


Computer Listing of Kurtosis Programme.

```basic
0 OPTION BASE 0
10 PRINT "LIST OF APPENDICES"
20 PRINT "APPENDIX 1" 337
30 PRINT "COMPUTER LISTING OF KURTOSIS PROGRAMME."
40 DIM Diam(100), Diameter(100), Em(100), Emass(100), Del(100), Delta(100)
50 DIM Deltax(100), V(100), Summ(100), Deltax(100), Deltam(100), Thirdm(100)
60 DIM Deltay2(100), D(100), Beta(100), Ei(100), Sum1(100), Sum2(100), Sum3(100)
70 INPUT "NUMBER OF SECTIONS?", Ns
80 FOR I=1 TO Ns
90 PRINT "DELTA X IN mm OF SECTION", I
100 INPUT "DELTA x IN mm OF SECTION", Del(I)
110 PRINT CHR$(12)
120 NEXT I
130 FOR I=1 TO Ns
140 PRINT "DIAMETER IN mm OF SECTION", I
150 INPUT "DIAMETER IN mm OF SECTION", Diam(I)
160 PRINT CHR$(12)
170 NEXT I
180 FOR I=1 TO Ns
190 PRINT "EQUIVALENT MASS IN kg OF SECTION", I
200 INPUT "EQUIVALENT MASS IN kg OF SECTION", Em(I)
210 Emass(I)=Em(I)
220 PRINT CHR$(12)
230 NEXT I
240 FOR I=1 TO Ns
250 PRINT "DELTA Y2 IN mm OF SECTION", I
260 INPUT "DELTA Y2 IN mm OF SECTION", D(I)
270 Deltay2(I)=D(I)/1000
280 PRINT CHR$(12)
290 NEXT I
300 FOR I=1 TO Ns
310 PRINT "DELTA Y2 IN mm OF SECTION (PART 2)", I
320 INPUT "DELTA Y2 IN mm OF SECTION (PART 2)", D1(I)
330 Deltay2z(I)=D1(I)/1000
340 PRINT CHR$(12)
350 NEXT I
360 FOR J=0 TO 500
370 Omega(J)=1*.J
380 Deltay(0)=0
390 V(0)=1
400 Summ(0)=0
410 FOR I=0 TO Ns
420 Deltax(I+1)=Deltay(I)*Emass(I)*Omega(J)*2
430 V(I+1)=Deltay(I+1)+0(I)
```

338 De lta m(I+1)=De lta(I+1)*V(I+1)
340 Summ(I+1)=Summ(I)+Delta m(I+1)
342 Thir dm(I)=Summ(I)/3
343 Sixthm(I)=Summ(I)/6
344 Thir dm(I+1)=Summ(I+1)/3
345 Sixthm(I+1)=Summ(I+1)/6
346 M1(I+1)=Thir dm(I)+Sixthm(I+1)
347 M2(I+1)=Thir dm(I+1)+Sixthm(I)
348 Bm1(I+1)=Beta(I+1)*M1(I+1)
349 Bm2(I+1)=Beta(I+1)*M2(I+1)
350 Del tayx(0)=0
351 Bm1(0)=0
352 Delta x(I+1)=Delta x(I)+Bm2(I)+Bm1(I+1)
354 Del tay(I+1)=Delta(I+1)*Del tayx(I+1)
355 Delta y(I+1)=Delta y(I)+Delta y2(I+1)+Delta y1(I+1)
357 NEXT I
358 Sum1(0)=0
359 Sum2(0)=0
360 Sum3(0)=0
361 FOR I=0 TO Ns-
362 !Sum1(I+1)=Summ(I)+Sum1(I)
363 Sum2(I+1)=Bm1(I+1)+Bm2(I+1)+Summ(I)
364 !Sum3(I+1)=Delta y(I)+Sum3(I)
365 NEXT I
366 !PRINT USING "30A.2X,15D.9D";"SUM OF THE MOMENTS =",Summ(Ns)
367 !PRINT USING "30A.2X.K";"SUM OF THE BM =",Sum2(Ns-1)
368 !PRINT USING "30A.2X.K";"SUM OF THE DEFLECTIONS=";Delta y(Ns)
369 Next y(I+1)=Delta y(I)*Delta y2(I)*Delta y1(I)
370 Delta y2(I+1)=Delta y2(I)+Vz(I+1)
371 Delta y1(I+1)=Delta y1(I)+Bm2(I+1)*Omega(I+1)
372 Delta y3(I+1)=Delta y3(I)+Summ(I)/3
373 Sixthm(I)=Summ(I)/6
374 Thir dm(I+1)=Summ(I+1)/3
375 Sixthm(I+1)=Summ(I+1)/6
376 M1z(I+1)=Thir dmz(I)+Sixthmz(I+1)
377 M2z(I+1)=Thir dmz(I+1)+Sixthmz(I)
378 Bm1z(I+1)=Beta(I+1)*M1z(I+1)
379 Bm2z(I+1)=Beta(I+1)*M2z(I+1)
380 Delta xz(0)=0
381 Delta xz(I+1)=Beta(I+1)*Delta xz(I+1)
384 Delta yz(I+1)=Delta yz(I)+Delta yz2(I)+Delta yz1(I)
385 NEXT I
386 !PRINT USING "30A.2X,15D.9D";"SUM OF THE 2ND MOMENTS =",Summz(Ns)
387 !PRINT USING "30A.2X.K";"SUM OF THE 2ND BM =",Sum2z(Ns-1)
388 !PRINT USING "30A.2X.K";"SUM OF THE 2ND DEFLECTIONS =";Delta yz(Ns)
389 M(J)=(Summ(Ns+1)+Delta yz(Ns+1))/Delta yz(Ns+1))
390 Summz(0)=0
391 Sum2z(0)=0
392 Sum3z(0)=0
393 FOR I=0 TO Ns-
394 !Sum1z(I)=Summz(I)+Sum1z(I)
395 Sum2z(I+1)=Bm1z(I+1)+Bm2z(I+1)+Summz(I)
396 !Sum3z(I+1)=Delta yz1(I)+Sum3z1(I)
397 NEXT I
398 !PRINT USING "30A.2X,15D.9D";"SUM OF 2ND MOMENTS =",Summz(Ns)
811  PRINT IS 701
815  PRINT USING "10A.2X,K.5X,20A.2X.K":"OMEGA=".Omega(J),"SUM OF MOMENTS=".
825  NEXT J
835  END
APPENDIX 2

DERIVATION OF THE COMPUTER SIMULATED SHAFT ORBITS

If vibrations of a rotor are considered as a two degree of freedom system then the amplitudes and phases of the horizontal and vertical forced vibrations are:

\[ X = \frac{mrw^2}{\sqrt{(k_x - Mw^2)^2 + (cw)^2} \} } \]

\[ Y = \frac{mrw^2}{\sqrt{(k_y - Mw^2)^2 + (cw)^2} \} } \]

and

\[ \theta_x = \arctan \left( \frac{cw}{k_x - Mw^2} \right) \]

\[ \theta_y = \arctan \left( \frac{cw}{k_y - Mw^2} \right) \]

The orbits were generated by plotting the following variables as one versus the other:

\[ x = X \cos (wt + \theta_x) \]

\[ y = Y \sin (wt + \theta_y) \]
Appropriate values were chosen for:

- \( i \) = Rotor Mass, \( i = 100 \text{ kg} \)
- \( m \) = unbalance mass, \( m = 0.001 \text{ kg} \)
- \( r \) = radius of unbalance, \( r = 0.001 \text{ m} \)
- \( k_x = 800000 \text{ N/m} \)

The damping coefficient \( c = 2 \sqrt{\frac{k}{m}} \)

Assuming a damping factor of 0.1

\[
c = (0.1)(2) \sqrt{(800000)(100)}
\]

\[
= 1789 \text{ kg/s}
\]
APPENDIX 3

CALCULATION OF THE SHAFT ATTITUDE AND SHAFT MOVEMENT VECTOR

Given the DC voltages from the two proximity probes at the relevant speeds and the bearing clearance, generation of the shaft centre line plot proceeds as follows. Refer to Figure 6.29.

The inner circle diameter represents the bearing clearance. The displacement of the shaft centre line, in the vertical and horizontal directions, at each speed increment is calculated as follows:

\[ d_v = \frac{(V_{0v} - V_v)}{S_v} \times 1000 \]

\[ d_h = \frac{(V_{0h} - V_h)}{S_h} \times 1000 \]

Where:

\[ d_v, d_h \] are the vertical and horizontal displacements respectively in micrometers.

\[ S_v, S_h \] are the sensitivities of the vertical and horizontal probes respectively in V/mm.

\[ V_v, V_h \] are the DC voltages of the vertical and horizontal probes respectively, at various speed increments in volts.

\[ V_{0v}, V_{0h} \] are the DC voltages of the vertical and horizontal probes respectively at zero RPM in volts.

The proximity probe's polarity is such that a negative decrease in DC voltage implies that the shaft is moving towards it.
Once $d_v$ and $d_h$ have been calculated, these values are then scaled and plotted commencing from the bottom of the inner circle.

The shaft attitude angle is the angle the vertical and the line joining the centre of bearing to the last point, point seven in this case. Refer to Figure 6.29.

The shaft movement vector is the vector joining points 1 and 7.
Vibrations of a rotor can be considered as a two degree of freedom system. The mathematical model of the rotor is as follows:

\[ \dot{x} + c \dot{x} + K_x x = m r^2 \cos \omega t \]  
\[ \dot{y} + c \dot{y} + K_y y = m r^2 \sin \omega t - Mg \]

Where \( M \) is the rotor mass  
\( c \) is the coefficient of damping  
\( m, r \) are mass and radius of unbalance  
\( g \) acceleration of gravity  
\( K_x, K_y \) are stiffness coefficients which depend on the shaft \( K \) and pedestal stiffnesses \( k_x \) and \( k_y \).

Let:

\[ \frac{1}{k_x} = \frac{1}{k} + \frac{1}{K} \]

\[ K_x = \frac{k_x K}{k_x + K} \text{ and } K_y = \frac{k_y K}{k_y + K} \]

Generally, the values of \( K_x \) and \( K_y \) are different, because the bearing pedestals are more flexible in the horizontal direction than in the vertical or vice versa. Sometimes, the pedestals are much more rigid than the shaft, then \( k_x = k_y \approx \infty \) and then

\[ K_x = K_y = K = \infty \]
The solution for equations (A4.1) and (A4.2) will contain natural frequencies of:

\[ \omega_{nx} = \sqrt{\frac{k_x}{M}} \quad \text{and} \quad \omega_{ny} = \sqrt{\frac{k_y}{M}} \]

These natural frequencies are different but very often their values are close, especially when the rigidity of the supports significantly exceeds the rigidity of the shaft. If for example, \( k_x = 3K \) and \( k_x = 4K \), then \( \omega_{nx} \) differs from \( \omega_{ny} \) by less than 33%.

Solving for the displacements \( X \) and \( Y \) in equations (A4.1) and (A4.2) results in:

\[
X = \frac{m\omega^2}{\sqrt{(K_x - M\omega^2)^2 + (c\omega)^2}}
\]

\[
Y = \frac{m\omega^2}{\sqrt{(K_y - M\omega^2)^2 + (c\omega)^2}}
\]

\[
\theta_x = \arctan \left( \frac{c\omega}{K_x - M\omega^2} \right)
\]

\[
\theta_y = \arctan \left( \frac{-c\omega}{K_y - M\omega^2} \right)
\]

where \( \theta_x \) and \( \theta_y \) are the phase lag angles for horizontal and vertical forced vibrations respectively.

To simplify the interpretation of Bode diagrams consider a single degree of freedom system.
Input force = $mrw^2 \sin(\omega t + \phi)$

Restraining force = $M\ddot{x} + c\dot{x} + kx$ using D'Alembert's principle.

Where $\phi$ = angular position of unbalance Figure A4.1 shows the model of the rotor.

The restraining force = input force so

$M\ddot{x} + c\dot{x} + kx = mrw^2 \sin(\omega t + \phi)$

Solving for the displacement $X$

$$X = \frac{mrw^2 \sin(\omega t + \phi - \theta)}{\sqrt{(k - \omega^2M^2)^2 + (cw)^2}}$$  \hspace{1cm} (A4.3)

where $\theta = \arctan\left(\frac{cw}{k - \omega^2M}\right)$

= phase lag angle.

The relation between the terms $Mw^2$, $cw$, $k$ and the total restraining force at low speeds, at $\omega_n = \sqrt[k]{\frac{k}{M}}$ and at high speeds can be seen in Figure A4.2.
RELATIONSHIP BETWEEN $Mw^2_{cw}$, $K$ AND THE TOTAL RESTRAINING FORCE

AT LOW SPEEDS
WHERE $\omega < \omega_0$
$\omega$ and $\omega^2 M$ terms have negligible contribution
So that
$$X = \frac{m\omega^2 \sin(\omega t - \beta - \phi)}{\omega}$$
Amplitude therefore increases with speed squared $\omega^4$. Heavy Spot and High Spot Coincident.

AT RESONANCE
$K = \frac{\omega^2 M}{\omega}$
$$X = \frac{m\omega^2 \sin(\omega t + \theta - \phi)}{\omega^2 M}$$
$\beta = 90^\circ$

AT HIGH SPEEDS
$\omega^4 > \omega_0$
$\omega$ and $K$ terms are negligible compared to $\omega^2 M \omega$
$$X = \frac{m\omega^2 \sin(\omega t - \beta - \phi)}{\omega^2 M}$$
$\beta = 180^\circ$

FIGURE A4.2
The Prohl calculation method combines generality with comparative simplicity. Any critical speed, first, second, or higher, may be calculated. The rotor may have variable cross-section, provided circular symmetry is maintained. The shaft journals may be considered to be elastically supported in the bearing with respect to both deflection and fitting of the journals.

For a balanced shaft of variable diameter which is rotating steadily at a critical speed with its central axis in a bowed shape, the following differential equation applies:

\[ \frac{\partial^2}{\partial x^2} (EI \frac{\partial^2 y}{\partial x^2}) + \mu \frac{\partial^2 y}{\partial x^2} \]

from elementary beam theory

\[ EI \frac{\partial^2 y}{\partial x^2} = M \]

substituting equation (A5.2) into equation (A5.1), (A5.1) becomes

\[ \frac{\partial^4 M}{\partial x^4} = \mu \frac{\partial^2 y}{\partial x^2} \]

Equation (A5.1) is a fourth order differential equation, and hence there are four boundary conditions to be satisfied. Any speed of rotation for which it is possible to effect a solution of this
equation which satisfies these four boundary conditions constitutes a critical speed. Equations (A5.2) and (A5.3) form the basis for making such a solution by a simultaneous construction of the bending moment and deflection diagrams using a step-by-step integration process. Let the shaft be divided into a series of appropriate sections. Equations (A5.2) and (A5.3) may be rewritten as follows:

\[
\Delta \left( \frac{d\gamma}{dx} \right) = \frac{\Delta x}{EI} M_{avg} \tag{A5.4}
\]
\[
\Delta \left( \frac{dM}{dx} \right) = (\mu w^2 \Delta x)y_{avg} \tag{A5.5}
\]

where \( \Delta x \) is the length of a given section and \( M_{avg} \) and \( y_{avg} \) are the average values of bending moment and deflection for that section. Equation (A5.4) states that the change in slope of the deflection curve which occurs at the given section is proportional to the average bending moment, the proportionality factor being the flexibility constant \( \Delta x/EI \). Equation (A5.5) states that the change in slope of the bending moment diagram, which occurs at the given section, is proportional to the average deflection, the proportionality factor being the inertia force of the section mass per unit of deflection \( \mu w^2 \Delta x \).

In order to apply the numerical-calculation method, the actual rotor must be transformed into an idealised equivalent system consisting of a series of disks connected by sections of elastic but massless shaft. The mass of these disks and their spacing are chosen so as to approximate the distribution of mass in the actual rotor. Likewise, the bending flexibility of the connecting sections of shaft is taken so as to correspond to the actual flexibility of the rotor. In the discussion that follows, it will be assumed that the moment of inertia of the disks is negligible so that the disks may be treated as mass points.
Assume an idealised system, shown below in Figure (A5.1) is whirling in the deflected position at some speed \( w \).

Since each section of shaft is massless, the shearing force is constant between two masses, and hence the bending moment diagram has a constant slope directly equal to the shearing force.

\[
\frac{dM}{dx} = V \quad (A5.6)
\]

A finite change in shearing force occurs at each mass, which is equal to the inertia force of the mass.

\[
\Delta V = mw^2y \quad (A5.7)
\]

This finite change in shearing force results in a finite change in slope of the moment diagram at each mass. Because of the continuity of the shaft, the deflection diagram is a smooth curve with no breaks or discontinuities.
Assume that the following quantities are known at the left-hand end of the idealised system.

\[ V_0 = \text{shearing force} \]
\[ M_0 = \text{bending moment} \]
\[ \theta_0 = \text{slop of deflection curve} \]
\[ y_0 = \text{deflection} \]

There will be a change in shearing force at point 0 due to the inertia force of the mass \( m_0 \) according to equation (A5.7) and hence the shearing force \( V_1 \) for the section of shaft between point 0 and point 1 is

\[ V_1 = V_0 + m_0 \omega^2 y \]

(A5.8)

and the bending moment \( M_1 \) at point 1 is

\[ M_1 = M_0 + V_1(\Delta x)_1 \]

(A5.9)

The bending moment \( M \) at any distance \( x \) from the left-hand end of section 1 is

\[ M = M_0 + \frac{(M_1 - M_0)x}{(\Delta x)_1} \]

(A5.10)

The slope \( \theta \) of the deflection diagram for first section is obtained as follows:

\[ \theta = \frac{1}{(EI)_1} \int_0^X Mdx + c \]

(A5.11)

Substituting equation (A5.10) in equation (A5.11)

\[ \theta = \frac{1}{(EI)_1} \int_0^X \left( M_0 + \frac{(M_1 - M_0)x}{(\Delta x)_1} \right) dx + c \]
and integrating gives:

\[ \theta = \frac{1}{(EI)_1} \left( M_0 x + \frac{(M_1 - M_0)x^2}{\Delta x_1} \right) + c \]

at \( x = 0 \), \( \theta = \theta_0 \) so \( c = \theta_0 \), hence

\[ \theta = \frac{1}{(EI)_1} \left( M_0 x + \frac{(M_1 - M_0)x^2}{\Delta x_1} \right) + \theta_0 \] (A5.12)

The deflection \( y \) is obtained as follows:

\[ y = \int_0^x \theta \, dx + c_1 \] (A5.13)

substituting equation (A5.12) in equation (A5.13)

\[ y = \int_0^x \frac{1}{(EI)_1} \left( M_0 x + \frac{(M_1 - M_0)x^2}{\Delta x_1} \right) + \theta_0 \right] dx + c_1 \]

and integrating gives

\[ y = \frac{1}{(EI)_1} \left( M_0 x^2 + \frac{(M_1 - M_0)x^3}{\Delta x_1} \right) + \theta_0 x + c_1 \] (A5.14)

at \( x = 0 \), \( y = y_0 \) so \( c_1 = y_0 \), hence

\[ y = \frac{1}{(EI)_1} \left( M_0 x^2 + \frac{(M_1 - M_0)x^3}{\Delta x_1} \right) + \theta_0 x + y_0 \]
It is only necessary to know the slope $\theta$ and the deflection $y$ at the end of the section i.e. point 1 to solve for the slope and deflection at any other point. Substituting $(\Delta x)_1$ for $x$ and $B_1 = (\Delta/EI)_1$ in equation (A5.12) and (A5.14) gives

$$\theta_1 = \frac{B_1}{2} (M_0 + M_1) + \theta_0 \quad (A5.15)$$

$$y_1 = B_1 (M_0 + M_1) (\Delta x)_1 + \theta_0 (\Delta x)_1 + y_0 \quad (A5.16)$$

From the value of the deflection at point 1, as given in (A5.16) it is now possible to evaluate the change in shearing force at point 1, and hence the shearing force $V_2$ in shaft section 2.

$$V_2 = V_1 + w_2 y_1 \quad (A5.17)$$

The bending moment $M_2$ at point 2 is

$$M_2 = M_1 + V_2 (\Delta x)_2 \quad (A5.18)$$

The bending moment in shaft section 2 is now completely specified, and the slope $\theta_2$, and the deflection $y_2$ at point 2 may be evaluated by equations similar to (A5.13) and (A5.16). By repeating the process for successive sections, the bending moment and deflection diagrams for the remainder of the span may be calculated.
The equations for the \( n^{th} \) section and point are listed as follows:

\[
V_n = V_{n-1} + m_{n-1} \omega^2 y_{n-1} \tag{A5.19}
\]

\[
M_n = M_{n-1} + V_n (\Delta x)_n \tag{A5.20}
\]

\[
\theta_n = \frac{B_n}{2} (M_{n-1} + M_n) + \theta_{n-1} \tag{A5.21}
\]

\[
y_n = B_n \left( \frac{M_{n-1} + M_n}{3} \right) (\Delta x)_n + \theta_{n-1} (\Delta x)_n + y_{n-1} \tag{A5.22}
\]

where \( B_n \) = flexibility constant = \( (\Delta x/EI)_n \)

From the foregoing equations, it can be demonstrated that the shearing force, bending moment, slope and deflection at any point in the span will be linear functions of the four assumed quantities at the left-hand end of the span, i.e. point \( o \). Hence, the deflection, for example, at point \( n \) may be expressed by the following equation.

\[
y_n = A_n y_o + C_n M_o + D_n \theta_o + F_n y_o \tag{A5.23}
\]

where \( A_n, C_n, D_n \) and \( F_n \) represent numerical coefficients which may be determined by using matrices. Since two boundary conditions must always be known at the end of the system from which the calculations originate, two of the forms in equation (A5.23) can be eliminated. Thus only two coefficients need to be evaluated and this is done in two parts in matrix form.
Effect of Mass Moment of Inertia of Disks

Assume that values of mass moment of inertia, both polar and diametral, have been assigned to the disks of the idealised system. It is necessary, however, to differentiate between the nonrotating and rotating cases.

NONROTATING CASE

For a rotor which is vibrating in a transverse plane at a natural frequency, rotary inertia effect is involved. Consider a disk on a vibrating shaft at the instant of maximum amplitude as shown below in Figure (A5.2)

Let the deflection of the disk centre of gravity be denoted by $y$, the angle of rotation by $\phi$, and the frequency of vibration by $w$. The force $F$ and the moment $M$ which must be exerted on the disk with respect to its centre of gravity are:

\begin{align*}
F &= mw^2y \\
M &= Bw^2\phi
\end{align*}

\text{(A5.24)} \quad \text{(A5.25)}
where $B$ is the mass moment of inertia of the disk about an axis through the centre of gravity and normal to the axis of symmetry. The force and moment which the disk exerts on the shaft will have magnitudes equal to the foregoing values but will have directions opposite to those shown above in Figure A5.2. Hence for small angles the slope $\theta$ has been substituted for $\phi$, equations (A5.24) and (A5.25) become

$$F = mw^2y$$  \hspace{1cm} (A5.26)

$$\Delta M = -Bw^2\theta$$  \hspace{1cm} (A5.27)

**Rotating Case**

For a rotor which is whirling at a critical speed, a gyroscopic effect is involved. Consider a balanced disk, as shown below in Figure A5.3, the centre of gravity of which is whirling in a circular path of radius $y$ at a speed $w$.

**Effect of Mass Moment of Inertia of Disks**

![Diagram of rotating case](image-url)
Let the axis of the disk make a constant angle $\phi$ with the rotation axis. Assume that there is no rotation of the disk relative to the rotating plane formed by the disk axis and the rotation axis. This means that a point $Q$, which is the outside point on the periphery of the disk will remain the outside point. The components of rotation about the axis $a$ and $b$ are $\omega \cos \phi$ and $\omega \sin \phi$ respectively. Since axis $a$ and $b$ are principal axes of inertia for the disk, the components of angular momentum $H_a$ and $H_b$ are given by

$$
H_a = A \omega \cos \phi \\
H_b = A \omega \sin \phi
$$

The moment which must act on the disk about the centre of gravity to sustain the prescribed motion is equal to the time rate of change of the resultant angular momentum $H$.

$$
M = \frac{dH}{dt} = H_y \omega \quad (A5.28)
$$

where $H_y$ is the component of $H$ normal to the axis of rotation

$$
H_y = H_a \sin \phi - H_b \cos \phi \\
H_y = (A-B) \omega \sin \phi \cos \phi \quad (A5.29)
$$

Substituting equation (A5.29) into (A5.25) and equating $\sin \phi = \phi$ and $\cos \phi = 1$ for small

$$
M = (A-B) \omega^2 \phi \quad (A5.29)
$$

Again substituting $\phi$ for slope $\Theta$

$$
\Delta M = (A-B) \omega^2 \Theta \quad (A5.30)
$$
The $\Delta M$ due to inertia and gyroscopic effects gives rise to another $\Delta M$, besides the $\Delta M$ due to the shearing force, equation (A5.30), which are added in the matrix. Note that from equation (A5.30) the gyroscopic effect has the opposite effect on the $\Delta M$, and tends to stiffen the shaft as compared to the inertia effect.

The mass moments of inertia $A$ and $B$ for a flat solid disk are

$$A = \frac{\epsilon \pi h D^4}{32} \quad (A5.31)$$

$$B = \frac{\epsilon \pi h D^4 \left(1 + \frac{4}{3} \left(\frac{h}{D}\right)^2\right)}{64} \quad (A5.32)$$

where $h$ = the thickness of the disk.

For the non-rotating case, the natural frequencies obtained by neglecting the mass moment of inertia are decreased if the effect of the mass moment of inertia is considered. For the rotating case, the critical speeds are increased by taking into account the mass moment of inertia provided that $A > B$. 
APPENDIX 6

THE TRANSFER MATRIX METHOD FOR CALCULATING CRITICAL SPEEDS OF FLEXIBLE ROTORS

The transfer matrix method is another method whereby shaft critical speeds can be calculated. In the Prohl method the shaft rotational speed is changed so as to satisfy the end boundary conditions. The transfer matrix method initially satisfies the end boundary conditions and then iterates to solve for the end polynomial.

It is necessary to define three terms: the system, state vector, and the transfer matrix. The idealised system in this case is composed of a number of elements, either springs or point masses; each element is situated between two points in the system. The state vector is a column matrix which specifies the displacements and internal forces at a point in the system, this point being a boundary of an element. The transfer matrix relates the state vector at two points in the system, and generally the two chosen points are those forming the boundaries of an element. If the transfer matrix relates conditions on the two sides of a massless spring element, it is called the field matrix, and if it relates conditions on the sides of point mass, it is called a point matrix.

The state vector of a point $i$ is usually expressed by

$$
(Z)_{i} = \begin{bmatrix}
y \\
\Theta \\
M \\
Q
\end{bmatrix}
$$

Vertical deflection (m)  
Slope  
Bending moment (Nm)  
Dimensionless shear force

The field matrix is determined by the elastic properties of the shaft in a region between points of force application. It transfers the state vectors at station $i$ to the vector at station $i+1$ across a spatial distance called a field.
The displacement and equilibrium equations in matrix form yield:

\[
\begin{bmatrix}
Y \\
\Theta \\
M \\
Q_{i+1}
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & L^2/2EI & L^3/6EI \\
0 & 1 & L/EI & L^2/2EI \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
Y \\
\Theta \\
M \\
Q_i
\end{bmatrix}
\]

where the field matrix is

\[
(T_f)_{(i+1)-i} =
\begin{bmatrix}
1 & L & L^2/2EI & L^3/6EI \\
0 & 1 & L/EI & L^2/2EI \\
0 & 0 & 0 & L \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

A point transfer matrix transfers a state vector from a location one side of a mass to the other side, at the same designated point. The displacement and equilibrium equations in matrix form yield:

\[
\begin{bmatrix}
Y \\
\Theta \\
M \\
Q_{i-1}
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
w^2m & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
Y \\
\Theta \\
M \\
Q_{i+1}
\end{bmatrix}
\]

where the point matrix is

\[
(T_p)_{i+1} =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
w^2m & 0 & 0 & 1
\end{bmatrix}
\]
The rotational and gyroscopic effects can readily be included in the point matrix

\[
(T_p)_{i+1} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & (A-B)\omega^2 & 1 & 0 \\
\omega^2m-k & 0 & 0 & 1
\end{bmatrix}
\]

where \(A\) and \(B\) are the inertia constants as described in the Prohl method and \(k\) is the stiffness at each point.

The shaft is considered as a series of mass and spring elements as shown below in Figure A6.1.

**MODEL OF SHAFT**

![Diagram of shaft model](image)

Starting from the left hand end, one has:

\[
(Z)_n = (T_f)_{n-(n-1)}(T_p)_{n-1} \ldots \ldots (T_p)_2(T_f)_{2-1} (Z)_1
\]

or \((Z)_n = (N)(Z_1)\)
The product of all the field and point matrices leads to a (4x4) matrix, called the total transfer matrix for the complete beam. The total transfer matrix is denoted by \((N)\), which is such that

\[
\begin{bmatrix}
Y \\
\theta \\
M \\
Q
\end{bmatrix} =
\begin{bmatrix}
N_{11} & N_{12} & N_{13} & N_{14} & Y \\
N_{21} & N_{22} & N_{23} & N_{24} & \theta \\
N_{31} & N_{32} & N_{33} & N_{34} & M \\
N_{41} & N_{42} & N_{43} & N_{44} & Q
\end{bmatrix}
\]

(A6.1)

In the above matrix (A6.1) all the elements of \((N)\) are known and all eight elements of the two column vectors are unknown. Thus the boundary conditions must specify four of these eight unknowns.

For a simply supported shaft on rigid bearings the boundary conditions are \(Y_1 = Y_n = M_1 = M_n = 0\). Substituting these boundary conditions in matrix (A6.1) and carrying out the multiplication gives for a non-trivial solution:

\[
\text{DET} = \begin{bmatrix}
N_{12} & N_{14} \\
N_{32} & N_{34}
\end{bmatrix} = 0
\]

Similarly for a simply supported shaft on non rigid bearings the boundary conditions are \(M_1 = M_n = Q_1 = Q_2 = 0\) and the determinant becomes, for a non-trivial solution:

\[
\text{DET} = \begin{bmatrix}
N_{31} & N_{32} \\
N_{41} & N_{42}
\end{bmatrix} = 0
\]
The determinant will be a function of the unknown $w$. When in a given frequency range, the values of the determinant are plotted against $w$, and natural frequencies will be those values where DET equals zero.
APPENDIX 7

SHAFT CRITICAL SAMPLE CALCULATIONS FOR FD ROTOR

Having divided each constant diameter section into numerous sections of approximately 50 mm long, the calculation of equivalent masses proceeds as follows:

The sample calculations are for sections 50, 51 and 52.

<table>
<thead>
<tr>
<th>Δx (mm)</th>
<th>DIAMETER (mm)</th>
<th>MASS (kg)</th>
<th>SECTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>51,30</td>
<td>600</td>
<td>113,62</td>
<td>50</td>
</tr>
<tr>
<td>51,75</td>
<td>610</td>
<td>118,47</td>
<td>51</td>
</tr>
<tr>
<td>51,75</td>
<td>610</td>
<td>118,47</td>
<td>52</td>
</tr>
</tbody>
</table>

\[
\text{Mass} = (\text{density})(\text{volume}) \text{ (for section 50)} \\
= \varepsilon (\pi D^2/4) \Delta x \\
= (7833,43)(\pi)(0,6)^2(0,05130)/4 \\
= 113,62 \text{ kg}
\]

The impeller has a mass of 8 310 kg and is distributed over 40 sections. Mass contribution of the impeller to each of the 40 sections is 8 310/40 = 207,75 kg.
The dimensions for sections 49 to 52 are shown below. Section 49 has been included because it contributes to the equivalent point mass as station 49. See Figure A6.1 below.
Let $m_i =$ mass of section $i$ in kg

$E_{m_i} =$ equivalent mass of station $i$ in kg

$D_{m_i} =$ distributed impeller mass of section $i$ in kg.

$$E_{m49} = \frac{1}{2}(m_{49} + m_{50})$$
$$= \frac{1}{2}(113,62 + 113,62)$$
$$= 113,62 \text{ kg}$$

$$E_{m50} = \frac{1}{2}(m_{50} + m_{51} + D_{m51})$$
$$= \frac{1}{2}(113,62 + 118,47 + 207,75)$$
$$= 219,92 \text{ kg}$$

$$E_{m51} = \frac{1}{2}(m_{51} + m_{52} + D_{m51} + D_{m52})$$
$$= \frac{1}{2}(113,62 + 118,47 + 207,75 + 207,75)$$
$$= 326,22 \text{ kg}$$
CALCULATION OF THE GYROSCOPIC CONSTANTS A AND B

The distributed impeller mass at each section was considered a thin disk of equivalent mass, having a specified diameter and calculated thickness \( h \).

If the effective diameter of the shaft is taken as 5.5 m, then a disk with this diameter and mass of 207.75 kg must be 1.12 mm thick.

\[
\text{mass} = (\text{density})(\text{volume})
\]

\[
207.75 = (7833.4)(\pi D^2)(h)/4
\]

\[
h = (207.75)(4)/(7833.4)(\pi)(5.5)^2
\]

\[
h = 1.12 \text{ mm}
\]

\[
A = \varepsilon \pi h D^4/32
\]

\[
= (\pi)(7833.4)(1.12 \times 10^{-3})(5.5)^4/32
\]

\[
= 785.55 \text{ kg.m}^2
\]

If the thickness \( h \) of the disk is small compared to the diameter \( D \) then \( A \approx 2B \). Hence \( B = 785.55/2 = 392.78 \text{ kg.m}^2 \).