed non-supersymmetric YM theories.

2.2 Supersymmetry

Supersymmetry (SUSY) is a hypothetical symmetry relating fermions and bosons. In a supersymmetric theory every fermion (boson) should have a corresponding bosonic (fermionic) superpartner. As yet no direct evidence for SUSY has been discovered, although several high energy experiments, which search for these superpartners or signatures of their existence, are currently underway. Nevertheless, SUSY remains an appealing concept within the theoretical community due to the comparatively simple nature of supersymmetric theories.

Now SUSY transformations are generated by the $\mathcal{N}$ supercharges Q^I, with conjugates $\bar{Q}^I$, contained within a supersymmetric theory. These supercharges are spinors and satisfy a SUSY algebra. Our original ten dimensional $\mathcal{N} = 1$ SYM theory contains only one 16-component Weyl-Majorana spinor supercharge, while there are four 2-component Weyl spinor supercharges with an internal $SU(4)$ R-symmetry in the reduced $\mathcal{N} = 4$ SYM theory. This is the maximum number of supercharges possible in a non-gravitational theory and hence $\mathcal{N} = 4$ SYM theory is called 'maximally supersymmetric'.

The most convenient way of formulating a supersymmetric theory involves the introduction of superspace, which is an extension of spacetime using non-commuting spinor coordinates and was invented by Salam and Strathdee [30]. In this section, we first explain how to rigorously describe a SUSY transformation in superspace. Chiral superfields, vector superfields and the Wess-Zumino gauge are also discussed, and we demonstrate that it is possible to construct a SUSY invariant action in $\mathcal{N} = 1$ superspace using F-terms, D-terms and a field strength term. Finally, we show that the original ten dimensional SYM action can be written in $\mathcal{N} = 1$ superspace and the implications for the reduced four dimensional SYM theory are mentioned. The form of the $\mathcal{N} = 4$ SYM action in $\mathcal{N} = 1$ superspace is also stated. This review is based on discussions in [25, 31, 32, 33, 34, 35, 36].

2.2.1 $\mathcal{N} = 1$ superspace and superfields

SUSY transformations change fermions into bosons and vice versa. The generators of $\mathcal{N} = 1$ SUSY transformations in four spacetime dimensions are the supercharge Q and its conjugate $\bar{Q}$, which are 2-component Weyl spinors and satisfy the SUSY algebra [25, 31]

$$\left\{Q_\alpha, \bar{Q}_{\dot{\beta}}\right\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu, \qquad \left\{Q_\alpha, Q_\beta\right\} = \left\{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\right\} = 0, \qquad \left[Q_\alpha, P_\mu\right] = \left[\bar{Q}_{\dot{\alpha}}, P_\mu\right] = 0, \tag{2.29}$$

where $P_\mu = i\partial_\mu$ is the momentum operator and σ^μ is defined just after (A.25).

In order to construct a SUSY transformation from these generators, we need to introduce a pair of Grassmannian[3] 2-component Weyl spinor coordinates θ and $\bar{\theta}$ upon which our supercharge and its conjugate can act. This leads us to define the superfield $\Phi(x, \theta, \bar{\theta})$ as a field in the extended superspace $(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$, where x^μ are the usual four dimensional Minkowski spacetime coordinates. A finite SUSY transformation, which acts on this superfield, is then $e^{i(\xi Q + \bar{Q}\bar{\xi})}$, where ξ and $\bar{\xi}$ are a pair of finite spinor parameters. The SUSY variation of the superfield Φ is thus given by

$$\delta\Phi(x, \theta, \bar{\theta}) = i(\xi Q + \bar{Q}\bar{\xi}) \; \Phi(x, \theta, \bar{\theta}), \tag{2.30}$$

where ξ and $\bar{\xi}$ are now infinitesimal spinor parameters.

The supercharges can be expressed in differential operator form in terms of the superspace coordinates. Specifically we see that[4]

$$Q_\alpha \equiv \frac{\partial}{\partial \theta^\alpha} - i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \qquad \text{and} \qquad \bar{Q}_{\dot{\alpha}} \equiv -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + i\theta^\beta(\sigma^\mu)_{\beta\dot{\alpha}}\partial_\mu, \tag{2.31}$$

satisfy our supersymmetric algebra (2.29). Furthermore, we shall define a set of covariant derivatives, which anticommute with the supercharges, as follows:

$$D_\alpha \equiv \frac{\partial}{\partial \theta^\alpha} + i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \qquad \text{and} \qquad \bar{D}_{\dot{\alpha}} \equiv -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i\theta^\beta(\sigma^\mu)_{\beta\dot{\alpha}}\partial_\mu. \tag{2.32}$$

The fact that these derivatives anticommute with Q_β and $\bar{Q}_{\dot{\beta}}$ means that they will commute with any SUSY variation.

2.2.2 Chiral superfields and F-terms

To construct the F-terms in a SUSY invariant Lagrangian, we must first introduce the concept of a chiral superfield. If $\Phi_L(x, \theta, \bar{\theta})$ and $\Phi_R(x, \theta, \bar{\theta})$ are left-handed and right-handed chiral superfields respectively, then [31, 32, 34]

$$\bar{D}_{\dot{\alpha}}\Phi_L\left(x, \theta, \bar{\theta}\right) = 0 \qquad \text{and} \qquad D_\alpha\Phi_R\left(x, \theta, \bar{\theta}\right) = 0. \tag{2.33}$$

[3]These spinor coordinates anticommute so that $\{\theta^\alpha, \theta^\beta\} = \{\bar{\theta}^{\dot{\alpha}}, \bar{\theta}^{\dot{\beta}}\} = \{\theta^\alpha, \bar{\theta}^{\dot{\beta}}\} = 0$.

[4]Differentiation in terms of Grassmannian coordinates is defined as follows:

$$\left\{\frac{\partial}{\partial\theta^\alpha}, \theta^\beta\right\} = \delta^\beta_\alpha, \qquad \left\{\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}, \bar{\theta}^{\dot{\beta}}\right\} = \delta^{\dot{\beta}}_{\dot{\alpha}}, \qquad \left\{\frac{\partial}{\partial\theta^\alpha}, \bar{\theta}^{\dot{\beta}}\right\} = 0, \qquad \left\{\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}, \theta^\beta\right\} = 0.$$

In other words, in the case of anticommuting coordinates, one must simply remember that derivatives also anticommute. The product rule will therefore change slightly - when differentiating the 2[nd], 4[th], etc terms in a product, we pick up a minus sign.

These names originate in the left-handed and right-handed chiral nature of the spinor fields $\psi_L(x)$ and $\psi_R(x)$, which we shall observe to be contained in these superfields.

We shall concentrate for now on left-handed chiral superfields. Let us define a new set of superspace coordinates y, θ, and $\bar{\theta}$, with $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$, in which the covariant derivatives are given by

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + 2i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \qquad \text{and} \qquad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}. \tag{2.34}$$

Notice that any left-handed chiral superfield $\Phi(x,\theta,\bar{\theta}) = \Phi(y,\theta)$ is now independent of $\bar{\theta}$. Expanding $\Phi(y,\theta)$ in a Taylor series in terms of θ yields

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y), \tag{2.35}$$

where ϕ and F are scalar fields and ψ is a spinor field. (The factor $\sqrt{2}$ has been included in front of ψ for convenience.) This is an exact expression - all terms higher than second order vanish because θ^α and θ^β anticommute. We can expand each of the terms $\phi(y)$, $\psi(y)$ and $F(y)$ around $y = x$ to obtain [25, 32]

$$\begin{aligned}\Phi(x,\theta,\bar{\theta}) = \phi(x) &+ i(\theta\sigma^\mu\bar{\theta})\partial_\mu\phi(x) - \frac{1}{2}(\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta})\partial_\mu\partial_\nu\phi(x) \\ &+ \sqrt{2}\theta\psi(x) + \sqrt{2}i\theta(\theta\sigma^\mu\bar{\theta})\partial_\mu\psi(x) + \theta\theta F(x),\end{aligned} \tag{2.36}$$

which is, again, an exact expansion.

Let us now calculate the SUSY variations of the fields ϕ, ψ and F. The SUSY variation of the left-handed chiral superfield $\Phi(y,\theta)$ can be expressed in terms of $\delta\phi$, $\delta\psi$ and δF as follows:

$$\delta\Phi(y,\theta) = \delta\phi(y) + \sqrt{2}\theta\delta\psi(y) + \theta\theta\delta F(y), \tag{2.37}$$

but also, writing the supercharge Q and its conjugate $\bar{Q}$ in (2.30) in terms of the coordinates y, θ and $\bar{\theta}$, we find that

$$\begin{aligned}\delta\Phi(y,\theta) &= i\left(\xi Q + \bar{Q}\bar{\xi}\right)\Phi(y,\theta) \\ &= i\left(\xi\frac{\partial}{\partial\theta} - \frac{\partial}{\partial\bar{\theta}}\bar{\xi} + 2i\theta\sigma^\mu\bar{\xi}\frac{\partial}{\partial y^\mu}\right)\left[\phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y)\right] \\ &= \sqrt{2}i\xi\psi(y) + 2i\theta\xi F(y) - 2\theta\sigma^\mu\bar{\xi}\partial_\mu\phi(y) + \sqrt{2}\theta\theta\partial_\mu\psi(y)\sigma^\mu\bar{\xi}.\end{aligned} \tag{2.38}$$

Hence, equating different orders of θ, we obtain

$$\delta\phi = \sqrt{2}i\xi\psi, \tag{2.39}$$

$$\delta\psi = \sqrt{2}i\xi F - \sqrt{2}\sigma^\mu\bar{\xi}\partial_\mu\phi, \tag{2.40}$$

$$\delta F = \sqrt{2}\partial_\mu\psi\sigma^\mu\bar{\xi}. \tag{2.41}$$

Now, clearly, (2.41) indicates that the SUSY variation of the scalar field F is a total derivative. This can also be seen simply using dimensional analysis [25]. Since the momentum operator P^μ has mass dimension[5] $+1$, we observe, from the SUSY algebra (2.29), that the supercharge Q and its conjugate $\bar{Q}$ must have mass dimension $+\frac{1}{2}$. Furthermore, the coordinates θ and $\bar{\theta}$ must have mass dimension $-\frac{1}{2}$ for the term in the exponential of our finite SUSY transformation to be dimensionless. Now, assuming that the scalar field ϕ has mass dimension $+1$ (as is the case for any physically meaningful scalar field in four spacetime dimensions), we see that ψ and F must therefore have mass dimensions $\frac{3}{2}$ and $+2$ respectively. Thus the only possible object that can produce the required mass dimension of $+2$ for the SUSY variation of the field F is the total derivative $\delta F \sim \partial_\mu \psi \sigma^\mu \bar{\xi}$. This argument is, perhaps, less rigorous than the previous explicit calculation, but it has the advantage of being more generally applicable.

The scalar field $F(x)$ is therefore an ideal candidate for a SUSY invariant Lagrangian, since $\int d^4x\ F(x)$ is invariant under SUSY transformations (if we ignore surface terms). This is the origin of the name 'F-terms'. Furthermore, any function of any number of left-handed chiral superfields Φ_i is also a left-handed chiral superfield (it depends only on y and θ). Hence the F-terms in a SUSY invariant action can be written as[6] [31, 34]

$$S_F = -\int d^4x \left\{ \int d^2\theta\ f(\Phi_i) + \int d^2\bar{\theta}\ f^*(\Phi_i^\dagger) \right\}, \tag{2.42}$$

where f is some function[7] of the left-handed chiral superfields Φ_i. Here we have included in our Lagrangian the hermitean conjugate of the relevant expression, which is obviously also SUSY invariant. This can also be seen as the analogous F-term for a function f^* of the right-handed chiral superfields $\Phi_i^\dagger$. These F-terms result in the mass terms in the Lagrangian as well as further interaction terms, but there are no kinetic terms contained in this expression.

[5]The mass dimension x of a quantity Q is defined such that $[Q] = M^x$. Note also that we are using units in which $c \equiv \hbar \equiv 1$.

[6]Integration of Grassmannian coordinates is defined as follows [33]:

$$\int d\theta^1 \theta^1 = \int d\theta^2 \theta^2 = 1 \qquad \text{and} \qquad \int d\theta^1 1 = \int d\theta^2 1 = 0.$$

Note also that $d^2\theta \equiv d\theta^1 d\theta^2$ and $d^2\bar{\theta} \equiv d\bar{\theta}^1 d\bar{\theta}^2$.

[7]This function is usually a polynomial of maximum degree three - higher order superpotentials lead to non-renormalizable theories [31].

2.2.3 Vector superfields, the Wess-Zumino gauge and D-terms

Another possible contribution to a SUSY invariant action are the so-called D-terms. These can be obtained from any vector superfield $V(x,\theta,\bar{\theta})$, which is defined as a self-conjugate superfield satisfying

$$V(x,\theta,\bar{\theta}) = V^{\dagger}(x,\theta,\bar{\theta}). \tag{2.43}$$

A general vector superfield can be written as [31, 32]

$$\begin{aligned} V(x,\theta,\bar{\theta}) = {} & C(x) + \theta\chi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta M(x) + \bar{\theta}\bar{\theta}M^*(x) \\ & - \theta\sigma^{\mu}\bar{\theta}A_{\mu}(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x), \end{aligned} \tag{2.44}$$

where $C(x)$ and $D(x)$ are real scalar fields, $M(x)$ is a complex scalar field, $A_{\mu}(x)$ is a real vector field, and $\chi(x)$ and $\lambda(x)$ are complex spinor fields.

We shall now demonstrate that the SUSY variation of the field $D(x)$ is a total derivative. This is to be expected, since $\frac{1}{2}D(x)$ is the coefficient of the highest order term in the above expression and has the highest mass dimension. Its SUSY variation should therefore be proportional to derivatives of the coefficients of the lower order terms. More explicitly, we can calculate

$$\delta V = \delta C + \theta\delta\chi + \bar{\theta}\delta\bar{\chi} + \theta\theta\delta M + \bar{\theta}\bar{\theta}\delta M^* - \theta\sigma^{\mu}\bar{\theta}\delta A_{\mu} + i\theta\theta\bar{\theta}\delta\bar{\lambda} - i\bar{\theta}\bar{\theta}\theta\delta\lambda + \tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\delta D, \tag{2.45}$$

and also

$$\begin{aligned} \delta V = {} & i\left[\xi\frac{\partial}{\partial\theta} - \frac{\partial}{\partial\bar{\theta}}\bar{\xi} - i\left(\xi\sigma^{\mu}\bar{\theta} - \theta\sigma^{\mu}\bar{\xi}\right)\partial_{\mu}\right] \\ & \times\left[C + \theta\chi + \bar{\theta}\bar{\chi} + \theta\theta M + \bar{\theta}\bar{\theta}M^* - \theta\sigma^{\mu}\bar{\theta}A_{\mu} + i\theta\theta\bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\theta\lambda + \tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D\right]. \end{aligned} \tag{2.46}$$

We need only determine the highest order term in the last expression, which can be equated to the corresponding term in the first equation as follows:

$$\begin{aligned} \tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) &= \left[\xi^{\alpha}(\sigma^{\mu})_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\right]\left[i\,(\theta\theta)\,\bar{\theta}_{\dot{\gamma}}\partial_{\mu}\bar{\lambda}^{\dot{\gamma}}\right] - \left[\theta^{\alpha}(\sigma^{\mu})_{\alpha\dot{\beta}}\bar{\xi}^{\dot{\beta}}\right]\left[-i\left(\bar{\theta}\bar{\theta}\right)\theta^{\gamma}\partial_{\mu}\lambda_{\gamma}\right] \\ &= -\tfrac{i}{2}\theta\theta\bar{\theta}\bar{\theta}\left[\xi\sigma^{\mu}\partial_{\mu}\bar{\lambda} + \partial_{\mu}\lambda\sigma^{\mu}\bar{\xi}\right]. \end{aligned} \tag{2.47}$$

Hence the SUSY variation of the real scalar field $D(x)$ is given by

$$\delta D = -i\partial_{\mu}\left(\xi\sigma^{\mu}\bar{\lambda} + \lambda\sigma^{\mu}\bar{\xi}\right), \tag{2.48}$$

which is a total derivative. The D-term $\int d^4x\ D(x)$ is therefore another possible candidate for our action, since it is SUSY invariant.

We shall now introduce the supersymmetric generalization of a non-abelian gauge transformation, which acts on the vector superfield $V(x, \theta, \bar{\theta})$ as follows:

$$e^{gV} \longrightarrow e^{ig\Lambda} e^{gV} e^{-ig\Lambda^\dagger}, \tag{2.49}$$

where $i\Lambda(x, \theta, \bar{\theta})$ is a left-handed chiral superfield and g is the gauge coupling constant. The left-handed chiral superfield $i\Lambda(x, \theta, \bar{\theta})$ contains two complex scalar fields and one complex spinor field, which we are at liberty to choose. It turns out to be possible to choose these fields so as to eliminate $C(x)$, $\chi(x)$ and $M(x)$ in the general expression (2.44) - this is called the Wess-Zumino gauge[8]. There also remains one unspecified degree of freedom, since $C(x)$ is a real scalar field and both the scalar fields in $i\Lambda$ are complex. This last degree of freedom results in the usual gauge freedom of the vector field A_μ, which can be changed in such a way as to leave the field strength $F_{\mu\nu}$ invariant. Thus a general vector superfield in the Wess-Zumino gauge is given by

$$V_{WZ}(x, \theta, \bar{\theta}) = -\theta\sigma^\mu\bar{\theta}A_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \tfrac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x), \tag{2.50}$$

where the vector field A_μ still maintains its usual gauge freedom.

Now we can generally use any vector superfield or function of vector superfields to construct the D-terms in our action. It is often convenient, however, to make use of the Kähler potential $K(\Phi_i, \Phi_i^\dagger)$, which is required to be a vector superfield and is constructed from the left-handed chiral superfields Φ_i. The D-terms in the SUSY invariant action can then be written as [33, 34]

$$S_D = \int d^4x \int d^2\theta \; d^2\bar{\theta} \; K(\Phi_i, \Phi_i^\dagger). \tag{2.51}$$

These D-terms contain fermionic and bosonic kinetic terms as well as interaction terms. There are no kinetic terms corresponding to the auxillary fields F and D, which have purely algebraic equations of motion and can be eliminated from the action.

[8]Choosing $i\Lambda$ so as to obtain the Wess-Zumino gauge in the general non-abelian case, in which our fields do not commute, is a highly non-linear problem and, as such, shall not be further discussed. There is a detailed description in [32] of the solution to the abelian problem, in which the supersymmetric gauge transformation becomes $V \to V + i(\Lambda - \Lambda^\dagger)$.

2.2.4 Field strength term

The last possible SUSY invariant term in our action is the field strength term. This is constructed from the field strength superfield

$$W_\alpha \equiv \frac{1}{8}\left(\bar{D}\bar{D}\right) e^{2gV} D_\alpha e^{-2gV}, \tag{2.52}$$

where D and $\bar{D}$ are the covariant derivatives in superspace and $V = V_{WZ}$ is a vector superfield in the Wess-Zumino gauge (2.50). The field strength superfield W_α is clearly a spinor and, moreover, is also a left-handed chiral superfield ($\bar{D}_{\dot\beta} W_\alpha = 0$, since $\bar{D}_{\dot\alpha}$ and $\bar{D}_{\dot\beta}$ anticommute).

We shall now, as in [34], consider the action of the supersymmetric gauge transformation (2.49) on W_α. This field strength superfield transforms as

$$W_\alpha \longrightarrow e^{2gi\Lambda} W_\alpha e^{-2gi\Lambda}, \tag{2.53}$$

which can be shown as follows:

$$\begin{aligned} W_\alpha \longrightarrow & \frac{1}{8}\left(\bar{D}\bar{D}\right)\left(e^{2gi\Lambda} e^{2gV} e^{-2gi\Lambda^\dagger}\right) D_\alpha \left(e^{2gi\Lambda^\dagger} e^{-2gV} e^{-2gi\Lambda}\right) \\ &= \frac{1}{8}\left(\bar{D}\bar{D}\right) e^{2gi\Lambda} e^{2gV}\left(D_\alpha e^{-2gV}\right) e^{-2gi\Lambda} + \frac{1}{8}\left(\bar{D}\bar{D}\right) e^{2gi\Lambda} D_\alpha e^{-2gi\Lambda} \\ &= \frac{1}{8} e^{2gi\Lambda}\left[\left(\bar{D}\bar{D}\right) e^{2gV}\left(D_\alpha e^{-2gV}\right)\right] e^{-2gi\Lambda} + \frac{1}{8} e^{2gi\Lambda}\left(\bar{D}\bar{D}\right) D_\alpha e^{-2gi\Lambda}. \end{aligned} \tag{2.54}$$

Notice that $-i\Lambda^\dagger$ and D_α commute, since $-i\Lambda^\dagger$ is a right-handed chiral superfield, as do the left-handed chiral superfield $i\Lambda$ and $\bar{D}_{\dot\alpha}$. The last term in this expression can be manipulated as follows:

$$\begin{aligned} \left(\bar{D}\bar{D}\right) D_\alpha e^{-2gi\Lambda} &= \left(\varepsilon^{\dot\beta\dot\gamma} \bar{D}_{\dot\beta} \bar{D}_{\dot\gamma}\right) D_\alpha e^{-2gi\Lambda} \\ &= \varepsilon^{\dot\beta\dot\gamma} \bar{D}_{\dot\beta} \left\{\bar{D}_{\dot\gamma}, D_\alpha\right\} e^{-2gi\Lambda} \\ &= -2\varepsilon^{\dot\beta\dot\gamma} \left(\sigma^\mu\right)_{\alpha\dot\gamma} \bar{D}_{\dot\beta} P_\mu \ e^{-2gi\Lambda} \\ &= -2\varepsilon^{\dot\beta\dot\gamma} \left(\sigma^\mu\right)_{\alpha\dot\gamma} \left[\bar{D}_{\dot\beta}, P_\mu\right] e^{-2gi\Lambda} \\ &= 0. \end{aligned} \tag{2.55}$$

Here we have used the fact that $\bar{D}_{\dot\alpha} e^{-2gi\Lambda} = 0$, together with the identities $\left\{\bar{D}_{\dot\alpha}, D_\beta\right\} = -2\left(\sigma^\mu\right)_{\beta\dot\alpha} P_\mu$ and $\left[\bar{D}_{\dot\alpha}, P_\mu\right] = 0$, which can easily be obtained from the definitions (2.32) of the covariant derivatives D_α and $\bar{D}_{\dot\alpha}$. Thus, since the last term in (2.54) vanishes, we obtain the result (2.53).

This simple behaviour of the field strength superfield W_α under supersymmetric gauge transformations immediately implies that $\mathrm{Tr}\left\{W^\alpha W_\alpha\right\}$ is gauge invariant. Moreover, $W^\alpha W_\alpha$ is also a left-handed chiral superfield from which one can construct SUSY invariant F-terms. Hence we shall define the field strength term in our action as [34]

$$S_W = \frac{1}{2g^2}\int d^4x \int d^2\theta \;\mathrm{Tr}\left\{W^\alpha W_\alpha\right\}, \tag{2.56}$$

which is invariant under both supersymmetric gauge transformations and SUSY transformations.

We would now like to rewrite this field strength term using the fields A_μ, λ, $\bar{\lambda}$ and D, which are contained in the vector superfield $V = V_{WZ}$. We shall, following [34], perform this calculation in the coordinates y, θ and $\bar{\theta}$, and therefore let us first rewrite our vector superfield as follows:

$$V\left(y,\theta,\bar{\theta}\right) = -\left(\theta\sigma^\mu\bar{\theta}\right)A_\mu(y) + i\theta\theta\bar{\theta}\bar{\lambda}(y) - i\bar{\theta}\bar{\theta}\theta\lambda(y) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\left[D(y) + i\partial^\mu A_\mu(y)\right]. \tag{2.57}$$

Here we have substituted $x^\mu = y^\mu - i\theta\sigma^\mu\bar{\theta}$ into the expression (2.50) and expanded around $x^\mu = y^\mu$, making use of the identity $\left(\theta\sigma^\mu\bar{\theta}\right)\left(\theta\sigma^\nu\bar{\theta}\right) = \frac{1}{2}\eta^{\mu\nu}\theta\theta\bar{\theta}\bar{\theta}$. We can hence calculate

$$\begin{aligned} e^{2gV} &= 1 + 2gV + 2g^2V^2 \qquad (2.58)\\ &= 1 - 2g\left(\theta\sigma^\mu\bar{\theta}\right)A_\mu + 2ig\theta\theta\bar{\theta}\bar{\lambda} - 2ig\bar{\theta}\bar{\theta}\theta\lambda + g\theta\theta\bar{\theta}\bar{\theta}\left[D + i\partial^\mu A_\mu + gA^\mu A_\mu\right], \end{aligned}$$

and

$$e^{-2gV} = 1 + 2g\left(\theta\sigma^\mu\bar{\theta}\right)A_\mu - 2ig\theta\theta\bar{\theta}\bar{\lambda} + 2ig\bar{\theta}\bar{\theta}\theta\lambda - g\theta\theta\bar{\theta}\bar{\theta}\left[D + i\partial^\mu A_\mu - gA^\mu A_\mu\right]. \tag{2.59}$$

Notice that the Taylor series for the exponential has been truncated at second order because higher order terms must be either higher than second order in θ or in $\bar{\theta}$, and therefore vanish.

The above expressions, together with the covariant derivatives (2.34), imply that

$$\begin{aligned} e^{2gV}D_\alpha e^{-2gV} = {} & 2g\left(\sigma^\mu\right)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}A_\mu - 4ig\theta_\alpha\bar{\theta}\bar{\lambda} + 2ig\bar{\theta}\bar{\theta}\lambda_\alpha - 2g\theta_\alpha\bar{\theta}\bar{\theta}\left[D + i\partial^\mu A_\mu - gA^\mu A_\mu\right]\\ & - 2ig\bar{\theta}\bar{\theta}\theta^\gamma\varepsilon^{\dot{\beta}\dot{\delta}}\left(\sigma^\nu\right)_{\alpha\dot{\beta}}\left(\sigma^\mu\right)_{\gamma\dot{\delta}}\left[\partial_\nu A_\mu + igA_\mu A_\nu\right]\\ & - 2g\theta\theta\bar{\theta}\bar{\theta}\left(\sigma^\mu\right)_{\alpha\dot{\beta}}\left\{\partial_\mu\lambda^{\dot{\beta}} - ig[A_\mu, \bar{\lambda}^{\dot{\beta}}]\right\}, \end{aligned} \tag{2.60}$$

and, substituting this result into (2.52) and using the identity $\bar{D}\bar{D}\left(\bar{\theta}\bar{\theta}\right) = -4$, we obtain as explicit expression for the field strength as follows:

$$\begin{aligned} W_\alpha = & -ig\lambda_\alpha + g\theta_\alpha \left[D + i\partial^\mu A_\mu - gA^\mu A_\mu\right] \\ & + ig\theta^\gamma \varepsilon^{\dot\beta\dot\delta} \left(\sigma^\nu\right)_{\alpha\dot\beta} \left(\sigma^\mu\right)_{\gamma\dot\delta} \left[\partial_\nu A_\mu + igA_\mu A_\nu\right] \\ & + g\theta\theta \left(\sigma^\mu\right)_{\alpha\dot\beta} \left\{\partial_\mu \lambda^{\dot\beta} - ig[A_\mu, \bar\lambda^{\dot\beta}]\right\}. \end{aligned} \tag{2.61}$$

Raising the spinor index then yields

$$\begin{aligned} W^\alpha = & -ig\lambda^\alpha + g\theta^\alpha \left[D + i\partial^\mu A_\mu - gA^\mu A_\mu\right] \\ & + ig\theta^\gamma \varepsilon^{\alpha\epsilon} \left(\sigma^\nu\right)_{\epsilon\dot\beta} \varepsilon^{\dot\beta\dot\delta} \left(\sigma^\mu\right)_{\gamma\dot\delta} \left[\partial_\nu A_\mu + igA_\mu A_\nu\right] \\ & + g\theta\theta\varepsilon^{\alpha\epsilon} \left(\sigma^\mu\right)_{\epsilon\dot\beta} \varepsilon^{\dot\beta\dot\delta} \left\{\partial_\mu \lambda_{\dot\delta} - ig\left[A_\mu, \bar\lambda_{\dot\delta}\right]\right\}, \end{aligned} \tag{2.62}$$

which can be simplified using the identity $\varepsilon^{\alpha\epsilon} \left(\sigma^\mu\right)_{\epsilon\dot\beta} \varepsilon^{\dot\beta\dot\delta} = -\left(\bar\sigma^\mu\right)^{\dot\delta\alpha}$ and hence also

$$\begin{aligned} & \theta^\gamma \varepsilon^{\alpha\epsilon} \left(\sigma^\nu\right)_{\epsilon\dot\beta} \varepsilon^{\dot\beta\dot\delta} \left(\sigma^\mu\right)_{\gamma\dot\delta} \left[\partial_\nu A_\mu + igA_\mu A_\nu\right] \\ & = \theta^\gamma \left(\sigma^{\mu\nu}\right)_\gamma{}^\alpha \left\{\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) - ig[A_\mu, A_\nu]\right\} - \theta^\alpha \left[\partial^\mu A_\mu + igA^\mu A_\mu\right], \end{aligned} \tag{2.63}$$

where $\sigma^{\mu\nu}$ is defined in (A.27). This last result was derived by separately manipulating the parts of the original expression symmetric and anti-symmetric in μ and ν. Thus we obtain

$$W^\alpha = -ig\lambda^\alpha + g\theta^\alpha D + ig\theta^\beta \left(\sigma^{\mu\nu}\right)_\beta{}^\alpha F_{\mu\nu} - g\theta\theta D_\mu \bar\lambda_{\dot\beta} \left(\bar\sigma^\mu\right)^{\dot\beta\alpha}, \tag{2.64}$$

where

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right] \quad \text{and} \quad D_\mu \bar\lambda_{\dot\beta} \equiv \partial_\mu \bar\lambda_{\dot\beta} - ig\left[A_\mu, \bar\lambda_{\dot\beta}\right]. \tag{2.65}$$

This expression (2.64) shall now be used to calculate $W^\alpha W_\alpha = \varepsilon_{\alpha\beta} W^\alpha W^\beta$ and hence the field strength term in the action. Actually, we only need to determine the coefficient of the $\theta\theta$ term in $W^\alpha W_\alpha$, which is given by

$$\begin{aligned} W^\alpha W_\alpha \Big|_{\theta\theta} = & -\frac{1}{2} g^2 \varepsilon_{\alpha\beta} \left(\sigma^{\mu\nu}\right)_\gamma{}^\alpha \varepsilon^{\gamma\delta} \left(\sigma^{\rho\tau}\right)_\delta{}^\beta F_{\mu\nu} F_{\rho\tau} + g^2 D^2 \\ & - ig^2 \lambda_\alpha D_\mu \lambda_{\dot\beta} \left(\sigma^\mu\right)^{\dot\beta\alpha} + ig^2 D_\mu \bar\lambda_{\dot\beta} \left(\bar\sigma^\mu\right)^{\dot\beta\alpha} \lambda_\alpha. \end{aligned} \tag{2.66}$$

This result can be simplified using the identities $\varepsilon^{\alpha\beta} \left(\sigma^{\mu\nu}\right)_\beta{}^\gamma \varepsilon_{\gamma\delta} = -\left(\sigma^{\mu\nu}\right)_\delta{}^\alpha$ and $\left(\sigma^{\mu\nu}\right)_\alpha{}^\beta \left(\sigma^{\rho\tau}\right)_\beta{}^\alpha = \frac{1}{2}\left(-\eta^{\mu\rho}\eta^{\nu\tau} + \eta^{\mu\tau}\eta^{\nu\rho} + i\varepsilon^{\mu\nu\rho\tau}\right)$. Hence we obtain

$$\frac{1}{g^2} W^\alpha W_\alpha \Big|_{\theta\theta} = -\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \frac{i}{4} \varepsilon^{\mu\nu\rho\tau} F_{\mu\nu} F_{\rho\tau} + D^2 - i\lambda_\alpha D_\mu \lambda_{\dot\beta} \left(\sigma^\mu\right)^{\dot\beta\alpha} + iD_\mu \bar\lambda_{\dot\beta} \left(\bar\sigma^\mu\right)^{\dot\beta\alpha} \lambda_\alpha, \tag{2.67}$$

Substituting this expression into (2.56) yields the field strength term in our SUSY invariant action, which is given by [34]

$$S_W = \int d^4x \ \mathrm{Tr} \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} + \frac{1}{2} D^2 - \frac{i}{2} \lambda \sigma^\mu \left(D_\mu \bar{\lambda} \right) + \frac{i}{2} \left(D_\mu \bar{\lambda} \right) \bar{\sigma}^\mu \lambda \right\}, \tag{2.68}$$

where $\tilde{F}^{\mu\nu} \equiv \frac{1}{2} \, \varepsilon^{\mu\nu\rho\tau} F_{\rho\tau}$ is the dual of the field strength $F_{\mu\nu}$.

Let us now consider the term containing the dual field strength $\tilde{F}^{\mu\nu}$, which is proportional to the topological charge [36]

$$Q \equiv -\frac{1}{16\pi^2} \int d^4x \ \mathrm{Tr} \left\{ F_{\mu\nu} \tilde{F}^{\mu\nu} \right\} = \int d^4x \ \partial_\mu J^\mu, \tag{2.69}$$

where

$$J^\mu \equiv -\frac{1}{8\pi^2} \ \varepsilon^{\mu\nu\rho\sigma} \ \mathrm{Tr} \left\{ A_\nu \left(\partial_\rho A_\sigma \right) - \frac{2}{3} i g A_\nu A_\rho A_\sigma \right\}. \tag{2.70}$$

This topological quantity is similar to a winding number and plays an important role in the quantized theory. It does not, however, have any effect on the classical equations of motion and is therefore sometimes neglected. To include this term correctly, we must make a slight change to the original field strength action (2.56) in superspace as follows [34]:

$$S_W = \frac{1}{8\pi} \ \mathrm{Im} \left[\tau \int d^4x \int d^2\theta \ \mathrm{Tr} \left\{ W^\alpha W_\alpha \right\} \right], \tag{2.71}$$

in terms of the complex coupling constant $\tau = \frac{4\pi i}{g^2} + \frac{\theta_{YM}}{2\pi}$. This yields the result

$$\begin{aligned} S_W = & \int d^4x \ \mathrm{Tr} \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D^2 - \frac{i}{2} \lambda \sigma^\mu \left(D_\mu \bar{\lambda} \right) + \frac{i}{2} \left(D_\mu \bar{\lambda} \right) \bar{\sigma}^\mu \lambda \right\} \\ & - \frac{\theta_{YM}}{32\pi^2} \ g^2 \int d^4x \ \mathrm{Tr} \left\{ F_{\mu\nu} \tilde{F}^{\mu\nu} \right\}, \end{aligned} \tag{2.72}$$

where the coefficient of the Yang-Mills theta term θ_{YM} is a topological quantity.

Neglecting the topological part of the field strength action and using the definition (A.25) of the gamma matrices γ^μ in terms of the off-diagonal elements σ^μ and $\bar{\sigma}^\mu$, we find that

$$S_W = \frac{1}{4} \int d^4x \ \mathrm{Tr} \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D^2 - \frac{i}{2} \bar{\Psi} \gamma^\mu D_\mu \Psi \right\}, \tag{2.73}$$

where $\bar{\Psi} \equiv \Psi^\dagger \gamma^0$ and $D_\mu \Psi_\alpha \equiv \partial_\mu \Psi_\alpha - ig \left[A_\mu, \Psi_\alpha \right]$, with $\Psi \equiv \begin{pmatrix} \lambda_\alpha \\ \bar{\lambda}^{\dot{\alpha}} \end{pmatrix}$ a Majorana spinor. This field strength term contains kinetic terms associated with the gauge field A_μ and spinor field λ, as well as further interaction terms. We again notice that there are no kinetic terms associated with the auxillary field D.

2.2.5 Super Yang-Mills theories

We shall now argue that our original ten dimensional SYM action, corresponding to the Lagrangian (2.12), can be written in $\mathcal{N}=1$ superspace. Only a field strength term analogous to (2.71) is required and this yields a result similar to (2.73). There is a slight complication in that we are now working in ten spacetime dimensions and therefore our supercharges, and the coordinates θ and $\bar{\theta}$, are 16-component Weyl-Majorana spinors. Furthermore, the gamma matrices Γ^M must now be written in the block form of [29] with off-diagonal components Σ^M and $\bar{\Sigma}^M$. However, we can see that this field strength term yields the correct two terms in the ten dimensional SYM Lagrangian, since the auxillary field D is zero as a direct result of its algebraic equation of motion.

Now let us consider the four dimensional reduced SYM theory described by the Lagrangian (2.26). This must also be invariant under SUSY transformations and, moreover, we can understand its $\mathcal{N}=4$ supersymmetric nature by considering the ten dimensional SYM theory from which it was derived. (Writing the action in $\mathcal{N}=4$ superspace is not a viable option - even writing it in $\mathcal{N}=1$ superspace is somewhat tricky.) The supercharge corresponding to our ten dimensional $\mathcal{N}=1$ SYM theory is a 16-component Weyl-Majorana spinor consisting of four 4-component Majorana spinors, which are equivalent to four 2-component Weyl spinors. There is an inherent $SU(4)$ symmetry amongst these Majorana spinors. Therefore, when we reduce our ten dimensional SYM theory to four spacetime dimensions, we are left with four supercharges, which are invariant under $SU(4)$ R-symmetry transformations.

Finally, we shall mention the $\mathcal{N}=1$ superspace representation of the $\mathcal{N}=4$ SYM action. The F-terms in this action are constructed from the superpotential

$$f(\Phi_i)=\frac{1}{2}g\mathrm{Tr}\left(\Phi_1\Phi_2\Phi_3-\Phi_1\Phi_3\Phi_2\right), \tag{2.74}$$

where Φ_1, Φ_2 and Φ_3 are superfields in $\mathcal{N}=1$ superspace. This leads to the contribution

$$-\frac{1}{4}g^2\mathrm{Tr}\left\{|\Phi_1\Phi_2-\Phi_2\Phi_1|^2+|\Phi_2\Phi_3-\Phi_3\Phi_2|^2+|\Phi_3\Phi_1-\Phi_1\Phi_3|^2\right\}, \tag{2.75}$$

in the $\mathcal{N}=4$ SYM scalar potential (after we have eliminated the auxillary fields F_i using their algebraic equations of motion). Here the fields Φ_i now denote only the zeroth order scalar fields in the corresponding superfields. The second term in this scalar potential (2.28) is the result of the D-terms in the $\mathcal{N}=4$ SYM action in $\mathcal{N}=1$ superspace, which are constructed from a Kähler potential of the form

$$K(\Phi_i, \Phi_i^\dagger) = \mathrm{Tr}\left(\sum_{i=1}^{3} e^{2gV}\Phi_i^\dagger e^{-2gV}\Phi_i\right), \tag{2.76}$$

where $V = V_{WZ}$ is a vector superfield in the Wess-Zumino gauge[9]. The field strength term (2.71) also appears in this $\mathcal{N}=1$ superspace action.

2.3 Conformal Invariance and Marginal Deformations

A conformal field theory displays a symmetry known as conformal invariance. In other words, the Lagrangian is invariant under the action of the conformal group, which consists of all coordinate transformations $x \to x'$ that leave the metric invariant up to an arbitrary scale factor $\Omega(x)$ as follows [37]:

$$g_{\mu\nu}(x) \longrightarrow g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\, g_{\alpha\beta}(x) = \Omega(x) g_{\mu\nu}(x). \tag{2.77}$$

The Poincaré group is always a subgroup of the conformal group (with $\Omega(x) = 1$) - any reasonable metric is invariant under local Poincaré transformations. Furthermore, if we consider a non-gravitational theory in flat d-dimensional Minkowski spacetime with $d > 2$, then the conformal group consists of little more than the Poincaré group together with a set of scale transformations. Thus, to verify the conformal nature of any such non-gravitational field theory, we need to check for an exact scale invariance [5, 37].

[9]Notice that, not only the field strength term in the superspace action, but also the F-terms and D-terms, which are contructed from the superpotential (2.74) and Kähler potential (2.76), are invariant under the supersymmetric gauge transformation

$$e^{gV} \longrightarrow e^{ig\Lambda}e^{gV}e^{-ig\Lambda^\dagger},$$

if we assume that our superfields Φ_i in the adjoint representation of $SU(N)$ transform as follows:

$$\Phi_i \longrightarrow e^{2ig\Lambda}\Phi_i e^{-2ig\Lambda} \qquad \text{and} \qquad \Phi_i^\dagger \longrightarrow e^{2ig\Lambda^\dagger}\Phi_i^\dagger e^{-2ig\Lambda^\dagger}.$$

We shall now discuss the conformal nature of $\mathcal{N}=4$ SYM theory and review the construction of marginal deformations thereof. Towards this end, we start by describing Wilson's method of renormalizing a quantum field theory, based on discussions in [5, 26]. Hence the β-function associated with a specific coupling is defined. We mention, with reference to [26, 38, 39], the chiral and dilatation currents, and corresponding anomalies, which are associated with chiral and scale transformations respectively. It turns out that the conservation of the dilatation current, which is required for scale invariance, implies the vanishing of all the β-functions. Finally, following [5, 6, 40, 41, 42, 43], we construct $\mathcal{N}=1$ supersymmetric marginal deformations of $\mathcal{N}=4$ SYM theory, which are described by the Leigh-Strassler superpotential and include the so-called β-deformations [7]. The non-supersymmetric γ_i-deformations of [8] are also mentioned.

2.3.1 Renormalization and β-functions

The process of renormalization eliminates the divergences, with usually cause serious problems in quantum field theory. The idea behind renormalization is that the bare masses and couplings in the original Lagrangian are not the measured values. It is possible [38] to reformulate the theory in terms of the measured masses and couplings by introducing conveniently chosen counterterms into the Lagrangian.

There is also another approach to renormalization, which was invented by Wilson and shall now be described based on discussions in [5, 26]. This method requires us to formulate our quantum field theory in terms of functionals and path integrals, and, towards this end, we shall define the generating functional

$$Z[J] \equiv \int \mathcal{D}\phi \; e^{i\int d^4x \;\; [\mathcal{L}(\phi)+J\phi]}, \tag{2.78}$$

where $\int \mathcal{D}\phi$ denotes a path integral[10] over all possible real fields $\phi(x)$ satisfying the constraints $\phi(-T,\vec{x})=\phi_1(\vec{x})$ and $\phi(T,\vec{x})=\phi_2(\vec{x})$, with $T\to\infty$, which fix the initial

[10]The path integral measure can be expressed as [26]

$$\mathcal{D}\phi = \prod_i d\phi(x_i),$$

where we have discretized our spacetime into a large number of positions $\vec{x}_i$ separated by equal small time intervals ϵ. Our path integral then becomes the product of a large, but finite, number of ordinary integrals.

and final field configurations. Note that we have added a source term $J\phi$ to the Lagrangian. Hence correlations functions can be calculated as follows[11]:

$$\langle 0| T\left(\phi(x_1)\ ...\ \phi(x_N)\right)|0\rangle = \frac{1}{Z_0}\left(-i\frac{\delta}{\delta J(x_1)}\right)\ ...\ \left(-i\frac{\delta}{\delta J(x_N)}\right) Z[J]\,\bigg|_{J=0}, \quad (2.79)$$

with $Z_0 \equiv Z[0]$ the generating functional without a source term.

In order to avoid ultraviolet divergences, we shall now introduce a cutoff Λ on the momentum. The generating functional must first be written in terms of the Fourier components $\phi(k)$ of the fields and, furthermore, we shall perform the Wick rotation $k^0 \to ik^0$ so that we can write the cutoff condition in Euclidean space. Thus we obtain

$$Z[J] = \int\limits_{|k|<\Lambda} \mathcal{D}\phi\ e^{-\int d^4x\ [\mathcal{L}(\phi)+J\phi]}, \quad (2.80)$$

where we have imposed $\phi(k) = 0$ for all $|k| \geq \Lambda$. This cutoff condition sets to zero the contribution to our generating functional from the high momentum modes.

Now the question is: how was our generating functional effected by the high momentum modes which we have just cut off? To answer this question, let us define a slightly lower cutoff μ and rewrite (2.80) in terms of a new collection of low momentum ($|k| < \mu$) and high momentum ($\mu \leq |k| < \Lambda$) modes as follows:

$$Z[J] = \int \mathcal{D}\phi_- \int \mathcal{D}\phi_+\ e^{-\int d^4x\ [\mathcal{L}(\phi_-+\phi_+)+J(\phi_-+\phi_+)]}, \quad (2.81)$$

where the Fourier transforms of $\phi_-(x)$ and $\phi_+(x)$ are given by

$$\phi_-(k) = \begin{cases} \phi(k) & \text{if } |k| < \mu \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \phi_+(k) = \begin{cases} \phi(k) & \text{if } |k| \geq \mu \\ 0 & \text{otherwise} \end{cases}. \quad (2.82)$$

We now perform the integral $\int \mathcal{D}\phi_+$ over the high momentum modes to obtain

$$Z[J] = \int \mathcal{D}\phi_-\ e^{-\int d^4x\ [\mathcal{L}_{\text{eff}}(\phi_-)+J\phi_-]}, \quad (2.83)$$

where $\mathcal{L}_{\text{eff}}$ is the effective Lagrangian. In other words, by integrating out the high momentum modes, we have traded our original Lagrangian $\mathcal{L}(\phi)$ and cutoff Λ for a

[11] A functional derivative is defined as

$$\frac{\delta}{\delta J(x)} J(y) = \delta^{(4)}(x-y) \quad \text{or} \quad \frac{\delta}{\delta J(x)} \int d^4y\ J(y)\phi(y) = \phi(x),$$

and derivatives of composite functionals are calculated using the chain and product rules [26].

new effective Lagrangian $\mathcal{L}_{\text{eff}}(\phi)$ with a lower cutoff μ. It is therefore possible, by continuously decreasing μ, to arrive at a low energy effective Lagrangian with masses and couplings which might be totally different from those in the original theory[12].

Now we usually rewrite the field $\phi(x)$ in the effective Lagrangian so that the coefficient of the kinetic term $\partial^\mu\phi(x)\partial_\mu\phi(x)$ remains unchanged [5, 26]:

$$\phi(x) \longrightarrow \phi'(x) \equiv \sqrt{Z(\mu)}\ \phi(x), \tag{2.84}$$

where $Z(\mu)$ is known as the *wave function renormalization*. This is equivalent to insisting that the field $\phi(x)$ should always create a particle with probability one. We shall thus define the *anomalous dimension* of the field $\phi(x)$, as in [6], to be

$$\gamma \equiv -\frac{\partial \ln Z(\mu)}{\partial \ln \mu}. \tag{2.85}$$

It can be seen that γ is related to the dependence of $Z(\mu)$ on the length scale $\frac{1}{\mu}$ and hence the term 'dimension'. For example, if $Z(\mu) \sim (\frac{1}{\mu})^n = \mu^{-n}$, then $\gamma = n$.

The masses and couplings are generally also dependent on the energy scale μ, and are effected by our redefinition (2.84), so that $m(\mu) \to m'(\mu)$ and $g(\mu) \to g'(\mu)$. Thus, following [5, 6, 41], we shall define the β-function (or Gell-Mann-Low function) as

$$\beta(g) \equiv \frac{\partial g'(\mu)}{\partial \ln \mu} = \mu \frac{\partial g'(\mu)}{\partial \mu}, \tag{2.86}$$

which tells us how the redefined coupling changes as a function of the energy scale. A conformal field theory has an exact scale invariance and therefore cannot contain couplings which are dependent on an energy scale (energy $\sim$ 1/length). Hence it is intuitively clear that all the β-functions must vanish. Often theories are only scale invariant for certain specific values of the coupling g, corresponding to specific energy scales μ, at which $\beta(g) = 0$. These are known as 'fixed points'.

2.3.2 Conserved currents and anomalies

An important aspect of any quantum field theory are the symmetries inherent in the system - we are especially interested in the symmetry of scale invariance. There exists a Noether current $j^\mu(x)$ corresponding to any such symmetry and, classically,

[12]This continuous collection of effective Lagrangians is called the 'renormalization group' [26].

this current satisfies the conservation equation $\partial_\mu j^\mu(x) = 0$. When a field theory is quantized we often find that this conservation equation is spoilt by an anomalous term which appears on the right-hand side. This anomaly is usually an exact one-loop expression.

Hereafter, following [26, 38, 39], we shall briefly discuss chiral and scale transformations, together with the corresponding chiral and dilatation currents, and their associated anomalies. The dilatation current is of obvious importance, because it must be conserved for a theory to be scale invariant and hence conformal. Furthermore, it turns out [40] that, in certain supersymmetric theories (such as $\mathcal{N} = 4$ SYM theory), the chiral and dilatation currents are related by a SUSY transformation.

Let us consider some general $SU(N)$ gauge invariant field theory with N_f flavours of massless fermions. A chiral transformation is then given by [26]

$$\psi_k(x) \longrightarrow e^{i\alpha\gamma^5}\psi_k(x), \tag{2.87}$$

in terms of the real parameter α and the chirality matrix γ^5 defined in (A.11). Here $\psi_k(x)$, where k runs from 1 to N_f, are Dirac spinor fields in the fundamental representation of $SU(N)$. Note that we have taken our fermion fields to be massless because any mass term in the Lagrangian automatically breaks chiral invariance. Now, if we assume our field theory to be invariant under this chiral transformation, then there exists a conserved chiral current $j^{\mu 5}(x)$ satisfying $\partial_\mu j^{\mu 5}(x) = 0$. This conservation equation is broken at the quantum level by the chiral or Adler-Bell-Jackiw anomaly.

A scale transformation acts by scaling any length by a factor of $e^{-\alpha}$, where α is a real parameter. Therefore this scale transformation acts on some field $\phi(x)$, with mass dimension D, as follows:

$$\phi(x) \longrightarrow e^{-D\alpha}\phi(xe^{-\alpha}). \tag{2.88}$$

Note that an identical transformation applies to spinor and vector fields. Let us, again, consider a general field theory containing only massless fields and dimensionless couplings g_i. This theory will be classically scale invariant, with the corresponding conserved dilatation current [26, 38]

$$D^\mu = \theta^{\mu\nu}x_\nu, \qquad \text{so that} \quad \partial_\mu D^\mu = \theta^\mu{}_\mu = 0, \tag{2.89}$$

where $\theta^{\mu\nu}$ is the symmetric and gauge invariant energy-momentum tensor[13]. At the quantum level, a trace anomaly appears on the right-hand side of this conservation

[13]The usual energy-momentum tensor $T^{\mu\nu}$ is not necessarily symmetric or gauge invariant. It

equation to yield $\partial_\mu D^\mu \propto \sum_i \beta_i(g_j)$. Thus, as expected, we only obtain a scale invariant quantum field theory when all the β-functions vanish.

2.3.3 $\mathcal{N} = 4$ SYM theory and marginal deformations

$\mathcal{N} = 4$ SYM theory is a finite quantum field theory - there is no dependence on an energy scale at all and the theory is always conformal. This is a direct result of its maximally supersymmetric nature [6, 43]. Marginal deformations of $\mathcal{N} = 4$ SYM theory can be constructed by adding what [6] have referred to as an 'exactly marginal' operator to the $\mathcal{N} = 4$ SYM Lagrangian and this results in a theory, which is non-finite, but contains a manifold of fixed points (fixed lines, planes, etc). We shall now discuss the conformal nature of $\mathcal{N} = 4$ SYM theory and marginal deformations thereof based on [5, 6, 40, 41, 42, 43].

Let us begin by considering a slightly more general SYM theory out of which both $\mathcal{N} = 4$ SYM theory and marginal deformations can be constructed. We make use of the following generalized $\mathcal{N} = 1$ superpotential [6]:

$$f(\Phi_i) = \frac{1}{2}\mathrm{Tr}\left\{\sum_s h_s f_s(\Phi_i)\right\}, \qquad \text{with} \;\; f_s(\Phi_i) = \Phi_{i_1}\Phi_{i_2}...\Phi_{i_{N_s}}. \tag{2.90}$$

The Kähler potential and field strength term remain the same, except that we can redefine the Wess-Zumino vector superfield $gV \to V$ for convenience, so that only the field strength term contains the gauge coupling g. The $\mathcal{N} = 4$ SYM superpotential (2.74) is recovered when we consider two terms, $\Phi_1\Phi_2\Phi_3$ and $\Phi_1\Phi_3\Phi_2$ respectively, and set $h_1 = g$ and $h_2 = -g$.

We shall now construct the β-functions corresponding to the couplings g and h_s in this generalized SYM theory. It was shown in [40] that the there exists a supermultiplet which contains the spinor current, the chiral vector current and the dilatation current. (In other words, these currents are connected by a SUSY transformation.) We can also write them in the form of a single supercurrent $J_{\alpha\dot{\alpha}}$, which is not classically

is always possible, however, to construct a new energy-momentum tensor with these properties as follows [26]:

$$\theta^{\mu\nu} = T^{\mu\nu} + \partial_\rho \Sigma^{\mu\nu\rho},$$

where $\Sigma^{\mu\nu\rho}$ is anti-symmetric in μ and ρ. This new energy-momentum tensor satisfies the same conservation equation $\partial_\mu\theta^{\mu\nu} = 0$ and produces the same momenta $P^\nu = \int d^3x\ \theta^{0\nu} = \int d^3x\ T^{0\nu}$.

conserved, due to the presence of the superpotential, but satisfies the relation [6]

$$\bar{D}^{\dot{\alpha}} J_{\alpha\dot{\alpha}}\big|_{\text{classical}} = \frac{1}{3} D_\alpha \left(3f - \sum_{i=1}^{N_d} \Phi_i \frac{\partial f}{\partial \Phi_i} \right), \tag{2.91}$$

where N_d is the number of distinct superfields in the superpotential $f(\Phi_i)$. Notice that this expression vanishes for the $\mathcal{N} = 4$ SYM superpotential (2.74), so that $\mathcal{N} = 4$ SYM theory is classically scale invariant.

We now need to determine the anomalies that appear in this equation when we quantize the theory. It was shown in [6] that the full quantum expression is

$$\begin{aligned} \bar{D}^{\dot{\alpha}} J_{\alpha\dot{\alpha}} = -\frac{1}{3} D_\alpha \Bigg[\frac{N}{32\pi^2}\, W_\beta W^\beta \left(3 - N_d + \sum_{i=1}^{N_d} \gamma_i \right) \\ + \sum_s h_s \left((N_s - 3) f_s + \frac{1}{2} \sum_{i=1}^{N_d} \gamma_i \Phi_i \frac{\partial f_s}{\partial \Phi_i} \right) \Bigg], \end{aligned} \tag{2.92}$$

where γ_i is the anomalous dimension of the superfield Φ_i, which we have assumed to be in the adjoint representation of $SU(N)$. The coefficients of each of these terms must be proportional to the corresponding β-function, so we obtain [6, 43]

$$\beta_g \propto 3 - N_d + \sum_{i=1}^{N_d} \gamma_i \qquad \text{and} \qquad \beta_{h_s} \propto N_s - 3 + \frac{1}{2} \sum_{i=1}^{N_d} \gamma_i \, \frac{\partial \, \ln f_s(\Phi_i)}{\partial \, \ln \Phi_i}. \tag{2.93}$$

The expression $\frac{\partial \, \ln f_s(\Phi_i)}{\partial \, \ln \Phi_i}$ counts the number of times Φ_i appears in the s^{th} term in the superpotential. It is also possible [5, 44] to construct β_{h_s} based only on arguments relating to the holomorphy of the superpotential.

We would now like to find manifolds of fixed points (fixed lines, planes, etc) for the generalized SYM theory. We shall therefore look for a situation in which these β-functions are linearly dependent, so that the number of conditions p is less than the number of couplings n. In this case, if the conditions for zero β-functions are satisfied, the result is an $n - p$ dimensional manifold of fixed points [6]. With this in mind, we shall consider a theory with three distinct superfields in the adjoint representation of $SU(N)$. Moreover, we shall specify a superpotential in which each term is the product of three superfields, so that $N_s = N_d = 3$. This Leigh-Strassler superpotential is given by

$$f(\Phi_i) = \frac{1}{2} \mathrm{Tr} \left\{ h_1 \Phi_1 \Phi_2 \Phi_3 + h_2 \Phi_1 \Phi_3 \Phi_2 + h_3 \left(\Phi_1^3 + \Phi_2^3 + \Phi_3^3 \right) \right\}, \tag{2.94}$$

which contains an inherent Z_3 symmetry - it is invariant under the transformation $\Phi_1 \to \Phi_2$, $\Phi_2 \to \Phi_3$ and $\Phi_3 \to \Phi_1$. This last property means that the anomalous dimensions of the superfields must be the same [5, 6]. Hence we obtain

$$\beta_{h_s} \propto \beta_g \propto \frac{3}{2}\gamma, \tag{2.95}$$

so that the β-functions vanish if $\gamma(g, h_s) = 0$. This condition describes a three dimensional manifold of fixed points in our four dimensional space of couplings.

Now, if we further specify that $h_2 = -h_1$ and $h_3 = 0$, we find a fixed line corresponding to $\gamma(g, h_1) = 0$. It turns out that this fixed line in our coupling space is really at $h_1 = g$, which describes $\mathcal{N} = 4$ SYM theory [43]. It is thus clear that at any energy scale $\mathcal{N} = 4$ SYM theory is a conformal field theory.

Furthermore, setting $h_1 = ge^{i\pi\beta}$, $h_2 = -ge^{-i\pi\beta}$ and $h_3 = 0$, with β some complex parameter, we obtain the β-deformed superpotential of Lunin and Maldacena [7]

$$f(\Phi_i) = \frac{1}{2} g \mathrm{Tr}\left(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2\right), \tag{2.96}$$

which results in the β-deformed scalar potential

$$\begin{aligned} V^\beta = -\frac{1}{4}g^2 \Big\{ & \mathrm{Tr}\left[\left|\Phi_1\Phi_2 - e^{-2i\pi\beta}\Phi_2\Phi_1\right|^2 + \left|\Phi_2\Phi_3 - e^{-2i\pi\beta}\Phi_3\Phi_2\right|^2 + \left|\Phi_3\Phi_1 - e^{-2i\pi\beta}\Phi_1\Phi_3\right|^2\right] \\ & - \frac{1}{4}\mathrm{Tr}\left[\left([\Phi_1, \Phi_1^*] + [\Phi_2, \Phi_2^*] + [\Phi_3, \Phi_3^*]\right)^2\right]\Big\}. \end{aligned} \tag{2.97}$$

Only the F-terms in this scalar potential have been β-deformed, which is clearly what we should expect, as these are the terms arising from the superpotential.

Lastly, we should mention that there exists also another deformation of $\mathcal{N} = 4$ SYM theory, which was invented by Frolov [8] and upon which we shall concentrate in this thesis. This γ_i-deformed theory is non-supersymmetric and thus cannot be described by an $\mathcal{N} = 1$ superpotential, but contains the γ_i-deformed scalar potential

$$\begin{aligned} V^{\gamma_i} = -\frac{1}{4}g^2 \Big\{ & \mathrm{Tr}\left[\left|\Phi_1\Phi_2 - e^{-2i\pi\gamma_3}\Phi_2\Phi_1\right|^2 + \left|\Phi_2\Phi_3 - e^{-2i\pi\gamma_1}\Phi_3\Phi_2\right|^2 + \left|\Phi_3\Phi_1 - e^{-2i\pi\gamma_2}\Phi_1\Phi_3\right|^2\right] \\ & - \frac{1}{4}\mathrm{Tr}\left[\left([\Phi_1, \Phi_1^*] + [\Phi_2, \Phi_2^*] + [\Phi_3, \Phi_3^*]\right)^2\right]\Big\}. \end{aligned} \tag{2.98}$$

Here γ_i are three different real parameters - the case of equal γ_i is equivalent to the case of real $\beta = \gamma$ in the previous example. This γ_i-deformed non-supersymmetric YM theory is conformally invariant in the large N limit [8].

Chapter 3

Matrix of Anomalous Dimensions and Spin Chains

3.1 SYM Matrix of Anomalous Dimensions

AdS/CFT correspondence matches the energy spectrum of string states with the spectrum of dimensions of the corresponding gauge theory operators. In other words, the excitation energies must correspond to the eigenvalues of the matrix of anomalous dimensions. This conjecture was initially tested [11] for chiral primary (half-BPS) operators of the form $\mathrm{Tr}\left(\Phi_i^J\right)$, which have conformal dimension $\Delta = J$ protected by supersymmetry and are dual to point-like strings. The string energies were calculated in the large λ limit and matched to the (trivial) dimensions on the gauge theory side.

Due to the strong/weak coupling nature of the gauge/string duality, extending this test non-protected operators and their string duals posed a serious problem. Recently, a partial solution was proposed [12] by Berenstein, Maldacena and Nastase (BMN) for operators with large quantum numbers (such as R-charge and spin). They considered the specific case of 'nearly BPS' operators, which are obtained from 'long' chiral primary operators by adding a small number of 'impurities' (other real scalar fields) into the trace as follows:

$$\text{no impurites} \qquad \mathrm{Tr}\left(\Phi_i^J\right), \tag{3.1}$$

$$\text{one impurity} \qquad \mathrm{Tr}\left(\phi_j \Phi_i^J\right), \tag{3.2}$$

$$\text{two impurities} \qquad \sum_{l=1}^{J} e^{2\pi i l n/J} \mathrm{Tr}\left(\phi_j \Phi_i^l \phi_k \Phi_i^{J-l}\right), \tag{3.3}$$

and so on. These BMN operators have large R-charge[1] J. The deviations $\Delta - J$ of the conformal dimensions of these BMN operators from the original conformal (and bare) dimension J of our 'long' chiral primary operator were found to be finite in the BMN limit

$$J \to \infty \qquad \text{with} \quad \tilde{\lambda} = \frac{\lambda}{J^2} = \text{fixed} \ll 1, \tag{3.4}$$

and could be expanded as a function of $\tilde{\lambda}$. (Note that only planar diagrams were included in this calculation.) Thus it is possible to perform calculations in the gauge theory, even at large λ, by considering sufficiently 'long' operators. The dimensions of these BMN operators were matched to the dimensions of nearly point-like strings in a pp-wave background.

It is also possible [13] to extend this idea to operators with a large spin quantum number S. These are single trace operators of the form $\mathrm{Tr}\left(\Phi_i \nabla_{(\mu_1} \ldots \nabla_{\mu_S)} \Phi_i\right)$, which contain a large number S of derivatives and have bare dimension $S+2$. The dual string configurations move with spin S in the AdS_5 space. As before, there exists a similar large S BMN limit in which the anomalous dimensions are finite and string/gauge theory comparisons can be performed.

Now, in this chapter, we are interested in 'long' single trace operators in the 'scalar sector', which are constructed from our six real scalar fields (with no derivatives) and take the form $\mathrm{Tr}\left(\phi_{i_1}\phi_{i_2}\ldots\phi_{i_J}\right)$, where J is assumed to be large. These operators are dual to extended closed strings rotating with total angular momentum J in the S^5 space. It was shown in [16] that the one-loop planar[2] matrix of anomalous dimensions in the scalar sector of $\mathcal{N}=4$ SYM theory can be expressed as the Hamiltonian of a closed integrable $SO(6)$ spin chain. The Bethe ansatz technique can then be used to diagonalize this anomalous dimension matrix.

[1] This R-charge is actually the charge with respect to only an $SO(2)$ subgroup of the R-symmetry group. We consider only transformations which rotate the real scalar field components of the complex scalar field Φ_i out of which our original chiral primary operator was constructed.

[2] The effects of non-planar diagrams were not considered. It should be noted, however, that [45, 46] showed that non-planar diagrams are not necessarily negligible in the BMN limit. It turns out that a general non-planar diagram has both an effective coupling constant $\tilde{\lambda} = \frac{\lambda}{J^2}$ and a genus-counting parameter $g_2^2 = \left(\frac{J^2}{N}\right)^2$. Planar diagrams have genus zero so that the dependence on g_2^2 disappears, but non-planar diagrams are suppressed by factors of this genus-counting parameter.

In this section, we briefly review the identification of [16] of the matrix of anomalous dimensions in the scalar sector with the Hamiltonian of an $SO(6)$ spin chain. We then restrict ourselves to the $SU(3)$ sector, which corresponds to operators of the form $\mathrm{Tr}\left(\Phi_{i_1}\Phi_{i_2}...\Phi_{i_J}\right)$, constructed from our three complex scalar fields. This anomalous dimension matrix corresponds to the Hamiltonian of an $SU(3)$ spin chain, a formal description of which is given in appendix B.

3.1.1 Matrix of anomalous dimensions

We shall now define the matrix of anomalous dimensions based on discussions in [16]. Let us consider some collection of operators in a basis O^A, which mix amongst themselves under renormalization. The renormalized basis operators O^A_{ren} are a linear combination of the bare basis operators O^A as follows:

$$O^A_{\mathrm{ren}} = Z^A{}_B O^B, \tag{3.5}$$

where $Z^A{}_B$ is a matrix of renormalization factors dependent on the energy scale defined by our varying ultraviolet cutoff μ.

The *matrix of anomalous dimensions* is now defined as

$$\Gamma^A{}_C = \frac{\partial Z^A{}_B}{\partial \ \mathrm{ln}\mu} \left(Z^{-1}\right)^B{}_C \, . \tag{3.6}$$

in the neighbourhood of the fixed point. The eigenvalues of this matrix of anomalous dimensions correspond to the anomalous dimensions γ_n of the operator eigenstates O_n, which are multiplicatively renormalizable.

3.1.2 Scalar sector operators as $SO(6)$ spin chains

The scalar sector of $\mathcal{N} = 4$ SYM theory is composed of single trace operators constructed from our six real scalar fields as follows:

$$O\left[\psi\right] = \psi^{i_1 i_2 \dots i_J} \mathrm{Tr}\left(\phi_{i_1}\phi_{i_2}\dots\phi_{i_J}\right), \tag{3.7}$$

where $\psi^{i_1 i_2 \dots i_J}$ are real coefficients. These operators have bare dimension J and, considering only planar diagrams, mix amongst themselves under renormalization. The obvious basis of bare operators for the scalar sector is thus

$$O_{i_1 i_2 \dots i_J} = \mathrm{Tr}\left(\phi_{i_1}\phi_{i_2}\dots\phi_{i_J}\right), \tag{3.8}$$

which, under renormalization, becomes

$$(O_{\text{ren}})_{j_1 j_2 \dots j_J} = Z^{i_1 i_2 \dots i_J}_{j_1 j_2 \dots j_J} \; O_{i_1 i_2 \dots i_J}. \tag{3.9}$$

Here $Z^{i_1 i_2 \dots i_J}_{j_1 j_2 \dots j_J}$ is the matrix of renormalization factors. The renormalized scalar sector operator $O_{\text{ren}}[\psi]$ can now be constructed from these renormalized basis operators as follows:

$$O_{\text{ren}}[\psi] = \psi^{j_1 j_2 \dots j_J} \; Z^{i_1 i_2 \dots i_J}_{j_1 j_2 \dots j_J} \; O_{i_1 i_2 \dots i_J} = (\psi_{\text{ren}})^{i_1 i_2 \dots i_J} \; O_{i_1 i_2 \dots i_J}, \tag{3.10}$$

with $(\psi_{\text{ren}})^{i_1 i_2 \dots i_J} \equiv Z^{i_1 i_2 \dots i_J}_{j_1 j_2 \dots j_J} \; \psi^{j_1 j_2 \dots j_J}$. Hence we see that it is possible to view the renormalization of the scalar sector operator $O[\psi]$ as the renormalization of the real wavefunction $\psi^{i_1 i_2 \dots i_J}$ (rather than of the basis operators).

We can already see the analogy to an $SO(6)$ spin chain starting to appear. Our matrix of renormalization factors (and thus also our matrix of anomalous dimensions) acts on the real wavefunction $\psi^{i_1 i_2 \dots i_J}$, which is a state in the tensor product of J six dimensional real $\mathbb{R}^6$ vector spaces. Furthermore, cyclic permutations of the indices $i_1, i_2, \dots, i_J$ should result in an equivalent state, due to the cyclicity of the trace in our basis operators. Thus $\psi^{i_1 i_2 \dots i_J}$ can be identified with a closed $SO(6)$ spin chain.

Let us now briefly review the construction of the one-loop planar matrix of renormalization factors $Z^{i_1 i_2 \dots i_J}_{j_1 j_2 \dots j_J}$ and the corresponding matrix of anomalous dimensions $\Gamma^{i_1 i_2 \dots i_J}_{j_1 j_2 \dots j_J}$ based on discussions in [16]:

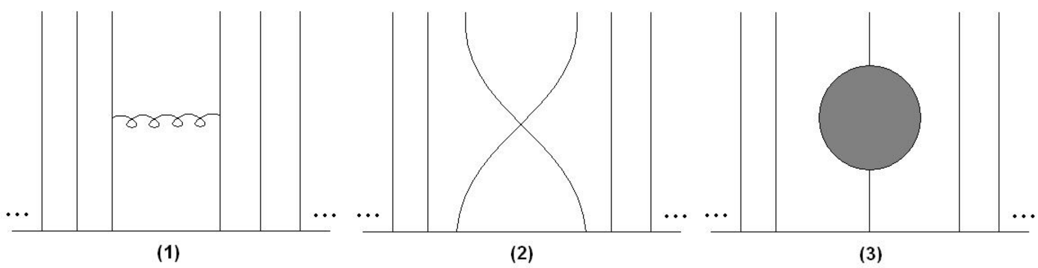

Figure 3.1: One-loop planar diagrams [16].

The bosonic part of the $\mathcal{N} = 4$ SYM Lagrangian (2.26) leads to three one-loop planar diagrams (see figure 3.1), which contribute to the matrix of anomalous dimensions. Here the notation of [16, 47] has been used: the horizontal line represents the renormalized operator $(O_{\text{ren}})_{i_1 i_2 \dots i_J}$ and the vertical lines link the real scalar fields ϕ_i to lattice sites along this operator (at the same spacetime point). We can easily see that the one-loop planar calculation involves only the mixing of fields at neighbouring lattice sites (sometimes referred to as 'nearest-neighbour interactions'). Diagrams (1) and (2) represent the mixing of operators due to gauge boson and scalar interactions respectively, whereas diagram (3) is the result of the self-energy correction to the scalar fields at each lattice site.

Thus, using these three diagrams, the matrix of renormalization factors corresponding to the mixing of two fields ϕ_{i_k} and $\phi_{i_{k+1}}$ at neighbouring lattice sites k and $k+1$ was calculated in [16] to be

$$Z_{k,k+1} = 1 + \frac{\lambda}{16\pi^2} \ln \mu \left(K_{k,k+1} + 2 - 2\mathcal{P}_{k,k+1} \right), \tag{3.11}$$

where the trace and permutation matrices are defined as

$$(K_{k,k+1})^{i_k i_{k+1}}_{j_k j_{k+1}} = \delta^{i_k i_{k+1}} \delta_{j_k j_{k+1}} \qquad \text{and} \qquad (\mathcal{P}_{k,k+1})^{i_k i_{k+1}}_{j_k j_{k+1}} = \delta^{i_k}_{j_{k+1}} \delta^{i_{k+1}}_{j_k}. \tag{3.12}$$

where the indices i_k, i_{k+1}, j_k and j_{k+1} run from 1 to 6. Here the action on the other lattice sites has been suppressed, since it is trivial.

The total renormalization matrix can now be expressed as a sum over all possible neighbouring lattice sites:

$$Z = \sum_{k=1}^{J} \left[1 + \frac{\lambda}{16\pi^2} \ln \mu \left(K_{k,k+1} + 2 - 2\mathcal{P}_{k,k+1} \right) \right], \tag{3.13}$$

with $J + 1 \equiv 1$. (The basis operators involve a cyclic trace over the real scalar fields and thus the first and last lattice sites are neighbours.)

Hence the one-loop planar matrix of anomalous dimensions in the scalar sector is

$$\Gamma = \frac{\lambda}{16\pi^2} \sum_{k=1}^{J} \left(K_{k,k+1} + 2 - 2\mathcal{P}_{k,k+1} \right), \tag{3.14}$$

which is the Hamiltonian of an integrable $SO(6)$ spin chain [16] acting on the closed $SO(6)$ spin chain state $\psi^{i_1 i_2 \ldots i_J}$. Notice that Γ does not depend on the energy scale, because it is calculated in the neighbourhood of the fixed point.

3.1.3 $SU(3)$ sector operators as $SU(3)$ spin chains

Let us now consider single trace operators of the form

$$O\left[\Psi\right] = \Psi^{i_1 i_2 \ldots i_J} \mathrm{Tr} \left(\Phi_{i_1} \Phi_{i_2} \ldots \Phi_{i_J} \right), \tag{3.15}$$

which are constructed from our complex scalar fields $\Phi_l = \phi_l + i\phi_{l+3}$, where l runs from 1 to 3, and span the $SU(3)$ sector of $\mathcal{N} = 4$ SYM theory. Here $\Psi^{i_1 i_2 \ldots i_J}$ is a complex wave function, which lives in a tensor product of J three dimensional complex $\mathbb{C}^3$ vector spaces. Our basis operators for this $SU(3)$ sector are thus

$$O_{i_1 i_2 \ldots i_J} = \mathrm{Tr}\left(\Phi_{i_1}\Phi_{i_2}\ldots\Phi_{i_J}\right) = \mathrm{Tr}\left[\left(\phi_{i_1} + i\phi_{i_1+3}\right)\left(\phi_{i_2} + i\phi_{i_2+3}\right)\ldots\left(\phi_{i_J} + i\phi_{i_J+3}\right)\right], \tag{3.16}$$

which can clearly be written in terms of the original basis scalar sector operators.

We would now like to understand the action of the trace and permutation matrices (and hence the matrix of anomalous dimensions) on these new basis operators. Firstly, we shall use the definitions (3.12) to write

$$\left(K_{k,k+1}\right)^{i_k i_{k+1}}_{j_k j_{k+1}} \mathrm{Tr}\left(\phi_{i_1}\ldots\phi_{i_k}\phi_{i_{k+1}}\ldots\phi_{i_J}\right) = \delta_{j_k j_{k+1}} \sum_{l=1}^{6} \mathrm{Tr}\left(\phi_{j_1}\ldots\phi_l\phi_l\ldots\phi_{j_J}\right), \tag{3.17}$$

$$\left(\mathcal{P}_{k,k+1}\right)^{i_k i_{k+1}}_{j_k j_{k+1}} \mathrm{Tr}\left(\phi_{i_1}\ldots\phi_{i_k}\phi_{i_{k+1}}\ldots\phi_{i_J}\right) = \mathrm{Tr}\left(\phi_{j_1}\ldots\phi_{j_{k+1}}\phi_{j_k}\ldots\phi_{j_J}\right). \tag{3.18}$$

Let us now calculate the action of the trace and permutation matrices on the new basis operators by expanding them in terms of the old basis operators as follows:

$$\begin{aligned}
&\left(K_{k,k+1}\right)^{i_k i_{k+1}}_{j_k j_{k+1}} \mathrm{Tr}\left(\Phi_{i_1}\ldots\Phi_{i_k}\Phi_{i_{k+1}}\ldots\Phi_{i_J}\right)\\
&= \left(K_{k,k+1}\right)^{i_k i_{k+1}}_{j_k j_{k+1}} \mathrm{Tr}\left[\Phi_{i_1}\ldots\left(\phi_{i_k} + i\phi_{i_k+3}\right)\left(\phi_{i_{k+1}} + i\phi_{i_{k+1}+3}\right)\ldots\Phi_{i_J}\right]\\
&= \left(K_{k,k+1}\right)^{i_k i_{k+1}}_{j_k j_{k+1}} \mathrm{Tr}\left[\Phi_{i_1}\ldots\left(\phi_{i_k}\phi_{i_{k+1}} + i\phi_{i_k}\phi_{i_{k+1}+3} + i\phi_{i_k+3}\phi_{i_{k+1}} - \phi_{i_k+3}\phi_{i_{k+1}+3}\right)\ldots\Phi_{i_J}\right]\\
&= \left(\delta_{j_k j_{k+1}} + 0 + 0 - \delta_{j_k j_{k+1}}\right) \sum_{l=1}^{6} \mathrm{Tr}\left(\Phi_{j_1}\ldots\phi_l\phi_l\ldots\Phi_{j_J}\right)\\
&= 0,
\end{aligned} \tag{3.19}$$

and, similarly,

$$\left(\mathcal{P}_{k,k+1}\right)^{i_k i_{k+1}}_{j_k j_{k+1}} \mathrm{Tr}\left(\Phi_{i_1}\ldots\Phi_{i_k}\Phi_{i_{k+1}}\ldots\Phi_{i_J}\right) = \mathrm{Tr}\left(\Phi_{j_1}\ldots\Phi_{j_{k+1}}\Phi_{j_k}\ldots\Phi_{j_J}\right). \tag{3.20}$$

Thus we see that the trace matrix $K_{k,k+1}$ annihilates any operator in the $SU(3)$ sector of $\mathcal{N} = 4$ SYM theory, whereas the permutation matrix $\mathcal{P}_{k,k+1}$ simply permutes the k^{th} and $(k+1)^{\mathrm{th}}$ complex scalar fields in our single trace operator.

Hence we can deduce from (3.14) that the one-loop planar matrix of anomalous dimensions for the $SU(3)$ sector of $\mathcal{N} = 4$ SYM theory is

$$\Gamma = \frac{\lambda}{8\pi^2} \sum_{k=1}^{J} (1 - \mathcal{P}_{k,k+1}) \,, \tag{3.21}$$

where we now define the permutation matrix as

$$(\mathcal{P}_{k,k+1})^{i_k i_{k+1}}_{j_k j_{k+1}} = \delta^{i_k}_{j_{k+1}} \delta^{i_{k+1}}_{j_k} , \qquad \text{with} \quad i_k, i_{k+1}, i_k, j_{k+1} \,\epsilon\, \{1,2,3\} \,. \tag{3.22}$$

This is the Hamiltonian of a closed $SU(3)$ spin chain, which is a $3^{\mathrm{J}} \times 3^{\mathrm{J}}$ matrix acting on the $SU(3)$ spin chain state $\Psi^{i_1 i_2 \ldots i_J}$ in the tensor product of J three dimensional complex vector spaces. A detailed review of the formal description of an $SU(3)$ spin chain is given in appendix B.

3.2 γ_i-deformed YM Matrix of Anomalous Dimensions

We would now like to extend the results of the previous section to deformations of $\mathcal{N} = 4$ SYM theory. An important question (studied in detail in [48]) is: for which deformations does the one-loop planar matrix of anomalous dimensions result in the Hamiltonian of an integrable spin chain? Our γ_i-deformed YM theory in the $SU(3)$ sector is one such example. The resulting γ_i-deformed $SU(3)$ spin chain was discussed in [9], while [49] extended these ideas to the $SU(2|3)$ sector.

In this section, we very briefly describe how the one-loop planar matrix of anomalous dimensions for the $SU(3)$ sector of the γ_i-deformed YM theory can be written as the Hamiltonian of a γ_i-deformed $SU(3)$ spin chain. Furthermore, we discuss such aspects of the γ_i-deformed $SU(3)$ spin chain formalism as the R-matrix, the Yang-Baxter equation, and the monodromy and transfer matrices. (Note that *any* integrable spin chain *always* has an R-matrix, which satisfies the Yang-Baxter equation.) The spin chain Hamiltonian and momentum operators are then written in terms of the transfer matrix. We make use of an algebraic Bethe ansatz to diagonalize this transfer matrix and hence the γ_i-deformed spin chain Hamiltonian. Finally, we discuss the γ_i-deformed vacuum states.

3.2.1 γ_i-deformed $SU(3)$ sector operators as γ_i-deformed $SU(3)$ spin chains

The $SU(3)$ sector of the γ_i-deformed YM theory is, again, composed of single trace scalar operators of the form

$$O\left[\Psi\right] = \Psi^{i_1 i_2 \ldots i_J} \mathrm{Tr}\left(\Phi_{i_1}\Phi_{i_2}\ldots\Phi_{i_J}\right), \tag{3.23}$$

as in our undeformed $\mathcal{N} = 4$ SYM theory. Thus the Hilbert space, in which our $SU(3)$ spin chain state $\Psi^{i_1 i_2 \ldots i_J}$ lives, is unchanged by the γ_i-deformation.

The one-loop planar matrix of anomalous dimensions (or $SU(3)$ spin chain Hamiltonian) does, however, depend on the deformation parameters. The same one-loop planar diagrams (see figure 3.1) are relevant, but the scalar interactions in the F-terms are now slightly different - when we exchange two fields Φ_i and Φ_j due to F-term interactions, our renormalization matrix picks up a factor of $e^{-2\pi i \varepsilon_{ijk}\gamma_k}$. This leads to the following one-loop expression of [9] for the planar matrix of anomalous dimensions in the $SU(3)$ sector of the γ_i-deformed YM theory:

$$\Gamma^{\gamma_i} = \frac{\lambda}{8\pi^2}\sum_{k=1}^{J}\left(1 - \mathcal{P}^{\gamma_i}_{k,k+1}\right), \tag{3.24}$$

where the action of the γ_i-deformed permutation matrix on the k^{th} and $(k+1)^{\text{th}}$ fields in our single trace basis operators (or, equivalently, on the k^{th} and $(k+1)^{\text{th}}$ indices of our $SU(3)$ spin chain state $\Psi^{i_1, i_2 \ldots i_J}$) is given by

$$\left(\mathcal{P}^{\gamma_i}_{k,k+1}\right)^{i_k i_{k+1}}_{j_k j_{k+1}} = e^{2\pi i \alpha_{i_k i_{k+1}}}\ \delta^{i_{k+1}}_{j_k}\delta^{i_k}_{j_{k+1}}, \qquad \text{with} \quad \alpha_{ij} \equiv -\varepsilon_{ijk}\gamma_k. \tag{3.25}$$

This matrix of anomalous dimensions is the Hamiltonian of a closed γ_i-deformed $SU(3)$ spin chain and reduces to the one-loop planar SYM matrix of anomalous dimensions (3.21), if we set the deformation parameters γ_i to zero.

3.2.2 γ_i-deformed $SU(3)$ spin chain formalism

We now briefly discuss the formal description of the γ_i-deformed $SU(3)$ spin chain in analogy to the more extensive review of the undeformed case in appendix B. This is based on the discussions in [9, 49].

γ_i-deformed spin chain Hamiltonian

The basic description of our closed $SU(3)$ spin chains states (see appendix B) does not change when we consider γ_i-deformed spin chains. The deformation is visible rather in the γ_i-deformed *Hamiltonian*

$$H^{\gamma_i} = \frac{\lambda}{8\pi^2}\sum_{k=1}^{J}\mathcal{H}^{\gamma_i}_{k,k+1} \qquad \text{with} \quad \mathcal{H}^{\gamma_i}_{k,k+1} = 1_{k,k+1} - \mathcal{P}^{\gamma_i}_{k,k+1}, \tag{3.26}$$

where $\mathcal{P}^{\gamma_i}_{k,k+1}$ is the γ_i-deformed permutation matrix (3.25), which is also given by

$$\mathcal{P}^{\gamma_i}_{k,k+1} = \sum_{m,n=1}^{3} e^{2\pi i\alpha_{mn}} e^m_n(k) e^n_m(k+1). \tag{3.27}$$

Here $e^m_n(k)$ are the basic observable states defined in (B.6) in terms of the matrices e^m_n in the k^{th} position of the tensor product. (The matrix e^m_n has a 1 in the m^{th} row and n^{th} column, and all the other elements are zero.)

Therefore we can write the action of $\mathcal{H}^{\gamma_i}_{k,k+1}$ on the k^{th} and $(k+1)^{\text{th}}$ complex vector spaces as an explicit sum over tensor products of the matrices e^m_n as follows:

$$\begin{aligned}\mathcal{H}^{\gamma_i}_{k,k+1} &= e^1_1\otimes e^2_2 + e^2_2\otimes e^1_1 - e^{2\pi i\alpha_{12}} e^1_2\otimes e^2_1 - e^{2\pi i\alpha_{21}} e^2_1\otimes e^1_2\\ &\quad + e^3_3\otimes e^1_1 + e^1_1\otimes e^3_3 - e^{2\pi i\alpha_{31}} e^3_1\otimes e^1_3 - e^{2\pi i\alpha_{13}} e^1_3\otimes e^3_1\\ &\quad + e^2_2\otimes e^3_3 + e^3_3\otimes e^2_2 - e^{2\pi i\alpha_{23}} e^2_3\otimes e^3_2 - e^{2\pi i\alpha_{32}} e^3_2\otimes e^2_3,\end{aligned} \tag{3.28}$$

or, more explicitly,

$$\mathcal{H}^{\gamma_i}_{k,k+1} = \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & -e^{2\pi i\alpha_{12}} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & -e^{2\pi i\alpha_{13}} & 0 & 0\\ \hline 0 & -e^{2\pi i\alpha_{21}} & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -e^{2\pi i\alpha_{23}} & 0\\ \hline 0 & 0 & -e^{2\pi i\alpha_{31}} & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -e^{2\pi i\alpha_{32}} & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right). \tag{3.29}$$

Now $\mathcal{H}^{\gamma_i}_{k,k+1}$ can also be expressed as follows:

$$\mathcal{H}^{\gamma_i}_{k,k+1} = \mathcal{U}_{k,k+1}\mathcal{H}_{k,k+1}\mathcal{U}^{-1}_{k,k+1}, \qquad \text{with} \ \ \mathcal{U}_{k,k+1} = \sum_{m,n=1}^{3} e^{i\pi\alpha_{mn}} e^m_m(k) e^n_n(k+1). \tag{3.30}$$

To prove this, we need to note that, since $\mathcal{U}_{k,k+1} 1_{k,k+1} \mathcal{U}^{-1}_{k,k+1} = 1_{k,k+1}$, it is equivalent to show that $\mathcal{P}^{\gamma_i}_{k,k+1} = \mathcal{U}_{k,k+1}\mathcal{P}_{k,k+1}\mathcal{U}^{-1}_{k,k+1}$ and therefore we shall calculate

$$\begin{aligned}
&\mathcal{U}_{k,k+1}\mathcal{P}_{k,k+1}\mathcal{U}^{-1}_{k,k+1} \\
&= \sum_{m,n=1}^{3} e^{i\pi\alpha_{mn}} e^m_m(k) e^n_n(k+1) \sum_{p,q=1}^{3} e^p_q(k) e^q_p(k+1) \sum_{r,s=1}^{3} e^{-i\pi\alpha_{rs}} e^r_r(k) e^s_s(k+1) \\
&= \sum_{m,n,p,q,r,s=1}^{3} e^{i\pi(\alpha_{mn}-\alpha_{rs})} \left[e^m_m(k) e^p_q(k) e^r_r(k)\right] \left[e^n_n(k+1) e^q_p(k+1) e^s_s(k+1)\right]. \qquad (3.31)
\end{aligned}$$

Since $e^m_m(k)e^p_q(k)e^r_r(k) = \delta^r_q \delta^p_m e^m_r(k)$ and $e^n_n(k+1)e^q_p(k+1)e^s_s(k+1) = \delta^s_p\delta^q_n e^n_s(k+1)$, we find that

$$\begin{aligned}
\mathcal{U}_{k,k+1}\mathcal{P}_{k,k+1}\mathcal{U}^{-1}_{k,k+1} &= \sum_{m,n,p,q,r,s=1}^{3} e^{i\pi(\alpha_{mn}-\alpha_{rs})} \, \delta^r_q \delta^p_m \delta^s_p \delta^q_n \; e^m_r(k) e^n_s(k+1) \\
&= \sum_{m,n=1}^{3} e^{2i\pi\alpha_{mn}} e^m_n(k) e^n_m(k+1) = \mathcal{P}^{\gamma_i}_{k,k+1}, \qquad (3.32)
\end{aligned}$$

which is sufficient to prove the statement (3.30).

γ_i-deformed R-matrix and the Yang-Baxter equation

The R-matrix for this γ_i-deformed $SU(3)$ spin chain is

$$R^{\gamma_i}_{i,j}(u) = u 1^{\gamma_i}_{i,j} + i\mathcal{P}_{i,j}, \qquad \text{with} \quad \left(1^{\gamma_i}_{i,j}\right)^{i_1 i_2}_{j_1 j_2} = e^{2\pi i \alpha_{i_1 i_2}} \delta^{i_1}_{j_1} \delta^{i_2}_{j_2}, \qquad (3.33)$$

which, although it is defined over the auxillary and quantum spaces, acts non-trivially only on the i^{th} and j^{th} complex vector spaces as follows:

$$R^{\gamma_i}_{i,j}(u) = \qquad (3.34)$$

$$\left(\begin{array}{ccc|ccc|ccc}
u+i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & ue^{2\pi i\alpha_{12}} & 0 & i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & ue^{2\pi i\alpha_{13}} & 0 & 0 & 0 & i & 0 & 0 \\
\hline
0 & i & 0 & ue^{2\pi i\alpha_{21}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & u+i & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & ue^{2\pi i\alpha_{23}} & 0 & i & 0 \\
\hline
0 & 0 & i & 0 & 0 & 0 & ue^{2\pi i\alpha_{31}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & i & 0 & ue^{2\pi i\alpha_{32}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u+i
\end{array}\right).$$

This γ_i-deformed R-matrix satisfies the Yang-Baxter equation

$$R^{\gamma_i}_{i,j}(u-v)R^{\gamma_i}_{i,k}(u)R^{\gamma_i}_{j,k}(v) = R^{\gamma_i}_{j,k}(v)R^{\gamma_i}_{i,k}(u)R^{\gamma_i}_{i,j}(u-v), \tag{3.35}$$

where $i \neq j \neq k$. The proof is similar to the undeformed case, which is discussed in appendix B.

γ_i-deformed monodromy and transfer matrices

We shall now introduce the γ_i-deformed *L-matrix* $L^{\gamma_i}_{i,j}(u) = R^{\gamma_i}_{i,j}\left(u - \frac{i}{2}\right)$, which shall be used to construct the γ_i-deformed *monodromy matrix* as follows:

$$T^{\gamma_i}_0(u) = L^{\gamma_i}_{0,J}(u)\dots L^{\gamma_i}_{0,2}(u)L^{\gamma_i}_{0,1}(u). \tag{3.36}$$

This monodromy matrix can also be expressed as

$$T^{\gamma_i}_0(u) \equiv \begin{pmatrix} \left(A^{\gamma_i}\right)(u) & \left(B^{\gamma_i}\right)_2(u) & \left(B^{\gamma_i}\right)_3(u) \\ \left(C^{\gamma_i}\right)^2(u) & \left(D^{\gamma_i}\right)^2_2(u) & \left(D^{\gamma_i}\right)^2_3(u) \\ \left(C^{\gamma_i}\right)^3(u) & \left(D^{\gamma_i}\right)^3_2(u) & \left(D^{\gamma_i}\right)^3_3(u) \end{pmatrix}, \tag{3.37}$$

which is a matrix in the auxillary space 0 with operators in the quantum spaces as components. (The algebraic Bethe ansatz is constructed from these operators.)

This γ_i-deformed monodromy matrix satisfies

$$R^{\gamma_i}_{a,b}(u-v)T^{\gamma_i}_a(u)T^{\gamma_i}_b(v) = T^{\gamma_i}_b(v)T^{\gamma_i}_a(u)R^{\gamma_i}_{a,b}(u-v), \tag{3.38}$$

where a and b are two different auxillary spaces. This is a direct result of the Yang-Baxter equation (3.35) and can be derived as in appendix B.

The γ_i-deformed *transfer matrix* is now defined as

$$t^{\gamma_i}(u) \equiv \mathrm{Tr}_0\left[T^{\gamma_i}_0(u)\right] = \left(A^{\gamma_i}\right)(u) + \left(D^{\gamma_i}\right)^l_l(u), \tag{3.39}$$

which is an operator on the quantum spaces.

γ_i-deformed momentum and Hamiltonian operators in terms of the γ_i-deformed transfer matrix

The *momentum operator* is given by

$$P^{\gamma_i} = \frac{1}{i}\log\left[i^{-J}\ t^{\gamma_i}\left(\tfrac{i}{2}\right)\right]. \tag{3.40}$$

As shown in appendix B for the momentum operator constructed from the undeformed transfer matrix, we find that $e^{iP^{\gamma_i}} = \mathcal{P}_{1,2}\mathcal{P}_{2,3}...\mathcal{P}_{J-1,J}$ and therefore P^{γ_i} generates translations along our spin chain. This verifies that P is, indeed, the momentum operator.

Furthermore, the γ_i-deformed *Hamiltonian operator* can be written as

$$H^{\gamma_i} = \frac{\lambda}{8\pi^2}\left[J - i\ \frac{d}{du}\ \log\, t^{\gamma_i}(u)\bigg|_{u=\frac{i}{2}}\right], \tag{3.41}$$

and thus we see that diagonalizing the γ_i-deformed Hamiltonian is equivalent to diagonalizing the γ_i-deformed transfer matrix.

3.2.3 γ_i-deformed algebraic Bethe ansatz

We now briefly discuss how to diagonlize the γ_i-deformed transfer matrix using the algebraic Bethe ansatz. This algebraic Bethe ansatz state is dependent on two sets of Bethe parameters, which must satisfy both the γ_i-deformed nested Bethe ansatz equations and a cyclicity condition. (This cyclicity condition is due to the fact that our spin chain is closed and thus any state should be invariant under cyclic permutations of the component spin states.) We also construct the energy and momentum eigenvalues in terms of the Bethe parameters. Note that a more extensive review of all these results, complete with derivations, is available in appendix B for the case of undeformed $SU(3)$ spin chains. We have based our discussions on the reviews [50, 51], which consider only $SU(2)$ spin chains, as well as the results in [9, 49].

γ_i-deformed fundamental commutation relations

An indirect result of the Yang-Baxter equation is that the operator components of the γ_i-deformed monodromy matrix $T_0^{\gamma_i}(u)$ in the auxillary space must satisfy a set of fundamental commutation relations:

$$\begin{aligned}&\left(A^{\gamma_i}\right)(u)\left(B^{\gamma_i}\right)_{i_1}(v)\\&= e^{2\pi i\alpha_{1i_1}}\left[\left(\frac{u-v-i}{u-v}\right)\left(B^{\gamma_i}\right)_{i_1}(v)\left(A^{\gamma_i}\right)(u) + \left(\frac{i}{u-v}\right)\left(B^{\gamma_i}\right)_{i_1}(u)\left(A^{\gamma_i}\right)(v)\right],\end{aligned} \tag{3.42}$$

$$\left(B^{\gamma_i}\right)_{i_1}(u)\left(B^{\gamma_i}\right)_{i_2}(v) = \left(\frac{1}{u-v+i}\right)\left(\tilde{R}^{\gamma_i}\right)^{j_1 j_2}_{i_1 i_2}(u-v)\left(B^{\gamma_i}\right)_{j_2}(v)\left(B^{\gamma_i}\right)_{j_1}(u), \quad (3.43)$$

$$\begin{aligned}\left(D^{\gamma_i}\right)^{k_1}_{i_1}(u)\left(B^{\gamma_i}\right)_{i_2}(v) = e^{2\pi i \alpha_{1k_1}} & \left[\left(\frac{1}{u-v}\right)\left(\tilde{R}^{\gamma_i}\right)^{j_1 j_2}_{i_1 i_2}(u-v)\left(B^{\gamma_i}\right)_{j_2}(v)\left(D^{\gamma_i}\right)^{k_1}_{j_1}(u)\right.\\ & \left. + \left(\frac{-i}{u-v}\right)\left(B^{\gamma_i}\right)_{i_1}(u)\left(D^{\gamma_i}\right)^{k_1}_{i_2}(v)\right],\end{aligned} \quad (3.44)$$

where $i_1, i_2, j_1, j_2, k_1 \,\epsilon\, \{2,3\}$ and we define the γ_i-deformed $SU(2)$ R-matrix as

$$\left(\tilde{R}^{\gamma_i}\right)^{i_1 i_2}_{j_1 j_2}(u) = \begin{cases} (u+i)\,\delta^{i_1}_{j_1}\delta^{i_2}_{j_2} & \text{if } i_1 = i_2 \\ u\; e^{2\pi i \alpha_{i_1 i_2}}\;\delta^{i_1}_{j_1}\delta^{i_2}_{j_2} + i\;\delta^{i_2}_{j_1}\delta^{i_1}_{j_2} & \text{if } i_1 \neq i_2. \end{cases} \quad (3.45)$$

In matrix form, this R-matrix is given by

$$\tilde{R}^{\gamma_i}(u) = \left(\begin{array}{cc|cc} u+i & 0 & 0 & 0 \\ 0 & u\; e^{2\pi i \alpha_{23}} & i & 0 \\ \hline 0 & i & u\; e^{2\pi i \alpha_{32}} & 0 \\ 0 & 0 & 0 & u+i \end{array}\right), \quad (3.46)$$

which clearly acts on a tensor product of two $\mathbb{C}^2$ complex vector spaces.

γ_i-deformed algebraic Bethe ansatz and the eigenvalues of the γ_i-deformed transfer matrix

Let us first define the ground state of our γ_i-deformed $SU(3)$ spin chain as the state of maximum $J_1 = J$, consisting of a tensor product of J spin-up vectors

$$\omega_+ = \begin{pmatrix}1\\0\\0\end{pmatrix} \otimes \ldots \otimes \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad (3.47)$$

which is clearly annihilated by the γ_i-deformed spin chain Hamiltonian. This ground state is also an eigenstate of the operators $\left(A^{\gamma_i}\right)(u)$ and $\left(D^{\gamma_i}\right)^i_j(u)$, and is annihilated by $\left(C^{\gamma_i}\right)^i(u)$. Most interestingly, however, the $\left(B^{\gamma_i}\right)_i(u)$ operators act by lowering the spin of one site in our ground state spin chain.

We shall now make the *first part of the γ_i-deformed algebraic Bethe ansatz* for the eigenstates of the γ_i-deformed transfer matrix as follows:

$$\Phi^{\gamma_i}(u_{1,1},\ldots,u_{1,M}) = \left(f^{\gamma_i}\right)^{i_1,\ldots,i_M}\left(B^{\gamma_i}\right)_{i_1}(u_{1,1})\ldots\left(B^{\gamma_i}\right)_{i_M}(u_{1,M})\,\omega_+, \quad (3.48)$$

where $i_1, \ldots, i_M \,\epsilon\, \{2,3\}$ and $(f^{\gamma_i})^{i_1,\ldots,i_M}$ are, for now, arbitrary complex coefficients. We have thus lowered the spin of $M = J_2 + J_3$ sites of our spin chain.

Let us consider the action of the γ_i-deformed transfer matrix $t^{\gamma_i}(u) = (A^{\gamma_i})(u) + (D^{\gamma_i})_l^l(u)$ on this algebraic Bethe ansatz state. The fundamental commutation relations allow us to move the operators $(A^{\gamma_i})(u)$ and $(D^{\gamma_i})_l^l(u)$ through the series of $(B^{\gamma_i})_i$ operators until they act on the ground state ω_+. Remembering that the ground state is an eigenstate of $(A^{\gamma_i})(u)$ and $(D^{\gamma_i})_j^i(u)$, we see that it is possible to obtain an explicit result for the action of the γ_i-deformed transfer matrix on the algebraic Bethe ansatz state.

Thus, assuming that the first nested Bethe ansatz equation is satisfied, we find that the algebraic Bethe ansatz state diagonalizes the γ_i-deformed transfer matrix if the state $(f^{\gamma_i})^{i_1\ldots i_M}$ in the basis $(B^{\gamma_i})_{i_1}(u_{1,1})\ldots(B^{\gamma_i})_{i_M}(u_{1,M})$ diagonlizes the matrix

$$e^{2\pi i J_1 \alpha_{l1}} \left(\tilde{R}^{\gamma_i}\right)^{lk_M}_{j_{M-1}i_M} \left(u - u_{1,M} - \tfrac{i}{2}\right) \ldots \left(\tilde{R}^{\gamma_i}\right)^{j_2k_2}_{j_1i_2} \left(u - u_{1,2} - \tfrac{i}{2}\right) \left(\tilde{R}^{\gamma_i}\right)^{j_1k_1}_{li_1} \left(u - u_{1,1} - \tfrac{i}{2}\right). \tag{3.49}$$

Note that we have made a redefinition $u \to u - \frac{i}{2}$, for convenience, in this calculation. This looks very much like the original transfer matrix, except that now the indices run over only 2 and 3, our spin chain state $(f^{\gamma_i})^{i_1\ldots i_M}$ is of length M, and there is a dependence on both the deformation parameters γ_i and the first set of Bethe parameters $\{u_{1,1}\ldots u_{1,M}\}$.

Therefore we have, in some sense, reduced the dimension of our problem by one and must now solve an $SU(2)$ spin chain problem. This $SU(2)$ spin chain has a γ_i-deformed R-matrix (3.45), which satisfies the Yang-Baxter equation, and a γ_i-deformed $SU(2)$ monodromy matrix, with component operators $\tilde{A}^{\gamma_i}(u)$, $\tilde{B}^{\gamma_i}(u)$, $\tilde{C}^{\gamma_i}(u)$ and $\tilde{D}^{\gamma_i}(u)$. These component operators depend also on the first set of Bethe parameters and satisfy a set of fundamental commutation relations. Our $SU(2)$ spin chain state $(f^{\gamma_i})^{i_1\ldots i_M}$ must now diagonalize a weighted combination of the states $\tilde{A}^{\gamma_i}(u)$ and $\tilde{D}^{\gamma_i}(u)$. We can, as before, define the ground state of our $SU(2)$ spin chain as a tensor product of M spin-up vectors

$$\tilde{\omega}_+ = \begin{pmatrix}1\\0\end{pmatrix} \otimes \ldots \otimes \begin{pmatrix}1\\0\end{pmatrix}, \tag{3.50}$$

which is an eigenstate of the operators $\tilde{A}^{\gamma_i}(u)$ and $\tilde{D}^{\gamma_i}(u)$, and is, again, annihilated by $\tilde{C}^{\gamma_i}(u)$. Our operator $\tilde{B}^{\gamma_i}(u)$ lowers the spin of one site of the ground state $\tilde{\omega}_+$.

We shall hence make the *second part of the γ_i-deformed algebraic Bethe ansatz* as follows:

$$(f^{\gamma_i})^{i_1,\ldots,i_M} \equiv \tilde{\Phi}^{\gamma_i}(u_{2,1},\ldots,u_{2,L}) = \tilde{B}^{\gamma_i}(u_{2,1})\ldots\tilde{B}^{\gamma_i}(u_{2,L})\,\tilde{\omega}_+. \tag{3.51}$$

Here we have lowered the spin of $L = J_3$ sites of our $SU(2)$ spin chain.

We can now operate with the weighted combinations of the operators $\tilde{A}^{\gamma_i}(u)$ and $\tilde{D}^{\gamma_i}(u)$ on our Bethe ansatz for $(f^{\gamma_i})^{i_1,\ldots,i_M}$ and use the $SU(2)$ fundemental commutation relations to move these operators through the series of $\tilde{B}^{\gamma_i}$ operators until they act on the $SU(2)$ ground state $\tilde{\omega}_+$. We thus find that (3.49) is diagonalized if the second nested Bethe ansatz equation is satisfied.

Therefore we finally determine that the algebraic Bethe ansatz state Φ^{γ_i} is an eigenstate of the γ_i-deformed transfer matrix if and only if our two sets of Bethe parameters satisfy the γ_i-deformed nested Bethe ansatz equations:

$$e^{-2\pi i J\gamma_3}\left(\frac{u_{1,j}+\frac{i}{2}}{u_{1,j}-\frac{i}{2}}\right)^J = e^{-2\pi i J_3(\gamma_1+\gamma_2+\gamma_3)}\left[\prod_{\substack{k=1\\k\neq j}}^{M}\left(\frac{u_{1,j}-u_{1,k}+i}{u_{1,j}-u_{1,k}-i}\right)\right]\left[\prod_{l=1}^{L}\left(\frac{u_{1,j}-u_{2,l}-\frac{i}{2}}{u_{1,j}-u_{2,l}+\frac{i}{2}}\right)\right], \tag{3.52}$$

for all $j \,\epsilon\, \{1,\ldots,M\}$, and

$$e^{2\pi i(J_2+J_3)(\gamma_1+\gamma_2+\gamma_3)}\left[\prod_{\substack{k=1\\k\neq j}}^{L}\left(\frac{u_{2,j}-u_{2,k}+i}{u_{2,j}-u_{2,k}-i}\right)\right]\left[\prod_{l=1}^{M}\left(\frac{u_{1,l}-u_{2,j}+\frac{i}{2}}{u_{1,l}-u_{2,j}-\frac{i}{2}}\right)\right] = e^{2\pi i J(\gamma_2+\gamma_3)}, \tag{3.53}$$

for every $j \,\epsilon\, \{1,\ldots,L\}$. These γ_i-deformed nested Bethe ansatz equations agree with the results quoted in [9].

Furthermore, the eigenvalue of the γ_i-deformed transfer matrix $t^{\gamma_i}(u)$ corresponding to the algebraic Bethe ansatz state Φ^{γ_i} is given by

$$\begin{aligned}\Lambda^{\gamma_i}(u) &= e^{-2\pi i(J_2\gamma_3-J_3\gamma_2)}\left[\prod_{k=1}^{M}\left(\frac{u-u_{1,k}-\frac{3i}{2}}{u-u_{1,k}-\frac{i}{2}}\right)\right]u^J \\ &+\left[\prod_{k=1}^{M}\left(\frac{1}{u-u_{1,k}-\frac{i}{2}}\right)\right](u-i)^J\left\{e^{-2\pi i(J_3\gamma_1-J_1\gamma_3)}\left[\prod_{k=1}^{L}\left(\frac{u-u_{2,k}-i}{u-u_k}\right)\right]\left[\prod_{l=1}^{M}(u-u_{1,l}+\tfrac{i}{2})\right]\right. \\ &\left.+\,e^{-2\pi i(J_1\gamma_2-J_2\gamma_1)}\left[\prod_{k=1}^{L}\left(\frac{u-u_{2,k}+i}{u-u_k}\right)\right]\left[\prod_{l=1}^{M}(u-u_{1,l}-\tfrac{i}{2})\right]\right\},\end{aligned} \tag{3.54}$$

in terms of our two sets of Bethe parameters $\{u_{1,1},\ldots,u_{1,M}\}$ and $\{u_{2,1},\ldots,u_{2,L}\}$, where $L = J_3$ and $M = J_2 + J_3$.

γ_i-deformed energy and momentum eigenvalues and the γ_i-deformed cyclicity condition

Taking into account our redefinition of $u \to u - \frac{i}{2}$ and using equation (3.41), we find that the energy eigenvalues are

$$\begin{aligned}
E^{\gamma_i} &= \frac{\lambda}{8\pi^2}\left[J - \frac{d}{du}\log \Lambda^{\gamma_i}(u)\bigg|_{u=i}\right] \\
&= \frac{\lambda}{8\pi^2}\left[J - i\frac{d}{du}\log\left\{e^{-2\pi i(J_2\gamma_3 - J_3\gamma_2)}\prod_{k=1}^{M}\left(\frac{u-u_{1,k}-\frac{3i}{2}}{u-u_{1,k}-\frac{i}{2}}\right)u^J\right\}\bigg|_{u=i}\right] \\
&= \frac{\lambda}{8\pi^2}\left[J - i\frac{d}{du}\left\{\sum_{k=1}^{M}\left[\log\left(u-u_{1,k}-\tfrac{3i}{2}\right)-\log\left(u-u_{1,k}-\tfrac{i}{2}\right)\right]+J\log u\right\}\bigg|_{u=i}\right] \\
&= \frac{\lambda}{8\pi^2}\, i\sum_{k=1}^{M}\left[\frac{1}{u_{1,k}+\frac{i}{2}}-\frac{1}{u_{1,k}-\frac{i}{2}}\right],
\end{aligned} \tag{3.55}$$

which gives

$$E^{\gamma_i} = \frac{\lambda}{8\pi^2}\sum_{k=1}^{M}\frac{1}{u_{1,k}^2+\frac{1}{4}}. \tag{3.56}$$

This agrees with the result in [9]. These energy eigenvalues appear to be independent of both the second set of Bethe parameters and our deformation parameters γ_i. However, we should remember that the first and second sets of Bethe parameters are related by the nested Bethe ansatz equations. Furthermore, these equations are γ_i-deformed. Thus the γ_i-deformed energy eigenvalues are indirectly dependent on both sets of Bethe parameters and the deformation parameters γ_i.

Finally, we shall derive the cyclicity condition using the γ_i-deformed momentum eigenvalues, which can be obtained using equation (3.40) as follows:

$$\begin{aligned}
P^{\gamma_i} &= \frac{1}{i}\log\left[i^{-J}\,\Lambda^{\gamma_i}(i)\right] \\
&= \frac{1}{i}\log\left[e^{-2\pi i(J_2\gamma_3-J_3\gamma_2)}\prod_{k=1}^{M}\left(\frac{u_{1,k}+\frac{i}{2}}{u_{1,k}-\frac{i}{2}}\right)\right] \\
&= -2\pi(J_2\gamma_3-J_3\gamma_2)+\frac{1}{i}\sum_{k=1}^{M}\log\left(\frac{u_{1,k}+\frac{i}{2}}{u_{1,k}-\frac{i}{2}}\right).
\end{aligned} \tag{3.57}$$

We must require that $e^{iP^{\gamma_i}} = 1$, so that a translation by one site along our closed spin chain results in no change. We thus obtain the γ_i-deformed cyclicity condition

$$e^{-2\pi i(J_2\gamma_3 - J_3\gamma_2)} \prod_{k=1}^{M} \left(\frac{u_{1,k} + \frac{i}{2}}{u_{1,k} - \frac{i}{2}} \right) = 1, \tag{3.58}$$

which agrees with the results in [9] and [49].

3.2.4 γ_i-deformed vacuum states

We shall now consider the 'angular momenta'[3] J_1, J_2 and J_3, which describe the γ_i-deformed vacuum states. These are the states with zero spin chain energy. The discussion given hereafter closely follows [9].

Consider the expression (3.56) for the energy eigenvalues of our γ_i-deformed $SU(3)$ spin chain. There are two ways in which we can obtain zero energy. Firstly, the energy of the spin chain is clearly zero if there are no excited modes ($J_2 + J_3 = 0$). Secondly, the energy is zero if the term $\frac{1}{u_{1,k}^2 + \frac{1}{4}}$ is zero for every parameter $u_{1,k}$, which occurs only when $u_{1,k}$ is infinite for all k. We shall then also assume that the difference between these parameters $|u_{1,j} - u_{1,k}|$ for $j \neq k$ is also infinite.

Now, in the first case, we obtain the state $(J, 0, 0)$, where $J_1 = J$ and $J_2 = J_3 = 0$. However, there is no real difference between the three 'angular momenta' J_1, J_2 and J_3, so we could just as easily have chosen a state of maximum J_2 or J_3 to be the ground state. Therefore we must also have vacuum states $(0, J, 0)$, where $J_2 = J$ and $J_1 = J_3 = 0$, and $(0, 0, J)$, where $J_3 = J$ and $J_1 = J_2 = 0$. These vacuum states are independent of the deformation parameters γ_i and are present in the undeformed case.

The second case of infinite Bethe parameters $u_{1,k}$ is only possible if certain constraints are satisfied by the 'angular momenta' J_1, J_2 and J_3. These constraints are the result of the first and second γ_i-deformed nested Bethe ansatz equations (3.52) and (3.53) respectively, and the cyclicity condition (3.58).

[3] The term 'angular momenta' in reference to J_1, J_2 and J_3 is, strictly speaking, inaccurate. These parameters describe our algebraic Bethe ansatz state and represent the number of different types of spin states in our tensor product. They are, however, dual to the angular momenta in the γ_i-deformed S^5 space of the corresponding string theory.

Applying the assumption that $u_{1,k} \to \infty$ to the cyclicity condition (3.58) yields

$$e^{-2\pi i(J_2\gamma_3 - J_3\gamma_2)} = 1, \tag{3.59}$$

and thus we must have[4]

$$J_2\gamma_3 - J_3\gamma_2 = 0. \tag{3.60}$$

Furthermore, the first nested Bethe ansatz equation (3.52) with $u_{1,k} \to \infty$ implies

$$e^{-2\pi i(J_1+J_2+J_3)\gamma_3} = e^{-2\pi i J_3(\gamma_1+\gamma_2+\gamma_3)}, \tag{3.61}$$

and, making use of the condition (3.60), we obtain

$$e^{2\pi i(J_3\gamma_1 - J_1\gamma_3)} = 1, \tag{3.62}$$

from which it follows that

$$J_3\gamma_1 - J_1\gamma_3 = 0. \tag{3.63}$$

Lastly, let us consider the second nested Bethe ansatz equation (3.53). We have only assumed that $u_{1,k} \to \infty$, but, as yet, have placed no conditions on $u_{2,k}$. Therefore we first need to get rid of the product dependent only on the latter set of parameters. We do this by taking the product of this equation for all values of $j \,\epsilon\, \{1, \ldots, L\}$. It can be seen that

$$\left(\frac{u_{2,j} - u_{2,k} + i}{u_{2,j} - u_{2,k} - i}\right)\left(\frac{u_{2,k} - u_{2,j} + i}{u_{2,k} - u_{2,j} - i}\right) = 1 \qquad \text{for any } j \neq k, \tag{3.64}$$

and thus, since each term in our product has a corresponding term with which it cancels, when we assume that $u_{1,k} \to \infty$ we obtain

$$e^{2\pi i J_3(J_2+J_3)(\gamma_1+\gamma_2+\gamma_3)} = e^{2\pi i J_3(J_1+J_2+J_3)(\gamma_2+\gamma_3)}. \tag{3.65}$$

This implies, using the constraint (3.63), that

$$e^{2\pi i J_3(J_1\gamma_2 - J_2\gamma_1)} = 1, \tag{3.66}$$

and hence

$$J_1\gamma_2 - J_2\gamma_1 = 0. \tag{3.67}$$

[4]One might expect to find that any integer on the right hand side of this equation would suffice. At this point, however, we would like to consider solutions which exist for all real deformation parameters γ_i (not necessarily rational). For irrational values of the γ_i, the only integer which will provide a valid solution is zero.

Thus, for the case of infinite Bethe parameters $u_{1,k}$ to be a valid vacuum state, the 'angular momenta' J_1, J_2 and J_3 must satisfy

$$\varepsilon_{ijk} J_j \gamma_k = 0, \tag{3.68}$$

which corresponds to $(J_1, J_2, J_3) \sim (\gamma_1, \gamma_2, \gamma_3)$. This vacuum state clearly has no undeformed analogy, since the constraints disappear when we set all our deformation parameters γ_i to zero.

Now in this derivation we have considered a general γ_i-deformed background in which the deformation parameters γ_i are any real numbers. If we confine ourselves to the case of rational deformation parameters, then our condition can be broadened into

$$\varepsilon_{ijk} J_j \gamma_k = n_i, \qquad \text{where} \;\; n_i \in \mathbb{Z} \quad \text{for} \;\; i \in \{1, 2, 3\}\,. \tag{3.69}$$

Chapter 4

String Theory

4.1 Classical String Worldsheet Action

A string is a one dimensional object with some fundamental constant string tension T. Any such string traces out a two dimensional surface, known as a worldsheet (analogous to the worldline of a point particle), in spacetime, which can be parameterized by the temporal and spatial coordinates τ and σ respectively. In other words, this worldsheet is the image of an embedding $X^{\mu}(\tau, \sigma)$ from the parameter space (τ, σ) into the target space, which is some d dimensional spacetime described by the coordinates x^{μ}, where μ runs from 0 to $d-1$ [4]. We are particularly interested in the $d = 10$ dimensional $AdS_5 \times S^5$ target space.

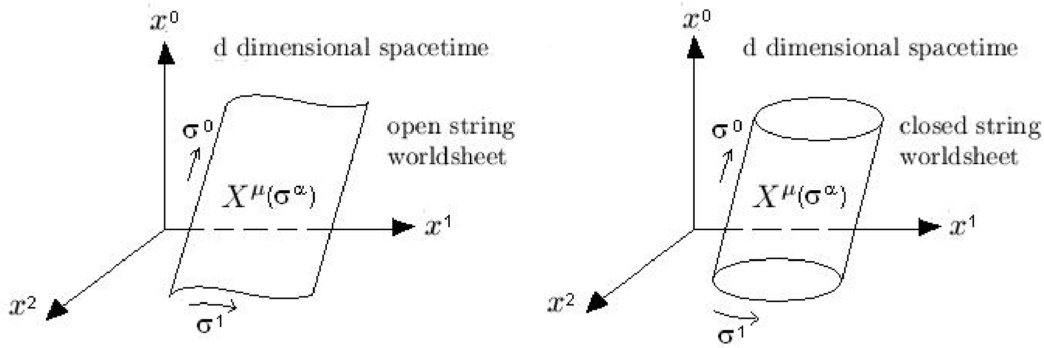

Figure 4.1: The worldsheets of open and closed strings.

The worldsheet of an open string is simply an open sheet, but a closed string must have its end-points identified at any time τ and thus the worldsheet becomes a tubelike surface (see figure 4.1). The temporal coordinate τ can take on any real value, but the spatial coordinate σ is generally confined to a finite interval [4]. We are especially interested in closed strings and, following [8], shall hence make use of the parameter space $\{(\tau, \sigma) : \tau \in \mathbb{R}, \ \sigma \in [0, 2\pi]\}$ and take the worldsheet to be periodic in σ, so that $X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi)$.

The classical[1] string worldsheet action is proportional to the proper area of the string worldsheet. (This is analogous to the classical point particle action, which is proportional to the length of the particle's worldline.) It can immediately be seen that this action is reparameterization invariant, since the area of a surface is independent of the parameters used to describe it. The proportionality constant can be obtained using dimensional arguments. In units of $c \equiv 1$ (so that L = T), the action must have dimensions $\frac{ML^2}{T} = ML$ and thus the proportionality constant has dimensions $\frac{[S]}{[A]} = \frac{ML}{L^2} = \frac{M}{L}$. These are the units of the string tension. Thus we take the classical string worldsheet action to be

$$S = -\frac{1}{2\pi\alpha'} A = -\frac{1}{2\pi\alpha'} \int_{\text{worldsheet}} dA, \tag{4.1}$$

where $T = \frac{1}{2\pi\alpha'}$ is the string tension [4, 52].

Now there are two common ways of expressing this classical worldsheet action in terms of the embedding $X^\mu(\tau, \sigma)$ and the metrics of the parameter and target spaces. These are known as the Nambu-Goto and Polyakov string actions. In this section, we describe the construction of these equivalent classical worldsheet actions.

4.1.1 The Nambu-Goto string action

Let us first derive an expression for the area of a surface in Euclidean space and then extend this result to the proper area of a worldsheet in some d dimensional spacetime following [4].

Consider an embedding $\vec{X}(\xi^i)$ from the parameter space (ξ^1, ξ^2) into d dimensional Euclidean space, which defines a surface in this target space. An infinitesimal square area, with side lengths $d\xi^1$ and $d\xi^2$, in the parameter space is mapped onto an infinitesimal parallelogram in the target space (see figure 4.2). This parallelogram

[1] By classical we mean that quantum effects have been neglected.

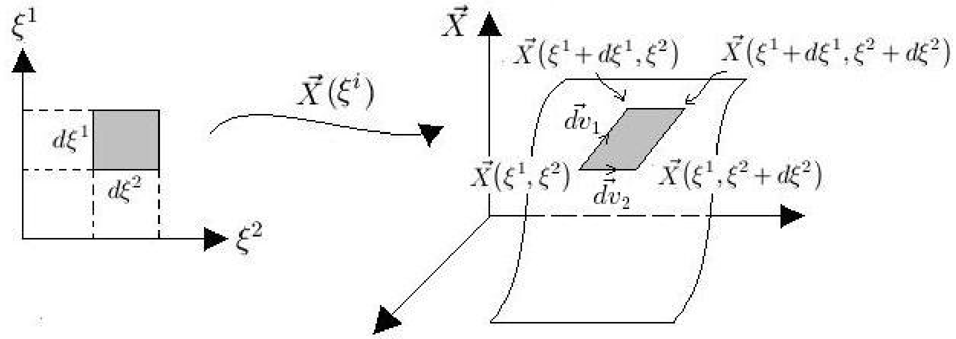

Figure 4.2: Infinitesimal area of the surface in Euclidean space [4].

has adjacent sides $\vec{dv}_1$ and $\vec{dv}_2$ as follows:

$$\vec{dv}_1 = \vec{X}\left(\xi^1 + d\xi^1, \xi^2\right) - \vec{X}\left(\xi^1, \xi^2\right) = \frac{\partial \vec{X}}{\partial \xi^1}\, d\xi^1, \tag{4.2}$$

$$\vec{dv}_2 = \vec{X}\left(\xi^1, \xi^2 + d\xi^2\right) - \vec{X}\left(\xi^1, \xi^2\right) = \frac{\partial \vec{X}}{\partial \xi^2}\, d\xi^2. \tag{4.3}$$

Now the area of this infinitesimal parallelogram is given by

$$dA = \left|\vec{dv}_1\right| \left|\vec{dv}_2\right| \sin\theta, \tag{4.4}$$

where $0 \leq \theta \leq \pi$ is the angle between $\vec{dv}_1$ and $\vec{dv}_2$. Thus, using the expressions (4.2) and (4.3) for the adjacent sides of the parallelogram, we find that the infinitesimal area in the target space is

$$\begin{aligned} dA &= \left|\frac{\partial \vec{X}}{\partial \xi^1} d\xi^1\right| \left|\frac{\partial \vec{X}}{\partial \xi^2} d\xi^2\right| \sin\theta \\ &= \sqrt{\left|\frac{\partial \vec{X}}{\partial \xi^1} d\xi^1\right|^2 \left|\frac{\partial \vec{X}}{\partial \xi^2} d\xi^2\right|^2 - \left|\frac{\partial \vec{X}}{\partial \xi^1} d\xi^1\right|^2 \left|\frac{\partial \vec{X}}{\partial \xi^2} d\xi^2\right|^2 \cos^2\theta} \\ &= \sqrt{\left(\frac{\partial \vec{X}}{\partial \xi^1} d\xi^1 \cdot \frac{\partial \vec{X}}{\partial \xi^1} d\xi^1\right)\left(\frac{\partial \vec{X}}{\partial \xi^2} d\xi^2 \cdot \frac{\partial \vec{X}}{\partial \xi^2} d\xi^2\right) - \left(\frac{\partial \vec{X}}{\partial \xi^1} d\xi^1 \cdot \frac{\partial \vec{X}}{\partial \xi^2} d\xi^2\right)^2} \\ &= d\xi^1 d\xi^2 \sqrt{\det\left[\frac{\partial \vec{X}}{\partial \xi^i} \cdot \frac{\partial \vec{X}}{\partial \xi^j}\right]}. \end{aligned} \tag{4.5}$$

Notice that the spatial interval along an infinitesimal vector $d\vec{X}$ on the surface is

$$ds^2 = d\vec{X} \cdot d\vec{X} = \left(\frac{\partial \vec{X}}{\partial \xi^1} d\xi^1 + \frac{\partial \vec{X}}{\partial \xi^2} d\xi^2\right) \cdot \left(\frac{\partial \vec{X}}{\partial \xi^1} d\xi^1 + \frac{\partial \vec{X}}{\partial \xi^2} d\xi^2\right) = \left(\frac{\partial \vec{X}}{\partial \xi^i} \cdot \frac{\partial \vec{X}}{\partial \xi^j}\right) d\xi^i d\xi^j, \tag{4.6}$$

so that $g_{ij} \equiv \frac{\partial \vec{X}}{\partial \xi^i} \cdot \frac{\partial \vec{X}}{\partial \xi^j}$ is the induced metric on our surface in d-dimensional Euclidean space. Hence the infinitesimal area can be written as

$$dA = d\xi^1 d\xi^2 \sqrt{g}, \tag{4.7}$$

in terms of the determinant $g \equiv \det(g_{ij})$ of the induced metric g_{ij}. We can obtain the area of the entire surface in the d dimensional target space by integrating this infinitesimal area over the parameter space as follows:

$$A = \int d\xi^0 d\xi^1 \sqrt{\det\left[\frac{\partial \vec{X}}{\partial \xi^j} \cdot \frac{\partial \vec{X}}{\partial \xi^j}\right]} = \int d\xi^0 d\xi^1 \sqrt{g}. \tag{4.8}$$

We shall now extend this result to the proper area of the string worldsheet in some d dimensional spacetime. Let us denote the full spacetime metric as $G_{\mu\nu}$, while the induced metric on the worldsheet is $\gamma_{\alpha\beta}$, which has a Minkowski signature. The spacetime interval on the worldsheet is then given by

$$\begin{aligned} ds^2 &= G_{\mu\nu} dx^\mu dx^\nu \\ &= \gamma_{\alpha\beta} d\sigma^\alpha d\sigma^\beta, \qquad \text{where } \sigma^0 = \tau \text{ and } \sigma^1 = \sigma. \end{aligned} \tag{4.9}$$

Here $\gamma \equiv \det(\gamma_{\alpha\beta}) < 0$ and we notice that the induced metric on the string worldsheet must thus be given by

$$\gamma_{\alpha\beta} = \partial_\alpha x^\mu \partial_\beta x^\nu G_{\mu\nu}. \tag{4.10}$$

Hence, in analogy to the previous result (4.8), the proper area of the string worldsheet is

$$A = \int d\sigma^0 d\sigma^1 \sqrt{-\gamma}. \tag{4.11}$$

Notice that this expression differs from the area of a surface in Euclidean space in that the term in the square root is $-\gamma$. This is due to the Minkowski signature of the induced worldsheet action $\gamma_{\alpha\beta}$. The classical string worldsheet action can thus be obtained from (4.1) as follows:

$$S = -\frac{1}{\alpha'} \int d\tau \frac{d\sigma}{2\pi} \sqrt{-\det\left[\partial_\alpha x^\mu \partial_\beta x^\nu G_{\mu\nu}\right]} = -\frac{1}{\alpha'} \int d\tau \frac{d\sigma}{2\pi} \sqrt{-\gamma}, \tag{4.12}$$

which is known as the Nambu-Goto string action [4, 52].

4.1.2 The Polyakov string action

Although the Nambu-Goto string action (4.12) has a reasonably simple form, it is often convenient to work with an action which does not contain only a square root. Thus we introduce the Polyakov string action

$$S = -\frac{1}{2\alpha'}\int d\tau \frac{d\sigma}{2\pi}\sqrt{-h}h^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu G_{\mu\nu}, \tag{4.13}$$

where $h_{\alpha\beta}$ is some symmetric invertible 2×2 matrix with inverse $h^{\alpha\beta}$ and determinant $h \equiv \det(h_{\alpha\beta})$. This Polyakov string action shall now be shown to be equivalent to the Nambu-Goto string action following [4, 52].

Let us first vary the Polyakov string action (4.13) with respect to $h^{\alpha\beta}$ to obtain

$$\delta S = -\frac{1}{2\alpha'}\int d\tau \frac{d\sigma}{2\pi}\left\{\sqrt{-h}\ \partial_\alpha x^\mu \partial_\beta x^\nu G_{\mu\nu}\delta h^{\alpha\beta} + h^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu G_{\mu\nu}\delta\sqrt{-h}\right\}. \tag{4.14}$$

We shall now make use of the identity[2] $\delta(\det A) = (\det A)\,\mathrm{Tr}\,(A^{-1}\delta A)$, which is valid for any invertible 2×2 matrix A, to calculate the variation of the determinant h as follows:

$$\delta h = hh^{\alpha\beta}\delta h_{\alpha\beta} = -hh_{\alpha\beta}\delta h^{\alpha\beta}, \tag{4.15}$$

since $\delta(h^{\alpha\beta}h_{\alpha\beta}) = \delta(2) = 0$, so that $h^{\alpha\beta}\delta h_{\alpha\beta} = -h_{\alpha\beta}\delta h^{\alpha\beta}$. Thus we find that

$$\delta\sqrt{-h} = -\frac{1}{2}\frac{\delta h}{\sqrt{-h}} = -\frac{1}{2}\frac{1}{\sqrt{-h}}\left(-hh_{\alpha\beta}\delta h^{\alpha\beta}\right) = -\frac{1}{2}\sqrt{-h}h_{\alpha\beta}\delta h^{\alpha\beta}. \tag{4.16}$$

The variation (4.14) of the Polyakov string action therefore becomes

$$\delta S = -\frac{1}{2\alpha'}\int d\tau \frac{d\sigma}{2\pi}\sqrt{-h}\left\{\partial_\alpha x^\mu \partial_\beta x^\nu G_{\mu\nu} - \frac{1}{2}h_{\alpha\beta}h^{\delta\varepsilon}\partial_\delta x^\mu \partial_\varepsilon x^\nu G_{\mu\nu}\right\}\delta h^{\alpha\beta}, \tag{4.17}$$

[2]This identity can be proved as follows:

Consider some matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}$ with inverse $A^{-1} = \frac{1}{\det A}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11}\end{pmatrix}$. Hence

$$\mathrm{Tr}\left(A^{-1}\delta A\right) = \mathrm{Tr}\left[\frac{1}{\det A}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11}\end{pmatrix}\begin{pmatrix} \delta a_{11} & \delta a_{12} \\ \delta a_{21} & \delta a_{22}\end{pmatrix}\right] = \frac{1}{\det A}\left(a_{22}\delta a_{11} - a_{12}\delta a_{21} + a_{11}\delta a_{22} - a_{21}\delta a_{12}\right),$$

and thus it follows that

$$(\det A)\ \mathrm{Tr}\left(A^{-1}\delta A\right) = (\delta a_{11})\,a_{22} + a_{11}\,(\delta a_{22}) - (\delta a_{12})\,a_{21} - a_{12}\,(\delta a_{21}) = \delta\left(a_{11}a_{22} - a_{12}a_{21}\right) = \delta\left(\det A\right).$$

and, setting this variation to zero, we obtain the equation of motion corresponding to the matrix $h^{\alpha\beta}$ as follows:

$$\partial_\alpha x^\mu \partial_\beta x^\nu G_{\mu\nu} = \frac{1}{2} h_{\alpha\beta} h^{\delta\varepsilon} \partial_\delta x^\mu \partial_\varepsilon x^\nu G_{\mu\nu}. \tag{4.18}$$

Now this equation of motion implies that $h_{\alpha\beta}$ and the induced metric $\gamma_{\alpha\beta} = \partial_\alpha x^\mu \partial_\beta x^\nu G_{\mu\nu}$ are conformally related, so that

$$h_{\alpha\beta} = f\,(\tau, \sigma)\,\gamma_{\alpha\beta}, \tag{4.19}$$

where $f\,(\tau, \sigma)$ is a proportionality constant, which is a function of the worldsheet coordinates τ and σ, and is assumed to be positive at every point on the worldsheet. Thus we find that[3]

$$\sqrt{-h}h^{\alpha\beta} = \sqrt{-f^2\gamma}\,\frac{1}{f}\gamma^{\alpha\beta} = \sqrt{-\gamma}\gamma^{\alpha\beta}. \tag{4.20}$$

The Polyakov string action (4.13) can therefore be written as

$$S = -\frac{1}{2\alpha'}\int d\tau \frac{d\sigma}{2\pi}\sqrt{-\gamma}\;\gamma^{\alpha\beta}\gamma_{\alpha\beta} = -\frac{1}{\alpha'}\int d\tau \frac{d\sigma}{2\pi}\sqrt{-\gamma}, \tag{4.21}$$

in terms of the induced metric $\gamma_{\alpha\beta}$, and is thus equivalent to the Nambu-Goto string action (4.12). Notice that the Polyakov string action (4.13) is clearly invariant under the Weyl transformation $h_{\alpha\beta} \to \Omega^2(\tau, \sigma) h_{\alpha\beta}$, where $\Omega(\tau, \sigma)$ is some real function of the worldsheet coordinates τ and σ.

4.2 Strings in an $AdS_5 \times S^5$ Background

The string theory in our AdS/CFT correspondence involves an $AdS_5 \times S^5$ target space. This is the product of five dimensional anti-de Sitter spacetime with that of a five-sphere. We are especially interested in strings stationary in the AdS_5 spacetime and therefore, after an initial description of both the AdS_5 and S^5 spaces, we shall confine our discussion to strings moving in an $\mathbb{R} \times S^5$ background. The Polyakov string worldsheet action is constructed and the $U(1)$ charges (angular momenta) corresponding to rotations on the five-sphere are derived.

[3]Notice that, if we had not chosen a positive proportionality constant $f\,(\tau, \sigma)$, this equation would contain a troublesome factor of sign (f).

4.2.1 $AdS_5 \times S^5$ Background

We now give a detailed description of anti-de Sitter spacetime and the five-sphere space, which can be viewed as five dimensional surfaces embedded in six dimensional flat spacetime and Euclidean space respectively. We define suitable sets of coordinates for the higher dimensional spaces and hence construct the AdS_5 and S^5 spacetime intervals confined to these surfaces. The anti-de Sitter and five-sphere metrics are clearly visible from this construction.

Anti-de Sitter spacetime

Anti-de Sitter spacetime in D dimensions (AdS_D) can be thought of as a hyperboloid embedded in a flat $D+1$ dimensional spacetime with metric $\eta = \mathrm{diag}\,(-1,-1,1,\ldots,1)$ and coordinates $X_{-1}, X_0, \ldots, X_{D-1}$. This hyperboloid satisfies

$$-X_{-1}^2 - X_0^2 + X_1^2 + \ldots + X_{D-2}^2 + X_{D-1}^2 = -R^2, \tag{4.22}$$

where R is the 'radius' of the anti-de Sitter spacetime. We shall now calculate the anti-de Sitter spacetime interval based on discussions in [3].

Let us first define a more convenient set of coordinates, which describe the higher dimensional spacetime, as follows:

$$x_\alpha \equiv \frac{X_\alpha R}{U}, \qquad U \equiv X_{-1} + X_{D-1}, \qquad V \equiv X_{-1} - X_{D-1}, \tag{4.23}$$

where α runs from 0 to $D-2$. The constraint equation (4.22) for the hyperboloid can then be written as

$$-UV + \frac{U^2}{R^2}\, x^\alpha x_\alpha = -R^2 \quad \Longrightarrow \quad V = \frac{U}{R^2}\, x^\alpha x_\alpha + \frac{R^2}{U}. \tag{4.24}$$

We shall now derive the spacetime interval confined to the hyperboloid in these new coordinates (4.23). The spacetime interval of the flat $D+1$ dimensional background in the original coordinates is given by

$$ds^2 = -dX_{-1}^2 + dX_{D-1}^2 + dX^\alpha dX_\alpha, \tag{4.25}$$

while we can also calculate

$$dX^\alpha dX_\alpha = d\left(\frac{Ux^\alpha}{R}\right) d\left(\frac{Ux_\alpha}{R}\right) = \frac{1}{R^2}\left(x^\alpha x_\alpha dU^2 + 2UdUx^\alpha dx_\alpha + U^2 dx^\alpha dx_\alpha\right) \tag{4.26}$$

and

$$-dX_{-1}^2 + dX_{D-1}^2 = -dUdV = -dUd\left(\frac{Ux^\alpha x_\alpha}{R^2} + \frac{R^2}{U}\right)$$
$$= -\frac{x^\alpha x_\alpha dU^2}{R^2} - \frac{2UdUx^\alpha dx_\alpha}{R^2} + \frac{R^2 dU^2}{U^2}. \tag{4.27}$$

Here we have made use of the hyperboloid constraint (4.24). The spacetime interval on the hyperboloid is thus

$$ds^2 = -\frac{U^2}{R^2}\left(dx^0\right)^2 + \frac{U^2}{R^2}\left(dx^1\right)^2 + \ldots + \frac{U^2}{R^2}\left(dx^{D-2}\right)^2 + \frac{R^2}{U^2}\,dU^2. \tag{4.28}$$

We now make one last redefinition $\tilde{U} = \frac{U}{R^2}$, so that we can express the 'radius' R as an overall scale factor as follows:

$$ds^2 = R^2\left[-\tilde{U}^2\left(dx^0\right)^2 + \tilde{U}^2\left(dx^1\right)^2 + \ldots + \tilde{U}^2\left(dx^{D-2}\right)^2 + \frac{d\tilde{U}^2}{\tilde{U}^2}\right], \tag{4.29}$$

which is the spacetime interval of our D dimensional anti-de Sitter spacetime in terms of the coordinates x^α and $\tilde{U}$. We are especially interested in the AdS_5 spacetime interval, which can be obtained by setting $D = 5$.

Lastly, we should mention that strings which are stationary in this AdS_5 spacetime are described only by a temporal coordinate $t = \tilde{U}x^0$. The other four coordinates x^1, x^2, x^3 and $\tilde{U}$ are constant. The spacetime interval then becomes $ds^2 = -R^2 dt^2$.

Five-sphere space

The five-sphere (S^5) space is that of a five dimensional sphere of radius R embedded in six dimensional Euclidean space. Our Euclidean coordinates x_i satisfy

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 = R^2. \tag{4.30}$$

We shall now write these six Euclidean coordinates as three sets of polar coordinates (r_i, ϕ_i) as follows:

$$x_i = Rr_i\cos\phi_i, \qquad x_{i+3} = Rr_i\sin\phi_i, \qquad \text{with} \;\; i \,\epsilon\, \{1,2,3\}\,. \tag{4.31}$$

Thus we find that $x_i^2 + x_{i+3}^2 = R^2 r_i^2$, so that the constraint (4.30) can be written in terms of the three radii r_i as

$$r_1^2 + r_2^2 + r_3^2 = 1. \tag{4.32}$$

The Euclidean spatial interval is now given by

$$ds^2 = \sum_{i=1}^{6} dx_i^2 = R^2 \sum_{i=1}^{3} \left(dr_i^2 + r_i^2 d\phi_i^2 \right), \tag{4.33}$$

and, when the constraint (4.32) is implemented, this also describes the spatial interval of our five-sphere space.

4.2.2 String worldsheet action in an $\mathbb{R} \times S^5$ background

We shall now construct the classical worldsheet action for a string in an $\mathbb{R} \times S^5$ background, which is stationary in the AdS_5 space and moves only on the five-sphere.

The only contribution to the worldsheet action from the AdS_5 spacetime involves the temporal variable t, since the other four coordinates have been assumed to be constant. As previously mentioned, the AdS_5 spacetime interval is then $ds^2 = -R^2 dt^2$. The spacetime interval on the five-sphere is shown in (4.33). We should also note that the radii of the AdS_5 spacetime and five-sphere space are taken to be the same. Substituting these results into the Polyakov action (4.13) then allows us to construct the string worldsheet action as follows:

$$\begin{aligned} \tilde{\tilde{S}} = -\frac{\sqrt{\lambda}}{2} \int d\tau \, \frac{d\sigma}{2\pi} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.34) \\ \times \left\{ \sqrt{-h} h^{\alpha\beta} \left[-\partial_\alpha t \partial_\beta t + \sum_{i=1}^{3} \left(\partial_\alpha r_i \partial_\beta r_i + r_i^2 \partial_\alpha \tilde{\tilde{\phi}}_i \partial_\beta \tilde{\tilde{\phi}}_i \right) \right] + \Lambda \left(\sum_{i=1}^{3} r_i^2 - 1 \right) \right\}. \end{aligned}$$

Here we have defined $\sqrt{\lambda} \equiv \frac{R^2}{\alpha'}$ as in [8]. We now also make use of the notation $\tilde{\tilde{\phi}}_i$ of [8] to describe the three angular coordinates, because it will later be useful to distinguish between the angular coordinates in undeformed and deformed $\mathbb{R} \times S^5$ backgrounds. The last term is a constraint term with corresponding Lagrange multiplier Λ, which ensures that the sum of the three S^5 radii squared is equal to one, so that we remain confined to the five-sphere.

4.2.3 $U(1)$ charges or angular momenta

The string worldsheet action (4.34) is clearly invariant under constant shifts of the angular coordinates $\tilde{\tilde{\phi}}_i \to \tilde{\tilde{\phi}}_i + \varepsilon_i$, which describe $U(1)$ transformations $R r_i e^{i\tilde{\tilde{\phi}}_i} \to$

$e^{i\varepsilon_i}\left(Rr_ie^{i\tilde{\tilde{\phi}}_i}\right)$ or rotations on our five-sphere. We shall now calculate the corresponding charge and current densities $\tilde{\tilde{p}}_i$ and $\tilde{\tilde{j}}_i$ respectively. We can hence integrate $\tilde{\tilde{p}}_i$ over the spatial worldsheet coordinate to obtain the $U(1)$ charges $\tilde{\tilde{J}}_i$, which are the angular momenta of a string moving in the S^5 space.

Consider an infinitesimal $U(1)$ transformation $\tilde{\tilde{\phi}}_i \longrightarrow \tilde{\tilde{\phi}}_i + \varepsilon_i$ of the i^{th} angular coordinate, with ε_i some small constant parameter. We see that

$$\mathcal{D}_i\tilde{\tilde{\mathcal{L}}} \equiv \left.\frac{\partial\tilde{\tilde{L}}}{\partial\varepsilon_i}\right|_{\varepsilon_i=0} = 0 \qquad \text{and} \qquad \mathcal{D}_i\tilde{\tilde{\phi}}_j \equiv \left.\frac{\partial\tilde{\tilde{\phi}}_j}{\partial\varepsilon_i}\right|_{\varepsilon_i=0} = \delta^i_j, \tag{4.35}$$

and thus, using the results of [38], we can calculate the i^{th} conserved 2-current as follows:

$$\tilde{\tilde{\mathcal{J}}}^{\alpha}_i = \sum_{j=1}^{3}\tilde{\tilde{\Pi}}^{\alpha}_j\mathcal{D}_i\tilde{\tilde{\phi}}_j = \tilde{\tilde{\Pi}}^{\alpha}_i, \qquad \text{with} \quad \tilde{\tilde{\Pi}}^{\alpha}_j \equiv \frac{\partial\tilde{\tilde{\mathcal{L}}}}{\partial\left(\partial_\alpha\tilde{\tilde{\phi}}_i\right)}, \tag{4.36}$$

so that

$$\tilde{\tilde{\mathcal{J}}}^{\alpha}_i = -\sqrt{\lambda}r_i^2\sqrt{-h}h^{\alpha\beta}\partial_\beta\tilde{\tilde{\phi}}_i. \tag{4.37}$$

These three 2-currents satisfy the conservation equations $\partial_\alpha\tilde{\tilde{\mathcal{J}}}^{\alpha}_i = 0$. The i^{th} charge and current densities are $\tilde{\tilde{p}}_i = \tilde{\tilde{\mathcal{J}}}^0_i$ and $\tilde{\tilde{j}}_i = \tilde{\tilde{\mathcal{J}}}^1_i$ respectively. Hence we finally obtain the $U(1)$ charges or angular momenta

$$J_i \equiv \int\frac{d\sigma}{2\pi}\,\tilde{\tilde{p}}_i = -\sqrt{\lambda}\int\frac{d\sigma}{2\pi}\left(r_i^2\sqrt{-h}h^{0\beta}\partial_\beta\tilde{\tilde{\phi}}_i\right), \tag{4.38}$$

where we have integrated the i^{th} charge density over the spatial worldsheet coordinate. The above results for the $U(1)$ charge and current densities, and the $U(1)$ charges agree with the expressions given in [8].

4.3 Strings in the Lunin-Maldacena Background

The task of finding string theories dual to deformations of $\mathcal{N}=4$ SYM theory is highly non-trivial. Lunin and Maldacena [7] showed this to be possible for certain Leigh-Strassler deformations, in which the superpotential is deformed as follows:

$$\frac{1}{2}g\text{Tr}\left(\Phi_1\Phi_2\Phi_3 - \Phi_1\Phi_3\Phi_2\right) \longrightarrow \frac{1}{2}g\text{Tr}\left(e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2\right), \tag{4.39}$$

where β is a complex parameter. The gravity dual of this $\mathcal{N}=1$ SYM theory is a string theory in a β-deformed Lunin-Maldacena background. We shall henceforth consider only the case of a real deformation parameter $\beta=\gamma$.

Let us first discuss how this γ-deformed string theory was originally constructed in [7]. The superpotential (4.39) is invariant under the $U(1)$ transformations:

$$U(1)_1: \quad \Phi_1 \to e^{i\alpha_1}\Phi_1, \qquad \Phi_2 \to e^{i\alpha_1}\Phi_2, \qquad \Phi_3 \to e^{-2i\alpha_1}\Phi_3, \tag{4.40}$$

$$U(1)_2: \quad \Phi_1 \to e^{-2i\alpha_2}\Phi_1, \qquad \Phi_2 \to e^{i\alpha_2}\Phi_2, \qquad \Phi_3 \to e^{i\alpha_2}\Phi_3. \tag{4.41}$$

On the string theory side, these $U(1)$ transformations correspond to the transformations $\tilde{\tilde{\varphi}}_1 \to \tilde{\tilde{\varphi}}_1 + \alpha_1$ and $\tilde{\tilde{\varphi}}_2 \to \tilde{\tilde{\varphi}}_2 + \alpha_2$ of the angular coordinates

$$\tilde{\tilde{\varphi}}_1 = \frac{1}{3}\left(\tilde{\tilde{\phi}}_1 + \tilde{\tilde{\phi}}_2 - 2\tilde{\tilde{\phi}}_3\right) \qquad \text{and} \qquad \tilde{\tilde{\varphi}}_2 = \frac{1}{3}\left(-2\tilde{\tilde{\phi}}_1 + \tilde{\tilde{\phi}}_2 + \tilde{\tilde{\phi}}_3\right), \tag{4.42}$$

which, together with the total angular coordinate $\tilde{\tilde{\psi}} = \frac{1}{3}\left(\tilde{\tilde{\phi}}_1 + \tilde{\tilde{\phi}}_2 + \tilde{\tilde{\phi}}_3\right)$, form an alternative set of angular coordinates describing our S^5 space. We thus observe that $\tilde{\tilde{\varphi}}_1$ and $\tilde{\tilde{\varphi}}_2$ define a special 2-torus $\left(\tilde{\tilde{\varphi}}_1, \tilde{\tilde{\varphi}}_2\right)$ on our five-sphere.

Let us now define a parameter τ, which describes the structure of the torus, as follows:

$$\tau = B_{12} + i\sqrt{g}, \tag{4.43}$$

where g is the determinant of the metric confined to the torus (so that $\sqrt{g}$ is the torus volume), and B_{12} is the B-field or coefficient of the $\partial_\alpha\tilde{\tilde{\phi}}_1\partial_\beta\tilde{\tilde{\phi}}_2$ term in the string action. The γ-deformation of the string theory is then implemented by making the transformation

$$\tau \longrightarrow \tau' = \frac{\tau}{1+\hat{\gamma}\tau}, \qquad \text{with} \quad \gamma = \frac{\hat{\gamma}}{\sqrt{\lambda}}. \tag{4.44}$$

This alters the volume of the torus and turns on a B-field in the string action.

It was shown by Frolov [8] that the γ-deformed worldsheet string action can also be obtained by performing a TsT-transformation on the original string worldsheet action (4.34). We discuss this perspective in detail in this section.

4.3.1 T-duality transformation

We shall begin by describing the notion of a T-duality transformation based on discussions in [8, 53]. Let us consider a general string action of the form

$$S = -\frac{\sqrt{\lambda}}{2}\int d\tau\,\frac{d\sigma}{2\pi}\left[\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N G_{MN}\left(X^i\right) - \varepsilon^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N B_{MN}\left(X^i\right)\right], \tag{4.45}$$

where $M, N \,\epsilon\, \{1,2,3\}$ and $i, j \,\epsilon\, \{2,3\}$, and $\varepsilon^{\alpha\beta}$ is defined in (A.22). Here G_{MN} is the symmetric metric of the background spacetime and B_{MN} is an anti-symmetric matrix. We shall assume that both G_{MN} and B_{MN} are independent of X^1.

Now this string action (4.45) turns out to be equivalent to a more complicated expression dependent on a new coordinate p^α, which is given by

$$\begin{aligned}S = -\sqrt{\lambda}\int d\tau\,\frac{d\sigma}{2\pi}\Bigg[&p^\alpha\left(\partial_\alpha X^M\frac{G_{1M}}{G_{11}} - \frac{h_{\alpha\beta}}{\sqrt{-h}}\varepsilon^{\beta\rho}\partial_\rho X^M\frac{B_{1M}}{G_{11}}\right)\\ &-\frac{1}{2G_{11}}\frac{h_{\alpha\beta}}{\sqrt{-h}}p^\alpha p^\beta + \frac{1}{2}\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j\left(G_{ij} - \frac{G_{1i}G_{1j} - B_{1i}B_{1j}}{G_{11}}\right)\\ &-\frac{1}{2}\,\varepsilon^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j\left(B_{ij} - \frac{G_{1i}B_{1j} - B_{1i}G_{1j}}{G_{11}}\right)\Bigg]\,. \end{aligned} \tag{4.46}$$

This shall be demonstrated by varying the new action (4.46) with respect to p^α to obtain the equation of motion

$$p^\alpha = \sqrt{-h}h^{\alpha\beta}\partial_\beta X^M G_{1M} - \varepsilon^{\alpha\beta}\partial_\beta X^M B_{1M}, \tag{4.47}$$

from which it follows that

$$\begin{aligned}&p^\alpha\left(\partial_\alpha X^M\frac{G_{1M}}{G_{11}} - \frac{h_{\alpha\beta}}{\sqrt{-h}}\varepsilon^{\beta\rho}\partial_\rho X^M\frac{B_{1M}}{G_{11}}\right)\\ &= \left(\sqrt{-h}h^{\alpha\delta}\partial_\delta X^M G_{1M} - \varepsilon^{\alpha\delta}\partial_\delta X^M B_{1M}\right)\left(\partial_\alpha X^N\frac{G_{1N}}{G_{11}} - \frac{h_{\alpha\beta}}{\sqrt{-h}}\varepsilon^{\beta\rho}\partial_\rho X^N\frac{B_{1N}}{G_{11}}\right)\\ &= \sqrt{-h}h^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N\left(\frac{G_{1M}G_{1N} - B_{1M}B_{1N}}{G_{11}}\right) - \varepsilon^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N\left(\frac{G_{1M}B_{1N} - B_{1M}G_{1N}}{G_{11}}\right),\end{aligned} \tag{4.48}$$

and also

$$\begin{aligned}&\frac{1}{2G_{11}}\frac{h_{\alpha\beta}}{\sqrt{-h}}p^\alpha p^\beta\\ &= \frac{1}{2G_{11}}\frac{h_{\alpha\beta}}{\sqrt{-h}}\left(\sqrt{-h}h^{\alpha\rho}\partial_\rho X^M G_{1M} - \varepsilon^{\alpha\rho}\partial_\rho X^M B_{1M}\right)\left(\sqrt{-h}h^{\beta\delta}\partial_\delta X^N G_{1N} - \varepsilon^{\beta\delta}\partial_\delta X^N B_{1N}\right)\\ &= \sqrt{-h}h^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N\left(\frac{G_{1M}G_{1N} - B_{1M}B_{1N}}{2G_{11}}\right) - \varepsilon^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N\left(\frac{G_{1M}B_{1N} - B_{1M}G_{1N}}{2G_{11}}\right).\end{aligned} \tag{4.49}$$

Here we have used the identity $\varepsilon^{\alpha\beta}\frac{h_{\beta\rho}}{\sqrt{-h}}\varepsilon^{\rho\delta} = \sqrt{-h}h^{\alpha\delta}$, which can be verified by writing out each side of the equation explicitly for all combinations of $\alpha, \delta \,\epsilon\, \{0,1\}$.

Substituting (4.48) and (4.49) into the new string action (4.46) now yields

$$
\begin{aligned}
S = -\frac{\sqrt{\lambda}}{2}\int d\tau\, \frac{d\sigma}{2\pi}\Bigg\{\sqrt{-h}h^{\alpha\beta}\Bigg[&\left(\frac{G_{1M}G_{1N}-B_{1M}B_{1N}}{G_{11}}\right)\partial_\alpha X^M\partial_\beta X^N \\
&+\left(G_{ij}-\frac{G_{1i}G_{1j}-B_{1i}B_{1j}}{G_{11}}\right)\partial_\alpha X^i\partial_\beta X^j\Bigg] \\
-\,\varepsilon^{\alpha\beta}\Bigg[&\left(\frac{G_{1M}B_{1N}-B_{1M}G_{1N}}{G_{11}}\right)\partial_\alpha X^M\partial_\beta X^N \\
&+\left(B_{ij}-\frac{G_{1i}B_{1j}-B_{1i}G_{1j}}{G_{11}}\right)\partial_\alpha X^i\partial_\beta X^j\Bigg]\Bigg\}, \qquad (4.50)
\end{aligned}
$$

When $M, N \neq 1$ we see that each of the two expressions in square brackets simplify to $G_{ij}\partial_\alpha X^i\partial_\beta X^j$ and $B_{ij}\partial_\alpha X^i\partial_\beta X^j$ respectively. The three cases $M = N = 1$, $M = 1$ and $N = i$, and $M = i$ and $N = 1$ can then be considered separately and similar results derived. We thus see that (4.46) reduces to the original string worldsheet action (4.45).

Let us consider varying the action (4.46) with respect to the first coordinate X^1. Since we have assumed that G_{MN} and B_{MN} are independent of X^1, so that there is no hidden dependence on the first coordinate, the equation of motion is simply

$$\partial_\alpha p^\alpha = \partial_\alpha\left(\sqrt{-h}h^{\alpha\beta}\partial_\beta X^M G_{1M} - \varepsilon^{\alpha\beta}\partial_\beta X^M B_{1M}\right) = 0. \qquad (4.51)$$

Thus we can generally write $p^\alpha = \varepsilon^{\alpha\beta}\partial_\beta\tilde{X}^1$, where $\tilde{X}^1$ is the first T-dual coordinate. The T-dual coordinates $\tilde{X}^M$ are therefore defined as follows:

$$
\begin{aligned}
&\varepsilon^{\alpha\beta}\partial_\beta\tilde{X}^1 \equiv \sqrt{-h}h^{\alpha\beta}\partial_\beta X^M G_{1M} - \varepsilon^{\alpha\beta}\partial_\beta X^M B_{1M}, \qquad (4.52)\\
&\tilde{X}^i \equiv X^i. \qquad (4.53)
\end{aligned}
$$

Substituting these T-dual coordinates into (4.46), we obtain

$$
\begin{aligned}
\tilde{S} = -\sqrt{\lambda}\int d\tau\,\frac{d\sigma}{2\pi}\Bigg[&\left(\varepsilon^{\alpha\delta}\partial_\delta\tilde{X}^1\right)\left(\partial_\alpha X^1 + \partial_\alpha\tilde{X}^i\frac{G_{1i}}{G_{11}} - \frac{h_{\alpha\beta}}{\sqrt{-h}}\varepsilon^{\beta\rho}\partial_\rho\tilde{X}^i\frac{B_{1i}}{G_{11}}\right) \\
&-\frac{1}{2G_{11}}\frac{h_{\alpha\beta}}{\sqrt{-h}}\left(\varepsilon^{\alpha\rho}\partial_\rho\tilde{X}^1\right)\left(\varepsilon^{\beta\delta}\partial_\delta\tilde{X}^1\right) \\
&+\frac{1}{2}\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j\left(G_{ij}-\frac{G_{1i}G_{1j}-B_{1i}B_{1j}}{G_{11}}\right) \\
&-\frac{1}{2}\,\varepsilon^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j\left(B_{ij}-\frac{G_{1i}B_{1j}-B_{1i}G_{1j}}{G_{11}}\right)\Bigg], \qquad (4.54)
\end{aligned}
$$

and, since $\varepsilon^{\alpha\delta}\partial_\delta\tilde{X}^1\partial_\alpha X^1 \sim \varepsilon^{\alpha\delta}\partial_\alpha\partial_\delta\tilde{X}^1 X^1 = 0$ up to a total derivative, the above expression can be written as

$$\tilde{S} = -\frac{\sqrt{\lambda}}{2}\int d\tau\,\frac{d\sigma}{2\pi}\Bigg\{\sqrt{-h}h^{\alpha\beta}\Bigg[\partial_\alpha\tilde{X}^1\partial_\beta\tilde{X}^1\frac{1}{G_{11}} + 2\partial_\alpha\tilde{X}^1\partial_\beta\tilde{X}^i\frac{B_{1i}}{G_{11}} + \partial_\alpha\tilde{X}^i\partial_\beta\tilde{X}^j\left(G_{ij} - \frac{G_{1i}G_{1j} - B_{1i}B_{1j}}{G_{11}}\right)\Bigg] - \varepsilon^{\alpha\beta}\left[2\partial_\alpha\tilde{X}^1\partial_\beta\tilde{X}^i\frac{G_{1i}}{G_{11}} + \partial_\alpha\tilde{X}^i\partial_\beta\tilde{X}^j\left(B_{ij} - \frac{G_{1i}B_{1j} - B_{1i}G_{1j}}{G_{11}}\right)\right]\Bigg\}, \tag{4.55}$$

up to surface terms. Thus we obtain the string worldsheet action in the T-dual space

$$\tilde{S} = -\frac{\sqrt{\lambda}}{2}\int d\tau\,\frac{d\sigma}{2\pi}\left[\sqrt{-h}h^{\alpha\beta}\partial_\alpha\tilde{X}^M\partial_\beta\tilde{X}^N\tilde{G}_{MN} - \varepsilon^{\alpha\beta}\partial_\alpha\tilde{X}^M\partial_\beta\tilde{X}^N\tilde{B}_{MN}\right], \tag{4.56}$$

where the symmetric and anti-symmetric matrices $\tilde{G}_{MN}$ and $\tilde{B}_{MN}$ respectively are defined as follows:

$$\tilde{G}_{11} = \frac{1}{G_{11}}, \qquad \tilde{G}_{1i} = \frac{B_{1i}}{G_{11}}, \qquad \tilde{G}_{ij} = G_{ij} - \frac{G_{1i}G_{1j} - B_{1i}B_{1j}}{G_{11}},$$
$$\tilde{B}_{1i} = \frac{G_{1i}}{G_{11}}, \qquad \tilde{B}_{ij} = B_{ij} - \frac{G_{1i}B_{1j} - B_{1i}G_{1j}}{G_{11}}. \tag{4.57}$$

4.3.2 Derivation of the γ-deformed string worldsheet action via a TsT-transformation

We now show how to derive the γ-deformed string worldsheet action by performing a TsT-transformation on the original string action following [8]. This involves a T-duality transformation on the first angular coordinate $\tilde{\tilde{\varphi}}_1$, a shift $\tilde{\varphi}_2 \to \tilde{\varphi}_2 + \hat{\gamma}\tilde{\varphi}_1$ of the second angular coordinate in the T-dual space and a further T-duality transformation on the T-dual coordinate $\tilde{\varphi}_1$. Since this deformation effects only the five-sphere space, it is sufficient, for our purposes, to consider strings in an $\mathbb{R}\times S^5$ background. Thus we begin with the string action (4.34), which, making a change

$$\tilde{\tilde{\phi}}_1 = \tilde{\tilde{\varphi}}_3 - \tilde{\tilde{\varphi}}_2, \qquad \tilde{\tilde{\phi}}_2 = \tilde{\tilde{\varphi}}_1 + \tilde{\tilde{\varphi}}_2 + \tilde{\tilde{\varphi}}_3, \qquad \tilde{\tilde{\phi}}_3 = \tilde{\tilde{\varphi}}_3 - \tilde{\tilde{\varphi}}_1 \tag{4.58}$$

to the new angular coordinates $\tilde{\tilde{\varphi}}_1$, $\tilde{\tilde{\varphi}}_2$ and $\tilde{\tilde{\varphi}}_3 = \tilde{\tilde{\psi}}$ (the coordinates of our torus, together with the total angular coordinate), can be written as

$$\tilde{\tilde{S}} = -\frac{\sqrt{\lambda}}{2}\int d\tau\,\frac{d\sigma}{2\pi} \times\left[\sqrt{-h}h^{\alpha\beta}\left(-\partial_\alpha t\partial_\beta t + \sum_{i=1}^{3}\partial_\alpha r_i\partial_\beta r_i + \sum_{i,j=1}^{3} g_{ij}\partial_\alpha\tilde{\tilde{\varphi}}_i\partial_\beta\tilde{\tilde{\varphi}}_j\right) + \Lambda\left(\sum_{i=1}^{3} r_i^2 - 1\right)\right], \tag{4.59}$$

where g_{ij} is defined as follows:

$$\begin{aligned} g_{11}^2 &= r_2^2 + r_3^2, & g_{22} &= r_1^2 + r_2^2, & g_{33} &= r_1^2 + r_2^2 + r_3^2 = 1, \\ g_{12} &= r_2^2, & g_{31} &= r_2^2 - r_3^2, & g_{23} &= r_2^2 - r_1^2. \end{aligned} \tag{4.60}$$

Let us now perform a T-duality transformation on the angular coordinate $\tilde{\tilde{\varphi}}_1$. In other words, we shall set $X_1 = \tilde{\tilde{\varphi}}_1$, $X_2 = \tilde{\tilde{\varphi}}_2$ and $X_3 = \tilde{\tilde{\varphi}}_3$ in our previous discussion of T-duality transformations. The metric g_{ij} is independent of $\tilde{\tilde{\varphi}}_1$ (and, indeed, of all the angular coordinates $\tilde{\tilde{\varphi}}_i$) as assumed in this discussion. We shall use $\tilde{\varphi}_1$, $\tilde{\varphi}_2$ and $\tilde{\varphi}_3$ to represent the T-dual coordinates, which must then satisfy

$$\varepsilon^{\alpha\beta}\partial_\beta\tilde{\varphi}_1 = \sqrt{-h}h^{\alpha\beta}\sum_{i=1}^{3}\partial_\beta\tilde{\tilde{\varphi}}_i\, g_{1i}, \qquad \tilde{\varphi}_2 = \tilde{\tilde{\varphi}}_2, \qquad \tilde{\varphi}_3 = \tilde{\tilde{\varphi}}_3, \tag{4.61}$$

and, using the results (4.56) and (4.57), the string action can hence be written in the T-dual space as follows:

$$\begin{aligned} \tilde{S} = -\frac{\sqrt{\lambda}}{2}\int d\tau\,\frac{d\sigma}{2\pi}\Bigg[\sqrt{-h}h^{\alpha\beta}\left(-\partial_\alpha t\partial_\beta t + \sum_{i=1}^{3}\partial_\alpha r_i\partial_\beta r_i + \sum_{i,j=1}^{3}\tilde{g}_{ij}\partial_\alpha\tilde{\varphi}_i\partial_\beta\tilde{\varphi}_j\right) \\ - \varepsilon^{\alpha\beta}\left(\sum_{i,j=1}^{3}\tilde{b}_{ij}\partial_\alpha\tilde{\varphi}_i\partial_\beta\tilde{\varphi}_j\right) + \Lambda\left(\sum_{i=1}^{3}r_i^2 - 1\right)\Bigg], \end{aligned} \tag{4.62}$$

where the symmetric and anti-symmetric matrices $\tilde{g}_{ij}$ and $\tilde{b}_{ij}$ respectively are

$$\begin{aligned} \tilde{g}_{11} &= \frac{1}{r_2^2 + r_3^2}, \quad \tilde{g}_{22} = \frac{r_1^2 r_2^2 + r_3^2 r_1^2 + r_2^2 r_3^2}{r_2^2 + r_3^2}, \quad \tilde{g}_{33} = \frac{r_1^2 r_2^2 + r_3^2 r_1^2 + 4r_2^2 r_3^2}{r_2^2 + r_3^2}, \\ \tilde{g}_{12} &= 0, \qquad \tilde{g}_{31} = 0, \qquad \tilde{g}_{23} = \frac{-r_1^2 r_2^2 - r_3^2 r_1^2 + 2r_2^2 r_3^2}{r_2^2 + r_3^2}. \end{aligned} \tag{4.63}$$

and

$$\tilde{b}_{12} = \frac{r_2^2}{r_2^2 + r_3^2}, \qquad \tilde{b}_{31} = \frac{r_2^2 - r_3^2}{r_2^2 + r_3^2}, \qquad \tilde{b}_{23} = 0. \tag{4.64}$$

The real parameter $\hat{\gamma}$ shall now be introduced by shifting the T-dual coordinates

$$\tilde{\varphi}_1 \longrightarrow \tilde{\varphi}_1, \qquad \tilde{\varphi}_2 \longrightarrow \tilde{\varphi}_2 + \hat{\gamma}\tilde{\varphi}_1, \qquad \tilde{\varphi}_3 \longrightarrow \tilde{\varphi}_3, \tag{4.65}$$

which causes a change $\tilde{g}_{ij} \longrightarrow \tilde{G}_{ij}$ in the T-dual metric as follows:

$$\begin{aligned} \tilde{G}_{11} &= \tilde{g}_{11} + \hat{\gamma}\left(\tilde{g}_{12} + \tilde{g}_{21}\right) + \hat{\gamma}^2\tilde{g}_{22}, & \tilde{G}_{22} &= \tilde{g}_{22}, & \tilde{G}_{33} &= \tilde{g}_{33}, \\ \tilde{G}_{12} &= \tilde{g}_{12} + \hat{\gamma}\tilde{g}_{22}, & \tilde{G}_{31} &= \tilde{g}_{31} + \hat{\gamma}\tilde{g}_{23}, & \tilde{G}_{23} &= \tilde{g}_{23}, \end{aligned} \tag{4.66}$$

The anti-symmetric matrix $\tilde{b}_{ij}$ can be seen to remain unchanged by this shift. Hence we obtain the shifted string action in the T-dual space, which is given by

$$\tilde{S}^\gamma = -\frac{\sqrt{\lambda}}{2}\int d\tau\,\frac{d\sigma}{2\pi}\left[\sqrt{-h}h^{\alpha\beta}\left(-\partial_\alpha t\partial_\beta t + \sum_{i=1}^{3}\partial_\alpha r_i\partial_\beta r_i + \sum_{i,j=1}^{3}\tilde{G}_{ij}\partial_\alpha\tilde{\varphi}_i\partial_\beta\tilde{\varphi}_j\right)\right.$$
$$\left. - \varepsilon^{\alpha\beta}\left(\sum_{i,j=1}^{3}\tilde{B}_{ij}\partial_\alpha\tilde{\varphi}_i\partial_\beta\tilde{\varphi}_j\right) + \Lambda\left(\sum_{i=1}^{3}r_i^2 - 1\right)\right], \quad (4.67)$$

where

$$\begin{aligned}
\tilde{G}_{11} &= \frac{G^{-1}}{r_2^2+r_3^2}, & \tilde{G}_{22} &= \frac{r_1^2r_2^2+r_3^2r_1^2+r_2^2r_3^2}{r_2^2+r_3^2},\\
\tilde{G}_{33} &= \frac{r_1^2r_2^2+r_3^2r_1^2+4r_2^2r_3^2}{r_2^2+r_3^2}, & \tilde{G}_{12} &= \hat{\gamma}\,\frac{r_1^2r_2^2+r_3^2r_1^2+r_2^2r_3^2}{r_2^2+r_3^2},\\
\tilde{G}_{31} &= \hat{\gamma}\,\frac{-r_1^2r_2^2-r_3^2r_1^2+2r_2^2r_3^2}{r_2^2+r_3^2}, & \tilde{G}_{23} &= \frac{-r_1^2r_2^2-r_3^2r_1^2+2r_2^2r_3^2}{r_2^2+r_3^2},
\end{aligned} \quad (4.68)$$

with $G^{-1} \equiv 1 + \hat{\gamma}^2\left(r_1^2r_2^2+r_3^2r_1^2+r_2^2r_3^2\right)$, and $\tilde{B}_{ij} = \tilde{b}_{ij}$ as shown in (4.64).

Finally, we shall perform another T-duality transformation on the coordinate $\tilde{\varphi}_1$, and the new T-dual coordinates φ_1, φ_2 and φ_3 satisfy

$$\varepsilon^{\alpha\beta}\partial_\beta\varphi_1 = \sqrt{-h}h^{\alpha\beta}\partial_\beta\tilde{\varphi}_i\tilde{G}_{1i} - \varepsilon^{\alpha\beta}\partial_\beta\tilde{\varphi}_i\tilde{B}_{1i}, \qquad \varphi_2 = \tilde{\varphi}_2, \qquad \varphi_3 = \tilde{\varphi}_3. \quad (4.69)$$

The string action in this new T-dual space is given by

$$S^\gamma = -\frac{\sqrt{\lambda}}{2}\int d\tau\,\frac{d\sigma}{2\pi}\left[\sqrt{-h}h^{\alpha\beta}\left(\sum_{i=1}^{3}\partial_\alpha r_i\partial_\beta r_i + \sum_{i,j=1}^{3}G_{ij}\partial_\alpha\varphi_i\partial_\beta\varphi_j\right)\right.$$
$$\left. - \varepsilon^{\alpha\beta}\left(\sum_{i,j=1}^{3}B_{ij}\partial_\alpha\varphi_i\partial_\beta\varphi_j\right) + \Lambda\left(\sum_{i=1}^{3}r_i^2 - 1\right)\right], \quad (4.70)$$

with

$$\begin{aligned}
G_{11} &= G\left(r_2^2+r_3^2\right), & G_{22} &= G\left(r_1^2+r_2^2\right), & G_{33} &= G + 9\hat{\gamma}^2Gr_1^2r_2^2r_3^2,\\
G_{12} &= Gr_2^2, & G_{31} &= G\left(r_2^2-r_3^2\right), & G_{23} &= G\left(r_2^2-r_1^2\right),
\end{aligned} \quad (4.71)$$

and

$$\begin{aligned}
B_{12} &= \hat{\gamma}\;G\left(r_1^2r_2^2+r_3^2r_1^2+r_2^2r_3^2\right), \qquad B_{31} = \hat{\gamma}\;G\left(r_1^2r_2^2+r_3^2r_1^2-2r_2^2r_3^2\right),\\
B_{23} &= \hat{\gamma}\;G\left(-2r_1^2r_2^2+r_3^2r_1^2+r_2^2r_3^2\right).
\end{aligned} \quad (4.72)$$

Switching back to the angular coordinates ϕ_i using the transformation

$$\varphi_1 = \frac{1}{3}\left(\phi_1+\phi_2-2\phi_3\right), \quad \varphi_2 = \frac{1}{3}\left(-2\phi_1+\phi_2+\phi_3\right), \quad \varphi_3 = \frac{1}{3}\left(\phi_1+\phi_2+\phi_3\right) \quad (4.73)$$

then yields the string worldsheet action in the γ-deformed Lunin-Maldacena background, which is given by [8]

$$
\begin{aligned}
S^{\gamma} = & -\frac{\sqrt{\lambda}}{2}\int d\tau\,\frac{d\sigma}{2\pi} \\
& \times\Bigg\{\sqrt{-h}h^{\alpha\beta}\left[-\partial_{\alpha}t\partial_{\beta}t+\sum_{i=1}^{3}\left(\partial_{\alpha}r_i\partial_{\beta}r_i+Gr_i^2\partial_{\alpha}\phi_i\partial_{\beta}\phi_i\right)+\hat{\gamma}^2Gr_1^2r_2^2r_3^2\left(\sum_{i=1}^{3}\partial_{\alpha}\phi_i\right)\left(\sum_{j=1}^{3}\partial_{\beta}\phi_j\right)\right] \\
& -2\varepsilon^{\alpha\beta}\hat{\gamma}G\left(r_1^2r_2^2\partial_{\alpha}\phi_1\partial_{\beta}\phi_2+r_3^2r_1^2\partial_{\alpha}\phi_3\partial_{\beta}\phi_1+r_2^2r_3^2\partial_{\alpha}\phi_2\partial_{\beta}\phi_3\right)+\Lambda\left(\sum_{i=1}^{3}r_i^2-1\right)\Bigg\},
\end{aligned}
\tag{4.74}
$$

where $G^{-1}=1+\hat{\gamma}^2\left(r_1^2r_2^2+r_3^2r_1^2+r_2^2r_3^2\right)$. This reproduces the Lunin-Maldacena string worldsheet action of [7] in the case of a real deformation parameter.

Let us now derive a relation between the original and γ-deformed angular coordinates $\tilde{\tilde{\phi}}_i$ and ϕ_i. The transformations (4.61), (4.65) and (4.69) can be used to relate the alternative angular coordinates $\tilde{\tilde{\varphi}}_i$ and φ_i. Firstly, (4.61) implies that

$$
\partial_{\alpha}\tilde{\tilde{\varphi}}_1=\frac{h_{\alpha\beta}}{\sqrt{-h}}\varepsilon^{\beta\rho}\partial_{\rho}\tilde{\varphi}_1\tilde{g}_{11}-\sum_{i=1}^{3}\partial_{\alpha}\tilde{\varphi}_i\tilde{b}_{1i},\qquad \partial_{\alpha}\tilde{\tilde{\varphi}}_2=\partial_{\alpha}\tilde{\varphi}_2,\qquad \partial_{\alpha}\tilde{\tilde{\varphi}}_3=\partial_{\alpha}\tilde{\varphi}_3, \tag{4.75}
$$

which, taking into account our shift by $\hat{\gamma}$ shown in (4.65), becomes

$$
\partial_{\alpha}\tilde{\tilde{\varphi}}_1=\frac{h_{\alpha\beta}}{\sqrt{-h}}\varepsilon^{\beta\rho}\partial_{\rho}\tilde{\varphi}_1\tilde{g}_{11}-\sum_{i=1}^{3}\partial_{\alpha}\tilde{\varphi}_i\tilde{b}_{1i}-\hat{\gamma}\partial_{\alpha}\tilde{\varphi}_1\tilde{b}_{12},\quad \partial_{\alpha}\tilde{\tilde{\varphi}}_2=\partial_{\alpha}\tilde{\varphi}_2+\hat{\gamma}\partial_{\alpha}\tilde{\varphi}_1,\quad \partial_{\alpha}\tilde{\tilde{\varphi}}_3=\partial_{\alpha}\tilde{\varphi}_3, \tag{4.76}
$$

and also, from (4.69), it follows that

$$
\partial_{\alpha}\tilde{\varphi}_1=\frac{h_{\alpha\beta}}{\sqrt{-h}}\varepsilon^{\beta\rho}\sum_{i=1}^{3}\partial_{\rho}\varphi_iG_{1i}-\sum_{i=1}^{3}\partial_{\alpha}\varphi_iB_{1i},\quad \partial_{\alpha}\tilde{\varphi}_2=\partial_{\alpha}\varphi_2,\quad \partial_{\alpha}\tilde{\varphi}_3=\partial_{\alpha}\varphi_3. \tag{4.77}
$$

Thus, making use of the above relations (4.76) and (4.77), we obtain

$$
\begin{aligned}
&\partial_{\alpha}\tilde{\tilde{\varphi}}_1=\sum_{i=1}^{3}\left(G_{1i}\tilde{g}_{11}+\hat{\gamma}B_{1i}\tilde{b}_{12}-\tilde{b}_{1i}\right)\partial_{\alpha}\varphi_i-\sum_{i=1}^{3}\left(B_{1i}\tilde{g}_{11}+\hat{\gamma}G_{1i}\tilde{b}_{12}\right)\frac{h_{\alpha\beta}}{\sqrt{-h}}\varepsilon^{\beta\rho}\partial_{\rho}\varphi_i,\\
&\partial_{\alpha}\tilde{\tilde{\varphi}}_2=\partial_{\alpha}\varphi_2+\hat{\gamma}\sum_{i=1}^{3}\left(-B_{1i}\partial_{\alpha}\varphi_i+G_{1i}\frac{h_{\alpha\beta}}{\sqrt{-h}}\varepsilon^{\beta\rho}\partial_{\rho}\varphi_i\right),\\
&\partial_{\alpha}\tilde{\tilde{\varphi}}_3=\partial_{\alpha}\varphi_3.
\end{aligned}
\tag{4.78}
$$

Now, changing back to our original coordinates $\tilde{\tilde{\phi}}_i$ and ϕ_i, and using the expressions (4.63), (4.64), (4.71) and (4.72) for $\tilde{g}_{ij}$, $\tilde{b}_{ij}$, G_{ij} and B_{ij} respectively, the angular coordinates in the original and γ-deformed backgrounds can be related as follows [8]:

$$\partial_\alpha \tilde{\tilde{\phi}}_1 = G\left[\partial_\alpha\phi_1 + \hat{\gamma}^2 r_2^2 r_3^2 \sum_{i=1}^{3} \partial_\alpha\phi_i - \hat{\gamma}\frac{h_{\alpha\beta}}{\sqrt{-h}}\varepsilon^{\beta\rho}\left(r_2^2\partial_\rho\phi_2 - r_3^2\partial_\rho\phi_3\right)\right],$$
$$\partial_\alpha \tilde{\tilde{\phi}}_2 = G\left[\partial_\alpha\phi_2 + \hat{\gamma}^2 r_3^2 r_1^2 \sum_{i=1}^{3} \partial_\alpha\phi_i - \hat{\gamma}\frac{h_{\alpha\beta}}{\sqrt{-h}}\varepsilon^{\beta\rho}\left(r_3^2\partial_\rho\phi_3 - r_1^2\partial_\rho\phi_1\right)\right],$$
$$\partial_\alpha \tilde{\tilde{\phi}}_3 = G\left[\partial_\alpha\phi_3 + \hat{\gamma}^2 r_1^2 r_2^2 \sum_{i=1}^{3} \partial_\alpha\phi_i - \hat{\gamma}\frac{h_{\alpha\beta}}{\sqrt{-h}}\varepsilon^{\beta\rho}\left(r_1^2\partial_\rho\phi_1 - r_2^2\partial_\rho\phi_2\right)\right]. \quad (4.79)$$

4.3.3 $U(1)$ charges or angular momenta

We can again see that the string action (4.74) in the Lunin-Maldacena background is invariant under shifts $\phi_i \to \phi_i + \varepsilon_i$ of the angular coordinates. The corresponding γ-deformed conserved $U(1)$ 2-currents are given by

$$\mathcal{J}_i^\alpha = \frac{\partial \mathcal{L}^\gamma}{\partial\left(\partial_\alpha\phi_i\right)}, \quad (4.80)$$

which can be explicitly calculated as

$$\mathcal{J}_1^\alpha = -\sqrt{\lambda} r_1^2 \sqrt{-h} h^{\alpha\delta} G\left[\partial_\delta\phi_1 + \hat{\gamma}^2 r_2^2 r_3^2 \sum_{i=1}^{3} \partial_\delta\phi_i - \hat{\gamma}\frac{h_{\delta\beta}}{\sqrt{-h}}\varepsilon^{\beta\rho}\left(r_2^2\partial_\rho\phi_2 - r_3^2\partial_\rho\phi_3\right)\right],$$
$$\mathcal{J}_2^\alpha = -\sqrt{\lambda} r_2^2 \sqrt{-h} h^{\alpha\delta} G\left[\partial_\delta\phi_2 + \hat{\gamma}^2 r_3^2 r_1^2 \sum_{i=1}^{3} \partial_\delta\phi_i - \hat{\gamma}\frac{h_{\delta\beta}}{\sqrt{-h}}\varepsilon^{\beta\rho}\left(r_3^2\partial_\rho\phi_3 - r_1^2\partial_\rho\phi_1\right)\right],$$
$$\mathcal{J}_3^\alpha = -\sqrt{\lambda} r_3^2 \sqrt{-h} h^{\alpha\delta} G\left[\partial_\delta\phi_3 + \hat{\gamma}^2 r_1^2 r_2^2 \sum_{i=1}^{3} \partial_\delta\phi_i - \hat{\gamma}\frac{h_{\delta\beta}}{\sqrt{-h}}\varepsilon^{\beta\rho}\left(r_1^2\partial_\rho\phi_1 - r_2^2\partial_\rho\phi_2\right)\right]. \quad (4.81)$$

These results agree with those quoted in [8]. The γ-deformed charge and current densities are $p_i = \mathcal{J}_i^0$ and $j_i = \mathcal{J}_i^1$ respectively, and the γ-deformed $U(1)$ charges or angular momenta $J_i = \int \frac{d\sigma}{2\pi}\, p_i$ are obtained by integrating the charge density p_i over the spatial worldsheet coordinate.

Now, comparing the γ-deformed conserved $U(1)$ 2-currents with their undeformed counterparts (4.37), we see that the relations (4.79) are simply a statement of the fact that the conserved $U(1)$ 2-currents are unaltered by the γ-deformation [8].

4.3.4 Twisted boundary conditions

We shall now further consider the expressions $p_i = \tilde{\tilde{p}}_i$ and $j_i = \tilde{\tilde{j}}_i$ based on discussions in [8]. It is possible to solve for $\dot{\tilde{\tilde{\phi}}}_i$ and $\dot{\phi}_i$ in terms of the (equal) charge densities p_i and $\tilde{\tilde{p}}_i$, and hence eliminate any dependence on the time derivatives of the angular coordinates in the current densities $\tilde{\tilde{j}}_i$ and j_i. Setting the current densities to be equal then yields

$$\tilde{\tilde{\phi}}'_1 = \phi'_1 + \gamma\left(p_2 - p_3\right), \qquad \tilde{\tilde{\phi}}'_2 = \phi'_2 + \gamma\left(p_3 - p_1\right), \qquad \tilde{\tilde{\phi}}'_1 = \phi'_1 + \gamma\left(p_1 - p_2\right), \tag{4.82}$$

where we have defined $\gamma \equiv \frac{\hat{\gamma}}{\sqrt{\lambda}}$. This is the real gauge theory deformation parameter $\beta = \gamma$, which appears in the deformed $\mathcal{N} = 1$ superpotential (4.39).

Let us assume that there exists some physical closed string solution in the γ-deformed background, the angular coordinates of which must satisfy the periodic boundary conditions

$$\phi_i\left(2\pi\right) - \phi_i\left(0\right) = 2\pi \int_0^{2\pi} \frac{d\sigma}{2\pi}\, \phi'_i = 2\pi n_i, \tag{4.83}$$

where the n_i are integer winding numbers. These correspond to solutions in the original undeformed background with angular coordinates with twisted boundary conditions

$$\begin{aligned}
\tilde{\tilde{\phi}}_1\left(2\pi\right) - \tilde{\tilde{\phi}}_1\left(0\right) &= 2\pi \int_0^{2\pi} \frac{d\sigma}{2\pi}\, \tilde{\tilde{\phi}}'_1 = 2\pi \int_0^{2\pi} \frac{d\sigma}{2\pi} \left[\phi'_1 + \gamma\left(p_2 - p_3\right)\right] = 2\pi n_1 + 2\pi\gamma\left(J_2 - J_3\right), \\
\tilde{\tilde{\phi}}_2\left(2\pi\right) - \tilde{\tilde{\phi}}_2\left(0\right) &= 2\pi \int_0^{2\pi} \frac{d\sigma}{2\pi}\, \tilde{\tilde{\phi}}'_2 = 2\pi \int_0^{2\pi} \frac{d\sigma}{2\pi} \left[\phi'_2 + \gamma\left(p_3 - p_1\right)\right] = 2\pi n_2 + 2\pi\gamma\left(J_3 - J_1\right), \\
\tilde{\tilde{\phi}}_3\left(2\pi\right) - \tilde{\tilde{\phi}}_3\left(0\right) &= 2\pi \int_0^{2\pi} \frac{d\sigma}{2\pi}\, \tilde{\tilde{\phi}}'_3 = 2\pi \int_0^{2\pi} \frac{d\sigma}{2\pi} \left[\phi'_1 + \gamma\left(p_1 - p_2\right)\right] = 2\pi n_3 + 2\pi\gamma\left(J_1 - J_2\right),
\end{aligned} \tag{4.84}$$

in terms of the $U(1)$ charges or angular momenta J_i.

Thus it is clear that solutions in the γ-deformed Lunin-Maldacena background correspond to solutions in the original undeformed background with twisted boundary conditions. These boundary conditions would usually be discarded as unphysical for closed string configurations, but now we can interpret them as physical solutions in a deformed background.

4.4 Strings in the γ_i-deformed Background

The string theory proposed to be dual to our non-supersymmetric γ_i-deformed YM theory was originally constructed by Frolov in [8]. A series of three TsT-transformations involving the torii $\left(\tilde{\tilde{\phi}}_1, \tilde{\tilde{\phi}}_2\right)$, $\left(\tilde{\tilde{\phi}}_2, \tilde{\tilde{\phi}}_3\right)$ and $\left(\tilde{\tilde{\phi}}_3, \tilde{\tilde{\phi}}_1\right)$, and with distinct real shift parameters $\hat{\gamma}_3$, $\hat{\gamma}_1$ and $\hat{\gamma}_2$ respectively, were performed on the string worldsheet action to obtain the action in the γ_i-deformed $\mathbb{R} \times S^5$ background. In this section, we briefly review the derivation and properties of this γ_i-deformed string theory.

4.4.1 Derivation of γ_i-deformed string worldsheet action via three TsT-transformations

We shall now demonstrate how to construct the γ_i-deformed string worldsheet action based on discussions in [8]. Consider first the undeformed string action (4.34) in an $\mathbb{R} \times S^5$ background. Our first TsT-transformation on the torus $\left(\tilde{\tilde{\phi}}_1, \tilde{\tilde{\phi}}_2\right)$ can be represented as follows:

$$\mathbf{T}_{\tilde{\tilde{\phi}}_1}\ \mathbf{S}_{\tilde{\phi}_2 \to \tilde{\phi}_2 + \hat{\gamma}_3 \tilde{\phi}_1}\ \mathbf{T}_{\tilde{\phi}_1}. \tag{4.85}$$

In other words, we perform a T-duality transformation on the first angular coordinate $\tilde{\tilde{\phi}}_1$, shift the second angular coordinate in the T-dual space $\tilde{\phi}_2 \to \tilde{\phi}_2 + \hat{\gamma}_3 \tilde{\phi}_1$ using the parameter $\hat{\gamma}_3$ and then make another T-duality transformation on the T-dual coordinate $\tilde{\phi}_1$. We thus obtain the intermediate string action

$$\begin{aligned} S = & -\frac{\sqrt{\lambda}}{2} \int d\tau\, \frac{d\sigma}{2\pi} \\ & \times \Bigg\{ \sqrt{-h} h^{\alpha\beta} \left[-\partial_\alpha t \partial_\beta t + \sum_{i=1}^{3} \left(\partial_\alpha r_i \partial_\beta r_i + A r_i^2 \partial_\alpha \phi_i \partial_\beta \phi_i \right) + A \hat{\gamma}_3^2 r_1^2 r_2^2 r_3^2 \partial_\alpha \phi_3 \partial_\beta \phi_3 \right] \\ & \qquad - 2A \varepsilon^{\alpha\beta} \left(\hat{\gamma}_3 r_1^2 r_2^2 \partial_\alpha \phi_1 \partial_\beta \phi_2 \right) + \Lambda \left(\sum_{i=1}^{3} r_i^2 - 1 \right) \Bigg\}, \end{aligned} \tag{4.86}$$

with $A^{-1} \equiv 1 + \hat{\gamma}_3^2 r_1^2 r_2^2$.

Redefining $\phi_i \to \tilde{\tilde{\phi}}_i$, we now perform the second TsT-transformation on the torus $\left(\tilde{\tilde{\phi}}_2, \tilde{\tilde{\phi}}_3\right)$, which is given by

$$\mathbf{T}_{\tilde{\tilde{\phi}}_2}\ \mathbf{S}_{\tilde{\phi}_3 \to \tilde{\phi}_3 + \hat{\gamma}_1 \tilde{\phi}_2}\ \mathbf{T}_{\tilde{\phi}_2}, \tag{4.87}$$

and yields the string action

$$S = -\frac{\sqrt{\lambda}}{2} \int d\tau \, \frac{d\sigma}{2\pi} \left\{ \sqrt{-h} \, h^{\alpha\beta} \, \big[-\partial_\alpha t \partial_\beta t + \sum_{i=1}^{3} \left(\partial_\alpha r_i \partial_\beta r_i + C r_i^2 \partial_\alpha \phi_i \partial_\beta \phi_i \right) \right. \tag{4.88}$$
$$+ C r_1^2 r_2^2 r_3^2 \left(\hat{\gamma}_1 \partial_\alpha \phi_1 + \hat{\gamma}_3 \partial_\alpha \phi_3 \right) \left(\hat{\gamma}_1 \partial_\beta \phi_1 + \hat{\gamma}_3 \partial_\beta \phi_3 \right) \big]$$
$$\left. - 2C \varepsilon^{\alpha\beta} \left(\hat{\gamma}_3 r_1^2 r_2^2 \partial_\alpha \phi_1 \partial_\beta \phi_2 + \hat{\gamma}_1 r_2^2 r_3^2 \partial_\alpha \phi_2 \partial_\beta \phi_3 \right) + \Lambda \left(\sum_{i=1}^{3} r_i^2 - 1 \right) \right\},$$

where $C^{-1} \equiv 1 + \hat{\gamma}_3^2 r_1^2 r_2^2 + \hat{\gamma}_1^2 r_2^2 r_3^2$.

Finally, we shall again redefine $\phi_i \to \tilde{\tilde{\phi}}_i$ and perform the last TsT-transformation on the torus $\left(\tilde{\tilde{\phi}}_3, \tilde{\tilde{\phi}}_1 \right)$ as follows:

$$\mathbf{T}_{\tilde{\tilde{\phi}}_3} \, \mathbf{S}_{\tilde{\phi}_1 \to \tilde{\phi}_1 + \hat{\gamma}_2 \tilde{\phi}_3} \, \mathbf{T}_{\tilde{\phi}_3}. \tag{4.89}$$

Hence we obtain the γ_i-deformed string worldsheet action, which is dependent on the three parameters $\hat{\gamma}_i$ and is given by

$$S^{\gamma_i} = -\frac{\sqrt{\lambda}}{2} \int d\tau \, \frac{d\sigma}{2\pi} \tag{4.90}$$
$$\times \left\{ \sqrt{-h} h^{\alpha\beta} \left[-\partial_\alpha t \partial_\beta t + \sum_{i=1}^{3} \left(\partial_\alpha r_i \partial_\beta r_i + G r_i^2 \partial_\alpha \phi_i \partial_\beta \phi_i \right) + G r_1^2 r_2^2 r_3^2 \left(\sum_{i=1}^{3} \hat{\gamma}_i \partial_\alpha \phi_i \right) \left(\sum_{j=1}^{3} \hat{\gamma}_j \partial_\beta \phi_j \right) \right] \right.$$
$$\left. -2G \varepsilon^{\alpha\beta} \left(\hat{\gamma}_3 r_1^2 r_2^2 \partial_\alpha \phi_1 \partial_\beta \phi_2 + \hat{\gamma}_2 r_3^2 r_1^2 \partial_\alpha \phi_3 \partial_\beta \phi_1 + \hat{\gamma}_1 r_2^2 r_3^2 \partial_\alpha \phi_2 \partial_\beta \phi_3 \right) + \Lambda \left(\sum_{i=1}^{3} r_i^2 - 1 \right) \right\},$$

where $G^{-1} \equiv 1 + \hat{\gamma}_3^2 r_1^2 r_2^2 + \hat{\gamma}_2^2 r_3^2 r_1^2 + \hat{\gamma}_1^2 r_2^2 r_3^2$. This γ_i-deformed string action agrees with the result quoted in [8]. Notice that, in the case of equal deformation parameters $\hat{\gamma}_i = \hat{\gamma}$, the γ_i-deformed string worldsheet action (4.90) simply reduces to the string worldsheet action (4.74) in the Lunin-Maldacena background.

Lastly, we should mention that the angular coordinates in the original and γ_i-deformed backgrounds can be related as follows:

$$\partial_\alpha \tilde{\tilde{\phi}}_1 = G \left[\partial_\alpha \phi_1 + \hat{\gamma}_1 r_2^2 r_3^2 \sum_{i=1}^{3} \hat{\gamma}_i \partial_\alpha \phi_i - \frac{h_{\alpha\beta}}{\sqrt{-h}} \varepsilon^{\beta\rho} \left(\hat{\gamma}_3 r_2^2 \partial_\rho \phi_2 - \hat{\gamma}_2 r_3^2 \partial_\rho \phi_3 \right) \right],$$
$$\partial_\alpha \tilde{\tilde{\phi}}_2 = G \left[\partial_\alpha \phi_2 + \hat{\gamma}_2 r_3^2 r_1^2 \sum_{i=1}^{3} \hat{\gamma}_i \partial_\alpha \phi_i - \frac{h_{\alpha\beta}}{\sqrt{-h}} \varepsilon^{\beta\rho} \left(\hat{\gamma}_1 r_3^2 \partial_\rho \phi_3 - \hat{\gamma}_3 r_1^2 \partial_\rho \phi_1 \right) \right],$$
$$\partial_\alpha \tilde{\tilde{\phi}}_3 = G \left[\partial_\alpha \phi_3 + \hat{\gamma}_3 r_1^2 r_2^2 \sum_{i=1}^{3} \hat{\gamma}_i \partial_\alpha \phi_i - \frac{h_{\alpha\beta}}{\sqrt{-h}} \varepsilon^{\beta\rho} \left(\hat{\gamma}_2 r_1^2 \partial_\rho \phi_1 - \hat{\gamma}_1 r_2^2 \partial_\rho \phi_2 \right) \right]. \tag{4.91}$$

4.4.2 $U(1)$ charges or angular momenta

We shall again consider the conserved $U(1)$ 2-currents corresponding to the transformations $\phi_i \to \phi_i + \varepsilon_i$ under which our γ_i-deformed string worldsheet action (4.90) is invariant. These can be calculated by taking derivatives of our γ_i-deformed Lagrangian with respect to $\partial_\alpha \phi_i$ to obtain

$$
\begin{aligned}
\mathcal{J}_1^\alpha &= -\sqrt{\lambda} r_1^2 \sqrt{-h} h^{\alpha\delta} G \left[\partial_\delta \phi_1 + \hat{\gamma}_1 r_2^2 r_3^2 \sum_{i=1}^{3} \hat{\gamma}_i \partial_\delta \phi_i - \frac{h_{\delta\beta}}{\sqrt{-h}} \varepsilon^{\beta\rho} \left(\hat{\gamma}_3 r_2^2 \partial_\rho \phi_2 - \hat{\gamma}_2 r_3^2 \partial_\rho \phi_3 \right) \right], \\
\mathcal{J}_2^\alpha &= -\sqrt{\lambda} r_2^2 \sqrt{-h} h^{\alpha\delta} G \left[\partial_\delta \phi_2 + \hat{\gamma}_2 r_3^2 r_1^2 \sum_{i=1}^{3} \hat{\gamma}_i \partial_\delta \phi_i - \frac{h_{\delta\beta}}{\sqrt{-h}} \varepsilon^{\beta\rho} \left(\hat{\gamma}_1 r_3^2 \partial_\rho \phi_3 - \hat{\gamma}_3 r_1^2 \partial_\rho \phi_1 \right) \right], \\
\mathcal{J}_3^\alpha &= -\sqrt{\lambda} r_3^2 \sqrt{-h} h^{\alpha\delta} G \left[\partial_\delta \phi_3 + \hat{\gamma}_3 r_1^2 r_2^2 \sum_{i=1}^{3} \hat{\gamma}_i \partial_\delta \phi_i - \frac{h_{\delta\beta}}{\sqrt{-h}} \varepsilon^{\beta\rho} \left(\hat{\gamma}_2 r_1^2 \partial_\rho \phi_1 - \hat{\gamma}_1 r_2^2 \partial_\rho \phi_2 \right) \right].
\end{aligned}
\tag{4.92}
$$

As before, the γ_i-deformed charge and current densities are given by $p_i = \mathcal{J}_i^0$ and $j_i = \mathcal{J}_i^1$ respectively. The $U(1)$ charges or angular momenta are thus $J_i = \int \frac{d\sigma}{2\pi}\, p_i$. It again turns out that these conserved $U(1)$ 2-currents remain unchanged by the γ_i-deformation.

4.4.3 Twisted boundary conditions

Now, the equivalence of the $U(1)$ 2-currents (4.37) and (4.92) in the undeformed and γ_i-deformed backgrounds again leads to a set of conditions connecting the spatial derivatives of the original and γ_i-deformed angular coordinates as follows:

$$
\tilde{\tilde{\phi}}'_1 = \phi'_1 + \gamma_3 p_2 - \gamma_2 p_3, \qquad \tilde{\tilde{\phi}}'_2 = \phi'_2 + \gamma_1 p_3 - \gamma_3 p_1, \qquad \tilde{\tilde{\phi}}'_1 = \phi'_1 + \gamma_2 p_1 - \gamma_1 p_2, \tag{4.93}
$$

with $\gamma_i \equiv \frac{\hat{\gamma}_i}{\sqrt{\lambda}}$. These are the deformation parameters in the non-supersymmetric γ_i-deformed YM gauge theory.

A closed string solution in the γ_i-deformed background with angular coordinates ϕ_i satisfying the periodic conditions

$$
\phi_i(2\pi) - \phi_i(0) = 2\pi \int_0^{2\pi} \frac{d\sigma}{2\pi}\, \phi'_i = 2\pi n_i, \tag{4.94}
$$

for the winding numbers n_i, then corresponds to a solution in the original background with twisted boundary conditions

$$\begin{aligned}
\tilde{\tilde{\phi}}_1(2\pi) - \tilde{\tilde{\phi}}_1(0) &= 2\pi \int_0^{2\pi} \frac{d\sigma}{2\pi}\, \tilde{\tilde{\phi}}'_1 = 2\pi \int_0^{2\pi} \frac{d\sigma}{2\pi}\left(\phi'_1 + \gamma_3 p_2 - \gamma_2 p_3\right) = 2\pi\left(n_1 + \gamma_3 J_2 - \gamma_2 J_3\right), \\
\tilde{\tilde{\phi}}_2(2\pi) - \tilde{\tilde{\phi}}_2(0) &= 2\pi \int_0^{2\pi} \frac{d\sigma}{2\pi}\, \tilde{\tilde{\phi}}'_2 = 2\pi \int_0^{2\pi} \frac{d\sigma}{2\pi}\left(\phi'_2 + \gamma_1 p_3 - \gamma_3 p_1\right) = 2\pi\left(n_2 + \gamma_1 J_3 - \gamma_3 J_1\right), \\
\tilde{\tilde{\phi}}_3(2\pi) - \tilde{\tilde{\phi}}_3(0) &= 2\pi \int_0^{2\pi} \frac{d\sigma}{2\pi}\, \tilde{\tilde{\phi}}'_3 = 2\pi \int_0^{2\pi} \frac{d\sigma}{2\pi}\left(\phi'_1 + \gamma_2 p_1 - \gamma_1 p_2\right) = 2\pi\left(n_3 + \gamma_2 J_1 - \gamma_1 J_2\right).
\end{aligned} \tag{4.95}$$

These results can be proved in a similar way to those in the Lunin-Maldacena background and agree with the expressions in [8].

4.5 Lax Pairs for Strings Moving on Undeformed and γ_i-deformed Five-spheres

The existence of a Lax pair in any theory is of great significance, as it is a demonstration of integrability. This Lax pair should satisfy a zero curvature condition, which is equivalent to the equations of motion and allows for the construction of an infinite number of conserved charges, which make the theory theoretically soluble. It was shown in [8] that there exists such a Lax pair for strings moving on a five-sphere space and, furthermore, that it is possible to extend this Lax pair to describe strings on a γ_i-deformed five-sphere (and thus also a Lunin-Maldacena γ-deformed five-sphere) by making use of the transformation between the original and γ_i-deformed angular coordinates.

In this section, following [8], we rewrite the original string worldsheet action in an $\mathbb{R}\times S^5$ background in terms of anti-symmetric $SU(4)$ matrices and hence calculate the five-sphere equation of motion. We then introduce a Lax pair and the corresponding zero curvature condition is shown to be equivalent to this equation of motion. Finally, this Lax pair is certainly not unique and an equivalent gauged Lax pair is defined, from which it is possible to construct a Lax pair for our γ_i-deformed string theory.

4.5.1 String worldsheet action and equations of motion in terms of anti-symmetric $SU(4)$ matrices

The string worldsheet action (4.34) shall now be written in terms of an anti-symmetric $SU(4)$ matrix, following [8], as

$$S = -\frac{\sqrt{\lambda}}{2} \int d\tau \, \frac{d\sigma}{2\pi} \sqrt{-h} h^{\alpha\beta} \left[-\partial_\alpha t \partial_\beta t + \frac{1}{4} \mathrm{Tr} \left(g^{-1} \partial_\alpha g g^{-1} \partial_\beta g \right) \right], \tag{4.96}$$

where we define

$$g \equiv \begin{pmatrix} 0 & X_3 & X_1 & X_2 \\ -X_3 & 0 & X_2^* & -X_1^* \\ -X_1 & -X_2^* & 0 & X_3^* \\ -X_2 & X_1^* & -X_3^* & 0 \end{pmatrix}, \qquad \text{with} \quad X_k \equiv r_k e^{i\phi_k}, \tag{4.97}$$

which must satisfy the constraint

$$\det g = \left(|X_1|^2 + |X_2|^2 + |X_3|^2 \right)^2 = 1. \tag{4.98}$$

This can be verified by noticing that $g^{-1} = -g^*$ and also $g^{-1} \left(\partial_\beta g \right) = - \left(\partial_\beta g^{-1} \right) g$, so that (4.96) becomes

$$S = -\frac{\sqrt{\lambda}}{2} \int d\tau \, \frac{d\sigma}{2\pi} \sqrt{-h} h^{\alpha\beta} \left[-\partial_\alpha t \partial_\beta t + \frac{1}{4} \mathrm{Tr} \left(\partial_\alpha g \partial_\beta g^* \right) \right], \tag{4.99}$$

which can be reduced to (4.34) by simply substituting (4.97) into this expression and multiplying out the matrices.

Now let us derive the equation describing the motion in the S^5 space by varying (4.99) with respect to g as follows:

$$S = -\frac{\sqrt{\lambda}}{8} \int d\tau \, \frac{d\sigma}{2\pi} \sqrt{-h} h^{\alpha\beta} \; \mathrm{Tr} \left\{ \partial_\alpha \left(\delta g \right) \partial_\beta g^* + \partial_\alpha g \partial_\beta \left(\delta g^* \right) \right\}. \tag{4.100}$$

We now make use of the identity $\delta g^* = g^* \left(\delta g \right) g^*$ to obtain

$$S = -\frac{\sqrt{\lambda}}{8} \int d\tau \, \frac{d\sigma}{2\pi} \; \mathrm{Tr} \left\{ \left[\partial_\alpha \left(\sqrt{-h} h^{\alpha\beta} \partial_\beta g^* \right) + g^* \partial_\beta \left(\sqrt{-h} h^{\alpha\beta} \partial_\alpha g \right) g^* \right] \delta g \right\}, \tag{4.101}$$

where surface terms have been discarded. Setting this variation to zero[4] and noting that $h^{\alpha\beta}$ is symmetric then implies

$$\partial_\alpha \left(\sqrt{-h} h^{\alpha\beta} \partial_\beta g^* \right) + g^* \partial_\alpha \left(\sqrt{-h} h^{\alpha\beta} \partial_\beta g \right) g^* = 0, \tag{4.102}$$

[4]To be more rigorous, we should write this expression out explicitly in terms of components and then set the variation with respect to g_i^j to zero. This would yield an identical result.

and right-multiplying by g gives

$$\partial_\alpha \left(\sqrt{-h}h^{\alpha\beta}\partial_\beta g^*\right) g - g^*\partial_\alpha \left(\sqrt{-h}h^{\alpha\beta}\partial_\beta g\right) = 2\partial_\alpha \left\{\sqrt{-h}h^{\alpha\beta} g^*\partial_\beta g\right\} = 0. \quad (4.103)$$

Thus, again using $g^* = -g^{-1}$, we finally obtain the equation of motion

$$\partial_\alpha \left(\sqrt{-h}h^{\alpha\beta} R_\beta\right) = 0, \qquad \text{with} \quad R_\beta \equiv g^{-1}\partial_\beta g, \quad (4.104)$$

which agrees with the result quoted in [8]. Note that the R_α is sometimes called the right-current.

4.5.2 Undeformed Lax pair and zero curvature condition

We shall now introduce the Lax pair for strings in an undeformed five-sphere space, based on discussions in [8], as follows:

$$D_\alpha \equiv \partial_\alpha - A_\alpha(x), \qquad \text{with} \quad A_\alpha(x) \equiv \frac{R_\alpha^+}{2(x-1)} - \frac{R_\alpha^-}{2(x+1)}, \quad (4.105)$$

where

$$R_\alpha^\pm \equiv \left(\delta_\alpha^\beta \mp \frac{h_{\alpha\rho}}{\sqrt{-h}}\varepsilon^{\rho\beta}\right) R_\beta = R_\alpha \mp \frac{h_{\alpha\rho}}{\sqrt{-h}}\varepsilon^{\rho\beta} R_\beta. \quad (4.106)$$

The new parameter x, which has been introduced in the above definition, takes on an infinite number of values and is know as the spectral parameter. We can now simply our undeformed Lax pair to the form

$$D_\alpha(x) = \partial_\alpha - \frac{R_\alpha - x\frac{h_{\alpha\rho}}{\sqrt{-h}}\varepsilon^{\rho\beta} R_\beta}{x^2-1}. \quad (4.107)$$

This Lax pair must satisfy the zero curvature condition

$$[D_\alpha, D_\beta] = \partial_\alpha A_\beta - \partial_\beta A_\alpha - [A_\alpha, A_\beta] = 0. \quad (4.108)$$

Substituting (4.107) into this expression and multiplying by $(x^2-1)^2$ then yields

$$\begin{aligned} &\left(x^2-1\right)\partial_\alpha \left(R_\beta - x\frac{h_{\beta\delta}}{\sqrt{-h}}\varepsilon^{\delta\lambda} R_\lambda\right) - \left(x^2-1\right)\partial_\beta \left(R_\alpha - x\frac{h_{\alpha\rho}}{\sqrt{-h}}\varepsilon^{\rho\tau} R_\tau\right) \\ &- \left[R_\alpha - x\frac{h_{\alpha\rho}}{\sqrt{-h}}\varepsilon^{\rho\tau} R_\tau,\ R_\beta - x\frac{h_{\beta\delta}}{\sqrt{-h}}\varepsilon^{\delta\lambda} R_\lambda\right] = 0, \end{aligned} \quad (4.109)$$

which, equating different orders of the spectral parameter x, results in the following four equations:

$$O(x^0): \quad \partial_\beta R_\alpha - \partial_\alpha R_\beta - [R_\alpha, R_\beta] = 0, \tag{4.110}$$

$$\begin{aligned} O(x^1): \quad & \partial_\alpha \left(\frac{h_{\beta\delta}}{\sqrt{-h}}\, \varepsilon^{\delta\lambda} R_\lambda \right) - \partial_\beta \left(\frac{h_{\alpha\rho}}{\sqrt{-h}}\, \varepsilon^{\rho\tau} R_\tau \right) \\ & + \frac{h_{\alpha\rho}}{\sqrt{-h}} \varepsilon^{\rho\tau} [R_\tau, R_\beta] + \frac{h_{\beta\delta}}{\sqrt{-h}} \varepsilon^{\delta\lambda} [R_\alpha, R_\lambda] = 0, \end{aligned} \tag{4.111}$$

$$O(x^2): \quad \partial_\alpha R_\beta - \partial_\beta R_\alpha - \frac{h_{\alpha\rho}}{\sqrt{-h}} \varepsilon^{\rho\tau} \frac{h_{\beta\delta}}{\sqrt{-h}} \varepsilon^{\delta\lambda} [R_\tau, R_\lambda] = 0, \tag{4.112}$$

$$O(x^3): \quad -\partial_\alpha \left(\frac{h_{\beta\delta}}{\sqrt{-h}} \varepsilon^{\delta\lambda} R_\lambda \right) + \partial_\beta \left(\frac{h_{\alpha\rho}}{\sqrt{-h}} \varepsilon^{\rho\tau} R_\tau \right) = 0. \tag{4.113}$$

Now it turns out, upon closer inspection, that the $O\left(x^0\right)$ and $O\left(x^2\right)$ equations are equivalent and, furthermore, are trivially satisfied by the expression $R_\alpha = g^{-1}\partial_\alpha g$. The $O\left(x^1\right)$ and $O\left(x^3\right)$ equations are also equivalent and are satisfied if and only if the equation of motion (4.104) is valid. Thus the zero curvature condition (4.108) is equivalent to the equation of motion (4.104), so that (4.107) is, indeed, a suitable Lax pair for the theory.

4.5.3 Gauged undeformed and γ_i-deformed Lax pairs

The Lax pair (4.107) for the theory describing strings moving in the undeformed S^5 space is by no means unique. The transformation $D_\alpha \to \tilde{D}_\alpha = M D_\alpha M^{-1}$, with M any invertible 4×4 matrix, results in an equivalent Lax pair, which also satisfies the zero curvature condition, since $[\tilde{D}_\alpha, \tilde{D}_\beta] = M[D_\alpha, D_\beta]M^{-1} = 0$. Now, while any Lax pair is good enough to prove the integrability of the theory, for the purpose of extending the Lax pair to the γ_i-deformed string theory, we shall choose (as in [8]) a specific gauged Lax pair, which depends only on the derivatives of the angular coordinates and not on the angular coordinates themselves.

Let us begin by writing

$$g = M(\tilde{\tilde{\phi}}_i)\ \hat{g}(r_i)\ M(\tilde{\tilde{\phi}}_i), \qquad \text{with} \quad M(\tilde{\tilde{\phi}}_i) = e^{i\tilde{\tilde{\Phi}}}, \tag{4.114}$$

where we have defined

$$\hat{g}(r_i) \equiv \begin{pmatrix} 0 & r_3 & r_1 & r_2 \\ -r_3 & 0 & r_2 & -r_1 \\ -r_1 & -r_2 & 0 & r_3 \\ -r_2 & r_1 & -r_3 & 0 \end{pmatrix}, \tag{4.115}$$

and

$$\tilde{\tilde{\Phi}} \equiv \frac{1}{2}\begin{pmatrix} \tilde{\tilde{\phi}}_1+\tilde{\tilde{\phi}}_2+\tilde{\tilde{\phi}}_3 & 0 & 0 & 0 \\ 0 & -\tilde{\tilde{\phi}}_1-\tilde{\tilde{\phi}}_2+\tilde{\tilde{\phi}}_3 & 0 & 0 \\ 0 & 0 & \tilde{\tilde{\phi}}_1-\tilde{\tilde{\phi}}_2-\tilde{\tilde{\phi}}_3 & 0 \\ 0 & 0 & 0 & -\tilde{\tilde{\phi}}_1+\tilde{\tilde{\phi}}_2-\tilde{\tilde{\phi}}_3 \end{pmatrix}. \tag{4.116}$$

This can be verified by multiplying out these matrices and noticing that the result is identical to the definition of g given in (4.97).

Now, using the above redefinition of g in terms of $\hat{g}(r_i)$ and $M(\tilde{\tilde{\phi}}_i)$, together with the identities $\hat{g}^{-1}(r_i) = -\hat{g}(r_i)$ and $M^{-1}(\tilde{\tilde{\phi}}_i) = e^{-i\tilde{\tilde{\Phi}}}$, we find that

$$\begin{aligned} R_\alpha(r_i, \tilde{\tilde{\phi}}_i) &= \left[M^{-1}(\tilde{\tilde{\phi}}_i)\ \hat{g}^{-1}(r_i)\ M^{-1}(\tilde{\tilde{\phi}}_i)\right] \partial_\alpha \left[M(\tilde{\tilde{\phi}}_i)\ \hat{g}(r_i)\ M(\tilde{\tilde{\phi}}_i)\right] \\ &= M^{-1}(\tilde{\tilde{\phi}}_i)\hat{R}_\alpha(r_i, \partial\tilde{\tilde{\phi}}_i)M(\tilde{\tilde{\phi}}_i), \end{aligned} \tag{4.117}$$

where

$$\hat{R}_\alpha(r_i, \partial\tilde{\tilde{\phi}}_i) \equiv -\hat{g}(r_i)\ \partial_\alpha \hat{g}(r_i) - \hat{g}(r_i)\ \partial_\alpha \tilde{\tilde{\Phi}}\ \hat{g}(r_i) + i\partial_\alpha \tilde{\tilde{\Phi}}. \tag{4.118}$$

This suggests a suitable gauge for our new Lax pair. We shall use the matrix $M(\tilde{\tilde{\phi}}_i)$ to define the gauged Lax pair, as in [8], as follows:

$$\tilde{D}_\alpha \equiv M(\tilde{\tilde{\phi}}_i)\ D_\alpha\ M^{-1}(\tilde{\tilde{\phi}}_i) = \partial_\alpha - \tilde{A}_\alpha(x), \tag{4.119}$$

with

$$\begin{aligned} \tilde{A}_\alpha(x) &\equiv M(\tilde{\tilde{\phi}}_i)A_\alpha(x)M^{-1}(\tilde{\tilde{\phi}}_i) - M(\tilde{\tilde{\phi}}_i)\partial_\alpha M^{-1}(\tilde{\tilde{\phi}}_i) \\ &= \frac{\hat{R}_\alpha(r_i, \partial\tilde{\tilde{\phi}}_i) - x\frac{h_{\alpha\rho}}{\sqrt{-h}}\varepsilon^{\rho\beta}\hat{R}_\beta(r_i, \partial\tilde{\tilde{\phi}}_i)}{x^2-1} + i\partial_\alpha \tilde{\tilde{\Phi}}. \end{aligned} \tag{4.120}$$

This gauged Lax pair clearly depends only on the radii r_i and derivatives thereof, and the derivatives of the angular coordinates $\partial_\alpha \tilde{\tilde{\phi}}_i$.

Now, finally, we know that the derivatives of the angular coordinates in the original and γ_i-deformed backgrounds are connected via the transformation (4.91) and that the radii are unchanged by the deformation. Thus it was observed in [8] that a Lax pair for the γ_i-deformed string theory is

$$\tilde{D}_\alpha^{\gamma_i} \equiv \partial_\alpha - \tilde{A}_\alpha^{\gamma_i}(x), \tag{4.121}$$

where $\tilde{A}_\alpha^{\gamma_i}$ is obtained by simply replacing all the undeformed derivative terms $\partial_\alpha \tilde{\tilde{\phi}}_i$ in (4.120) with the corresponding expressions in (4.91), which are written in terms of the γ_i-deformed derivatives $\partial_\alpha \phi_i$. This demonstrates that the theory describing strings moving in a γ_i-deformed five-sphere space (and hence also a γ-deformed Lunin-Maldacena five-sphere space) is integrable.

Chapter 5

γ_i-deformed Strings and Spin Chains in a Semiclassical Limit

5.1 Coherent State Action for γ_i-deformed $SU(3)$ Spin Chains in the Continuum Limit

It is our aim, in this chapter, to compare the γ_i-deformed gauge and string theories in a semiclassical limit at the level of the action. We shall first concentrate on the gauge theory or spin chain side of this comparison. The relevant semiclassical limit in which to consider our gauge theory operators is simply the BMN limit discussed in chapter 3. This corresponds to a continuum limit of our spin chain system: the length of the spin chain J becomes large and thus the ratio of the site spacing to the spin chain length becomes small, so that the spin chain forms a one dimensional continuum. We can hence perform expansions in terms of the small parameter $\tilde{\lambda} = \frac{\lambda}{J^2}$, which is taken to be fixed when J becomes large.

We are interested in the coherent state action describing a γ_i-deformed $SU(3)$ spin chain. We therefore review, based on discussions in [9, 20], the construction of the coherent state for an $SU(3)$ spin chain system. Hence, using an equivalent γ_i-deformed spin chain Hamiltonian, we derive the γ_i-deformed coherent state effective action in the continuum limit to leading order in $\tilde{\lambda}$ following [9].

5.1.1 Coherent state description

The coherent state $|\alpha\rangle$ of a harmonic oscillator is an eigenstate of the annihilation operator $\hat{a}$ with eigenvalue α. The expectation values of operators with respect to this coherent state can be viewed as a classical limit of the system. Now it is possible to extend these ideas to a finite spin-S system by introducing an analogous coherent state $|\mu\rangle$ such that, in the limit as $S \to \infty$, this coherent state is an eigenstate of the raising operator $\hat{S}_+$ with eigenvalue μ. The analogy to a harmonic oscillator then becomes an exact correspondence with the identifications $\hat{S}_+ \to (2S)^{1/2}\hat{a}$ and $\mu \to \frac{\alpha}{(2S)^{1/2}}$. (A detailed description of the construction of this analogous spin-S coherent state is available in [54].)

We can also consider a more complicated spin-S chain, which consists of a number of these spin-S systems. It is possible [55] to construct a coherent state for this spin-S chain by simply taking a tensor product of the individual spin-S coherent states. The coherent state description of this system in the continuum limit, in which the number of sites in the spin-S chain becomes large, was discussed in [56].

Now the Hamiltonian of a spin-S system is invariant under $SU(2)$ transformations. The construction of a coherent state for a general spin system, the Hamiltonian of which is invariant under the action of some arbitrary Lie group, was discussed in [20, 57, 58]. The general case of an $SU(3)$ Lie group, in which we are particularly interested, was mentioned in [57] and described in more detail in [9, 20].

We now briefly review the description of a general coherent state and construct the $SU(3)$ coherent state in detail. Lastly, we take a tensor product of these $SU(3)$ coherent states to form a coherent state describing an $SU(3)$ spin chain system.

General coherent state

We shall first, following [20, 57, 58], define a general coherent state corresponding to some Lie group G with Cartan basis $[H_i, E_\alpha, E_{-\alpha}]$, where H_i are elements of the commuting Cartan algebra, and E_α and $E_{-\alpha}$ represent the α^{th} raising and lower operators respectively. This group is the symmetry group of some Hamiltonian.

Consider an irreducible representation of this group G with elements $\Lambda(g)$, where

$g \epsilon G$, which act on the vector space V_Λ. We shall also make use of the ground state $|0\rangle$, which is generally chosen as a state annihilated by all the raising operators (the maximum spin state). The maximum stability group is the subgroup H, the elements of which leave the ground state invariant up to a phase, so that

$$\Lambda(h)|0\rangle = e^{i\phi(h)}|0\rangle, \qquad \text{for all } h \,\epsilon\, H. \tag{5.1}$$

The coherent state is then defined as

$$\Lambda(g)|0\rangle = \Lambda(\omega)\Lambda(h)|0\rangle \sim \Lambda(\omega)|0\rangle, \qquad \text{with } \omega \,\epsilon\, G/H, \tag{5.2}$$

up to a phase, since all the elements of G can be expressed as $g = \omega h$ and $\Lambda(g)$ is a homomorphism so that $\Lambda(\omega h) = \Lambda(\omega)\Lambda(h)$. The general coherent state is thus parameterized by elements of the coset group $\omega \,\epsilon\, G/H$.

$SU(3)$ coherent state

We shall now construct the $SU(3)$ coherent state, which corresponds to the coherent state for one site of our spin chain, based on [20]. In other words, we shall set $G = SU(3)$ in the above discussion. Any element of $SU(3)$ can be expressed as $\Lambda = e^{i\sum\limits_k a_k\lambda_k}$, with $k \,\epsilon\, \{1, \ldots, 8\}$, where the a_k are real parameters and the generators λ_k are the eight traceless Hermitean Gell-Mann matrices

$$\lambda_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \lambda_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \qquad \lambda_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\lambda_4 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_5 = \frac{1}{2}\begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.3}$$

The Cartan algebra consists of the two commuting Gell-Mann matrices λ_3 and λ_8. There are also three $SU(2)$ subgroups, $\{\lambda_1, \lambda_2, \lambda_3\}$, $\left\{\lambda_4, \lambda_5, -\frac{1}{2}\left(\lambda_3 + \sqrt{3}\lambda_8\right)\right\}$ and $\left\{\lambda_6, \lambda_7, \frac{1}{2}\left(\lambda_3 - \sqrt{3}\lambda_8\right)\right\}$, each of which results in a raising and lowering operator.

The ground state will now be chosen as the maximum spin state [1]

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{5.4}$$

which is annihilated by all the raising operators. This ground state is also an eigenstate of both elements of the Cartan algebra, specifically $\lambda_3|0\rangle = 0|0\rangle$ and $\lambda_8|0\rangle = -\frac{1}{\sqrt{3}}|0\rangle$, and is annihilated by λ_1 and λ_2. The subgroup H, which leaves the ground state $|0\rangle$ invariant up to a phase, is thus generated by the four Gell-Mann matrices λ_1, λ_2, λ_3 and λ_8. The coset group G/H is generated by the remaining four Gell-Mann matrices λ_4, λ_5, λ_6 and λ_7. Hence the coherent state is given by

$$|N\rangle = e^{i(a\lambda_4+b\lambda_5+c\lambda_6+d\lambda_7)}|0\rangle, \tag{5.5}$$

where a, b, c and d are real parameters.

Let us now calculate the coherent state more explicitely in terms of these parameters. We first find, using the definition (5.3) of the Gell-Mann matrices, that

$$L \equiv a\lambda_4 + b\lambda_5 + c\lambda_6 + d\lambda_7 = \frac{1}{2}\begin{pmatrix} 0 & a+ib & c+id \\ a-ib & 0 & 0 \\ c-id & 0 & 0 \end{pmatrix}, \tag{5.6}$$

$$L^2 = \frac{1}{4}\begin{pmatrix} a^2+b^2+c^2+d^2 & 0 & 0 \\ 0 & a^2+b^2 & (ac+bd)+i(ad-bc) \\ 0 & (ac+bd)-i(ad-bc) & c^2+d^2 \end{pmatrix}, \tag{5.7}$$

and

$$L^3 = \left(\frac{\Delta}{2}\right)^2 L, \qquad \text{with} \quad \Delta^2 \equiv a^2+b^2+c^2+d^2. \tag{5.8}$$

Expressing the exponential e^{iL} in a Taylor series and using the above results then gives

$$e^{iL} = 1 + \frac{2i}{\Delta}\left[\sin\left(\frac{\Delta}{2}\right)\right]L + \left(\frac{2}{\Delta}\right)^2\left[\cos\left(\frac{\Delta}{2}\right) - 1\right]L^2, \tag{5.9}$$

[1] Note that this choice of ground state for each site is consistent with the $SU(3)$ spin chain formalism described in appendix B.

which yields the coherent state

$$
|N\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \frac{i}{\Delta}\sin\left(\frac{\Delta}{2}\right)\begin{pmatrix} 0 & a+ib & c+id \\ a-ib & 0 & 0 \\ c-id & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{5.10}
$$

$$
+\frac{1}{\Delta^2}\left[\cos\left(\frac{\Delta}{2}\right)-1\right]\begin{pmatrix} \Delta^2 & 0 & 0 \\ 0 & a^2+b^2 & (ac+bd)+i(ad-bc) \\ 0 & (ac+bd)-i(ad-bc) & c^2+d^2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.
$$

Thus, finally, the $SU(3)$ coherent state $|N\rangle$ is given by

$$
|N\rangle = \begin{pmatrix} \cos\Delta \\ \frac{i}{\Delta}\sin\left(\frac{\Delta}{2}\right)(a-ib) \\ \frac{i}{\Delta}\sin\left(\frac{\Delta}{2}\right)(c-id) \end{pmatrix}, \tag{5.11}
$$

which is a function of the four real parameters a, b, c and d.

We shall now introduce a reparameterization of this state. The radial coordinates m_i are defined as

$$
m_1 = \cos\left(\frac{\Delta}{2}\right), \quad m_2 = \sin\left(\frac{\Delta}{2}\right)\frac{\sqrt{a^2+b^2}}{\Delta}, \quad m_3 = \sin\left(\frac{\Delta}{2}\right)\frac{\sqrt{c^2+d^2}}{\Delta}, \tag{5.12}
$$

whereas the angular coordinates h_i must satisfy

$$
\tan(h_2-h_1) = -\frac{a}{b}, \qquad \tan(h_3-h_1) = -\frac{c}{d}, \qquad h_1+h_2+h_3 = 0. \tag{5.13}
$$

Notice that $\sum_{i=1}^{3} m_i^2 = 1$ and $\sum_{i=1}^{3} h_i = 0$ from these definitions. The $SU(3)$ coherent state can hence be expressed as follows:

$$
|N\rangle = \begin{pmatrix} m_1 \\ m_2 e^{i(h_2-h_1)} \\ m_3 e^{i(h_3-h_1)} \end{pmatrix} = \begin{pmatrix} m_1 e^{ih_1} \\ m_2 e^{ih_2} \\ m_3 e^{ih_3} \end{pmatrix} e^{-ih_1} \sim \begin{pmatrix} m_1 e^{ih_1} \\ m_2 e^{ih_2} \\ m_3 e^{ih_3} \end{pmatrix}, \tag{5.14}
$$

where this last equivalence is up to the phase e^{-ih_1}. This reparameterization yields the $\mathbb{CP}^2$ representation[2] of the $SU(3)$ coherent state used in [9].

[2]The complex projective space $\mathbb{CP}^2$ is defined as $\mathbb{C}^3/\mathbb{C}^*$, where $\mathbb{C}^* = \mathbb{C} - \{0\}$. More simply put, it is a three dimensional complex vector space in which the elements (z_1, z_2, z_3) and $\lambda(z_1, z_2, z_3)$ are equivalent, for any non-zero complex number λ. Now any complex 3-vector can be written as $(z_1, z_2, z_3) = Me^{iH}(m_1e^{-h_1}, m_2e^{-h_2}, m_3e^{-h_3})$, where we have pulled out the magnitude M and total phase H. The equivalence class of this vector can be represented by $(m_1e^{-h_1}, m_2e^{-h_2}, m_3e^{-h_3})$, for which $\sum_{i=1}^{3} m_i^2 = 1$ and $\sum_{i=1}^{3} h_i = 0$. Thus we can see that our reparameterized $SU(3)$ coherent state is, indeed, an element of $\mathbb{CP}^2$.

$SU(3)$ spin chain coherent state

We can now construct the full $SU(3)$ spin chain coherent state as a tensor product of the $SU(3)$ coherent states corresponding to each site. Thus, as in [9], we obtain

$$|n\rangle\rangle = |n_1\rangle \otimes |n_2\rangle \otimes ... \otimes |n_J\rangle, \tag{5.15}$$

where the k^{th} coherent state in the spin chain is given by

$$|n_k\rangle = m_1(k)e^{ih_1(k)}|1\rangle + m_2(k)e^{ih_2(k)}|2\rangle + m_3(k)e^{ih_3}|3\rangle. \tag{5.16}$$

with $|1\rangle \equiv \begin{pmatrix}1\\0\\0\end{pmatrix}$, $|2\rangle \equiv \begin{pmatrix}0\\1\\0\end{pmatrix}$ and $|3\rangle \equiv \begin{pmatrix}0\\0\\1\end{pmatrix}$. The radial and angular coordinates $m_i(k)$ and $h_i(k)$ satisfy the constraints $\sum\limits_{i=1}^{3} m_i(k)^2 = 1$ and $\sum\limits_{i=1}^{3} h_i(k) = 0$ respectively.

5.1.2 Equivalent Hamiltonian

An important step in the derivation of the γ_i-deformed coherent state effective action is the construction of the coherent state Hamiltonian for our γ_i-deformed spin chain system. It would initially appear that this γ_i-deformed coherent state Hamiltonian can be calculated by simply taking the expectation value $\langle\langle n|H^{\gamma_i}|n\rangle\rangle$ with respect to the coherent state (5.15) of the γ_i-deformed Hamiltonian, which, from (3.26) and (3.30), is given by

$$H^{\gamma_i} = \frac{\lambda}{8\pi^2}\sum_{k=1}^{J}\mathcal{H}^{\gamma_i}_{k,k+1} \qquad \text{with} \quad \mathcal{H}^{\gamma_i}_{k,k+1} = \mathcal{U}_{k,k+1}\mathcal{H}_{k,k+1}\mathcal{U}^{-1}_{k,k+1}, \tag{5.17}$$

where $\mathcal{H}_{k,k+1} = 1_{k,k+1} - \mathcal{P}_{k,k+1}$ and the unitary operator $\mathcal{U}_{k,k+1}$ is defined as

$$\mathcal{U}_{k,k+1} \equiv \sum_{m,n=1}^{3} e^{i\pi\alpha_{mn}}e^m_m(k)e^n_n(k+1). \tag{5.18}$$

Taking a continuum limit of this γ_i-deformed coherent state Hamiltonian then yields an effective Hamiltonian, which contains kinetic terms (involving derivatives with respect to the now continuous spatial variable) as well as a 'potential' for the system. The zeros of this potential should correspond to the vacuum states of the γ_i-deformed spin chain.

Now it turns out [9] that the γ_i-deformed potential obtained in this way does not result in the correct vacuum states described in section 3.2.4 - the vacuum state $(J_1, J_2, J_3) \sim (\gamma_1, \gamma_2, \gamma_3)$ is absent. This is an indication that we may not use the $SU(3)$ coherent state (5.15) for the γ_i-deformed $SU(3)$ spin chain system.

It was, however, also pointed out in [9] that, instead of changing the coherent state basis, it is equivalent to alter the γ_i-deformed Hamiltonian by some unitary transformation $U(\xi)$ as follows:

$$H^{\gamma_i} \longrightarrow \tilde{H}^{\gamma_i} = U^{-1}(\xi) H^{\gamma_i} U(\xi) \tag{5.19}$$

This transformed Hamiltonian should have an energy spectrum equivalent to that of the original Hamiltonian, but, as shall later be seen, will result in a different coherent state Hamiltonian.

Let us now construct this equivalent γ_i-deformed Hamiltonian, based on discussions in [9], making use of the following ansatz for the unitary operators $U(\xi)$:

$$U(\xi) = \prod_{k=1}^{J} U_{k,k+1}(\xi) \qquad \text{with} \quad U_{k,k+1}(\xi) = \sum_{m,n=1}^{3} e^{i\pi\xi\alpha_{mn}} e_m^m(k) e_n^n(k+1), \tag{5.20}$$

where the complex parameter ξ shall be specified later so as to obtain the correct γ_i-deformed vacuum states.

We should first notice that this unitary transformation has the properties

1. $U_{k,k+1}^{-1}(\xi) = U_{k,k+1}(-\xi)$,
2. $U_{k,k+1}(1) = \mathcal{U}_{k,k+1}$,
3. $U_{k,k+1}(\xi) U_{k,k+1}(\lambda) = U_{k,k+1}(\xi + \lambda)$,
4. $U_{k,k+1}(\xi)$ and $U_{q,q+1}(\lambda)$ commute for all $k, q \,\epsilon\, \{1, 2, ..., J\}$ and $\xi, \lambda \,\epsilon\, \mathbb{C}$,

which are a direct result of the definition. These properties shall be used to rewrite the equivalent γ_i-deformed Hamiltonian as follows:

$$\tilde{H}^{\gamma_i} = \frac{\lambda}{8\pi^2} \sum_{k=1}^{J} U^{-1}(\xi) \mathcal{H}_{k,k+1}^{\gamma_i} U(\xi) \tag{5.21}$$

$$\begin{aligned} &= \frac{\lambda}{8\pi^2} \sum_{k=1}^{J} \left[U_{1,2}(-\xi) \ldots U_{k-1,k}(-\xi) U_{k,k+1}(-\xi) U_{k+1,k+2}(-\xi) \ldots U_{J,1}(-\xi)\right] \\ &\qquad \times \mathcal{H}_{k,k+1}^{\gamma_i} \left[U_{1,2}(\xi) \ldots U_{k-1,k}(\xi) U_{k,k+1}(\xi) U_{k+1,k+2}(\xi) \ldots U_{J,1}(\xi)\right] \end{aligned} \tag{5.22}$$

$$= \frac{\lambda}{8\pi^2} \sum_{k=1}^{J} \left[U_{k-1,k}(-\xi) U_{k,k+1}(-\xi) U_{k+1,k+2}(-\xi)\right] \mathcal{H}_{k,k+1}^{\gamma_i} \left[U_{k-1,k}(\xi) U_{k,k+1}(\xi) U_{k+1,k+2}(\xi)\right],$$

since the $U_{l,l+1}(\xi)$ terms on the right commute with $\mathcal{H}^{\gamma_i}_{k,k+1}$, for all $l \neq k-1, k, k+1$, and then cancel with the corresponding term $U_{l,l+1}(-\xi)$ on the left. Hence, using $\mathcal{H}^{\gamma_i}_{k,k+1} = \mathcal{U}_{k,k+1}\mathcal{H}_{k,k+1}\mathcal{U}^{-1}_{k,k+1} = U_{k,k+1}(1)\mathcal{H}_{k,k+1}U_{k,k+1}(-1)$, we find that the equivalent γ_i-deformed Hamiltonian is given by

$$\begin{aligned}\tilde{H}^{\gamma_i} = \frac{\lambda}{8\pi^2}\sum_{k=1}^{J}&\left[U_{k-1,k}(-\xi)U_{k,k+1}(-\xi+1)U_{k+1,k+2}(-\xi)\right]\mathcal{H}_{k,k+1} \\ &\times\left[U_{k-1,k}(\xi)U_{k,k+1}(\xi-1)U_{k+1,k+2}(\xi)\right],\end{aligned} \tag{5.23}$$

and, substituting the definition (5.20) of $U_{k,k+1}(\xi)$ into this expression, we obtain

$$\begin{aligned}\tilde{H}^{\gamma_i} = \frac{\lambda}{8\pi^2}\sum_{k=1}^{J}\sum_{\substack{m,n,l,p,u,t,\\ w,r,a,q,c,d=1}}^{3} &\left\{e^{-i\pi\xi\alpha_{mn}}e^{-i\pi(\xi-1)\alpha_{lp}}e^{-i\pi\xi\alpha_{ut}}e^{i\pi\xi\alpha_{wr}}e^{i\pi(\xi-1)\alpha_{aq}}e^{i\pi\xi\alpha_{cd}}\right. \\ \times&\left[e^m_m(k-1)e^w_w(k-1)\right]\left[e^t_t(k+2)e^d_d(k+2)\right] \\ \times&\left.\left[e^n_n(k)e^l_l(k)\right]\left[e^p_p(k+1)e^u_u(k+1)\right]\mathcal{H}_{k,k+1}\left[e^r_r(k)e^a_a(k)\right]\left[e^q_q(k+1)e^c_c(k+1)\right]\right\}.\end{aligned} \tag{5.24}$$

We shall now make use of the identity $e^m_m e^n_n = \delta^n_m e^m_m$ to derive the final expression for the equivalent γ_i-deformed Hamiltonian as follows:

$$\tilde{H}^{\gamma_i} = \frac{\lambda}{8\pi^2}\sum_{k=1}^{J}\tilde{\mathcal{H}}^{\gamma_i}_{[k]}, \tag{5.25}$$

where we define

$$\begin{aligned}\tilde{\mathcal{H}}^{\gamma_i}_{[k]} \equiv \sum_{m,n,p,r,q,t=1}^{3} &e^{i\pi\xi(\alpha_{mr}-\alpha_{mn})}e^{i\pi\xi(\alpha_{qt}-\alpha_{pt})}e^{i\pi(\xi-1)(\alpha_{rq}-\alpha_{np})} \\ &\times e^m_m(k-1)\left[e^n_n(k)e^p_p(k+1)\ \mathcal{H}_{k,k+1}\ e^r_r(k)e^q_q(k+1)\right]e^t_t(k+2).\end{aligned} \tag{5.26}$$

5.1.3 Derivation of the coherent state action in the continuum limit to $O(\tilde{\lambda})$

The coherent state effective action for our γ_i-deformed $SU(3)$ spin chain is

$$S^{\gamma_i} = \int d\tau\left\{\langle\langle n|i\partial_\tau|n\rangle\rangle - \langle\langle n|\tilde{H}^{\gamma_i}|n\rangle\rangle\right\}, \tag{5.27}$$

where $\tilde{H}^{\gamma_i}$ is the equivalent γ_i-deformed Hamiltonian (5.25). We must, furthermore, consider the above expression in the continuum limit, in which the number of states in our spin chain becomes large. This γ_i-deformed coherent state effective action shall now be derived closely following [9].

Let us first calculate the expression $\langle\langle n|i\partial_\tau|n\rangle\rangle$, which is known as the Wess-Zumino (WZ) term, as follows:

$$\begin{aligned}\langle\langle n|i\partial_\tau|n\rangle\rangle &= \sum_{k=1}^{J}\langle n_k|i\partial_\tau|n_k\rangle \\ &= i\sum_{k=1}^{J}\left\{\sum_{i=1}^{3} m_i(k)e^{-ih_i(k)}\ \ \partial_\tau\left[m_i(k)e^{ih_i(k)}\right]\right\} \\ &= -\sum_{k=1}^{J}\left\{\sum_{i=1}^{3} m_i^2(k)\dot{h}_i(k)\right\},\end{aligned} \tag{5.28}$$

since $\sum\limits_{i=1}^{3} m_i\dot{m}_i = \frac{1}{2}\partial_\tau\left(\sum\limits_{i=1}^{3} m_i^2\right) = \frac{1}{2}\partial_\tau(1) = 0$.

We shall now determine the continuum limit of this expression by taking the length of the spin chain J to be large. In other words, if we label the sites with the variable $0 \le \sigma \le 2\pi$, the site spacing $a \equiv \frac{2\pi}{J} \to 0$ and the spatial variable σ becomes continuous. Our discrete summation then becomes an integral $\sum\limits_{k=1}^{J} \to \int \frac{d\sigma}{a} = J\int\frac{d\sigma}{2\pi}$. Hence Wess-Zumino term in the continuum limit is given by

$$\langle\langle n|i\partial_\tau|n\rangle\rangle \longrightarrow -J\int\frac{d\sigma}{2\pi}\sum_{i=1}^{3} m_i(\sigma)^2\dot{h}_i(\sigma). \tag{5.29}$$

Let us now calculate the γ_i-deformed coherent state Hamiltonian $\langle\langle n|\tilde{H}^{\gamma_i}|n\rangle\rangle$. Making use of the γ_i-deformed equivalent Hamiltonian (5.25), together with the coherent state (5.15), we find that

$$\langle\langle n|\tilde{H}^{\gamma_i}|n\rangle\rangle = \frac{\lambda}{8\pi^2}\sum_{k=1}^{J}\langle\langle n|\tilde{\mathcal{H}}^{\gamma_i}_{[k]}|n\rangle\rangle, \tag{5.30}$$

where

$$\begin{aligned}\langle\langle n|\tilde{\mathcal{H}}^{\gamma_i}_{[k]}|n\rangle\rangle &= \langle n_{k+2}|\otimes\langle n_{k+1}|\otimes\langle n_k|\otimes\langle n_{k-1}|\ \tilde{\mathcal{H}}^{\gamma_i}_{k}\ |n_{k-1}\rangle\otimes|n_k\rangle\otimes|n_{k+1}\rangle\otimes|n_{k+2}\rangle \\ &= \sum_{m,n,p,r,q,t=1}^{3} e^{i\pi\xi(\alpha_{mr}-\alpha_{mn})}e^{i\pi\xi(\alpha_{qt}-\alpha_{pt})}e^{i\pi(\xi-1)(\alpha_{rq}-\alpha_{np})} \\ &\quad\times\langle n_{k-1}|e^m_m(k-1)|n_{k-1}\rangle\ \langle n_{k+2}|e^t_t(k+2)|n_{k+2}\rangle \\ &\quad\times\langle n_{k+1}|\otimes\langle n_k|e^n_n(k)e^p_p(k+1)\ \mathcal{H}_{k,k+1}\ e^r_r(k)e^q_q(k+1)|n_k\rangle\otimes|n_{k+1}\rangle.\end{aligned} \tag{5.31}$$

This expression can be simplified using the following identities

$$\langle n_{k-1}|e^m_m(k-1)|n_{k-1}\rangle = m^2_m(k-1), \tag{5.32}$$

$$\begin{aligned}&\langle n_{k+1}|\otimes\langle n_k|e^n_n(k)e^p_p(k+1)\ \mathcal{H}_{k,k+1}\ e^r_r(k)e^q_q(k+1)|n_k\rangle\otimes|n_{k+1}\rangle\\ &= m^2_n(k)m^2_p(k+1)\delta^r_n\delta^q_p - m_n(k)m_p(k)m_n(k+1)m_p(k+1)\delta^q_n\delta^r_p e^{-ih_n(k)}e^{ih_n(k+1)}e^{ih_p(k)}e^{-ih_p(k+1)},\end{aligned} \tag{5.33}$$

$$\langle n_{k+2}|e^t_t(k+2)|n_{k+2}\rangle = m^2_t(k+2). \tag{5.34}$$

Here we have made use of the components $(\mathcal{H}_{k,k+1})^{rq}_{np} = \delta^r_n\delta^q_p - \delta^q_n\delta^r_p$ of the undeformed $\mathcal{H}_{k,k+1}$ matrix. Thus we obtain

$$\begin{aligned}\langle\langle n|\tilde{\mathcal{H}}^{\gamma_i}_{[k]}|n\rangle\rangle = &\sum_{m,n,p,r,q,t=1}^{3} e^{i\pi\xi(\alpha_{mr}-\alpha_{mn})}e^{i\pi\xi(\alpha_{qt}-\alpha_{pt})}e^{i\pi(\xi-1)(\alpha_{rq}-\alpha_{np})}m^2_m(k-1)m^2_t(k+2)\\ &\quad\times\big\{m^2_n(k)m^2_p(k+1)\delta^r_n\delta^q_p\\ &\qquad - m_n(k)m_p(k)m_n(k+1)m_p(k+1)e^{i[h_n(k+1)-h_n(k)-h_p(k+1)+h_p(k)]}\delta^q_n\delta^r_p\big\}\\ = &\sum_{m,n,p,t=1}^{3} m^2_m(k-1)m^2_n(k)m^2_p(k+1)m^2_t(k+2)\\ &-\sum_{m,n,p,t=1}^{3} e^{i\pi\xi(\alpha_{mp}-\alpha_{mn})}e^{i\pi\xi(\alpha_{nr}-\alpha_{pr})}e^{-2i\pi(\xi-1)\alpha_{np}}m^2_m(k-1)m^2_t(k+2)\\ &\quad\times m_n(k)m_p(k)m_n(k+1)m_p(k+1)e^{i[h_n(k+1)-h_n(k)-h_p(k+1)+h_p(k)]}.\end{aligned} \tag{5.35}$$

Changing $m\to q$ and $t\to r$, and noting that $\sum_{q=1}^{3} m_q(k-1)^2 = \sum_{r=1}^{3} m_r(k+2)^2 = 1$, we find that

$$\begin{aligned}\langle\langle n|\tilde{\mathcal{H}}^{\gamma_i}_{[k]}|n\rangle\rangle = &\sum_{n,p=1}^{3} m^2_n(k)m^2_p(k+1)\\ &-\sum_{n,p,q,r=1}^{3} m^2_q(k-1)e^{i\pi\xi(\alpha_{qp}-\alpha_{qn})}m^2_r(k+2)e^{i\pi\xi(\alpha_{nr}-\alpha_{pr})}e^{-2i\pi(\xi-1)\alpha_{np}}\\ &\quad\times m_n(k)m_p(k)m_n(k+1)m_p(k+1)e^{i[h_n(k+1)-h_n(k)-h_p(k+1)+h_p(k)]}.\end{aligned} \tag{5.36}$$

We would now like to calculate the continuum limit of the γ_i-deformed coherent state Hamiltonian. Towards this end, we shall expand $\langle\langle n|\tilde{\mathcal{H}}^{\gamma_i}_{[k]}|n\rangle\rangle$ to 2nd order in α_{ij} and the site spacing a, which are both taken to be of $O(\frac{1}{J})$. We shall also make use of the definitions

$$\partial m_i(k) \equiv \frac{m_i(k+1)-m_i(k)}{a} \qquad \text{and} \qquad \partial h_i \equiv \frac{h_i(k+1)-h_i(k)}{a}, \tag{5.37}$$

which are the discrete versions of the spatial derivatives of m_i and h_i respectively. Thus we find that

$$
\begin{aligned}
\langle\langle n|\tilde{\mathcal{H}}^{\gamma_i}_{[k]}|n\rangle\rangle = & \sum_{n,p=1}^{3} m_n^2(k)m_p^2(k+1) \\
& - \sum_{n,p,q,r=1}^{3} m_q^2(k-1)m_r^2(k+2)m_n(k)m_p(k)m_n(k+1)m_p(k+1) \\
& \quad \times \left[1 + i\pi\xi(\alpha_{qp}-\alpha_{qn}) - \frac{1}{2}\pi^2\xi^2(\alpha_{qp}-\alpha_{qn})^2\right] \\
& \quad \times \left[1 + i\pi\xi(\alpha_{nr}-\alpha_{pr}) - \frac{1}{2}\pi^2\xi^2(\alpha_{nr}-\alpha_{pr})^2\right] \\
& \quad \times \left[1 - 2i\pi(\xi-1)\alpha_{np} - 2\pi^2(\xi-1)^2\alpha_{np}^2\right] \\
& \quad \times \left[1 + ia\left(\partial h_n(k) - \partial h_n(k)\right) - \frac{1}{2}a^2\left(\partial h_n(k) - \partial h_p(k)\right)^2\right]. \qquad (5.38)
\end{aligned}
$$

Multiplying out the brackets, keeping only terms up to 2[nd] order, and noticing that all the 1[st] order terms vanish, since they involve the contraction of a symmetric expression with an anti-symmetric one, gives

$$
\begin{aligned}
\langle\langle n|\tilde{\mathcal{H}}^{\gamma_i}_{[k]}|n\rangle\rangle = & \sum_{n,p=1}^{3} \left\{m_n^2(k)m_p^2(k+1) - m_n(k)m_p(k)m_n(k+1)m_p(k+1)\right\} \\
& - \sum_{n,p,q,r=1}^{3} m_q^2(k-1)m_r^2(k+2)m_n(k)m_p(k)m_n(k+1)m_p(k+1) \\
& \times \Big\{-\frac{1}{2}\pi^2\xi^2(\alpha_{qp}-\alpha_{qn})^2 - \frac{1}{2}\pi^2\xi^2(\alpha_{nr}-\alpha_{pr})^2 - 2\pi^2(\xi-1)^2\alpha_{np}^2 \\
& \quad - \frac{1}{2}a^2\left(\partial h_n(k) - \partial h_p(k)\right)^2 - \pi^2\xi^2(\alpha_{qp}-\alpha_{qn})(\alpha_{nr}-\alpha_{pr}) \\
& \quad + 2\pi^2\xi(\xi-1)\alpha_{np}(\alpha_{qp}-\alpha_{qn}) - \pi\xi a(\alpha_{qp}-\alpha_{qn})\left(\partial h_n(k) - \partial h_p(k)\right) \\
& \quad + 2\pi^2\xi(\xi-1)\alpha_{np}(\alpha_{nr}-\alpha_{pr}) - \pi\xi a(\alpha_{nr}-\alpha_{pr})\left(\partial h_n(k) - \partial h_p(k)\right) \\
& \quad + 2\pi(\xi-1)a\alpha_{np}\left(\partial h_n(k) - \partial h_p(k)\right)\Big\}, \qquad (5.39)
\end{aligned}
$$

which can be written in the simplified form

$$
\begin{aligned}
\langle\langle n|\tilde{\mathcal{H}}^{\gamma_i}_{[k]}|n\rangle\rangle = a^2 \Bigg\{ & \sum_{i=1}^{3} [\partial m_i(k)]^2 + \frac{1}{2}\sum_{i,j=1}^{3} m_i(k)^2 m_j(k)^2 \left[\partial h_i(k) - \partial h_j(k) + \frac{2\pi}{a}\alpha_{ij}\right]^2 \\
& - \left(\frac{2\pi}{a}\right)^2 2\xi(1-\xi)\; m_1(k)^2 m_2(k)^2 m_3(k)^2 (\alpha_{12}+\alpha_{31}+\alpha_{23})^2 \Bigg\}. \qquad (5.40)
\end{aligned}
$$

We can thus determine the γ_i-deformed coherent state Hamiltonian in the continuum limit by, again, labelling our spin chain sites with the continuous variable $0 \le \sigma \le 2\pi$ and changing $\sum_{k=1}^{J} \to \int \frac{d\sigma}{a}$, where $a = \frac{2\pi}{J}$ is the site spacing, as follows:

$$\langle\langle n|\tilde{H}^{\gamma_i}|n\rangle\rangle = \frac{\lambda}{8\pi^2}\sum_{k=1}^{J}\langle\langle n|\tilde{\mathcal{H}}^{\gamma_i}_{[k]}|n\rangle\rangle \longrightarrow \frac{\lambda}{8\pi^2}\int \frac{d\sigma}{a}\ \langle\langle n|\tilde{\mathcal{H}}^{\gamma_i}_{[k]}|n\rangle\rangle. \tag{5.41}$$

Hence, in the continuum limit, the coherent state Hamiltonian is given by

$$\begin{aligned}\langle\langle n|\tilde{H}^{\gamma_i}|n\rangle\rangle = {} & \frac{1}{2}\tilde{\lambda}J\int\frac{d\sigma}{2\pi} \\ & \times\left\{\sum_{i=1}^{3} m_i'(\sigma)^2 + \frac{1}{2}\sum_{i,j=1}^{3} m_i(\sigma)^2 m_j(\sigma)^2\left[h_i'(\sigma) - h_j'(\sigma) - \sum_{k=1}^{3}\varepsilon_{ijk}J\gamma_k\right]^2\right. \\ & \left. \quad - 2\xi(1-\xi)\ m_1(\sigma)^2 m_2(\sigma)^2 m_3(\sigma)^2\left(\sum_{i=1}^{3} J\gamma_i\right)^2\right\}, \end{aligned} \tag{5.42}$$

where we have set $\tilde{\lambda} = \frac{\lambda}{J^2}$ and noted that $\alpha_{12} = -\gamma_3$, $\alpha_{31} = -\gamma_2$ and $\alpha_{23} = -\gamma_1$.

We shall now choose the parameter ξ, as in [9], such that we recover the correct γ_i-deformed vacuum states from the γ_i-deformed potential. This potential, which is now a function of the parameter ξ, can be derived from the γ_i-deformed coherent state effective Hamiltonian (5.42) and is given by

$$V^{\gamma_i}(\xi) \sim m_1^2 m_2^2\gamma_3^2 + m_3^2 m_1^2\gamma_2^2 + m_2^2 m_3^2\gamma_1^2 - 2\xi\,(1-\xi)\, m_1^2 m_2^2 m_3^2\,(\gamma_1+\gamma_2+\gamma_3)^2\,. \tag{5.43}$$

Firstly, the three vacuum states with $J_i = J$ correspond to setting $m_j = \delta_{ij}$, which automatically results in the γ_i-deformed potential vanishing for all values of ξ. The vacuum state $(J_1, J_2, J_3) \sim (\gamma_1, \gamma_2, \gamma_3)$, however, does place a constraint on the parameter ξ. This state corresponds to $m_i = \sqrt{\frac{\gamma_i}{\gamma_1+\gamma_2+\gamma_3}}$ and, for this to yield $V^{\gamma_i}(\xi) = 0$, we must require that $2\xi(1-\xi) = 1$.

Therefore the coherent state Hamiltonian to leading order in $\tilde{\lambda}$, with the parameter ξ chosen so as to give the correct γ_i-deformed vacuum states, is

$$\langle\langle n|\tilde{H}^{\gamma_i}|n\rangle\rangle = \frac{1}{2}\tilde{\lambda}J\int\frac{d\sigma}{2\pi}\left\{\sum_{i=1}^{3}(m_i')^2 + \frac{1}{2}\sum_{i,j=1}^{3} m_i^2 m_j^2\left[h_i' - h_j' - \sum_{k=1}^{3}\varepsilon_{ijk}\bar{\gamma}_k\right]^2 - \bar{\gamma}^2 m_1^2 m_2^2 m_3^2\right\}, \tag{5.44}$$

where we have defined $\bar{\gamma}_i \equiv \frac{\gamma_i}{J}$ and $\bar{\gamma} \equiv \sum_{i=1}^{3}\bar{\gamma}_i$.

Finally, we can determine the coherent state effective action by substituting the results (5.28) and (5.44) into the expression (5.27). Hence we obtain

$$S^{\gamma_i} = -J \int d\tau \, \frac{d\sigma}{2\pi} \left[\mathcal{L}^{\gamma_i} + O\left(\tilde{\lambda}^2\right) \right], \tag{5.45}$$

where the Lagrangian to leading order in $\tilde{\lambda}$ is given by

$$\mathcal{L}^{\gamma_i} = \sum_{i=1}^{3} m_i^2 \dot{h}_i + \frac{1}{2}\tilde{\lambda} \left\{ \sum_{i=1}^{3} (m_i')^2 + \frac{1}{2} \sum_{i,j=1}^{3} m_i^2 m_j^2 \left[h_i' - h_j' - \sum_{k=1}^{3} \varepsilon_{ijk} \bar{\gamma}_k \right]^2 - \bar{\gamma}^2 m_1^2 m_2^2 m_3^2 \right\}. \tag{5.46}$$

5.2 γ_i-deformed String Worldsheet Action for Rotating Strings in the Fast Motion Limit

We would now like to construct the γ_i-deformed string worldsheet action in the relevant semiclassical limit and compare this with the coherent state effective action of a γ_i-deformed $SU(3)$ spin chain following [9]. This semiclassical or fast motion limit is obtained by considering strings in a γ_i-deformed $\mathbb{R} \times S^5$ background moving with a large total angular momentum J. Furthermore, we shall assume that the time derivatives of the radii r_i and 'slow' angular coordinates φ_1 and φ_2, and the deformation parameters $\hat{\gamma}_i$ are of $O(\tilde{\lambda})$ and $O(\sqrt{\tilde{\lambda}})$ respectively, where $\tilde{\lambda} = \frac{\lambda}{J}$ is a small fixed parameter. For comparison purposes, the string worldsheet metric shall be specified using the non-diagonal uniform gauge of [18], in which the total angular momentum is taken to be spread evenly along the string (as it is along the spin chain). It was shown in [21] that this gauge can be more easily implemented in the T-dual space of the 'fast' or total angular coordinate $\psi = \varphi_3$.

In this section, we construct the γ_i-deformed string worldsheet action in the fast motion limit to $O(\tilde{\lambda}^2)$. The $U(1)$ charge and current densities p_i and j_i corresponding to the angular coordinate ϕ_i shall also be derived to $O(\tilde{\lambda})$ from this action. The requirement that these charge and current densities should remain unchanged by the γ_i-deformation, even in the fast motion limit, then results in a relation between the differences of the undeformed and γ_i-deformed angular coordinates.

5.2.1 Derivation of the γ_i-deformed string worldsheet action in the fast motion limit to $O(\tilde{\lambda}^2)$

We shall begin by considering the action (4.90) describing strings moving in a γ_i-deformed $\mathbb{R}\times S^5$ background. Let us now rewrite this γ_i-deformed string worldsheet action in terms of the coordinates φ_1, φ_2 and φ_3, which satisfy

$$\phi_1 = \varphi_3 - \varphi_2, \qquad \phi_2 = \varphi_1 + \varphi_2 + \varphi_3, \qquad \phi_3 = \varphi_3 - \varphi_1, \tag{5.47}$$

where $\psi = \varphi_3$ is the 'fast' or total angular coordinate, as follows:

$$S^{\gamma_i} = -\frac{\sqrt{\lambda}}{2}\int d\tau\,\frac{d\sigma}{2\pi} \tag{5.48}$$
$$\times\left\{\sqrt{-h}h^{\alpha\beta}\left[-\partial_\alpha t\partial_\beta t + \sum_{i=1}^{3}\partial_\alpha r_i\partial_\beta r_i + \sum_{i,j=1}^{3}G_{ij}\partial_\alpha\varphi_i\partial_\beta\varphi_j\right] - \varepsilon^{\alpha\beta}\left[\sum_{i,j=1}^{3}B_{ij}\partial_\alpha\varphi_i\partial_\beta\varphi_j\right]\right\}.$$

Here G_{ij} is a symmetric metric with components

$$\begin{aligned}
G_{11} &= G\left[(r_2^2+r_3^2) + r_1^2r_2^2r_3^2\left(\hat\gamma_2-\hat\gamma_3\right)^2\right],\\
G_{22} &= G\left[(r_1^2+r_2^2) + r_1^2r_2^2r_3^2\left(\hat\gamma_2-\hat\gamma_1\right)^2\right],\\
G_{33} &= G\left[1 + r_1^2r_2^2r_3^2\left(\hat\gamma_1+\hat\gamma_2+\hat\gamma_3\right)^2\right],\\
G_{12} &= G\left[r_2^2 + r_1^2r_2^2r_3^2\left(\hat\gamma_2-\hat\gamma_3\right)\left(\hat\gamma_2-\hat\gamma_1\right)\right],\\
G_{31} &= G\left[(r_2^2-r_3^2) + r_1^2r_2^2r_3^2\left(\hat\gamma_1+\hat\gamma_2+\hat\gamma_3\right)\left(\hat\gamma_2-\hat\gamma_3\right)\right],\\
G_{23} &= G\left[(r_2^2-r_1^2) + r_1^2r_2^2r_3^2\left(\hat\gamma_1+\hat\gamma_2+\hat\gamma_3\right)\left(\hat\gamma_2-\hat\gamma_1\right)\right],
\end{aligned}\tag{5.49}$$

and B_{ij} is anti-symmetric matrix with

$$\begin{aligned}
B_{12} &= G\left(\hat\gamma_3r_1^2r_2^2 + \hat\gamma_2r_3^2r_1^2 + \hat\gamma_1r_2^2r_3^2\right),\\
B_{31} &= G\left(\hat\gamma_3r_1^2r_2^2 + \hat\gamma_2r_3^2r_1^2 - 2\hat\gamma_1r_2^2r_3^2\right),\\
B_{23} &= G\left(-2\hat\gamma_3r_1^2r_2^2 + \hat\gamma_2r_3^2r_1^2 + \hat\gamma_1r_2^2r_3^2\right).
\end{aligned}\tag{5.50}$$

Note also that, in the above action, we have neglected to mention the constraint term, which ensures that the square of the radii r_i sum to one. This term is of no importance in the present discussion and thus, for convenience, has been left out.

Now, at this point, it is necessary for us to make a specific choice for the string worldsheet metric $h_{\alpha\beta}$. One possibility is to simply use the diagonal conformal gauge in which $\sqrt{-h}h^{\alpha\beta} = \mathrm{diag}\,(-1,1)$. It was noted in [18], however, that, if one wants to compare the resulting string action to the coherent state effective action of a spin chain, one must rather use a non-diagonal uniform gauge in which the total angular momentum is spread evenly along the string. A procedure for implementing this gauge was described in [21] and involves first performing a T-duality transformation on the 'fast' angular coordinate $\psi = \varphi_3$. Hence we obtain

$$\begin{aligned} S^{\gamma_i} = -\frac{\sqrt{\lambda}}{2} \int d\tau \, \frac{d\sigma}{2\pi} \Big\{ \sqrt{-h} h^{\alpha\beta} \Big[& - \partial_\alpha t \partial_\beta t + \partial_\alpha r_i \partial_\beta r_i \\ & + \tilde{G}_{11} \partial_\alpha \varphi_1 \partial_\beta \varphi_1 + \tilde{G}_{22} \partial_\alpha \varphi_2 \partial_\beta \varphi_2 + \tilde{G}_{33} \partial_\alpha \tilde{\varphi}_3 \partial_\beta \tilde{\varphi}_3 \\ & + \tilde{G}_{12} \partial_\alpha \varphi_1 \partial_\beta \varphi_2 + \tilde{G}_{21} \partial_\alpha \varphi_2 \partial_\beta \varphi_1 + \tilde{G}_{31} \partial_\alpha \tilde{\varphi}_3 \partial_\beta \varphi_1 \\ & + \tilde{G}_{13} \partial_\alpha \varphi_1 \partial_\beta \tilde{\varphi}_3 + \tilde{G}_{23} \partial_\alpha \varphi_2 \partial_\beta \tilde{\varphi}_3 + \tilde{G}_{32} \partial_\alpha \tilde{\varphi}_3 \partial_\beta \varphi_2 \Big] \\ - 2\varepsilon^{\alpha\beta} \Big[& \tilde{B}_{12} \partial_\alpha \varphi_1 \partial_\beta \varphi_2 + \tilde{B}_{31} \partial_\alpha \tilde{\varphi}_3 \partial_\beta \varphi_1 + \tilde{B}_{23} \partial_\alpha \varphi_2 \partial_\beta \tilde{\varphi}_3 \Big] \Big\} , \end{aligned} \tag{5.51}$$

where we have used $\tilde{\varphi}_1 = \varphi_1$ and $\tilde{\varphi}_2 = \varphi_2$, together with the definitions

$$\begin{aligned} &\tilde{G}_{11} = G_{11} - \frac{G_{31}^2 - B_{31}^2}{G_{33}} & &\tilde{G}_{22} = G_{22} - \frac{G_{23}^2 - B_{23}^2}{G_{33}}, & &\tilde{G}_{33} = \frac{1}{G_{33}}, \\ &\tilde{G}_{12} = G_{12} - \frac{G_{31}G_{23} + B_{31}B_{23}}{G_{33}}, & &\tilde{G}_{31} = \frac{B_{31}}{G_{33}}, & &\tilde{G}_{23} = -\frac{B_{23}}{G_{33}}, \end{aligned} \tag{5.52}$$

and

$$\tilde{B}_{12} = B_{12} + \frac{G_{31}B_{23} + B_{31}G_{23}}{G_{33}}, \qquad \tilde{B}_{31} = \frac{G_{31}}{G_{33}}, \qquad \tilde{B}_{23} = -\frac{G_{23}}{G_{33}}. \tag{5.53}$$

We shall now convert to the Nambu-Goto form of this γ_i-deformed string worldsheet action in the T-dual space, which is given by

$$S^{\gamma_i} = -\sqrt{\lambda} \int d\tau \, \frac{d\sigma}{2\pi} \Big\{ \sqrt{-h} - \varepsilon^{\alpha\beta} \Big[\tilde{B}_{12} \partial_\alpha \varphi_1 \partial_\beta \varphi_2 + \tilde{B}_{31} \partial_\alpha \tilde{\varphi}_3 \partial_\beta \varphi_1 + \tilde{B}_{23} \partial_\alpha \varphi_2 \partial_\beta \tilde{\varphi}_3 \Big] \Big\} , \tag{5.54}$$

where

$$\begin{aligned} h_{\alpha\beta} = -\partial_\alpha t \partial_\beta t + \partial_\alpha r_i \partial_\beta r_i + \tilde{G}_{11} \partial_\alpha \varphi_1 \partial_\beta \varphi_1 + \tilde{G}_{22} \partial_\alpha \varphi_2 \partial_\beta \varphi_2 + \tilde{G}_{33} \partial_\alpha \tilde{\varphi}_3 \partial_\beta \tilde{\varphi}_3 \\ + \tilde{G}_{12} \partial_\alpha \varphi_1 \partial_\beta \varphi_2 + \tilde{G}_{21} \partial_\alpha \varphi_2 \partial_\beta \varphi_1 + \tilde{G}_{31} \partial_\alpha \tilde{\varphi}_3 \partial_\beta \varphi_1 + \tilde{G}_{13} \partial_\alpha \varphi_1 \partial_\beta \tilde{\varphi}_3 \\ + \tilde{G}_{23} \partial_\alpha \varphi_2 \partial_\beta \tilde{\varphi}_3 + \tilde{G}_{32} \partial_\alpha \tilde{\varphi}_3 \partial_\beta \varphi_2 . \end{aligned} \tag{5.55}$$

The original Polyakov action (5.51) can be proved to be equivalent to this Nambu-Goto action (5.54) in exactly the same way discussed in section 4.1.2. Since this derivation relies upon the equation of motion corresponding to $h_{\alpha\beta}$, the extra term $\varepsilon^{\alpha\beta}[\ldots]$, which is independent of $h_{\alpha\beta}$, is largely unimportant.

Let us now note that the total angular momentum 2-current, which corresponds to the 'fast' angular coordinate $\psi = \varphi_3$, is given by

$$p^\alpha = \varepsilon^{\alpha\beta}\partial_\beta\tilde{\psi}, \tag{5.56}$$

and thus the total angular momentum charge density is $p^0 = \partial_1\tilde{\psi}$. The non-diagonal uniform gauge, in which $p^0 = \mathcal{J}$ is constant along the string (in the spatial worldsheet coordinate σ), can be obtained by setting $\tilde{\psi} = \tilde{\varphi}_3 = \mathcal{J}\sigma$. We shall also take t to be simply the temporal worldsheet coordinate or proper time τ.

Thus the components of the string worldsheet metric in the non-diagonal uniform gauge can be calculated from (5.55) to be

$$\begin{aligned}
h_{00} &= -1 + \dot{r}_i^2 + \tilde{G}_{11}\dot{\varphi}_1^2 + \tilde{G}_{22}\dot{\varphi}_2^2 + 2\tilde{G}_{12}\dot{\varphi}_1\dot{\varphi}_2,\\
h_{11} &= (r_i')^2 + \tilde{G}_{11}(\varphi_1')^2 + \tilde{G}_{22}(\varphi_2')^2 + \mathcal{J}^2\tilde{G}_{33} + 2\tilde{G}_{12}\varphi_1'\varphi_2' + 2\mathcal{J}\tilde{G}_{31}\varphi_1' + 2\mathcal{J}\tilde{G}_{23}\varphi_2',\\
h_{01} &= h_{01} = \dot{r}_i r_i' + \tilde{G}_{11}\dot{\varphi}_1\varphi_1' + \tilde{G}_{22}\dot{\varphi}_2\varphi_2' + \tilde{G}_{12}(\dot{\varphi}_1\varphi_2' + \varphi_1'\dot{\varphi}_2) + \mathcal{J}\tilde{G}_{31}\dot{\varphi}_1 + \mathcal{J}\tilde{G}_{23}\dot{\varphi}_2,
\end{aligned} \tag{5.57}$$

The γ_i-deformed string worldsheet action can hence be written as

$$S^{\gamma_i} = -\sqrt{\lambda}\int d\tau\,\frac{d\sigma}{2\pi}\left\{\sqrt{-h} - \varepsilon^{\alpha\beta}\left[\tilde{B}_{12}\left(\dot{\varphi}_1\varphi_2' - \varphi_1'\dot{\varphi}_2\right) - \tilde{B}_{31}\mathcal{J}\dot{\varphi}_1 + \tilde{B}_{23}\mathcal{J}\dot{\varphi}_2\right]\right\}, \tag{5.58}$$

in terms of the determinant $h = h_{00}h_{11} - (h_{01})^2$.

We shall now define the small parameter $\tilde{\lambda} \equiv \frac{\lambda}{J^2} = \frac{1}{\mathcal{J}^2}$, since our total angular momentum is $J = \sqrt{\lambda}\int_0^{2\pi}\frac{d\sigma}{2\pi}\mathcal{J} = \sqrt{\lambda}\mathcal{J}$, and construct a semiclassical expansion of the action (5.58) to $O(\tilde{\lambda}^2)$. We shall assume that the radii and 'slow' angular coordinate φ_1 and φ_2 vary slowly with time. More specifically, we shall make the redefinition $\tau \to \frac{1}{\tilde{\lambda}}\tau$, so that $\frac{\partial}{\partial\tau} \to \tilde{\lambda}\frac{\partial}{\partial\tau}$ and therefore all the time derivatives of the remaining coordinates (the 'fast' angular coordinate has been eliminated) become of $O(\tilde{\lambda})$. We shall also assume the deformation parameters $\hat{\gamma}_i$ to be of $O(\sqrt{\tilde{\lambda}})$ and define the $\tilde{\lambda}$-independent parameters $\bar{\gamma}_i \equiv \mathcal{J}\hat{\gamma}_i = J\gamma_i$. (This is consistent with the earlier definition in section 5.1.3 of $\bar{\gamma}_i$ in terms of the gauge theory deformation parameters γ_i.) Hence we obtain the γ_i-deformed string worldsheet action

$$S^{\gamma_i} = -J\int d\tau\,\frac{d\sigma}{2\pi}\left\{\sqrt{-\frac{1}{\mathcal{J}^2}\,h} - \frac{1}{\mathcal{J}^3}\varepsilon^{\alpha\beta}\left[\tilde{B}_{12}\left(\dot{\varphi}_1\varphi_2' - \varphi_1'\dot{\varphi}_2\right) - \tilde{B}_{31}\mathcal{J}\dot{\varphi}_1 + \tilde{B}_{23}\mathcal{J}\dot{\varphi}_2\right]\right\}, \tag{5.59}$$

where

$$h = \left[-1 + \frac{1}{\mathcal{J}^4} \left(\dot{r}_i^2 + \tilde{G}_{11} \dot{\varphi}_1^2 + \tilde{G}_{22} \dot{\varphi}_2^2 + 2\tilde{G}_{12} \dot{\varphi}_1 \dot{\varphi}_2 \right) \right] \tag{5.60}$$

$$\times \left[(r_i')^2 + \tilde{G}_{11} (\varphi_1')^2 + \tilde{G}_{22} (\varphi_2')^2 + \mathcal{J}^2 \tilde{G}_{33} + 2\tilde{G}_{12} \varphi_1' \varphi_2' + 2\mathcal{J} \tilde{G}_{31} \varphi_1' + 2\mathcal{J} \tilde{G}_{23} \varphi_2' \right]$$

$$- \frac{1}{\mathcal{J}^4} \left[\dot{r}_i r_i' + \tilde{G}_{11} \dot{\varphi}_1 \varphi_1' + \tilde{G}_{22} \dot{\varphi}_2 \varphi_2' + \tilde{G}_{12} \left(\dot{\varphi}_1 \varphi_2' + \varphi_1' \dot{\varphi}_2 \right) + \mathcal{J} \tilde{G}_{31} \dot{\varphi}_1 + \mathcal{J} \tilde{G}_{23} \dot{\varphi}_2 \right]^2 .$$

We only want to consider terms up to $O(\frac{1}{\mathcal{J}^4})$ in our Lagrangian and thus, neglecting higher order terms, we find that

$$-\frac{1}{\mathcal{J}^2}\, h = \tilde{G}_{33} + \frac{1}{\mathcal{J}^2} \left[(r_i')^2 + \tilde{G}_{11} (\varphi_1')^2 + \tilde{G}_{22} (\varphi_2')^2 + 2\tilde{G}_{12} \varphi_1' \varphi_2' + 2\mathcal{J} \tilde{G}_{31} \varphi_1' + 2\mathcal{J} \tilde{G}_{23} \varphi_2' \right]$$

$$- \frac{1}{\mathcal{J}^4}\, \tilde{G}_{33} \left(\dot{r}_i^2 + \tilde{G}_{11} \dot{\varphi}_1^2 + \tilde{G}_{22} \dot{\varphi}_2^2 + 2\tilde{G}_{12} \dot{\varphi}_1 \dot{\varphi}_2 \right) + O\left(\frac{1}{\mathcal{J}^6} \right) . \tag{5.61}$$

Now we need to calculate $\tilde{G}_{ij}$ and $\tilde{B}_{ij}$ explicitly using the definitions (5.52) and (5.53), which are given in terms of the expressions (5.49) and (5.50). Keeping only the necessary orders in $\frac{1}{\mathcal{J}}$, we obtain

$$\begin{aligned}
\tilde{G}_{11} &= \left(r_1^2 r_2^2 + r_3^2 r_1^2 + 4 r_2^2 r_3^2 \right) - \frac{1}{\mathcal{J}^2} r_1^2 r_2^2 r_3^2 \bar{\gamma}^2 \left(r_1^2 r_2^2 + r_3^2 r_1^2 + 4 r_2^2 r_3^2 \right) + O\left(\frac{1}{\mathcal{J}^4} \right) , \\
\tilde{G}_{22} &= \left(4 r_1^2 r_2^2 + r_3^2 r_1^2 + r_2^2 r_3^2 \right) - \frac{1}{\mathcal{J}^2} r_1^2 r_2^2 r_3^2 \bar{\gamma}^2 \left(4 r_1^2 r_2^2 + r_3^2 r_1^2 + r_2^2 r_3^2 \right) + O\left(\frac{1}{\mathcal{J}^4} \right) , \\
\tilde{G}_{33} &= 1 + \frac{1}{\mathcal{J}^2} \left(\bar{\gamma}_3^2 r_1^2 r_2^2 + \bar{\gamma}_2^2 r_3^2 r_1^2 + \bar{\gamma}_1^2 r_2^2 r_3^2 - \bar{\gamma}^2 r_1^2 r_2^2 r_3^2 \right) \\
&\quad - \frac{1}{\mathcal{J}^4}\, r_1^2 r_2^2 r_3^2 \bar{\gamma}^2 \left(\bar{\gamma}_3^2 r_1^2 r_2^2 + \bar{\gamma}_2^2 r_3^2 r_1^2 + \bar{\gamma}_1^2 r_2^2 r_3^2 - \bar{\gamma}^2 r_1^2 r_2^2 r_3^2 \right) + O\left(\frac{1}{\mathcal{J}^6} \right) , \\
\tilde{G}_{12} &= \left(2 r_1^2 r_2^2 - r_3^2 r_1^2 + 2 r_2^2 r_3^2 \right) - \frac{1}{\mathcal{J}^2} r_1^2 r_2^2 r_3^2 \bar{\gamma}^2 \left(2 r_1^2 r_2^2 - r_3^2 r_1^2 + 2 r_2^2 r_3^2 \right) + O\left(\frac{1}{\mathcal{J}^4} \right) , \\
\tilde{G}_{31} &= \frac{1}{\mathcal{J}} \left(\bar{\gamma}_3 r_1^2 r_2^2 + \bar{\gamma}_2 r_3^2 r_1^2 - 2\bar{\gamma}_1 r_2^2 r_3^2 \right) - \frac{1}{\mathcal{J}^3}\, r_1^2 r_2^2 r_3^2 \bar{\gamma}^2 \left(\bar{\gamma}_3 r_1^2 r_2^2 + \bar{\gamma}_2 r_3^2 r_1^2 - 2\bar{\gamma}_1 r_2^2 r_3^2 \right) + O\left(\frac{1}{\mathcal{J}^5} \right) , \\
\tilde{G}_{23} &= \frac{1}{\mathcal{J}} \left(2\bar{\gamma}_3 r_1^2 r_2^2 - \bar{\gamma}_2 r_3^2 r_1^2 - \bar{\gamma}_1 r_2^2 r_3^2 \right) - \frac{1}{\mathcal{J}^3} r_1^2 r_2^2 r_3^2 \bar{\gamma}^2 \left(2\bar{\gamma}_3 r_1^2 r_2^2 - \bar{\gamma}_2 r_3^2 r_1^2 - \bar{\gamma}_1 r_2^2 r_3^2 \right) + O\left(\frac{1}{\mathcal{J}^5} \right) ,
\end{aligned} \tag{5.62}$$

and

$$\begin{aligned}
\tilde{B}_{12} &= \frac{1}{\mathcal{J}} \left(3 r_1^2 r_2^2 r_3^2 \bar{\gamma} \right) + O\left(\frac{1}{\mathcal{J}^3} \right) , \\
\tilde{B}_{31} &= \left(r_2^2 - r_3^2 \right) + \frac{1}{\mathcal{J}^2}\, r_1^2 r_2^2 r_3^2 \bar{\gamma} \left[(\bar{\gamma}_2 - \bar{\gamma}_3) - \bar{\gamma} \left(r_2^2 - r_3^2 \right) \right] + O\left(\frac{1}{\mathcal{J}^4} \right) , \\
\tilde{B}_{23} &= \left(r_1^2 - r_2^2 \right) + \frac{1}{\mathcal{J}^2}\, r_1^2 r_2^2 r_3^2 \bar{\gamma} \left[(\bar{\gamma}_1 - \bar{\gamma}_2) - \bar{\gamma} (r_1^2 - r_2^2) \right] + O\left(\frac{1}{\mathcal{J}^4} \right) .
\end{aligned} \tag{5.63}$$

with $\bar{\gamma} \equiv \sum_{i=1}^{3} \bar{\gamma}_i$. Hence, substituting these expressions into (5.61), we determine

$$-\frac{1}{\mathcal{J}^2}\ h = 1 + \tilde{\lambda} a + \tilde{\lambda}^2 b + O(\tilde{\lambda}^3), \tag{5.64}$$

where we define a and b explicitly as

$$\begin{aligned} a \equiv{} & \left(\bar{\gamma}_3^2 r_1^2 r_2^2 + \bar{\gamma}_2^2 r_3^2 r_1^2 + \bar{\gamma}_1^2 r_2^2 r_3^2 - \bar{\gamma}^2 r_1^2 r_2^2 r_3^2\right) + \sum_{i=1}^{3} \left(r_i'\right)^2 \\ & + \left(r_1^2 r_2^2 + r_3^2 r_1^2 + 4 r_2^2 r_3^2\right) \left(\varphi_1'\right)^2 + \left(4 r_1^2 r_2^2 + r_3^2 r_1^2 + r_2^2 r_3^2\right) \left(\varphi_2'\right)^2 \\ & + 2\left(2 r_1^2 r_2^2 - r_3^2 r_1^2 + 2 r_2^2 r_3^2\right) \varphi_1' \varphi_2' + 2\left(\bar{\gamma}_3 r_1^2 r_2^2 + \bar{\gamma}_2 r_3^2 r_1^2 - 2\bar{\gamma}_1 r_2^2 r_3^2\right) \varphi_1' \\ & + 2\left(2\bar{\gamma}_3 r_1^2 r_2^2 - \bar{\gamma}_2 r_3^2 r_1^2 - \bar{\gamma}_1 r_2^2 r_3^2\right) \varphi_2', \end{aligned} \tag{5.65}$$

$$\begin{aligned} b \equiv{} & -r_1^2 r_2^2 r_3^2 \bar{\gamma}^2 \left[\left(\bar{\gamma}_3^2 r_1^2 r_2^2 + \bar{\gamma}_2^2 r_3^2 r_1^2 + \bar{\gamma}_1^2 r_2^2 r_3^2 - \bar{\gamma}^2 r_1^2 r_2^2 r_3^2\right)\right. \\ & \qquad + \left(r_1^2 r_2^2 + r_3^2 r_1^2 + 4 r_2^2 r_3^2\right) \left(\varphi_1'\right)^2 + \left(4 r_1^2 r_2^2 + r_3^2 r_1^2 + r_2^2 r_3^2\right) \left(\varphi_2'\right)^2 \\ & \qquad + 2\left(2 r_1^2 r_2^2 - r_3^2 r_1^2 + 2 r_2^2 r_3^2\right) \varphi_1' \varphi_2' + 2\left(\bar{\gamma}_3 r_1^2 r_2^2 + \bar{\gamma}_2 r_3^2 r_1^2 - 2\bar{\gamma}_1 r_2^2 r_3^2\right) \varphi_1' \\ & \qquad \left. +2\left(2\bar{\gamma}_3 r_1^2 r_2^2 - \bar{\gamma}_2 r_3^2 r_1^2 - \bar{\gamma}_1 r_2^2 r_3^2\right) \varphi_2'\right] \\ & - \dot{r}_i^2 - \left(r_1^2 r_2^2 + r_3^2 r_1^2 + 4 r_2^2 r_3^2\right) \dot{\varphi}_1^2 - \left(4 r_1^2 r_2^2 + r_3^2 r_1^2 + r_2^2 r_3^2\right) \dot{\varphi}_2^2 \\ & - 2\left(2 r_1^2 r_2^2 - r_3^2 r_1^2 + 2 r_2^2 r_3^2\right) \dot{\varphi}_1 \dot{\varphi}_2. \end{aligned} \tag{5.66}$$

A binomial expansion of the square root then finally yields

$$\sqrt{-\frac{1}{\mathcal{J}^2}\ h} = 1 + \frac{1}{2}\tilde{\lambda} a + \tilde{\lambda}^2 \left(\frac{1}{2} b - \frac{1}{8} a^2\right) + O\left(\tilde{\lambda}^3\right). \tag{5.67}$$

Now, using (5.63), we calculate the second term in the Lagrangian to $O\left(\frac{1}{\mathcal{J}^4}\right)$ to be

$$\tilde{B}_{12}\left(\dot{\varphi}_1 \varphi_2' - \varphi_1' \dot{\varphi}_2\right) - \tilde{B}_{31} \mathcal{J} \dot{\varphi}_1 + \tilde{B}_{23} \mathcal{J} \dot{\varphi}_2 = \tilde{\lambda} c + \tilde{\lambda}^2 d + O\left(\tilde{\lambda}^3\right), \tag{5.68}$$

with

$$c \equiv -\left(r_2^2 - r_3^2\right) \dot{\varphi}_1 + \left(r_1^2 - r_2^2\right) \dot{\varphi}_2, \tag{5.69}$$

$$\begin{aligned} d \equiv{} & -r_1^2 r_2^2 r_3^2 \bar{\gamma} \left\{\bar{\gamma}\left[\left(\bar{\gamma}_2 - \bar{\gamma}_3\right) - \bar{\gamma}\left(r_2^2 - r_3^2\right)\right] \dot{\varphi}_1 + \bar{\gamma}\left[\left(\bar{\gamma}_1 - \bar{\gamma}_2\right) - \bar{\gamma}\left(r_1^2 - r_2^2\right)\right] \dot{\varphi}_2 \right. \\ & \qquad \left. + 3\left(\dot{\varphi}_1 \varphi_2' - \varphi_1' \dot{\varphi}_2\right)\right\}. \end{aligned} \tag{5.70}$$

The γ_i-deformed string worldsheet action to $O(\tilde{\lambda}^2)$ can thus be written as

$$S^{\gamma_i} = -J \int d\tau \frac{d\sigma}{2\pi} \left\{1 + \tilde{\lambda}\left(\frac{1}{2} a - c\right) + \tilde{\lambda}^2 \left(\frac{1}{2} b - \frac{1}{8} a^2 - d\right) + O\left(\tilde{\lambda}^3\right)\right\}, \tag{5.71}$$

where we now express the new variables a, b, c and d in terms of the original angular coordinates ϕ_i by substituting

$$\varphi_1 = \frac{1}{3}\left(\phi_1 + \phi_2 - 2\phi_3\right) \qquad \text{and} \qquad \varphi_2 = \frac{1}{3}\left(-2\phi_1 + \phi_2 + \phi_3\right) \tag{5.72}$$

into the expressions (5.65), (5.66), (5.69) and (5.70) so as to obtain

$$a = \sum_{i=1}^{3} (r_i')^2 + \frac{1}{2}\sum_{i,j=1}^{3} r_i^2 r_j^2 \left(\phi_i' - \phi_j' - \sum_{k=1}^{3} \varepsilon_{ijk}\bar{\gamma}_k\right)^2 - \bar{\gamma}^2 r_1^2 r_2^2 r_3^2, \tag{5.73}$$

$$\begin{aligned} b = &- \sum_{i=1}^{3} \dot{r}_i^2 - \frac{1}{2}\sum_{i,j=1}^{3} r_i^2 r_j^2 \left(\dot{\phi}_i - \dot{\phi}_j\right)^2 \\ &- r_1^2 r_2^2 r_3^2 \bar{\gamma}^2 \left[\frac{1}{2}\sum_{i,j=1}^{3} r_i^2 r_j^2 \left(\phi_i' - \phi_j' - \sum_{k=1}^{3}\varepsilon_{ijk}\bar{\gamma}_k\right)^2 - \bar{\gamma}^2 r_1^2 r_2^2 r_3^2\right], \end{aligned} \tag{5.74}$$

$$c = \frac{1}{3}\sum_{i=1}^{3}\dot{\phi}_i - \sum_{i=1}^{3} r_i^2 \dot{\phi}_i, \tag{5.75}$$

$$d = r_1^2 r_2^2 r_3^2 \bar{\gamma}^2 \sum_{i=1}^{3} r_i^2 \dot{\phi} - r_1^2 r_2^2 r_3^2 \bar{\gamma} \sum_{i=1}^{3} \bar{\gamma}_i \dot{\phi}_i + \frac{1}{2} r_1^2 r_2^2 r_3^2 \bar{\gamma} \sum_{i,j,k=1}^{3} \varepsilon_{ijk}\left(\dot{\phi}_i \phi_j' - \phi_i' \dot{\phi}_j\right). \tag{5.76}$$

Hence, changing back $\tau \to \tilde{\lambda}\tau$, and neglecting the total derivative term in the variable c and the constant 1 at the beginning of the Lagrangian, we obtain the γ_i-deformed string worldsheet action to $O(\tilde{\lambda}^2)$ in the fast motion limit

$$S^{\gamma_i} = -J \int d\tau \, \frac{d\sigma}{2\pi} \left[\mathcal{L}^{\gamma_i} + O\left(\tilde{\lambda}^3\right)\right], \tag{5.77}$$

where

$$\begin{aligned} \mathcal{L}^{\gamma_i} = &\left(1 - \tilde{\lambda} r_1^2 r_2^2 r_3^2 \bar{\gamma}^2\right) \\ &\times \left\{\sum_{i=1}^{3} r_i^2 \dot{\phi}_i + \frac{\tilde{\lambda}}{2}\left[\sum_{i=1}^{3}(r_i')^2 + \frac{1}{2}\sum_{i,j=1}^{3} r_i^2 r_j^2 \left(\phi_i' - \phi_j' - \sum_{k=1}^{3}\varepsilon_{ijk}\bar{\gamma}_k\right)^2 - \bar{\gamma}^2 r_1^2 r_2^2 r_3^2\right]\right\} \\ &- \frac{1}{2}\sum_{i=1}^{3}\dot{r}_i^2 - \frac{1}{4}\sum_{i,j=1}^{3} r_i^2 r_j^2 \left(\dot{\phi}_i - \dot{\phi}_j\right)^2 \\ &- \frac{\tilde{\lambda}^2}{8}\left[\sum_{i=1}^{3}(r_i')^2 + \frac{1}{2}\sum_{i,j=1}^{3} r_i^2 r_j^2\left(\phi_i' - \phi_j' - \sum_{k=1}^{3}\varepsilon_{ijk}\bar{\gamma}_k\right)^2 - \bar{\gamma}^2 r_1^2 r_2^2 r_3^2\right]^2 \\ &+ \tilde{\lambda} r_1^2 r_2^2 r_3^2 \bar{\gamma} \sum_{i=1}^{3}\bar{\gamma}_i \dot{\phi}_i - \frac{1}{2}\tilde{\lambda} r_1^2 r_2^2 r_3^2 \bar{\gamma} \sum_{i,j,k=1}^{3}\varepsilon_{ijk}\left(\dot{\phi}_i \phi_j' - \phi_i' \dot{\phi}_j\right) \end{aligned} \tag{5.78}$$

Notice that the $O(\tilde{\lambda})$ part of this γ_i-deformed string worldsheet action, which involves simply the expression in curly brackets $\{\ldots\}$, agrees with the coherent state effective action (5.45) for a γ_i-deformed spin chain, if we make the identifications $r_i \to m_i$ and $\phi_i \to h_i$. This agreement between the γ_i-deformed spin chain/string first order semiclassical actions was first observed in [9][3].

5.2.2 $U(1)$ charges densities and currents to $O(\tilde{\lambda})$

This γ_i-deformed Lagrangian, which describes semiclassical strings moving in a fast motion limit, can be seen to still be invariant under rotations on our γ_i-deformed five-sphere. Let us now calculate the corresponding $U(1)$ charge and current densities to $O(\tilde{\lambda})$ as follows:

$$p_i = \frac{\partial \mathcal{L}^{\gamma_i}}{\partial \dot{\phi}_i} = \left(1 - r_1^2 r_2^2 r_3^2 \bar{\gamma}^2 \tilde{\lambda}\right) r_i^2 - \tilde{\lambda} r_i^2 \sum_{j=1}^{3} r_j^2 \left(\dot{\phi}_i - \dot{\phi}_j\right) \\ + \tilde{\lambda} r_1^2 r_2^2 r_3^2 \bar{\gamma}\bar{\gamma}_i - \tilde{\lambda} r_1^2 r_2^2 r_3^2 \bar{\gamma} \sum_{j,k=1}^{3} \varepsilon_{ijk} \phi_j', \tag{5.79}$$

and also

$$j_i = \frac{\partial \mathcal{L}^{\gamma_i}}{\partial \phi_i'} = \tilde{\lambda} r_i^2 \sum_{j=1}^{3} r_j^2 \left(\phi_i' - \phi_j' - \sum_{k=1}^{3} \varepsilon_{ijk} \bar{\gamma}_k\right). \tag{5.80}$$

Note that, in the case of the current densities j_i, we have kept only the $O(\tilde{\lambda})$ terms, but no higher order terms have been neglected when calculating the charge densities p_i. The reason for this is that we need to take a time derivative to obtain the charge densities and thus we are implicitly reducing the order of the expression by one. In other words, the $O(\tilde{\lambda}^2)$ Lagrangian automatically results in $O(\tilde{\lambda})$ charge densities.

Furthermore, we can determine the $U(1)$ charge and current densities to $O(\tilde{\lambda})$ for similar semiclassical strings in an undeformed $\mathbb{R} \times S^5$ background by simply setting $\bar{\gamma}_i = 0$. We thus obtain

$$\tilde{\tilde{p}}_i = r_i^2 - \tilde{\lambda} r_i^2 \sum_{j=1}^{3} \left(\dot{\tilde{\tilde{\phi}}}_i - \dot{\tilde{\tilde{\phi}}}_j\right) \qquad \text{and} \qquad \tilde{\tilde{j}}_i = \tilde{\lambda} r_i^2 \sum_{j=1}^{3} r_j^2 \left(\tilde{\tilde{\phi}}_i' - \tilde{\tilde{\phi}}_j'\right). \tag{5.81}$$

[3] Note that the expressions obtained in [9] for both the leading order string and spin chain actions differ from those derived above in that the time derivative term appears with an extra negative sign. This is equivalent to a redefinition of time $\tau \to -\tau$ or, alternatively, to a redefinition of both the angular coordinates $\phi_i \to -\phi_i$ and the deformations parameters $\gamma_i \to -\gamma_i$.

Now we demonstrated in chapter 4 (based on discussions in [9]) that the $U(1)$ charge and current densities remain unchanged by the γ_i-deformation. This should still be true in the fast motion limit. Thus, setting $\tilde{\tilde{p}}_i = p_i$ and $\tilde{\tilde{j}}_i = j_i$, it is possible to obtain the following expressions:

$$\begin{aligned}
\dot{\tilde{\tilde{\phi}}}_1 - \dot{\tilde{\tilde{\phi}}}_2 &= \dot{\phi}_1 - \dot{\phi}_2 - r_1^2 r_3^2 \bar{\gamma} \left(\phi_3' - \phi_1' - \bar{\gamma}_2\right) + r_2^2 r_3^2 \bar{\gamma} \left(\phi_2' - \phi_3' - \bar{\gamma}_1\right) \\
\dot{\tilde{\tilde{\phi}}}_3 - \dot{\tilde{\tilde{\phi}}}_1 &= \dot{\phi}_3 - \dot{\phi}_1 - r_2^2 r_3^2 \bar{\gamma} \left(\phi_2' - \phi_3' - \bar{\gamma}_1\right) + r_1^2 r_2^2 \bar{\gamma} \left(\phi_1' - \phi_2' - \bar{\gamma}_3\right) \\
\dot{\tilde{\tilde{\phi}}}_2 - \dot{\tilde{\tilde{\phi}}}_3 &= \dot{\phi}_2 - \dot{\phi}_3 - r_1^2 r_2^2 \bar{\gamma} \left(\phi_1' - \phi_2' - \bar{\gamma}_3\right) + r_1^2 r_3^2 \bar{\gamma} \left(\phi_3' - \phi_1' - \bar{\gamma}_2\right),
\end{aligned} \tag{5.82}$$

and

$$\begin{aligned}
\tilde{\tilde{\phi}}_1' - \tilde{\tilde{\phi}}_2' &= \phi_1' - \phi_2' + \bar{\gamma} r_3^2 - \bar{\gamma}_3 \\
\tilde{\tilde{\phi}}_3' - \tilde{\tilde{\phi}}_1' &= \phi_3' - \phi_1' + \bar{\gamma} r_2^2 - \bar{\gamma}_2 \\
\tilde{\tilde{\phi}}_2' - \tilde{\tilde{\phi}}_3' &= \phi_2' - \phi_3' + \bar{\gamma} r_1^2 - \bar{\gamma}_1.
\end{aligned} \tag{5.83}$$

These equations describe the connection between the undeformed and γ_i-deformed angular coordinates $\tilde{\tilde{\phi}}_i$ and ϕ_i respectively to $O(\tilde{\lambda})$ in the fast motion limit.

5.3 Lax Pair for the γ_i-deformed Spin Chain/String Semiclassical Action to $O(\tilde{\lambda})$

We shall now demonstrate that the γ_i-deformed semiclassical spin chain/string action to leading order in $\tilde{\lambda}$ admits a Lax pair representation. In other words, the γ_i-deformed string worldsheet action remains integrable in the fast motion limit. This new result was published in [23] and is presented with minimal changes.

We begin by considering an undeformed semiclassical spin chain/string system and show, following [9, 20], that the equations of motion are equivalent to a Landau-Lifshitz equation for which there is a known Lax pair. We then derive the γ_i-deformed equations of motion and construct a transformation on the angular coordinates that takes the undeformed equations of motion into the γ_i-deformed equations of motion. A γ_i-deformed Lax pair is hence constructed and the corresponding zero curvature condition is shown to be equivalent to the γ_i-deformed equations of motion. Further details of these calculations are presented in appendix C.

5.3.1 Undeformed semiclassical Lax pair representation

Undeformed equations of motion

Let us consider the undeformed semiclassical spin chain/string action to leading order in $\tilde{\lambda}$, which can be obtained from (5.77) or, alternatively, from (5.45) by setting $\bar{\gamma}_i = 0$. We shall now make explicit mention of the constraint term, and redefine $\tau \to -\frac{1}{\tilde{\lambda}}\tau$ and $\mathcal{L} \to \frac{1}{\tilde{\lambda}}\mathcal{L}$, so as to obtain

$$S = -J \int d\tau \, \frac{d\sigma}{2\pi} \left[\tilde{\lambda}\mathcal{L} + O\left(\tilde{\lambda}^2\right) \right], \tag{5.84}$$

where the undeformed first order Lagrangian is given by

$$\mathcal{L} = -\sum_{i=1}^{3} r_i^2 \dot{\tilde{\tilde{\phi}}}_i + \frac{1}{2}\sum_{i=1}^{3} \left(r_i'\right)^2 + \frac{1}{4}\sum_{i,j=1}^{3} r_i^2 r_j^2 \left(\tilde{\tilde{\phi}}_i' - \tilde{\tilde{\phi}}_j'\right)^2 + \frac{1}{2}\Lambda\left(\sum_{i=1}^{3} r_i^2 - 1\right). \tag{5.85}$$

We can now derive the undeformed equations of motion by varying with respect to the radial and angular coordinates r_i and $\tilde{\tilde{\phi}}_i$ respectively to obtain

$$r_i'' = -2r_i \dot{\tilde{\tilde{\phi}}}_i + r_i \sum_{k=1}^{3} r_k^2 \left(\tilde{\tilde{\phi}}_i' - \tilde{\tilde{\phi}}_k'\right)^2 + \Lambda r_i, \tag{5.86}$$

$$\dot{r}_i = \sum_{k=1}^{3} r_k \left(r_i r_k\right)' \left(\tilde{\tilde{\phi}}_i' - \tilde{\tilde{\phi}}_k'\right) + \frac{1}{2} r_i \sum_{k=1}^{3} r_k^2 \left(\tilde{\tilde{\phi}}_i'' - \tilde{\tilde{\phi}}_k''\right), \tag{5.87}$$

while varying with respect to the Lagrange multiplier Λ yields the constraint equation $\sum_{i=1}^{3} r_i^2 = 1$. Now, assuming this constraint is satisfied, the equations of motion (5.86) and (5.87) are equivalent to

$$r_j r_i'' - r_i r_j'' = 2r_i r_j \left(\dot{\tilde{\tilde{\phi}}}_j - \dot{\tilde{\tilde{\phi}}}_i\right) + r_i r_j \sum_{k=1}^{3} r_k^2 \left(\tilde{\tilde{\phi}}_i' - \tilde{\tilde{\phi}}_k'\right)^2 - r_i r_j \sum_{k=1}^{3} r_k^2 \left(\tilde{\tilde{\phi}}_j' - \tilde{\tilde{\phi}}_k'\right)^2, \tag{5.88}$$

$$\begin{aligned} \dot{r}_i r_j + r_i \dot{r}_j = {} & r_j \sum_{k=1}^{3} r_k \left(r_i r_k\right)' \left(\tilde{\tilde{\phi}}_i' - \tilde{\tilde{\phi}}_k'\right) + r_i \sum_{k=1}^{3} r_k \left(r_j r_k\right)' \left(\tilde{\tilde{\phi}}_j' - \tilde{\tilde{\phi}}_k'\right) \\ & + \frac{1}{2} r_i r_j \sum_{k=1}^{3} r_k^2 \left(\tilde{\tilde{\phi}}_i'' - \tilde{\tilde{\phi}}_k''\right) + \frac{1}{2} r_i r_j \sum_{k=1}^{3} r_k^2 \left(\tilde{\tilde{\phi}}_j'' - \tilde{\tilde{\phi}}_k''\right). \end{aligned} \tag{5.89}$$

Notice that the constraint term cancels out of equation (5.88).

Undeformed Lax Pair

The Landau-Lifshitz Lax pair, which is a function of the spectral parameter x, is

$$\mathcal{D}_\alpha = \partial_\alpha - A_\alpha, \qquad \text{with} \;\; \alpha = 0, 1, \tag{5.90}$$

where

$$A_0 = \frac{1}{6}\left[N, \partial_1 N\right] x + \frac{3i}{2} N x^2, \tag{5.91}$$
$$A_1 = iNx, \tag{5.92}$$

and we have defined $N_{ij} = 3U_i^* U_j - \delta_{ij}$, where $U_i = r_i e^{i\tilde{\tilde{\phi}}_i}$ and $\sum_{i=1}^{3} r_i^2 = 1$ [9].

This satisfies the zero curvature condition $[\mathcal{D}_\alpha, \mathcal{D}_\beta] = 0$, which is equivalent to

$$\partial_0 A_1 - \partial_1 A_0 - [A_0, A_1] = 0, \tag{5.93}$$

since the only non-trivial independent equation comes from setting $\alpha = 0$ and $\beta = 1$. This condition results in the Landau-Lifshitz equation of motion [9, 20]

$$i\partial_0 N = \frac{1}{6}\left[N, \partial_1^2 N\right], \tag{5.94}$$

and, upon substitution of $N_{ij} = 3U_i^* U_j - \delta_{ij} = 3r_i r_j e^{i\left(\tilde{\tilde{\phi}}_j - \tilde{\tilde{\phi}}_i\right)} - \delta_{ij}$ into this equation, one obtains (5.88) and (5.89), which are equivalent to the undeformed equations of motion (see appendix C.1 for details).

In terms of r_i and $\tilde{\tilde{\phi}}_i$, the undeformed Lax pair is $\mathcal{D}_\alpha = \partial_\alpha - A_\alpha$, where

$$\left(A_\alpha\right)_{ij} = \left(B_\alpha\right)_{ij} \; e^{i\left(\tilde{\tilde{\phi}}_j - \tilde{\tilde{\phi}}_i\right)}, \qquad \text{with} \;\; \alpha = 0, 1, \tag{5.95}$$

and we define

$$\begin{aligned}(B_0)_{ij} = &\left[\frac{3}{2}\left(r_i r_j' - r_i' r_j\right) + \frac{3i}{2} r_i r_j \left(\tilde{\tilde{\phi}}_i' + \tilde{\tilde{\phi}}_j'\right) - 3i r_i r_j \sum_{k=1}^{3} r_k^2 \tilde{\tilde{\phi}}_k'\right] x \\ &+ \frac{3i}{2}\left(3r_i r_j - \delta_{ij}\right) x^2,\end{aligned} \tag{5.96}$$

$$\left(B_1\right)_{ij} = i\left(3r_i r_j - \delta_{ij}\right) x. \tag{5.97}$$

5.3.2 γ_i-deformed semiclassical Lax pair representation

γ_i-deformed equations of motion

We shall now consider the more general γ_i-deformed semiclassical spin chain/string action, which, from (5.77) or (5.45), is given by

$$S^{\gamma_i} = -J \int d\tau \, \frac{d\sigma}{2\pi} \left[\tilde{\lambda} \mathcal{L}^{\gamma_i} + O\left(\tilde{\lambda}^2\right) \right]. \tag{5.98}$$

Here we have again redefined $\tau \to -\frac{1}{\tilde{\lambda}}\tau$ and $\mathcal{L}^{\gamma_i} \to \frac{1}{\tilde{\lambda}}\mathcal{L}^{\gamma_i}$, and included the constraint term in the γ_i-deformed Lagrangian, which to first order in $\tilde{\lambda}$ is

$$\begin{aligned} \mathcal{L}^{\gamma_i} = &- \sum_{i=1}^{3} r_i^2 \dot{\phi}_i + \frac{1}{2} \sum_{i=1}^{3} (r_i')^2 + \frac{1}{4} \sum_{i,j=1}^{3} r_i^2 r_j^2 \left(\phi_i' - \phi_j' - \sum_{k=1}^{3} \varepsilon_{ijk} \bar{\gamma}_k \right)^2 \\ &- \frac{1}{2} \bar{\gamma}^2 r_1^2 r_2^2 r_3^2 + \frac{1}{2} \Lambda \left(\sum_{i=1}^{3} r_i^2 - 1 \right). \end{aligned} \tag{5.99}$$

This expression can be rewritten using the constraint $\sum_{i=1}^{3} r_i^2 = 1$ as follows:

$$\begin{aligned} \mathcal{L}^{\gamma_i} = &- \sum_{i=1}^{3} r_i^2 \dot{\phi}_i + \frac{1}{2} \sum_{i=1}^{3} (r_i')^2 + \frac{1}{2} \Lambda \left(\sum_{i=1}^{3} r_i^2 - 1 \right) \\ &+ \frac{1}{4} \sum_{i,j=1}^{3} r_i^2 r_j^2 \left[\left(\phi_i' + \sum_{l,m=1}^{3} \varepsilon_{ilm} \bar{\gamma}_l r_m^2 \right) - \left(\phi_j' + \sum_{l,m=1}^{3} \varepsilon_{jlm} \bar{\gamma}_l r_m^2 \right) \right]^2. \end{aligned} \tag{5.100}$$

Varying the above Lagrangian with respect to the radial and angular coordinates r_i and ϕ_i respectively, and using the constraint equation $\sum_{i=1}^{3} r_i^2 = 1$, which can be obtained by varying with respect to Λ, yields the γ_i-deformed equations of motion

$$\begin{aligned} r_i'' = -2 r_i \Bigg\{ &\dot{\phi}_i + \sum_{l,m=1}^{3} \varepsilon_{ilm} r_i^2 r_l^2 \bar{\gamma}_m \left(\phi_i' - \phi_l' - \varepsilon_{ilm} \bar{\gamma}_m \right) \\ &- \frac{1}{2} \sum_{l,m=1}^{3} \varepsilon_{ilm} r_l^2 r_m^2 \left(\bar{\gamma}_l + \bar{\gamma}_m \right) \left(\phi_l' - \phi_m' - \varepsilon_{ilm} \bar{\gamma}_i \right) \Bigg\} \\ &+ r_i \sum_{k=1}^{3} r_k^2 \left[\left(\phi_i' + \sum_{l,m=1}^{3} \varepsilon_{ilm} \bar{\gamma}_l r_m^2 \right) - \left(\phi_k' + \sum_{l,m=1}^{3} \varepsilon_{klm} \bar{\gamma}_l r_m^2 \right) \right]^2 + \Lambda r_i, \end{aligned} \tag{5.101}$$

$$\dot{r}_i = \sum_{k=1}^{3} r_k \left(r_i r_k\right)' \left[\left(\phi_i' + \sum_{l,m=1}^{3} \varepsilon_{ilm}\bar{\gamma}_l r_m^2\right) - \left(\phi_k' + \sum_{l,m=1}^{3} \varepsilon_{klm}\bar{\gamma}_l r_m^2\right)\right] \tag{5.102}$$
$$+ \frac{1}{2}\, r_i \sum_{k=1}^{3} r_k^2 \left[\left(\phi_i'' + 2\sum_{l,m=1}^{3} \varepsilon_{ilm}\bar{\gamma}_l r_m r_m'\right) - \left(\phi_k'' + 2\sum_{l,m=1}^{3} \varepsilon_{klm}\bar{\gamma}_l r_m r_m'\right)\right].$$

Now, assuming the constraint is satisfied, the above equations of motion (5.101) and (5.102) are equivalent to

$$\begin{aligned}
&r_j r_i'' - r_i r_j'' = 2 r_i r_j \left(\dot{\phi}_j - \dot{\phi}_i\right) \\
&-2 r_i r_j \left\{\sum_{l,m=1}^{3} \varepsilon_{ilm} r_i^2 r_l^2 \bar{\gamma}_m \left(\phi_i' - \phi_l' - \varepsilon_{ilm}\bar{\gamma}_m\right) - \frac{1}{2}\sum_{l,m=1}^{3} \varepsilon_{ilm} r_l^2 r_m^2 \left(\bar{\gamma}_l + \bar{\gamma}_m\right)\left(\phi_l' - \phi_m' - \varepsilon_{ilm}\bar{\gamma}_i\right)\right\} \\
&+2 r_i r_j \left\{\sum_{l,m=1}^{3} \varepsilon_{jlm} r_j^2 r_l^2 \bar{\gamma}_m \left(\phi_j' - \phi_l' - \varepsilon_{jlm}\bar{\gamma}_m\right) - \frac{1}{2}\sum_{l,m=1}^{3} \varepsilon_{jlm} r_l^2 r_m^2 \left(\bar{\gamma}_l + \bar{\gamma}_m\right)\left(\phi_l' - \phi_m' - \varepsilon_{jlm}\bar{\gamma}_j\right)\right\} \\
&+ r_i r_j \sum_{k=1}^{3} r_k^2 \left[\left(\phi_i' + \sum_{l,m=1}^{3} \varepsilon_{ilm}\bar{\gamma}_l r_m^2\right) - \left(\phi_k' + \sum_{l,m=1}^{3} \varepsilon_{klm}\bar{\gamma}_l r_m^2\right)\right]^2 \\
&- r_i r_j \sum_{k=1}^{3} r_k^2 \left[\left(\phi_j' + \sum_{l,m=1}^{3} \varepsilon_{jlm}\bar{\gamma}_l r_m^2\right) - \left(\phi_k' + \sum_{l,m=1}^{3} \varepsilon_{klm}\bar{\gamma}_l r_m^2\right)\right]^2,
\end{aligned} \tag{5.103}$$

$$\begin{aligned}
\dot{r}_i r_j + r_i \dot{r}_j = {}& r_j \sum_{k=1}^{3} r_k \left(r_i r_k\right)' \left[\left(\phi_i' + \sum_{l,m=1}^{3} \varepsilon_{ilm}\bar{\gamma}_l r_m^2\right) - \left(\phi_k' + \sum_{l,m=1}^{3} \varepsilon_{klm}\bar{\gamma}_l r_m^2\right)\right] \\
&+ r_i \sum_{k=1}^{3} r_k \left(r_j r_k\right)' \left[\left(\phi_j' + \sum_{l,m=1}^{3} \varepsilon_{jlm}\bar{\gamma}_l r_m^2\right) - \left(\phi_k' + \sum_{l,m=1}^{3} \varepsilon_{klm}\bar{\gamma}_l r_m^2\right)\right] \\
&+ \frac{1}{2} r_i r_j \sum_{k=1}^{3} r_k^2 \left[\left(\phi_i'' + 2\sum_{l,m=1}^{3} \varepsilon_{ilm}\bar{\gamma}_l r_m r_m'\right) - \left(\phi_k'' + 2\sum_{l,m=1}^{3} \varepsilon_{klm}\bar{\gamma}_l r_m r_m'\right)\right] \\
&+ \frac{1}{2} r_i r_j \sum_{k=1}^{3} r_k^2 \left[\left(\phi_j'' + 2\sum_{l,m=1}^{3} \varepsilon_{jlm}\bar{\gamma}_l r_m r_m'\right) - \left(\phi_k'' + 2\sum_{l,m=1}^{3} \varepsilon_{klm}\bar{\gamma}_l r_m r_m'\right)\right].
\end{aligned} \tag{5.104}$$

Transformation from the undeformed equations of motion to the γ_i-deformed equations of motion

We now observe that a transformation which takes the undeformed equations of motion into the γ_i-deformed equations of motion is

$$\dot{\tilde{\phi}}_i = \dot{\phi}_i + \sum_{l,m=1}^{3} \varepsilon_{ilm} r_i^2 r_l^2 \bar{\gamma}_m \left(\phi_i' - \phi_l' - \varepsilon_{ilm}\bar{\gamma}_m\right) - \frac{1}{2}\sum_{l,m=1}^{3} \varepsilon_{ilm} r_l^2 r_m^2 \left(\bar{\gamma}_l + \bar{\gamma}_m\right)\left(\phi_l' - \phi_m' - \varepsilon_{ilm}\bar{\gamma}_i\right), \tag{5.105}$$

$$\tilde{\phi}_i' = \phi_i' + \sum_{l,m=1}^{3} \varepsilon_{ilm}\bar{\gamma}_l r_m^2. \tag{5.106}$$

Taking into account our redefinition of τ, this transformation satisfies the equations (5.82) and (5.83), which were derived by equating the undeformed and γ_i-deformed $U(1)$ charge and current densities. Thus the relation we have observed between the undeformed and γ_i-deformed angular coordinates is simply the result of the $U(1)$ charge and current densities remaining unaltered by the γ_i-deformation (as observed in [8] for the general string theory before the fast motion limit was taken). Notice also that this transformation is only one of an entire set of possible transformations, because it is only the difference of the angular coordinates which effects both (5.82) and (5.83), and the undeformed and γ_i-deformed equations of motion.

Now, for this transformation to be valid, we must have $(\dot{\tilde{\phi}}_i)' = (\tilde{\phi}_i')^{\cdot}$. Therefore the compatibility condition, which must be satisfied, is

$$2\sum_{l,m=1}^{3} \varepsilon_{ilm}\bar{\gamma}_l r_m \dot{r}_m = \partial_1\left\{\sum_{l,m=1}^{3} \varepsilon_{ilm} r_i^2 r_l^2 \bar{\gamma}_m \left(\phi_i' - \phi_l' - \varepsilon_{ilm}\bar{\gamma}_m\right) - \frac{1}{2}\sum_{l,m=1}^{3} \varepsilon_{ilm} r_l^2 r_m^2 \left(\bar{\gamma}_l + \bar{\gamma}_m\right)\left(\phi_l' - \phi_m' - \varepsilon_{ilm}\bar{\gamma}_i\right)\right\}. \tag{5.107}$$

However, from the equation of motion (5.102), we know that

$$r_i \dot{r}_i = \frac{1}{2}\,\partial_1\left\{\sum_{k=1}^{3} r_i^2 r_k^2 \left[\left(\phi_i' + \sum_{n,s=1}^{3} \varepsilon_{ins}\bar{\gamma}_n r_s^2\right) - \left(\phi_k' + \sum_{n,s=1}^{3} \varepsilon_{kns}\bar{\gamma}_n r_s^2\right)\right]\right\}, \tag{5.108}$$

and thus

$$2\sum_{l,m=1}^{3} \varepsilon_{ilm}\bar{\gamma}_l r_m \dot{r}_m = \partial_1\left\{\sum_{k,l,m=1}^{3} \varepsilon_{ilm}\bar{\gamma}_l r_m^2 r_k^2 \left[\left(\phi_m' + \sum_{n,s=1}^{3} \varepsilon_{mns}\bar{\gamma}_n r_s^2\right) - \left(\phi_k' + \sum_{n,s=1}^{3} \varepsilon_{kns}\bar{\gamma}_n r_s^2\right)\right]\right\}. \tag{5.109}$$

By setting $i = 1, 2$ and 3, and evaluating equations (5.107) and (5.109) separately (see appendix C.2), these equations can be shown to be the same. Thus the compatibility condition is automatically satisfied if the γ_i-deformed equations of motion (and the constraint equation) are valid.

γ_i-deformed Lax Pair

The γ_i-deformed Lax pair shall now be derived from the undeformed one following a similar procedure to that discussed in section 4.5.3.

First the $\tilde{\tilde{\phi}}_i$-dependence of the undeformed Lax pair will be gauged away. More specifically, we shall now change to an equivalent Lax pair as follows:

$$\mathcal{D}_\alpha \longrightarrow \tilde{\mathcal{D}}_\alpha = M\mathcal{D}_\alpha M^{-1} = \partial_\alpha - \mathcal{R}_\alpha, \qquad \text{with } \mathcal{R}_\alpha = MA_\alpha M^{-1} - M\partial_\alpha M^{-1}, \tag{5.110}$$

where $M \equiv ie^{i\tilde{\tilde{\phi}}_i}\delta_{ij}$ and thus $M^{-1} = -ie^{-i\tilde{\tilde{\phi}}_i}\delta_{ij}$.

Now we can make use of the definition (5.95) of the undeformed Lax pair, in which the dependence on the angular coordinates is entirely in the exponential, to derive the gauged undeformed Lax pair as follows:

$$\tilde{\mathcal{D}}_\alpha = \partial_\alpha - \mathcal{R}_\alpha, \qquad \text{where} \quad (\mathcal{R}_\alpha)_{ij} = (B_\alpha)_{ij} + i\partial_\alpha\tilde{\tilde{\phi}}_i \; \delta_{ij}, \tag{5.111}$$

and thus, substituting the definitions of $(B_0)_{ij}$ and $(B_1)_{ij}$ into the above equation, we obtain the explicit expressions

$$\begin{aligned}(\mathcal{R}_0)_{ij} = &\left[\frac{3}{2}\left(r_i r_j' - r_i' r_j\right) + \frac{3i}{2} r_i r_j \left(\tilde{\tilde{\phi}}_i' + \tilde{\tilde{\phi}}_j'\right) - 3ir_i r_j \sum_{k=1}^{3} r_k^2 \tilde{\tilde{\phi}}_k'\right] x \\ &+ \frac{3i}{2}\left(3r_i r_j - \delta_{ij}\right) x^2 + i\dot{\tilde{\tilde{\phi}}}_i \; \delta_{ij}\end{aligned} \tag{5.112}$$

$$(\mathcal{R}_1)_{ij} = i\left(3r_i r_j - \delta_{ij}\right) x + i\tilde{\tilde{\phi}}_i' \; \delta_{ij}. \tag{5.113}$$

This undeformed gauged Lax pair depends only on the radii and the derivatives thereof, and the derivatives of the undeformed angular coordinates. It can therefore be expressed in terms of the γ_i-deformed angular coordinates using the transformation (5.105) and (5.106). Hence we can construct the γ_i-deformed gauged Lax pair

$$\tilde{\mathcal{D}}_\alpha^{\gamma_i} = \partial_\alpha - \mathcal{R}_\alpha^{\gamma_i}, \tag{5.114}$$

where

$$
\begin{aligned}
(\mathcal{R}_0^{\gamma_i})_{ij} = {} & \frac{3}{2}\left(r_i r_j' - r_i' r_j\right) x + \frac{3i}{2} r_i r_j \left[\left(\phi_i' + \sum_{l,m=1}^{3} \varepsilon_{ilm}\bar{\gamma}_l r_m^2\right) + \left(\phi_j' + \sum_{l,m=1}^{3} \varepsilon_{jlm}\bar{\gamma}_l r_m^2\right)\right] x \\
& - 3i r_i r_j \sum_{k=1}^{3} r_k^2 \left(\phi_k' + \sum_{l,m=1}^{3} \varepsilon_{klm}\bar{\gamma}_l r_m^2\right) x + \frac{3i}{2}\left(3 r_i r_j - \delta_{ij}\right) x^2 \\
& + i \left\{ \dot{\phi}_i + \sum_{l,m=1}^{3} \varepsilon_{ilm} r_i^2 r_l^2 \bar{\gamma}_m \left(\phi_i' - \phi_l' - \varepsilon_{ilm}\bar{\gamma}_m\right) \right. \\
& \left. \qquad - \frac{1}{2} \sum_{l,m=1}^{3} \varepsilon_{ilm} r_l^2 r_m^2 \left(\bar{\gamma}_l + \bar{\gamma}_m\right)\left(\phi_l' - \phi_m' - \varepsilon_{ilm}\bar{\gamma}_i\right) \right\} \delta_{ij}, \quad (5.115)
\end{aligned}
$$

$$
(\mathcal{R}_1^{\gamma_i})_{ij} = i\left(3 r_i r_j - \delta_{ij}\right) x + i\left(\phi_i' + \sum_{l,m=1}^{3} \varepsilon_{ilm}\bar{\gamma}_l r_m^2\right)\delta_{ij}. \quad (5.116)
$$

Now the zero curvature condition $\left[\tilde{\mathcal{D}}_\alpha^{\gamma_i}, \tilde{\mathcal{D}}_\beta^{\gamma_i}\right] = 0$ is equivalent to

$$
\partial_0 \mathcal{R}_1^{\gamma_i} - \partial_1 \mathcal{R}_0^{\gamma_i} - \left[\mathcal{R}_0^{\gamma_i}, \mathcal{R}_1^{\gamma_i}\right] = 0, \quad (5.117)
$$

and the equations thus obtained from this gauged γ_i-deformed Lax pair (see appendix C.3) are equations (5.103) and (5.104), which are equivalent to the γ_i-deformed equations of motion, and the compatibility condition, which follows directly from these equations motion. Thus $\tilde{\mathcal{D}}_\alpha^{\gamma_i}$ is a valid Lax pair representation for our γ_i-deformed semiclassical spin chain/string system. We have therefore shown that the γ_i-deformed spin chain/string action is integrable to leading order in the semiclassical limit.

Chapter 6

Summary and Conclusion

A non-supersymmetric γ_i-deformed extension of AdS/CFT correspondence, which was originally conjectured by Frolov, has been studied. Both sides of this proposed γ_i-deformed gauge/string duality were described.

On the gauge theory side, the original maximally supersymmetric $\mathcal{N} = 4$ SYM conformal field theory was discussed in detail. $\mathcal{N} = 1$ supersymmetric marginal deformations of $\mathcal{N} = 4$ SYM theory were then constructed and the non-supersymmetric γ_i-deformed YM theory was mentioned. We reviewed the identification of the γ_i-deformed matrix of anomalous dimensions in the $SU(3)$ sector with the Hamiltonian of an integrable γ_i-deformed $SU(3)$ spin chain.

We then turned our attention to the string theory side of the proposed duality. Due consideration was given to $AdS_5 \times S^5$ spacetime before we confined our discussion to strings moving only in the five-sphere space. The γ_i-deformed string worldsheet action was constructed by performing three TsT-transformations on the original string worldsheet action. Frolov's Lax pair representation for strings moving on a γ_i-deformed five-sphere was also mentioned.

We then reviewed Frolov, Roiban and Tseytlin's semiclassical leading order comparison between the γ_i-deformed spin chain and string actions. The coherent state effective action for a γ_i-deformed $SU(3)$ spin chain was constructed to first order in the continuum limit. The string worldsheet action describing strings moving in a γ_i-deformed $\mathbb{R} \times S^5$ background was calculated to second order in the fast motion limit, together with the first order conserved γ_i-deformed $U(1)$ charges or angular momentua. Agreement was thus shown at leading order between the γ_i-deformed gauge and string theories in the semiclassical limit.

Furthermore, we demonstrated that there exists a Lax pair representation for the leading order semiclassical γ_i-deformed spin chain/string action, so that both systems remain integrable in the semiclassical limit. This result relied upon a transformation relating the undeformed and γ_i-deformed angular coordinates, which was seen to arise from the requirement that the first order semiclassical conserved $U(1)$ charges remain unchanged by the γ_i-deformation.

Possible extensions to this thesis include the construction of the conserved quantities associated with this Lax pair. Specifically, one could attempt to calculate the monodromy matrix and conserved quasi-momenta as a function of the spectral parameter for this γ_i-deformed semiclassical system.

Appendix A

Representations of the Lorentz Group

A.1 The Lorentz Group

The Lorentz group is a group of rotations and boosts under which any reasonable relativistic Lagrangian should be locally invariant. In this section, we begin by writing the generators of the Lorentz group as differential operators and hence construct the Lorentz algebra. The standard representation of the Lorentz group, in which the generators take the form of 4×4 matrices, is then described, together with more general n dimensional representations. This review is based on discussions in [26].

A.1.1 The Lorentz algebra

The group of Lorentz transformations describing rotations and boosts in four dimensional Minkowski spacetime can be viewed as an extension of the group of rotations in three dimensional Euclidean space. Therefore, let us first consider the generators of the rotation group, which are simply the three components of the angular momentum vector $\vec{J} = \vec{x} \times \vec{p}$. Since the momentum can be written as $\vec{p} = -i\vec{\nabla}$, the generators of the rotation group in differential operator form are given by

$$J^{jk} = -i \left(x^j \nabla^k - x^k \nabla^j \right) . \tag{A.1}$$

It is now possible to extend this result to the generators of the Lorentz group by simply changing the spatial derivatives to spacetime derivatives as follows:

$$J^{\mu\nu} = i\left(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}\right). \tag{A.2}$$

There are six generators of the Lorentz group corresponding to the six independent components of the anti-symmetric tensor $J^{\mu\nu}$. While J^{jk} still describe rotations, the new generators J^{0k} with temporal indices describe boosts. Finally, we find that these generators $J^{\mu\nu}$ satisfy the Lorentz algebra

$$\left[J^{\mu\nu}, J^{\rho\sigma}\right] = i\left(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}\right), \tag{A.3}$$

where $\eta^{\mu\nu} = \mathrm{diag}\,(+1, -1, -1, -1)$ is the Minkowski metric.

A.1.2 Standard representation of the Lorentz group

The standard representation of the Lorentz group is four dimensional: the Lorentz transformations are 4×4 matrices acting on the 4-vectors in Minkowski spacetime. The generators of the Lorentz group in this representation are given by

$$\left(\mathcal{J}^{\mu\nu}\right)^{\alpha}{}_{\beta} = i\left(\eta^{\alpha\mu}\delta^{\nu}{}_{\beta} - \eta^{\alpha\nu}\delta^{\mu}{}_{\beta}\right), \tag{A.4}$$

which satisfy the Lorentz algebra (A.3). These generators yield the finite Lorentz transformation $\Lambda = e^{-\frac{i}{2}\,\omega_{\mu\nu}\mathcal{J}^{\mu\nu}}$, where $\omega_{\mu\nu}$ is an anti-symmetric matrix of coefficients.

We can now derive the familiar Lorentz transformation matrices by making specific choices for the coefficients $\omega_{\mu\nu}$. For example, a rotation by an angle θ around the z-axis corresponds to all the components of $\omega_{\mu\nu}$ being zero, except $\omega_{12} = -\omega_{21} = \theta$. Thus we obtain

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{A.5}$$

which is obviously a rotation in the xy-plane. Rotation matrices in the xz and yz-planes can similarly be determined by setting $\omega_{31} = -\omega_{13} = \theta$ and $\omega_{23} = -\omega_{32} = \theta$ respectively.

Furthermore, a boost of rapidity y in the x-direction corresponds to $\omega_{01} = -\omega_{10} = \mathrm{y}$ as the non-zero coefficients. Hence

$$\Lambda = \begin{pmatrix} \cosh \mathrm{y} & \sinh \mathrm{y} & 0 & 0 \\ \sinh \mathrm{y} & \cosh \mathrm{y} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{A.6}$$

which is the usual Lorentz transformation in terms of the rapidity. We can similarly obtain the boosts in the y and z-directions using $\omega_{02} = -\omega_{20} = \mathrm{y}$ and $\omega_{03} = -\omega_{30} = \mathrm{y}$ respectively.

A.1.3 General representations of the Lorentz group

A general n dimensional representation $M(\Lambda)$ of the Lorentz group is an $n \times n$ matrix, which is a homomorphism of the Lorentz transformations Λ in four dimensional Minkowski spacetime. In other words, $M(\Lambda)$ has the following property:

$$M(\Lambda)M(\Lambda') = M(\Lambda\Lambda'). \tag{A.7}$$

Since this homomorphism maps the identity 1_4 onto the identity 1_n, the generators of the Lorentz group in this representation can be obtained by considering the image of an infinitesimal Lorentz transformation.

A.2 The Dirac Equation and Spin-$\frac{1}{2}$ Representation of the Lorentz Group

We shall now describe the spin-$\frac{1}{2}$ representation of the Lorentz group following [25, 26]. It is first necessary to mention the Dirac equation and the gamma matrices, with their corresponding Clifford algebra. The generators of the spin-$\frac{1}{2}$ representation are then constructed from these gamma matrices and it turns out that the Lorentz algebra is a direct result of the Clifford algebra. Lastly, the Dirac equation and corresponding Dirac action are shown to be Lorentz invariant.

A.2.1 The Dirac equation and Clifford algebra

The Dirac equation was developed by Dirac in 1928 as a relativistic and linear wave equation, which also contains the second order Klein-Gordon equation. He realized that one could obtain such a linear equation within a non-commutative framework. This Dirac equation is given by

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0, \tag{A.8}$$

where the gamma matrices γ^μ satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}. \tag{A.9}$$

This last condition is necessary for the Dirac equation to automatically contain the Klein-Gordon equation. In other words, if $\Psi(x)$ is a solution to the Dirac equation then it also satisfies $\left(\partial^\mu \partial_\mu + m^2\right)\Psi = 0$.

Now there are many possible representations of this Clifford algebra. The most common is the lowest dimensional representation in terms of 4×4 matrices, which, in the Dirac basis, is

$$\gamma^0 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \tag{A.10}$$

where σ^i are the usual Pauli matrices and the chirality matrix, defined as $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$, has been included for convenience. This is the only distinct four dimensional representation of the Clifford Algebra. In other words, if there exist any other 4×4 matrices γ^μ satisfying (A.9), then they are equivalent to the above gamma matrices by a change of basis.

It is also common, however, to write these matrices in the Weyl or chiral basis, in which the chirality matrix γ^5 is diagonal, as follows:

$$\gamma^0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} -1_2 & 0 \\ 0 & 1_2 \end{pmatrix}. \tag{A.11}$$

It is especially convenient to work in this Weyl basis when one is dealing with relativistic or massless particles (which is the case in $\mathcal{N} = 4$ SYM theory). For massless particles, it turns out that solutions to the Dirac equation are also eigenstates of the chirality operator (since the gamma matrices γ^μ anti-commute with the chirality matrix γ^5).

A.2.2 The spin-$\frac{1}{2}$ representation of the Lorentz group

Now, if we choose γ^μ so as to satisfy the Clifford algebra (A.9), then it turns out that we automatically obtain a spinor representation of the Lorentz group with generators

$$S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right]. \tag{A.12}$$

It can easily be shown that $S^{\mu\nu}$ satisfies the Lorentz algebra (A.3). More explicitly, the generators of the spin-$\frac{1}{2}$ representation of the Lorentz group are given by

$$S^{0i} = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \qquad S^{ij} = \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \tag{A.13}$$

using the 4×4 gamma matrices (A.11) in the Weyl basis. A finite Lorentz transformation in this spin-$\frac{1}{2}$ representation is $\Lambda_{\frac{1}{2}} = e^{-\frac{i}{2}\,\omega_{\mu\nu}S^{\mu\nu}}$, where $\omega_{\mu\nu}$ is, again, an anti-symmetric matrix of coefficients.

Let us now demonstrate, as in [26], that the Dirac equation is Lorentz invariant by showing that the matrices γ^μ are invariant under a simultaneous Lorentz transformation of both their spinor and spacetime indices. We first make use of the Clifford algebra (A.9) to calculate

$$\left[\gamma^\rho, S^{\mu\nu}\right] = \frac{i}{4}\left[\gamma^\rho, \left[\gamma^\mu, \gamma^\nu\right]\right] = \frac{i}{2}\left[\gamma^\rho, \gamma^\mu\gamma^\nu\right] = i\left(\eta^{\rho\mu}\delta^\nu{}_\sigma - \eta^{\rho\nu}\delta^\nu{}_\sigma\right)\gamma^\sigma = \left(\mathcal{J}^{\mu\nu}\right)^\rho{}_\sigma\,\gamma^\sigma, \tag{A.14}$$

and hence

$$\left(1 + \tfrac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\gamma^\rho\left(1 - \tfrac{i}{2}\omega_{\lambda\sigma}S^{\lambda\sigma}\right) \approx \gamma^\rho - \tfrac{i}{2}\omega_{\mu\nu}\left[\gamma^\rho, S^{\mu\nu}\right] = \left(1 - \tfrac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right)^\rho{}_\sigma\,\gamma^\sigma, \tag{A.15}$$

which is the infinitesimal form of $\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu\gamma^\nu$. In other words, under a Lorentz transformation of the spinor and spacetime indices $\gamma^\mu \to \Lambda^\mu{}_\nu\Lambda_{\frac{1}{2}}\gamma^\nu\Lambda_{\frac{1}{2}}^{-1} = \gamma^\mu$. The Dirac equation therefore transforms under a Lorentz transformation as follows:

$$\begin{aligned}
\left[i\gamma^\mu\partial_\mu - m\right]\Psi(x) \;&\longrightarrow \left[i\gamma^\mu\left(\Lambda^{-1}\right)^\nu{}_\mu\partial'_\nu - m\right]\Lambda_{\frac{1}{2}}\Psi(x') \\
&= \Lambda_{\frac{1}{2}}\left[i\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}}\left(\Lambda^{-1}\right)^\nu{}_\mu\partial'_\nu - m\right]\Psi(x') \\
&= \Lambda_{\frac{1}{2}}\left[i\Lambda^\mu{}_\rho\left(\Lambda^{-1}\right)^\nu{}_\mu\gamma^\rho\partial'_\nu - m\right]\Psi(x') \\
&= \Lambda_{\frac{1}{2}}\left[i\gamma^\mu\partial'_\mu - m\right]\Psi(x') \\
&= 0,
\end{aligned} \tag{A.16}$$

where we define $x' = \Lambda^{-1}x$ and $\partial'_\mu = \frac{\partial}{\partial x'^\mu}$. The Dirac equation is thus invariant under Lorentz transformations.

Finally, we should mention that the Dirac action is given by

$$S_{\text{Dirac}} = \int d^4x \; \mathcal{L}_{\text{Dirac}}(x), \qquad \text{where} \quad \mathcal{L}_{\text{Dirac}} = \bar{\Psi}(x) \left[i\gamma^\mu \partial_\mu - m \right] \Psi(x) \tag{A.17}$$

is the Dirac Lagrangian and we define $\bar{\Psi} \equiv \Psi^\dagger \gamma^0$. We must be careful to make use of $\bar{\Psi}$ rather than $\Psi^\dagger$ because the S^{0i} are anti-hermitean and therefore $\Lambda_{\frac{1}{2}}$ is not unitary. One can show, as mentioned in [26], that this action is Lorentz invariant making use of the identity $\Lambda^\dagger_{\frac{1}{2}}\gamma^0 = \gamma^0 \Lambda^{-1}_{\frac{1}{2}}$, which implies that

$$\bar{\Psi}(x) = \Psi^\dagger(x)\gamma^0 \longrightarrow \Psi^\dagger(x')\Lambda^\dagger_{\frac{1}{2}}\gamma^0 = \Psi^\dagger(x')\gamma^0\Lambda^{-1}_{\frac{1}{2}} = \bar{\Psi}(x')\Lambda^{-1}_{\frac{1}{2}}, \qquad \text{where} \;\; x' = \Lambda^{-1}x, \tag{A.18}$$

under the action of a Lorentz transformation Λ. Thus, noting from (A.16) that

$$\left[i\gamma^\mu \partial_\mu - m \right] \Psi(x) \longrightarrow \Lambda_{\frac{1}{2}} \left[i\gamma^\mu \partial'_\mu - m \right] \Psi(x'), \tag{A.19}$$

we obtain $\mathcal{L}_{\text{Dirac}}(x) \to \mathcal{L}_{\text{Dirac}}(x')$. Since the Jacobian of the coordinate transformation $x \to x'$ is one (Λ^{-1} has determinant one), we therefore observe that the Dirac action is Lorentz invariant.

A.3 Weyl Spinors

We now discuss the reducible nature of the four dimensional spin-$\frac{1}{2}$ representation of the Lorentz group based on [25]. It turns out to be possible to write this representation as the product of two $SU(2)$ groups by splitting the Dirac spinor into two Weyl spinors. The dotted and undotted notation, which can be used to describe these Weyl spinors, is also discussed. These ideas are especially important as a background for the understanding of supersymmetry.

A.3.1 Reducibility and Weyl spinors

The block diagonal form (A.13) of the generators of the four dimensional spin-$\frac{1}{2}$ representation of the Lorentz group is a clear indication of reducibility. Furthermore, since the block diagonal components are simply multiples of the Pauli matrices, which are the generators of $SU(2)$, this spin-$\frac{1}{2}$ representation is equivalent to $SU(2) \times SU(2)$. Hence we can split up any 4-component Dirac spinor as follows:

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}, \tag{A.20}$$

with ψ_α and $\bar{\chi}^{\dot{\alpha}}$ two 2-component Weyl spinors[1], where α and $\dot{\alpha}$ take on the values 1 and 2. Each of these Weyl spinors lives in a different $SU(2)$.

A.3.2 Dotted and undotted notation

Let us now briefly discuss the dotted and undotted notation describing these two Weyl spinors. The idea is simply to distinguish between the two $SU(2)$'s in the spin-$\frac{1}{2}$ representation of the Lorentz group. Weyl spinors with undotted indices live in the first $SU(2)$, whereas Weyl spinors with dotted indices live in the second $SU(2)$.

The Weyl spinors ψ_α and $\bar{\chi}^{\dot{\alpha}}$ were introduced when we rewrote the Dirac spinor Ψ in a reducible form (A.20). We shall also define

$$\bar{\psi}_{\dot{\alpha}} \equiv (\psi_\alpha)^* \qquad \text{and} \qquad \chi^\alpha \equiv \left(\bar{\chi}^{\dot{\alpha}}\right)^*, \tag{A.21}$$

and note that we can raise and lower indices using the anti-symmetric matrices

$$\varepsilon_{\alpha\beta} = \varepsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \varepsilon^{\alpha\beta} = \varepsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.22}$$

Hence $\psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta$ and $\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta$, and similarly for the dotted coordinates.

Now, at this point, we should notice that, due to the anti-symmetric nature of the above matrices, the contraction of two spinors $\psi\chi$ is ambiguous because $\psi^\alpha\chi_\alpha = \varepsilon^{\alpha\beta}\psi_\beta\chi_\alpha = -\varepsilon^{\beta\alpha}\psi_\beta\chi_\alpha = -\psi_\beta\chi^\beta$. Thus we define

$$\psi\chi \equiv \psi^\alpha\chi_\alpha \qquad \text{and} \qquad \bar{\psi}\bar{\chi} \equiv \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}. \tag{A.23}$$

If the components of the spinors are simply commuting complex numbers, then $\psi\chi = -\chi\psi$ and $\bar{\psi}\bar{\chi} = -\bar{\chi}\bar{\psi}$. However, if these components are Grassmannian numbers which anti-commute (for example, when we are working with spinor supercharges or the coordinates in superspace), then the two effects cancel and we find that $\psi\chi = \chi\psi$ and $\bar{\psi}\bar{\chi} = \bar{\chi}\bar{\psi}$.

[1]Technically, it is not quite accurate to call ψ_α and $\bar{\chi}^{\dot{\alpha}}$ Weyl spinors, although it appears to be common jargon. A Weyl spinor is an eigenstate of the chirality operator γ^5, which is diagonal in the Weyl basis. Therefore, in this basis, we find that left-handed and right-handed Weyl spinors take the form $\begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$ respectively.

We shall now briefly mention the idea of a Majorana spinor, which is a Dirac spinor Ψ equal to its charge conjugate $\Psi^C \equiv -i\gamma^0\gamma^2\bar{\Psi}^T$. Any Majorana spinor takes the form

$$\Psi_M = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}, \tag{A.24}$$

where ψ_α is a 2-component Weyl spinor. Therefore any Weyl spinor can be used to construct a Majorana spinor and vice versa.

Let us consider the gamma matrices in the Weyl basis, which can be rewritten as

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \tag{A.25}$$

where $\sigma^\mu = (1_2, \vec{\sigma})$ and $\bar{\sigma}^\mu = (1_2, -\vec{\sigma})$. These matrices σ^μ and $\bar{\sigma}^\mu$ carry mixed dotted and undotted indices because they take a spinor in one $SU(2)$ to a spinor in the other $SU(2)$. More explicitly, σ^μ and $\bar{\sigma}^\mu$ carry the indices $(\sigma^\mu)_{\alpha\dot{\beta}}$ and $(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}$.

The generators $S^{\mu\nu}$ of the four dimensional spin-$\frac{1}{2}$ representation of the Lorentz group can also be rewritten as follows:

$$S^{\mu\nu} = i \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix}, \tag{A.26}$$

in terms of the matrices

$$\sigma^{\mu\nu} \equiv \frac{1}{4}\left(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu\right) \qquad \text{and} \qquad \bar{\sigma}^{\mu\nu} \equiv \frac{1}{4}\left(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu\right), \tag{A.27}$$

which carry the unmixed indices $(\sigma^{\mu\nu})_\alpha{}^\beta$ and $(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}$ respectively.

Appendix B

$SU(3)$ Spin Chains and the Algebraic Bethe Ansatz

B.1 $SU(3)$ Spin Chain Formalism

It is our aim, in this section, to review the formal description of a closed $SU(3)$ spin chain based on discussions in [9, 16, 48, 49, 50, 51]. We shall first construct the Hilbert space in which such a spin chain lives, together with the relevant Hamiltonian. The R-matrix shall then be introduced and shown to satisfy the Yang-Baxter equation (which results in the integrability of the system). We shall hence define the monodromy and transfer matrices. The Hamiltonian and momentum operators can be written in terms of this transfer matrix and thus all three operators can be simultaneously diagonalized.

B.1.1 Hilbert space and observables

A spin chain of length J is an ordered collection of J vector spin states. We are especially interested in the case of an $SU(3)$ spin chain[1], which consists of a collection of 3-component complex vectors (spin-1 states). A natural way in which to rigorously

[1]The Hamiltonian is invariant under $SU(3)$ transformations of the component spin states.

describe such a spin chain is in terms of a tensor product

$$x_1 \otimes x_2 \otimes \ldots \otimes x_J, \qquad \text{with} \;\; x_i \,\epsilon\, \mathbb{C}^3. \tag{B.1}$$

For example, the tensor product of two vectors x and y is defined as $(x \otimes y)^{i_1 i_2} = x^{i_1} y^{i_2}$, where the first index indicates the block row and the second the row within the block. More explicitly,

$$x \otimes y = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \otimes \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix} = \left(\begin{array}{c} x^1 y^1 \\ x^1 y^2 \\ x^1 y^3 \\ \hline x^2 y^1 \\ x^2 y^2 \\ x^2 y^3 \\ \hline x^3 y^1 \\ x^3 y^2 \\ x^3 y^3 \end{array}\right). \tag{B.2}$$

This definition can be generalized in the obvious way to tensor products of an arbitrary number of 3-component complex vectors.

Thus an $SU(3)$ spin chain can be represented by a state in the Hilbert space $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \ldots \otimes \mathbb{C}^3$, which consists of a tensor product of J three dimensional complex vector spaces. Each of these $\mathbb{C}^3$ vector spaces represents a site in the spin chain. For a closed spin chain, the identification of the first site is arbitrary and thus cyclic permutations of our vectors should result in an equivalent state.

We usually work in a basis made up of tensor products of different numbers, J_1, J_2 and J_3 respectively, and different combinations of the $\mathbb{C}^3$ basis states

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}. \tag{B.3}$$

The total spin chain length is $J = J_1 + J_2 + J_3$. We shall find that the eigenstates of the spin chain Hamiltonian (the algebraic Bethe ansatz states) have well-defined J_1, J_2 and J_3. In other words, they consist of combinations of states, each of which involves a tensor product containing a fixed number of each basis state.

Operators acting on any spin chain state can be represented by a tensor product of 3×3 matrices. For example, the tensor product of two matrices M and N is defined as $(M \otimes N)^{i_1 i_2}_{j_1 j_2} = M^{i_1}_{j_1} N^{i_2}_{j_2}$, so that

$$
M \otimes N = \begin{pmatrix} M^1_1 & M^1_2 & M^1_3 \\ M^2_1 & M^2_2 & M^2_3 \\ M^3_1 & M^3_2 & M^3_3 \end{pmatrix} \otimes \begin{pmatrix} N^1_1 & N^1_2 & N^1_3 \\ N^2_1 & N^2_2 & N^2_3 \\ N^3_1 & N^3_2 & N^3_3 \end{pmatrix}
$$

$$
= \left(\begin{array}{ccc|ccc|ccc}
M^1_1 N^1_1 & M^1_1 N^1_2 & M^1_1 N^1_3 & M^1_2 N^1_1 & M^1_2 N^1_2 & M^1_2 N^1_3 & M^1_3 N^1_1 & M^1_3 N^1_2 & M^1_3 N^1_3 \\
M^1_1 N^2_1 & M^1_1 N^2_2 & M^1_1 N^2_3 & M^1_2 N^2_1 & M^1_2 N^2_2 & M^1_2 N^2_3 & M^1_3 N^2_1 & M^1_3 N^2_2 & M^1_3 N^2_3 \\
M^1_1 N^3_1 & M^1_1 N^3_2 & M^1_1 N^3_3 & M^1_2 N^3_1 & M^1_2 N^3_2 & M^1_2 N^3_3 & M^1_3 N^3_1 & M^1_3 N^3_2 & M^1_3 N^3_3 \\
\hline
M^2_1 N^1_1 & M^2_1 N^1_2 & M^2_1 N^1_3 & M^2_2 N^1_1 & M^2_2 N^1_2 & M^2_2 N^1_3 & M^2_3 N^1_1 & M^2_3 N^1_2 & M^2_3 N^1_3 \\
M^2_1 N^2_1 & M^2_1 N^2_2 & M^2_1 N^2_3 & M^2_2 N^2_1 & M^2_2 N^2_2 & M^2_2 N^2_3 & M^2_3 N^2_1 & M^2_3 N^2_2 & M^2_3 N^2_3 \\
M^2_1 N^3_1 & M^2_1 N^3_2 & M^2_1 N^3_3 & M^2_2 N^3_1 & M^2_2 N^3_2 & M^2_2 N^3_3 & M^2_3 N^3_1 & M^2_3 N^3_2 & M^2_3 N^3_3 \\
\hline
M^3_1 N^1_1 & M^3_1 N^1_2 & M^3_1 N^1_3 & M^3_2 N^1_1 & M^3_2 N^1_2 & M^3_2 N^1_3 & M^3_3 N^1_1 & M^3_3 N^1_2 & M^3_3 N^1_3 \\
M^3_1 N^2_1 & M^3_1 N^2_2 & M^3_1 N^2_3 & M^3_2 N^2_1 & M^3_2 N^2_2 & M^3_2 N^2_3 & M^3_3 N^2_1 & M^3_3 N^2_2 & M^3_3 N^2_3 \\
M^3_1 N^3_1 & M^3_1 N^3_2 & M^3_1 N^3_3 & M^3_2 N^3_1 & M^3_2 N^3_2 & M^3_2 N^3_3 & M^3_3 N^3_1 & M^3_3 N^3_2 & M^3_3 N^3_3
\end{array} \right) . \tag{B.4}
$$

Again, for each pair of indices, the first index is a block index and the second is the index within the block.

A basic set of observables consists of the identity matrix, together with

$$
\lambda^i_n = \underset{1^{\text{st}}}{1} \otimes \ldots \otimes \underset{n^{\text{th}}}{\lambda^i} \otimes \ldots \otimes \underset{J^{\text{th}}}{1}. \tag{B.5}
$$

Here λ^i is the i^{th} Gell-Mann matrix in the n^{th} position, with $i \,\epsilon\, \{1, \ldots, 8\}$ and $n \,\epsilon\, \{1, \ldots, J\}$. The Gell-Mann matrices are the generators of $SU(3)$. The spin chain Hamiltonian can be constructed out of these basic observables. For convenience, however, and following [9], we shall rather make use of the states $e^m_n(k)$ to describe the system. These are given by

$$
e^m_n(k) = \underset{1^{\text{st}}}{1} \otimes \ldots \otimes \underset{k^{\text{th}}}{e^m_n} \otimes \ldots \otimes \underset{J^{\text{th}}}{1}, \tag{B.6}
$$

where $(e^m_n)^i_j = \delta^{mi}\delta_{nj}$, with $m, n \,\epsilon\, \{1, 2, 3\}$, is a 3×3 matrix with a 1 in the m^{th} row and n^{th} column as its only non-zero component.

B.1.2 Hamiltonian

The *Hamiltonian* of our closed $SU(3)$ spin chain is

$$H = \frac{\lambda}{8\pi^2} \sum_{k=1}^{J} \mathcal{H}_{k,k+1} \qquad \text{with} \quad \mathcal{H}_{k,k+1} = 1_{k,k+1} - \mathcal{P}_{k,k+1}, \tag{B.7}$$

where $J+1 \equiv 1$ (since our spin chain is closed), and $1_{k,k+1}$ and $\mathcal{P}_{k,k+1}$ are the identity and permutation matrices respectively[2]. These can be written in terms of our basic observables $e^m_n(k)$ and $e^m_n(k+1)$ as follows:

$$1_{k,k+1} = \sum_{n,m=1}^{3} e^m_m(k) e^n_n(k+1) \quad \text{and} \quad \mathcal{P}_{k,k+1} = \sum_{n,m=1}^{3} e^m_n(k) e^n_m(k+1). \tag{B.8}$$

Thus, using the definition (B.6), each part of our spin chain Hamiltonian can be written as

$$\begin{aligned} \mathcal{H}_{k,k+1} = {} & e^1_1 \otimes e^2_2 + e^2_2 \otimes e^1_1 - e^1_2 \otimes e^2_1 - e^2_1 \otimes e^1_2 \\ & + e^3_3 \otimes e^1_1 + e^1_1 \otimes e^3_3 - e^3_1 \otimes e^1_3 - e^1_3 \otimes e^3_1 \\ & + e^2_2 \otimes e^3_3 + e^3_3 \otimes e^2_2 - e^2_3 \otimes e^3_2 - e^3_2 \otimes e^2_3, \end{aligned} \tag{B.9}$$

which, explicitly, gives

$$\mathcal{H}_{k,k+1} = \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ \hline 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ \hline 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \tag{B.10}$$

where the action on all but the k^{th} and $(k+1)^{\text{th}}$ spaces has been suppressed (since it is trivial). The components of this matrix are $(\mathcal{H}_{k,k+1})^{i_k i_{k+1}}_{j_k j_{k+1}} = \delta^{i_k}_{j_k} \delta^{i_{k+1}}_{j_{k+1}} - \delta^{i_{k+1}}_{j_k} \delta^{i_k}_{j_{k+1}}$.

[2]The components of these identity and permutation matrices in the k^{th} and $(k+1)^{\text{th}}$ spaces are $(1_{k,k+1})^{i_k i_{k+1}}_{j_k j_{k+1}} = \delta^{i_k}_{j_k} \delta^{i_{k+1}}_{j_{k+1}}$ and $(\mathcal{P}_{k,k+1})^{i_k i_{k+1}}_{j_k j_{k+1}} = \delta^{i_{k+1}}_{j_k} \delta^{i_k}_{j_{k+1}}$ respectively.

B.1.3 R-matrix and the Yang-Baxter equation

We shall now discuss the *R-matrix*, which is given by

$$R_{i,j}(u) \equiv u 1_{i,j} + i\mathcal{P}_{i,j}, \tag{B.11}$$

where u is a complex parameter and $i, j \in 0, 1, \ldots, J$. This matrix is defined on the tensor product $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \ldots \otimes \mathbb{C}^3$ of $J+1$ three dimensional complex vector spaces. The 0^{th} space is called the *auxillary space* and is an extra $\mathbb{C}^3$ vector space that we have included in our tensor product. The other J vector spaces are called *quantum spaces*. The R-matrix acts non-trivially only on the sites i and j.

More explicitly, the action of the R-matrix on the i^{th} and j^{th} spaces is

$$R_{i,j}(\lambda) = \left(\begin{array}{ccc|ccc|ccc} u+i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & u & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 & 0 & i & 0 & 0 \\ \hline 0 & i & 0 & u & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & u+i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u & 0 & i & 0 \\ \hline 0 & 0 & i & 0 & 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & 0 & u & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u+i \end{array}\right). \tag{B.12}$$

Furthermore, it satisfies the *Yang-Baxter equation*

$$R_{i,j}(u-v)\, R_{i,k}(u)\, R_{j,k}(v) = R_{j,k}(v)\, R_{i,k}(u)\, R_{i,j}(u-v), \tag{B.13}$$

where $i \neq j \neq k$. This can be proved as follows:

$$\begin{aligned} & R_{i,j}(u-v)\, R_{i,k}(u)\, R_{j,k}(v) \\ & = \left[(u-v)\, 1_{i,j} + i\mathcal{P}_{i,j}\right]\left[u 1_{i,k} + i\mathcal{P}_{i,k}\right]\left[v 1_{j,k} + i\mathcal{P}_{j,k}\right] \\ & = (u-v)\, uv 1 + iuv\mathcal{P}_{i,j} + i(u-v)\, v\mathcal{P}_{i,k} + i(u-v)\, u\mathcal{P}_{j,k} - v\mathcal{P}_{i,j}\mathcal{P}_{i,k} \\ & \quad - u\mathcal{P}_{i,j}\mathcal{P}_{j,k} - (u-v)\,\mathcal{P}_{i,k}\mathcal{P}_{j,k} - i\mathcal{P}_{i,j}\mathcal{P}_{i,k}\mathcal{P}_{j,k}. \end{aligned} \tag{B.14}$$

$$\begin{aligned} & R_{j,k}(v)\, R_{i,k}(u)\, R_{i,j}(u-v) \\ & = \left[v 1_{j,k} + i\mathcal{P}_{j,k}\right]\left[u 1_{i,k} + i\mathcal{P}_{i,k}\right]\left[(u-v)\, 1_{i,j} + i\mathcal{P}_{i,j}\right] \\ & = vu(u-v)\, 1 + iu(u-v)\,\mathcal{P}_{j,k} + iv(u-v)\,\mathcal{P}_{i,k} + ivu\mathcal{P}_{i,j} - (u-v)\,\mathcal{P}_{j,k}\mathcal{P}_{i,k} \\ & \quad - u\mathcal{P}_{j,k}\mathcal{P}_{i,j} - v\mathcal{P}_{i,k}\mathcal{P}_{i,j} - i\mathcal{P}_{j,k}\mathcal{P}_{i,k}\mathcal{P}_{i,j}. \end{aligned} \tag{B.15}$$

Looking at the terms depending on u and v, and also at the constant term, one finds that the above expressions are identically equal if and only if the following three equations are satisfied:

$$\mathcal{P}_{i,j}\mathcal{P}_{j,k} + \mathcal{P}_{i,k}\mathcal{P}_{j,k} = \mathcal{P}_{j,k}\mathcal{P}_{i,k} + \mathcal{P}_{j,k}\mathcal{P}_{i,j}, \tag{B.16}$$

$$\mathcal{P}_{i,j}\mathcal{P}_{i,k} - \mathcal{P}_{i,k}\mathcal{P}_{j,k} = \mathcal{P}_{i,k}\mathcal{P}_{i,j} - \mathcal{P}_{j,k}\mathcal{P}_{i,k}, \tag{B.17}$$

$$\mathcal{P}_{i,j}\mathcal{P}_{i,k}\mathcal{P}_{j,k} = \mathcal{P}_{j,k}\mathcal{P}_{i,k}\mathcal{P}_{i,j}. \tag{B.18}$$

We can easily check these equations are valid by considering their action on an arbitrary state $x_0 \otimes x_1 \otimes ... \otimes x_J$, which, using short-hand similar to the notation of the S^3 permutation group, we shall call our base state (ijk)[3]. The permutation operators act on this state by permuting x_i, x_j and x_k (or i, j and k in our short-hand). Either side of the first equation becomes $(kij) + (jki)$, the right-hand side and left-hand side of the second equation are both zero, and the last equation results in the state (kji) on both sides.

B.1.4 Monodromy and transfer matrices

We shall now introduce the *L-matrix*, which is defined as $L_{i,j}(u) = R_{i,j}\left(u - \frac{i}{2}\right)$. We are particularly interested in those L-matrices which act non-trivially on the auxillary space and one of the quantum spaces. (It is these L-matrices which will be used to construct the monodromy matrix.) Thus, setting $i = 0$ and $j = n$, where $n \in \{1, \ldots, J\}$, we can write

$$L_{0,n}(u) = R_{0,n}\left(u - \tfrac{i}{2}\right) \equiv \begin{pmatrix} (\alpha_n)(u) & (\beta_n)_2(u) & (\beta_n)_3(u) \\ (\gamma_n)^2(u) & (\chi_n)^2_2(u) & (\chi_n)^2_3(u) \\ (\gamma_n)^3(u) & (\chi_n)^3_2(u) & (\chi_n)^3_3(u) \end{pmatrix}, \tag{B.19}$$

which is a matrix in the auxillary space 0. The components are operators in the quantum spaces and are given by

$$\begin{aligned} (\alpha_n)(u) &= 1 \otimes \ldots \otimes \alpha(u) \otimes \ldots \otimes 1, \\ (\beta_n)_i(u) &= 1 \otimes \ldots \otimes \beta_i(u) \otimes \ldots \otimes 1, \\ (\gamma_n)^i(u) &= 1 \otimes \ldots \otimes \gamma^i(u) \otimes \ldots \otimes 1, \\ (\chi_n)^i_j(u) &= \underset{1^{\text{st}}}{1} \otimes \ldots \otimes \underset{n^{\text{th}}}{\chi^i_j(u)} \otimes \ldots \otimes \underset{J^{\text{th}}}{1}, \end{aligned} \tag{B.20}$$

[3] Elements of the permutation group are usually written in terms of 1, 2 and 3, instead of i, j and k.

where

$$\alpha(u) = \begin{pmatrix} u+\frac{i}{2} & 0 & 0 \\ 0 & u-\frac{i}{2} & 0 \\ 0 & 0 & u-\frac{i}{2} \end{pmatrix}, \tag{B.21}$$

$$\beta_2(u) = \begin{pmatrix} 0 & 0 & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \beta_3(u) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \tag{B.22}$$

$$\gamma^2(u) = \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^3(u) = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{B.23}$$

$$\chi_2^2(u) = \begin{pmatrix} u-\frac{i}{2} & 0 & 0 \\ 0 & u+\frac{i}{2} & 0 \\ 0 & 0 & u-\frac{i}{2} \end{pmatrix}, \quad \chi_3^2(u) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & i & 0 \end{pmatrix},$$

$$\chi_2^3(u) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}, \qquad \chi_3^3(u) = \begin{pmatrix} u-\frac{i}{2} & 0 & 0 \\ 0 & u-\frac{i}{2} & 0 \\ 0 & 0 & u+\frac{i}{2} \end{pmatrix}. \tag{B.24}$$

Now let us consider two L-matrices $L_{a,n}$ and $L_{b,n}$, which act on different auxillary spaces a and b[4]. Setting $i = a$, $j = b$ and $k = n$ in the Yang-Baxter equation, and changing $u \to u - \frac{i}{2}$ and $v \to v - \frac{i}{2}$, we obtain

$$R_{a,b}(u-v)L_{a,n}(u)L_{b,n}(v) = L_{b,n}(v)L_{a,n}(u)R_{a,b}(u-v). \tag{B.25}$$

The *monodromy matrix* is now defined as

$$T_0(u) = L_{0,J}(u)\dots L_{0,2}(u)L_{0,1}(u), \tag{B.26}$$

and thus, using our previous results for $L_{0,n}(u)$, we find that

$$T_0(u) \equiv \begin{pmatrix} A(u) & B_2(u) & B_3(u) \\ C^2(u) & D_2^2(u) & D_3^2(u) \\ C^3(u) & D_2^3(u) & D_3^3(u) \end{pmatrix} \tag{B.27}$$

$$= \begin{pmatrix} (\alpha_J)(u) & (\beta_J)_2(u) & (\beta_J)_3(u) \\ (\gamma_J)^2(u) & (\chi_J)_2^2(u) & (\chi_J)_3^2(u) \\ (\gamma_J)^3(u) & (\chi_J)_2^3(u) & (\chi_J)_3^3(u) \end{pmatrix} \dots \begin{pmatrix} (\alpha_1)(u) & (\beta_1)_2(u) & (\beta_1)_3(u) \\ (\gamma_1)^2(u) & (\chi_1)_2^2(u) & (\chi_1)_3^2(u) \\ (\gamma_1)^3(u) & (\chi_1)_2^3(u) & (\chi_1)_3^3(u) \end{pmatrix}.$$

[4]We extend the space on which they act to a tensor product over both auxillary spaces a and b, and the quantum spaces i.e. a tensor product of $J+2$ $\mathbb{C}^3$ vector spaces.

This monodromy matrix satisfies

$$R_{a,b}(u-v)T_a(u)T_b(v) = T_b(v)T_a(u)R_{a,b}(u-v), \tag{B.28}$$

where a and b are again two different auxillary spaces. This can be proved using the Yang-Baxter equation (B.25) for the L-matrix and the fact that two L-matrices commute if the subscripts are distinct. The proof is as follows:

$$\begin{aligned}
&R_{a,b}(u-v)T_a(u)T_b(v) \\
&= R_{a,b}(u-v)L_{a,J}(u)\ldots L_{a,2}(u)L_{a,1}(u)L_{b,J}(v)\ldots L_{b,2}(v)L_{b,1}(v) \\
&= R_{a,b}(u-v)L_{a,J}(u)L_{b,J}(v)\ldots L_{a,2}(u)L_{b,2}(v)L_{a,1}(u)L_{b,1}(v) \\
&= L_{b,J}(v)L_{a,J}(u)R_{a,b}(u-v)L_{a,J-1}(u)L_{b,J-1}(v)\ldots L_{a,1}(u)L_{b,1}(v) \\
&= L_{b,J}(v)L_{a,J}(u)L_{b,J-1}(v)L_{a,J-1}(u)R_{a,b}(u-v)L_{a,J-2}(u)L_{b,J-2}(v)\ldots L_{a,1}(u)L_{b,1}(v) \\
&= \ \ldots \\
&= L_{b,J}(v)L_{a,J}(u)\ldots L_{b,2}(v)L_{a,2}(u)L_{b,1}(v)L_{a,1}(u)R_{a,b}(u-v) \\
&= L_{b,J}(v)\ldots L_{b,2}(v)L_{b,1}(v)L_{a,J}(u)\ldots L_{a,2}(u)L_{a,1}(u)R_{a,b}(u-v) \\
&= T_b(v)T_a(u)R_{a,b}(u-v).
\end{aligned} \tag{B.29}$$

The *transfer matrix* is finally defined by taking the trace of the monodromy matrix over the auxillary space as follows:

$$t(u) \equiv \mathrm{Tr}_0\left[T_0(u)\right] = A(u) + D^l_l(u), \tag{B.30}$$

to obtain an operator, which acts only on the quantum spaces.

B.1.5 Momentum and Hamiltonian operators in terms of the transfer matrix

The *momentum operator* can be written in terms of the transfer matrix as

$$P = \frac{1}{i}\,\log\left[i^{-J}\ t\left(\tfrac{i}{2}\right)\right], \tag{B.31}$$

which shall be checked as follows:

$$\begin{aligned}
e^{iP} &= i^{-J}\ t\left(\tfrac{i}{2}\right) \\
&= \mathrm{Tr}_0\left[\mathcal{P}_{0,J}\ldots\mathcal{P}_{0,2}\mathcal{P}_{0,1}\right] \\
&= \mathrm{Tr}_0\left[\mathcal{P}_{1,2}\mathcal{P}_{2,3}\ldots\mathcal{P}_{J-1,J}\mathcal{P}_{J,0}\right] \\
&= \mathcal{P}_{1,2}\mathcal{P}_{2,3}\ldots\mathcal{P}_{J-1,J},
\end{aligned} \tag{B.32}$$

since $\mathcal{P}_{0,J} \dots \mathcal{P}_{0,2}\mathcal{P}_{0,1} = \mathcal{P}_{1,2}\mathcal{P}_{2,3} \dots \mathcal{P}_{J-1,J}\mathcal{P}_{J,0}$ and $\mathrm{Tr}_0\left(\mathcal{P}_{0,J}\right) = 1$. The first result can be verified simply by looking at the action of either side of the expression on an arbitrary state $x_0 \otimes x_1 \otimes \dots \otimes x_J$, which in our short-hand we shall call $(0, 1, \dots, J-1, J)$. Both sides of the equation change this state into $(1, 2, \dots, J, 0)$. The last result can be trivially checked by writing the action of the permutation operator on the 0^{th} and J^{th} spaces in matrix form.

We see that $e^{iP} = \mathcal{P}_{12}\mathcal{P}_{23} \dots \mathcal{P}_{J-1,J}$ is a translation by one site along the (closed) spin chain. Thus P generates translations and is, indeed, the momentum operator.

The *Hamiltonian operator* can also be written in terms of the transfer matrix as

$$H = \frac{\lambda}{8\pi^2}\left[J - i\,\frac{d}{du}\,\log\,t(u)\bigg|_{u=\frac{i}{2}}\right], \tag{B.33}$$

which is proved as follows:

First let us calculate

$$\begin{aligned}
\left[t\left(\tfrac{i}{2}\right)\right]^{-1} &= \left\{i^J\ \mathrm{Tr}_0[\mathcal{P}_{0,J} \dots \mathcal{P}_{0,2}\mathcal{P}_{0,1}]\right\}^{-1} \\
&= i^{-J}\left\{\mathrm{Tr}_0\left[\mathcal{P}_{1,2}\mathcal{P}_{2,3} \dots \mathcal{P}_{J-1,J}\mathcal{P}_{J,0}\right]\right\}^{-1} \\
&= i^{-J}\left\{\mathcal{P}_{1,2}\mathcal{P}_{2,3} \dots \mathcal{P}_{J-1,J}\right\}^{-1} \\
&= i^{-J}\ \mathcal{P}_{J,J-1} \dots \mathcal{P}_{3,2}\mathcal{P}_{2,1},
\end{aligned} \tag{B.34}$$

and also

$$\begin{aligned}
i\,\frac{dt(u)}{du}\bigg|_{u=\frac{i}{2}} &= i\,\frac{d}{du}\left[\mathrm{Tr}_0\left\{\left[\left(u-\tfrac{i}{2}\right)1_{0,J} + i\mathcal{P}_{0,J}\right] \dots \left[\left(u-\tfrac{i}{2}\right)1_{0,1} + i\mathcal{P}_{0,1}\right]\right\}\right]\bigg|_{u=\frac{i}{2}} \\
&= i\sum_{n=1}^{J}\ \mathrm{Tr}_0\left\{\left[\left(u-\tfrac{i}{2}\right)1_{0,J} + i\mathcal{P}_{0,J}\right] \dots \left[\widehat{\left(u-\tfrac{i}{2}\right)1_{0,n}} + i\mathcal{P}_{0,n}\right]\right. \\
&\qquad\qquad \left.\dots\left[\left(u-\tfrac{i}{2}\right)1_{0,1} + i\mathcal{P}_{0,1}\right]\right\}\Big|_{u=\frac{i}{2}} \\
&= i^J\ \sum_{n=1}^{J}\ \mathrm{Tr}_0\left[\mathcal{P}_{0,J} \dots \widehat{\mathcal{P}}_{0,n} \dots \mathcal{P}_{0,2}\mathcal{P}_{0,1}\right] \\
&= i^J\left[\mathcal{P}_{2,3}\mathcal{P}_{3,4} \dots \mathcal{P}_{J-1,J}\right] + i^J\sum_{n=2}^{J-1}\left[\mathcal{P}_{1,2}\mathcal{P}_{2,3} \dots \mathcal{P}_{n-1,n+1} \dots \mathcal{P}_{J-1,J}\right] \\
&\quad + i^J\left[\mathcal{P}_{1,2}\mathcal{P}_{23} \dots \mathcal{P}_{J-2,J-1}\right].
\end{aligned} \tag{B.35}$$

Hence we obtain

$$
\begin{aligned}
i\,\frac{d}{du}\,\log t(u)\bigg|_{u=\frac{i}{2}} &= i\,\frac{dt(u)}{du}\bigg|_{u=\frac{i}{2}}\,\left[t\left(\tfrac{i}{2}\right)\right]^{-1} \\
&= \left[\mathcal{P}_{2,3}\mathcal{P}_{3,4}\ldots\mathcal{P}_{J-1,J}\right]\left[\mathcal{P}_{J,J-1}\ldots\mathcal{P}_{3,2}\mathcal{P}_{2,1}\right] \\
&\quad+\sum_{n=2}^{J-1}\left[\mathcal{P}_{1,2}\mathcal{P}_{2,3}\ldots\mathcal{P}_{n-1,n+1}\ldots\mathcal{P}_{J-1,J}\right]\left[\mathcal{P}_{J,J-1}\ldots\mathcal{P}_{3,2}\mathcal{P}_{2,1}\right] \\
&\quad+\left[\mathcal{P}_{1,2}\mathcal{P}_{2,3}\ldots\mathcal{P}_{J-2,J-1}\right]\left[\mathcal{P}_{J,J-1}\ldots\mathcal{P}_{3,2}\mathcal{P}_{2,1}\right] \\
&= \mathcal{P}_{1,2}+\mathcal{P}_{J,1}+\sum_{n=2}^{J-1}\,\mathcal{P}_{n,n+1}=\sum_{n=1}^{J}\mathcal{P}_{n,n+1},
\end{aligned}
\tag{B.36}
$$

where we define $J+1\equiv 1$ (since our spin chain is closed). In moving from line 2 to line 3, we check the first and last terms explicitly (by considering their action on an arbitrary state), and then notice that $\mathcal{P}_{n-1,n+1}\mathcal{P}_{n+1,n}\mathcal{P}_{n,n-1}=\mathcal{P}_{n,n+1}$ for $n=2,\ldots,J-1$ and the other terms commute around this expression and cancel.

Therefore (B.33) implies that

$$
H=\frac{\lambda}{8\pi^2}\left\{J-\sum_{n=1}^{J}\mathcal{P}_{n,n+1}\right\}=\frac{\lambda}{8\pi^2}\left\{\sum_{n=1}^{J}\left[1_{n,n+1}-\mathcal{P}_{n,n+1}\right]\right\}, \tag{B.37}
$$

which agrees with our original expression (B.7) for the spin chain Hamiltonian.

B.2 Algebraic Bethe ansatz

In this section, we construct states which diagonalize the transfer matrix (and thus also the Hamiltonian and momentum operators) using the so-called algebraic Bethe ansatz. These states are dependent, as shall be seen, on two sets of parameters $\{u_{1,1},\ldots,u_{1,M}\}$ and $\{u_{2,1},\ldots,u_{2,L}\}$, where $M=J_2+J_3$ and $L=J_3$. It is demonstrated that these Bethe parameters must satisfy the nested Bethe ansatz equations. Furthermore, we construct the momentum and energy eigenvalues, and also show that there is a cyclicity condition, which is due to the fact that a translation by one site along our closed spin chain must be equivalent to the identity. These ideas are described in the excellent reviews [50, 51] for $SU(2)$ spin chains. We have extended these concepts to $SU(3)$ spin chains, with the aid of the more terse results in [9, 49].

B.2.1 Fundamental commutation relations

The fundamental commutation relations are a set of constraints satisfied by the operator components $A(u)$, $B_i(u)$, $C^i(u)$ and $D^i_j(u)$ of the monodromy matrix (B.27), and are an indirect result of the Yang-Baxter equation. Let us now briefly review the derivation of these relations:

Our monodromy matrix (B.27) must satisfy (B.28). The R-matrix $R_{a,b}(u-v)$ can be written out in matrix form over the spaces a and b, together with the expressions $T_a(u)T_b(v)$ and $T_b(v)T_a(u)$. Plugging these matrices into (B.28) yields a number of constraints on $A(u)$, $B_i(u)$, $C^i(u)$ and $D^i_j(u)$, among which are the fundamental commutation relations

$$A(u)B_{i_1}(v) = \left(\frac{u-v-i}{u-v}\right) B_{i_1}(v)A(u) + \left(\frac{i}{u-v}\right) B_{i_1}(u)A(v), \tag{B.38}$$

$$B_{i_1}(u)B_{i_2}(v) = \begin{cases} \left(\frac{u-v}{u-v+i}\right) B_{i_1}(v)B_{i_2}(u) + \left(\frac{i}{u-v+i}\right) B_{i_2}(u)B_{i_1}(v) & \text{if } i_1 \neq i_2 \\ B_{i_2}(v)B_{i_1}(u) & \text{if } i_1 = i_2, \end{cases} \tag{B.39}$$

$$D^{k_1}_{i_1}(u)B_{i_2}(v) = \begin{cases} B_{i_2}(v)D^{k_1}_{i_1}(u) + \left(\frac{i}{u-v}\right)\left[B_{i_1}(v)D^{k_1}_{i_2}(u) - B_{i_1}(u)D^{k_1}_{i_2}(v)\right] & \text{if } i_1 \neq i_2 \\ \left(\frac{u-v+i}{u-v}\right) B_{i_2}(v)D^{k_1}_{i_1}(u) + \left(\frac{-i}{u-v}\right) B_{i_2}(u)D^{k_1}_{i_1}(v) & \text{if } i_1 = i_2, \end{cases} \tag{B.40}$$

where our indices may take on the possible values 2 and 3.

We shall now define the matrix $\tilde{R}^{i_1 i_2}_{j_1 j_2}(u)$, which acts on a tensor product of two complex two dimensional $\mathbb{C}^2$ vector spaces, as follows:

$$\tilde{R}^{i_1 i_2}_{j_1 j_2}(u) = \begin{cases} (u+i)\ \delta^{i_1}_{j_1}\delta^{i_2}_{j_2} & \text{if } i_1 = i_2 \\ u\ \delta^{i_1}_{j_1}\delta^{i_2}_{j_2} + i\ \delta^{i_2}_{j_1}\delta^{i_1}_{j_2} & \text{if } i_1 \neq i_2, \end{cases} \tag{B.41}$$

or, alternatively, in matrix form

$$\tilde{R}(u) = \left(\begin{array}{cc|cc} u+i & 0 & 0 & 0 \\ 0 & u & i & 0 \\ \hline 0 & i & u & 0 \\ 0 & 0 & 0 & u+i \end{array}\right). \tag{B.42}$$

This can be written as $\tilde{R}(u) = u1 + i\mathcal{P}$ and is thus, by analogy, the R-matrix of an $SU(2)$ spin chain (confined to the two spaces on which it acts non-trivially).

Hence the fundamental commutation relations can be written as

$$A(u)B_{i_1}(v) = \left(\frac{u-v-i}{u-v}\right) B_{i_1}(v)A(u) + \left(\frac{i}{u-v}\right) B_{i_1}(u)A(v), \tag{B.43}$$

$$B_{i_1}(u)B_{i_2}(v) = \left(\frac{1}{u-v+i}\right) \tilde{R}^{j_1 j_2}_{i_1 i_2}(u-v)\, B_{j_2}(v)B_{j_1}(u), \tag{B.44}$$

$$D^{k_1}_{i_1}(u)B_{i_2}(v) = \left(\frac{1}{u-v}\right) \tilde{R}^{j_1 j_2}_{i_1 i_2}(u-v)\, B_{j_2}(v)D^{k_1}_{j_1}(u) + \left(\frac{-i}{u-v}\right) B_{i_1}(u)D^{k_1}_{i_2}(v), \tag{B.45}$$

with $i_1, i_2, j_1, j_2, k_1 \,\epsilon\, \{2,3\}$.

B.2.2 Algebraic Bethe ansatz and the eigenvalues of the transfer matrix

Let us first define the ground state of our $SU(3)$ spin chain. This consists of a chain of J spin-up vectors as follows:

$$\omega_+ = \begin{pmatrix}1\\0\\0\end{pmatrix} \otimes \ldots \otimes \begin{pmatrix}1\\0\\0\end{pmatrix}. \tag{B.46}$$

This is an eigenstate of the spin chain Hamiltonian (B.7) with eigenvalue zero. In fact, this is true for *any* combination of J identical states, but we have specifically chosen this ground state ω_+ from which to construct our algebraic Bethe ansatz.

We would now like to establish the action of the operator components of the monodromy matrix on this ground state. Thus let us write

$$\begin{aligned} T_0(u)\,\omega_+ &= \begin{pmatrix} A(u) & B_1(u) & B_1(u) \\ C^1(u) & D^2_2(u) & D^2_3(u) \\ C^2(u) & D^3_2(u) & D^3_3(u) \end{pmatrix} \omega_+ \\ &= \begin{pmatrix} (\alpha_J)(u) & (\beta_J)_2(u) & (\beta_J)_3(u) \\ (\gamma_J)^2(u) & (\chi_J)^2_2(u) & (\chi_J)^2_3(u) \\ (\gamma_J)^3(u) & (\chi_J)^3_2(u) & (\chi_J)^3_3(u) \end{pmatrix} \cdots \begin{pmatrix} (\alpha_1)(u) & (\beta_1)_2(u) & (\beta_1)_3(u) \\ (\gamma_1)^2(u) & (\chi_1)^2_2(u) & (\chi_1)^2_3(u) \\ (\gamma_1)^3(u) & (\chi_1)^3_2(u) & (\chi_1)^3_3(u) \end{pmatrix} \omega_+. \end{aligned} \tag{B.47}$$

We can see from the definition (B.20) that $(\alpha_n)(u)$, $(\beta_n)_i(u)$, $(\gamma_n)^i(u)$ and $(\chi_n)^i_j(u)$ act non-trivially only on the n^{th} quantum space and we shall thus look at the action of $\alpha(u)$, $\beta_i(u)$, $\gamma^i(u)$ and $\chi^i_j(u)$, shown explicitly in (B.21)-(B.24), on the n^{th} site of the ground state:

$$\alpha(u)\begin{pmatrix}1\\0\\0\end{pmatrix} = \left(u+\tfrac{i}{2}\right)\begin{pmatrix}1\\0\\0\end{pmatrix}, \tag{B.48}$$

$$\beta_2(u)\begin{pmatrix}1\\0\\0\end{pmatrix} = i\begin{pmatrix}0\\1\\0\end{pmatrix}, \qquad \beta_3(u)\begin{pmatrix}1\\0\\0\end{pmatrix} = i\begin{pmatrix}0\\0\\1\end{pmatrix}, \tag{B.49}$$

$$\gamma^i(u)\begin{pmatrix}1\\0\\0\end{pmatrix} = 0 \tag{B.50}$$

$$\chi^i_j(u)\begin{pmatrix}1\\0\\0\end{pmatrix} = \left(u-\tfrac{i}{2}\right)\delta^i_j\begin{pmatrix}1\\0\\0\end{pmatrix}. \tag{B.51}$$

Multiplying out the matrices in equation (B.47) and applying the result to the ground state, we therefore find that

$$A\,(u)\,\omega_+ = \left(u+\tfrac{i}{2}\right)^J\omega_+, \tag{B.52}$$

$$C^i\,(u)\,\omega_+ = 0, \tag{B.53}$$

$$D^i_j\,(u)\,\omega_+ = \left(u-\tfrac{i}{2}\right)^J\delta^i_j\;\omega_+, \tag{B.54}$$

and $B_i\,(u)$ act by lowering the spin of one site in our ground state. In other words, $B_2\,(u)\,\omega_+$ and $B_3\,(u)\,\omega_+$ are combinations of states involving the tensor product of one $\begin{pmatrix}0\\1\\0\end{pmatrix}$ vector and $J-1$ $\begin{pmatrix}1\\0\\0\end{pmatrix}$ vectors, and one $\begin{pmatrix}0\\0\\1\end{pmatrix}$ vector and $J-1$ $\begin{pmatrix}1\\0\\0\end{pmatrix}$ vectors respectively.

These results now lead us to introduce the *first part of the algebraic Bethe ansatz* for the eigenstates of the transfer matrix as follows:

$$\Phi\,(u_{1,1},\ldots,u_{1,M}) = f^{i_1,\ldots,i_M}B_{i_1}\,(u_{1,1})\ldots B_{i_M}\,(u_{1,M})\,\omega_+, \tag{B.55}$$

where $f^{i_1,\ldots,i_M}$ are complex coefficients and $\{u_{1,1},\ldots,u_{1,M}\}$ is the first set of Bethe parameters. Here we have used our operators $B_i\,(u)$ to lower the spin of M sites in the ground state and thus $M = J_2 + J_3$.

Now let us apply each part of the transfer matrix $t(u) = A(u)+D_l^l(u)$ to this algebraic Bethe ansatz state:

Firstly,

$$A\left(u\right)\Phi\left(u_{1,1},\ldots,u_{1,M}\right) = f^{i_1,\ldots,i_M} A\left(u\right) B_{i_1}\left(u_{1,1}\right)\ldots B_{i_M}\left(u_{1,M}\right)\omega_+, \tag{B.56}$$

and hence, using the fundamental commutation relation (B.43), we find that

$$\begin{aligned}
&A\left(u\right)\Phi\left(u_{1,1},\ldots,u_{1,M}\right)\\
&= f^{i_1,\ldots,i_M}\left(\frac{u-u_{1,1}-i}{u-u_{1,1}}\right) B_{i_1}\left(u_{1,1}\right) A\left(u\right) B_{i_2}\left(u_{1,2}\right)\ldots B_{i_M}\left(u_{1,M}\right)\omega_+\\
&+f^{i_1,\ldots,i_M}\left(\frac{i}{u-u_{1,1}}\right) B_{i_1}\left(u\right) A\left(u_{1,1}\right) B_{i_2}\left(u_{1,2}\right)\ldots B_{i_M}\left(u_{1,M}\right)\omega_+\\
&\\
&= f^{i_1,\ldots,i_M}\left(\frac{u-u_{1,1}-i}{u-u_{1,1}}\right)\left(\frac{u-u_{1,2}-i}{u-u_{1,2}}\right) B_{i_1}(u_{1,1}) B_{i_2}\left(u_{1,2}\right) A\left(u\right)\ldots B_{i_M}\left(u_{1,M}\right)\omega_+\\
&+f^{i_1,\ldots,i_M}\left(\frac{i}{u-u_{1,1}}\right)\left(\frac{u_{1,1}-u_{1,2}-i}{u_{1,1}-u_{1,2}}\right) B_{i_1}\left(u\right) B_{i_2}\left(u_{1,2}\right) A\left(u_{1,1}\right)\ldots B_{i_M}\left(u_{1,M}\right)\omega_+\\
&+f^{i_1,\ldots,i_M}\left(\frac{u-u_{1,1}-i}{u-u_{1,1}}\right)\left(\frac{i}{u-u_{1,2}}\right) B_{i_1}\left(u_{1,1}\right) B_{i_2}\left(u\right) A\left(u_{1,2}\right)\ldots B_{i_M}\left(u_{1,M}\right)\omega_+\\
&+f^{i_1,\ldots,i_M}\left(\frac{i}{u-u_{1,1}}\right)\left(\frac{i}{u_{1,1}-u_{1,2}}\right) B_{i_1}\left(u\right) B_{i_2}\left(u_{1,1}\right) A\left(u_{1,2}\right)\ldots B_{i_M}\left(u_{1,M}\right)\omega_+.
\end{aligned} \tag{B.57}$$

In this way, we can continue to move A through the B_i operators until it is the right-most operator and we can use the expression (B.52) to determine its action on the ground state ω_+. This gives the following result

$$\begin{aligned}
&A\left(u\right)\Phi\left(u_{1,1},\ldots,u_{1,M}\right)\\
&= f^{i_1,\ldots,i_M}\left[\prod_{k=1}^{M}\left(\frac{u-u_{1,k}-i}{u-u_{1,k}}\right)\right]\left(u+\tfrac{i}{2}\right)^J B_{i_1}\left(u_{1,1}\right)\ldots B_{i_M}\left(u_{1,M}\right)\omega_+\\
&+f^{i_1,\ldots,i_M}\left[\prod_{k=2}^{M}\left(\frac{u_{1,1}-u_{1,k}-i}{u_{1,1}-u_{1,k}}\right)\right]\left(\frac{i}{u-u_{1,1}}\right)\left(u_{1,1}+\tfrac{i}{2}\right)^J B_{i_1}\left(u\right) B_{i_2}\left(u_{1,2}\right)\ldots B_{i_M}\left(u_{1,M}\right)\omega_+
\end{aligned}$$

$$
\begin{aligned}
&+f^{i_1,\ldots,i_M}\left[\prod_{k=3}^{M}\left(\frac{u_{1,2}-u_{1,k}-i}{u_{1,2}-u_{1,k}}\right)\right]\left(u_{1,2}+\tfrac{i}{2}\right)^J \\
&\quad\times\left\{\left(\frac{u-u_{1,1}-i}{u-u_{1,1}}\right)\left(\frac{i}{u-u_{1,2}}\right)B_{i_1}(u_{1,1})\,B_{i_2}(u)\,B_{i_3}(u_{1,3})\ldots B_{i_M}(u_{1,M})\,\omega_+\right. \\
&\qquad\left.+\left(\frac{i}{u-u_{1,1}}\right)\left(\frac{i}{u_{1,1}-u_{1,2}}\right)B_{i_1}(u)\,B_{i_2}(u_{1,1})\,B_{i_3}(u_{1,3})\ldots B_{i_M}(u_{1,M})\,\omega_+\right\} \\
&\quad+\ \ldots . \qquad\qquad (B.58)
\end{aligned}
$$

The other terms involve the operators $A(u_{1,3})$, $A(u_{1,4})$, $\ldots$, $A(u_{1,M})$ acting on the ground state ω_+.

Secondly,

$$
D_l^l(u)\,\Phi(u_{1,1},\ldots,u_{1,M}) = f^{i_1,\ldots,i_M}D_l^l(u)\,B_{i_1}(u_{1,1})\ldots B_{i_M}(u_{1,M})\,\omega_+, \qquad (B.59)
$$

and hence, using the fundamental commutation relation (B.45), it follows that

$$
\begin{aligned}
&D_l^l(u)\,\Phi(u_{1,1},\ldots,u_{1,M}) \\
&= f^{i_1,\ldots,i_M}\left(\frac{1}{u-u_{1,1}}\right)\tilde{R}_{li_1}^{j_1k_1}(u-u_{1,1}) \\
&\qquad\times B_{k_1}(u_{1,1})\,D_{j_1}^l(u)\,B_{i_2}(u_{1,2})\ldots B_{i_M}(u_{1,M})\,\omega_+ \\
&\quad+f^{i_1,\ldots,i_M}\left(\frac{-i}{u-u_{1,1}}\right)B_l(u)\,D_{i_1}^l(u_{1,1})\,B_{i_2}(u_{1,2})\ldots B_{i_M}(u_{1,M})\,\omega_+ \\
&= f^{i_1,\ldots,i_M}\left(\frac{1}{u-u_{1,1}}\right)\left(\frac{1}{u-u_{1,2}}\right)\tilde{R}_{j_1i_2}^{j_2k_2}(u-u_{1,2})\,\tilde{R}_{li_1}^{j_1k_1}(u-u_{1,1}) \\
&\qquad\times B_{k_1}(u_{1,1})\,B_{k_2}(u_{1,2})\,D_{j_2}^l(u)\,B_{i_3}(u_{1,3})\ldots B_{i_M}(u_{1,M})\,\omega_+ \\
&\quad+f^{i_1,\ldots,i_M}\left(\frac{-i}{u-u_{1,1}}\right)\left(\frac{1}{u_{1,1}-u_{1,2}}\right)\tilde{R}_{i_1i_2}^{j_1k_2}(u_{1,1}-u_{1,2}) \\
&\qquad\times B_l(u)\,B_{k_2}(u_{1,2})\,D_{j_1}^l(u_{1,1})\,B_{i_3}(u_{1,3})\ldots B_{i_M}(u_{1,M})\,\omega_+ \\
&\quad+f^{i_1,\ldots,i_M}\left(\frac{1}{u-u_{1,1}}\right)\left(\frac{-i}{u-u_{1,2}}\right)\tilde{R}_{li_1}^{j_1k_1}(u-u_{1,1}) \\
&\qquad\times B_{k_1}(u_{1,1})\,B_{j_1}(u)\,D_{i_2}^l(u_{1,2})\,B_{i_3}(u_{1,3})\ldots B_{i_M}(u_{1,M})\,\omega_+ \\
&\quad+f^{i_1,\ldots,i_M}\left(\frac{-i}{u-u_{1,1}}\right)\left(\frac{-i}{u_{1,1}-u_{1,2}}\right) \\
&\qquad\times B_l(u)\,B_{i_1}(u_{1,1})\,D_{i_2}^l(u_{1,2})\,B_{i_3}(u_{1,3})\ldots B_{i_M}(u_{1,M})\,\omega_+, \qquad (B.60)
\end{aligned}
$$

and so on. Again we move the D_l^l operator through all the B_i operators in our algebraic Bethe ansatz state and then use equation (B.54) to determine how it acts on the ground state ω_+. Thus we find that

$$
\begin{aligned}
& D_l^l\,(u)\,\Phi\,(u_{1,1},\ldots,u_{1,M}) \\
& \\
&= f^{i_1,\ldots,i_M}\left[\prod_{k=1}^{M}\left(\frac{1}{u-u_{1,k}}\right)\right]\left(u-\tfrac{i}{2}\right)^J \\
&\quad\times \tilde{R}^{lk_M}_{j_{M-1}i_M}\,(u-u_{1,M})\ldots\tilde{R}^{j_2k_2}_{j_1i_2}\,(u-u_{1,2})\,\tilde{R}^{j_1k_1}_{li_1}\,(u-u_{1,1}) \\
&\quad\times B_{k_1}\,(u_{1,1})\,B_{k_2}\,(u_{1,2})\ldots B_{k_M}\,(u_{1,M})\,\omega_+ \\
& \\
&\quad+ f^{i_1,\ldots,i_M}\left[\prod_{k=2}^{M}\left(\frac{1}{u_{1,1}-u_{1,k}}\right)\right]\left(u_{1,1}-\tfrac{i}{2}\right)^J\left(\frac{-i}{u-u_{1,1}}\right) \\
&\quad\times \tilde{R}^{lk_M}_{j_{M-1}i_M}(u_{1,1}-u_{1,M})\ldots\tilde{R}^{j_3k_3}_{j_2i_3}\,(u_{1,1}-u_{1,3})\,\tilde{R}^{j_2k_2}_{i_1i_2}\,(u_{1,1}-u_{1,2}) \\
&\quad\times B_l\,(u)\,B_{k_2}\,(u_{1,2})\,B_{k_3}\,(u_{1,3})\ldots B_{k_M}\,(u_{1,M})\,\omega_+ \\
& \\
&\quad+ f^{i_1,\ldots,i_M}\left[\prod_{k=3}^{M}\left(\frac{1}{u_{1,2}-u_{1,k}}\right)\right]\left(u_{1,2}-\tfrac{i}{2}\right)^J \\
&\quad\times\left\{\left(\frac{1}{u-u_{1,1}}\right)\left(\frac{-i}{u-u_{1,2}}\right)\right. \\
&\qquad\times \tilde{R}^{lk_M}_{j_{M-1}i_M}\,(u_{1,2}-u_{1,M})\ldots\tilde{R}^{j_4k_4}_{j_3i_4}\,(u_{1,2}-u_{1,4})\,\tilde{R}^{j_3k_3}_{i_2i_3}\,(u_{1,2}-u_{1,3})\,\tilde{R}^{j_1k_1}_{li_1}\,(u-u_{1,1}) \\
&\qquad\times B_{k_1}\,(u_{1,1})\,B_{j_1}\,(u)\,B_{k_3}\,(u_{1,3})\,B_{k_4}\,(u_{1,4})\ldots B_{k_M}\ldots(u_{1,M})\,\omega_+ \\
& \\
&\qquad+\left(\frac{-i}{u-u_{1,1}}\right)\left(\frac{-i}{u_{1,1}-u_{1,2}}\right) \\
&\qquad\quad\times \tilde{R}^{lk_M}_{j_{M-1}i_M}\,(u_{1,2}-u_{1,M})\ldots\tilde{R}^{j_4k_4}_{j_3i_4}\,(u_{1,2}-u_{1,4})\,\tilde{R}^{j_3k_3}_{i_2i_3}\,(u_{1,2}-u_{1,3}) \\
&\qquad\quad\left.\times B_l\,(u)\,B_{i_1}\,(u_{1,1})\,B_{k_3}\,(u_{1,3})\,B_{k_4}\,(u_{1,4})\ldots B_{k_M}\,(u_{1,M})\,\omega_+\right\} \\
&\quad+\ \ldots.
\end{aligned}
\tag{B.61}
$$

The first terms in the expressions for $A(u)\Phi\,(u_{1,1},\ldots,u_{1,M})$ and $D_l^l(u)\Phi\,(u_{1,1},\ldots,u_{1,M})$ will be used to derive the eigenvalues of the transfer matrix $t(u)=A(u)+D_l^l(u)$. The requirement that the sum of all the other terms should cancel will give us the first nested Bethe ansatz equation. This is really a collection of M equations, which will be discussed in detail later and will ensure that each two corresponding 'extra terms' sum to zero.

For now, let us assume that only the first terms are relevant and try to determine the eigenvalues $\Lambda(u)$ of the transfer matrix. We shall also redefine $u \to u - \frac{i}{2}$ at this point for convenience. Thus we obtain

$$
\begin{aligned}
& t(u)\,\Phi(u_{1,1},\ldots,u_{1,M}) = \left[A(u) + D_l^l(u)\right]\Phi\left(u_{1,1},\ldots,u_{1,M}\right) \\
& = f^{i_1,\ldots,i_M}\left[\prod_{k=1}^{M}\frac{u-u_{1,k}-\frac{3i}{2}}{u-u_{1,k}-\frac{i}{2}}\right]u^J\;B_{i_1}\left(u_{1,1}\right)B_{i_2}\left(u_{1,2}\right)\ldots B_{i_M}\left(u_{1,M}\right)\omega_+ \\
& \quad + f^{i_1,\ldots,i_M}\left[\prod_{k=1}^{M}\frac{1}{u-u_{1,k}-\frac{i}{2}}\right](u-i)^J \\
& \quad\quad \times \tilde{R}^{lk_M}_{j_{M-1}i_M}\left(u-u_{1,M}-\tfrac{i}{2}\right)\ldots\tilde{R}^{j_2k_2}_{j_1i_2}\left(u-u_{1,2}-\tfrac{i}{2}\right)\tilde{R}^{j_1k_1}_{li_1}\left(u-u_{1,1}-\tfrac{i}{2}\right) \\
& \quad\quad \times B_{k_1}\left(u_{1,1}\right)B_{k_2}\left(u_{1,2}\right)\ldots B_{k_M}\left(u_{1,M}\right)\omega_+.
\end{aligned}
\tag{B.62}
$$

We can see immediately that for the right-hand side of this equation to be proportional to $\Phi\left(u_{1,1},\ldots,u_{1,M}\right)$, we must require that

$$
\begin{aligned}
& \tilde{R}^{lk_M}_{j_{M-1}i_M}\left(u-u_{1,M}-\tfrac{i}{2}\right)\ldots\tilde{R}^{j_2k_2}_{j_1i_2}\left(u-u_{1,2}-\tfrac{i}{2}\right)\tilde{R}^{j_1k_1}_{li_1}\left(u-u_{1,1}-\tfrac{i}{2}\right) \\
& \times f^{i_1,\ldots,i_M}B_{k_1}\left(u_{1,1}\right)B_{k_2}\left(u_{1,2}\right)\ldots B_{k_M}\left(u_{1,M}\right)\omega_+ \\
& = \tilde{\Lambda}(u)\,f^{i_1,\ldots,i_M}B_{i_1}\left(u_{1,1}\right)B_{i_2}\left(u_{1,2}\right)\ldots B_{i_M}\left(u_{1,M}\right)\omega_+,
\end{aligned}
\tag{B.63}
$$

where $\tilde{\Lambda}(u)$ is some complex function.

The trick is to think of $f^{i_1,\ldots,i_M}$ (in the basis $B_{i_1}\left(u_{1,1}\right)B_{i_2}\left(u_{1,2}\right)\ldots B_{i_M}\left(u_{1,M}\right)\omega_+$) as an $SU(2)$ spin chain state of length M on the tensor product $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\ldots\otimes\mathbb{C}^2$ of M two dimensional complex vector spaces. The k^{th} site of the $SU(2)$ spin chain is represented by the index i_k, so that $i_k = 2$ and 3 represent spin-up and spin-down states. The number of spin-up and spin-down states in the $SU(2)$ spin chain correspond to J_2 and J_3 respectively. Our problem now becomes to diagonalize

$$
\tilde{t}(u) = \mathrm{Tr}_0\left[\tilde{R}_{0,M}\left(u-u_{1,M}-\tfrac{i}{2}\right)\ldots\tilde{R}_{0,2}\left(u-u_{1,2}-\tfrac{i}{2}\right)\tilde{R}_{0,1}\left(u-u_{1,1}-\tfrac{i}{2}\right)\right], \tag{B.64}
$$

where $\tilde{R}_{0,n}(u)$ is the R-matrix on $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\ldots\otimes\mathbb{C}^2$ acting non-trivially on the auxillary space 0 and the quantum space n. We have thus reduced our problem by one dimension and must deal with almost the same equation now on a tensor product of $\mathbb{C}^2$ complex vector spaces.

As in the $SU(3)$ case, we find that $\tilde{t}(u)$ is the trace of the $SU(2)$ monodromy matrix, which can be written as a matrix over the auxillary space as follows:

$$\tilde{T}(u) = \begin{pmatrix} \tilde{A}(u) & \tilde{B}(u) \\ \tilde{C}(u) & \tilde{D}(u) \end{pmatrix} \tag{B.65}$$

$$= \begin{pmatrix} \tilde{\alpha}_M(u-u_{1,M}) & \tilde{\beta}_M(u-u_{1,M}) \\ \tilde{\gamma}_M(u-u_{1,M}) & \tilde{\delta}_M(u-u_{1,M}) \end{pmatrix} \ldots \begin{pmatrix} \tilde{\alpha}_1(u-u_{1,1}) & \tilde{\beta}_1(u-u_{1,1}) \\ \tilde{\gamma}_1(u-u_{1,1}) & \tilde{\chi}_1(u-u_{1,1}) \end{pmatrix},$$

where

$$\begin{aligned}
\tilde{\alpha}_n(u) &= 1 \otimes \ldots \otimes \tilde{\alpha}(u) \otimes \ldots \otimes 1, \\
\tilde{\beta}_n(u) &= 1 \otimes \ldots \otimes \tilde{\beta}(u) \otimes \ldots \otimes 1, \\
\tilde{\gamma}_n(u) &= 1 \otimes \ldots \otimes \tilde{\gamma}(u) \otimes \ldots \otimes 1, \\
\tilde{\chi}_n(u) &= \underset{1^{\mathrm{st}}}{1} \otimes \ldots \otimes \underset{n^{\mathrm{th}}}{\tilde{\chi}(u)} \otimes \ldots \otimes \underset{M^{\mathrm{th}}}{1},
\end{aligned} \tag{B.66}$$

and we define

$$\tilde{\alpha}(u) = \begin{pmatrix} u+\frac{i}{2} & 0 \\ 0 & u-\frac{i}{2} \end{pmatrix}, \qquad \tilde{\beta}(u) = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix},$$

$$\tilde{\gamma}(u) = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \qquad \tilde{\chi}(u) = \begin{pmatrix} u-\frac{i}{2} & 0 \\ 0 & u+\frac{i}{2} \end{pmatrix}. \tag{B.67}$$

Now, as before, we shall define the ground state of our $SU(2)$ spin chain to consist entirely of spin-up states as follows:

$$\tilde{\omega}_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \ldots \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{B.68}$$

Thus we obtain

$$\tilde{A}(u)\,\tilde{\omega}_+ = \prod_{k=1}^{M}\left(u-u_{1,k}+\frac{i}{2}\right)\tilde{\omega}_+, \tag{B.69}$$

$$\tilde{C}(u)\,\tilde{\omega}_+ = 0, \tag{B.70}$$

$$\tilde{D}(u)\,\tilde{\omega}_+ = \prod_{k=1}^{M}\left(u-u_{1,k}-\frac{i}{2}\right)\tilde{\omega}_+, \tag{B.71}$$

and $\tilde{B}(u)\,\tilde{\omega}_+$ gives a combination of states with one spin-down vector.

We shall again use the $\tilde{B}$ operator to construct the *second part of the algebraic Bethe ansatz* for the eigenstates $f^{i_1,\ldots,i_M}$ of our $SU(2)$ transfer matrix as follows:

$$f^{i_1,\ldots,i_M} \equiv \tilde{\Phi}(u_{2,1},\ldots,u_{2,L}) = \tilde{B}(u_{2,1})\ldots\tilde{B}(u_{2,L})\,\tilde{\omega}_+, \tag{B.72}$$

where $\{u_{2,1}, u_{2,2}, \ldots, u_{2,L}\}$ is the second set of Bethe parameters. Here we have lowered the spin of L sites in our $SU(2)$ spin chain so that $L = J_3$.

The fundamental commutation relations in this $SU(2)$ case are

$$\tilde{A}(u)\,\tilde{B}(v) = \left(\frac{u-v-i}{u-v}\right)\tilde{B}(v)\,\tilde{A}(u) + \left(\frac{i}{u-v}\right)\tilde{B}(u)\,\tilde{A}(v)\,, \tag{B.73}$$

$$\tilde{B}(u)\,\tilde{B}(v) = \tilde{B}(v)\,\tilde{B}(u)\,, \tag{B.74}$$

$$\tilde{D}(u)\,\tilde{B}(v) = \left(\frac{u-v+i}{u-v}\right)\tilde{B}(v)\,\tilde{D}(u) + \left(\frac{-i}{u-v}\right)\tilde{B}(u)\,\tilde{D}(v)\,. \tag{B.75}$$

The derivation of the above relations for $SU(2)$ spin chains is identical to the derivation in the $SU(3)$ case. We have a similar R-matrix, which satisfies an identical Yang-Baxter equation and so on.

The action of each part of the $SU(2)$ transfer matrix $\tilde{t}(u) = \tilde{A}(u) + \tilde{D}(u)$ on the state $\tilde{\Phi}(u_{2,1}, \ldots, u_{2,L})$ shall now be determined:

We first find that

$$\tilde{A}(u)\,\tilde{\Phi}(u_{2,1}, \ldots, u_{2,L}) = \tilde{A}(u)\,\tilde{B}(u_{2,1})\ldots\tilde{B}(u_{2,L})\,\tilde{\omega}_+, \tag{B.76}$$

and thus, using the fundamental commutation relation (B.73),

$$\begin{aligned}
&\tilde{A}(u)\,\tilde{\Phi}(u_{2,1}, \ldots, u_{2,L}) \\
&= \left(\frac{u-u_{2,1}-i}{u-u_{2,1}}\right)\tilde{B}(u_{2,1})\,\tilde{A}(u)\,\tilde{B}(u_{2,2})\ldots\tilde{B}(u_{2,L})\,\tilde{\omega}_+ \\
&\quad + \left(\frac{i}{u-u_{2,1}}\right)\tilde{B}(u)\,\tilde{A}(u_{2,1})\,\tilde{B}(u_{2,2})\ldots\tilde{B}(u_{2,L})\,\tilde{\omega}_+ \\
&= \left(\frac{u-u_{2,1}-i}{u-u_{2,1}}\right)\left(\frac{u-u_{2,2}-i}{u-u_{2,2}}\right)\tilde{B}(u_{2,1})\,\tilde{B}(u_{2,2})\,\tilde{A}(u)\,\tilde{B}(u_{2,3})\ldots\tilde{B}(u_{2,L})\,\tilde{\omega}_+ \\
&\quad + \left(\frac{i}{u-u_{2,1}}\right)\left(\frac{u_{2,1}-u_{2,2}-i}{u_{2,1}-u_{2,2}}\right)\tilde{B}(u)\,\tilde{B}(u_{2,2})\,\tilde{A}(u_{2,1})\,\tilde{B}(u_{2,3})\ldots\tilde{B}(u_{2,L})\,\tilde{\omega}_+ \\
&\quad + \left(\frac{i}{u-u_{2,2}}\right)\left(\frac{u_{2,2}-u_{2,1}-i}{u_{2,2}-u_{2,1}}\right)\tilde{B}(u)\,\tilde{B}(u_{2,2})\,\tilde{A}(u_{2,1})\,\tilde{B}(u_{2,3})\ldots\tilde{B}(u_{2,L})\,\tilde{\omega}_+.
\end{aligned} \tag{B.77}$$

Continuing in this way and finally using (B.69), we obtain

$$
\begin{aligned}
&\tilde{A}(u)\tilde{\Phi}\left(u_{2,1},\ldots,u_{2,L}\right)\\
&=\left[\prod_{k=1}^{L}\left(\frac{u-u_{2,k}-i}{u-u_{2,k}}\right)\right]\left[\prod_{l=1}^{M}\left(u-u_{1,l}+\tfrac{i}{2}\right)\right]\tilde{\Phi}\left(u_{2,1},\ldots,u_{2,L}\right)\\
&\quad+\sum_{j=1}^{L}\left(\frac{i}{u-u_{2,j}}\right)\left[\prod_{\substack{k=1\\k\neq j}}^{L}\left(\frac{u_{2,j}-u_{2,k}-i}{u_{2,j}-u_{2,k}}\right)\right]\left[\prod_{l=1}^{M}\left(u_{2,j}-u_{1,l}+\tfrac{i}{2}\right)\right]\\
&\quad\times\tilde{B}\left(u_{2,L}\right)\ldots\widehat{\tilde{B}}\left(u_{2,j}\right)\ldots\tilde{B}\left(u_{2,1}\right)\tilde{B}\left(u\right)\tilde{\omega}_{+}.
\end{aligned}
\tag{B.78}
$$

We can similarly determine, using the fundamental commutation relation (B.75) and (B.71) for the action of the operator $\tilde{D}(u)$ on the ground state $\tilde{\omega}_{+}$, that

$$
\begin{aligned}
&\tilde{D}\left(u\right)\tilde{\Phi}\left(u_{2,1},\ldots,u_{2,L}\right)\\
&=\left[\prod_{k=1}^{L}\left(\frac{u-u_{2,k}+i}{u-u_{2,k}}\right)\right]\left[\prod_{l=1}^{M}\left(u-u_{1,l}-\tfrac{i}{2}\right)\right]\tilde{\Phi}\left(u_{2,1},\ldots,u_{2,L}\right)\\
&\quad+\sum_{j=1}^{L}\left(\frac{-i}{u-u_{2,j}}\right)\left[\prod_{\substack{k=1\\k\neq j}}^{L}\left(\frac{u_{2,j}-u_{2,k}+i}{u_{2,j}-u_{2,k}}\right)\right]\left[\prod_{l=1}^{M}\left(u_{2,j}-u_{1,l}-\tfrac{i}{2}\right)\right]\\
&\quad\times\tilde{B}\left(u_{2,L}\right)\ldots\widehat{\tilde{B}}\left(u_{2,j}\right)\ldots\tilde{B}(u_{2,1})\tilde{B}\left(u\right)\tilde{\omega}_{+}.
\end{aligned}
\tag{B.79}
$$

Again it is only the first terms in equations (B.78) and (B.79) that are relevant when calculating the eigenvalue of our $SU(2)$ transfer matrix, which is given by

$$
\begin{aligned}
\tilde{\Lambda}(u)\;&=\left[\prod_{k=1}^{L}\left(\frac{u-u_{2,k}-i}{u-u_{2,k}}\right)\right]\left[\prod_{l=1}^{M}\left(u-u_{1,l}+\tfrac{i}{2}\right)\right]\\
&\quad+\left[\prod_{k=1}^{L}\left(\frac{u-u_{2,k}+i}{u-u_{2,k}}\right)\right]\left[\prod_{l=1}^{M}\left(u-u_{1,l}-\tfrac{i}{2}\right)\right].
\end{aligned}
\tag{B.80}
$$

We shall ensure that the other terms not proportional to $\tilde{\Phi}\left(u_{2,1},\ldots,u_{2,L}\right)$ cancel out by enforcing the second nested Bethe ansatz equation (which is actually a collection of L equations - one for each unwanted pair of terms in our sum).

Now, substituting this expression into (B.62), we find that the eigenvalues of the

$SU(3)$ transfer matrix are

$$\Lambda(u) = \left[\prod_{k=1}^{M}\left(\frac{u-u_{1,k}-\frac{3i}{2}}{u-u_{1,k}-\frac{i}{2}}\right)\right]u^J \tag{B.81}$$

$$+\left[\prod_{k=1}^{M}\left(\frac{1}{u-u_{1,k}-\frac{i}{2}}\right)\right](u-i)^J\left\{\left[\prod_{k=1}^{L}\left(\frac{u-u_{2,k}-i}{u-u_k}\right)\right]\left[\prod_{l=1}^{M}\left(u-u_{1,l}+\tfrac{i}{2}\right)\right]\right.$$

$$\left.+\left[\prod_{k=1}^{L}\left(\frac{u-u_{2,k}+i}{u-u_k}\right)\right]\left[\prod_{l=1}^{M}\left(u-u_{1,l}-\tfrac{i}{2}\right)\right]\right\},$$

in terms of the two sets of Bethe parameters $\{u_{1,1},\ldots,u_{1,M}\}$ and $\{u_{1,2},\ldots,u_{1,L}\}$, which define our algebraic Bethe ansatz state.

B.2.3 Energy and momentum eigenvalues and the cyclicity condition

We find, using (B.33) for the Hamiltonian in terms of the transfer matrix and taking into account our previous redefinition of u, that the energy eigenvalues are

$$\begin{aligned}
E &= \frac{\lambda}{8\pi^2}\left[J - i\,\frac{d}{du}\,\log\,\Lambda(u)\Big|_{u=i}\right] \\
&= \frac{\lambda}{8\pi^2}\left[J - i\,\frac{d}{du}\,\log\left\{\prod_{k=1}^{M}\left(\frac{u-u_{1,k}-\frac{3i}{2}}{u-u_{1,k}-\frac{i}{2}}\right)u^J\right\}\Bigg|_{u=i}\right] \\
&= \frac{\lambda}{8\pi^2}\left[J + J\,\log\,u - i\frac{d}{du}\left\{\left[\sum_{k=1}^{M}\log\left(u-u_{1,k}-\tfrac{3i}{2}\right)-\log\left(u-u_{1,k}-\tfrac{i}{2}\right)\right]\right\}\Bigg|_{u=i}\right] \\
&= \frac{\lambda}{8\pi^2}\;i\;\sum_{k=1}^{M}\left[\frac{1}{u_{1,k}+\frac{i}{2}}-\frac{1}{u_{1,k}-\frac{i}{2}}\right],
\end{aligned} \tag{B.82}$$

which finally gives

$$E = \frac{\lambda}{8\pi^2}\sum_{k=1}^{M}\frac{1}{u_{1,k}^2+\frac{1}{4}}. \tag{B.83}$$

The energy eigenvalues corresponding to our algebraic Bethe ansatz states are thus directly dependent only on the first set of Bethe parameters. Indirectly, however, the two sets of Bethe parameters are interdependent, as shall be seen, due to the nested Bethe ansatz equations.

Lastly, we should mention that there exists also a cyclicity condition. This can be explained by considering the eigenvalues of the momentum operator (B.31), which are given by

$$P = \frac{1}{i}\,\log\left[i^{-J}\,\Lambda\left(i\right)\right] = \frac{1}{i}\sum_{k=1}^{M}\,\log\left(\frac{u_{1,k}+\frac{i}{2}}{u_{1,k}-\frac{i}{2}}\right). \tag{B.84}$$

We must insist that there should be no change to any spin chain state when we translate by one site along our closed spin chain. Therefore we shall require that $e^{iP}=1$, which implies the cyclicity condition

$$\prod_{k=1}^{M}\left(\frac{u_{1,k}+\frac{i}{2}}{u_{1,k}-\frac{i}{2}}\right) = 1, \tag{B.85}$$

on our first set of Bethe parameters $\{u_{1,1},\ldots,u_{1,M}\}$.

B.2.4 Nested Bethe ansatz equations

The two Bethe ansatz equations shall now be derived by requiring that the sum of the 'extra terms' in (B.58) and (B.61), and (B.78) and (B.79), which are not proportional to $\Phi\left(u_{1,1},\ldots,u_{1,M}\right)$ and $\tilde{\Phi}\left(u_{2,1},\ldots,u_{2,L}\right)$ respectively, cancel out.

The *first nested Bethe ansatz equation* comes from looking at (B.58) and (B.61). Let us consider the $j=1$ equation, which corresponds to our first pair of 'extra terms'. For these first terms to cancel out, we must require that

$$\begin{aligned}
&\left[\prod_{k=2}^{M}\left(\frac{u_{1,1}-u_{1,k}-i}{u_{1,1}-u_{1,k}}\right)\right]\left(\frac{i}{u-u_{1,1}-\frac{i}{2}}\right)\left(u_{1,1}+\tfrac{i}{2}\right)^{J}\\
&\quad\times f^{i_1,\ldots,i_M}B_{i_1}\left(u-\frac{i}{2}\right)B_{i_2}(u_{1,2})\ldots B_{i_M}(u_{1,M})\\
&+\left[\prod_{k=2}^{M}\left(\frac{1}{u_{1,1}-u_{1,k}}\right)\right]\left(u_{1,1}-\tfrac{i}{2}\right)^{J}\left(\frac{-i}{u-u_{1,1}-\frac{i}{2}}\right)\\
&\quad\times \tilde{R}^{lk_M}_{j_{M-1}i_M}\left(u_{1,1}-u_{1,M}\right)\ldots\tilde{R}^{j_3k_3}_{j_2i_3}\left(u_{1,1}-u_{1,2}\right)\tilde{R}^{j_2k_2}_{i_1i_2}\left(u_{1,1}-u_{1,2}\right)\\
&\quad\times f^{i_1,\ldots,i_M}B_{l}\left(u-\frac{i}{2}\right)B_{k_2}\left(u_{1,2}\right)B_{k_3}\left(u_{1,3}\right)\ldots B_{k_M}\left(u_{1,M}\right)\omega_{+}=0,
\end{aligned} \tag{B.86}$$

where we have taken into account our redefinition of u. To calculate the last term in this expression we need to determine

$$\tilde{R}_{1,M}\left(u_{1,1}-u_{1,M}\right)\ldots\tilde{R}_{1,3}\left(u_{1,1}-u_{1,3}\right)\tilde{R}_{1,2}\left(u_{1,1}-u_{1,2}\right)\tilde{\Phi}\left(u_{2,1},\ldots,u_{2,L}\right)\Big|_{u=u_{1,1}+\frac{i}{2}}. \tag{B.87}$$

With this in mind, let us consider us consider the $SU(2)$ transfer matrix evaluated at $u = u_{1,1} + \frac{i}{2}$ as follows:

$$(-i)\,\tilde{t}\left(u_{1,1} + \tfrac{i}{2}\right) = -i\ \mathrm{Tr}_0 \left\{\tilde{R}_{0,M}\left(u_{1,1} - u_{1,M}\right)\ldots\tilde{R}_{0,2}\left(u_{1,1} - u_{1,2}\right)\tilde{R}_{0,1}\left(0\right)\right\}$$

$$= \mathrm{Tr}_0 \left\{\left[\left(u_{1,1} - u_{1,M}\right)1_{0,M} + i\mathcal{P}_{0,M}\right]\ldots\left[\left(u_{1,1} - u_{1,2}\right)1_{0,2} + i\mathcal{P}_{0,2}\right]\mathcal{P}_{0,1}\right\}$$

$$\begin{aligned}= \mathrm{Tr}_0 \{&\left(u_{1,1} - u_{1,M}\right)\ldots\left(u_{1,1} - u_{1,2}\right)\mathcal{P}_{0,1}\\ &+ \sum_{k=2}^{M}\left(u_{1,1} - u_{1,M}\right)\ldots\left(\widehat{u_{1,1} - u_{1,k}}\right)\ldots\left(u_{1,1} - u_{1,2}\right)\mathcal{P}_{0,k}\mathcal{P}_{0,1} + \ldots\}\end{aligned}$$

$$\begin{aligned}= &\left(u_{1,1} - u_{1,M}\right)\ldots\left(u_{1,1} - u_{1,2}\right)\\ &+ \sum_{k=2}^{M}\left(u_{1,1} - u_{1,M}\right)\ldots\left(\widehat{u_{1,1} - u_{1,k}}\right)\ldots\left(u_{1,1} - u_{1,2}\right)\mathcal{P}_{1,k} + \ldots\end{aligned}$$

$$= \left[\left(u_{1,1} - u_{1,M}\right)1_{1,M} + i\mathcal{P}_{1,M}\right]\ldots\left[\left(u_{1,1} - u_{1,3}\right)1_{1,3} + i\mathcal{P}_{1,3}\right]\left[\left(u_{1,1} - u_{1,2}\right)1_{1,2} + i\mathcal{P}_{1,2}\right]$$

$$= \tilde{R}_{1,M}\left(u_{1,1} - u_{1,M}\right)\ldots\tilde{R}_{1,3}\left(u_{1,1} - u_{1,3}\right)\tilde{R}_{1,2}\left(u_{1,1} - u_{1,2}\right). \tag{B.88}$$

Thus we see that the expression (B.87) can be written as

$$\begin{aligned}&\tilde{R}_{1,M}\left(u_{1,1} - u_{1,M}\right)\ldots\tilde{R}_{1,3}\left(u_{1,1} - u_{1,3}\right)\tilde{R}_{1,2}\left(u_{1,1} - u_{1,2}\right)\tilde{\Phi}\left(u_{2,1},\ldots,u_{2,L}\right)\Big|_{u=u_{1,1}+\frac{i}{2}}\\ &= \left\{(-i)\,\tilde{t}\left(u_{1,1} + \tfrac{i}{2}\right)\tilde{\Phi}\left(u_{2,1},\ldots,u_{2,L}\right)\right\}\Big|_{u=u_{1,1}+\frac{i}{2}}\\ &= (-i)\,\tilde{\Lambda}\left(u_{1,1} + \tfrac{i}{2}\right)\tilde{\Phi}\left(u_{2,1},\ldots,u_{2,L}\right)\Big|_{u=u_{1,1}+\frac{i}{2}},\end{aligned} \tag{B.89}$$

so that both sides of (B.86) are proportional to the $SU(2)$ algebraic Bethe ansatz state $\tilde{\Phi}\left(u_{2,1},\ldots,u_{2,L}\right)$ evaluated at $u = u_{1,1} + \frac{i}{2}$ and thus we need only equate the coefficients of this state. Therefore, substituting the eigenvalues of the $SU(2)$ transfer matrix (B.80) into our constraint (B.86), we obtain

$$\begin{aligned}&\left[\prod_{k=2}^{M}\left(u_{1,1} - u_{1,k} - i\right)\right]\left(u_{1,1} + \tfrac{i}{2}\right)^J = \left(u_{1,1} - \tfrac{i}{2}\right)^J(-i)\,\tilde{\Lambda}\left(u_{1,1} + \tfrac{i}{2}\right)\\ &= \left(u_{1,1} - \tfrac{i}{2}\right)^J\left[\prod_{l=1}^{L}\left(\frac{u_{1,1} - u_{2,l} - \frac{i}{2}}{u_{1,1} - u_{2,l} + \frac{i}{2}}\right)\right]\left[\prod_{k=2}^{M}\left(u_{1,1} - u_{1,k} + i\right)\right].\end{aligned} \tag{B.90}$$

Hence the first Bethe ansatz equation for $j = 1$ is

$$\left(\frac{u_{1,1}+\frac{i}{2}}{u_{1,1}-\frac{i}{2}}\right)^J = \left[\prod_{k=2}^{M}\left(\frac{u_{1,1}-u_{1,k}+i}{u_{1,1}-u_{1,k}-i}\right)\right]\left[\prod_{l=1}^{L}\left(\frac{u_{1,1}-u_{2,l}-\frac{i}{2}}{u_{1,1}-u_{2,l}+\frac{i}{2}}\right)\right]. \tag{B.91}$$

I shall not prove this equation for the other $j = 2, \ldots, M$ values, since the expressions get more and more complicated. However, a similar equation can be obtained in these cases. The first nested Bethe ansatz equation, for every $j \in \{1, \ldots, M\}$, is thus given by

$$\left(\frac{u_{1,j}+\frac{i}{2}}{u_{1,j}-\frac{i}{2}}\right)^J = \left[\prod_{\substack{k=1\\k\neq j}}^{M}\left(\frac{u_{1,j}-u_{1,k}+i}{u_{1,j}-u_{1,k}-i}\right)\right]\left[\prod_{l=1}^{L}\left(\frac{u_{1,j}-u_{2,l}-\frac{i}{2}}{u_{1,j}-u_{2,l}+\frac{i}{2}}\right)\right]. \tag{B.92}$$

The *second nested Bethe ansatz equation* shall now be derived from (B.78) and (B.79). For the 'extra terms' to be zero, we must require that, for all $j \in \{1, \ldots, L\}$,

$$\begin{aligned}
&\left(\frac{i}{u-u_{2,j}}\right)\left[\prod_{\substack{k=1\\k\neq j}}^{L}\left(\frac{u_{2,j}-u_{2,k}-i}{u_{2,j}-u_{2,k}}\right)\right]\left[\prod_{l=1}^{M}\left(u_{2,j}-u_{1,l}+\tfrac{i}{2}\right)\right]\\
&\quad\times \tilde{B}\left(u_{2,1}\right)\ldots\widehat{\tilde{B}}\left(u_{2,j}\right)\ldots\tilde{B}\left(u_{2,L}\right)\tilde{B}\left(u\right)\tilde{\omega}_+\\
&+\left(\frac{-i}{u-u_{2,j}}\right)\left[\prod_{\substack{k=1\\k\neq j}}^{L}\left(\frac{u_{2,j}-u_{2,k}+i}{u_{2,j}-u_{2,k}}\right)\right]\left[\prod_{l=1}^{M}\left(u_{2,j}-u_{1,l}-\tfrac{i}{2}\right)\right]\\
&\quad\times \tilde{B}\left(u_{2,1}\right)\ldots\widehat{\tilde{B}}\left(u_{2,j}\right)\ldots\tilde{B}\left(u_{2,L}\right)\tilde{B}\left(u\right)\tilde{\omega}_+ = 0,
\end{aligned} \tag{B.93}$$

We can now equate the coefficients of the states in the above set of equations to zero. This leads to the second nested Bethe ansatz equation, which is given by

$$\left[\prod_{\substack{k=1\\k\neq j}}^{L}\left(\frac{u_{2,j}-u_{2,k}+i}{u_{2,j}-u_{2,k}-i}\right)\right]\left[\prod_{l=1}^{M}\left(\frac{u_{1,l}-u_{2,j}+\frac{i}{2}}{u_{1,l}-u_{2,j}-\frac{i}{2}}\right)\right] = 1, \tag{B.94}$$

for every $j \in \{1, \ldots, L\}$.

These nested Bethe ansatz equations, together with the cyclicity condition, are the constraints which must be satisfied by the two sets of Bethe parameters for our algebraic Bethe ansatz state to be an eigenstate of the closed $SU(3)$ spin chain Hamiltonian.

Appendix C

Details of Calculations for Semiclassical Lax Pairs

C.1 Derivation of the Undeformed Equations of Motion from the Landau-Lifshitz Lax Pair

C.1.1 Derivation of the Landau-Lifshitz equation

We shall now derive the Landau-Lifshitz equation from the relevant Lax pair representation based on discussions in [9, 20]. The Landau-Lifshitz Lax pair is given by

$$\mathcal{D}_\alpha = \partial_\alpha - A_\alpha, \tag{C.1}$$

where

$$A_0 = \frac{1}{6}\left[N, \partial_1 N\right] x + \frac{3i}{2} N x^2, \tag{C.2}$$

$$A_1 = iNx. \tag{C.3}$$

The zero curvature condition, which must now be satisfied, is

$$\partial_0 A_1 - \partial_1 A_0 - [A_0, A_1] = 0. \tag{C.4}$$

Note also that N satisfies the constraints $\mathrm{Tr}(N) = 0$ and $N^2 = N + 2$, due to the definition $N_{ij} = 3U_i^* U_j - \delta_{ij}$, where $U_i = r_i e^{i\tilde{\tilde{\phi}}_i}$, and the constraint $\sum_{i=1}^{3} r_i^2 = 1$.

Now this equation (C.4) can be written in terms of N as follows:

$$i\partial_0 N x - \frac{1}{6}\partial_1 \left[N, \partial_1 N\right] x - \frac{3i}{2}\partial_1 N x^2 - \frac{i}{6}\left[\left[N, \partial_1 N\right], N\right] x^2 = 0, \tag{C.5}$$

and thus, equating different orders in x,

$$i\partial_0 N = \frac{1}{6}\partial_1 \left[N, \partial_1 N\right], \tag{C.6}$$

$$\frac{3}{2}\partial_1 N = -\frac{1}{6}\left[\left[N, \partial_1 N\right], N\right]. \tag{C.7}$$

Equation (C.7) follows from the constraint $N^2 = N + 2$, whereas (C.6) is equivalent to the Landau-Lifshitz equation of motion

$$i\partial_0 N = \frac{1}{6}\left[N, \partial_1^2 N\right]. \tag{C.8}$$

C.1.2 Derivation of the undeformed equations of motion from the Landau-Lifshitz equation

Let us now express this Landau-Lifshitz equation in terms of the undeformed radial and angular coordinates r_i and $\tilde{\tilde{\phi}}_i$ respectively. The definition of N in component form is

$$N_{ij} = 3 r_i r_j e^{i\left(\tilde{\tilde{\phi}}_j - \tilde{\tilde{\phi}}_i\right)} - \delta_{ij}. \tag{C.9}$$

Thus we obtain

$$\partial_0 N_{ij} = 3\left[\left(\dot{r}_i r_j + r_i \dot{r}_j\right) + i r_i r_j \left(\dot{\tilde{\tilde{\phi}}}_j - \dot{\tilde{\tilde{\phi}}}_i\right)\right] e^{i\left(\tilde{\tilde{\phi}}_j - \tilde{\tilde{\phi}}_i\right)}, \tag{C.10}$$

$$\partial_1 N_{ij} = 3\left[\left(r_i' r_j + r_i r_j'\right) + i r_i r_j \left(\tilde{\tilde{\phi}}_j' - \tilde{\tilde{\phi}}_i'\right)\right] e^{i\left(\tilde{\tilde{\phi}}_j - \tilde{\tilde{\phi}}_i\right)}, \tag{C.11}$$

and hence also

$$\begin{aligned}\partial_1^2 N_{ij} = 3 e^{i\left(\tilde{\tilde{\phi}}_j - \tilde{\tilde{\phi}}_i\right)} \Big[&\left(r_i'' r_j + 2 r_i' r_j' + r_i r_j''\right) + 2i \left(r_i r_j\right)' \left(\tilde{\tilde{\phi}}_j' - \tilde{\tilde{\phi}}_i'\right) \\ &+ i r_i r_j \left(\tilde{\tilde{\phi}}_j'' - \tilde{\tilde{\phi}}_i''\right) - r_i r_j \left(\tilde{\tilde{\phi}}_j' - \tilde{\tilde{\phi}}_i'\right)^2\Big].\end{aligned} \tag{C.12}$$

Therefore, using the explicit expression

$$\left[N, \partial_1^2 N\right]_{ij} = \sum_{k=1}^{3} N_{ik} \partial_1^2 N_{kj} - \sum_{k=1}^{3} \partial_1^2 N_{ik} N_{kj}, \tag{C.13}$$

it follows that

$$\begin{aligned}\left[N,\partial_1^2 N\right]_{ij} = 9e^{i\left(\tilde{\tilde{\phi}}_j-\tilde{\tilde{\phi}}_i\right)}\Bigg\{ & \left[r_i r_j'' - r_j r_i''\right] \\ & + 2i\left[r_i \sum_{k=1}^{3} r_k\left(r_k r_j\right)'\left(\tilde{\tilde{\phi}}_j' - \tilde{\tilde{\phi}}_k'\right) - r_j \sum_{k=1}^{3} r_k\left(r_i r_k\right)'\left(\tilde{\tilde{\phi}}_k' - \tilde{\tilde{\phi}}_i'\right)\right] \\ & + i\left[r_i r_j \sum_{k=1}^{3} r_k^2\left(\tilde{\tilde{\phi}}_j'' - \tilde{\tilde{\phi}}_k''\right) - r_i r_j \sum_{k=1}^{3} r_k^2\left(\tilde{\tilde{\phi}}_k'' - \tilde{\tilde{\phi}}_i''\right)\right] \\ & - \left[r_i r_j \sum_{k=1}^{3} r_k^2\left(\tilde{\tilde{\phi}}_j' - \tilde{\tilde{\phi}}_k'\right)^2 - r_i r_j \sum_{k=1}^{3} r_k^2\left(\tilde{\tilde{\phi}}_k' - \tilde{\tilde{\phi}}_i'\right)^2\right]\Bigg\}. \end{aligned} \tag{C.14}$$

We can now substitute the definition of N, together with (C.10) and (C.14), into the Landau-Lifshitz equation (C.8) to obtain

$$\begin{aligned} & i\left(\dot{r}_i r_j + r_i \dot{r}_j\right) - r_i r_j\left(\dot{\tilde{\tilde{\phi}}}_j - \dot{\tilde{\tilde{\phi}}}_i\right) \\ &= \frac{1}{2}\Bigg\{\left[r_i r_j'' - r_j r_i''\right] + 2i\left[r_i \sum_{k=1}^{3} r_k\left(r_k r_j\right)'\left(\tilde{\tilde{\phi}}_j' - \tilde{\tilde{\phi}}_k'\right) - r_j \sum_{k=1}^{3} r_k\left(r_i r_k\right)'\left(\tilde{\tilde{\phi}}_k' - \tilde{\tilde{\phi}}_i'\right)\right] \\ & \quad + i\left[r_i r_j \sum_{k=1}^{3} r_k^2\left(\tilde{\tilde{\phi}}_j'' - \tilde{\tilde{\phi}}_k''\right) - r_i r_j \sum_{k=1}^{3} r_k^2\left(\tilde{\tilde{\phi}}_k'' - \tilde{\tilde{\phi}}_i''\right)\right] \\ & \quad - \left[r_i r_j \sum_{k=1}^{3} r_k^2\left(\tilde{\tilde{\phi}}_j' - \tilde{\tilde{\phi}}_k'\right)^2 - r_i r_j \sum_{k=1}^{3} r_k^2\left(\tilde{\tilde{\phi}}_k' - \tilde{\tilde{\phi}}_i'\right)^2\right]\Bigg\}. \end{aligned} \tag{C.15}$$

Therefore, separating the real and imaginary parts of the above expression,

$$\text{Re:}\quad r_j r_i'' - r_i r_j'' = 2 r_i r_j\left(\dot{\tilde{\tilde{\phi}}}_j - \dot{\tilde{\tilde{\phi}}}_i\right) + r_i r_j \sum_{k=1}^{3} r_k^2\left(\tilde{\tilde{\phi}}_i' - \tilde{\tilde{\phi}}_k'\right)^2 - r_i r_j \sum_{k=1}^{3} r_k^2 (\tilde{\tilde{\phi}}_j' - \tilde{\tilde{\phi}}_k')^2, \tag{C.16}$$

$$\begin{aligned}\text{Im:}\quad \dot{r}_i r_j + r_i \dot{r}_j = & \; r_j \sum_{k=1}^{3} r_k\left(r_i r_k\right)'\left(\tilde{\tilde{\phi}}_i' - \tilde{\tilde{\phi}}_k'\right) + r_i \sum_{k=1}^{3} r_k\left(r_j r_k\right)'\left(\tilde{\tilde{\phi}}_j' - \tilde{\tilde{\phi}}_k'\right) \\ & + \frac{1}{2}\, r_i r_j \sum_{k=1}^{3} r_k^2\left(\tilde{\tilde{\phi}}_i'' - \tilde{\tilde{\phi}}_k''\right) + \frac{1}{2}\, r_i r_j \sum_{k=1}^{3} r_k^2\left(\tilde{\tilde{\phi}}_j'' - \tilde{\tilde{\phi}}_k''\right), \end{aligned} \tag{C.17}$$

which can be compared with equations (5.88) and (5.89), and seen to be equivalent to the undeformed equations of motion.

C.2 Compatibility Condition

The compatibility condition for the transformation (5.105) and (5.106) relating the undeformed and γ_i-deformed angular coordinates ϕ_i and $\tilde{\tilde{\phi}}_i$ respectively is

$$2\sum_{l,m=1}^{3}\varepsilon_{ilm}\bar{\gamma}_l r_m \dot{r}_m = \partial_1\left\{\sum_{l,m=1}^{3}\varepsilon_{ilm}r_i^2 r_l^2\bar{\gamma}_m\left(\phi_i'-\phi_l'-\varepsilon_{ilm}\bar{\gamma}_m\right)\right. \\ \left.-\frac{1}{2}\sum_{l,m=1}^{3}\varepsilon_{ilm}r_l^2 r_m^2\left(\bar{\gamma}_l+\bar{\gamma}_m\right)\left(\phi_l'-\phi_m'-\varepsilon_{ilm}\bar{\gamma}_i\right)\right\}, \tag{C.18}$$

whereas the γ_i-deformed equation of motion (5.102) gives

$$2\sum_{l,m=1}^{3}\varepsilon_{ilm}\bar{\gamma}_l r_m \dot{r}_m \\ =\partial_1\left\{\sum_{k,l,m=1}^{3}\varepsilon_{ilm}\bar{\gamma}_l r_m^2 r_k^2\left[\left(\phi_m'+\sum_{n,s=1}^{3}\varepsilon_{mns}\bar{\gamma}_n r_s^2\right)-\left(\phi_k'+\sum_{n,s=1}^{3}\varepsilon_{kns}\bar{\gamma}_n r_s^2\right)\right]\right\}. \tag{C.19}$$

First we shall evaluate

$$\{1\}\equiv\left\{\sum_{l,m=1}^{3}\varepsilon_{ilm}r_i^2 r_l^2\bar{\gamma}_m\left(\phi_i'-\phi_l'-\varepsilon_{ilm}\bar{\gamma}_m\right)-\frac{1}{2}\sum_{l,m=1}^{3}\varepsilon_{ilm}r_l^2 r_m^2(\bar{\gamma}_l+\bar{\gamma}_m)(\phi_l'-\phi_m'-\varepsilon_{ilm}\bar{\gamma}_i)\right\}, \tag{C.20}$$

for $i=1,2$ and 3 as follows:

$$\{1\}_{i=1}=r_1^2 r_2^2\bar{\gamma}_3\left(\phi_1'-\phi_2'-\bar{\gamma}_3\right)+r_1^2 r_3^2\bar{\gamma}_2\left(\phi_3'-\phi_1'-\bar{\gamma}_2\right)-r_2^2 r_3^2\left(\bar{\gamma}_2+\bar{\gamma}_3\right)(\phi_2'-\phi_3'-\bar{\gamma}_1), \tag{C.21}$$

$$\{1\}_{i=2}=r_1^2 r_2^2\bar{\gamma}_3\left(\phi_1'-\phi_2'-\bar{\gamma}_3\right)+r_2^2 r_3^2\bar{\gamma}_1\left(\phi_2'-\phi_3'-\bar{\gamma}_1\right)-r_1^2 r_3^2\left(\bar{\gamma}_1+\bar{\gamma}_3\right)\left(\phi_3'-\phi_1'-\bar{\gamma}_2\right), \tag{C.22}$$

$$\{1\}_{i=3}=r_1^2 r_3^2\bar{\gamma}_2\left(\phi_3'-\phi_1'-\bar{\gamma}_2\right)+r_2^2 r_3^2\bar{\gamma}_1\left(\phi_2'-\phi_3'-\bar{\gamma}_1\right)-r_1^2 r_2^2\left(\bar{\gamma}_1+\bar{\gamma}_2\right)\left(\phi_1'-\phi_2'-\bar{\gamma}_3\right). \tag{C.23}$$

We shall now determine

$$\{2\}\equiv\left\{\sum_{k,l,m=1}^{3}\varepsilon_{ilm}\bar{\gamma}_l r_m^2 r_k^2\left[\left(\phi_m'+\sum_{n,s=1}^{3}\varepsilon_{mns}\bar{\gamma}_n r_s^2\right)-\left(\phi_k'+\sum_{n,s=1}^{3}\varepsilon_{kns}\bar{\gamma}_n r_s^2\right)\right]\right\}, \tag{C.24}$$

for $i = 1, 2$ and 3 as follows:

$$\begin{aligned}
\{2\}_{i=1} &= \bar{\gamma}_2 r_3^2 \left(\phi_3' + \bar{\gamma}_1 r_2^2 - \bar{\gamma}_2 r_1^2\right) - \bar{\gamma}_3 r_2^2 \left(\phi_2' + \bar{\gamma}_3 r_1^2 - \bar{\gamma}_1 r_3^2\right) - \bar{\gamma}_2 r_3^2 \left(r_1^2 \phi_1' + r_2^2 \phi_2' + r_3^2 \phi_3'\right) \\
&\quad + \bar{\gamma}_3 r_2^2 \left(r_1^2 \phi_1' + r_2^2 \phi_2' + r_3^2 \phi_3'\right) \\
&= \bar{\gamma}_2 r_3^2 \left[\left(r_1^2 + r_2^2 + r_3^2\right) \phi_3' + \bar{\gamma}_1 r_2^2 - \bar{\gamma}_2 r_1^2\right] - \bar{\gamma}_3 r_2^2 \left[\left(r_1^2 + r_2^2 + r_3^2\right) \phi_2' + \bar{\gamma}_3 r_1^2 - \bar{\gamma}_1 r_3^2\right] \\
&\quad - \bar{\gamma}_2 r_3^2 \left(r_1^2 \phi_1' + r_2^2 \phi_2' + r_3^2 \phi_3'\right) + \bar{\gamma}_3 r_2^2 \left(r_1^2 \phi_1' + r_2^2 \phi_2' + r_3^2 \phi_3'\right) \\
&= r_1^2 r_2^2 \bar{\gamma}_3 \left(\phi_1' - \phi_2' - \bar{\gamma}_3\right) + r_1^2 r_3^2 \bar{\gamma}_2 \left(\phi_3' - \phi_1' - \bar{\gamma}_2\right) - r_2^2 r_3^2 \left(\bar{\gamma}_2 + \bar{\gamma}_3\right)(\phi_2' - \phi_3' - \bar{\gamma}_1)\,,
\end{aligned} \tag{C.25}$$

$$\begin{aligned}
\{2\}_{i=2} &= \bar{\gamma}_3 r_1^2 \left(\phi_1' + \bar{\gamma}_2 r_3^2 - \bar{\gamma}_3 r_2^2\right) - \bar{\gamma}_1 r_3^2 \left(\phi_3' + \bar{\gamma}_1 r_2^2 - \bar{\gamma}_2 r_1^2\right) - \bar{\gamma}_3 r_1^2 \left(r_1^2 \phi_1' + r_2^2 \phi_2' + r_3^2 \phi_3'\right) \\
&\quad + \bar{\gamma}_1 r_3^2 \left(r_1^2 \phi_1' + r_2^2 \phi_2' + r_3^2 \phi_3'\right) \\
&= \bar{\gamma}_3 r_1^2 \left[\left(r_1^2 + r_2^2 + r_3^2\right) \phi_1' + \bar{\gamma}_2 r_3^2 - \bar{\gamma}_3 r_2^2\right] - \bar{\gamma}_1 r_3^2 \left[\left(r_1^2 + r_2^2 + r_3^2\right) \phi_3' + \bar{\gamma}_1 r_2^2 - \bar{\gamma}_2 r_1^2\right] \\
&\quad - \bar{\gamma}_3 r_1^2 \left(r_1^2 \phi_1' + r_2^2 \phi_2' + r_3^2 \phi_3'\right) + \bar{\gamma}_1 r_3^2 \left(r_1^2 \phi_1' + r_2^2 \phi_2' + r_3^2 \phi_3'\right) \\
&= r_1^2 r_2^2 \bar{\gamma}_3 \left(\phi_1' - \phi_2' - \bar{\gamma}_3\right) + r_2^2 r_3^2 \bar{\gamma}_1 \left(\phi_2' - \phi_3' - \bar{\gamma}_1\right) - r_1^2 r_3^2 \left(\bar{\gamma}_1 + \bar{\gamma}_3\right) \left(\phi_3' - \phi_1' - \bar{\gamma}_2\right),
\end{aligned} \tag{C.26}$$

$$\begin{aligned}
\{2\}_{i=3} &= \bar{\gamma}_1 r_2^2 \left(\phi_2' + \bar{\gamma}_3 r_1^2 - \bar{\gamma}_1 r_3^2\right) - \bar{\gamma}_2 r_1^2 \left(\phi_1' + \bar{\gamma}_2 r_3^2 - \bar{\gamma}_3 r_2^2\right) - \bar{\gamma}_1 r_2^2 \left(r_1^2 \phi_1' + r_2^2 \phi_2' + r_3^2 \phi_3'\right) \\
&\quad + \bar{\gamma}_2 r_1^2 \left(r_1^2 \phi_1' + r_2^2 \phi_2' + r_3^2 \phi_3'\right) \\
&= \bar{\gamma}_1 r_2^2 \left[\left(r_1^2 + r_2^2 + r_3^2\right) \phi_2' + \bar{\gamma}_3 r_1^2 - \bar{\gamma}_1 r_3^2\right] - \bar{\gamma}_2 r_1^2 \left[\left(r_1^2 + r_2^2 + r_3^2\right) \phi_1' + \bar{\gamma}_2 r_3^2 - \bar{\gamma}_3 r_2^2\right] \\
&\quad - \bar{\gamma}_1 r_2^2 \left(r_1^2 \phi_1' + r_2^2 \phi_2' + r_3^2 \phi_3'\right) + \bar{\gamma}_2 r_1^2 \left(r_1^2 \phi_1' + r_2^2 \phi_2' + r_3^2 \phi_3'\right) \\
&= r_1^2 r_3^2 \bar{\gamma}_2 \left(\phi_3' - \phi_1' - \bar{\gamma}_2\right) + r_2^2 r_3^2 \bar{\gamma}_1 \left(\phi_2' - \phi_3' - \bar{\gamma}_1\right) - r_1^2 r_2^2 \left(\bar{\gamma}_1 + \bar{\gamma}_2\right) \left(\phi_1' - \phi_2' - \bar{\gamma}_3\right),
\end{aligned} \tag{C.27}$$

using the constraint $\sum_{i=1}^{3} r_i^2 = 1$. We have thus show that $\{1\} = \{2\}$ and therefore the compatibility condition follows from the γ_i-deformed equations of motion.

C.3 Derivation of the γ_i-deformed Equations of Motion from the γ_i-deformed Lax Pair

The gauged γ_i-deformed Lax pair is given by

$$\tilde{\mathcal{D}}_\alpha^{\gamma_i} = \partial_\alpha - \mathcal{R}_\alpha^{\gamma_i}, \tag{C.28}$$

where

$$\begin{aligned}
(\mathcal{R}_0^{\gamma_i})_{ij} = {} & \frac{3}{2}\left(r_i r_j' - r_i' r_j\right) x + \frac{3i}{2} r_i r_j \left[\left(\phi_i' + \sum_{l,m=1}^{3} \varepsilon_{ilm}\bar{\gamma}_l r_m^2\right) + \left(\phi_j' + \sum_{l,m=1}^{3} \varepsilon_{jlm}\bar{\gamma}_l r_m^2\right)\right] x \\
& - 3i r_i r_j \sum_{k=1}^{3} r_k^2 \left(\phi_k' + \sum_{l,m=1}^{3} \varepsilon_{klm}\gamma_l r_m^2\right) x + \frac{3i}{2}\left(3 r_i r_j - \delta_{ij}\right) x^2 \\
& + i \left\{ \dot{\phi}_i + \sum_{l,m=1}^{3} \varepsilon_{ilm} r_i^2 r_l^2 \bar{\gamma}_m \left(\phi_i' - \phi_l' - \varepsilon_{ilm}\bar{\gamma}_m\right) \right. \\
& \left. \qquad - \frac{1}{2}\sum_{l,m=1}^{3} \varepsilon_{ilm} r_l^2 r_m^2 \left(\bar{\gamma}_l + \bar{\gamma}_m\right)\left(\phi_l' - \phi_m' - \varepsilon_{ilm}\bar{\gamma}_i\right) \right\} \delta_{ij},
\end{aligned} \tag{C.29}$$

$$(\mathcal{R}_1^{\gamma_i})_{ij} = i\left(3 r_i r_j - \delta_{ij}\right) x + i\left(\phi_i' + \sum_{l,m=1}^{3} \varepsilon_{ilm}\bar{\gamma}_l r_m^2\right)\delta_{ij}. \tag{C.30}$$

We must now insist that the zero curvature condition

$$\partial_0 \mathcal{R}_1^{\gamma_i} - \partial_1 \mathcal{R}_0^{\gamma_i} - [\mathcal{R}_0^{\gamma_i}, \mathcal{R}_1^{\gamma_i}] = 0 \tag{C.31}$$

is satisfied. Let us substitute (C.29) and (C.30) into this condition and equate different orders of the spectral parameter x as follows:

$\underline{O(x^0):}$ At zeroth order in the spectral parameter we obtain

$$\begin{aligned}
& i\partial_0\left(\phi_i' + \sum_{l,m=1}^{3} \varepsilon_{ilm}\bar{\gamma}_l r_m^2\right)\delta_{ij} - i\partial_1\left\{\dot{\phi}_i + \sum_{l,m=1}^{3} \varepsilon_{ilm} r_i^2 r_l^2 \bar{\gamma}_m \left(\phi_i' - \phi_l' - \varepsilon_{ilm}\bar{\gamma}_m\right)\right. \\
& \left. -\frac{1}{2}\sum_{l,m=1}^{3} \varepsilon_{ilm} r_l^2 r_m^2 \left(\bar{\gamma}_l + \bar{\gamma}_m\right)\left(\phi_l' - \phi_m' - \varepsilon_{ilm}\bar{\gamma}_i\right)\right\}\delta_{ij} = 0
\end{aligned} \tag{C.32}$$

and therefore

$$\partial_0\left(\sum_{l,m=1}^{3}\varepsilon_{ilm}\bar{\gamma}_l r_m^2\right)=\partial_1\left\{\sum_{l,m=1}^{3}\varepsilon_{ilm}r_i^2 r_l^2\bar{\gamma}_m\left(\phi_i'-\phi_l'-\varepsilon_{ilm}\bar{\gamma}_m\right)\right.$$
$$\left.-\frac{1}{2}\sum_{l,m=1}^{3}\varepsilon_{ilm}r_l^2 r_m^2\left(\bar{\gamma}_l+\bar{\gamma}_m\right)\left(\phi_l'-\phi_m'-\varepsilon_{ilm}\bar{\gamma}_i\right)\right\}. \quad \text{(C.33)}$$

This is just the compatibility condition for the transformation from the undeformed equations of motion to the γ_i-deformed equations of motion.

$\underline{O(x^1):}$ At first order in the spectral parameter we find that

$$3i\left(\dot{r}_i r_j+r_i\dot{r}_j\right)-\frac{3}{2}\left(r_i r_j''-r_i'' r_j\right)$$
$$-\frac{3i}{2}\partial_1\left\{r_i r_j\left[\left(\phi_i'+\sum_{l,m=1}^{3}\varepsilon_{ilm}\bar{\gamma}_l r_m^2\right)+\left(\phi_j'+\sum_{l,m=1}^{3}\varepsilon_{jlm}\bar{\gamma}_l r_m^2\right)\right]\right\}$$
$$+3i\partial_1\left\{r_i r_j\sum_{k=1}^{3}r_k^2\left(\phi_k'+\sum_{l,m=1}^{3}\varepsilon_{klm}\bar{\gamma}_l r_m^2\right)\right\}+3r_i r_j\left(\dot{\phi}_i-\dot{\phi}_j\right)$$
$$+3r_i r_j\left\{\sum_{l,m=1}^{3}\varepsilon_{ilm}r_i^2 r_l^2\bar{\gamma}_m\left(\phi_i'-\phi_l'-\varepsilon_{ilm}\bar{\gamma}_m\right)-\frac{1}{2}\sum_{l,m=1}^{3}\varepsilon_{ilm}r_l^2 r_m^2\left(\bar{\gamma}_l+\bar{\gamma}_m\right)\left(\phi_l'-\phi_m'-\varepsilon_{ilm}\bar{\gamma}_i\right)\right\}$$
$$-3r_i r_j\left\{\sum_{l,m=1}^{3}\varepsilon_{jlm}r_j^2 r_l^2\bar{\gamma}_m\left(\phi_j'-\phi_l'-\varepsilon_{jlm}\bar{\gamma}_m\right)-\frac{1}{2}\sum_{l,m=1}^{3}\varepsilon_{jlm}r_l^2 r_m^2\left(\bar{\gamma}_l+\bar{\gamma}_m\right)\left(\phi_l'-\phi_m'-\varepsilon_{jlm}\bar{\gamma}_j\right)\right\}$$
$$+\frac{3i}{2}\left(r_i r_j'-r_i' r_j\right)\left[\left(\phi_i'+\sum_{l,m=1}^{3}\varepsilon_{ilm}\bar{\gamma}_l r_m^2\right)-\left(\phi_j'+\sum_{l,m=1}^{3}\varepsilon_{jlm}\bar{\gamma}_l r_m^2\right)\right]$$
$$-\frac{3}{2}r_i r_j\left[\left(\phi_i'+\sum_{l,m=1}^{3}\varepsilon_{ilm}\bar{\gamma}_l r_m^2\right)+\left(\phi_j'+\sum_{l,m=1}^{3}\varepsilon_{jlm}\bar{\gamma}_l r_m^2\right)\right]$$
$$\times\left[\left(\phi_i'+\sum_{n,s=1}^{3}\varepsilon_{ins}\bar{\gamma}_n r_s^2\right)-\left(\phi_j'+\sum_{n,s=1}^{3}\varepsilon_{jns}\bar{\gamma}_n r_s^2\right)\right]$$
$$+3r_i r_j\sum_{k=1}^{3}r_k^2\left(\phi_k'+\sum_{l,m=1}^{3}\varepsilon_{klm}\bar{\gamma}_l r_m^2\right)\left[\left(\phi_i'+\sum_{n,s=1}^{3}\varepsilon_{ins}\bar{\gamma}_n r_s^2\right)-\left(\phi_j'+\sum_{n,s=1}^{3}\varepsilon_{jns}\bar{\gamma}_n r_s^2\right)\right]$$
$$=0. \quad \text{(C.34)}$$

Thus, considering the real and imaginary parts of the above expression separately, we obtain

$$\text{Re:}\;\; r_i'' r_j - r_i r_j'' = 2r_i r_j\left(\dot{\phi}_j - \dot{\phi}_i\right)$$

$$-2r_ir_j\left\{\sum_{l,m=1}^{3}\varepsilon_{ilm}r_i^2r_l^2\bar{\gamma}_m\left(\phi_i'-\phi_l'-\varepsilon_{ilm}\bar{\gamma}_m\right)-\frac{1}{2}\sum_{l,m=1}^{3}\varepsilon_{ilm}r_l^2r_m^2\left(\bar{\gamma}_l+\bar{\gamma}_m\right)\left(\phi_l'-\phi_m'-\varepsilon_{ilm}\bar{\gamma}_i\right)\right\}$$

$$+2r_ir_j\left\{\sum_{l,m=1}^{3}\varepsilon_{jlm}r_j^2r_l^2\bar{\gamma}_m\left(\phi_j'-\phi_l'-\varepsilon_{jlm}\bar{\gamma}_m\right)-\frac{1}{2}\sum_{l,m=1}^{3}\varepsilon_{jlm}r_l^2r_m^2(\bar{\gamma}_l+\bar{\gamma}_m)\left(\phi_l'-\phi_m'-\varepsilon_{jlm}\bar{\gamma}_j\right)\right\}$$

$$+r_ir_j\left[\left(\phi_i'+\sum_{l,m=1}^{3}\varepsilon_{ilm}\bar{\gamma}_lr_m^2\right)^2-\left(\phi_j'+\sum_{l,m=1}^{3}\varepsilon_{jlm}\bar{\gamma}_lr_m^2\right)^2\right]$$

$$-2r_ir_j\sum_{k=1}^{3}r_k^2\left(\phi_k'+\sum_{l,m=1}^{3}\varepsilon_{klm}\bar{\gamma}_lr_m^2\right)\left[\left(\phi_i'+\sum_{n,s=1}^{3}\varepsilon_{ins}\bar{\gamma}_nr_s^2\right)-\left(\phi_j'+\sum_{n,s=1}^{3}\varepsilon_{jns}\bar{\gamma}_nr_s^2\right)\right], \quad \text{(C.35)}$$

$$\text{Im:}\;\; \dot{r}_ir_j+r_i\dot{r}_j=\frac{1}{2}\partial_1\left\{r_ir_j\left[\left(\phi_i'+\sum_{l,m=1}^{3}\varepsilon_{ilm}\bar{\gamma}_lr_m^2\right)+\left(\phi_j'+\sum_{l,m=1}^{3}\varepsilon_{jlm}\bar{\gamma}_lr_m^2\right)\right]\right\}$$

$$-\partial_1\left\{r_ir_j\sum_{k=1}^{3}r_k^2\left(\phi_k'+\sum_{l,m=1}^{3}\varepsilon_{klm}\bar{\gamma}_lr_m^2\right)\right\}$$

$$-\frac{1}{2}\left(r_ir_j'-r_i'r_j\right)\left[\left(\phi_i'+\sum_{l,m=1}^{3}\varepsilon_{ilm}\bar{\gamma}_lr_m^2\right)-\left(\phi_j'+\sum_{l,m=1}^{3}\varepsilon_{jlm}\bar{\gamma}_lr_m^2\right)\right]. \quad \text{(C.36)}$$

Now the real equation (C.35) is equivalent to

$$r_jr_i''-r_ir_j''=2r_ir_j\left(\dot{\phi}_j-\dot{\phi}_i\right)$$

$$-2r_ir_j\left\{\sum_{l,m=1}^{3}\varepsilon_{ilm}r_i^2r_l^2\bar{\gamma}_m\left(\phi_i'-\phi_l'-\varepsilon_{ilm}\bar{\gamma}_m\right)-\frac{1}{2}\sum_{l,m=1}^{3}\varepsilon_{ilm}r_l^2r_m^2\left(\bar{\gamma}_l+\bar{\gamma}_m\right)\left(\phi_l'-\phi_m'-\varepsilon_{ilm}\bar{\gamma}_i\right)\right\}$$

$$+2r_ir_j\left\{\sum_{l,m=1}^{3}\varepsilon_{jlm}r_j^2r_l^2\bar{\gamma}_m\left(\phi_j'-\phi_l'-\varepsilon_{jlm}\bar{\gamma}_m\right)-\frac{1}{2}\sum_{l,m=1}^{3}\varepsilon_{jlm}r_l^2r_m^2\left(\bar{\gamma}_l+\bar{\gamma}_m\right)\left(\phi_l'-\phi_m'-\varepsilon_{jlm}\bar{\gamma}_j\right)\right\}$$

$$+r_ir_j\sum_{k=1}^{3}r_k^2\left[\left(\phi_i'+\sum_{l,m=1}^{3}\varepsilon_{ilm}\bar{\gamma}_lr_m^2\right)-\left(\phi_k'+\sum_{l,m=1}^{3}\varepsilon_{klm}\bar{\gamma}_lr_m^2\right)\right]^2$$

$$-r_ir_j\sum_{k=1}^{3}r_k^2\left[\left(\phi_j'+\sum_{l,m=1}^{3}\varepsilon_{jlm}\bar{\gamma}_lr_m^2\right)-\left(\phi_k'+\sum_{l,m=1}^{3}\varepsilon_{klm}\bar{\gamma}_lr_m^2\right)\right]^2, \quad \text{(C.37)}$$

which can be seen by multiplying out the last two squared terms and making use of the constraint $\sum_{i=1}^{3} r_i^2 = 1$. Furthermore, the imaginary equation (C.36) can be written as

$$\begin{aligned}
\dot{r}_i r_j + r_i \dot{r}_j = &\frac{1}{2}\left(r_i r_j' + r_i' r_j\right)\left[\left(\phi_i' + \sum_{l,m=1}^{3} \varepsilon_{ilm}\bar{\gamma}_l r_m^2\right) + \left(\phi_j' + \sum_{l,m=1}^{3} \varepsilon_{jlm}\bar{\gamma}_l r_m^2\right)\right] \\
&+ \frac{1}{2} r_i r_j \left[\left(\phi_i'' + 2\sum_{l,m=1}^{3} \varepsilon_{ilm}\bar{\gamma}_l r_m r_m'\right) + \left(\phi_j'' + 2\sum_{l,m=1}^{3} \varepsilon_{jlm}\bar{\gamma}_l r_m r_m'\right)\right] \\
&- \left(r_i r_j' + r_i' r_j\right)\sum_{k=1}^{3} r_k^2 \left(\phi_k' + \sum_{l,m=1}^{3} \varepsilon_{klm}\bar{\gamma}_l r_m^2\right) \\
&- 2 r_i r_j \sum_{k=1}^{3} r_k r_k' \left(\phi_k' + \sum_{l,m=1}^{3} \varepsilon_{klm}\bar{\gamma}_l r_m^2\right) - r_i r_j \sum_{k=1}^{3} r_k^2 \left(\phi_k'' + 2\sum_{l,m=1}^{3} \varepsilon_{klm}\bar{\gamma}_l r_m r_m'\right) \\
&- \frac{1}{2}\left(r_i r_j' - r_i' r_j\right)\left[\left(\phi_i' + \sum_{l,m=1}^{3} \varepsilon_{ilm}\bar{\gamma}_l r_m^2\right) - \left(\phi_j' + \sum_{l,m=1}^{3} \varepsilon_{jlm}\bar{\gamma}_l r_m^2\right)\right]
\end{aligned} \tag{C.38}$$

$$\begin{aligned}
&= r_i' r_j \left(\phi_i' + \sum_{l,m=1}^{3} \varepsilon_{ilm}\bar{\gamma}_l r_m^2\right) + r_i r_j' \left(\phi_j' + \sum_{l,m=1}^{3} \varepsilon_{jlm}\bar{\gamma}_l r_m^2\right) \\
&- 2 r_i r_j \sum_{k=1}^{3} r_k r_k' \left(\phi_k' + \sum_{l,m=1}^{3} \varepsilon_{klm}\bar{\gamma}_l r_m^2\right) - r_i' r_j \sum_{k=1}^{3} r_k^2 \left(\phi_k' + \sum_{l,m=1}^{3} \varepsilon_{klm}\bar{\gamma}_l r_m^2\right) \\
&- r_i r_j' \sum_{k=1}^{3} r_k^2 \left(\phi_k' + \sum_{l,m=1}^{3} \varepsilon_{klm}\bar{\gamma}_l r_m^2\right) + \frac{1}{2} r_i r_j \left(\phi_i'' + 2\sum_{l,m=1}^{3} \varepsilon_{ilm}\bar{\gamma}_l r_m r_m'\right) \\
&+ \frac{1}{2} r_i r_j \left(\phi_j'' + 2\sum_{l,m=1}^{3} \varepsilon_{jlm}\bar{\gamma}_l r_m r_m'\right) - r_i r_j \sum_{k=1}^{3} r_k^2 \left(\phi_k'' + 2\sum_{l,m=1}^{3} \varepsilon_{klm}\bar{\gamma}_l r_m r_m'\right),
\end{aligned} \tag{C.39}$$

which implies, if one uses the constraint $\sum_{i=1}^{3} r_i^2 = 1$ and thus $\sum_{i=1}^{3} r_i r_i' = 0$, that

$$\begin{aligned}
\dot{r}_i r_j + r_i \dot{r}_j = &\, r_j \sum_{k=1}^{3} r_k \left(r_i r_k\right)' \left[\left(\phi_i' + \sum_{l,m=1}^{3} \varepsilon_{ilm}\bar{\gamma}_l r_m^2\right) - \left(\phi_k' + \sum_{l,m=1}^{3} \varepsilon_{klm}\bar{\gamma}_l r_m^2\right)\right] \\
&+ r_i \sum_{k=1}^{3} r_k \left(r_j r_k\right)' \left[\left(\phi_j' + \sum_{l,m=1}^{3} \varepsilon_{jlm}\bar{\gamma}_l r_m^2\right) - \left(\phi_k' + \sum_{l,m=1}^{3} \varepsilon_{klm}\bar{\gamma}_l r_m^2\right)\right] \\
&+ \frac{1}{2} r_i r_j \sum_{k=1}^{3} r_k^2 \left[\left(\phi_i'' + 2\sum_{l,m=1}^{3} \varepsilon_{ilm}\bar{\gamma}_l r_m r_m'\right) - \left(\phi_k'' + 2\sum_{l,m=1}^{3} \varepsilon_{klm}\bar{\gamma}_l r_m r_m'\right)\right] \\
&+ \frac{1}{2} r_i r_j \sum_{k=1}^{3} r_k^2 \left[\left(\phi_j'' + 2\sum_{l,m=1}^{3} \varepsilon_{jlm}\bar{\gamma}_l r_m r_m'\right) - \left(\phi_k'' + 2\sum_{l,m=1}^{3} \varepsilon_{klm}\bar{\gamma}_l r_m r_m'\right)\right].
\end{aligned} \tag{C.40}$$

Finally, we notice that equations (C.37) and (C.40) are the same as (5.103) and (5.104), and are thus equivalent to the γ_i-deformed equations of motion.

$\underline{O(x^2):}$ At second order in the spectral parameter, one obtains an equation which is trivially satisfied, again using the constraint $\sum_{i=1}^{3} r_i^2 = 1$ and hence that $\sum_{i=1}^{3} r_i r_i' = 0$.

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