Figure 3.6 - Stress distribution over the outer rim surface (Rope tension = 250 kN)
The particular material parameters required as input by this model and the values selected are listed below:

- Young's Modulus - 96 GPa
- Poisson's Ratio - 0.3
- Yield stress - 180 MPa
- Linear strain hardening modulus - 70 GPa

3.4.3 Finite Element Results

In the finite element simulation a uniform compressive load, equivalent to that generated by a 250 KN tensile rope load, was applied in a single step. The load acted on nodes 1-11. Equilibrium iteration was employed to track the specified nonlinear behaviour.

The results of the analysis confirm that the maximum stress occurs at the centre of the arc on the line of symmetry and at the junction of the spoke to the rim. The stress distribution predicted by the analysis over the inner and outer rim surface, is illustrated in Figs 3.5 and 3.6 respectively.

The maximum tensile and compressive stresses achieved at the centre of the arc were 337 MPa and -520 MPa, whilst at the junction of the spoke to the rim, stresses of 548 MPa and -592 MPa were predicted. In both cases the tensile stress exceeds the ultimate tensile stress of the material, 220 MPa, as determined experimentally. Since compressive failure was not achieved in the compression tests (loading was discontinued when barreling became noticeable), it is likely that rim failure will be initiated by tensile cracks and hence the tensile stress will determine failure.

3.4.4 Conclusions

Although this analysis was simplistic it indicated to some extent that the reduced rim section proposed, would result in failure before reaching the maximum load which the rig could withstand.

1. The loading frame was designed to operate up to a rope tension of 250 KN, with a safety factor of -2, [17].
The major limitations of this analysis are listed below:

- the rim was approximated grossly by a rectangular section
- the radii at the junction of the spoke to the rim were neglected
- the nonlinear material properties of the cast iron were inadequately represented.

3.5 BASIS FOR A MORE DETAILED STUDY

Ideally an accurate finite element simulation of the sheave would involve a three-dimensional analysis incorporating the nonlinear material behaviour. Limitations on computer resources prevent such an approach, however a good approximation may be achieved by modelling the sheave adequately. The approach adopted was to model the rim of the sheave with two-dimensional plane stress elements of varying thickness in areas of high stress and with beam elements in areas where the stresses were likely to be low.

A number of nonlinear material models are available in ADINA for two-dimensional solid elements. Although material model 7 (Drucker Prager yield criterion with hardening cap and tension cut off) is capable of modelling materials exhibiting higher strength in compression than tension, it predicts volumetric plastic dilatency, which is not common in metal plasticity and which was considered original; as inapplicable to cast iron.

Accordingly, it was decided to utilise the 'user supplied' material model option for developing a nonlinear material law which would account for the cast iron properties. The law adopted was similarly based on the Drucker Prager yield criterion, however initially it did not predict volumetric plastic dilatency.

The formulation of the nonlinear material law and the subsequent development of a material model subroutine based on this law, is discussed in Chapter 4.
CHAPTER 4

PLASTICITY

4.1 INTRODUCTION

The basic laws governing the elastoplastic behaviour of the cast iron material used in the manufacture of the sheave must be derived before any numerical aspects of a finite element analysis can be considered. Using existing classical hypotheses it is possible to derive a two-dimensional stress-strain relationship from experimental data acquired from material tests. In order to formulate such a relationship it is necessary to postulate the existence of a yield criterion which describes the stress level at which plastic flow commences. Ideally biaxial tests are required to determine such a criterion experimentally. The facilities for performing these tests were not available hence a yield criterion had to be selected on the grounds of the uniaxial test results.

The uniaxial behaviour of the cast iron was studied by testing specimens machined from a similar sheave manufactured from the same cast (see Appendix 1). The results of these tests confirmed that the yield strength of the cast iron is higher in compression than it is in tension. This property is characteristic of a frictional material conforming to the Mohr-Coulomb failure criterion or the Drucker-Prager yield criterion. The latter is computationally easier to implement and it was thus postulated that it represents the yield criterion of cast iron.

In order to derive the relationship between the increments of plastic strain and stress a further assumption regarding the material behaviour must be made. Classically it is assumed that the plastic
strain component is normal to the yield surface. This is termed the normality condition which leads to an associated theory of plasticity. When applied to the Drucker Prager criterion this theory predicts material dilatancy during plastic deformation.

Initially the cast iron material was assumed to be plastically incompressible. To implement this assumption a von Mises plastic potential was defined leading to a non associated theory of plasticity. Additional experimental tests were performed on both tensile and compressive specimens to verify the above assumption. The specimens tested were machined from lightly loaded portions of the actual sheave after the destructive test. This ensured that the uniaxial test results would represent the virgin properties of the material as closely as possible. These results revealed that the cast iron dilates substantially during plastic deformation hence the initial assumption proved to be invalid. Subsequently the associated constitutive law which accounts for plastic dilatancy was formulated.

Both the associated and non associated constitutive relationships based on the Drucker Prager yield criterion are discussed and developed in subsequent sections.

4.2 THE MATHEMATICAL THEORY OF PLASTICITY.\(^1\) [13]

'The object of the mathematical theory of plasticity is to provide a theoretical description of the relationship between stress and strain for a material which exhibits an elastic plastic response. In essence this relationship is characterized by an irreversible

---

\(^1\) It is the author's opinion that the mathematical theory of plasticity has been concisely expressed in the literature by various researchers. For this reason the author has chosen to quote directly from other literature sources where a more concise explanation could not be produced without loss of meaning.
straining which is independent of time and which can only be sustained once a certain level of stress has been reached. In order to formulate a theory which models elasto plastic material deformation three requirements have to be met.

- An explicit constitutive relationship must be formulated to describe material behaviour under elastic conditions.
- A yield criterion must be postulated to indicate the stress level at which plastic flow commences.
- A relationship describing both elastic and plastic behaviour must be developed for post yield conditions.

The strain is linearly related to stress in the elastic region by Hooke's Law. This results in a unique relationship between stress and strain which is independent of the stress level. In the plastic region however the relationship is nonlinear. Furthermore the relationship is non unique [11] since the total plastic strains are path dependent. Before presenting the theoretical derivation of the elastoplastic relationship the concept of a yield surface is examined.

4.2.1 The Yield Criterion

It is postulated that yielding can only occur if the state of stress satisfies the yield criterion [13]

\[ f(\sigma) = \kappa(k) \]  \hspace{1cm} (4.1)

where \( f \) is a function of the stress \( \sigma \) and \( \kappa \) is a material parameter determined experimentally. This parameter may be a function of a hardening parameter \( k \) (refer to Section 4.2.6), which governs the expansion of the yield surface in the principal stress space. Since the yield criterion is invariant in this space it must consist of the three stress invariants, [11,13]
Thus the yield criterion can be visualised as a surface in the n-dimensional stress space with the position of the surface dependent on the instantaneous value of $k$.

As expressed by eqn (4.1), $f(\sigma) = k$ represents plastic behaviour, whilst $f(\sigma) < k$ characterises elastic behaviour. By evaluating the incremental change in the yield function $f$, due to an incremental stress change, it is possible to determine the type of loading occurring, i.e.

$$df = \frac{\partial f}{\partial \sigma} \delta \sigma$$  \hspace{1cm} (4.3)$$

if: $df < 0$ elastic unloading occurs and the stress point returns inside the yield surface

$df = 0$ neutral loading occurs and the stress point remains on the stationary yield surface (plastic behaviour for a perfectly plastic material)

$df > 0$ plastic loading and the stress point remains on the expanding yield surface (plastic behaviour for a hardening material).

1. Einstein's summation convention is invoked, whereby it is implicitly assumed that a summation from 1 to 3 is performed over any index which is repeated in any term of an expression. Indices 1, 2, 3 refer to cartesian coordinates $x, y, z$ respectively, i.e. $\sigma_{11} = \sigma_{xx} = c_{xx}^1$. 
Thus the yield criterion can be visualised as a surface in the n-dimensional stress space with the position of the surface dependent on the instantaneous value of $\kappa$.

As expressed by eqn (4.1), $f(\sigma) = \kappa$ represents plastic behaviour, whilst $f(\sigma) < \kappa$ characterises elastic behaviour. By evaluating the incremental change in the yield function $f$, due to an incremental stress change, it is possible to determine the type of loading occurring, ie [13]

\[
\frac{df}{d\sigma} = \frac{\partial f}{\partial \sigma} \quad (4.3)
\]

if: $df < 0$ elastic unloading occurs and the stress point returns inside the yield surface

$df = 0$ neutral loading occurs and the stress point remains on the stationary yield surface (plastic behaviour for a perfectly plastic material)

$df > 0$ plastic loading and the stress point remains on the expanding yield surface (plastic behaviour for a hardening material).

---

1. Einstein's summation convention is invoked, whereby it is implicitly assumed that a summation from 1 to 3 is performed over any index which is repeated in any term of an expression. Indices 1, 2, 3 refer to cartesian coordinates $x, y, z$ respectively, ie $\sigma_{ii} = \sigma_{xx} = \sigma_x$
4.2.2 The Drucker-Prager Yield Criterion

The Drucker-Prager yield criterion is extensively employed in the field of soil mechanics as an approximation to the Mohr Coulomb failure criterion. The yield surface is a right circular cone inclined along the mean stress axis in the principal stress space. For this case the yield criterion expressed in eqn (4.1) is rewritten as

\[ F(\sigma, k) = f(\sigma) - \kappa(k) = 0 \]  \(4.4\)

where \( F(\sigma, k) \) now represents a function of both the stress state and the hardening parameter \( k \). The Drucker-Prager criterion is expressed as

\[ F(\sigma, k) = \alpha \sqrt{J_1} + \sqrt[2]{\frac{J_1}{2}} - K \]  \(4.5\)

where \( \alpha, K \) are material parameters determined experimentally.

In order to ensure that the Drucker-Prager criterion coincides with the outer apices of the Mohr Coulomb hexagon at any section along the mean stress axis, it can be shown that \(13\)

\[ \alpha = \frac{3 \sin \phi}{\sqrt{3(3 - \sin \phi)}} \]  \(4.6\)

\[ \gamma = \frac{6 \cos \phi}{\sqrt{3(3 - \sin \phi)}} \]

where \( c \) and \( \phi \) represent the cohesion and the angle of internal friction of the material respectively. Furthermore both \( c \) and \( \phi \) may be functions of the hardening parameter \( k \), i.e. \( c(k), \phi(k) \).

A geometrical representation of both the Mohr Coulomb and Drucker-Prager yield surfaces in the principal stress space is illustrated in Fig 4.1.
4.2.3 The Definition of Work Hardening [10]

A precise definition of work hardening was first proposed by Drucker.

Drucker postulated that for a work hardening material at a given state of stress; if an external agency slowly applied additional stresses and then slowly removed them, the material remained in equilibrium and:

1) 'Positive work is done by the external agency during the application of the added set of stresses.'

2) 'The net work performed by the external agency over the cycle of application and removal is positive if plastic deformation occurred in the cycle. The net work is zero if and only if elastic changes in strain alone are produced.'

These conditions may be stated in mathematical terms as

1) \[ \delta W = (\sigma_{ij} - \sigma^p_{ij}) \varepsilon_{ij}^P + \delta \sigma_{ij} \varepsilon_{ij}^P > 0 \] (4.7)

2) \[ \delta W = \delta \sigma_{ij} \varepsilon_{ij}^P > 0 \] (4.8)
where $\sigma_{ij}$ represents a state of stress lying on the yield surface

$\sigma^*_{ij}$ represents a state of stress lying on or inside the yield surface

$\mathrm{d}\sigma_{ij}$ represents an increment of stress which produces a small increment of plastic strain $\mathrm{d}e^p_{ij}$

It has been shown by Drucker that both of these conditions are satisfied only if

(i) the yield surface is convex

(ii) the plastic strain increment is normal to the yield surface.

4.2.4 The Elasto Plastic Stress Strain relations

After initial yielding the material behaviour will be partly elastic and partly plastic. The changes of strain due to an increment of stress $\mathrm{d}\sigma_{ij}$ exceeding the yield limit are assumed to be decomposable into elastic and plastic components

$$\mathrm{d}e_{ij} = \mathrm{d}e^e_{ij} + \mathrm{d}e^p_{ij} \quad (4.9)$$

where the elastic increment of strain is related to the stress increment by

$$\mathrm{d}e^e_{ij} = D_{ijkl}\mathrm{d}\varepsilon_{kl} \quad (4.10)$$

where $D_{ijkl}$ represents the elastic constitutive matrix.

The plastic increment of strain is assumed to be proportional to the gradient of a quantity termed the plastic potential, $Q$, i.e

$$\mathrm{d}e^p_{ij} = \lambda \frac{\partial Q}{\partial \sigma_{ij}} \quad (4.11)$$
where \( dA \) represents a positive proportionality constant; the plastic multiplier. This relationship determines the flow rule of the material since it governs plastic flow after yielding has occurred. The relation \( Q = F \) has special significance as it implies normality of the plastic strain increment to the yield surface and results in certain uniqueness and variational theorems, [1,3,11,13].

A further assumption that is implied in this derivation is that the increments of stress and plastic strain are linearly related.

\[
d\sigma_{ij} = D_{ijkl}^p \, d\varepsilon_{kl}^p
\]  

(4.12)

where \( D_{ijkl}^p \) represents the plastic constitutive law governing the relation between the increment of stress and plastic strain. Thus it is assumed that \( D_{ijkl}^p \) is dependent only on the current stress and total strain levels but not on their increments. In the special case of a hardening material where the yield function is only dependent on the current stress level and total plastic strain no assumption is necessary, [10]. That is when

\[
F(\varepsilon,\varepsilon^p) = f(\sigma) - \kappa(\varepsilon^P)
\]

differentiating gives

\[
dF = 0 = \frac{\partial F}{\partial \sigma_{ij}} \, d\sigma_{ij} + \frac{\partial F}{\partial \varepsilon_{ij}^p} \, d\varepsilon_{ij}^p
\]

In this case \( \partial F/\partial \sigma_{ij} \) and \( \partial F/\partial \varepsilon_{ij}^p \) are independent of \( d\sigma_{ij} \), \( d\varepsilon_{ij}^p \). A linear relationship between the increment of stress \( d\sigma_{ij} \) and the

1. This is the requirement to ensure that the stress point remains on the expanding yield surface.
Increment of plastic strain clearly exists, [10].

**Matrix Formulation of the Stress Strain Relationship**

The theoretical expressions are now converted into matrix form. In the following section vectors and matrices are denoted by \{ \} and \[ \], respectively.

As stated in equation (4.9)

\[ \frac{de^p}{d\tau} = \theta \] \hspace{1cm} (4.13)

From

\[ \dot{e}^p = \int_{\partial S} \frac{d\sigma}{d\tau} \] \hspace{1cm} (4.9)

Then

\[ \dot{\sigma} = [0] \cdot \dot{\sigma} + dx \left( \frac{\partial \sigma}{\partial \sigma} \right) \] \hspace{1cm} (4.14)

When plastic yield occurs the stresses remain on the expanding yield surface, i.e.

\[ \dot{\sigma}^p = \dot{\sigma} \] \hspace{1cm} (4.15)

and

\[ \dot{\sigma}^p \cdot \frac{d\sigma}{d\tau} + \frac{d\sigma}{d\tau} \cdot \frac{d\sigma}{d\tau} = 0 \] \hspace{1cm} (4.15)

where

\[ \dot{\sigma}^p \cdot \frac{d\sigma}{d\tau} + \frac{d\sigma}{d\tau} = 0 \] \hspace{1cm} (4.16)

where

\[ \dot{\sigma}^p = \frac{\partial \sigma}{\partial \sigma} \cdot \frac{d\sigma}{d\tau} \] \hspace{1cm} (4.17)

Equation (4.14) and (4.16) can be written in matrix form as, [1]
The constant $d\lambda$ is eliminated resulting in an explicit expansion which determines the stress increments in terms of the imposed strain increments.

\[
\{d\sigma\} = [D_{ep}]{de}
\]  \hspace{1cm} (4.18)

where

\[
[D_{ep}] = [D] - \frac{[D]\left(\frac{\partial \mathbf{F}}{\partial \sigma}\right)^T[D]}{\lambda + \left(\frac{\partial \mathbf{F}}{\partial \sigma}\right)^T[D]\left(\frac{\partial \mathbf{F}}{\partial \sigma}\right)}
\]  \hspace{1cm} (4.19)

This matrix is symmetric only when plasticity is associated, that is, when $Q \equiv F$.

By re-examining the second condition stated when defining a work hardening material, a mathematical explanation of this restriction to symmetry is obtained, i.e.

\[
\delta W = d\varepsilon_{ij} d\varepsilon_{ij}^P > 0
\]

\[
= 0 \text{ only if } d\varepsilon_{ij}^P = 0
\]

This may be stated in vector form as

\[
{d\varepsilon^P}^T{d\sigma} \leq 0
\]  \hspace{1cm} (4.20)

From eqn (4.13)

\[
{d\varepsilon^P} = {d\varepsilon} - {d\varepsilon^e}
\]
Equation (4.20) may be restated as

\[
\{d\varepsilon\}^T \{d\sigma\} \geq \{d\varepsilon^e\}^T \{d\sigma\} \tag{4.21}
\]

substituting eqn (4.10), eqn (4.18) into eqn (4.20)

\[
\{d\varepsilon\}^T \{D_{ep}\} \{d\varepsilon\} \geq \{d\varepsilon^e\}^T \{0\} \{d\varepsilon^e\} \tag{4.22}
\]

Since the right hand side of eqn (4.22) is always real and greater than zero, unless \( \{d\varepsilon^e\} \equiv 0 \)

\[
\{d\varepsilon\}^T \{D_{ep}\} \{d\varepsilon\} > 0 \tag{4.25}
\]

\[
= 0 \text{ if and only if } \{d\varepsilon\} \equiv 0
\]

For this quantity to be real and greater than zero for any \( \{d\varepsilon\} \), the elastoplastic matrix must be Hermite and hence symmetric and positive definite. Hence by deduction a nonsymmetric elastoplastic matrix implies that inequality (4.20) may be violated for certain strain paths.

This concept is illustrated in Fig 4.2 where a non associated plastic potential defines the direction of the plastic strain increment.

\[
\{d\sigma\}^T \{d\varepsilon^P\} < 0
\]

Figure 4.2 - A Graphical Representation of a Non Associated Plastic Potential Surface
The foregoing discussion illustrates that if a non associated plastic potential is employed during a cycle of loading it is possible to recover energy which would otherwise be lost. Although this is unacceptable from a thermodynamic consideration the non associated formulation is still employed in plasticity.

4.2.6 The Hardening Parameter \( k \)

A material may be idealised as behaving perfectly plastically when once the yield limit has been reached the material deforms plastically at constant stress. Alternatively it may exhibit a rising stress strain curve in the plastic region. In the latter case the material behaviour is termed work- or strain-hardening.

![Perfectly plastic material](a) ![Hardening material](b)

**Figure 4.3 - Idealised Material Behaviour**

The degree of hardening is calculated in one of two ways. Firstly it may be postulated to be a function of the total plastic work \( W_p \) only. Hence the hardening parameter \( k \) is interpreted as

\[
k = W_p
\]

\[k = \left\{ \sigma \right\}^T d\varepsilon^P\]  

(4.24)

This postulate is termed work hardening.
Secondly, $k$ may be postulated to represent the total plastic deformation incurred during the loading history. The total plastic deformation is related to the effective or generalised plastic strain which is given by: [13]

$$ k = \varepsilon_p $$

(4.25)

$$ \varepsilon_p = \sqrt{\frac{2}{3}(d\varepsilon^P)^T(d\varepsilon^P)} $$

For the purpose of further discussion, $k$ will be calculated according to the work hardening postulate.

The significance of parameter $k$ is that it provides a means of monitoring the extent of plastic deformation as the yield surface progressively expands. In essence, since $k$ is an experimentally determined function dependent on the plastic properties of the material, it directly relates the theoretical behaviour of the model to experimental observations. This concept is implemented by equating the theoretical increment of work performed during an incremental expansion of the yield surface with the incremental change in $k$. Since the theoretical work performed is a function of the increment of plastic strain,

$$(d\varepsilon^P) = d\lambda \frac{2\varepsilon^e}{J_2},$$

the plastic multiplier $d\lambda$ may be evaluated. This approach is implied in eqn (4.16). In this context parameter $A$ is interpreted as a weighting factor ensuring compliance of the model with the work hardening properties of the material.

4.3 THE DRUCKER-PRAGER MATERIAL MODEL

The elasto-plastic stress strain relations for a material conforming to the Drucker-Prager yield criterion were developed for both an
associated and a non-associated plastic potential surface. Both models presented were based on the assumption of plane stress and have been implemented successfully in a computer code compatible with the 1981 version of the ADINA programme.

The Drucker-Prager yield criterion as stated in eqn (4.5) is repeated below

\[
F(\sigma, k) = \alpha(k) J_1 + \frac{1}{2} \sigma_2^2 - K(k) = 0
\]  

(4.26)

This is a more general form where \( \alpha \) and \( K \) may be dependent in some way on the hardening properties of the surface. It is possible to show that if the parameter \( \alpha \) maintains a constant value the yield surface expands isotropically as illustrated in Fig 4.4a.

On the other hand if \( \alpha \) is a function of \( k \), then the yield surface expands in an isotropic-kinematic fashion (Fig 4.4b).

![Diagrams](image)

\[
F(\sigma) = \alpha J_1 + \frac{1}{2} \sigma_2^2 - K(k) = 0
\]

(a) Isotropic expansion

\[
F(\sigma) = \alpha(k) J_1 + \frac{1}{2} \sigma_2^2 - K(k) = 0
\]

(b) Isotropic-kinematic expansion

Figure 4.4 - Alternative Modes of Expansion of the Yield Locus in Plane Stress

1. By isotropic it is meant that the yield surface expands gradually and uniformly [22] in all directions as straining proceeds.

2. By kinematic it is meant that the yield surface does not expand but translates as a rigid body. In this context isotropic-kinematic behaviour implies that the yield surface translates along the mean stress axis and simultaneously expands from the initial yield surface to the failure surface [22].
The final computer code incorporates both cases. The more general case is required if the material properties obtained from experiment indicate that the initial yield surface (defined from the values of the yield limit in tension and compression) has a different cone angle to that of the final failure surface (defined by the values of the rupture stress in tension and compression). This case is illustrated in Fig 4.5 below.

Figure 4.5

4.3.1 The Elasto-plastic Matrix

The elasto plastic matrix given by eqn (4.19) defines the elasto plastic response of the material

\[
[D_{ep}] = [D] - [D]\left(\frac{\partial \sigma}{\partial e}\right)^T [D]\left[\lambda + \left(\frac{\partial F}{\partial e}\right)^T [D] \left(\frac{\partial Q}{\partial e}\right)^{-1}\right]^{-1}
\]
where \( F(\sigma, k) = a(k)J_1 + J_2^{1/2} - K(k) \) \( (4.27) \)

\( Q \) represents the plastic potential surface

\( A \) as defined in eqn 4.17

\[
[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}
\]

\( E \) represents the elastic modulus of the material

\( \nu \) represents Poisson's ratio

The explicit form of the elastoplastic matrix is obtained from an evaluation of the plastic potential and yield function gradients in the principal stress space. Since the model of the sheave, as discussed in the latter part of Chapter 3, is based on the assumption of plane stress three components of stress and four components of strain are considered. An explicit form of the elastoplastic matrix, for both a non associated (von Mises) and associated plastic potential surface, is presented in Appendix 2.1.

4.3.2 Parameter A

Parameter A has been introduced previously where it was stated that its function is to ensure compliance of the theoretical model with the work hardening properties of the material. The formulation of parameter A is presented in Appendix 2.4 where it is shown that its evaluation depends on the determination of the tangent modulus \( E_T \) for each subsequent yield surface.

In order to evaluate the tangent modulus and hence parameter A, which varies continuously as straining proceeds, the stress state was related to a single parameter; The ratio \( R^1 \), which defines the current yield surface as illustrated in Fig 4.6. The initial yield surface is defined by \( R = 0 \), whilst the final surface at fracture.

---

1. The derivation of this ratio is presented in Appendix A2.5.
Figure 4.6 - The Dimensionless Ratio $R$

attains a value of $R = 1$. In the interval $0 < R < 1$ the uniaxial stress is defined by

$$\sigma = f(R).$$

This made it possible to fit the tangent modulus (determined from uniaxial tension and compression experiments) at any post yield stress state to a single function, $E_T = f(R)$.

To implement the technique outlined above a certain amount of manipulation was required. Essentially the tension part of the curve must be geometrically similar to the compression part as illustrated in Fig 4.6.

4.4 IMPLEMENTATION OF THE DRUCKER-PRAGER CONSTITUTIVE RELATIONS

The ADINA finite element package has the facility to incorporate a user supplied material model applicable to two-dimensional solid elements. This facility enables the user to model the nonlinear properties of materials which cannot be adequately represented by
Figure 4.6 - The Dimensionless Ratio R attains a value of $R = 1$. In the interval $0 < R < 1$ the uniaxial stress is defined by

$$
s = f(R) .
$$

This made it possible to fit the tangent modulus (determined from uniaxial tension and compression experiments) at any post yield stress state to a single function, $E_T = f(R)$.

To implement the technique outlined above a certain amount of manipulation was required. Essentially the tension part of the curve must be geometrically similar to the compression part as illustrated in Fig. 4.6.

4.4 IMPLEMENTATION OF THE DRUCKER-PRAGER CONSTITUTIVE RELATIONS

The ADINA finite element package has the facility to incorporate a user supplied material model applicable to two-dimensional solid elements. This facility enables the user to model the nonlinear properties of materials which cannot be adequately represented by
the standard models available in the ADINA two-dimensional solid element library.

An insight into the function of the material model may be obtained by reviewing the options available in the package for the solution of elastoplastic problems. Three techniques are employed which are briefly outlined in the following section.

4.4.1 Procedures Employed in the Solution of Elastoplastic Problems

The finite element method applied to elasticity problems results in a set of linear simultaneous equations of the form

\[ [K] \{a\} + \{f\} = 0 \]  \hspace{1cm} (4.28)

In the linear region these equations can be solved directly to obtain an approximation to the unknown nodal displacements \( \{a\} \).

In the post yield region however the stress strain relationship is nonlinear and hence the stiffness of the material varies continuously as the applied loads increase. This results in the stiffness matrix being dependent in some way on the displacements \( \{a\} \), and their derivatives. The governing equations become nonlinear and are generally of the form

\[ \psi(a) = [K(a)]\{a\} + \{f\} \]  \hspace{1cm} (4.29)

where \( \psi(a) \) represents the residual load vector.

The equations are theoretically satisfied once the residual load vector \( \psi(a) = 0 \). Owing to the method of formulating the stiffness matrix it is not possible to determine explicitly the dependence of the stiffness coefficient matrix on the nodal displacements, ie [K(a)] and hence an iterative procedure is employed. In essence these procedures linearise and solve the governing equations (eqn 4.29) repetitively by maintaining a constant stiffness.
coefficient matrix over each iterative interval, until the residual load vector \( \psi(a) \) is within an acceptable tolerance.

In order to evaluate the residual load vector for the purpose of tracking solution convergence a technique termed 'stress transfer' [1] is applied. This technique transforms eqn (4.29) into a form whereby \( \psi(a) \) may be determined explicitly during the iterative procedure, i.e.

\[
\psi_n(a) = \{P(a^n)\} + \{f\} \quad (4.30)
\]

where \( \{P(a^n)\} \) represents the equivalent nodal loads required to balance the actual stresses generated within the element by the nodal displacements \( \{a\}^n \), calculated up to iteration \( n \).

\( \phi_n(a) \) represents the discrepancy between the external load vector \( \{f\} \), and the load vector \( \{P(a^n)\} \) which would in practice generate the actual stresses within the element.

In this context eqn (4.30) represents the requirement of equilibrium between the external load vector \( \{f\} \) and the internal stress state which is only satisfied when \( \phi_n(a) = 0 \). By treating the total stress induced in the element during iteration \( n \) in an analogous manner to distributed loads \( \{P(a^n)\} \) may be evaluated, i.e.

\[
\{P(a^n)\} = \int_V [B]^T\{\sigma^n\}dV \quad (4.31)
\]

whilst the total stress incurred up to iteration \( n \), \( \{\sigma^n\} \) may be calculated by integrating the constitutive stress strain relations in an accumulative fashion for each iterative interval, i.e.

\[
\{\sigma\} = [D_{qp}][\epsilon]
\]

Thus the total stress at the end of iteration \( n \) is given by

1. The evaluation of the constant coefficient matrix employed during the iterative process is dependent on the particular technique applied as discussed in Section 4.4.2 - 4.4.4.
This concept is illustrated diagrammatically in Fig 4.7.

\[(\sigma^1) = \begin{bmatrix} 0 & \{D_{sp}\}|(d\varepsilon) \end{bmatrix} + \{\sigma^{n-1}\} \]

\[(f^*)\] represents the external load vector  
\[(a^*), (\varepsilon^*), (\sigma^*)\] represents the converged solution

Figure 4.7 - A Graphical Representation of an Iterative Scheme Employed in the Solution of Non-Linear Problems

The flow chart illustrating the overall solution procedure is presented in Fig 4.9 at the end of this section.

4.3. The Newton Raphson Method

If an approximate solution to eqn (4.29) is available, \(\{a\} = \{a\}^N\), for which \(\psi(a^N) \neq 0\), then an improved estimate \(\{a\}^{n+1}\) can be found by means of a curtailed Taylor series expansion about \(\psi(a^N)\).
\[ \psi(a^{n+1}) = \psi(a^n) + \left( \frac{d\psi}{da} \right)_n \Delta a^n = 0 \]  

where the superscript \( n, n+1 \) refers to the \( n^{th} \) or \((n+1)^{th}\) iteration.

The improved estimate is given by

\[ (a)^{n+1} = (a)^n + (\Delta a)^n \]

where

\[ (\Delta a)^n = - \left( \frac{d\psi}{da} \right)_n^{-1} \psi(a^n) \]  

and

\[ \left( \frac{d\psi}{da} \right)_n = [K]^n_T \]  

\([K]^n_T\) represents the tangent stiffness matrix for the \( n^{th} \) iteration.

It is shown in [1], that the tangent stiffness may be calculated at any step in the solution by substituting the current value of \([D_{ep}]\) for \([D]\) when evaluating the stiffness coefficients.

Thus

\[ [K] = \int_{V} [B]^T [D] [B] dV \]  

\[ [K]^n_T = \int_{V} [B]^T [D_{ep}] [B] dV \]  

4.4.3 The Modified Newton Raphson Method

In the previous method the tangent stiffness matrix is reformulated at each stage of iteration and consequently a complete factorisation of the stiffness matrix is necessary for each iteration. In order to reduce the computational effort involved in this approach, the
linear elastic stiffness matrix is substituted for the tangent stiffness matrix in eqn (4.35). Thus

$$\Delta a^h = -([K]^0)^{-1} \psi(a^h)$$  \hspace{1cm} (4.37)

The Newton Raphson, Modified Newton Raphson techniques applied to plasticity problems are termed the tangent and initial stiffness methods respectively. Although the tangent stiffness matrix results in a more rapid convergence, the initial stiffness method may provide an overall economy of solution. [1]

It is important to note that the modified Newton Raphson technique is the only method capable of attaining a solution to non associated plasticity problems [1].

4.4.4 The BFGS Matrix Update Method, [3]

The BFGS technique belongs to a class of methods known as matrix update or quasi-newton methods. These methods seek to refine the existing stiffness coefficient matrix (without having to formulate the tangential stiffness matrix explicitly) to provide a secant approximation from iteration $n$ to $n+1$. The technique operates on the increments of displacement as well as the increment of the residual load vector during iteration $n$ to $n+1$, in order to refine the stiffness coefficient matrix by means of a matrix multiplication. This technique is well documented in the literature [3]. The three methods presented are illustrated in Fig 4.8 on the next page.
Figure 4.8 - Iterative Schemes Available in the ADINA Library
Refonnulate stiffness matrix according to equilibrium scheme chosen: is Newton Raphson, Modified Newton Raphson, BFGS technique.

\[
\begin{align*}
\{K\}^n & = \{a\} - \{a\}^{n-1} - \{a\}^n \\
\{c\}^n & = [B]^T \{s\} \\
\{f\}^n & = \{a\}^{n-1} - \{f\}^n
\end{align*}
\]

Call material model subroutine above.

Figure 4.9 - Flow Chart of Elastic-Plastic Solution Procedures
MATERIAL MODEL SUBROUTINE

calculate the total elastic stress
\( \sigma_I = [D] \varepsilon_I \)

elastic behaviour
return for the next time step

elastic behaviour
< 0
\( F(\sigma, k) > 0 \)
plastic behaviour
calculate the elasto plastic stress

elastic
STRESS STATE IN PREVIOUS TIME STEP
plastic

calculate the ratio of elastic to plastic stress by scaling the stress state to the yield surface
ie. \( F(\sigma, k) = 0 \)

\( \Rightarrow \) elastic strain increment
\( (\Delta \varepsilon)_I^e = R \times (\Delta \varepsilon)_I \)

TOTAL ELASTIC STRAIN \( \varepsilon^e = (\varepsilon)_{I-1} + (\varepsilon)_I^e \)

\( \Rightarrow \) strain increment causing elasto plastic behaviour

\( (\Delta \varepsilon)_I^{ep} = (1-R) \times (\Delta \varepsilon)_I \)

elasto plastic behaviour between the limits \( (\varepsilon)^e, (\varepsilon) \)

calculate elasto plastic stress change

\( (\sigma)_I^{ep} = \int [D_{ep}] (\varepsilon) \)

\( \Rightarrow \) \( (\sigma)_I = [D] (\Delta \varepsilon)_I^e + (\Delta \varepsilon)_I^{ep} + (\sigma)_{I-1} \)

Figure 4.9b - Material Model Subroutine
4.5 FINITE ELEMENT SIMULATION OF THE UNIAXIAL BEHAVIOUR OF CAST IRON

A uniaxial tension/compression test was simulated by loading the surface of a single plane stress two-dimensional element with a compressive/tensile pressure load. This verified that the subroutine was operating correctly and provided a means of comparing the material model behaviour with experimentally observed results.

Figure 4.10 - Five Noded 2-D Plane Stress Element 2 x 2 Gaussian Integration

Before executing this simulation the material properties of the cast iron, as determined from the experimental tests, were manipulated so as to evaluate the parameters required by the material model. It was decided to select these parameters on the basis of the results obtained from tensile test 5 and compression test 2 since these specimens were machined from the sheave in question.

The procedure for selecting these parameters relies on manipulating the yield surface and failure surface parameters so as to obtain a single function describing the tangent modulus at any uniaxial tensile/compression stress state. This procedure is summarised in point form below:

- select the yield limit in tension; \( (R = 0) \)

1. The material parameters required by the model are presented with a detailed flow chart of the subroutine in Appendix 5.
• select the yield limit in compression; \( (R = 0) \)
• evaluate Young's modulus between these limits
• select the failure stress in tension; \( (R = 1) \)
• calculate the tangent modulus at the tensile failure stress defined above
• select the failure stress in compression on the basis that the tangent modulus should correspond to that defined at the tensile failure stress
• evaluate the parameters, \( a_y, K_y, a_f, K_f \), which define yield and failure surfaces
• employ these parameters to map the tangent moduli obtained from the tension/compression tests on a graph of \( E_T \) versus \( R \). (The method of calculating \( R \) is presented in Appendix 2.)
• if it is possible to fit a single curve \( E_T = f(R) \) through the data, then satisfactory parameters have been selected.

The material parameters selected are listed below; these were employed in all subsequent analyses.

- tensile yield stress 60 MPa
- ultimate tensile stress 222 MPa
- compressive yield stress -120 MPa
- compressive failure stress -540 MPa
- Young's modulus 95 GPa
- yield surface parameters
  \[ a_y = 0.192 \ 45 \]
  \[ K_y = 46.183 \]
- failure surface parameters
  \[ a_f = 0.121 \ 223 \]
  \[ K_f = 155.083 \]
- the function selected for the purpose of describing the tangent modulus is illustrated in Plot 4.1.
select the yield limit in compression; \((R = 0)\)
evaluate Young's modulus between these limits
select the failure stress in tension; \((R = 1)\)
calculate the tangent modulus at the tensile failure stress defined above
select the failure stress in compression on the basis that the tangent modulus should correspond to that defined at the tensile failure stress
evaluate the parameters, \(a_y, K_y, a_F, K_F\), which define yield and failure surfaces
employ these parameters to map the tangent moduli obtained from the tension/compression tests on a graph of \(E_t\) versus \(R\). (The method of calculating \(R\) is presented in Appendix 2.)
if it is possible to fit a single curve \(E_t = f(R)\) through the data, then satisfactory parameters have been selected.

The material parameters selected are listed below; these were employed in all subsequent analyses.
- tensile yield stress 60 MPa
- ultimate tensile stress 222 MPa
- compressive yield stress -120 MPa
- compressive failure stress -340 MPa
- Young's modulus 95 GPa
- yield surface parameters
  \(a_y = 0.192\ 45\)
  \(K_y = 46,188\)
- failure surface parameters
  \(a_F = 0.121\ 223\)
  \(K_F = 155,083\)
- the function selected for the purpose of describing the tangent modulus is illustrated in Plot 4.1.
The simulated uniaxial tension/compression behaviour of the material conforming to both an associated and non associated material law is illustrated in Plots 4.2 and 4.3.

The results of this analysis indicate that the uniaxial behaviour of the cast iron is adequately modelled. The essential difference between the associated and non associated formulation is apparent in Plots 4.4 and 4.5 where volumetric strain effects are considered.

The associated material law characterises the cast iron material better, but not accurately (the volumetric strain is under evaluated in tension and over evaluated in compression). In view of the relationship between the volumetric strain component and the inclination of the plastic strain rate vector to the potential surface, it is likely that a parabaloid potential surface would characterise the volumetric behaviour of the material more accurately.

Figure 4.11 - Parabaloid Plastic Potential Surface

It was decided that a redevelopment of the constitutive laws based on a parabaloid plastic potential surface would be beyond the purpose of this dissertation. However an attempt was made to define a non associated Drucker Prager plastic potential surface requiring minor modifications to the subroutine. This was unsuccessful but for the sake of completeness the concept is presented in Appendix 2.7.
PLOT 4.1 - Specification of the Tangent Modulus as a Function of the Stress Ratio R
Plot 4.2 - Uniaxial Tension

Plot 4.3 - Uniaxial Compression
Plot 4.4 - Tensile Volumetric Strain

Plot 4.5 - Compressive Volumetric Strain
4.6 CONCLUDING REMARKS

4.6.1 Material Modeling

Bilinear or trilinear variations in the tangent modulus are commonly employed to model the uniaxial stress strain characteristics of a material as depicted in Fig 4.6. The ratio R was employed to model this variation more accurately, although this was achieved for the uniaxial stress strain behaviour (see Plot 4.2, 4.3) new difficulties became apparent when comparing dilatency in tension and compression, which limits the accuracy of the model.

![Material modeling technique](image)

Figure 4.12 - Techniques of modelling the tangent modulus

4.6.2 Subroutine

Although the user supplied material model is a listed option available in the ADINA package, difficulties were experienced when attempting to utilise the option.

- The source program contained statements preventing the user from erroneously defining such an option. This difficulty was overcome by deleting these statements and recompiling the source program.

- Information regarding the operation of such a model in the context of the ADINA program, is scarce. This necessitated a study of similar material models in the ADINA source, i.e. material model 7.
(elastoplastic, Drucker-Prager yield criterion with a hardening cap and tension cut-off); Material Model 8 (elastoplastic, von Mises yield criterion, isotropic hardening), in order to define arrays and variables common to the source program, which facilitate data transfer during solution. More information regarding these requirements and the structure of the subroutine would reduce the effort and the time required to utilise this option.

This exercise provided an introduction to the basic concepts of plasticity as well as finite element techniques employed in the solution to elastoplastic problems. In addition an overall view of the structure of the ADINA source was obtained through the implementation of the elastoplastic relations derived.
CHAPTER 5

TWO-DIMENSIONAL SOLID ELEMENT ANALYSIS OF THE SHEAVE

5.1 INTRODUCTION

The preliminary study of the sheave initiated in Chapter 3 was extended by applying the nonlinear constitutive stress/strain relationship, developed in the previous chapter, so that a more comprehensive analysis of the sheave could be executed.

An experimental study of the sheave behaviour when loaded to failure was undertaken for the purpose of comparison with the finite element simulation. In the experiment electrical resistance strain gauges were mounted at significant positions on the sheave, monitoring the strain incurred when the sheave was loaded to failure. These readings enabled the overall effectiveness of the finite element simulation to be evaluated.

5.2 EXPERIMENTAL PROCEDURE

The sheave had been machined in accordance to the dimensions specified in Chapter 3. The dimensions are illustrated in Appendix 4.3. The sheave was mounted on a shaft and then placed on a pedestal. In order to apply a rope load a special loading rig (see Fig 5.1) was inserted between the upper and lower platens of an Amsler compression testing machine. The upper platen and hence the loading rig was raised and the assembly of the sheave and pedestal inserted. Before testing commenced the loading rig was lowered and the alignment of the loading rig with the sheave was checked. This arrangement is illustrated in Plates 5.1 and 5.2.
Twenty four electrical resistance strain gauges were mounted at positions on the sheave as illustrated in Fig 5.1. From this figure it can be seen that pairs of gauges were placed on either side of the outer rim. The reason for this was to ascertain whether asymmetrical loading occurred during the test. The gauges were connected to Wheatstone bridges excited by a constant voltage source. Dummy gauges were included in the bridges to compensate for changes in temperature. The output voltage from each bridge was monitored by Esterline Angus data loggers.

On a headgear and under steady running conditions the rope is fully tensioned on contact with the sheave and little friction or creep is present. Since the test performed in the experiment was concerned with the static response of the sheave, pre-tension does not exist in the rope and considerable friction may develop between the rope and the tread of the sheave as the load is increased. This was obviated by ringing a series of brass strips along the length of the rope and using grease liberally prior to testing.

The purpose of the analysis undertaken in Chapter 3 was to select the dimensions of the rim section of the sheave. The sheave was remachined in accordance to these specifications and failure was expected to occur in the 400 KN range. Hence a range from 0 - 500 KN was selected on the Amsler testing machine. However at the maximum load of 500 KN, failure had not occurred. Subsequently the sheave was unloaded; on reloading, in the 0 - 1 000 KN range, failure occurred at 680 KN (see Plate 5.3). This corresponds to a rope tension of 402.26 KN.

1. Since each data logger provides a twenty channel facility, two instruments were employed.
Plate 5.1 - The sheave mounted in the loading rig (Esterline Angus data loggers in lower left corner)
Plate 5.1 The sheave mounted in the loading rig
(Esterline Angus data loggers in lower left corner)
PLATE 5.4 - Primary and Secondary Failure of the Rim Section
A - primary failure       B - secondary failure