Stochastic Portfolio Theory
and its Applications to Equity Management

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Declaration

I declare that this dissertation is my own, unaided work, except where otherwise acknowledged. It is being submitted for the Degree of Master of Science to the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination to any other University.

(Signature)

(Date)

Abstract

Stochastic portfolio theory is a novel methodology, developed by Fernholz (2002), for analysing stock and portfolio behaviour, and equity market structure, constructing portfolios and understanding the structure of equity markets. It thus has immediate applications to equity portfolio management and performance measurement. This theory successfully generalises well-known models for the stock price to provide models for portfolios and markets, leading to a better and more precise understanding of equity market structure. The aim of this dissertation is to present an exhaustive review of stochastic portfolio theory by imitating the work done and contributions made by Fernholz (2002) thus far. A detailed discussion of stochastic portfolio theory as well as how the implications differ from the conclusions and results of classic portfolio theory will be provided. In this dissertation, we will undertake a thorough investigation into stochastic portfolio theory; by focusing on the central, innovative ideas of the excess growth rate, long-term stock market and portfolio behaviour, stock market diversity of equity markets, portfolio generating functions, the concept of how to select stocks by their rank and the existence of relative arbitrage opportunities within the context of stochastic portfolio theory. Thus, we shall review the central concepts of stochastic portfolio theory, this will include a detailed explanation of the excess growth rate, long-term behaviour of portfolios, stock market diversity, portfolio generating functions and stocks selected by rank. We will also present examples of portfolios and markets with a wide variety of different properties. We will also show how this new and fast-evolving theory can be applied, in particular, to equity management, by considering the performance of certain functionally generated portfolios. Furthermore, several results and implications of stochastic portfolio theory will be discussed, and in this dissertation, we shall examine these results in far greater depth.

Keywords and Phrases: Stochastic portfolio theory, Portfolios, Stock market and portfolio behaviour, Stock market diversity, Portfolio generating functions, Functionally generated portfolios, Rank-dependent portfolio generating functions, Local time, Relative arbitrage opportunities, Performance of functionally generated portfolios.

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Lisa Bonney
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For my loving parents Mark and Magda
and my beloved sister Siân.

"As far as the laws of Mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality."

– Albert Einstein (1879 - 1955)

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Chapter 1

Introduction

As with the rest of mathematical finance, *Stochastic Portfolio Theory* (SPT) descended from the paper 'Portfolio selection' by Markowitz (1952). Stochastic portfolio theory had its genesis in 1982, where Fernholz & Shay (1982) presented their ideas in the paper 'Stochastic portfolio theory and stock market equilibrium' and later Fernholz (1999a) in his paper 'On the diversity of equity markets'. Since then, the evolution of stochastic portfolio theory has been quite remarkable, developing into a rich and flexible framework. Stochastic portfolio theory is a new and exciting mathematical methodology with its implementation in analysing portfolio behaviour and the structure of equity markets. It also enables the construction of certain portfolios that have desirable investment properties. Moreover, stochastic portfolio theory provides a better understanding of the structure of equity markets.

Stochastic portfolio theory has its origin in classical portfolio theory, the work of Markowitz (1952). However, it differs from some of the determining aspects of the current theory of dynamic asset pricing. The theory of dynamic asset pricing is a *normative* theory that emanated from the general equilibrium model of mathematical economics for financial markets [Arrow (1953)], evolved through the capital asset pricing models [Sharpe (1964), Merton (1969)], and is currently predicated on the absence of arbitrage and on the existence of an equivalent martingale measure [Harrison & Kreps (1979)]. These theories of finance just mentioned: equilibrium, capital asset pricing and no-arbitrage, are the normative theories that are based on assumed ideal behaviour regarding the interaction of the participants and agents in the markets under consideration [Fernholz (2003a)]. Furthermore, this ideal behaviour frequently represents a significant departure from the actual observed behaviour. Stochastic portfolio theory differs from these central theories of finance in that it is a *descriptive* theory, as opposed to a normative theory, and hence is a distinct area of mathematical finance that is in contrast to the current precept of mathematical and quantitative finance. Since stochastic portfolio theory is descriptive, it is not built on strong normative assumptions and is applicable under a wide range of assumptions and conditions that may hold in real equity markets [Fernholz (2003a)]. Descriptive theories are employed in the study of observable phenomena and impart explanations for observed phenomena that occur in real equity markets. Furthermore, it provides predictions for the outcomes of future experiments. Unlike models currently used in mathematical finance, stochastic portfolio theory is consistent with either equilibrium or disequilibrium, arbitrage or no-arbitrage, market completeness or incompleteness and remains valid regardless of the existence of an equivalent martingale measure.

Stochastic portfolio theory is useful and effective in both a theoretical and practical context. In fact, the consistency of stochastic portfolio theory with the observable characteristics of actual equity markets lends itself to being a beneficial tool when considering practical applications. Within the theoretical setting, this framework offers a greater comprehension of and fresh insights into inquiries of equity market structure and the concept of arbitrage. Furthermore, the theory is particularly useful in constructing portfolios with controlled behaviour. The controlled behaviour that is of interest is one that generates portfolios that have profitable outcomes. The practical implementation of stochastic portfolio theory involves the study, analysis and optimisation of portfolio performance. In fact, this use of stochastic portfolio theory has been the basis of many successful investment

strategies adopted by Enhanced Investment Technologies, LLC or INTECH.

INTECH has managed institutional portfolios since 1987, establishing one of the industry's longest continuous records of mathematically-driven equity investing strategies. INTECH has pioneered a unique investment process based on a rigorous mathematical theory that does not depend on fundamental forecasts, and attempts to capitalise on the random nature of stock price movements. The goal of the investment process is to achieve long-term returns that outperform the benchmark index, while controlling risks and trading costs. In controlling risks, INTECH attempts to reduce the risk of significant underperformance. INTECH's mathematical process searches for stocks with high relative volatility and low correlation to build portfolios whose total return will exceed the return of the component stocks.

1.1 The Structure of the Dissertation

This dissertation is organised as follows: The essence of stochastic portfolio theory will be captured in Chapter 2. It is in this chapter where we shall provide a comprehensive treatment of this theory. In particular, the exhaustive survey, will introduce the fundamental structures of stochastic portfolio theory, including stocks, portfolios and the renowned market portfolio. The definitions of these such structures, although technically equivalent to the conventional definitions, differ from a philosophical perspective. It is precisely this difference that sets the theory, inspired by Fernholz, apart from the conventional theories. We also encounter the quantity termed the excess growth rate, which is an essential component that pervades stochastic portfolio theory.

Chapter 3 is devoted to the stock market and portfolio behaviour, where we analyse the long-term behaviour of stocks, portfolios, or the market itself.

The stock market diversity of capital distributions or the diversity of the distribution of capital is explored in Chapter 4. Here we introduce a pinnacle concept in stochastic portfolio theory: diversity. Diversity establishes a market in which the capital distribution of the market exhibits and maintains a stability over time, this occurs when the entire capital of the equity market is not concentrated into a single stock but is rather evenly distributed among all the stocks in the equity market in some fashion. We also determine conditions that will be compatible with equity market stability, as well as the nature of the consequences that stability engenders.

In Chapter 5 we introduce one of the central concepts of stochastic portfolio theory, that being, portfolio generating functions and functionally generated portfolios. These functions are capable of constructing many different types of portfolios. Furthermore, the relative return of these functionally generated portfolios can be decomposed into its constituents that each have specific characteristics and properties associated with them.

Chapter 6 extends the generating function methodology of the previous chapter to stocks that are identified according to their relative ranking within the equity market, rather than being identified by their name, which is what was broached in the previous chapter. Thus, in this chapter we take a look at rank-dependent portfolio generating functions and rank-based functionally generated portfolios.

Relative arbitrage opportunities in equity markets along with relevant consequences are considered next in Chapter 7. Here we shall introduce the concept of relative arbitrage, together with the allied notions of weak relative arbitrage and strong relative arbitrage. In this chapter we demonstrate that there exist strong relative arbitrage opportunities that exist in nondegenerate and (weakly) diverse equity markets. These strong relative arbitrage opportunities are borne out of the measures of diversity. As a result, the portfolios generated using measures of diversity will exhibit a dominance relationship with the market portfolio.

We also provide the following Appendices: Appendix A considers stochastic calculus, Appendix B considers Itô calculus, we present a few auxiliary proofs of selected results in Appendix C, higher order derivatives are given a brief mention in Appendix D, in Appendix E we present concave and convex functions along with some useful results, and finally, in Appendix F, we meet the man behind stochastic portfolio theory and who made all this possible.

Chapter 2

Stochastic Portfolio Theory

2.1 Introduction

In this chapter we introduce the basic structures of stochastic portfolio theory: stocks and portfolios. The definitions for stocks and portfolios will be stated along with their dynamics. Furthermore, beneficial results will be proved that will be required throughout the later chapters. The definitions, notation and stock price model used in this theory are all fairly standard in current mathematical finance. A number of certain simplifying assumptions are made that are again fairly standard in current mathematical finance. These basic assumptions are:

- The number of companies in the market is finite and fixed. Neither are there new companies founded nor do existing companies go bankrupt.
- We assume that companies neither enter nor leave the market, the total number of shares of each company remains constant and companies do not merge or break up.
- Trading is continuous in time. We shall assume that we operate in a continuously-traded, frictionless market in which the stock prices vary continuously.
- Continuous trading is also possible in the amount of shares. Thus, there are no issues regarding the indivisibility of shares (i.e. infinitesimally small fractions of shares can be bought or sold). Furthermore, since the shares are infinitely divisible, there is no loss of generality in assuming that each company has a single share of stock outstanding. Thus, since each company has only one share of its stock outstanding, this outstanding sole share of stock represents the total market capitalisation of that company.
- There are no transaction costs or taxes.
- Dividends are paid continuously rather than discretely, when not needed, we assume that there are no dividends.

Throughout this dissertation, we shall assume that all stock prices and portfolio values follow random processes. These random processes are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider a standard $d-$dimensional Brownian motion, W, for some positive integer d,

$$W \quad = \quad \big\{W(t) = \big(W_1(t), W_2(t), \ldots, W_d(t)\big), \mathcal{F}_t, t \in [0, \infty)\big\}.$$

This $d-$dimensional standard Brownian motion is defined on the induced filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathcal{F}_t, t \in [0, \infty)\}$ represents all the available information in the market. This filtration contains the Brownian motion filtration $\mathbb{F}^W = \big\{\mathcal{F}_t^W, t \in [0, \infty)\big\}$, which is simply the natural filtration generated by W, and is defined by

$$\mathcal{F}_t^W \quad = \quad \sigma\big(W(s); 0 \leq s \leq t\big) \quad \subseteq \quad \mathcal{F}_t, \qquad \text{for all } \ t \in [0, \infty).$$

Moreover, $\mathbb{F} = \{\mathcal{F}_t, t \in [0, \infty)\}$ is the augmentation under $\mathbb{P}$ of the natural filtration $\{\mathcal{F}_t^W = \sigma(W(s); 0 \leq s \leq t)\}$. The source of randomness in the model that will be presented is given by W. Moreover, the filtration $\mathbb{F}$ is the only filtration that will be considered when we mention adapted[1] processes, i.e., all adapted processes, martingales, etc., are defined with respect to this filtration. Furthermore, this filtration satisfies the "usual" conditions of right continuity and augmentation by $\mathbb{P}-$negligible sets.

Remark 2.1.1. In this dissertation, for the most part, we shall take $d = n$, i.e., equivalent to the number of stocks in the market. Thus, we shall, for the most part, consider an $n-$dimensional standard Brownian motion W for some positive integer n,

$$W = \{W(t) = (W_1(t), W_2(t), \ldots, W_n(t)), \mathcal{F}_t, t \in [0, \infty)\}.$$

Remark 2.1.2. The time, t, is always specified within the infinite time domain given by $[0, \infty)$. However, the finite time domain $[0, T]$, where $T > 0$, is commonly used in mathematical finance due to the need for Girsanov's theorem. Since we do not depend on this theorem, all our results hold for the time domain $[0, \infty)$. In fact, the infinite time domain will prove to be a necessary and convenient construct when we start to examine asymptotic events. However, in a non-asymptotic setting we restrict our consideration to the finite time domain $[0, T]$, so as to comply with convention. Thus, the time domain is always defined by the interval $[0, \infty)$, unless otherwise stated.

This chapter borrows heavily from and is largely based on the work of Fernholz (2002) and several collaborators, in terms of concepts and notation. The aim of this chapter is to present the reader to the introductory world of stochastic portfolio theory and all of its proponents. The intent here is to bring all readers up to the same level in respect of stochastic portfolio theory that has been the work of Fernholz (2002). In Section 2.2, the notion of stocks and portfolios are reviewed. It is with these basic tenets that we introduce the basic financial equity market model postulated by Fernholz (2002). This financial equity market model is based on the geometric rate of return (alternatively, the logarithmic rate of return) of stocks and portfolios, which is unlike the arithmetic rate of return employed by the traditional equity market models. These two approaches do, however, have a close connection which will be explored here. This section shall also serve as the introduction for the covariance process, the variance process, the portfolio variance process and the fundamental market conditions. Armed with the knowledge of the covariance process, we shall also explore some of its key characteristics. We shall also examine the consequences attributable to the equity market model developed by Fernholz (2002), namely the inclusion of the excess growth rate process. Moreover, the interpretation of the excess growth rate as a measure of a stock's or a portfolio's intrinsic volatility will be divulged. The latter part of this section is devoted to a brief discussion of the dividend rate and dividend-paying stocks, which will necessitate an investigation pertaining to the total return process and the total return process of portfolios. We want to be able to measure the performance of stocks or portfolios in the equity market relative to some given benchmark reference portfolio or index. This is of extreme importance, since one of our aspirations is to collate the performance of two different portfolios so as to derive potential profitable outcomes. To this end, in Section 2.3, we shall consider the concept of the relative return, as well as the relative covariance process and the relative variance process, within the context of stochastic portfolio theory. We shall establish several results for this relative return process which shall include descriptions of the dynamics of the relative return process. More precisely, the relative return process of a stock versus an arbitrary benchmark portfolio can be expressed as a weighted average of the relative return processes of the individual stocks and an additional component, the excess growth rate. The quadratic variance and the covariance of these relative return processes shall also be derived. The relative return process of stocks is firstly presented and then secondly we shall also provide the relative return process of a stock relative to an arbitrary benchmark portfolio. In what follows, we shall develop certain fundamental and useful properties of the relative covariance process and of the excess growth rate process that are essential to our analysis, moreover crucial upper and lower bounds on both the excess growth rate as well as the relative covariance and relative variance will be imposed in Section 2.4. This will require us to introduce the reverse-order-statistics notation for the weights of a portfolio, which is the ranking of the portfolio weights in decreasing order, from the largest

[1] A process $X = \{X(t), \mathcal{F}_t, t \in [0, \infty)\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is *adapted* if $X(t)$ is $\mathcal{F}_t-$measurable for $t \in [0, \infty)$. Roughly speaking, this suggests that X does not depend on future events.

portfolio weight to the smallest portfolio weight. One such property of the excess growth rate process that will be of substantial importance and will be put to considerable use in this dissertation is the so-called numéraire invariance property of the excess growth rate of a particular portfolio. This property, namely that the excess growth rate of a portfolio is numéraire invariant, is of keen interest when the numéraire proposed is the canonical benchmark market portfolio. This property demonstrates the fact that the excess growth rate of a portfolio does not depend on the choice of the benchmark numéraire portfolio, the portfolio with which the performance of the portfolio is to be contrasted. The numéraire invariance property offers an expression that calculates the excess growth rate, by estimating the relative variances as opposed to estimating the variances themselves. Thus, by employing a change of numéraire technique we can essentially replace the variances by the relative variances without changing the value of the excess growth rate. Another property of the excess growth rate establishes the positivity of the excess growth rate, i.e., the excess growth rate of a portfolio will be strictly positive if the portfolio holds at least two or more stocks with no short sales; and will be nonnegative for strictly long-only portfolios. Consequently, the portfolio growth rate will always exceed the weighted average of the growth rates of the individual component stocks. In this section, we shall also formalise the notion of the relative portfolio variance process by providing a formal definition for the relative variance process of two arbitrary portfolios. This relative portfolio variance process also exhibits some rather enlightening properties, which are displayed next. In this regard, it shall be evident that the relative variance process of an arbitrary portfolio versus itself is zero. The next few sections are devoted to the inclusion of various processes into the stochastic portfolio theory arena. Many results affiliated with these processes shall also be presented, as well as determining the associated representation of the dynamics of such processes together with the resultant quadratic variation and covariation (or, cross-variation) processes corresponding to these processes. We start this off with Section 2.5, in which we shall introduce the notion of the quotient process, as well as establish several results for this process. In particular, we shall offer representations for the dynamics of the quotient process. The quadratic variance and the covariance of these quotient processes shall also be derived. The quotient process of stocks is firstly presented and then secondly we shall also provide the quotient process of a stock relative to an arbitrary portfolio. Section 2.6 segues into a discussion of the relative return of portfolios, in which we shall relate its structure. The structure of the relative return process of one arbitrary portfolio versus another, is given through numerous representations of the dynamics of the relative return of these two arbitrary portfolios. In addition, we shall determine the quadratic variation of this relative return process. It is in this section that we shall define the portfolio covariance process, which is simply the covariance process of an arbitrary portfolio with another and indicates how the one arbitrary portfolio varies in line with another. Moreover, we offer an alternative definition for the relative variance of an arbitrary portfolio versus another. Then, the quotient process of portfolios is handled in Section 2.7. We shall also offer representations for the dynamics of the quotient process of an arbitrary portfolio with respect to another. The quadratic variance of these quotient processes shall also be derived. We shall now combine the relative return process with dividends, to establish the relative total return in Section 2.8. Representations for the dynamics of the relative total return process shall also be studied. The quadratic variation and the cross-variation processes for these relative total return processes shall also be derived. It shall be revealed that the cross-variation process is unaltered when dividends are introduced into the financial equity market model, this is what we would expect since dividends are only captured in the drift process. The relative total return process of stocks is firstly presented and then secondly we shall also provide the total return process process of a stock relative to an arbitrary portfolio. This brings us to the concept of the total quotient process defined in Section 2.9. The total quotient process of stocks is firstly presented and then secondly we shall also provide the total quotient process of a stock relative to an arbitrary portfolio. In particular, we shall offer representations for the dynamics of the total quotient process. The quadratic variance and the covariance of these total quotient processes shall also be derived. In Section 2.10, we again move onto portfolios, to consider the relative total return of portfolios. The structure of the relative total return process of one arbitrary portfolio versus another, is given through numerous representations of the dynamics of the relative total return of these two arbitrary portfolios. The relative total return process of one arbitrary portfolio against another arbitrary portfolio can be expressed in terms of the individual stock relative total returns against the other arbitrary portfolio. In addition, we shall determine the quadratic variation corresponding to this relative total return process. The last of these processes to be considered is the total quotient process of portfolios which is provided in Section 2.11. In this respect, we shall define the total quotient process of one arbitrary portfolio

versus another, and obtain its associated dynamics and derive its quadratic variation process. In Section 2.12, the revered market portfolio is introduced, this portfolio plays a central role in the analysis of relative portfolio performance. Thus far, we have considered the Fernholz (2002) framework in general, the primary advantage of this framework is that it allows for the effective evaluation of long-term portfolio performance. The value of the market portfolio represents the combined capitalisation of all the stocks in the entire market. We shall also supply a brief explanation of the market's intrinsic volatility in this section. The excess growth rate of the market portfolio measures, at any time, the amount of available volatility in the market, namely, the relative variation of the stocks in the market. This gives the interpretation of the excess growth rate of the market portfolio, as a measure of the market's available "intrinsic" volatility, i.e., the intrinsic volatility available in the market at any given time. A total return of a slightly different kind is put forth in Section 2.13, that being what is referred to as the alternative total return of a portfolio which signifies the return process of a portfolio in which the dividends of each stock are reinvested in the exact same stock. We shall examine how this framework extends to portfolio optimisation in Section 2.14. In addition, we shall contrast the classical portfolio optimisation approach inspired by Markowitz (1952) to the portfolio optimisation approach adopting stochastic portfolio theory, i.e., the stochastic portfolio optimisation approach. This entire chapter is concluded with a summary and conclusion in Section 2.15.

2.2 The Basic Equity Market Model

2.2.1 Stocks

Stochastic portfolio theory differs from the conventional theories of mathematical finance in that it makes use of the logarithmic representation for stocks and portfolios rather than the usual arithmetic representation. This idea was first presented by Fernholz & Shay (1982). These two perspectives are essentially one and the same, yet differ in their portrayal of certain aspects of portfolio behaviour. It is precisely this difference that makes the logarithmic perspective much more appealing, in particular to long-term investors, as we shall see at a later stage. Thus, stochastic portfolio theory considers *logarithmic* returns and therefore focuses on what is termed the **growth rate**, sometimes referred to as the geometric rate of return or the logarithmic rate of return. The logarithmic return is the alternative to the arithmetic return (or simply, the rate of return), and it sometimes yields a clearer picture of asset behaviour than is available from the usual rate of return. Also, the logarithmic model is advantageous for analysing long-term events, because the log-price processes resemble ordinary linear random walks rather than the exponential random walks of the standard representation. The growth rate intuitively explains long-term investing, in fact, it will be shown that the growth rate of a portfolio determines the long-term behaviour of the portfolio value. Since our primary concern is long-term investing, it is the growth rate and not the rate of return that is of interest to us and to long-term investors. In fact, the rate of return is shown to become irrelevant over time. Unlike the rate of return, the growth rate of a portfolio does not *only* depend on the growth rates of the component stocks in the portfolio, but also on an additional component, known as the **excess growth rate**. The excess growth rate is determined by the stocks' variances and covariances. This addition suggests portfolio optimisation in which the covariances of the stocks play a greatly increased role. This has practical benefit, since the variances and covariances are more amenable to statistical analysis than are the rates of return.

Remark 2.2.1. The use of the logarithmic model does in no way imply a preference for the logarithmic utility function. Utility functions are a minor feature of stochastic portfolio theory, indeed the framework of stochastic portfolio theory is not concerned with the notion of expected utility maximisation at all.

Now we present a definition that allows us to place ourselves in a financial equity market model consisting of n stocks, whose price processes $X_1(t), \ldots, X_n(t)$, at some time $t \in [0, \infty)$, are driven by the $d-$dimensional standard Brownian motion W. Contrary to the usual assumption imposed on such models, here it is not crucial that the filtration $\mathbb{F} = \{\mathcal{F}_t, t \in [0, \infty)\}$ be the one generated by the Brownian motion itself. Thus, and until further notice, we shall take $\mathbb{F}$ to contain the Brownian motion filtration $\mathbb{F}^W = \left\{\mathcal{F}_t^W, t \in [0, \infty)\right\}$ [Fernholz & Karatzas (2009)].

Definition 2.2.2 (Stock Price Process with $d-$dimensional Brownian motion)**.** *Let d be a positive integer, $d \in \mathbb{N}$ where $d \geq 2$. A **stock price process** $X = \{X(t), t \in [0, \infty)\}$ is a process that satisfies*

$$d \log X(t) \quad = \quad \gamma(t)\, dt + \sum_{\nu=1}^{d} \xi_\nu(t)\, dW_\nu(t), \qquad t \in [0, \infty), \tag{2.2.1}$$

where $W(t) = \big(W_1(t), W_2(t), \ldots, W_d(t)\big)$ is a $d-$dimensional standard Brownian motion, $\gamma = \{\gamma(t), t \in [0, \infty)\}$ is a measurable and adapted process, and is of bounded variation and thus satisfies[2]

$$\int_0^T \big|\gamma(t)\big|\, dt \quad < \quad \infty, \qquad \textit{for all } T \in [0, \infty), \qquad \textit{a.s.}$$

The $(1 \times d)-$vector-valued process $\boldsymbol{\xi} = \{\boldsymbol{\xi}(t) = \big(\xi_1(t), \ldots, \xi_d(t)\big), t \in [0, \infty)\}$, i.e., $\xi_\nu = \{\xi_\nu(t), t \in [0, \infty)\}$, $\nu = 1, 2, \ldots, d$, are also measurable and adapted processes that satisfy

$$(i) \quad \int_0^T \big(\xi_1^2(t) + \cdots + \xi_d^2(t)\big)\, dt \quad < \quad \infty, \qquad T \in [0, \infty), \qquad \textit{a.s.;}$$

$$(ii) \quad \lim_{t \to \infty} \frac{1}{t} \big(\xi_1^2(t) + \cdots + \xi_d^2(t)\big) \log \log t \quad = \quad 0, \qquad \textit{a.s.,} \quad \textit{and;}$$

$$(iii) \quad \xi_1^2(t) + \cdots + \xi_d^2(t) \quad > \quad 0, \qquad t \in [0, \infty), \qquad \textit{a.s.}$$

However, as previously stated and for the purposes of this dissertation, we shall take $d = n$. Thus, we shall redefine the stock price process so that it is governed by an $n-$dimensional standard Brownian motion rather than by the $d-$dimensional counterpart. Reasons for this will be made more apparent as we progress through the dissertation.

Definition 2.2.3 (Stock Price Process)**.** *Let n be a positive integer, $n \in \mathbb{N}$ where $n \geq 2$. A **stock price process** $X = \{X(t), t \in [0, \infty)\}$ is a process that satisfies*

$$d \log X(t) \quad = \quad \gamma(t)\, dt + \sum_{\nu=1}^{n} \xi_\nu(t)\, dW_\nu(t), \qquad t \in [0, \infty), \tag{2.2.2}$$

where $W(t) = \big(W_1(t), W_2(t), \ldots, W_n(t)\big)$ is an $n-$dimensional standard Brownian motion, $\gamma = \{\gamma(t), t \in [0, \infty)\}$ is a measurable and adapted process, and is of bounded variation and thus satisfies

$$\int_0^T \big|\gamma(t)\big|\, dt \quad < \quad \infty, \qquad \textit{for all } T \in [0, \infty), \qquad \textit{a.s.}$$

The $(1 \times n)-$vector-valued process $\boldsymbol{\xi} = \big\{\boldsymbol{\xi}(t) = \big(\xi_1(t), \ldots, \xi_n(t)\big), t \in [0, \infty)\big\}$, i.e., $\xi_\nu = \{\xi_\nu(t), t \in [0, \infty)\}$, $\nu = 1, 2, \ldots, n$, are also measurable and adapted processes that satisfy

$$(i) \quad \int_0^T \big(\xi_1^2(t) + \cdots + \xi_n^2(t)\big)\, dt \quad < \quad \infty, \qquad T \in [0, \infty), \qquad \textit{a.s.;}$$

$$(ii) \quad \lim_{t \to \infty} \frac{1}{t} \big(\xi_1^2(t) + \cdots + \xi_n^2(t)\big) \log \log t \quad = \quad 0, \qquad \textit{a.s.,} \quad \textit{and;}$$

$$(iii) \quad \xi_1^2(t) + \cdots + \xi_n^2(t) \quad > \quad 0, \qquad t \in [0, \infty), \qquad \textit{a.s.}$$

Equations (2.2.1) and (2.2.2) are both referred to as the logarithmic representation of the stock price process.

Remark 2.2.4. The integrability conditions above admit a rich class of continuous-path Itô processes, which have very general distributions. In particular, no Markovian or Gaussian assumption is imposed. This characteristic allows for the extension of this theory to very general semimartingale settings.

[2] a.s. is notation for "almost surely" which indicates that an event occurs with probability one, as measured by $\mathbb{P}$.

Given that $X(0) := X_0 > 0$ signifies the initial value of the stock, integrating equation (2.2.2) directly yields the following integral form

$$\log X(t) = \log X_0 + \int_0^t \gamma(s)\,ds + \int_0^t \sum_{\nu=1}^n \xi_\nu(s)\,dW_\nu(s), \qquad t \in [0,\infty). \tag{2.2.3}$$

Hence, in its exponential form, the stock price process can be expressed as

$$X(t) = X_0 \exp\left(\int_0^t \gamma(s)\,ds + \int_0^t \sum_{\nu=1}^n \xi_\nu(s)\,dW_\nu(s)\right), \qquad t \in [0,\infty) \tag{2.2.4}$$

$$= x \exp\left(\int_0^t \gamma(s)\,ds + \int_0^t \sum_{\nu=1}^n \xi_\nu(s)\,dW_\nu(s)\right), \qquad t \in [0,\infty). \tag{2.2.5}$$

In Definition 2.2.3, $X(t)$ represents the price of the stock at time $t \geq 0$, and it follows from equation (2.2.4) that $X(t) > 0$ for all $t \in [0,\infty)$. Also, the stock X has initial value $X(0)$, given by X_0 (where X_0 is a positive constant), i.e., $X(0) = X_0 = x > 0$. The assumptions listed at the inception of this chapter mentioned that each company has a single share of stock outstanding. In accordance with this assumption, the stock price $X(t)$ at time t also represents the total capitalisation of the company at time t. Thus, the initial value of the stock is also the initial value of the total capitalisation of the company.

In the stochastic differential equation (2.2.2), the first component $\gamma(t)\,dt$ is the deterministic component of the stock price process and the process γ is called the *growth rate process* of X, whereas the second component in equation (2.2.2) $\sum_{\nu=1}^n \xi_\nu(t)\,dW_\nu(t)$ is the stochastic component. The ξ_ν are called the *volatility processes* of X. Again, the components of W can be viewed as the n sources of uncertainty or "noise" factors. Thus, the process ξ_ν represents the sensitivity of the stock X to the νth source of uncertainty or "noise" component, given by the νth Brownian motion component W_ν.

- **Condition (i)** of Definition 2.2.3 ensures that the variance of $X(t)$ at time t is a.s. of bounded variation, i.e.,

$$\int_0^t \sum_{\nu=1}^n \xi_\nu(t)\,dW_\nu(t) < \infty.$$

- **Condition (ii)** guarantees that the volatility of the stock does not increase too quickly so as to render meaningless the growth rate of the stock, i.e., the impact of the volatility diminishes relative to the growth rate.
- **Condition (iii)** assures that the variance of $X(t)$ at time t is nondegenerate, i.e., there exists $\nu = 1, 2, \ldots, n$, such that $\xi_\nu(t) > 0$, for all $t \in [0,\infty)$. This condition simply implies that at least one of the ξ_ν must be positive over the entire time domain $[0,\infty)$.

We now demonstrate the link between this logarithmic representation to the one adopted in classical asset pricing theory. The arithmetic rate of return used in the standard financial equity market model of classical asset pricing theory was usually considered, instead here the growth rate used in the logarithmic equity market model will be of focus.

Corollary 2.2.5 ([**Fernholz (2002)**]). *The stock price process $X = \{X(t), t \in [0,\infty)\}$ satisfies*

$$dX(t) = \alpha(t)\,X(t)\,dt + X(t)\sum_{\nu=1}^n \xi_\nu(t)\,dW_\nu(t), \qquad t \in [0,\infty), \tag{2.2.6}$$

$$= X(t)\left[\alpha(t)\,dt + \sum_{\nu=1}^n \xi_\nu(t)\,dW_\nu(t)\right], \qquad t \in [0,\infty), \tag{2.2.7}$$

where $W(t) = \big(W_1(t), W_2(t), \ldots, W_n(t)\big)$ is an $n-$dimensional standard Brownian motion, $\alpha = \{\alpha(t), t \in [0,\infty)\}$ is a measurable and adapted process that satisfies

$$\alpha(t) = \gamma(t) + \frac{1}{2}\sum_{\nu=1}^n \xi_\nu^2(t), \qquad t \in [0,\infty). \tag{2.2.8}$$

Moreover, the instantaneous rate of return (or, arithmetic return) for the classical equity market model is given by

$$\frac{dX(t)}{X(t)} = \alpha(t)\,dt + \sum_{\nu=1}^{n} \xi_\nu(t)\,dW_\nu(t), \qquad t \in [0,\infty). \tag{2.2.9}$$

Proof. From equation (2.2.2), we immediately notice that $\log X$ is a *continuous semimartingale* (see Appendix A) with bounded variation component

$$\gamma(t)\,dt, \qquad t \in [0,\infty),$$

and local martingale component

$$\sum_{\nu=1}^{n} \xi_\nu(t)\,dW_\nu(t), \qquad t \in [0,\infty).$$

Alternatively, from the equivalent integral form (2.2.3), $\log X$ has a bounded variation component given in integral form by

$$\int_0^t \gamma(s)\,ds, \qquad t \in [0,\infty),$$

and a local martingale component given in integral form by

$$\int_0^t \sum_{\nu=1}^{n} \xi_\nu(s)\,dW_\nu(s), \qquad t \in [0,\infty).$$

Since we know the dynamics for the process $\log X$, given by equation (2.2.2), we can apply Itô's formula (see Appendix B) to $X = \exp\big(\log X(t)\big)$. Thus, by setting $Y(t) := \log X(t)$, the form of the function to be used in Itô's formula is given by $F\big(t, Y(t)\big) = \exp\big(Y(t)\big)$, and the following are easily obtained

$$\begin{aligned}
\frac{\partial F}{\partial t}(t,y) &= 0, \quad \text{and,} \\
\frac{\partial F}{\partial y}(t,y) &= \exp\big(Y(t)\big), \quad \text{and,} \\
\frac{\partial^2 F}{\partial y^2}(t,y) &= \exp\big(Y(t)\big).
\end{aligned}$$

We then arrive at the following, for $t \in [0,\infty)$, a.s.,

$$dF\big(t, Y(t)\big) = \exp\big(Y(t)\big)\,dY(t) + \frac{1}{2}\exp\big(Y(t)\big)\,d\langle Y\rangle_t.$$

Since $Y(t) = \log X(t)$, a.s., for $t \in [0,\infty)$, the following formula for dX is obtained

$$dX(t) = X(t)\,d\log X(t) + \frac{1}{2}X(t)\,d\langle \log X\rangle_t. \tag{2.2.10}$$

Consequently, for the instantaneous rate of return, for $t \in [0,\infty)$, we a.s. have

$$\frac{dX(t)}{X(t)} = d\log X(t) + \frac{1}{2}\,d\langle \log X\rangle_t. \tag{2.2.11}$$

Suppose that X and Y are two real-valued continuous functions, then $\langle X, Y\rangle$ denotes their *cross-variation*, while $\langle X\rangle \triangleq \langle X, X\rangle$ is the *quadratic variation* function for X. The differential of the quadratic variation process

$d\langle \log X\rangle$ in equation (2.2.10) can be determined from (2.2.3) as follows[3]

$$\begin{aligned}
\langle \log X\rangle_t &= \left\langle \int_0^{\cdot} \sum_{\nu=1}^{n} \xi_{\nu,s}\, dW_{\nu,s} \right\rangle_t \\
&= \left\langle \sum_{\nu=1}^{n} \int_0^{\cdot} \xi_{\nu,s}\, dW_{\nu,s} \right\rangle_t \\
&= \sum_{\nu=1}^{n} \left\langle \int_0^{\cdot} \xi_{\nu,s}\, dW_{\nu,s} \right\rangle_t \\
&= \sum_{\nu=1}^{n} \int_0^{t} \xi_{\nu}^2(s)\, d\langle W_\nu\rangle_s \\
&= \int_0^{t} \sum_{\nu=1}^{n} \xi_{\nu}^2(s)\, ds. \qquad (2.2.12)
\end{aligned}$$

Thus,

$$\begin{aligned}
d\langle \log X\rangle_t &= d\left\langle \int_0^{\cdot} \sum_{\nu=1}^{n} \xi_{\nu,s}\, dW_{\nu,s} \right\rangle_t \\
&= d\left(\int_0^{t} \sum_{\nu=1}^{n} \xi_{\nu}^2(s)\, ds \right) \\
&= \sum_{\nu=1}^{n} \xi_{\nu}^2(t)\, dt. \qquad (2.2.13)
\end{aligned}$$

By applying a simple substitution of equations (2.2.2) and (2.2.13) into equation (2.2.10), we obtain a.s., for $t \in [0,\infty)$, the dynamics for the stock price process as follows

$$\begin{aligned}
dX(t) &= \left(\gamma(t)\, dt + \sum_{\nu=1}^{n} \xi_\nu(t)\, dW_\nu(t) \right) X(t) + \frac{1}{2} X(t) \sum_{\nu=1}^{n} \xi_\nu^2(t)\, dt \\
&= \left(\gamma(t) + \frac{1}{2} \sum_{\nu=1}^{n} \xi_\nu^2(t) \right) X(t)\, dt + X(t) \sum_{\nu=1}^{n} \xi_\nu(t)\, dW_\nu(t) \qquad (2.2.14) \\
&= X(t) \left[\left(\gamma(t) + \frac{1}{2} \sum_{\nu=1}^{n} \xi_\nu^2(t) \right) dt + \sum_{\nu=1}^{n} \xi_\nu(t)\, dW_\nu(t) \right]. \qquad (2.2.15)
\end{aligned}$$

Since (2.2.14) can be resolved into two components, a finite variation component and a local martingale component, the process X is a *continuous semimartingale* whose evolution is governed by equation (2.2.14). Let the term in parentheses, that is revealed in the finite variation component of (2.2.14), be represented by the process $\alpha = \{\alpha(t), t \in [0,\infty)\}$, i.e., define

$$\alpha(t) \triangleq \gamma(t) + \frac{1}{2} \sum_{\nu=1}^{n} \xi_\nu^2(t), \qquad (2.2.16)$$

where α signifies the **rate of return process** of the stock X. Equation (2.2.16) relates the growth rate process to the rate of return process, adjusted by a variance component. By defining α in this fashion, the following standard representation for the stock price process, used within the classical asset pricing domain, is established and is represented as

$$\begin{aligned}
dX(t) &= \alpha(t)\, X(t)\, dt + X(t) \sum_{\nu=1}^{n} \xi_\nu(t)\, dW_\nu(t), \qquad t \in [0,\infty), \qquad \text{a.s.}, \qquad (2.2.17) \\
&= X(t) \left[\alpha(t)\, dt + \sum_{\nu=1}^{n} \xi_\nu(t)\, dW_\nu(t) \right], \qquad t \in [0,\infty), \qquad \text{a.s.} \qquad (2.2.18)
\end{aligned}$$

[3]See Appendix A, where it is stated that the Brownian motion processes are characterised by their cross-variation processes $\langle W_\nu, W_\upsilon\rangle_t = \rho_{\nu\upsilon} t$, where $\rho_{\nu\upsilon} = 1$ if $\nu = \upsilon$ and 0 otherwise. Thus, $d\langle W_\nu, W_\upsilon\rangle_t = dW_\nu(t) dW_\upsilon(t) = 0$ if $\nu \neq \upsilon$, and the quadratic variation process of Brownian motion reveals that $d\langle W_\nu, W_\upsilon\rangle_t = d\langle W_\nu\rangle_t = \left(dW_\nu(t)\right)^2 = dt$ if $\nu = \upsilon$. Furthermore, we know that $dW_\nu(t) dt = 0$ for all $t \in [0,\infty)$.

Equation (2.2.17) can be written in the following more explicable manner as

$$\frac{dX(t)}{X(t)} = \alpha(t)\,dt + \sum_{\nu=1}^{n} \xi_\nu(t)\,dW_\nu(t), \qquad t \in [0,\infty), \qquad \text{a.s,} \tag{2.2.19}$$

where $\frac{dX}{X}$ can be interpreted as the "instantaneous" return on the stock X, more precisely, the instantaneous *arithmetic* return. ■

In a discrete time setting, the arithmetic return for an infinitesimally small time horizon dt, is given by

$$\mu^{\mathsf{arith}}(t, t+dt) = \frac{X(t+dt) - X(t)}{X(t)} = \frac{dX(t)}{X(t)}.$$

In a similar manner, the expression $d\log X$ can be construed as the instantaneous *logarithmic* return (also, log-return), or *geometric* return on the stock X. The logarithmic (geometric) return, in a discrete time setting, is given by

$$\mu^{\mathsf{geom}}(t, t+dt) = \log\left(\frac{X(t+dt)}{X(t)}\right) = \log X(t+dt) - \log X(t) = d\log X(t).$$

The sum of the log-returns over two consecutive time intervals is the log-return over the union of the intervals. This, however, is not the case for the arithmetic return. Therefore, the log-return lends itself more to the evaluation of the long-term behaviour of stocks. Indeed, the growth rate of the stock, as will be shown later, is a more appropriate tool for analysing long-term trends of the stock. Thus, for our purposes, we shall consider the growth rate process γ and not the rate of return process α. Thus, the dynamics of the stock given by (2.2.2) (i.e., the logarithmic return), rather than the dynamics of the stock given by (2.2.19) (i.e., the arithmetic return), will be of primary concern to us in the type of analysis that we shall perform.

2.2.2 The General Equity Market Model

It is here that we shall set up the financial equity market model to be used throughout in our investigation. We shall place ourselves in the standard Itô process model for a financial market which goes back to Samuelson (1965).

2.2.2.1 The Basic Logarithmic Equity Market Model (with n Sources of Uncertainty)

Suppose that we have a family of n non-dividend paying stocks, each represented by their *stock price processes* $X_i = \{X_i(t), t \in [0,\infty)\}$, for $i = 1, 2, \ldots, n$, that are defined in differential form as per (2.2.2)

$$d\log X_i(t) = \gamma_i(t)\,dt + \sum_{\nu=1}^{n} \xi_{i\nu}(t)\,dW_\nu(t), \qquad t \in [0,\infty). \tag{2.2.20}$$

The above equation (2.2.20) can be integrated to obtain the following integral form

$$\log X_i(t) = \log X_i(0) + \int_0^t \gamma_i(s)\,ds + \int_0^t \sum_{\nu=1}^{n} \xi_{i\nu}(s)\,dW_\nu(s), \qquad t \in [0,\infty). \tag{2.2.21}$$

Equivalently, in exponential form as

$$X_i(t) = x_i\,\exp\left(\int_0^t \gamma_i(s)\,ds + \int_0^t \sum_{\nu=1}^{n} \xi_{i\nu}(s)\,dW_\nu(s)\right), \qquad t \in [0,\infty), \tag{2.2.22}$$

where $X_i(0) = x_i > 0$ is the initial value of the ith stock (alternatively expressed as X_i^0, which is the convention that Fernholz (2002) adopts). The stock prices $X_1, \ldots, X_n$ are driven by the standard $n-$dimensional Brownian motion $W(t) = \big(W_1(t), \ldots, W_n(t)\big)$. The quantity $X_i(t)$ represents the price of the ith stock or asset at time

$t \in [0, \infty)$. Once again, it is assumed that each company has exactly one share of stock outstanding, i.e., there is a single share of each stock, so that $X_i(t)$ can also be interpreted as the total capitalisation of the ith company at time $t \in [0, \infty)$.

In equation (2.2.20), $d \log X_i$ represents the log-return (the logarithmic or geometric rate of return) of X_i over the infinitesimal time period dt. We shall also assume that the $(1 \times n)-$vector-valued process $\boldsymbol{\gamma} = \big\{\boldsymbol{\gamma}(t) = \big(\gamma_1(t), \ldots, \gamma_n(t)\big), t \in [0, \infty)\big\}$, of growth rates for the various stocks (i.e., the process $\gamma_i = \{\gamma_i(t), t \in [0, \infty)\}$ is called the *growth rate process* for the ith stock X_i and $\boldsymbol{\gamma}$ is the $n-$dimensional row vector of growth rates), and the $(n \times n)-$matrix-valued process[4] $\boldsymbol{\xi} = \big\{\boldsymbol{\xi}(t) = \big(\xi_{i\nu}(t)\big)_{1 \le i, \nu \le n}, t \in [0, \infty)\big\}$, of stock price volatilities (i.e., the process $\xi_{i\nu} = \{\xi_{i\nu}(t), t \in [0, \infty)\}$ is the *volatility process* of the ith stock X_i with respect to the νth source of uncertainty), are all $\mathbb{F}-$progressively measurable and adapted processes that satisfy the following integrability conditions, for each $i = 1, 2, \ldots, n$,

$$\text{(i)} \quad \int_0^T \big(\xi_{i1}^2(t) + \cdots + \xi_{in}^2(t)\big)\, dt \;=\; \int_0^T \sum_{\nu=1}^n \xi_{i\nu}^2(t)\, dt \;<\; \infty, \qquad T \in [0, \infty), \qquad \text{a.s.;}$$

$$\text{(ii)} \quad \lim_{t \to \infty} \frac{1}{t} \big(\xi_{i1}^2(t) + \cdots + \xi_{in}^2(t)\big) \log \log t \;=\; \lim_{t \to \infty} \frac{1}{t} \left(\sum_{\nu=1}^n \xi_{i\nu}^2(t) \right) \log \log t \;=\; 0, \qquad \text{a.s.;}$$

$$\text{(iii)} \quad \xi_{i1}^2(t) + \cdots + \xi_{in}^2(t) \;=\; \sum_{\nu=1}^n \xi_{i\nu}^2(t) \;>\; 0, \qquad t \in [0, \infty), \qquad \text{a.s., and;}$$

$$\text{(iv)} \quad \int_0^T \big|\gamma_i(t)\big|\, dt \;<\; \infty, \qquad \text{for all } T \in [0, \infty), \qquad \text{a.s.}$$

Thus, in accordance with the aforementioned integrability conditions, the following expressions must also hold

$$\int_0^T \sum_{i=1}^n \big|\gamma_i(t)\big|\, dt \;<\; \infty, \qquad \text{for all } T \in [0, \infty), \qquad \text{a.s., and,} \tag{2.2.23}$$

$$\int_0^T \sum_{i,\nu=1}^n \xi_{i\nu}^2(t)\, dt \;<\; \infty, \qquad \text{for all } T \in [0, \infty), \qquad \text{a.s.} \tag{2.2.24}$$

Karatzas (2006) expresses these integrability conditions in the following compact form

$$\int_0^T \sum_{i=1}^n \left(\big|\gamma_i(t)\big| + \sum_{\nu=1}^n \xi_{i\nu}^2(t) \right) dt \;<\; \infty, \qquad \text{for all } T \in [0, \infty), \qquad \text{a.s., or,} \tag{2.2.25}$$

$$\sum_{i=1}^n \int_0^T \left(\big|\gamma_i(t)\big| + \sum_{\nu=1}^n \xi_{i\nu}^2(t) \right) dt \;<\; \infty, \qquad \text{for all } T \in [0, \infty), \qquad \text{a.s.} \tag{2.2.26}$$

The second integrability condition imposed above (2.2.24) is a consequence of the bounded market variance condition and stipulates that the volatility coefficients are square integrable processes.

The logarithmic representation (2.2.20) for each stock in the market is quite general and, as we have already shown, it is related to the usual arithmetic representation commonly used in mathematical finance where the price of each stock is described by the dynamics $dX_i(t) = \alpha_i(t)\, X_i(t)\, dt + X_i(t) \sum_{\nu=1}^n \xi_{i\nu}(t)\, dW_\nu(t)$ for $i = 1, 2, \ldots, n$ [see, for example, Karatzas & Shreve (1998)]. The arithmetic representation is a fairly standard model for the stock price processes and, when only one source of uncertainty is permitted, this equity market model is the familiar **geometric Brownian motion** model for the stock price process, $dX_i(t) = \alpha_i(t)\, X_i(t)\, dt + \sigma_i(t)\, X_i(t)\, dW(t)$, for $i = 1, 2, \ldots, n$. Furthermore, we use the logarithmic representation because it brings to light certain aspects of portfolio behaviour that remain obscure with the conventional arithmetic representation [Fernholz (2005)].

[4] This matrix is symmetric, so that $\boldsymbol{\xi}(t) = \boldsymbol{\xi}^{\mathsf{T}}(t)$.

From the integral form (2.2.21), $\log X_i$ for all $i = 1, 2, \ldots, n$, has a bounded variation component given in integral form by

$$\int_0^t \gamma_i(s)\, ds, \qquad t \in [0, \infty),$$

and a local martingale component given in integral form by

$$\int_0^t \sum_{\nu=1}^{n} \xi_{i\nu}(s)\, dW_\nu(s), \qquad t \in [0, \infty).$$

Remark 2.2.6. Note that, in equation (2.2.20), the number of stocks in the market is equal to the dimension of the Brownian motion process $W = \big\{W(t) = \big(W_1(t), \ldots, W_n(t)\big), \mathcal{F}_t, t \in [0, \infty)\big\}$ driving the stocks, i.e., defining d as the dimension of the Brownian motion process, we have $d = n$. However, in general, we just need to have at least as many sources of uncertainty in the market as there are stocks, i.e., $d \geq n$. Throughout this dissertation, we shall consider the case where $d = n$. In this regard, the dimension n is chosen to be large enough so as to avoid any unnecessary dependencies among the stocks we define. However, for the sake of completeness and generality, presented below is the model of the financial market where there are d sources of uncertainty, where $d \geq n$.

2.2.2.2 The General Logarithmic Equity Market Model (with d Sources of Uncertainty)

Suppose, in this particular case, that we have a family of n non-dividend paying stocks $X_i = \{X_i(t), t \in [0, \infty)\}$, for $i = 1, 2, \ldots, n$, that are defined in differential form as per (2.2.2), but now with d independent sources of uncertainty, i.e., we have $d, n \in \mathbb{N}$ with $d \geq n \geq 2$,

$$d \log X_i(t) \quad = \quad \gamma_i(t)\, dt + \sum_{\nu=1}^{d} \xi_{i\nu}(t)\, dW_\nu(t), \qquad t \in [0, \infty). \tag{2.2.27}$$

Thus, integrating directly we arrive at the following

$$\log X_i(t) \quad = \quad \log X_i(0) + \int_0^t \gamma_i(s)\, ds + \int_0^t \sum_{\nu=1}^{d} \xi_{i\nu}(s)\, dW_\nu(s), \qquad t \in [0, \infty). \tag{2.2.28}$$

Equivalently, in exponential form as

$$X_i(t) \quad = \quad x_i \, \exp\left(\int_0^t \gamma_i(s)\, ds + \int_0^t \sum_{\nu=1}^{d} \xi_{i\nu}(s)\, dW_\nu(s) \right), \qquad t \in [0, \infty), \tag{2.2.29}$$

where $X_i(0) = x_i > 0$ is the initial value of the ith stock. The stock prices $X_1, \ldots, X_n$ are driven by the standard $d-$dimensional Brownian motion $W(t) = \big(W_1(t), \ldots, W_d(t)\big)$ with $d \geq n$. We shall assume that the $(1 \times n)-$vector-valued process $\boldsymbol{\gamma} = \big\{\boldsymbol{\gamma}(t) = \big(\gamma_1(t), \ldots, \gamma_n(t)\big), t \in [0, \infty)\big\}$, of growth rates for the various stocks, and the $(n \times d)-$matrix-valued process $\boldsymbol{\xi} = \big\{\boldsymbol{\xi}(t) = \big(\xi_{i\nu}(t)\big)_{1 \leq i \leq n, 1 \leq \nu \leq d}, t \in [0, \infty)\big\}$, of stock price volatilities, are all $\mathbb{F}-$progressively measurable and adapted processes that satisfy the following integrability conditions, for each $i = 1, 2, \ldots, n$,

(i) $$\int_0^T \big(\xi_{i1}^2(t) + \cdots + \xi_{id}^2(t)\big)\, dt \quad = \quad \int_0^T \sum_{\nu=1}^{d} \xi_{i\nu}^2(t)\, dt \quad < \quad \infty, \qquad T \in [0, \infty), \qquad \text{a.s.;}$$

(ii) $$\lim_{t \to \infty} \frac{1}{t} \big(\xi_{i1}^2(t) + \cdots + \xi_{id}^2(t)\big) \log \log t \quad = \quad \lim_{t \to \infty} \frac{1}{t} \left(\sum_{\nu=1}^{d} \xi_{i\nu}^2(t) \right) \log \log t \quad = \quad 0, \qquad \text{a.s.;}$$

(iii) $$\xi_{i1}^2(t) + \cdots + \xi_{id}^2(t) \quad = \quad \sum_{\nu=1}^{d} \xi_{i\nu}^2(t) \quad > \quad 0, \qquad t \in [0, \infty), \qquad \text{a.s., and;}$$

(iv) $\displaystyle\int_0^T \big|\gamma_i(t)\big|\,dt \quad < \quad \infty, \qquad \text{for all } T \in [0,\infty), \qquad \text{a.s.}$

Thus, in accordance with the aforementioned integrability conditions, the following expressions must also hold

$$\int_0^T \sum_{i=1}^n \big|\gamma_i(t)\big|\,dt \quad < \quad \infty, \qquad \text{for all } T \in [0,\infty), \qquad \text{a.s., and,} \tag{2.2.30}$$

$$\int_0^T \sum_{i=1}^n \sum_{\nu=1}^d \xi_{i\nu}^2(t)\,dt \quad < \quad \infty, \qquad \text{for all } T \in [0,\infty), \qquad \text{a.s.} \tag{2.2.31}$$

From the integral form (2.2.28), $\log X_i$ for $i = 1, 2, \ldots, n$, has a bounded variation component given in integral form by

$$\int_0^t \gamma_i(s)\,ds, \qquad t \in [0,\infty),$$

and a local martingale component given in integral form by

$$\int_0^t \sum_{\nu=1}^d \xi_{i\nu}(s)\,dW_\nu(s), \qquad t \in [0,\infty).$$

Remark 2.2.7. Note that we have included no **riskless asset** (e.g. the money market account) in the model of the financial market. The reason for this exclusion is simply that our purpose here is to study the behaviour of stocks and stock portfolios, and thus, the existence of a riskless asset is irrelevant. For this reason, we shall only concern ourselves with the analysis of risky assets. However, if one chooses to include such an asset, one does so by incorporating the following **money-market account process** $B = \{B(t), t \in [0,\infty)\}$ into the financial market model, as follows

$$dB(t) \quad = \quad r(t)B(t)\,dt, \qquad B(0) \quad = \quad 1. \tag{2.2.32}$$

The process $r = \{r(t), t \in [0,\infty)\}$ is the continuously compounded risk-free interest rate process for the money market account and is assumed to be an $\mathbb{F}-$progressively measurable and adapted process that satisfies the following integrability condition [refer to Fernholz & Karatzas (2009)],

$$\int_0^T \big|r(t)\big|\,dt \quad < \quad \infty, \qquad \text{for all } T \in [0,\infty), \qquad \text{a.s.} \tag{2.2.33}$$

In which case, the integrability condition (2.2.26) becomes

$$\int_0^T \big|r(t)\big|\,dt + \sum_{i=1}^n \int_0^T \left(\big|\gamma_i(t)\big| + \sum_{\nu=1}^n \xi_{i\nu}^2(t)\right) dt \quad < \quad \infty, \qquad \text{for all } T \in [0,\infty), \qquad \text{a.s.} \tag{2.2.34}$$

2.2.2.3 The Covariance Process

To evaluate how well-behaved the stock prices are, we study their variability through the induced covariance matrix. The covariance matrix is primarily used to measure the risk of achieving a particular goal inherent to the stocks as a whole. This leads us to the following definition of the **covariance process**.

Definition 2.2.8 (Covariance Process)**.** *Consider the matrix-valued process of the volatility processes* $\boldsymbol{\xi}$ *given by* $\boldsymbol{\xi}(t) = \big(\xi_{i\nu}(t)\big)_{1\le i,\nu\le n}$. *The* ***covariance process*** $\boldsymbol{\sigma} = \big\{\boldsymbol{\sigma}(t) = \big(\sigma_{ij}(t)\big)_{1\le i,j\le n}, t \in [0,\infty)\big\}$ *is defined by*

$$\boldsymbol{\sigma}(t) \quad \triangleq \quad \boldsymbol{\xi}(t)\boldsymbol{\xi}^{\mathsf{T}}(t), \qquad t \in [0,\infty). \tag{2.2.35}$$

Therefore, $\boldsymbol{\sigma}(t) = \big(\sigma_{ij}(t)\big)_{1\le i,j\le n}$ denotes the $(n \times n)-$matrix-valued covariance process of the stocks in the market, i.e., the covariance of the ith stock X_i and the jth stock X_j, for all $i, j = 1, \ldots, n$. More specifically,

it represents how the ith stock X_i varies relative to the jth stock X_j. Furthermore, let $\mathbf{e}_i$ denote the ith unit vector in $\mathbb{R}^n$, i.e., $\mathbf{e}_i \triangleq (0,\ldots,0,1,0,\ldots,0)$, where the ith coordinate is 1 and the other $n-1$ coordinates are all 0, and let $\mathbf{e}_j$ denote the jth unit vector in $\mathbb{R}^n$. For all $i,j=1,2,\ldots,n$, the covariance process of X_i and X_j, $\sigma_{ij}=\big\{\sigma_{ij}(t), t\in[0,\infty)\big\}$, is represented in matrix form by

$$\sigma_{ij}(t) \quad = \quad \mathbf{e}_i\boldsymbol{\sigma}(t)\mathbf{e}_j^{\mathsf{T}}, \qquad t\in[0,\infty). \tag{2.2.36}$$

The following is apparent from equation (2.2.35) of the preceding definition

$$\sigma_{ij}(t) \quad = \quad \big(\boldsymbol{\sigma}(t)\big)_{ij} \quad = \quad \Big(\boldsymbol{\xi}(t)\boldsymbol{\xi}^{\mathsf{T}}(t)\Big)_{ij} \quad = \quad \sum_{\nu=1}^{n}\xi_{i\nu}(t)\xi_{j\nu}(t), \tag{2.2.37}$$

for $i,j=1,2,\ldots,n$, and $t\in[0,\infty)$. Since $\boldsymbol{\xi}(t)$ is an $n\times n$ symmetric matrix, the covariance matrix is also an $n\times n$ symmetric matrix, and for all $t\in[0,\infty)$, we have $\boldsymbol{\sigma}(t)=\boldsymbol{\sigma}^{\mathsf{T}}(t)$ or $\sigma_{ij}(t)=\sigma_{ji}(t)$, for all $i,j=1,2,\ldots,n$. This obviously implies that

$$\sigma_{ji}(t) \quad = \quad \sum_{\nu=1}^{n}\xi_{j\nu}(t)\xi_{i\nu}(t) \quad = \quad \sum_{\nu=1}^{n}\xi_{i\nu}(t)\xi_{j\nu}(t) \quad = \quad \sigma_{ij}(t).$$

2.2.2.4 Characteristics of the Covariance Process

For any $\mathbf{x}\in\mathbb{R}^n$ and $t\in[0,\infty)$, from (2.2.35) of Definition 2.2.8, we acquire the following

$$\mathbf{x}\boldsymbol{\sigma}(t)\mathbf{x}^{\mathsf{T}} \quad = \quad \mathbf{x}\boldsymbol{\xi}(t)\boldsymbol{\xi}^{\mathsf{T}}(t)\mathbf{x}^{\mathsf{T}} \quad = \quad \mathbf{x}\boldsymbol{\xi}(t)\big(\mathbf{x}\boldsymbol{\xi}(t)\big)^{\mathsf{T}} \quad = \quad \big\|\mathbf{x}\boldsymbol{\xi}(t)\big\|^2 \quad \geq \quad 0, \tag{2.2.38}$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$.

From above, we can conclude that the covariance matrix-valued process $\boldsymbol{\sigma}(t)$ is positive semidefinite[5] for all $t\in[0,\infty)$. For $i,j=1,2,\ldots,n$ and $i\neq j$, the covariance process of X_i and X_j, at time $t\in[0,\infty)$, is $\sigma_{ij}=\big\{\sigma_{ij}(t), t\in[0,\infty)\big\}$, while the process $\sigma_{ii}=\{\sigma_{ii}(t), t\in[0,\infty)\}$ is called the covariance process of the ith stock X_i, or more succinctly, the *variance process* of X_i. We shall provide a formal definition of this process shortly.

The quadratic variation processes for $\log X_i$, for all $i=1,2,\ldots,n$, can be obtained as follows

$$\begin{aligned}
\big\langle \log X_i\big\rangle_t &= \left\langle \int_0^{\cdot}\sum_{\nu=1}^{n}\xi_{i\nu,s}\,dW_{\nu,s}\right\rangle_t \\
&= \left\langle \sum_{\nu=1}^{n}\int_0^{\cdot}\xi_{i\nu,s}\,dW_{\nu,s}\right\rangle_t \\
&= \sum_{\nu=1}^{n}\left\langle \int_0^{\cdot}\xi_{i\nu,s}\,dW_{\nu,s}\right\rangle_t \\
&= \sum_{\nu=1}^{n}\int_0^{t}\xi_{i\nu}^2(s)\,d\big\langle W_\nu\big\rangle_s \\
&= \int_0^{t}\sum_{\nu=1}^{n}\xi_{i\nu}^2(s)\,ds.
\end{aligned} \tag{2.2.39}$$

Therefore, for $i=1,2,\ldots,n$, we have

$$\begin{aligned}
d\big\langle \log X_i\big\rangle_t &= d\left\langle \int_0^{\cdot}\sum_{\nu=1}^{n}\xi_{i\nu,s}\,dW_{\nu,s}\right\rangle_t \\
&= d\left(\int_0^{t}\sum_{\nu=1}^{n}\xi_{i\nu}^2(s)\,ds\right) \\
&= \sum_{\nu=1}^{n}\xi_{i\nu}^2(t)\,dt.
\end{aligned} \tag{2.2.40}$$

[5] A matrix $\mathbf{A}$ is said to be positive semidefinite if $\mathbf{x}\mathbf{A}\mathbf{x}^{\mathsf{T}}\geq 0$, for all $\mathbf{x}\in\mathbb{R}^n$.

Consider, now, the cross-variation processes for $\log X_i$ and $\log X_j$, for all $i, j = 1, 2, \ldots, n$. Employing equation (2.2.20), the cross-variation processes are related to $\boldsymbol{\sigma}(t)$ by

$$\begin{aligned}
\left\langle \log X_i, \log X_j \right\rangle_t &= \left\langle \int_0^{\cdot} \sum_{\nu=1}^{n} \xi_{i\nu,s}\, dW_{\nu,s}, \int_0^{\cdot} \sum_{\upsilon=1}^{n} \xi_{j\upsilon,s}\, dW_{\upsilon,s} \right\rangle_t \\
&= \left\langle \sum_{\nu=1}^{n} \int_0^{\cdot} \xi_{i\nu,s}\, dW_{\nu,s}, \sum_{\upsilon=1}^{n} \int_0^{\cdot} \xi_{j\upsilon,s}\, dW_{\upsilon,s} \right\rangle_t \\
&= \sum_{\nu,\upsilon=1}^{n} \left\langle \int_0^{\cdot} \xi_{i\nu,s}\, dW_{\nu,s}, \int_0^{\cdot} \xi_{j\upsilon,s}\, dW_{\upsilon,s} \right\rangle_t \\
&= \sum_{\nu,\upsilon=1}^{n} \int_0^{t} \xi_{i\nu}(s)\xi_{j\upsilon}(s)\, d\left\langle W_\nu, W_\upsilon \right\rangle_s \\
&= \int_0^{t} \sum_{\nu,\upsilon=1}^{n} \xi_{i\nu}(s)\xi_{j\upsilon}(s)\, \rho_{\nu\upsilon}(s)\, ds \\
&= \int_0^{t} \sum_{\nu=1}^{n} \xi_{i\nu}(s)\xi_{j\nu}(s)\, ds. \qquad (2.2.41)
\end{aligned}$$

Thus, from equation (2.2.37), the cross-variation processes can be expressed in the following form

$$\left\langle \log X_i, \log X_j \right\rangle_t = \int_0^{t} \sigma_{ij}(s)\, ds. \qquad (2.2.42)$$

Hence, for the differential of the cross-variation process, we have

$$\begin{aligned}
d\left\langle \log X_i, \log X_j \right\rangle_t &= d\left\langle \int_0^{\cdot} \sum_{\nu=1}^{n} \xi_{i\nu,s}\, dW_{\nu,s}, \int_0^{\cdot} \sum_{\upsilon=1}^{n} \xi_{j\upsilon,s}\, dW_{\upsilon,s} \right\rangle_t \\
&= d\left(\int_0^{t} \sum_{\nu=1}^{n} \xi_{i\nu}(s)\xi_{j\nu}(s)\, ds \right) \\
&= \sum_{\nu=1}^{n} \xi_{i\nu}(t)\xi_{j\nu}(t)\, dt. \qquad (2.2.43)
\end{aligned}$$

Consequently, the cross-variation process can be expressed in the form

$$d\left\langle \log X_i, \log X_j \right\rangle_t = \sigma_{ij}(t)\, dt. \qquad (2.2.44)$$

In conclusion, for $i, j = 1, \ldots, n$ and $t \in [0, \infty)$, we have

$$\sigma_{ij}(t)\, dt = d\left\langle \log X_i, \log X_j \right\rangle_t = \sum_{\nu=1}^{n} \xi_{i\nu}(t)\xi_{j\nu}(t)\, dt, \quad \text{and,} \qquad (2.2.45)$$

$$\sigma_{ij}(t) = \frac{d}{dt}\left\langle \log X_i, \log X_j \right\rangle_t = \sum_{\nu=1}^{n} \xi_{i\nu}(t)\xi_{j\nu}(t). \qquad (2.2.46)$$

2.2.2.5 The Variance Process

Definition 2.2.9 (Variance Process). *Consider the matrix-valued process of the volatility processes $\boldsymbol{\xi}$ given by $\boldsymbol{\xi}(t) = \left(\xi_{i\nu}(t)\right)_{1 \le i,\nu \le n}$. The **variance process** of X_i, $\sigma_{ii} = \{\sigma_{ii}(t), t \in [0, \infty)\}$, for $i = 1, 2, \ldots, n$, is defined by*

$$\sigma_{ii}(t) = \left(\boldsymbol{\sigma}(t)\right)_{ii} = \left(\boldsymbol{\xi}(t)\boldsymbol{\xi}^{\mathsf{T}}(t)\right)_{ii} = \sum_{\nu=1}^{n} \xi_{i\nu}^2(t), \qquad (2.2.47)$$

for $t \in [0, \infty)$, which are the diagonal elements of the covariance matrix, $\boldsymbol{\sigma}(t)$.

Furthermore, the variance process of X_i is represented in matrix form by

$$\sigma_{ii}(t) \quad = \quad \mathbf{e}_i \boldsymbol{\sigma}(t) \mathbf{e}_i^{\mathsf{T}}, \qquad t \in [0, \infty). \tag{2.2.48}$$

Therefore, from (2.2.45), for all $i = 1, 2, \ldots, n$ and $t \in [0, \infty)$, we have

$$\sigma_{ii}(t)\, dt \quad = \quad d\left\langle \log X_i \right\rangle_t \quad = \quad \sum_{\nu=1}^{n} \xi_{i\nu}^2(t)\, dt, \quad \text{and,} \tag{2.2.49}$$

$$\sigma_{ii}(t) \quad = \quad \frac{d}{dt} \left\langle \log X_i \right\rangle_t \quad = \quad \sum_{\nu=1}^{n} \xi_{i\nu}^2(t). \tag{2.2.50}$$

The latter equation we recognise from the integrability conditions, (i), (ii) and (iii), for the basic logarithmic equity market model with n sources of uncertainty. Therefore, in accordance with the equation above, these integrability conditions, in terms of the volatility processes, yield equivalent integrability conditions, in terms of the variance processes. Thus, for $i = 1, 2, \ldots, n$, the integrability conditions are given by

(i) $\displaystyle\int_0^T \sigma_{ii}(t)\, dt \quad < \quad \infty, \qquad T \in [0, \infty), \qquad \text{a.s.;}$

(ii) $\displaystyle\lim_{t \to \infty} \frac{1}{t}\, \sigma_{ii}(t) \log\log t \quad = \quad 0, \qquad \text{a.s., and;}$

(iii) $\sigma_{ii}(t) \quad > \quad 0, \qquad t \in [0, \infty), \qquad \text{a.s.}$

In summary, for all $i, j = 1, 2, \ldots, n$ and $t \in [0, \infty)$, we have

$$\begin{aligned} \left\langle \log X_i, \log X_j \right\rangle_t \quad &= \quad \int_0^t \sigma_{ij}(s)\, ds, \quad \text{and,} \\ \left\langle \log X_i \right\rangle_t \quad &= \quad \int_0^t \sigma_{ii}(s)\, ds. \end{aligned}$$

Since the volatility processes $\xi_{i\nu}$ are assumed to be locally square-integrable in condition (i) of Definition 2.2.3, it follows that for all $i, j = 1, 2, \ldots, n$ and $t \in [0, \infty)$, we have

$$\begin{aligned} \int_0^t \left|\sigma_{ij}(s)\right| ds \quad &= \quad \int_0^t \left| \sum_{\nu=1}^{n} \xi_{i\nu}(s) \xi_{j\nu}(s) \right| ds \\ &\leq \quad \int_0^t \sum_{\nu=1}^{n} \left|\xi_{i\nu}(s) \xi_{j\nu}(s)\right| ds \qquad (2.2.51) \\ &= \quad \int_0^t \sum_{\nu=1}^{n} \left|\xi_{i\nu}(s)\right| \left|\xi_{j\nu}(s)\right| ds \qquad (2.2.52) \\ &< \quad \infty, \qquad (2.2.53) \end{aligned}$$

where the inequality (2.2.51) follows from the triangle inequality[6] and the equality (2.2.52) follows from the fact that the absolute value of a product is the product of the absolute values. Therefore, for all $i, j = 1, 2, \ldots, n$ and $t \in [0, \infty)$, we have

$$\int_0^t \left|\sigma_{ij}(s)\right| ds \quad < \quad \infty, \qquad \text{a.s.} \tag{2.2.54}$$

2.2.3 The Financial Equity Market

Here we introduce the concept of a financial market, along with certain conditions that will be a requirement for use in later chapters.

[6]The triangle inequality or the subadditivity property is given by $|x + y| \leq |x| + |y|$, $x, y \in \mathbb{R}$, which simply states that the absolute value of a sum is less than or equal to the sum of the absolute values.

Definition 2.2.10 (Financial Market). *A **financial market** is a family $\mathcal{M} = \{X_1, X_2, \ldots, X_n\}$ of stocks, each defined as in* (2.2.22)*, such that the covariance process $\boldsymbol{\sigma}(t)$ is nonsingular,*[7] *for all $t \in [0, \infty)$, a.s.*

Remark 2.2.11. The covariance process $\boldsymbol{\sigma}(t)$ is assumed to be nonsingular in Definition 2.2.10. However, nonsingularity of $\boldsymbol{\sigma}(t)$ is not always necessary for our purposes, but the slight generality gained by removing it is not especially relevant.

2.2.3.1 The Fundamental Market Conditions

Within the context of this financial market, there are certain criteria that the market needs to satisfy, that will be required at a later stage. The market conditions on the volatility structure that are crucial for our purposes are provided in the following definitions. The first condition we impose is that of *nondegeneracy*.

Definition 2.2.12 (Strong Nondegeneracy). *The financial market $\mathcal{M}$ is **nondegenerate** (ND) if there exists a constant number $\varepsilon > 0$ such that*

$$\mathbf{x}\boldsymbol{\sigma}(t)\mathbf{x}^{\mathsf{T}} \quad \geq \quad \varepsilon \left\|\mathbf{x}\right\|^2, \qquad \mathbf{x} \in \mathbb{R}^n, \qquad t \in [0, \infty), \qquad a.s. \tag{2.2.55}$$

An alternative formulation of this definition, in terms of the eigenvalues of the covariance matrix $\boldsymbol{\sigma}(t)$, can also be considered.

Definition 2.2.13 (Strong Nondegeneracy). *The financial market $\mathcal{M}$ is **nondegenerate** (ND) if all the eigenvalues of the covariance matrix $\boldsymbol{\sigma}(t)$ are bounded away from zero. Let ε be the minimum eigenvalue of the covariance matrix $\boldsymbol{\sigma}(t)$, then the market satisfies the strong nondegeneracy condition if*

$$\mathbf{x}\boldsymbol{\sigma}(t)\mathbf{x}^{\mathsf{T}} \quad \geq \quad \varepsilon \left\|\mathbf{x}\right\|^2, \qquad \mathbf{x} \in \mathbb{R}^n, \qquad t \in [0, \infty), \qquad a.s. \tag{2.2.56}$$

The strong ND market condition on the volatility structure simply ensures that the variances of all the stocks in the market $\mathcal{M}$ do not diminish. Roughly speaking, this ND condition suggests that it is not possible to construct a portfolio of stocks that has zero variance. This fundamental market condition is somewhat stronger than that of nonsingularity. It is fairly common in the literature and can be found as the *uniform ellipticity* condition[8] [see e.g., Karatzas & Shreve (1991), Karatzas & Kou (1996) and Duffie (1992)].

Since the processes $\xi_{i\nu}$ are assumed to be bounded on $[0, \infty)$, the same will hold for σ_{ij}. This leads us to the following market condition on the volatility structure, namely uniform boundedness (UB).

Definition 2.2.14 (Uniform Boundedness). *The financial market $\mathcal{M}$ has **uniform bounded variance** (UB) if there exists some constant number $K > 0$ such that*

$$\mathbf{x}\boldsymbol{\sigma}(t)\mathbf{x}^{\mathsf{T}} \quad \leq \quad K \left\|\mathbf{x}\right\|^2, \qquad \mathbf{x} \in \mathbb{R}^n, \qquad t \in [0, \infty), \qquad a.s. \tag{2.2.57}$$

We shall refer to the inequality (2.2.57) as the uniform boundedness (UB) market condition on the volatility structure of the financial equity market $\mathcal{M}$. The UB market condition on the volatility structure is satisfied when all the eigenvalues of the covariance matrix $\boldsymbol{\sigma}(t)$ are uniformly bounded away from infinity. This condition requires that the variances of the stocks in the market do not snowball. Roughly speaking, this condition infers that it is not possible to construct portfolios with arbitrarily large variances as a proportion of the magnitude of their allocation proportions.

Lemma 2.2.15 ([**Fernholz (2002)**]). *The market covariance process $\boldsymbol{\sigma}(t)$ is positive definite for all $t \in [0, \infty)$, a.s.*

[7] If $\mathbf{A}$ is a square matrix, and if a matrix $\mathbf{B}$ of the same size can be found such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$ (where $\mathbf{I}$ is the identity matrix), then $\mathbf{A}$ is said to be invertible or nonsingular and $\mathbf{B}$ is called an inverse of $\mathbf{A}$. In this particular case, $\boldsymbol{\sigma}(t)$ is nonsingular if an inverse matrix $\boldsymbol{\sigma}^{-1}(t)$ exists.

[8] The uniform ellipticity condition is as follows: there exists a positive constant λ such that, $\sum_{i=1}^{d}\sum_{k=1}^{d} a_{ik}(t,x)\xi_i\xi_k \geq \lambda \left\|\boldsymbol{\xi}\right\|^2$ holds for every $\boldsymbol{\xi} \in \mathbb{R}^d$ and $(t,x) \in [0,\infty) \times \mathbb{R}^d$, in accordance with the notation adopted in Karatzas & Shreve (1991). In this text, the nondegeneracy condition (ND) is given by the following relation: $\sigma^2(x) > 0$, for all $x \in \mathbb{R}$.

Proof. In expression (2.2.38), it was shown for all $t \in [0, \infty)$, that the covariance matrix-valued process $\boldsymbol{\sigma}(t)$ is positive semidefinite. A direct consequence of a matrix being positive semidefinite is that the eigenvalues of the matrix are all nonnegative and all its principal submatrices have nonnegative determinants. Since Definition 2.2.10 states that $\boldsymbol{\sigma}(t)$ is nonsingular,[9] then all the eigenvalues of the covariance matrix $\boldsymbol{\sigma}(t)$ are non-zero. Thus, all the eigenvalues of $\boldsymbol{\sigma}(t)$ are strictly positive, it follows then that $\boldsymbol{\sigma}(t)$ is positive definite[10] for all $t \in [0, \infty)$, a.s. ■

2.2.4 Portfolios

Definition 2.2.16 (Extended Portfolios). *A **portfolio** in the market $\mathcal{M}$ is an $\mathbb{F}-$progressively measurable, adapted, $(1 \times n)-$vector-valued process $\pi = \big\{\boldsymbol{\pi}(t) = \big(\pi_1(t), \ldots, \pi_n(t)\big), t \in [0, \infty)\big\}$, such that $\boldsymbol{\pi}(t)$ is a.s. uniformly bounded on $[0, \infty) \times \Omega$, and*

$$\pi_1(t) + \cdots + \pi_n(t) \quad = \quad 1, \qquad t \in [0, \infty), \qquad a.s.$$

Thus, a portfolio takes values in the set

$$\bigcup_{\kappa \in \mathbb{N}} \Big\{\boldsymbol{\pi}(t) = \big(\pi_1(t), \ldots, \pi_n(t)\big) \in \mathbb{R}^n \;\Big|\; \pi_1^2(t) + \cdots + \pi_n^2(t) \leq \kappa^2,\ \pi_1(t) + \cdots + \pi_n(t) = 1\Big\}, \qquad t \in [0, \infty).$$

*Such a portfolio is termed an **extended portfolio** in Karatzas (2006).*

Definition 2.2.17 (Long-only Portfolios). *An $n-$dimensional $\mathbb{F}-$progressively measurable, adapted, vector-valued process $\boldsymbol{\pi}(t) = \big(\pi_1(t), \ldots, \pi_n(t)\big)$, is called a **long-only portfolio** if the vector $\boldsymbol{\pi}(t)$ is nonnegative. More precisely, if for all $i = 1, 2, \ldots, n$, we have*

$$\begin{aligned} 0 \quad \leq \quad \pi_i(t) \quad &\leq \quad 1, \qquad t \in [0, \infty), \qquad a.s., \quad and, \\ \pi_1(t) + \cdots + \pi_n(t) \quad &= \quad 1, \qquad t \in [0, \infty), \qquad a.s. \end{aligned}$$

*Thus, a **long-only portfolio** $\boldsymbol{\pi}(t) = \big(\pi_1(t), \ldots, \pi_n(t)\big)$, for $t \in [0, \infty)$, is a portfolio that takes values in the **closed nonnegative unit $(n-1)-$simplex** (i.e., the $(n-1)-$dimensional simplex)[11] $\overline{\Delta}^{n-1} \subset \mathbb{R}^n$, given by*

$$\begin{aligned} \overline{\Delta}^{n-1} \quad &\triangleq \quad \Big\{\boldsymbol{\pi}(t) = \big(\pi_1(t), \ldots, \pi_n(t)\big) \in \mathbb{R}^n \;\Big|\; \pi_1(t) \geq 0, \ldots, \pi_n(t) \geq 0,\ \pi_1(t) + \cdots + \pi_n(t) = 1\Big\}, \\ &= \quad \Big\{\boldsymbol{\pi}(t) = \big(\pi_1(t), \ldots, \pi_n(t)\big) \in \mathbb{R}^n \;\Big|\; \pi_1(t) + \cdots + \pi_n(t) = 1,\ 0 \leq \pi_i(t) \leq 1,\ i = 1, 2, \ldots, n\Big\}. \end{aligned} \tag{2.2.58}$$

*For future reference, we shall also introduce the notation of the following **open positive unit $(n-1)-$simplex** $\Delta^{n-1} \subset \mathbb{R}^n$, given by*

$$\begin{aligned} \Delta^{n-1} \quad &\triangleq \quad \Big\{\boldsymbol{\pi}(t) = \big(\pi_1(t), \ldots, \pi_n(t)\big) \in \mathbb{R}^n \;\Big|\; \pi_1(t) > 0, \ldots, \pi_n(t) > 0,\ \pi_1(t) + \cdots + \pi_n(t) = 1\Big\}, \\ &= \quad \Big\{\boldsymbol{\pi}(t) = \big(\pi_1(t), \ldots, \pi_n(t)\big) \in \overline{\Delta}^{n-1} \;\Big|\; \pi_1(t) > 0, \ldots, \pi_n(t) > 0\Big\}. \end{aligned}$$

More specifically, we have

$$\Delta^{n-1} \quad \triangleq \quad \Big\{\boldsymbol{\pi}(t) = \big(\pi_1(t), \ldots, \pi_n(t)\big) \in \mathbb{R}^n \;\Big|\; \pi_1(t) + \cdots + \pi_n(t) = 1,\ 0 < \pi_i(t) < 1,\ i = 1, 2, \ldots, n\Big\}, \tag{2.2.59}$$

$$= \quad \Big\{\boldsymbol{\pi}(t) = \big(\pi_1(t), \ldots, \pi_n(t)\big) \in \overline{\Delta}^{n-1} \;\Big|\; 0 < \pi_i(t) < 1,\ i = 1, 2, \ldots, n\Big\}. \tag{2.2.60}$$

[9] A square matrix $\mathbf{A}$ is nonsingular if and only if $\lambda = 0$ is not an eigenvalue of $\mathbf{A}$.

[10] A symmetric matrix $\mathbf{A}$ is said to be positive definite if $\mathbf{x}\mathbf{A}\mathbf{x}^{\mathsf{T}} > 0$, for all $\mathbf{x} \in \mathbb{R}^n$. A symmetric matrix $\mathbf{A}$ is positive definite if and only if all the eigenvalues of $\mathbf{A}$ are strictly positive.

[11] Alternatively, we can consider the unit (or standard) $(n-1)-$simplex which is the $(n-1)-$dimensional subset of $\mathbb{R}^n$ defined in Fernholz (2002) by: $\Delta^n = \big\{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \;\big|\; x_1 > 0, \ldots, x_n > 0,\ \sum_{i=1}^n x_i = 1\big\}$. Thus, with this notation, Δ^n is taken to mean the $(n-1)-$dimensional open unit simplex that comprises n elements (or vertices) and is referred to as the unit $(n-1)-$simplex. This is in accordance with the notation that Fernholz (2002) adopts. This notation, however, does not correspond to the usual convention adopted. The notation adopted henceforward, herein, complies with usual convention which expresses the open unit $n-$simplex as the $n-$dimensional subset of $\mathbb{R}^{n+1}$ comprising $n+1$ points (or vertices) and is given by: $\Delta^n = \big\{\mathbf{x} = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \;\big|\; x_1 > 0, \ldots, x_{n+1} > 0,\ \sum_{i=1}^{n+1} x_i = 1\big\}$. Thus, in line with this convention, we have $\Delta^{n-1} = \big\{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \;\big|\; x_1 > 0, \ldots, x_n > 0,\ \sum_{i=1}^n x_i = 1\big\}$, is the open unit $(n-1)-$simplex, a subset of $\mathbb{R}^n$. However, in an effort to maintain consistency with the related literature, one can simply replace Δ^{n-1} and $\overline{\Delta}^{n-1}$(adopted herein) by Δ^n and $\overline{\Delta}^n$, respectively (adopted in Fernholz (2002)), throughout.

Hence, $\Delta^{n-1} \subset \overline{\Delta}^{n-1}$.

Consequently, an extended portfolio corresponds to a trading strategy that is fully invested at all times in the equity market, that can sell one or more stocks short, though certainly not all of the stocks, yet is never allowed to borrow from, or invest in, the money market account (the riskless asset). Whereas, a long-only portfolio corresponds to a trading strategy that is, again, fully invested in the equity market, but never sells stocks short. Thus, a long-only portfolio trading strategy does not permit any form of short selling. Unless otherwise specified, we shall take a portfolio π to represent an extended portfolio. When the need arises to differentiate portfolios from their extended counterparts, we shall add the adjective "long-only" for emphasis.

Remark 2.2.18. The notation appointed to represent the unit $(n-1)-$simplex in Fernholz (2002), Karatzas (2006) and Fernholz & Karatzas (2009), are at odds with each other and can be the cause of some confusion. Thus, to dispel some of the confusion, the notation appointed herein is similar to that of Fernholz (2002). In Fernholz & Karatzas (2009),[12] the convention is to use Δ^n_+ to represent the open positive unit $(n-1)-$simplex and in Karatzas (2006), the convention is to use Δ^n_{++} to represent the open positive unit $(n-1)-$simplex. Whereas Fernholz (2002) adopts the representation Δ^n, herein represented as Δ^{n-1}. Analogously, in Fernholz & Karatzas (2009), Δ^n is used to represent the closure of the corresponding open unit simplex Δ^n_+, i.e., the closed unit $(n-1)-$simplex and in Karatzas (2006), Δ^n_+ is used to represent the closure of the corresponding open unit simplex Δ^n_{++}, i.e., the closed unit $(n-1)-$simplex. Whereas Fernholz (2002) adopts the notation $\overline{\Delta}^n$, herein represented as $\overline{\Delta}^{n-1}$.

The component processes $\pi_i = \{\pi_i(t), t \in [0,\infty)\}$, for all $i = 1, 2, \ldots, n$, of a portfolio represent the *proportions* or *weights* of the corresponding stocks in the portfolio. More precisely, $\pi_i(t)$ denotes the weight of the ith stock in the portfolio π. We say that two portfolios are *equal* if their weights are equal for all $t \in [0,\infty)$, a.s. We shall also say that a stock is *held* in a portfolio if the corresponding weight is positive (i.e., $\pi_i(t) > 0$, for all $t \in [0,\infty)$). However, in the case where the portfolio holds no shares of a given stock, then the weight of that stock in the portfolio is zero. A negative value for π_i indicates a short sale in the ith stock (i.e., $\pi_i(t) < 0$, for all $t \in [0,\infty)$). Consequently, the weights of a portfolio that admit no short sales are all nonnegative. At this stage, it is necessary to emphasise the fact that π_i is not the number of shares of the ith stock held in the portfolio, but rather the proportion of the ith stock of the total portfolio value.

Let $Z_{w,\pi} = \big\{Z_{w,\pi}(t), t \in [0,\infty)\big\}$, where $Z_{w,\pi}(t) > 0$ for all $t \in [0,\infty)$, denote the value of an investment in the portfolio π. We shall refer to $Z_{w,\pi}$ as the *portfolio value process* (alternatively, as the *wealth process* [see Karatzas (2006)]) corresponding to the portfolio π and to an initial wealth or capital $Z_{w,\pi}(0) = w > 0$. Thus, the interpretation is that $\pi_i(t)$ represents the proportion of wealth $Z_{w,\pi}(t)$ invested at time t in the ith stock. As a consequence, the cash amount $h_i = \{h_i(t), t \in [0,\infty)\}$ (in some unit of currency) invested in the ith stock $X_i(t)$ at any given time $t \in [0,\infty)$, is represented by the quantity

$$h_i(t) \quad \triangleq \quad \pi_i(t)\, Z_{w,\pi}(t),$$

for $i = 1, 2, \ldots, n$.

Corollary 2.2.19 ([**Fernholz (2002)**]). *Let* $\pi = \big\{\boldsymbol{\pi}(t) = \big(\pi_1(t), \ldots, \pi_n(t)\big), t \in [0,\infty)\big\}$ *be a portfolio in the financial market* $\mathcal{M}$. *Then the portfolio value process* $Z_{w,\pi} = \big\{Z_{w,\pi}(t), t \in [0,\infty)\big\}$, *with initial capital* $Z_{w,\pi}(0) = w > 0$, *satisfies the stochastic differential equation*

$$\frac{dZ_{w,\pi}(t)}{Z_{w,\pi}(t)} \quad = \quad \sum_{i=1}^{n} \pi_i(t)\, \frac{dX_i(t)}{X_i(t)}, \qquad t \in [0,\infty), \qquad a.s. \tag{2.2.61}$$

[12] Therefore, Fernholz & Karatzas (2009), offer the following notation:

$$\begin{aligned}
\Delta^n \quad &\triangleq \quad \Big\{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \;\Big|\; x_1 \geq 0, \ldots, x_n \geq 0, \sum_{i=1}^{n} x_i = 1\Big\}, \quad \text{and,} \\
\Delta^n_+ \quad &\triangleq \quad \Big\{\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \;\Big|\; x_1 > 0, \ldots, x_n > 0, \sum_{i=1}^{n} x_i = 1\Big\} \\
&= \quad \Big\{\mathbf{x} = (x_1, \ldots, x_n) \in \Delta^n \;\Big|\; x_1 > 0, \ldots, x_n > 0\Big\}.
\end{aligned}$$

Proof. In a heuristic fashion, suppose that the price of the ith stock X_i changes by an infinitesimal amount $dX_i(t) = X_i(t+dt) - X_i(t)$, so that the stock incurs a percentage change $\frac{dX_i(t)}{X_i(t)}\%$. Then, the *induced change in the portfolio value* due solely to the change in the ith stock is given by

$$h_i(t)\left(\frac{X_i(t+dt)-X_i(t)}{X_i(t)}\right) = h_i(t)\left(\frac{dX_i(t)}{X_i(t)}\right),$$

where $h_i(t) = \pi_i(t)Z_{w,\pi}(t)$ is the cash amount invested in the ith stock. Thus, $h_i(t)\frac{dX_i(t)}{X_i(t)}$ represents the adjustment made to the cash amount invested in the ith stock. Hence, if the price of the ith stock X_i, for all $i = 1, 2, \ldots, n$, changes by an amount $dX_i(t)$, then the *total change in the portfolio value* at time $t \in [0, \infty)$, $dZ_{w,\pi}(t)$, due to changes incurred in all the stock prices, is given by

$$\begin{aligned} dZ_{w,\pi}(t) &= \sum_{i=1}^{n} h_i(t)\,\frac{dX_i(t)}{X_i(t)} \\ &= \sum_{i=1}^{n} \pi_i(t)\,Z_{w,\pi}(t)\,\frac{dX_i(t)}{X_i(t)} \qquad (2.2.62) \\ &= Z_{w,\pi}(t)\sum_{i=1}^{n} \pi_i(t)\,\frac{dX_i(t)}{X_i(t)}, \qquad (2.2.63) \end{aligned}$$

which can be equivalently expressed as the instantaneous arithmetic return of the portfolio π at time t. Hence, the wealth process $Z_{w,\pi}(t)$, that corresponds to the portfolio π and to an initial capital $Z_{w,\pi}(0) = w > 0$, also satisfies

$$\frac{dZ_{w,\pi}(t)}{Z_{w,\pi}(t)} = \sum_{i=1}^{n} \pi_i(t)\,\frac{dX_i(t)}{X_i(t)}. \qquad (2.2.64)$$

■

The form in equation (2.2.64) reveals that the instantaneous arithmetic return of the portfolio π, $\frac{dZ_{w,\pi}}{Z_{w,\pi}}$, is the weighted average of the instantaneous arithmetic returns of the individual component stocks in the portfolio, $\frac{dX_i}{X_i}$, for $i = 1, 2, \ldots, n$, weighted by the proportions of the component stocks, π_i, $i = 1, 2, \ldots, n$. Since our primary concern is in analysing the behaviour of stocks, we are likewise concerned with the behaviour of equity portfolios. As a result, we are intent in examining solutions to equation (2.2.64). In this study, however, we would like to study the nature of solutions to equation (2.2.64) within the context of the logarithmic model, as opposed to the arithmetic model indicated here. This leads us to the following proposition which provides an equivalent expression to equation (2.2.64) for the portfolio value process $Z_{w,\pi}(t)$ in differential form for use within the framework of the logarithmic model.

Proposition 2.2.20 ([**Fernholz (2002)**]). *Let $\pi = \{\boldsymbol{\pi}(t) = (\pi_1(t), \ldots, \pi_n(t)), t \in [0, \infty)\}$ be a portfolio in the financial market $\mathcal{M}$. Then the portfolio value process $Z_{w,\pi} = \{Z_{w,\pi}(t), t \in [0, \infty)\}$ satisfies*

$$d\log Z_{w,\pi}(t) = \gamma_\pi(t)\,dt + \sum_{i,\nu=1}^{n} \pi_i(t)\xi_{i\nu}(t)\,dW_\nu(t), \qquad t \in [0, \infty), \qquad a.s., \qquad (2.2.65)$$

where

$$\gamma_\pi(t) = \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^{n} \pi_i(t)\sigma_{ij}(t)\pi_j(t)\right). \qquad (2.2.66)$$

The properties of γ_i, π_i and $\xi_{i\nu}$, ensure that for $t \in [0, \infty)$, $\log Z_{w,\pi}(t)$ is a continuous semimartingale. For any initial value $Z_{w,\pi}(0) = w > 0$, equation (2.2.65) can be integrated directly to yield the integral form

$$\log Z_{w,\pi}(t) = \log Z_{w,\pi}(0) + \int_0^t \gamma_\pi(s)\,ds + \int_0^t \sum_{i,\nu=1}^{n} \pi_i(s)\xi_{i\nu}(s)\,dW_\nu(s), \qquad (2.2.67)$$

or, equivalently, the exponential form

$$Z_{w,\pi}(t) \quad = \quad w \, \exp\left(\int_0^t \gamma_\pi(s)\, ds + \int_0^t \sum_{i,\nu=1}^n \pi_i(s)\xi_{i\nu}(s)\, dW_\nu(s) \right), \tag{2.2.68}$$

which is a strong solution of equation (2.2.64). From the last equation, it is clear that $Z_{w,\pi}(t) > 0$ for all $t \in [0,\infty)$, a.s.

Proof of Proposition 2.2.20. It is evident from equation (2.2.68) that the portfolio value process is adapted, since for any $t \in [0,\infty)$, the process depends only on the events that have occurred up to time t and the process is independent of future events. The proof requires the confirmation that $Z_{w,\pi}$ satisfies equation (2.2.64), given the information posited in the proposition. Consider the following form for the portfolio value process $Z_{w,\pi}(t) = \exp\big(\log Z_{w,\pi}(t)\big)$. Thus, we can apply Itô's formula to $\exp\big(\log Z_{w,\pi}(t)\big)$. Hence, by setting $Y(t) := \log Z_{w,\pi}(t)$, the form of the function to be used in Itô's formula is given by $F\big(t, Y(t)\big) = \exp\big(Y(t)\big)$, and the following are easily obtained

$$\begin{aligned} \frac{\partial F}{\partial t}(t,y) &= 0, \quad \text{and,} \\ \frac{\partial F}{\partial y}(t,y) &= \exp\big(Y(t)\big), \quad \text{and,} \\ \frac{\partial^2 F}{\partial y^2}(t,y) &= \exp\big(Y(t)\big). \end{aligned}$$

We then arrive at the following for $t \in [0,\infty)$, a.s.,

$$dF\big(t, Y(t)\big) \quad = \quad \exp\big(Y(t)\big)\, dY(t) + \frac{1}{2} \exp\big(Y(t)\big)\, d\langle Y \rangle_t.$$

Since $Y(t) = \log Z_{w,\pi}(t)$, the following formula for $dZ_{w,\pi}$, for $t \in [0,\infty)$, a.s., is obtained

$$dZ_{w,\pi}(t) \quad = \quad Z_{w,\pi}(t)\, d\log Z_{w,\pi}(t) + \frac{1}{2} Z_{w,\pi}(t)\, d\big\langle \log Z_{w,\pi} \big\rangle_t. \tag{2.2.69}$$

Consequently, for the instantaneous return on the portfolio, for $t \in [0,\infty)$, a.s., we have

$$\frac{dZ_{w,\pi}(t)}{Z_{w,\pi}(t)} \quad = \quad d\log Z_{w,\pi}(t) + \frac{1}{2}\, d\big\langle \log Z_{w,\pi} \big\rangle_t. \tag{2.2.70}$$

Thus, substituting equation (2.2.65) for $d\log Z_{w,\pi}$ in the above equation, for $t \in [0,\infty)$, a.s., we have

$$\frac{dZ_{w,\pi}(t)}{Z_{w,\pi}(t)} \quad = \quad \gamma_\pi(t)\, dt + \sum_{i,\nu=1}^n \pi_i(t)\xi_{i\nu}(t)\, dW_\nu(t) + \frac{1}{2}\, d\big\langle \log Z_{w,\pi} \big\rangle_t. \tag{2.2.71}$$

The quadratic variation process $\big\langle \log Z_{w,\pi} \big\rangle$ in equation (2.2.71) can be determined from equation (2.2.65) as

follows

$$\begin{aligned}
\left\langle \log Z_{w,\pi} \right\rangle_t &= \left\langle \int_0^{\cdot} \sum_{i,\nu=1}^{n} \pi_{i,s}\xi_{i\nu,s}\, dW_{\nu,s} \right\rangle_t \\
&= \left\langle \sum_{\nu=1}^{n} \int_0^{\cdot} \sum_{i=1}^{n} \pi_{i,s}\xi_{i\nu,s}\, dW_{\nu,s} \right\rangle_t \\
&= \sum_{\nu=1}^{n} \left\langle \int_0^{\cdot} \sum_{i=1}^{n} \pi_{i,s}\xi_{i\nu,s}\, dW_{\nu,s} \right\rangle_t \\
&= \sum_{\nu=1}^{n} \int_0^{t} \left(\sum_{i=1}^{n} \pi_i(s)\xi_{i\nu}(s) \right)^2 d\left\langle W_\nu \right\rangle_s \\
&= \sum_{\nu=1}^{n} \int_0^{t} \sum_{i,j=1}^{n} \pi_i(s)\pi_j(s)\xi_{i\nu}(s)\xi_{j\nu}(s)\, ds \\
&= \int_0^{t} \sum_{i,j=1}^{n} \pi_i(s)\pi_j(s) \sum_{\nu=1}^{n} \xi_{i\nu}(s)\xi_{j\nu}(s)\, ds \qquad (2.2.72) \\
&= \int_0^{t} \sum_{i,j=1}^{n} \pi_i(s)\sigma_{ij}(s)\pi_j(s)\, ds. \qquad (2.2.73)
\end{aligned}$$

Hence, by making use of equation (2.2.45), the differential of the quadratic variation process is obtained as follows

$$\begin{aligned}
d\left\langle \log Z_{w,\pi} \right\rangle_t &= d\left\langle \int_0^{\cdot} \sum_{i,\nu=1}^{n} \pi_{i,s}\xi_{i\nu,s}\, dW_{\nu,s} \right\rangle_t \\
&= d\left(\int_0^{t} \sum_{i,j=1}^{n} \pi_i(s)\pi_j(s) \sum_{\nu=1}^{n} \xi_{i\nu}(s)\xi_{j\nu}(s)\, ds \right) \\
&= \sum_{i,j=1}^{n} \pi_i(t)\pi_j(t) \sum_{\nu=1}^{n} \xi_{i\nu}(t)\xi_{j\nu}(t)\, dt \qquad (2.2.74) \\
&= \sum_{i,j=1}^{n} \pi_i(t)\sigma_{ij}(t)\pi_j(t)\, dt. \qquad (2.2.75)
\end{aligned}$$

In the statement of the proposition, the following form for γ_π is postulated

$$\begin{aligned}
\gamma_\pi(t) &= \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^{n} \pi_i(t)\sigma_{ij}(t)\pi_j(t) \right) \\
&= \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \frac{1}{2}\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - \frac{1}{2}\sum_{i,j=1}^{n} \pi_i(t)\sigma_{ij}(t)\pi_j(t). \qquad (2.2.76)
\end{aligned}$$

By substituting equation (2.2.76) and equation (2.2.75), the simplified form for $d\left\langle \log Z_{w,\pi} \right\rangle$, into equation (2.2.71), it follows that for all $t \in [0, \infty)$, a.s., we have

$$\begin{aligned}
\frac{dZ_{w,\pi}(t)}{Z_{w,\pi}(t)} &= \gamma_\pi(t)\, dt + \sum_{i,\nu=1}^{n} \pi_i(t)\xi_{i\nu}(t)\, dW_\nu(t) + \frac{1}{2}\sum_{i,j=1}^{n} \pi_i(t)\sigma_{ij}(t)\pi_j(t)\, dt \\
&= \sum_{i=1}^{n} \pi_i(t)\gamma_i(t)\, dt + \frac{1}{2}\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t)\, dt + \sum_{i,\nu=1}^{n} \pi_i(t)\xi_{i\nu}(t)\, dW_\nu(t) \\
&= \sum_{i=1}^{n} \pi_i(t)\left(\gamma_i(t) + \frac{1}{2}\sigma_{ii}(t) \right) dt + \sum_{i,\nu=1}^{n} \pi_i(t)\xi_{i\nu}(t)\, dW_\nu(t) \\
&= \sum_{i=1}^{n} \pi_i(t)\left[\left(\gamma_i(t) + \frac{1}{2}\sigma_{ii}(t) \right) dt + \sum_{\nu=1}^{n} \xi_{i\nu}(t)\, dW_\nu(t) \right]. \qquad (2.2.77)
\end{aligned}$$

Now, for $i = 1, 2, \ldots, n$ and $t \in [0, \infty)$, equation (2.2.14) in conjunction with (2.2.47) yields

$$\begin{aligned} dX_i(t) &= \left(\gamma_i(t) + \frac{1}{2}\sum_{\nu=1}^{n} \xi_{i\nu}^2(t)\right) X_i(t)\, dt + X_i(t) \sum_{\nu=1}^{n} \xi_{i\nu}(t)\, dW_\nu(t) \\ &= \left(\gamma_i(t) + \frac{1}{2}\,\sigma_{ii}(t)\right) X_i(t)\, dt + X_i(t) \sum_{\nu=1}^{n} \xi_{i\nu}(t)\, dW_\nu(t) \qquad (2.2.78) \\ &= X_i(t)\left[\left(\gamma_i(t) + \frac{1}{2}\,\sigma_{ii}(t)\right) dt + \sum_{\nu=1}^{n} \xi_{i\nu}(t)\, dW_\nu(t)\right]. \qquad (2.2.79) \end{aligned}$$

Consequently, the instantaneous arithmetic return for the ith component stock in the portfolio π, a.s. has the form for $t \in [0, \infty)$,

$$\frac{dX_i(t)}{X_i(t)} = \left(\gamma_i(t) + \frac{1}{2}\,\sigma_{ii}(t)\right) dt + \sum_{\nu=1}^{n} \xi_{i\nu}(t)\, dW_\nu(t). \qquad (2.2.80)$$

Thus, from the above equation, for $t \in [0, \infty)$, a.s., equation (2.2.77) becomes

$$\frac{dZ_{w,\pi}(t)}{Z_{w,\pi}(t)} = \sum_{i=1}^{n} \pi_i(t)\, \frac{dX_i(t)}{X_i(t)}.$$

Now that we have confirmed the expression (2.2.64) using the information posited in the proposition, this validates the expressions given in the proposition. From equation (2.2.20), a.s., for $t \in [0, \infty)$, this produces an expression analogous to (2.2.10)

$$\begin{aligned} dX_i(t) &= X_i(t)\, d\log X_i(t) + \frac{1}{2}\, X_i(t)\, d\left\langle \log X_i \right\rangle_t \qquad (2.2.81) \\ &= X_i(t)\, d\log X_i(t) + \frac{1}{2}\, X_i(t)\, \sigma_{ii}(t)\, dt, \qquad (2.2.82) \end{aligned}$$

and produces an expression analogous to (2.2.11)

$$\begin{aligned} \frac{dX_i(t)}{X_i(t)} &= d\log X_i(t) + \frac{1}{2}\, d\left\langle \log X_i \right\rangle_t \qquad (2.2.83) \\ &= d\log X_i(t) + \frac{1}{2}\, \sigma_{ii}(t)\, dt. \qquad (2.2.84) \end{aligned}$$

Therefore, we also have

$$d\log X_i(t) = \frac{dX_i(t)}{X_i(t)} - \frac{1}{2}\, \sigma_{ii}(t)\, dt. \qquad (2.2.85)$$

In passing, we notice

$$d\left\langle X_i \right\rangle_t = X_i^2(t)\, d\left\langle \log X_i \right\rangle_t = \sigma_{ii}(t)\, X_i^2(t)\, dt, \qquad (2.2.86)$$

and,

$$d\left\langle X_i, X_j \right\rangle_t = X_i(t) X_j(t)\, d\left\langle \log X_i, \log X_j \right\rangle_t = \sigma_{ij}(t)\, X_i(t) X_j(t)\, dt. \qquad (2.2.87)$$

Alternatively, for $i = 1, \ldots, n$, the following processes $\alpha_i = \{\alpha_i(t), t \in [0, \infty)\}$ are defined as follows

$$\alpha_i(t) \triangleq \gamma_i(t) + \frac{1}{2}\sum_{\nu=1}^{n} \xi_{i\nu}^2(t) = \gamma_i(t) + \frac{1}{2}\, \sigma_{ii}(t), \qquad (2.2.88)$$

where $\boldsymbol{\alpha} = \left\{\boldsymbol{\alpha}(t) = \left(\alpha_1(t), \ldots, \alpha_n(t)\right), t \in [0, \infty)\right\}$ is the $(1 \times n)-$vector-valued process of the arithmetic rates of returns, i.e., the *rate of return process*. Consequently, for $i = 1, \ldots, n$ and $t \in [0, \infty)$, a.s., analogous to (2.2.17), we obtain

$$\begin{aligned} dX_i(t) &= \alpha_i(t)\, X_i(t)\, dt + X_i(t) \sum_{\nu=1}^{n} \xi_{i\nu}(t)\, dW_\nu(t) \qquad (2.2.89) \\ &= X_i(t)\left[\alpha_i(t)\, dt + \sum_{\nu=1}^{n} \xi_{i\nu}(t)\, dW_\nu(t)\right]. \qquad (2.2.90) \end{aligned}$$

Thus, for $t \in [0, \infty)$, a.s., the instantaneous arithmetic return is given by

$$\frac{dX_i(t)}{X_i(t)} = \alpha_i(t)\,dt + \sum_{\nu=1}^{n} \xi_{i\nu}(t)\,dW_\nu(t). \tag{2.2.91}$$

This is the *arithmetic representation* of the stock price processes that was mentioned earlier, typically used in mathematical finance as the model for the stocks in the financial market, which is in contrast to the model adopted and endorsed by Fernholz (2002), namely, the logarithmic equity market model. This logarithmic representation considers the growth rate γ_i, which is interpreted as the expected rate of change of the logarithm of the stock price at time t, whereas, in the arithmetic representation, the rate of return α_i, rather than the growth rate γ_i, is considered for each stock X_i. Moreover, the relation between these two variables is provided by the expression (2.2.88). Therefore, from (2.2.77), (2.2.88) and (2.2.91), for $t \in [0, \infty)$, a.s., we obtain

$$\begin{aligned} \frac{dZ_{w,\pi}(t)}{Z_{w,\pi}(t)} &= \sum_{i=1}^{n} \pi_i(t) \left[\alpha_i(t)\,dt + \sum_{\nu=1}^{n} \xi_{i\nu}(t)\,dW_\nu(t) \right] \\ &= \sum_{i=1}^{n} \pi_i(t)\,\frac{dX_i(t)}{X_i(t)}. \end{aligned}$$

For an alternative proof, refer to Appendix C. ■

The process $\gamma_\pi = \{\gamma_\pi(t), t \in [0, \infty)\}$ in equation (2.2.66) is referred to as the **portfolio growth rate process** for the portfolio π.

Definition 2.2.21 (Portfolio Growth Rate). *For an arbitrary portfolio $\pi = \big\{\boldsymbol{\pi}(t) = \big(\pi_1(t), \ldots, \pi_n(t)\big), t \in [0, \infty)\big\}$, the process $\gamma_\pi = \{\gamma_\pi(t), t \in [0, \infty)\}$ defined for $t \in [0, \infty)$, a.s., by*

$$\gamma_\pi(t) \triangleq \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^{n} \pi_i(t)\sigma_{ij}(t)\pi_j(t) \right), \tag{2.2.92}$$

*is called the **portfolio growth rate process** for the portfolio π.*

2.2.4.1 The Portfolio Variance Process

In equation (2.2.92) we observe the following term

$$\sum_{i,j=1}^{n} \pi_i(t)\sigma_{ij}(t)\pi_j(t) = \boldsymbol{\pi}(t)\boldsymbol{\sigma}(t)\boldsymbol{\pi}^{\mathsf{T}}(t), \qquad t \in [0, \infty), \tag{2.2.93}$$

which will be formally introduced in the following definition.

Definition 2.2.22 (Portfolio Variance). *For an arbitrary portfolio $\pi = \big\{\boldsymbol{\pi}(t) = \big(\pi_1(t), \ldots, \pi_n(t)\big), t \in [0, \infty)\big\}$, the process $\sigma_{\pi\pi} = \{\sigma_{\pi\pi}(t), t \in [0, \infty)\}$ defined by*

$$\sigma_{\pi\pi}(t) \triangleq \sum_{i,j=1}^{n} \pi_i(t)\sigma_{ij}(t)\pi_j(t) = \boldsymbol{\pi}(t)\boldsymbol{\sigma}(t)\boldsymbol{\pi}^{\mathsf{T}}(t), \qquad t \in [0, \infty), \qquad \text{a.s.}, \tag{2.2.94}$$

*is called the **portfolio variance process** for the portfolio π.*

Thus, from (2.2.75), for $t \in [0, \infty)$, a.s., we have

$$d\left\langle \log Z_{w,\pi} \right\rangle_t = \sigma_{\pi\pi}(t)\,dt, \quad \text{and}, \tag{2.2.95}$$

$$\frac{d}{dt}\left\langle \log Z_{w,\pi} \right\rangle_t = \sigma_{\pi\pi}(t). \tag{2.2.96}$$

In addition, for $t \in [0, \infty)$, a.s., we have

$$\left\langle \log Z_{w,\pi} \right\rangle_t = \int_0^t \sigma_{\pi\pi}(s)\,ds. \tag{2.2.97}$$

From (2.2.69) we notice

$$d\left\langle Z_{w,\pi}\right\rangle_t \;=\; Z^2_{w,\pi}(t)\, d\left\langle \log Z_{w,\pi}\right\rangle_t \;=\; \sigma_{\pi\pi}(t)\, Z^2_{w,\pi}(t)\, dt.$$

Alternatively, from equations (2.2.62) and (2.2.87), we equivalently obtain

$$\begin{aligned}
d\left\langle Z_{w,\pi}\right\rangle_t &= d\left\langle \int_0^{\cdot} Z_{w,\pi,s} \sum_{i=1}^n \pi_{i,s}\, \frac{dX_{i,s}}{X_{i,s}}, \int_0^{\cdot} Z_{w,\pi,s} \sum_{j=1}^n \pi_{j,s}\, \frac{dX_{j,s}}{X_{j,s}} \right\rangle_t \\
&= d\left\langle \sum_{i=1}^n \int_0^{\cdot} Z_{w,\pi,s}\, \pi_{i,s}\, \frac{dX_{i,s}}{X_{i,s}}, \sum_{j=1}^n \int_0^{\cdot} Z_{w,\pi,s}\, \pi_{j,s}\, \frac{dX_{j,s}}{X_{j,s}} \right\rangle_t \\
&= \sum_{i,j=1}^n d\left\langle \int_0^{\cdot} Z_{w,\pi,s}\, \pi_{i,s}\, \frac{dX_{i,s}}{X_{i,s}}, \int_0^{\cdot} Z_{w,\pi,s}\, \pi_{j,s}\, \frac{dX_{j,s}}{X_{j,s}} \right\rangle_t \\
&= \sum_{i,j=1}^n Z^2_{w,\pi}(t)\, \pi_i(t)\pi_j(t) \frac{1}{X_i(t)X_j(t)}\, d\left\langle X_i, X_j\right\rangle_t \qquad (2.2.98)\\
&= \sum_{i,j=1}^n Z^2_{w,\pi}(t)\, \pi_i(t)\pi_j(t)\, d\left\langle \log X_i, \log X_j\right\rangle_t \qquad (2.2.99)\\
&= Z^2_{w,\pi}(t) \sum_{i,j=1}^n \pi_i(t)\sigma_{ij}(t)\pi_j(t)\, dt \qquad (2.2.100)\\
&= \sigma_{\pi\pi}(t)\, Z^2_{w,\pi}(t)\, dt. \qquad (2.2.101)
\end{aligned}$$

Thus, analogously for the individual component stocks in the portfolio, the portfolio variance process is the differential of the quadratic variation of the logarithm of the portfolio value process. This is in a similar fashion to the individual component stocks in the portfolio, whose covariance process is the differential of the cross-variation (or, covariation) of the logarithm of the stocks, i.e., for $t \in [0,\infty)$, a.s., we have

$$\begin{aligned}
\sigma_{\pi\pi}(t) &= \frac{d}{dt}\left\langle \log Z_{w,\pi}\right\rangle_t, \quad \text{and,} \qquad (2.2.102)\\
\sigma_{ij}(t) &= \frac{d}{dt}\left\langle \log X_i, \log X_j\right\rangle_t. \qquad (2.2.103)
\end{aligned}$$

Furthermore, the expression for the portfolio variance process (or, portfolio variance rate) (2.2.94) within the logarithmic context is identical to the portfolio variance rate within the arithmetic representation.

Remark 2.2.23. If the market $\mathcal{M}$ is of bounded variance, as per Definition 2.2.14, then the portfolio variance process $\sigma_{\pi\pi}$ is a.s. bounded on $[0,\infty)$ for any portfolio π.

Thus, for $t \in [0,\infty)$, a.s., equation (2.2.69) can be written as

$$dZ_{w,\pi}(t) \;=\; Z_{w,\pi}(t)\, d\log Z_{w,\pi}(t) + \frac{1}{2}\, Z_{w,\pi}(t)\, \sigma_{\pi\pi}(t)\, dt, \qquad (2.2.104)$$

and, equation (2.2.70) can be written as

$$\frac{dZ_{w,\pi}(t)}{Z_{w,\pi}(t)} \;=\; d\log Z_{w,\pi}(t) + \frac{1}{2}\, \sigma_{\pi\pi}(t)\, dt. \qquad (2.2.105)$$

Consequently, we have

$$d\log Z_{w,\pi}(t) \;=\; \frac{dZ_{w,\pi}(t)}{Z_{w,\pi}(t)} - \frac{1}{2}\, \sigma_{\pi\pi}(t)\, dt. \qquad (2.2.106)$$

By employing the notation used at the beginning of this chapter, equation (2.2.105) can be written in the following more interpretable manner

$$\mu^{\mathsf{arith}}_{\pi}(t, t+dt) \;=\; \mu^{\mathsf{geom}}_{\pi}(t, t+dt) + \frac{1}{2}\, \sigma^2_{\pi}(t)\, dt,$$

where σ_π^2 is the portfolio variance. Or, equivalently, as

$$\mu_\pi^{\mathsf{geom}}(t, t+dt) \quad = \quad \mu_\pi^{\mathsf{arith}}(t, t+dt) - \frac{1}{2}\,\sigma_\pi^2(t)\,dt.$$

From this we ascertain that the instantaneous arithmetic return exceeds the instantaneous geometric return by some variance component $\frac{1}{2}\,\sigma_\pi^2$. Here we have an intuitive relationship between the two returns. Furthermore, it is precisely this relationship that results in the overoptimism inherent in the arithmetic return. Such a difference will not be as prominent over the short term. Since the variance component which depends on dt, the length of the time interval under consideration, is smaller in comparison to that over the long term. Yet, over the long term, the difference between the two returns is much more apparent, and so the difference increases as the time horizon lengthens. It is indubitable that the difference is an increasing function of the portfolio variance σ_π^2 and of the length of the time horizon.

2.2.4.2 The Excess Growth Rate Process

The second term in equation (2.2.66) for $t \in [0, \infty)$, a.s., can be set as follows

$$\frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^{n} \pi_i(t)\sigma_{ij}(t)\pi_j(t)\right) \quad \triangleq \quad \gamma_\pi^*(t). \tag{2.2.107}$$

A formal definition of this term is provided next.

Definition 2.2.24 (Excess Growth Rate). *For an arbitrary portfolio* $\pi = \big\{\boldsymbol{\pi}(t) = \big(\pi_1(t), \ldots, \pi_n(t)\big), t \in [0, \infty)\big\}$, *the process in* (2.2.66), $\gamma_\pi^* = \{\gamma_\pi^*(t), t \in [0, \infty)\}$, *defined by*

$$\gamma_\pi^*(t) \quad \triangleq \quad \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^{n} \pi_i(t)\sigma_{ij}(t)\pi_j(t)\right), \qquad t \in [0, \infty), \qquad a.s., \tag{2.2.108}$$

is called the ***excess growth rate process*** *of the portfolio* π. *This is also equivalent to*

$$\gamma_\pi^*(t) \quad = \quad \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - \sigma_{\pi\pi}(t)\right), \qquad t \in [0, \infty), \qquad a.s. \tag{2.2.109}$$

From equation (2.2.108), we can see that the excess growth rate is contingent on the weights of the stocks in the portfolio and on the stock variances and covariances. Thus, with this notation of the excess growth rate, for $t \in [0, \infty)$, a.s., the portfolio growth rate can be expressed in the following forms

$$\gamma_\pi(t) \quad = \quad \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - \sigma_{\pi\pi}(t)\right), \tag{2.2.110}$$

and

$$\gamma_\pi(t) \quad = \quad \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \gamma_\pi^*(t). \tag{2.2.111}$$

Recall that we have assumed that shares are infinitely divisible and thus infinitesimally small fractions of shares can be bought or sold. We then, in turn, assumed that we could normalise in such a way that each stock has always just one share outstanding, or more simply put, that the number of shares for a particular stock is always one. Hence, we shall set the initial wealth $Z_{w,\pi}(0) = w$, invested in the portfolio π, to $w := 1$ unit of currency. In which case, the number of shares held of the individual stocks corresponds to the proportion or weight of the stocks in the portfolio π. Henceforth, and without further comment, we shall write $Z_\pi \equiv Z_{1,\pi}$, to represent the portfolio value process or wealth process that corresponds to the portfolio π and to an initial capital or wealth of $Z_{w,\pi}(0) = w = 1$ unit of currency.

Thus, let $Z_\pi = \{Z_\pi(t), t \in [0, \infty)\}$, where $Z_\pi(t) > 0$ for all $t \in [0, \infty)$, denote the value of an investment in the portfolio π with initial capital or wealth $w = 1$ unit of currency. Therefore, all of the results derived for the

portfolio value process $Z_{w,\pi}$ with initial capital $Z_{w,\pi}(0) = w > 0$ can be similarly derived for the portfolio value process Z_π with initial capital or wealth $Z_\pi(0) = w = 1$ unit of currency, by simply substituting $Z_{w,\pi}$ with $Z_{1,\pi}$, and adjusting appropriately. The exact initial capital amount is not of immense importance at this stage as it can be adjusted to suit our purposes and needs. Hereinafter, we shall adopt the notation of Z_π to represent the portfolio value process with a specified initial capital $Z_\pi(0) = 1$, purely for the purposes of compactness.

Corollary 2.2.25 ([**Fernholz (2002)**]). *Let π be a portfolio and let $Z_\pi = \{Z_\pi(t), t \in [0,\infty)\}$ be its associated value process. Then the portfolio value process $Z_\pi = \{Z_\pi(t), t \in [0,\infty)\}$ satisfies*

$$d\log Z_\pi(t) = \sum_{i=1}^n \pi_i(t)\, d\log X_i(t) + \gamma_\pi^*(t)\, dt, \qquad t \in [0,\infty), \qquad a.s. \tag{2.2.112}$$

Proof. By equation (2.2.20), we have

$$\sum_{i=1}^n \pi_i(t)\, d\log X_i(t) = \sum_{i=1}^n \pi_i(t)\gamma_i(t)\, dt + \sum_{i,\nu=1}^n \pi_i(t)\xi_{i\nu}(t)\, dW_\nu(t). \tag{2.2.113}$$

From equations (2.2.65), (2.2.111) and (2.2.113), for $t \in [0,\infty)$, a.s., we have

$$\begin{aligned} d\log Z_\pi(t) &= \gamma_\pi(t)\, dt + \sum_{i,\nu=1}^n \pi_i(t)\xi_{i\nu}(t)\, dW_\nu(t) \\ &= \sum_{i=1}^n \pi_i(t)\gamma_i(t)\, dt + \gamma_\pi^*\, dt + \sum_{i,\nu=1}^n \pi_i(t)\xi_{i\nu}(t)\, dW_\nu(t) \\ &= \sum_{i=1}^n \pi_i(t)\left[\gamma_i(t)\, dt + \sum_{\nu=1}^n \pi_i(t)\xi_{i\nu}(t)\, dW_\nu(t)\right] + \gamma_\pi^*\, dt \\ &= \sum_{i=1}^n \pi_i(t)\, d\log X_i(t) + \gamma_\pi^*(t)\, dt. \end{aligned}$$

■

Therefore, by appealing to equation (2.2.112), we can further substantiate equation (2.2.95) by applying the results, (2.2.44) and (2.2.94), as follows

$$\begin{aligned} d\left\langle \log Z_\pi \right\rangle_t &= d\left\langle \int_0^\cdot \sum_{i=1}^n \pi_{i,s}\, d\log X_{i,s}, \int_0^\cdot \sum_{j=1}^n \pi_{j,s}\, d\log X_{j,s} \right\rangle_t \\ &= d\left\langle \sum_{i=1}^n \int_0^\cdot \pi_{i,s}\, d\log X_{i,s}, \sum_{j=1}^n \int_0^\cdot \pi_{j,s}\, d\log X_{j,s} \right\rangle_t \\ &= \sum_{i,j=1}^n d\left\langle \int_0^\cdot \pi_{i,s}\, d\log X_{i,s}, \int_0^\cdot \pi_{j,s}\, d\log X_{j,s} \right\rangle_t \\ &= \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\, d\left\langle \log X_i, \log X_j \right\rangle_t \qquad (2.2.114) \\ &= \sum_{i,j=1}^n \pi_i(t)\sigma_{ij}(t)\pi_j(t)\, dt \qquad (2.2.115) \\ &= \sigma_{\pi\pi}(t)\, dt. \qquad (2.2.116) \end{aligned}$$

Remark 2.2.26. Note equation (2.2.65) has the following matrix representation

$$\begin{aligned} d\log Z_\pi(t) &= \left(\boldsymbol{\pi}(t)\boldsymbol{\gamma}^\mathsf{T}(t) + \gamma_\pi^*(t)\right) dt + \boldsymbol{\pi}(t)\boldsymbol{\xi}(t)\, dW^\mathsf{T}(t) \\ &= \boldsymbol{\pi}(t)\left(\boldsymbol{\gamma}^\mathsf{T}(t)\, dt + \boldsymbol{\xi}(t)\, dW^\mathsf{T}(t)\right) + \gamma_\pi^*(t)\, dt. \end{aligned}$$

Furthermore, employing equation (2.2.112) of Corollary 2.2.25, equation (2.2.85) and equation (2.2.109) gives

$$\begin{aligned} d\log Z_\pi(t) &= \sum_{i=1}^n \pi_i(t)\left[\frac{dX_i(t)}{X_i(t)} - \frac{1}{2}\,\sigma_{ii}(t)\,dt\right] + \frac{1}{2}\left(\sum_{i=1}^n \pi_i(t)\sigma_{ii}(t) - \sigma_{\pi\pi}(t)\right)dt \\ &= \sum_{i=1}^n \pi_i(t)\frac{dX_i(t)}{X_i(t)} - \frac{1}{2}\sum_{i=1}^n \pi_i(t)\sigma_{ii}(t)\,dt + \frac{1}{2}\sum_{i=1}^n \pi_i(t)\sigma_{ii}(t)\,dt - \frac{1}{2}\,\sigma_{\pi\pi}(t)\,dt \\ &= \sum_{i=1}^n \pi_i(t)\frac{dX_i(t)}{X_i(t)} - \frac{1}{2}\,\sigma_{\pi\pi}(t)\,dt. \end{aligned} \tag{2.2.117}$$

In addition, from equation (2.2.104) and equation (2.2.112) of Corollary 2.2.25, we obtain

$$\begin{aligned} dZ_\pi(t) &= Z_\pi(t)\left[\sum_{i=1}^n \pi_i(t)\,d\log X_i(t) + \gamma_\pi^*(t)\,dt\right] + \frac{1}{2}\,Z_\pi(t)\,\sigma_{\pi\pi}(t)\,dt \\ &= Z_\pi(t)\left[\sum_{i=1}^n \pi_i(t)\,d\log X_i(t) + \left(\gamma_\pi^*(t) + \frac{1}{2}\,\sigma_{\pi\pi}(t)\right)dt\right], \end{aligned} \tag{2.2.118}$$

which is also identically expressed as

$$\frac{dZ_\pi(t)}{Z_\pi(t)} = \sum_{i=1}^n \pi_i(t)\,d\log X_i(t) + \left(\gamma_\pi^*(t) + \frac{1}{2}\,\sigma_{\pi\pi}(t)\right)dt. \tag{2.2.119}$$

From equation (2.2.112), we observe that the instantaneous logarithmic portfolio return $d\log Z_\pi$ is the weighted average of the instantaneous logarithmic returns of the individual component stocks $d\log X_i$, with an additional, essential constituent, the excess growth rate γ_π^*. It follows from equation (2.2.109) that the excess growth rate is half the difference between the weighted-average variance of the individual component stocks and the portfolio variance. Consequently, in a heuristic sense, the excess growth rate can be construed as a measure of the efficacy of diversification in reducing the volatility of the portfolio value Z_π compared to that of its component stocks. Hence, it measures the efficacy of portfolio diversification in reducing portfolio risk. This suggests that γ_π^* might be used as an indicator of **market diversity** and we shall investigate this possibility in Chapter 4. However, the excess growth rate measures more than just the reduction of portfolio risk. The fact that diversification can lower portfolio volatility is a well-known result of classical portfolio theory, but it may be less universally recognised that diversification also influences the portfolio growth rate [Fernholz (2002)], since it influences the excess growth rate. It can be shown (as will be revealed in Section 2.4) that the excess growth rate of a portfolio is always strictly positive, at least if the portfolio holds two or more stocks (i.e., more than a single stock) with no short sales (i.e., as soon as there are long positions of more than two stocks in the portfolio) and will be nonnegative for strictly long-only portfolios. It is necessary to bring this result to light at this stage since it makes apparent a crucial observation, namely, that the portfolio π, under these circumstances, will have a higher growth rate than the weighted average of the growth rates of its individual component stocks. Thus, under the circumstances when $\gamma_\pi^*(t) > 0$ for all $t \in [0, \infty)$, we have

$$\gamma_\pi(t) > \sum_{i=1}^n \pi_i(t)\gamma_i(t).$$

Hence, superior portfolio diversification not only reduces the risk level of the portfolio, but also increases the growth rate of the portfolio. Thus, portfolio diversification strives at minimising the risk level and maximising the rate of growth of the portfolio. Also, the notion of the excess growth being positive suggests that, for any such portfolio, the weighted-average variance of the individual component stocks in the portfolio is *greater* than the portfolio variance. This will no longer be true in general if the portfolio permits short sales. Henceforth, we shall make the assumption that the portfolios we consider permit no short sales.

This behaviour is not evident when we consider the arithmetic portfolio return (2.2.64), due to the fact that the excess growth rate term is concealed in the expression for the portfolio rate of return,

$$\frac{dZ_\pi(t)}{Z_\pi(t)} = \sum_{i=1}^n \pi_i(t)\,\frac{dX_i(t)}{X_i(t)}.$$

Within the logarithmic approach, the portfolio growth rate exceeds the weighted average of the growth rates of the individual component stocks in the portfolio, and the amount by which these two differ is precisely this excess growth rate. This is in contrast to the classical approach in which the portfolio rate of return is expressed as the weighted average of the rates of return of the individual component stocks in the portfolio, in which case the excess growth rate term is absent. The excess growth rate would appear to be an inconvenient complication, however, it is unequivocally this complication that provides insight into aspects of portfolio behaviour that remain obscure under the standard representation within the classical approach. Let us define the *portfolio rate of return process* for the portfolio π, $\alpha_\pi = \{\alpha_\pi(t), t \in [0,\infty)\}$, as follows.

Definition 2.2.27 (Portfolio Rate of Return). *For an arbitrary portfolio* $\pi = \big\{\boldsymbol{\pi}(t) = \big(\pi_1(t), \ldots, \pi_n(t)\big), t \in [0,\infty)\big\}$, *the process* $\alpha_\pi = \{\alpha_\pi(t), t \in [0,\infty)\}$ *defined by*

$$\alpha_\pi(t) \triangleq \sum_{i=1}^{n} \pi_i(t)\alpha_i(t) = \boldsymbol{\pi}(t)\boldsymbol{\alpha}^\mathsf{T}(t), \qquad t \in [0,\infty), \qquad a.s., \tag{2.2.120}$$

is called the ***portfolio rate of return process*** *for the portfolio* π *(i.e., the rate of return process associated with the portfolio* π*), where* $\alpha_i = \{\alpha_i(t), t \in [0,\infty)\}$ *for* $i = 1, 2, \ldots, n$*, is the rate of return of the* i*th stock* X_i*, defined as in* (2.2.88).

This is the classical equation for the portfolio rate of return akin to the expression (2.2.64). Recall, for all $i = 1, 2, \ldots, n$ and $t \in [0,\infty)$, from equation (2.2.91), we have

$$\frac{dX_i(t)}{X_i(t)} = \alpha_i(t)\,dt + \sum_{\nu=1}^{n} \xi_{i\nu}(t)\,dW_\nu(t). \tag{2.2.121}$$

Thus, by making use of (2.2.121) and (2.2.120), for $t \in [0,\infty)$, a.s., equation (2.2.64) becomes

$$\begin{aligned}
\frac{dZ_\pi(t)}{Z_\pi(t)} &= \sum_{i=1}^{n} \pi_i(t)\,\frac{dX_i(t)}{X_i(t)} \\
&= \sum_{i=1}^{n} \pi_i(t)\Big[\alpha_i(t)\,dt + \sum_{\nu=1}^{n} \xi_{i\nu}(t)\,dW_\nu(t)\Big] \\
&= \sum_{i=1}^{n} \pi_i(t)\alpha_i(t)\,dt + \sum_{i,\nu=1}^{n} \pi_i(t)\xi_{i\nu}(t)\,dW_\nu(t) && (2.2.122)\\
&= \alpha_\pi(t)\,dt + \sum_{i,\nu=1}^{n} \pi_i(t)\xi_{i\nu}(t)\,dW_\nu(t). && (2.2.123)
\end{aligned}$$

Hence, we also have

$$\begin{aligned}
dZ_\pi(t) &= \alpha_\pi(t)\,Z_\pi(t)\,dt + Z_\pi(t)\sum_{i,\nu=1}^{n} \pi_i(t)\xi_{i\nu}(t)\,dW_\nu(t) && (2.2.124)\\
&= Z_\pi(t)\Big[\alpha_\pi(t)\,dt + \sum_{i,\nu=1}^{n} \pi_i(t)\xi_{i\nu}(t)\,dW_\nu(t)\Big]. && (2.2.125)
\end{aligned}$$

Equation (2.2.123) has the matrix representation

$$\begin{aligned}
\frac{dZ_\pi(t)}{Z_\pi(t)} &= \boldsymbol{\pi}(t)\boldsymbol{\alpha}^\mathsf{T}(t)\,dt + \boldsymbol{\pi}(t)\boldsymbol{\xi}(t)\,dW^\mathsf{T}(t) \\
&= \boldsymbol{\pi}(t)\Big(\boldsymbol{\alpha}^\mathsf{T}(t)\,dt + \boldsymbol{\xi}(t)\,dW^\mathsf{T}(t)\Big).
\end{aligned}$$

Consequently, we also have

$$dZ_\pi(t) = Z_\pi(t)\Big(\boldsymbol{\pi}(t)\boldsymbol{\alpha}^\mathsf{T}(t)\,dt + \boldsymbol{\pi}(t)\boldsymbol{\xi}(t)\,dW^\mathsf{T}(t)\Big).$$

In order to proceed, we shall require the following useful definition that enables us to simplify most of the formulae above, thus providing a more compact form for such formulae for ease of exposition.

Definition 2.2.28 (Portfolio Volatility). *The following process* $\xi_{\pi\nu} = \{\xi_{\pi\nu}(t), t \in [0,\infty)\}$, *for* $\nu = 1,\ldots,n$, *defined by*

$$\xi_{\pi\nu}(t) \triangleq \sum_{i=1}^{n} \pi_i(t)\xi_{i\nu}(t), \tag{2.2.126}$$

represents the ***volatility process of the portfolio*** π *to the* ν*th source of uncertainty* (*i.e., the volatility coefficients associated with the portfolio* π).

By letting $\mathbf{e}_\nu^\mathsf{T}$ denote the νth unit row vector in $\mathbb{R}^n$ and noticing $\big(\xi_{\pi 1}(t), \xi_{\pi 2}(t), \ldots, \xi_{\pi n}(t)\big) = \boldsymbol{\pi}(t)\boldsymbol{\xi}(t)$, for the portfolio volatility process above, we have the corresponding matrix form

$$\xi_{\pi\nu}(t) = \boldsymbol{\pi}(t)\boldsymbol{\xi}(t)\mathbf{e}_\nu^\mathsf{T}. \tag{2.2.127}$$

Thus, in accordance with equation (2.2.126) of Definition 2.2.28, equation (2.2.123) simplifies to

$$\begin{aligned}
\frac{dZ_\pi(t)}{Z_\pi(t)} &= \alpha_\pi(t)\,dt + \sum_{\nu=1}^{n}\left[\sum_{i=1}^{n}\pi_i(t)\xi_{i\nu}(t)\right]dW_\nu(t) \\
&= \alpha_\pi(t)\,dt + \sum_{\nu=1}^{n}\xi_{\pi\nu}(t)\,dW_\nu(t), \quad \text{and,} && (2.2.128)\\
dZ_\pi(t) &= \alpha_\pi(t)\,Z_\pi(t)\,dt + Z_\pi(t)\sum_{\nu=1}^{n}\xi_{\pi\nu}(t)\,dW_\nu(t) && (2.2.129)\\
&= Z_\pi(t)\left[\alpha_\pi(t)\,dt + \sum_{\nu=1}^{n}\xi_{\pi\nu}(t)\,dW_\nu(t)\right]. && (2.2.130)
\end{aligned}$$

Similarly, by applying equation (2.2.126) to the logarithmic representation, the expression in equation (2.2.65) has the form

$$\begin{aligned}
d\log Z_\pi(t) &= \gamma_\pi(t)\,dt + \sum_{i,\nu=1}^{n}\pi_i(t)\xi_{i\nu}(t)\,dW_\nu(t) \\
&= \gamma_\pi(t)\,dt + \sum_{\nu=1}^{n}\left[\sum_{i=1}^{n}\pi_i(t)\xi_{i\nu}(t)\right]dW_\nu(t) \\
&= \gamma_\pi(t)\,dt + \sum_{\nu=1}^{n}\xi_{\pi\nu}(t)\,dW_\nu(t). && (2.2.131)
\end{aligned}$$

The preceding expression can be integrated directly to yield the integral form

$$\log Z_\pi(t) = \log Z_\pi(0) + \int_0^t \gamma_\pi(s)\,ds + \int_0^t \sum_{\nu=1}^{n}\xi_{\pi\nu}(s)\,dW_\nu(s), \tag{2.2.132}$$

or, equivalently, with an initial capital of $Z_{w,\pi}(0) = w > 0$, the expression has the exponential form

$$Z_{w,\pi}(t) = w\,\exp\left(\int_0^t \gamma_\pi(s)\,ds + \int_0^t \sum_{\nu=1}^{n}\xi_{\pi\nu}(s)\,dW_\nu(s)\right). \tag{2.2.133}$$

Upon substitution of equation (2.2.65) into equation (2.2.105), for $t \in [0,\infty)$, a.s., we obtain

$$\frac{dZ_\pi(t)}{Z_\pi(t)} = \gamma_\pi(t)\,dt + \frac{1}{2}\,\sigma_{\pi\pi}(t)\,dt + \sum_{i,\nu=1}^{n}\pi_i(t)\xi_{i\nu}(t)\,dW_\nu(t).$$

Furthermore, upon comparison of equations (2.2.65) and (2.2.123), the following relation holds

$$\frac{dZ_\pi(t)}{Z_\pi(t)} - \alpha_\pi(t)\,dt = d\log Z_\pi(t) - \gamma_\pi(t)\,dt.$$

Therefore, we have

$$\frac{dZ_\pi(t)}{Z_\pi(t)} = d\log Z_\pi(t) + \big(\alpha_\pi(t) - \gamma_\pi(t)\big)\,dt,$$

which, when compared with equation (2.2.105), implies for $t \in [0,\infty)$, a.s., that

$$\alpha_\pi(t) = \gamma_\pi(t) + \frac{1}{2}\,\sigma_{\pi\pi}(t). \tag{2.2.134}$$

The relation above for the portfolio is akin to that obtained in (2.2.88) for the individual component stocks, where

$$\sigma_{ii}(t) = \sum_{\nu=1}^{n} \xi_{i\nu}^2(t),$$

is effectively replaced by the portfolio version thereof

$$\sigma_{\pi\pi}(t) = \sum_{\nu=1}^{n} \xi_{\pi\nu}^2(t). \tag{2.2.135}$$

Alternatively, recalling (2.2.94) and using (2.2.126), the variance of the portfolio can be derived in terms of the portfolio volatility processes

$$\begin{aligned}
\sigma_{\pi\pi}(t) &= \sum_{i,j=1}^{n} \pi_i(t)\sigma_{ij}(t)\pi_j(t) \\
&= \sum_{i,j=1}^{n} \pi_i(t)\pi_j(t) \sum_{\nu=1}^{n} \xi_{i\nu}(t)\xi_{j\nu}(t) \\
&= \sum_{\nu=1}^{n} \left[\sum_{i=1}^{n} \pi_i(t)\xi_{i\nu}(t)\right]\left[\sum_{j=1}^{n} \pi_j(t)\xi_{j\nu}(t)\right] \\
&= \sum_{\nu=1}^{n} \xi_{\pi\nu}(t)\xi_{\pi\nu}(t) \\
&= \sum_{\nu=1}^{n} \xi_{\pi\nu}^2(t).
\end{aligned} \tag{2.2.136}$$

Thus, from equations (2.2.131) and (2.2.136), we obtain

$$\begin{aligned}
d\left\langle \log Z_\pi \right\rangle_t &= d\left\langle \int_0^{\cdot} \sum_{\nu=1}^{n} \xi_{\pi\nu,s}\,dW_{\nu,s} \right\rangle_t \\
&= d\left\langle \sum_{\nu=1}^{n} \int_0^{\cdot} \xi_{\pi\nu,s}\,dW_{\nu,s} \right\rangle_t \\
&= \sum_{\nu=1}^{n} d\left\langle \int_0^{\cdot} \xi_{\pi\nu,s}\,dW_{\nu,s} \right\rangle_t \\
&= \sum_{\nu=1}^{n} \xi_{\pi\nu}^2(t)\,d\left\langle W_\nu \right\rangle_t \\
&= \sum_{\nu=1}^{n} \xi_{\pi\nu}^2(t)\,dt \\
&= \sigma_{\pi\pi}(t)\,dt.
\end{aligned} \tag{2.2.137}$$

Remark 2.2.29. Since the market is nondegenerate, we know that the market covariance process $\boldsymbol{\sigma}(t)$ is positive definite. This, along with the results established heretofore, suggests that $\sigma_{\pi\pi} > 0$ holds for the variance of the portfolio π.

By adopting the form provided in equation (2.2.134), the equations (2.2.123), (2.2.124) and (2.2.125), can, respectively, be represented by

$$\frac{dZ_\pi(t)}{Z_\pi(t)} = \left(\gamma_\pi(t) + \frac{1}{2}\,\sigma_{\pi\pi}(t)\right) dt + \sum_{i,\nu=1}^{n} \pi_i(t)\xi_{i\nu}(t)\, dW_\nu(t), \quad \text{and,} \tag{2.2.138}$$

$$dZ_\pi(t) = \left(\gamma_\pi(t) + \frac{1}{2}\,\sigma_{\pi\pi}(t)\right) Z_\pi(t)\, dt + Z_\pi(t) \sum_{i,\nu=1}^{n} \pi_i(t)\xi_{i\nu}(t)\, dW_\nu(t) \tag{2.2.139}$$

$$= Z_\pi(t)\left[\left(\gamma_\pi(t) + \frac{1}{2}\,\sigma_{\pi\pi}(t)\right) dt + \sum_{i,\nu=1}^{n} \pi_i(t)\xi_{i\nu}(t)\, dW_\nu(t)\right]. \tag{2.2.140}$$

In addition, by using either (2.2.126) of Definition 2.2.28 or (2.2.134), the equations (2.2.128), (2.2.129) and (2.2.130), can, respectively, be represented by

$$\frac{dZ_\pi(t)}{Z_\pi(t)} = \left(\gamma_\pi(t) + \frac{1}{2}\,\sigma_{\pi\pi}(t)\right) dt + \sum_{\nu=1}^{n} \xi_{\pi\nu}(t)\, dW_\nu(t), \quad \text{and,} \tag{2.2.141}$$

$$dZ_\pi(t) = \left(\gamma_\pi(t) + \frac{1}{2}\,\sigma_{\pi\pi}(t)\right) Z_\pi(t)\, dt + Z_\pi(t) \sum_{\nu=1}^{n} \xi_{\pi\nu}(t)\, dW_\nu(t) \tag{2.2.142}$$

$$= Z_\pi(t)\left[\left(\gamma_\pi(t) + \frac{1}{2}\,\sigma_{\pi\pi}(t)\right) dt + \sum_{\nu=1}^{n} \xi_{\pi\nu}(t)\, dW_\nu(t)\right]. \tag{2.2.143}$$

2.2.5 Total Return

Any gains made by the company are either reinvested back into the company or are shared to their stockholders through the employment of dividends. Dividends are used by companies to distribute earnings back to their stockholders. In our context, dividend payments allow a stock to have returns without affecting the capitalisation or the weight of the stock. In the mathematical finance literature, for reasons of convenience, it is often assumed that companies pay no dividends. Of course, this assumption is not realistic. Although it is frequently unnecessary to include dividends in our discussion, sometimes they play an important role, and accordingly, we introduce them at this stage [Fernholz (2002)]. We shall assume that dividends are paid continuously. Let us now formally introduce dividends into our model.

Definition 2.2.30 (Dividend Rate). *The **dividend rate process** is an $\mathbb{F}-$progressively measurable, adapted process $\delta = \{\delta(t), t \in [0,\infty)\}$ that satisfies*

$$\int_0^t \left|\delta(s)\right| ds \quad < \quad \infty, \qquad t \in [0,\infty), \qquad a.s.$$

Usually dividend rates are assumed to be nonnegative, but this assumption is not necessary.

2.2.5.1 Dividend-Paying Stocks

For a stock X with dividends, i.e., with an associated dividend rate process $\delta = \{\delta(t), t \in [0,\infty)\}$, we have the following definition for the total return process.

Definition 2.2.31 (Total Return of a Dividend-Paying Stock). *Consider a stock X that pays dividends, with an associated dividend rate process as stipulated above. Then the process $\widehat{X} = \left\{\widehat{X}(t), t \in [0,\infty)\right\}$ defined by*

$$\widehat{X}(t) \quad \triangleq \quad X(t)\exp\left(\int_0^t \delta(s)\, ds\right), \qquad t \in [0,\infty), \tag{2.2.144}$$

*is referred to as the **total return process** for the stock X.*

The total return process $\widehat{X}$ represents the value of an investment in the stock X with all dividends continuously reinvested. Notice that if $\delta(t) = 0$ for all $t \in [0, \infty)$ (i.e., for a market without dividends), then $\widehat{X}(t) = X(t)$ for all $t \in [0, \infty)$.

For $t = 0$, equation (2.2.144) implies that $\widehat{X}(0) = X(0) \triangleq x > 0$. Moreover, equation (2.2.144) implies for $t \in [0, \infty)$, that

$$\log \widehat{X}(t) = \log X(t) + \int_0^t \delta(s)\, ds, \quad \text{and,} \tag{2.2.145}$$

$$d \log \widehat{X}(t) = d \log X(t) + \delta(t)\, dt. \tag{2.2.146}$$

Definition 2.2.32 (Augmented Growth Rate). *It is convenient to define the* ***augmented growth rate process*** *(alternatively, the* ***total growth rate process****)* $\vartheta = \{\vartheta(t), t \in [0, \infty)\}$ *by setting*

$$\vartheta(t) \triangleq \gamma(t) + \delta(t), \qquad t \in [0, \infty). \tag{2.2.147}$$

Observe that, for the dynamics of the logarithm of X, given by equation (2.2.2), and by making use of equations (2.2.146) and (2.2.147), the total return process $\widehat{X}$ satisfies

$$\begin{aligned} d \log \widehat{X}(t) &= d \log X(t) + \delta(t)\, dt \\ &= \left(\gamma(t)\, dt + \sum_{\nu=1}^{n} \xi_\nu(t)\, dW_\nu(t) \right) + \delta(t)\, dt \\ &= \big(\gamma(t) + \delta(t)\big)\, dt + \sum_{\nu=1}^{n} \xi_\nu(t)\, dW_\nu(t) \\ &= \vartheta(t)\, dt + \sum_{\nu=1}^{n} \xi_\nu(t)\, dW_\nu(t). \end{aligned} \tag{2.2.148}$$

This is comparable to the form given in expression (2.2.2) for X, in which the γ term is effectively replaced with the ϑ term, i.e., the augmented growth rate for the total return process replaces the growth rate for the stock price process.

Now, in the market $\mathcal{M}$, let $\delta_i = \{\delta_i(t), t \in [0, \infty)\}\}$ for $i = 1, 2, \ldots, n$, be the respective dividend rates of the stocks $X_i = \{X_i(t), t \in [0, \infty)\}$, then $\boldsymbol{\delta} = \big\{\boldsymbol{\delta}(t) = \big(\delta_1(t), \ldots, \delta_n(t)\big), t \in [0, \infty)\big\}$ is the $(1 \times n)-$vector-valued process of dividend rates. In addition, let $\vartheta_i = \{\vartheta_i(t), t \in [0, \infty)\}$ be the respective augmented growth rates of the stocks $X_i = \{X_i(t), t \in [0, \infty)\}$, then $\boldsymbol{\vartheta} = \big\{\boldsymbol{\vartheta}(t) = \big(\vartheta_1(t), \ldots, \vartheta_n(t)\big), t \in [0, \infty)\big\}$ is the $(1 \times n)-$vector-valued process of the augmented growth rates. Therefore, for $i = 1, 2, \ldots, n$, we have the following equivalent results.

Definition 2.2.33 (Dividend Rate). *Consider a market $\mathcal{M}$ that comprises n stocks, $X_1, \ldots, X_n$. The* ***dividend rate process*** *for the ith stock $X_i = \{X_i(t), t \in [0, \infty)\}$ is an $\mathbb{F}-$progressively measurable, adapted process $\delta_i = \{\delta_i(t), t \in [0, \infty)\}$ that satisfies the following, for $i = 1, \ldots, n$,*

$$\int_0^t \big|\delta_i(s)\big|\, ds < \infty, \qquad t \in [0, \infty), \qquad a.s.$$

For a stock X_i with dividends, i.e., with an associated dividend rate process $\delta_i = \{\delta_i(t), t \in [0, \infty)\}$, we have the following definition for the total return process of the ith stock X_i, for $i = 1, \ldots, n$.

Definition 2.2.34 (Total Return of Dividend-Paying Stocks). *Consider a market comprising n stocks, $X_1, \ldots, X_n$, that pay dividends, with associated dividend rate processes as stipulated above. Then the process $\widehat{X}_i = \big\{\widehat{X}_i(t), t \in [0, \infty)\big\}$, for $i = 1, \ldots, n$, defined by*

$$\widehat{X}_i(t) \triangleq X_i(t) \exp\left(\int_0^t \delta_i(s)\, ds \right), \qquad t \in [0, \infty), \tag{2.2.149}$$

is referred to as the ***total return process*** *for the ith stock X_i.*

Again, for $t = 0$, equation (2.2.149) implies that $\widehat{X}_i(0) = X_i(0) \triangleq x_i > 0$, for $i = 1, \ldots, n$, and for $t \in [0, \infty)$ that

$$\log \widehat{X}_i(t) = \log X_i(t) + \int_0^t \delta_i(s)\, ds, \quad \text{and,} \tag{2.2.150}$$

$$d \log \widehat{X}_i(t) = d \log X_i(t) + \delta_i(t)\, dt. \tag{2.2.151}$$

Definition 2.2.35 (Augmented Growth Rate). *It is convenient to define the* ***augmented growth rate process*** *(or,* ***total growth rate process****)* $\vartheta_i = \{\vartheta_i(t), t \in [0, \infty)\}$*, for* $i = 1, \ldots, n$*, by setting*

$$\vartheta_i(t) \triangleq \gamma_i(t) + \delta_i(t), \qquad t \in [0, \infty). \tag{2.2.152}$$

Thus, following the same methodology as was employed in deriving equation (2.2.148), for $i = 1, \ldots, n$, we have

$$d \log \widehat{X}_i(t) = \vartheta_i(t)\, dt + \sum_{\nu=1}^{n} \xi_{i\nu}(t)\, dW_\nu(t), \tag{2.2.153}$$

which is comparable to equation (2.2.20), given below by

$$d \log X_i(t) = \gamma_i(t)\, dt + \sum_{\nu=1}^{n} \xi_{i\nu}(t)\, dW_\nu(t).$$

Thus, the processes above differ only in their respective *drift processes*.

2.2.6 Total Return of Portfolios

2.2.6.1 Portfolios Comprising Dividend-Paying Stocks

Definition 2.2.36 (Portfolio Dividend Rate). *Let* $\delta_1, \delta_2, \ldots, \delta_n$ *be the respective dividend rates of the stocks* $X_1, X_2, \ldots, X_n$ *in the market* $\mathcal{M}$*. Consider a portfolio* π*, we define the dividend rate process for the portfolio* π *(i.e., the* ***portfolio dividend rate process****),* $\delta_\pi = \{\delta_\pi(t), t \in [0, \infty)\}$*, by*

$$\delta_\pi(t) \triangleq \sum_{i=1}^{n} \pi_i(t)\delta_i(t), \qquad t \in [0, \infty). \tag{2.2.154}$$

Furthermore, we can define the total return process of the portfolio π much in the same way as was done for a stock.

Definition 2.2.37 (Total Portfolio Return). *Let* $\pi = \big\{\boldsymbol{\pi}(t) = \big(\pi_1(t), \ldots, \pi_n(t)\big), t \in [0, \infty)\big\}$ *represent a portfolio in the market* $\mathcal{M}$*, and let* $\delta_\pi = \{\delta_\pi(t), t \in [0, \infty)\}$ *be the portfolio dividend rate process associated with the portfolio* π*, as defined in the preceding definition. Then the process* $\widehat{Z}_{w,\pi} = \big\{\widehat{Z}_{w,\pi}(t), t \in [0, \infty)\big\}$*, with initial capital* $\widehat{Z}_{w,\pi}(0) = w > 0$*, defined by*

$$\widehat{Z}_{w,\pi}(t) \triangleq Z_{w,\pi}(t) \exp\left(\int_0^t \delta_\pi(s)\, ds\right), \qquad t \in [0, \infty), \tag{2.2.155}$$

is referred to as the ***total return process*** *of the portfolio* π*.*

For $t = 0$, equation (2.2.155) implies that $\widehat{Z}_{w,\pi}(0) = Z_{w,\pi}(0) = w := 1 > 0$, and that for $t \in [0, \infty)$, as for individual stocks, we have

$$\log \widehat{Z}_\pi(t) = \log Z_\pi(t) + \int_0^t \delta_\pi(s)\, ds, \quad \text{and,} \tag{2.2.156}$$

$$d \log \widehat{Z}_\pi(t) = d \log Z_\pi(t) + \delta_\pi(t)\, dt. \tag{2.2.157}$$

Alternatively, recalling equation (2.2.112), if we reinvest all the dividends for each of the stocks in the portfolio, then the total investment is given by

$$\begin{aligned} d\log \widehat{Z}_\pi(t) &= \sum_{i=1}^n \pi_i(t)\, d\log X_i(t) + \gamma_\pi^*(t)\, dt + \sum_{i=1}^n \pi_i(t)\delta_i(t)\, dt \\ &= d\log Z_\pi(t) + \sum_{i=1}^n \pi_i(t)\delta_i(t)\, dt \\ &= d\log Z_\pi(t) + \delta_\pi(t)\, dt. \end{aligned} \tag{2.2.158}$$

Integrating the above expression precisely yields the total return process

$$\log \widehat{Z}_\pi(t) - \log \widehat{Z}_\pi(0) = \log Z_\pi(t) - \log Z_\pi(0) + \int_0^t \delta_\pi(s)\, ds, \quad \text{and,}$$

$$\log \widehat{Z}_\pi(t) = \log Z_\pi(t) + \int_0^t \delta_\pi(s)\, ds, \quad \text{and,} \tag{2.2.159}$$

$$\widehat{Z}_\pi(t) = Z_\pi(t) \exp\left(\int_0^t \delta_\pi(s)\, ds\right), \tag{2.2.160}$$

where equation (2.2.159) follows from the fact that $\log \widehat{Z}_\pi(0) = \log Z_\pi(0)$. The process $\widehat{Z}_\pi$ represents the value of a portfolio with the same weights as π but in which all dividends are reinvested proportionally across the entire portfolio according to the weight of each stock. Hence, the reinvestment of dividends alters the value of Z_π to $\widehat{Z}_\pi$ whilst preserving the weights of the portfolio π.

Definition 2.2.38 (Augmented Portfolio Growth Rate). *The **augmented portfolio growth rate process** (or, **total portfolio growth rate process**) $\vartheta_\pi = \{\vartheta_\pi(t), t \in [0,\infty)\}$ for the portfolio π, is defined by*

$$\vartheta_\pi(t) \triangleq \gamma_\pi(t) + \delta_\pi(t), \qquad t \in [0,\infty). \tag{2.2.161}$$

From equations (2.2.157), (2.2.65) and (2.2.161), we deduce

$$\begin{aligned} d\log \widehat{Z}_\pi(t) &= d\log Z_\pi(t) + \delta_\pi(t)\, dt \\ &= \left(\gamma_\pi(t)\, dt + \sum_{i,\nu=1}^n \pi_i(t)\xi_{i\nu}(t)\, dW_\nu(t)\right) + \delta_\pi(t)\, dt \\ &= \big(\gamma_\pi(t) + \delta_\pi(t)\big)\, dt + \sum_{i,\nu=1}^n \pi_i(t)\xi_{i\nu}(t)\, dW_\nu(t) \\ &= \vartheta_\pi(t)\, dt + \sum_{i,\nu=1}^n \pi_i(t)\xi_{i\nu}(t)\, dW_\nu(t), \end{aligned} \tag{2.2.162}$$

for all $t \in [0,\infty)$, a.s., which can be directly contrasted to equation (2.2.65), given by

$$d\log Z_\pi(t) = \gamma_\pi(t)\, dt + \sum_{i,\nu=1}^n \pi_i(t)\xi_{i\nu}(t)\, dW_\nu(t).$$

Once again, the drift processes are the only difference between the two value processes. Clearly, for a market $\mathcal{M}$ without dividends (i.e., for which $\delta_i = 0$, for all $i = 1, 2, \ldots, n$), $Z_\pi = \widehat{Z}_\pi$, for all portfolios π. By applying equation (2.2.126) of Definition 2.2.28, we can rewrite equation (2.2.162) as

$$\begin{aligned} d\log \widehat{Z}_\pi(t) &= \vartheta_\pi(t)\, dt + \sum_{\nu=1}^n \left[\sum_{i=1}^n \pi_i(t)\xi_{i\nu}(t)\right] dW_\nu(t) \\ &= \vartheta_\pi(t)\, dt + \sum_{\nu=1}^n \xi_{\pi\nu}(t)\, dW_\nu(t). \end{aligned} \tag{2.2.163}$$

We shall now derive a result for the augmented growth rate of the portfolio π analogous to the result (2.2.111) obtained for the growth rate of the portfolio π.

Lemma 2.2.39. *For a portfolio π in the market $\mathcal{M}$, the augmented growth rate process of the portfolio π, $\vartheta_\pi = \{\vartheta_\pi(t), t \in [0,\infty)\}$, satisfies the analogue to* (2.2.111),

$$\vartheta_\pi(t) = \sum_{i=1}^{n} \pi_i(t)\vartheta_i(t) + \gamma_\pi^*(t), \qquad t \in [0,\infty), \qquad a.s. \tag{2.2.164}$$

Proof. We commence the proof by recalling equation (2.2.161) of Definition 2.2.38, and we have

$$\begin{aligned}
\vartheta_\pi(t) &= \gamma_\pi(t) + \delta_\pi(t) \\
&= \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \gamma_\pi^*(t) + \sum_{i=1}^{n} \pi_i(t)\delta_i(t) \qquad (2.2.165)\\
&= \sum_{i=1}^{n} \pi_i(t)\big(\gamma_i(t) + \delta_i(t)\big) + \gamma_\pi^*(t) \\
&= \sum_{i=1}^{n} \pi_i(t)\vartheta_i(t) + \gamma_\pi^*(t), \qquad (2.2.166)
\end{aligned}$$

where equation (2.2.165) follows from equation (2.2.111) and equation (2.2.154) of Definition 2.2.36, and equation (2.2.166) follows by the definition of the augmented growth rate of the ith stock, X_i, given in equation (2.2.152) of Definition 2.2.35. ■

From the above expression, we observe that the total growth rate of the portfolio π is a weighted average of the total growth rates of the individual (dividend-paying) component stocks and an additional component, γ_π^*. It is interesting to find that the excess growth rate of the portfolio comprised of dividend-paying stocks is exactly the excess growth rate of the non-dividend paying market, which we have already encountered. This leads us to the following definition.

Definition 2.2.40 (Excess Augmented Growth Rate). *The process $\vartheta_\pi^* = \{\vartheta_\pi^*(t), t \in [0,\infty)\}$ defined by*

$$\vartheta_\pi^*(t) \triangleq \gamma_\pi^*(t), \qquad t \in [0,\infty), \qquad a.s., \tag{2.2.167}$$

will be referred to as the ***excess augmented growth rate process*** *of the portfolio π.*

Thus, for the augmented growth rate of the portfolio π, which now takes the excess augmented growth rate into account, we have the following analogue to equation (2.2.164),

$$\vartheta_\pi(t) = \sum_{i=1}^{n} \pi_i(t)\vartheta_i(t) + \vartheta_\pi^*(t). \tag{2.2.168}$$

Recall from equation (2.2.162), that for $t \in [0,\infty)$, a.s., the total return process of the portfolio π satisfies

$$d\log \widehat{Z}_\pi(t) = \vartheta_\pi(t)\,dt + \sum_{i,\nu=1}^{n} \pi_i(t)\xi_{i\nu}(t)\,dW_\nu(t).$$

Employing equation (2.2.164) for the augmented portfolio growth rate ϑ_π, in the above equation, yields the following for the total return process of the portfolio π,

$$\begin{aligned}
d\log \widehat{Z}_\pi(t) &= \sum_{i=1}^{n} \pi_i(t)\vartheta_i(t)\,dt + \gamma_\pi^*(t)\,dt + \sum_{i,\nu=1}^{n} \pi_i(t)\xi_{i\nu}(t)\,dW_\nu(t) \\
&= \sum_{i=1}^{n} \pi_i(t)\left[\vartheta_i(t)\,dt + \sum_{\nu=1}^{n} \xi_{i\nu}(t)\,dW_\nu(t)\right] + \gamma_\pi^*(t)\,dt.
\end{aligned}$$

From equation (2.2.153), we notice that the expression enclosed in parentheses above is simply the total return process for the ith stock. Thus, the result presented below is the analogue to equation (2.2.112) of Corollary

2.2.25,

$$d\log \widehat{Z}_\pi(t) = \sum_{i=1}^n \pi_i(t)\, d\log \widehat{X}_i(t) + \gamma_\pi^*(t)\, dt, \quad \text{or,} \tag{2.2.169}$$

$$= \sum_{i=1}^n \pi_i(t)\, d\log \widehat{X}_i(t) + \vartheta_\pi^*(t)\, dt, \tag{2.2.170}$$

where, equation (2.2.170) follows from the equality (2.2.167) of Definition 2.2.40. Alternatively, the derivation of equation (2.2.170) can be carried out as follows

$$d\log \widehat{Z}_\pi(t) = d\log Z_\pi(t) + \delta_\pi(t)\, dt \tag{2.2.171}$$

$$= \left(\sum_{i=1}^n \pi_i(t)\, d\log X_i(t) + \gamma_\pi^*(t)\, dt\right) + \sum_{i=1}^n \pi_i(t)\delta_i(t)\, dt \tag{2.2.172}$$

$$= \sum_{i=1}^n \pi_i(t)\Big[d\log X_i(t) + \delta_i(t)\, dt\Big] + \gamma_\pi^*(t)\, dt$$

$$= \sum_{i=1}^n \pi_i(t)\, d\log \widehat{X}_i(t) + \gamma_\pi^*(t)\, dt \tag{2.2.173}$$

$$= \sum_{i=1}^n \pi_i(t)\, d\log \widehat{X}_i(t) + \vartheta_\pi^*(t)\, dt, \tag{2.2.174}$$

where equation (2.2.171) follows from equation (2.2.157), equation (2.2.172) follows by employing both equation (2.2.112) of Corollary 2.2.25 and equation (2.2.154) of Definition 2.2.36, equation (2.2.173) follows from equation (2.2.151), and the final expression follows from (2.2.167). Since the drift component is the only aspect that separates the above processes, for $i = 1, 2, \ldots, n$ and for all $t \in [0, \infty)$, the quadratic variation of the augmented value processes are established as follows

$$\left\langle \log \widehat{X}_i \right\rangle_t = \left\langle \log X_i \right\rangle_t = \int_0^t \sigma_{ii}(s)\, ds, \quad \text{and,} \tag{2.2.175}$$

$$\left\langle \log \widehat{Z}_\pi \right\rangle_t = \left\langle \log Z_\pi \right\rangle_t = \int_0^t \sigma_{\pi\pi}(s)\, ds, \quad \text{or, resp.,} \tag{2.2.176}$$

$$d\left\langle \log \widehat{X}_i \right\rangle_t = d\left\langle \log X_i \right\rangle_t = \sigma_{ii}(t)\, dt, \tag{2.2.177}$$

$$d\left\langle \log \widehat{Z}_\pi \right\rangle_t = d\left\langle \log Z_\pi \right\rangle_t = \sigma_{\pi\pi}(t)\, dt. \tag{2.2.178}$$

Similarly, we can derive expressions analogous to (2.2.42) and (2.2.44). Hence, for all $i, j = 1, 2, \ldots, n$ and for $t \in [0, \infty)$, we have

$$\left\langle \log \widehat{X}_i, \log \widehat{X}_j \right\rangle_t = \left\langle \log X_i, \log X_j \right\rangle_t = \int_0^t \sigma_{ij}(s)\, ds, \quad \text{or,} \tag{2.2.179}$$

$$d\left\langle \log \widehat{X}_i, \log \widehat{X}_j \right\rangle_t = d\left\langle \log X_i, \log X_j \right\rangle_t = \sigma_{ij}(t)\, dt. \tag{2.2.180}$$

An application of Itô's formula to $\exp\big(\log \widehat{X}_i(t)\big)$, together with equations (2.2.177) and (2.2.153), yields the following

$$d\widehat{X}_i(t) = \widehat{X}_i(t)\, d\log \widehat{X}_i(t) + \frac{1}{2}\, \widehat{X}_i(t)\, d\left\langle \log \widehat{X}_i \right\rangle_t \tag{2.2.181}$$

$$= \widehat{X}_i(t)\, d\log \widehat{X}_i(t) + \frac{1}{2}\, \widehat{X}_i(t)\, \sigma_{ii}(t)\, dt \tag{2.2.182}$$

$$= \left(\vartheta_i(t) + \frac{1}{2}\, \sigma_{ii}(t)\right) \widehat{X}_i(t)\, dt + \widehat{X}_i(t) \sum_{\nu=1}^n \xi_{i\nu}(t)\, dW_\nu(t) \tag{2.2.183}$$

$$= \widehat{X}_i(t)\left[\left(\vartheta_i(t) + \frac{1}{2}\, \sigma_{ii}(t)\right) dt + \sum_{\nu=1}^n \xi_{i\nu}(t)\, dW_\nu(t)\right]. \tag{2.2.184}$$

Therefore, we also have

$$\frac{d\widehat{X}_i(t)}{\widehat{X}_i(t)} = d\log\widehat{X}_i(t) + \frac{1}{2}\,d\left\langle\log\widehat{X}_i\right\rangle_t \tag{2.2.185}$$

$$= d\log\widehat{X}_i(t) + \frac{1}{2}\,\sigma_{ii}(t)\,dt \tag{2.2.186}$$

$$= \left(\vartheta_i(t) + \frac{1}{2}\,\sigma_{ii}(t)\right)dt + \sum_{\nu=1}^{n}\xi_{i\nu}(t)\,dW_\nu(t). \tag{2.2.187}$$

Consequently, from equations (2.2.182) and (2.2.177), we have

$$d\left\langle\widehat{X}_i\right\rangle_t = \widehat{X}_i^2(t)\,d\left\langle\log\widehat{X}_i\right\rangle_t = \sigma_{ii}(t)\,\widehat{X}_i^2(t)\,dt, \tag{2.2.188}$$

and, from equation (2.2.180), we have

$$d\left\langle\widehat{X}_i,\widehat{X}_j\right\rangle_t = \widehat{X}_i(t)\widehat{X}_j(t)\,d\left\langle\log\widehat{X}_i,\log\widehat{X}_j\right\rangle_t = \sigma_{ij}(t)\,\widehat{X}_i(t)\widehat{X}_j(t)\,dt. \tag{2.2.189}$$

Another application of Itô's formula to $\exp\big(\log\widehat{Z}_\pi(t)\big)$, together with (2.2.178) and (2.2.162), yields the following

$$d\widehat{Z}_\pi(t) = \widehat{Z}_\pi(t)\,d\log\widehat{Z}_\pi(t) + \frac{1}{2}\,\widehat{Z}_\pi(t)\,d\left\langle\log\widehat{Z}_\pi\right\rangle_t \tag{2.2.190}$$

$$= \widehat{Z}_\pi(t)\,d\log\widehat{Z}_\pi(t) + \frac{1}{2}\,\widehat{Z}_\pi(t)\,\sigma_{\pi\pi}(t)\,dt \tag{2.2.191}$$

$$= \left(\vartheta_\pi(t) + \frac{1}{2}\,\sigma_{\pi\pi}(t)\right)\widehat{Z}_\pi(t)\,dt + \widehat{Z}_\pi(t)\sum_{i,\nu=1}^{n}\pi_i(t)\xi_{i\nu}(t)\,dW_\nu(t) \tag{2.2.192}$$

$$= \widehat{Z}_\pi(t)\left[\left(\vartheta_\pi(t) + \frac{1}{2}\,\sigma_{\pi\pi}(t)\right)dt + \sum_{i,\nu=1}^{n}\pi_i(t)\xi_{i\nu}(t)\,dW_\nu(t)\right]. \tag{2.2.193}$$

Therefore, we also have

$$\frac{d\widehat{Z}_\pi(t)}{\widehat{Z}_\pi(t)} = \left(\vartheta_\pi(t) + \frac{1}{2}\,\sigma_{\pi\pi}(t)\right)dt + \sum_{i,\nu=1}^{n}\pi_i(t)\xi_{i\nu}(t)\,dW_\nu(t). \tag{2.2.194}$$

Alternatively, utilising (2.2.163), yields the following

$$d\widehat{Z}_\pi(t) = \left(\vartheta_\pi(t) + \frac{1}{2}\,\sigma_{\pi\pi}(t)\right)\widehat{Z}_\pi(t)\,dt + \widehat{Z}_\pi(t)\sum_{\nu=1}^{n}\xi_{\pi\nu}(t)\,dW_\nu(t) \tag{2.2.195}$$

$$= \widehat{Z}_\pi(t)\left[\left(\vartheta_\pi(t) + \frac{1}{2}\,\sigma_{\pi\pi}(t)\right)dt + \sum_{\nu=1}^{n}\xi_{\pi\nu}(t)\,dW_\nu(t)\right]. \tag{2.2.196}$$

Therefore, we also have

$$\frac{d\widehat{Z}_\pi(t)}{\widehat{Z}_\pi(t)} = \left(\vartheta_\pi(t) + \frac{1}{2}\,\sigma_{\pi\pi}(t)\right)dt + \sum_{\nu=1}^{n}\xi_{\pi\nu}(t)\,dW_\nu(t). \tag{2.2.197}$$

Consequently, from equations (2.2.191) and (2.2.178), we have

$$d\left\langle\widehat{Z}_\pi\right\rangle_t = \widehat{Z}_\pi^2(t)\,d\left\langle\log\widehat{Z}_\pi\right\rangle_t = \sigma_{\pi\pi}(t)\,\widehat{Z}_\pi^2(t)\,dt. \tag{2.2.198}$$

Additionally, from equation (2.2.190), we have

$$\frac{d\widehat{Z}_\pi(t)}{\widehat{Z}_\pi(t)} = d\log\widehat{Z}_\pi(t) + \frac{1}{2}\,d\left\langle\log\widehat{Z}_\pi\right\rangle_t \tag{2.2.199}$$

$$= d\log\widehat{Z}_\pi(t) + \frac{1}{2}\,\sigma_{\pi\pi}(t)\,dt, \tag{2.2.200}$$

which, combined with equation (2.2.170), gives

$$\frac{d\widehat{Z}_\pi(t)}{\widehat{Z}_\pi(t)} = \left(\vartheta^*_\pi(t) + \frac{1}{2}\,\sigma_{\pi\pi}(t)\right) dt + \sum_{i=1}^n \pi_i(t)\, d\log \widehat{X}_i(t), \tag{2.2.201}$$

or, in the following form

$$d\widehat{Z}_\pi(t) = \widehat{Z}_\pi(t)\left[\left(\vartheta^*_\pi(t) + \frac{1}{2}\,\sigma_{\pi\pi}(t)\right) dt + \sum_{i=1}^n \pi_i(t)\, d\log \widehat{X}_i(t)\right]. \tag{2.2.202}$$

Now, adopting equation (2.2.185) in the following form

$$d\log \widehat{X}_i(t) = \frac{d\widehat{X}_i(t)}{\widehat{X}_i(t)} - \frac{1}{2}\,\sigma_{ii}(t)\, dt, \tag{2.2.203}$$

we obtain, together with equation (2.2.109) and equation (2.2.167) of Definition 2.2.40, the following

$$\begin{aligned}
\frac{d\widehat{Z}_\pi(t)}{\widehat{Z}_\pi(t)} &= \left(\vartheta^*_\pi(t) + \frac{1}{2}\,\sigma_{\pi\pi}(t) - \frac{1}{2}\sum_{i=1}^n \pi_i(t)\sigma_{ii}(t)\right) dt + \sum_{i=1}^n \pi_i(t)\,\frac{d\widehat{X}_i(t)}{\widehat{X}_i(t)} \\
&= \left(\vartheta^*_\pi(t) - \gamma^*_\pi(t)\right) dt + \sum_{i=1}^n \pi_i(t)\,\frac{d\widehat{X}_i(t)}{\widehat{X}_i(t)} \\
&= \sum_{i=1}^n \pi_i(t)\,\frac{d\widehat{X}_i(t)}{\widehat{X}_i(t)}.
\end{aligned} \tag{2.2.204}$$

Therefore, we get the following result

$$d\widehat{Z}_\pi(t) = \sum_{i=1}^n \pi_i(t)\,\widehat{Z}_\pi(t)\,\frac{d\widehat{X}_i(t)}{\widehat{X}_i(t)} \tag{2.2.205}$$

$$= \widehat{Z}_\pi(t)\sum_{i=1}^n \pi_i(t)\,\frac{d\widehat{X}_i(t)}{\widehat{X}_i(t)}. \tag{2.2.206}$$

We can also write equation (2.2.200) in the following form

$$d\log \widehat{Z}_\pi(t) = \frac{d\widehat{Z}_\pi(t)}{\widehat{Z}_\pi(t)} - \frac{1}{2}\,\sigma_{\pi\pi}(t)\, dt. \tag{2.2.207}$$

Moreover, employing equations (2.2.174), (2.2.167) and (2.2.203) gives

$$\begin{aligned}
d\log \widehat{Z}_\pi(t) &= \sum_{i=1}^n \pi_i(t)\left[\frac{d\widehat{X}_i(t)}{\widehat{X}_i(t)} - \frac{1}{2}\,\sigma_{ii}(t)\, dt\right] + \frac{1}{2}\left(\sum_{i=1}^n \pi_i(t)\sigma_{ii}(t) - \sigma_{\pi\pi}(t)\right) dt \\
&= \sum_{i=1}^n \pi_i(t)\frac{d\widehat{X}_i(t)}{\widehat{X}_i(t)} - \frac{1}{2}\sum_{i=1}^n \pi_i(t)\sigma_{ii}(t)\, dt + \frac{1}{2}\sum_{i=1}^n \pi_i(t)\sigma_{ii}(t)\, dt - \frac{1}{2}\,\sigma_{\pi\pi}(t)\, dt \\
&= \sum_{i=1}^n \pi_i(t)\frac{d\widehat{X}_i(t)}{\widehat{X}_i(t)} - \frac{1}{2}\,\sigma_{\pi\pi}(t)\, dt.
\end{aligned} \tag{2.2.208}$$

2.3 Relative Return

As will be apparent in later chapters, it is frequently of interest to measure the performance of stocks or portfolios relative to a given benchmark portfolio or index. This is of extreme importance, since one of our aspirations is to collate the performance of two different portfolios so as to derive potential profitable outcomes. It must remain clear in our minds though that our primary objective is to collate the performance of a particular portfolio with the canonical benchmark portfolio, to wit, the market portfolio, which will be reviewed in Section 2.12. In this section, the notion of the **relative return** is discussed.

Before we get to the quintessential definition of the relative return, we first offer a definition of a relative return of a slightly different kind.

Definition 2.3.1 (Relative Return of Stocks). *For stock $X_i = \{X_i(t), t \in [0,\infty)\}$, for $i = 1, 2, \ldots, n$, and for a stock $X_j = \{X_j(t), t \in [0,\infty)\}$, $j \in \{1, 2, \ldots, n\}$, the process $\log\big(X_i/X_j\big) = \big\{\log\big(X_i(t)/X_j(t)\big), t \in [0,\infty)\big\}$, is called the **relative return process** of X_i versus X_j.*

Thus, by equation (2.2.20), the return of the ith stock relative to the jth stock can be described by the following dynamics

$$\begin{aligned} d\log\Big(X_i(t)/X_j(t)\Big) &= d\log X_i(t) - d\log X_j(t) \\ &= \Big(\gamma_i(t)\,dt + \sum_{\nu=1}^{n} \xi_{i\nu}(t)\,dW_\nu(t)\Big) - \Big(\gamma_j(t)\,dt + \sum_{\nu=1}^{n} \xi_{j\nu}(t)\,dW_\nu(t)\Big) \\ &= \big(\gamma_i(t) - \gamma_j(t)\big)\,dt + \sum_{\nu=1}^{n} \big(\xi_{i\nu}(t) - \xi_{j\nu}(t)\big)\,dW_\nu(t). \end{aligned} \tag{2.3.1}$$

Definition 2.3.2 (Relative Covariance of Stocks). *For stocks $X_i = \{X_i(t), t \in [0,\infty)\}$ and $X_j = \big\{X_j(t), t \in [0,\infty)\big\}$, for all $i, j = 1, 2, \ldots, n$, and for a fixed stock $X_k = \{X_k(t), t \in [0,\infty)\}$, where $k \in \{1, 2, \ldots, n\}$, the $(n \times n)-$matrix-valued process $\boldsymbol{\tau}^{\chi_k}(t) = \big(\tau_{ij}^{\chi_k}(t)\big)_{1\le i,j\le n}$, defined by*

$$\tau_{ij}^{\chi_k}(t) \triangleq \frac{d}{dt}\big\langle \log\big(X_i/X_k\big), \log\big(X_j/X_k\big)\big\rangle_t, \qquad t \in [0,\infty), \qquad a.s., \tag{2.3.2}$$

*represents the **relative covariance process**, i.e., the covariance of the ith stock X_i and the jth stock X_j, both relative to some fixed stock X_k. More specifically, the covariance of the relative return processes $\log\big(X_i/X_k\big)$ and $\log\big(X_j/X_k\big)$. We can think of X_k as representing a portfolio χ_k that comprises only of the single stock X_k, where $k \in \{1, 2, \ldots, n\}$.*

Remark 2.3.3. Thus, if the "benchmark stock" is X_1, i.e., in this case $k = 1$, then this corresponds to the portfolio χ_1 consisting only of the single stock X_1. Consequently, for all $i, j = 1, 2, \ldots, n$, the relative covariance process of interest to us is given by

$$\tau_{ij}^{\chi_1}(t) \triangleq \frac{d}{dt}\big\langle \log\big(X_i/X_1\big), \log\big(X_j/X_1\big)\big\rangle_t, \qquad t \in [0,\infty), \qquad \text{a.s.}$$

The same can be said for $k = 2$, which corresponds to the portfolio χ_2 consisting only of the single stock X_2, and, for all $i, j = 1, 2, \ldots, n$, the relative covariance of interest is

$$\tau_{ij}^{\chi_2}(t) = \frac{d}{dt}\big\langle \log\big(X_i/X_2\big), \log\big(X_j/X_2\big)\big\rangle_t, \qquad t \in [0,\infty), \qquad \text{a.s.},$$

and so forth,

$$\begin{aligned} \tau_{ij}^{\chi_3}(t) &= \frac{d}{dt}\big\langle \log\big(X_i/X_3\big), \log\big(X_j/X_3\big)\big\rangle_t, \qquad t \in [0,\infty), \qquad \text{a.s.}, \\ &\vdots \\ \tau_{ij}^{\chi_n}(t) &= \frac{d}{dt}\big\langle \log\big(X_i/X_n\big), \log\big(X_j/X_n\big)\big\rangle_t, \qquad t \in [0,\infty), \qquad \text{a.s.} \end{aligned}$$

Therefore, from (2.3.2), for all $i, j = 1, 2, \ldots, n$, we have

$$d\big\langle \log\big(X_i/X_k\big), \log\big(X_j/X_k\big)\big\rangle_t = \tau_{ij}^{\chi_k}(t)\,dt, \qquad t \in [0,\infty), \qquad \text{a.s., and,} \tag{2.3.3}$$

$$\big\langle \log\big(X_i/X_k\big), \log\big(X_j/X_k\big)\big\rangle_t = \int_0^t \tau_{ij}^{\chi_k}(s)\,ds, \qquad t \in [0,\infty), \qquad \text{a.s.} \tag{2.3.4}$$

From equation (2.3.1), for the covariation of this relative return, we obtain for all $i, j = 1, 2, \ldots, n$,

$$\begin{aligned}
\left\langle \log\left(X_i/X_k\right), \log\left(X_j/X_k\right)\right\rangle_t &= \left\langle \int_0^{\cdot} \sum_{\nu=1}^{n} \left(\xi_{i\nu,s} - \xi_{k\nu,s}\right) dW_{\nu,s}, \int_0^{\cdot} \sum_{\upsilon=1}^{n} \left(\xi_{j\upsilon,s} - \xi_{k\upsilon,s}\right) dW_{\upsilon,s} \right\rangle_t \\
&= \left\langle \sum_{\nu=1}^{n} \int_0^{\cdot} \left(\xi_{i\nu,s} - \xi_{k\nu,s}\right) dW_{\nu,s}, \sum_{\upsilon=1}^{n} \int_0^{\cdot} \left(\xi_{j\upsilon,s} - \xi_{k\upsilon,s}\right) dW_{\upsilon,s} \right\rangle_t \\
&= \sum_{\nu,\upsilon=1}^{n} \left\langle \int_0^{\cdot} \left(\xi_{i\nu,s} - \xi_{k\nu,s}\right) dW_{\nu,s}, \int_0^{\cdot} \left(\xi_{j\upsilon,s} - \xi_{k\upsilon,s}\right) dW_{\upsilon,s} \right\rangle_t \\
&= \sum_{\nu,\upsilon=1}^{n} \int_0^{t} \Big(\xi_{i\nu}(s) - \xi_{k\nu}(s)\Big)\Big(\xi_{j\upsilon}(s) - \xi_{k\upsilon}(s)\Big)\, d\left\langle W_\nu, W_\upsilon\right\rangle_s \\
&= \int_0^{t} \sum_{\nu,\upsilon=1}^{n} \Big(\xi_{i\nu}(s) - \xi_{k\nu}(s)\Big)\Big(\xi_{j\upsilon}(s) - \xi_{k\upsilon}(s)\Big)\, \rho_{\nu\upsilon}(s)\, ds \\
&= \int_0^{t} \sum_{\nu=1}^{n} \Big(\xi_{i\nu}(s) - \xi_{k\nu}(s)\Big)\Big(\xi_{j\nu}(s) - \xi_{k\nu}(s)\Big)\, ds. \qquad (2.3.5)
\end{aligned}$$

Therefore, we have

$$\frac{d}{dt}\left\langle \log\left(X_i/X_k\right), \log\left(X_j/X_k\right)\right\rangle_t = \sum_{\nu=1}^{n} \Big(\xi_{i\nu}(t) - \xi_{k\nu}(t)\Big)\Big(\xi_{j\nu}(t) - \xi_{k\nu}(t)\Big). \qquad (2.3.6)$$

Expanding this expression out gives

$$\begin{aligned}
d\left\langle \log\left(X_i/X_k\right), \log\left(X_j/X_k\right)\right\rangle_t &= d\left\langle \int_0^{\cdot} \sum_{\nu=1}^{n} \left(\xi_{i\nu,s} - \xi_{k\nu,s}\right) dW_{\nu,s}, \int_0^{\cdot} \sum_{\upsilon=1}^{n} \left(\xi_{j\upsilon,s} - \xi_{k\upsilon,s}\right) dW_{\upsilon,s} \right\rangle_t \\
&= d\left(\int_0^{t} \sum_{\nu=1}^{n} \Big(\xi_{i\nu}(s) - \xi_{k\nu}(s)\Big)\Big(\xi_{j\nu}(s) - \xi_{k\nu}(s)\Big)\, ds \right) \\
&= \sum_{\nu=1}^{n} \Big(\xi_{i\nu}(t) - \xi_{k\nu}(t)\Big)\Big(\xi_{j\nu}(t) - \xi_{k\nu}(t)\Big)\, dt \qquad (2.3.7) \\
&= \sum_{\nu=1}^{n} \Big(\xi_{i\nu}(t)\xi_{j\nu}(t) - \xi_{i\nu}(t)\xi_{k\nu}(t) - \xi_{j\nu}(t)\xi_{k\nu}(t) + \xi_{k\nu}^2(t)\Big)\, dt \\
&= \sum_{\nu=1}^{n} \xi_{i\nu}(t)\xi_{j\nu}(t)\, dt - \sum_{\nu=1}^{n} \xi_{i\nu}(t)\xi_{k\nu}(t)\, dt - \sum_{\nu=1}^{n} \xi_{j\nu}(t)\xi_{k\nu}(t)\, dt + \sum_{\nu=1}^{n} \xi_{k\nu}^2(t)\, dt,
\end{aligned}$$

which, in conjunction with (2.2.37) and (2.2.47), yields

$$\begin{aligned}
d\left\langle \log\left(X_i/X_k\right), \log\left(X_j/X_k\right)\right\rangle_t &= \sigma_{ij}(t)\, dt - \sigma_{ik}(t)\, dt - \sigma_{jk}(t)\, dt + \sigma_{kk}(t)\, dt \\
&= \big(\sigma_{ij}(t) - \sigma_{ik}(t) - \sigma_{jk}(t) + \sigma_{kk}(t)\big)\, dt. \qquad (2.3.8)
\end{aligned}$$

Consequently, we have

$$\tau_{ij}^{\chi_k}(t) = \frac{d}{dt}\left\langle \log\left(X_i/X_k\right), \log\left(X_j/X_k\right)\right\rangle_t = \sigma_{ij}(t) - \sigma_{ik}(t) - \sigma_{jk}(t) + \sigma_{kk}(t). \qquad (2.3.9)$$

Alternatively, for $t \in [0, \infty)$, a.s., we have

$$\begin{aligned}
\left\langle \log\left(X_i/X_k\right), \log\left(X_j/X_k\right)\right\rangle_t &= \left\langle \log X_i - \log X_k, \log X_j - \log X_k \right\rangle_t \\
&= \left\langle \log X_i, \log X_j \right\rangle_t - \left\langle \log X_i, \log X_k \right\rangle_t - \left\langle \log X_k, \log X_j \right\rangle_t + \left\langle \log X_k \right\rangle_t \\
&= \left\langle \log X_i, \log X_j \right\rangle_t - \left\langle \log X_i, \log X_k \right\rangle_t - \left\langle \log X_j, \log X_k \right\rangle_t + \left\langle \log X_k \right\rangle_t. \qquad (2.3.10)
\end{aligned}$$

Thus, we have

$$d\left\langle \log\left(X_i/X_k\right), \log\left(X_j/X_k\right)\right\rangle_t = d\left\langle \log X_i, \log X_j\right\rangle_t - d\left\langle \log X_i, \log X_k\right\rangle_t - d\left\langle \log X_j, \log X_k\right\rangle_t + d\left\langle \log X_k\right\rangle_t. \tag{2.3.11}$$

Therefore, by appealing to (2.3.3) and (2.2.45), we equivalently obtain

$$\begin{aligned} \tau_{ij}^{\mathsf{X}_k}(t)\,dt &= d\left\langle \log X_i, \log X_j\right\rangle_t - d\left\langle \log X_i, \log X_k\right\rangle_t - d\left\langle \log X_j, \log X_k\right\rangle_t + d\left\langle \log X_k\right\rangle_t \\ &= \sigma_{ij}(t)\,dt - \sigma_{ik}(t)\,dt - \sigma_{jk}(t)\,dt + \sigma_{kk}(t)\,dt \\ &= \big(\sigma_{ij}(t) - \sigma_{ik}(t) - \sigma_{jk}(t) + \sigma_{kk}(t)\big)\,dt, \end{aligned}$$

and, we have

$$\tau_{ij}^{\mathsf{X}_k}(t) = \sigma_{ij}(t) - \sigma_{ik}(t) - \sigma_{jk}(t) + \sigma_{kk}(t). \tag{2.3.12}$$

In matrix form, we have

$$\tau_{ij}^{\mathsf{X}_k}(t) = \mathbf{e}_i \boldsymbol{\tau}^{\mathsf{X}_k}(t)\mathbf{e}_j^{\mathsf{T}} \tag{2.3.13}$$

$$= \big(\mathbf{e}_i - \mathbf{e}_k\big)\boldsymbol{\sigma}(t)\big(\mathbf{e}_j - \mathbf{e}_k\big)^{\mathsf{T}}. \tag{2.3.14}$$

Thus, in summary, the relative covariance for the stocks can be represented as follows

$$\tau_{ij}^{\mathsf{X}_k}(t) = \frac{d}{dt}\left\langle \log\left(X_i/X_k\right), \log\left(X_j/X_k\right)\right\rangle_t = \sum_{\nu=1}^{n}\Big(\xi_{i\nu}(t) - \xi_{k\nu}(t)\Big)\Big(\xi_{j\nu}(t) - \xi_{k\nu}(t)\Big), \quad \text{and,} \tag{2.3.15}$$

$$\tau_{ij}^{\mathsf{X}_k}(t)\,dt = d\left\langle \log\left(X_i/X_k\right), \log\left(X_j/X_k\right)\right\rangle_t = \sum_{\nu=1}^{n}\Big(\xi_{i\nu}(t) - \xi_{k\nu}(t)\Big)\Big(\xi_{j\nu}(t) - \xi_{k\nu}(t)\Big)\,dt. \tag{2.3.16}$$

Note that, for all $i, j = 1, 2, \ldots, n$ and for fixed $k, m \in \{1, 2, \ldots, n\}$, we shall also require the following notation

$$\tau_{ij}^{\mathsf{X}_k\mathsf{X}_m}(t) \triangleq \frac{d}{dt}\left\langle \log\left(X_i/X_k\right), \log\left(X_j/X_m\right)\right\rangle_t = \sum_{\nu=1}^{n}\Big(\xi_{i\nu}(t) - \xi_{k\nu}(t)\Big)\Big(\xi_{j\nu}(t) - \xi_{m\nu}(t)\Big), \quad \text{and,} \tag{2.3.17}$$

$$\tau_{ij}^{\mathsf{X}_k\mathsf{X}_m}(t)\,dt \triangleq d\left\langle \log\left(X_i/X_k\right), \log\left(X_j/X_m\right)\right\rangle_t = \sum_{\nu=1}^{n}\Big(\xi_{i\nu}(t) - \xi_{k\nu}(t)\Big)\Big(\xi_{j\nu}(t) - \xi_{m\nu}(t)\Big)\,dt. \tag{2.3.18}$$

Thus, from (2.3.17), we have

$$\begin{aligned} \tau_{ij}^{\mathsf{X}_k\mathsf{X}_m}(t) &= \sum_{\nu=1}^{n}\xi_{i\nu}(t)\xi_{j\nu}(t) - \sum_{\nu=1}^{n}\xi_{i\nu}(t)\xi_{m\nu}(t) - \sum_{\nu=1}^{n}\xi_{j\nu}(t)\xi_{k\nu}(t) + \sum_{\nu=1}^{n}\xi_{k\nu}(t)\xi_{m\nu}(t) \\ &= \sigma_{ij}(t) - \sigma_{im}(t) - \sigma_{jk}(t) + \sigma_{km}(t), \end{aligned} \tag{2.3.19}$$

and, equivalently

$$\begin{aligned} \tau_{ij}^{\mathsf{X}_k\mathsf{X}_m}(t)\,dt &= \sigma_{ij}(t)\,dt - \sigma_{im}(t)\,dt - \sigma_{jk}(t)\,dt + \sigma_{km}(t)\,dt \\ &= \big(\sigma_{ij}(t) - \sigma_{im}(t) - \sigma_{jk}(t) + \sigma_{km}(t)\big)\,dt. \end{aligned} \tag{2.3.20}$$

Definition 2.3.4 (Relative Variance of Stocks)**.** *For a stock $X_i = \{X_i(t), t \in [0,\infty)\}$, for all $i = 1, 2, \ldots, n$, and portfolio χ_k comprising a single fixed stock $X_k = \{X_k(t), t \in [0,\infty)\}$, $k \in \{1, 2, \ldots, n\}$, the process $\tau_{ii}^{\mathsf{X}_k} = \big\{\tau_{ii}^{\mathsf{X}_k}(t), t \in [0,\infty)\big\}$, for $i = 1, 2, \ldots, n$, defined by*

$$\tau_{ii}^{\mathsf{X}_k}(t) \triangleq \frac{d}{dt}\left\langle \log\left(X_i/X_k\right)\right\rangle_t, \qquad t \in [0,\infty), \qquad a.s., \tag{2.3.21}$$

*represents the **relative variance process** of the ith stock X_i versus the kth stock X_k, i.e., the variance of the relative return process $\log\left(X_i/X_k\right)$, or the variance of the ith stock X_i relative to the kth stock X_k.*

Therefore, from (2.3.21), we have

$$d\left\langle \log\left(X_i/X_k\right)\right\rangle_t = \tau_{ii}^{\chi_k}(t)\,dt, \qquad t\in[0,\infty), \qquad \text{a.s., and,} \tag{2.3.22}$$

$$\left\langle \log\left(X_i/X_k\right)\right\rangle_t = \int_0^t \tau_{ii}^{\chi_k}(s)\,ds, \qquad t\in[0,\infty), \qquad \text{a.s.} \tag{2.3.23}$$

From equation (2.3.1), the quadratic variation of this relative return is obtained as follows, for $i=1,2,\ldots,n$,

$$\begin{aligned}
\left\langle \log\left(X_i/X_k\right)\right\rangle_t &= \left\langle \int_0^{\cdot} \sum_{\nu=1}^n \left(\xi_{i\nu,s}-\xi_{k\nu,s}\right) dW_{\nu,s}\right\rangle_t \\
&= \left\langle \sum_{\nu=1}^n \int_0^{\cdot} \left(\xi_{i\nu,s}-\xi_{k\nu,s}\right) dW_{\nu,s}\right\rangle_t \\
&= \sum_{\nu=1}^n \left\langle \int_0^{\cdot} \left(\xi_{i\nu,s}-\xi_{k\nu,s}\right) dW_{\nu,s}\right\rangle_t \\
&= \sum_{\nu=1}^n \int_0^t \Big(\xi_{i\nu}(s)-\xi_{k\nu}(s)\Big)^2 d\left\langle W_\nu\right\rangle_s \\
&= \int_0^t \sum_{\nu=1}^n \Big(\xi_{i\nu}(s)-\xi_{k\nu}(s)\Big)^2 ds.
\end{aligned} \tag{2.3.24}$$

Therefore, we have

$$\frac{d}{dt}\left\langle \log\left(X_i/X_k\right)\right\rangle_t = \sum_{\nu=1}^n \Big(\xi_{i\nu}(t)-\xi_{k\nu}(t)\Big)^2. \tag{2.3.25}$$

Thus, expanding out gives

$$\begin{aligned}
d\left\langle \log\left(X_i/X_k\right)\right\rangle_t &= d\left\langle \int_0^{\cdot} \sum_{\nu=1}^n \left(\xi_{i\nu,s}-\xi_{k\nu,s}\right) dW_{\nu,s}\right\rangle_t \\
&= d\left(\int_0^t \sum_{\nu=1}^n \Big(\xi_{i\nu}(s)-\xi_{k\nu}(s)\Big)^2 ds\right) \\
&= \sum_{\nu=1}^n \Big(\xi_{i\nu}(t)-\xi_{k\nu}(t)\Big)^2 dt \\
&= \sum_{\nu=1}^n \Big(\xi_{i\nu}^2(t)-2\,\xi_{i\nu}(t)\xi_{k\nu}(t)+\xi_{k\nu}^2(t)\Big)\,dt \\
&= \sum_{\nu=1}^n \xi_{i\nu}^2(t)\,dt - 2\sum_{\nu=1}^n \xi_{i\nu}(t)\xi_{k\nu}(t)\,dt + \sum_{\nu=1}^n \xi_{k\nu}^2(t)\,dt,
\end{aligned}$$

which, in conjunction with (2.2.37) and (2.2.47), yields

$$\begin{aligned}
d\left\langle \log\left(X_i/X_k\right)\right\rangle_t &= \sigma_{ii}(t)\,dt - 2\,\sigma_{ik}(t)\,dt + \sigma_{kk}(t)\,dt \\
&= \big(\sigma_{ii}(t)-2\,\sigma_{ik}(t)+\sigma_{kk}(t)\big)\,dt.
\end{aligned} \tag{2.3.26}$$

Consequently, we have

$$\tau_{ii}^{\chi_k}(t) = \frac{d}{dt}\left\langle \log\left(X_i/X_k\right)\right\rangle_t = \sigma_{ii}(t)-2\,\sigma_{ik}(t)+\sigma_{kk}(t). \tag{2.3.27}$$

Alternatively, for $t\in[0,\infty)$, a.s., we have

$$\begin{aligned}
\left\langle \log\left(X_i/X_k\right)\right\rangle_t &= \left\langle \log X_i - \log X_k\right\rangle_t \\
&= \left\langle \log X_i\right\rangle_t - \left\langle \log X_i, \log X_k\right\rangle_t - \left\langle \log X_k, \log X_i\right\rangle_t + \left\langle \log X_k\right\rangle_t \\
&= \left\langle \log X_i\right\rangle_t - 2\left\langle \log X_i, \log X_k\right\rangle_t + \left\langle \log X_k\right\rangle_t,
\end{aligned} \tag{2.3.28}$$

or, equivalently

$$d\left\langle \log\left(X_i/X_k\right)\right\rangle_t = d\left\langle \log X_i\right\rangle_t - 2\,d\left\langle \log X_i, \log X_k\right\rangle_t + d\left\langle \log X_k\right\rangle_t. \tag{2.3.29}$$

Therefore, by appealing to equations (2.3.21) and (2.2.45), we equivalently obtain

$$\begin{aligned} \tau_{ii}^{\mathsf{X}_k}(t)\,dt &= d\left\langle \log X_i\right\rangle_t - 2\,d\left\langle \log X_i, \log X_k\right\rangle_t + d\left\langle \log X_k\right\rangle_t \\ &= \sigma_{ii}(t)\,dt - 2\,\sigma_{ik}(t)\,dt + \sigma_{kk}(t)\,dt \\ &= \big(\sigma_{ii}(t) - 2\,\sigma_{ik}(t) + \sigma_{kk}(t)\big)\,dt, \quad \text{or,} \qquad (2.3.30) \\ \tau_{ii}^{\mathsf{X}_k}(t) &= \sigma_{ii}(t) - 2\,\sigma_{ik}(t) + \sigma_{kk}(t). \qquad (2.3.31) \end{aligned}$$

In matrix form, we have

$$\begin{aligned} \tau_{ii}^{\mathsf{X}_k}(t) &= \mathbf{e}_i \boldsymbol{\tau}^{\mathsf{X}_k}(t)\mathbf{e}_i^{\mathsf{T}} \qquad (2.3.32) \\ &= \big(\mathbf{e}_i - \mathbf{e}_k\big)\boldsymbol{\sigma}(t)\big(\mathbf{e}_i - \mathbf{e}_k\big)^{\mathsf{T}}. \qquad (2.3.33) \end{aligned}$$

Thus, in summary, the relative variance for the stocks can be represented as follows

$$\tau_{ii}^{\mathsf{X}_k}(t) = \frac{d}{dt}\left\langle \log\left(X_i/X_k\right)\right\rangle_t = \sum_{\nu=1}^{n}\Big(\xi_{i\nu}(t) - \xi_{k\nu}(t)\Big)^2, \quad \text{or,} \tag{2.3.34}$$

$$\tau_{ii}^{\mathsf{X}_k}(t)\,dt = d\left\langle \log\left(X_i/X_k\right)\right\rangle_t = \sum_{\nu=1}^{n}\Big(\xi_{i\nu}(t) - \xi_{k\nu}(t)\Big)^2\,dt. \tag{2.3.35}$$

Moreover, armed with equations (2.3.21), (2.3.31) and the symmetry property of the covariance process, for $i, k = 1, 2, \ldots, n$, we obtain

$$\begin{aligned} \tau_{kk}^{\mathsf{X}_i}(t) &= \frac{d}{dt}\left\langle \log\left(X_k/X_i\right)\right\rangle_t \\ &= \sigma_{kk}(t) - 2\,\sigma_{ki}(t) + \sigma_{ii}(t) \\ &= \sigma_{ii}(t) - 2\,\sigma_{ik}(t) + \sigma_{kk}(t) \\ &= \frac{d}{dt}\left\langle \log\left(X_i/X_k\right)\right\rangle_t. \end{aligned}$$

Therefore, we have the identity

$$\tau_{kk}^{\mathsf{X}_i}(t) = \tau_{ii}^{\mathsf{X}_k}(t). \tag{2.3.36}$$

The identity above implies that the relative variance process of X_i versus X_k is equivalent to the relative variance process of X_k versus X_i, as we would naturally expect. In addition, we shall later require the following notation, for all $i = 1, 2, \ldots, n$ and for fixed $k, m \in \{1, 2, \ldots, n\}$, we have

$$\tau_{ii}^{\mathsf{X}_k\mathsf{X}_m}(t) \triangleq \frac{d}{dt}\left\langle \log\left(X_i/X_k\right), \log\left(X_i/X_m\right)\right\rangle_t = \sum_{\nu=1}^{n}\Big(\xi_{i\nu}(t) - \xi_{k\nu}(t)\Big)\Big(\xi_{i\nu}(t) - \xi_{m\nu}(t)\Big), \quad \text{and,} \tag{2.3.37}$$

$$\tau_{ii}^{\mathsf{X}_k\mathsf{X}_m}(t)\,dt \triangleq d\left\langle \log\left(X_i/X_k\right), \log\left(X_i/X_m\right)\right\rangle_t = \sum_{\nu=1}^{n}\Big(\xi_{i\nu}(t) - \xi_{k\nu}(t)\Big)\Big(\xi_{i\nu}(t) - \xi_{m\nu}(t)\Big)\,dt. \tag{2.3.38}$$

Thus, from (2.3.37), we have

$$\begin{aligned} \tau_{ii}^{\mathsf{X}_k\mathsf{X}_m}(t) &= \sum_{\nu=1}^{n}\xi_{i\nu}^2(t) - \sum_{\nu=1}^{n}\xi_{i\nu}(t)\xi_{m\nu}(t) - \sum_{\nu=1}^{n}\xi_{i\nu}(t)\xi_{k\nu}(t) + \sum_{\nu=1}^{n}\xi_{k\nu}(t)\xi_{m\nu}(t) \\ &= \sigma_{ii}(t) - \sigma_{im}(t) - \sigma_{ik}(t) + \sigma_{km}(t), \qquad (2.3.39) \end{aligned}$$

or, equivalently

$$\begin{aligned} \tau_{ii}^{\mathsf{X}_k\mathsf{X}_m}(t)\,dt &= \sigma_{ii}(t)\,dt - \sigma_{im}(t)\,dt - \sigma_{ik}(t)\,dt + \sigma_{km}(t)\,dt \\ &= \big(\sigma_{ii}(t) - \sigma_{im}(t) - \sigma_{ik}(t) + \sigma_{km}(t)\big)\,dt. \qquad (2.3.40) \end{aligned}$$

Now, we offer the following definition for the relative return to which Fernholz (2002) refers.

Definition 2.3.5 (Relative Return). *For any given stock $X_i = \{X_i(t), t \in [0,\infty)\}$ with initial capital $X_i(0) = x_i$, for $i = 1, 2, \ldots, n$, arbitrary portfolio $\eta = \big\{\boldsymbol{\eta}(t) = \big(\eta_1(t), \ldots, \eta_n(t)\big), t \in [0,\infty)\big\}$ and portfolio value process $Z_{w,\eta} = \big\{Z_{w,\eta}(t), t \in [0,\infty)\big\}$ with initial capital $Z_{w,\eta}(0) = w > 0$, with $w := X_i(0)$, the process $\log\big(X_i/Z_\eta\big) = \big\{\log\big(X_i(t)/Z_\eta(t)\big), t \in [0,\infty)\big\}$, i.e.,*

$$\log\Big(X_i(t)/Z_\eta(t)\Big) \;=\; \log\Big(X_i(t)/Z_{w,\eta}(t)\Big)\Big|_{w=X_i(0)}, \qquad t \in [0,\infty), \tag{2.3.41}$$

*is called the **relative return process** of the ith stock X_i versus η, for all $i = 1, 2, \ldots, n$.*

Thus, the process $\log\big(X_i/Z_\eta\big)$ provides a measure of the relative performance of the stocks in the market to some benchmark portfolio η. In Fernholz & Karatzas (2009), the relative return process is denoted by $R_i^\eta(t) := \log\big(X_i(t)/Z_\eta(t)\big) = \log X_i(t) - \log Z_\eta(t)$. The reader should consult this text for a full account. By adopting this notation, and comparing equations (2.2.20) and (2.2.65) (or, (2.2.131)), we arrive at the following

$$\begin{aligned}
dR_i^\eta(t) &= d\log\Big(X_i(t)/Z_\eta(t)\Big) \\
&= d\log X_i(t) - d\log Z_\eta(t) \\
&= \Big(\gamma_i(t)\,dt + \sum_{\nu=1}^n \xi_{i\nu}(t)\,dW_\nu(t)\Big) - \Big(\gamma_\eta(t)\,dt + \sum_{j,\nu=1}^n \eta_j(t)\xi_{j\nu}(t)\,dW_\nu(t)\Big) \\
&= \Big(\gamma_i(t)\,dt + \sum_{j,\nu=1}^n \eta_j(t)\xi_{i\nu}(t)\,dW_\nu(t)\Big) - \Big(\gamma_\eta(t)\,dt + \sum_{j,\nu=1}^n \eta_j(t)\xi_{j\nu}(t)\,dW_\nu(t)\Big) \\
&= \big(\gamma_i(t) - \gamma_\eta(t)\big)\,dt + \sum_{j,\nu=1}^n \eta_j(t)\big(\xi_{i\nu}(t) - \xi_{j\nu}(t)\big)\,dW_\nu(t).
\end{aligned} \tag{2.3.42}$$

Moreover, by comparing equations (2.2.20) and (2.2.65) (or, (2.2.131)), we can also arrive at the following

$$\begin{aligned}
dR_i^\eta(t) &= d\log\Big(X_i(t)/Z_\eta(t)\Big) \\
&= d\log X_i(t) - d\log Z_\eta(t) \\
&= \Big(\gamma_i(t)\,dt + \sum_{\nu=1}^n \xi_{i\nu}(t)\,dW_\nu(t)\Big) - \Big(\gamma_\eta(t)\,dt + \sum_{j,\nu=1}^n \eta_j(t)\xi_{j\nu}(t)\,dW_\nu(t)\Big) \\
&= \Big(\gamma_i(t)\,dt + \sum_{\nu=1}^n \xi_{i\nu}(t)\,dW_\nu(t)\Big) - \Big(\gamma_\eta(t)\,dt + \sum_{\nu=1}^n \Big[\sum_{j=1}^n \eta_j(t)\xi_{j\nu}(t)\Big]\,dW_\nu(t)\Big) \\
&= \Big(\gamma_i(t)\,dt + \sum_{\nu=1}^n \xi_{i\nu}(t)\,dW_\nu(t)\Big) - \Big(\gamma_\eta(t)\,dt + \sum_{\nu=1}^n \xi_{\eta\nu}(t)\,dW_\nu(t)\Big) \\
&= \big(\gamma_i(t) - \gamma_\eta(t)\big)\,dt + \sum_{\nu=1}^n \big(\xi_{i\nu}(t) - \xi_{\eta\nu}(t)\big)\,dW_\nu(t).
\end{aligned} \tag{2.3.43}$$

It is evident that the relative return process (2.3.41) is a continuous semimartingale, since both $\log X_i$ and $\log Z_\eta$ are continuous semimartingales provided that the requisite integrability conditions are satisfied.

Hence, using this portfolio representation, (2.3.1) can be reformulated as

$$d\log\Big(X_i(t)/Z_{\chi_j}(t)\Big) \;=\; \big(\gamma_i(t) - \gamma_{\chi_j}(t)\big)\,dt + \sum_{\nu=1}^n \big(\xi_{i\nu}(t) - \xi_{\chi_j\nu}(t)\big)\,dW_\nu(t). \tag{2.3.44}$$

Lemma 2.3.6. *Let X_i denote the ith stock in the market $\mathcal{M}$ and let η be any arbitrary portfolio in the market $\mathcal{M}$. Then, a.s., for $t \in [0,\infty)$ and for $i = 1, 2, \ldots, n$, we obtain the dynamics*

$$d\log\Big(X_i(t)/Z_\eta(t)\Big) \;=\; \sum_{j=1}^n \eta_j(t)\,d\log\Big(X_i(t)/X_j(t)\Big) - \gamma_\eta^*(t)\,dt, \tag{2.3.45}$$

for the relative return of stock X_i versus portfolio η.

Proof. The proof is a straightforward application of Corollary 2.2.25. Setting $\pi := \eta$ in (2.2.112) we obtain

$$\begin{aligned}
d\log\Big(X_i(t)/Z_\eta(t)\Big) &= d\log X_i(t) - d\log Z_\eta(t) \\
&= d\log X_i(t)\Big(\sum_{j=1}^n \eta_j(t)\Big) - \Big(\sum_{j=1}^n \eta_j(t)\, d\log X_j(t) + \gamma_\eta^*(t)\, dt\Big) \\
&= \sum_{j=1}^n \eta_j(t)\, d\log X_i(t) - \sum_{j=1}^n \eta_j(t)\, d\log X_j(t) - \gamma_\eta^*(t)\, dt \\
&= \sum_{j=1}^n \eta_j(t)\Big[d\log X_i(t) - d\log X_j(t)\Big] - \gamma_\eta^*(t)\, dt \\
&= \sum_{j=1}^n \eta_j(t)\, d\log\Big(X_i(t)/X_j(t)\Big) - \gamma_\eta^*(t)\, dt.
\end{aligned}$$

■

Here, $d\log\big(X_i/X_j\big)$ represents the return of the ith stock X_i relative to the jth stock X_j for all stocks $i = 1, 2, \ldots, n$, with $i \neq j$. Thus, the relative return of a stock with respect to a portfolio η can be expressed in terms of the individual stock relative returns (against another single stock in the portfolio), and the excess growth rate of the portfolio η. More precisely, the relative return of X_i versus η can be expressed as a weighted average of the relative returns of the individual stocks and an additional component, the excess growth rate, γ_η^*. Again, by appealing to the portfolio representation, for all $i = 1, 2, \ldots, n$, we can reformulate equation (2.3.45) as

$$d\log\Big(X_i(t)/Z_\eta(t)\Big) = \sum_{j=1}^n \eta_j(t)\, d\log\Big(X_i(t)/Z_{\chi_j}(t)\Big) - \gamma_\eta^*(t)\, dt. \tag{2.3.46}$$

The result of (2.3.42) can also be derived by substituting (2.3.1) into equation (2.3.45), obtained as follows

$$\begin{aligned}
d\log\Big(X_i(t)/Z_\eta(t)\Big) &= \sum_{j=1}^n \eta_j(t)\, d\log\Big(X_i(t)/X_j(t)\Big) - \gamma_\eta^*(t)\, dt \\
&= \sum_{j=1}^n \eta_j(t)\Big[\big(\gamma_i(t) - \gamma_j(t)\big)\, dt + \sum_{\nu=1}^n \big(\xi_{i\nu}(t) - \xi_{j\nu}(t)\big)\, dW_\nu(t)\Big] - \gamma_\eta^*(t)\, dt \\
&= \Big(\gamma_i(t) - \sum_{j=1}^n \eta_j(t)\gamma_j(t) - \gamma_\eta^*(t)\Big)\, dt + \sum_{j,\nu=1}^n \eta_j(t)\big(\xi_{i\nu}(t) - \xi_{j\nu}(t)\big)\, dW_\nu(t) \\
&= \big(\gamma_i(t) - \gamma_\eta(t)\big)\, dt + \sum_{j,\nu=1}^n \eta_j(t)\big(\xi_{i\nu}(t) - \xi_{j\nu}(t)\big)\, dW_\nu(t),
\end{aligned} \tag{2.3.47}$$

where equation (2.3.47) is obtained by utilising equation (2.2.111). Moreover, the result of (2.3.43) can also be derived by substituting (2.3.1) into equation (2.3.45), obtained as follows

$$\begin{aligned}
d\log\Big(X_i(t)/Z_\eta(t)\Big) &= \sum_{j=1}^n \eta_j(t)\, d\log\Big(X_i(t)/X_j(t)\Big) - \gamma_\eta^*(t)\, dt \\
&= \sum_{j=1}^n \eta_j(t)\Big[\big(\gamma_i(t) - \gamma_j(t)\big)\, dt + \sum_{\nu=1}^n \big(\xi_{i\nu}(t) - \xi_{j\nu}(t)\big)\, dW_\nu(t)\Big] - \gamma_\eta^*(t)\, dt \\
&= \Big(\gamma_i(t) - \sum_{j=1}^n \eta_j(t)\gamma_j(t) - \gamma_\eta^*(t)\Big)\, dt + \sum_{\nu=1}^n \Big[\xi_{i\nu}(t) - \sum_{j=1}^n \eta_j(t)\xi_{j\nu}(t)\Big]\, dW_\nu(t) \\
&= \big(\gamma_i(t) - \gamma_\eta(t)\big)\, dt + \sum_{\nu=1}^n \big(\xi_{i\nu}(t) - \xi_{\eta\nu}(t)\big)\, dW_\nu(t),
\end{aligned} \tag{2.3.48}$$

where equation (2.3.48) is obtained by recalling equations (2.2.111) and (2.2.126).

Furthermore, for $t \in [0, \infty)$, a.s., we have for $i, j = 1, 2, \ldots, n$,

$$\begin{aligned}
\left\langle \log\left(X_i/Z_\eta\right), \log\left(X_j/Z_\eta\right)\right\rangle_t &= \left\langle \log X_i - \log Z_\eta, \log X_j - \log Z_\eta\right\rangle_t \\
&= \left\langle \log X_i, \log X_j\right\rangle_t - \left\langle \log X_i, \log Z_\eta\right\rangle_t - \left\langle \log Z_\eta, \log X_j\right\rangle_t + \left\langle \log Z_\eta\right\rangle_t \\
&= \left\langle \log X_i, \log X_j\right\rangle_t - \left\langle \log X_i, \log Z_\eta\right\rangle_t - \left\langle \log X_j, \log Z_\eta\right\rangle_t + \left\langle \log Z_\eta\right\rangle_t. \quad (2.3.49)
\end{aligned}$$

The preceding expression is the covariation process for continuous semimartingales $\log\left(X_i/Z_\eta\right)$ and $\log\left(X_j/Z_\eta\right)$. Therefore, we have

$$d\left\langle \log\left(X_i/Z_\eta\right), \log\left(X_j/Z_\eta\right)\right\rangle_t = d\left\langle \log X_i, \log X_j\right\rangle_t - d\left\langle \log X_i, \log Z_\eta\right\rangle_t - d\left\langle \log X_j, \log Z_\eta\right\rangle_t + d\left\langle \log Z_\eta\right\rangle_t. \quad (2.3.50)$$

Let us define the process $\sigma_{i\eta} = \left\{\sigma_{i\eta}(t), t \in [0, \infty)\right\}$, the covariance process of the ith stock X_i and η, for $i = 1, 2, \ldots, n$, by

$$\sigma_{i\eta}(t) \triangleq \sum_{j=1}^{n} \eta_j(t)\sigma_{ij}(t) = \sum_{j=1}^{n} \eta_j(t)\sigma_{ji}(t), \qquad t \in [0, \infty), \quad (2.3.51)$$

and the process $\sigma_{j\eta} = \left\{\sigma_{j\eta}(t), t \in [0, \infty)\right\}$, the covariance process of the jth stock X_j and η, for $j = 1, 2, \ldots, n$, by

$$\sigma_{j\eta}(t) \triangleq \sum_{i=1}^{n} \eta_i(t)\sigma_{ji}(t) = \sum_{i=1}^{n} \eta_i(t)\sigma_{ij}(t), \qquad t \in [0, \infty), \quad (2.3.52)$$

where the second equality in last two expressions above follows from the symmetry of $\boldsymbol{\sigma}(t)$. We can express $\sigma_{i\eta}$ in terms of the volatility processes. Thus, from (2.3.51) and by setting $\pi := \eta$ in equation (2.2.126), we obtain

$$\begin{aligned}
\sigma_{i\eta}(t) &= \sum_{j=1}^{n} \eta_j(t)\left[\sum_{\nu=1}^{n} \xi_{i\nu}(t)\xi_{j\nu}(t)\right] \\
&= \sum_{\nu=1}^{n} \xi_{i\nu}(t)\left[\sum_{j=1}^{n} \eta_j(t)\xi_{j\nu}(t)\right] \\
&= \sum_{\nu=1}^{n} \xi_{i\nu}(t)\xi_{\eta\nu}(t). \quad (2.3.53)
\end{aligned}$$

Likewise, we can express $\sigma_{j\eta}$ in terms of the volatility processes as follows

$$\begin{aligned}
\sigma_{j\eta}(t) &= \sum_{i=1}^{n} \eta_i(t)\left[\sum_{\nu=1}^{n} \xi_{i\nu}(t)\xi_{j\nu}(t)\right] \\
&= \sum_{\nu=1}^{n} \xi_{j\nu}(t)\left[\sum_{i=1}^{n} \eta_i(t)\xi_{i\nu}(t)\right] \\
&= \sum_{\nu=1}^{n} \xi_{j\nu}(t)\xi_{\eta\nu}(t). \quad (2.3.54)
\end{aligned}$$

Recalling equation (2.2.94), the portfolio variance process of portfolio η, $\sigma_{\eta\eta} = \left\{\sigma_{\eta\eta}(t), t \in [0, \infty)\right\}$, for $t \in [0, \infty)$, is given by

$$\sigma_{\eta\eta}(t) = \sum_{i,j=1}^{n} \eta_i(t)\sigma_{ij}(t)\eta_j(t) = \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t). \quad (2.3.55)$$

Alternatively, for $t \in [0, \infty)$, we have

$$\sigma_{\eta\eta}(t) = \sum_{\nu=1}^{n} \xi_{\eta\nu}^2(t). \quad (2.3.56)$$

By making use of equations (2.2.46) and (2.3.51), for all $i = 1, 2, \ldots, n$, we have

$$\begin{aligned}
\left\langle \log X_i, \log Z_\eta \right\rangle_t &= \left\langle \int_0^\cdot \sum_{\nu=1}^n \xi_{i\nu,s}\, dW_{\nu,s}, \int_0^\cdot \sum_{j,\upsilon=1}^n \eta_{j,s}\xi_{j\upsilon,s}\, dW_{\upsilon,s} \right\rangle_t \\
&= \left\langle \sum_{\nu=1}^n \int_0^\cdot \xi_{i\nu,s}\, dW_{\nu,s}, \sum_{\upsilon=1}^n \int_0^\cdot \sum_{j=1}^n \eta_{j,s}\xi_{j\upsilon,s}\, dW_{\upsilon,s} \right\rangle_t \\
&= \sum_{\nu,\upsilon=1}^n \left\langle \int_0^\cdot \xi_{i\nu,s}\, dW_{\nu,s}, \int_0^\cdot \sum_{j=1}^n \eta_{j,s}\xi_{j\upsilon,s}\, dW_{\upsilon,s} \right\rangle_t \\
&= \sum_{\nu,\upsilon=1}^n \int_0^t \sum_{j=1}^n \eta_j(s)\xi_{i\nu}(s)\xi_{j\upsilon}(s)\, d\left\langle W_\nu, W_\upsilon \right\rangle_s \\
&= \int_0^t \sum_{j=1}^n \eta_j(s) \sum_{\nu,\upsilon=1}^n \xi_{i\nu}(s)\xi_{j\upsilon}(s)\, \rho_{\nu\upsilon}(s)\, ds \\
&= \int_0^t \sum_{j=1}^n \eta_j(s) \sum_{\nu=1}^n \xi_{i\nu}(s)\xi_{j\nu}(s)\, ds \qquad (2.3.57) \\
&= \int_0^t \sum_{j=1}^n \eta_j(s)\sigma_{ij}(s)\, ds \qquad (2.3.58) \\
&= \int_0^t \sigma_{i\eta}(s)\, ds. \qquad (2.3.59)
\end{aligned}$$

Hence,

$$\begin{aligned}
d\left\langle \log X_i, \log Z_\eta \right\rangle_t &= d\left\langle \int_0^\cdot \sum_{\nu=1}^n \xi_{i\nu,s}\, dW_{\nu,s}, \int_0^\cdot \sum_{j,\upsilon=1}^n \eta_{j,s}\xi_{j\upsilon,s}\, dW_{\upsilon,s} \right\rangle_t \\
&= d\left(\int_0^t \sum_{j=1}^n \eta_j(s) \sum_{\nu=1}^n \xi_{i\nu}(s)\xi_{j\nu}(s)\, ds \right) \\
&= \sum_{j=1}^n \eta_j(t) \sum_{\nu=1}^n \xi_{i\nu}(t)\xi_{j\nu}(t)\, dt \qquad (2.3.60) \\
&= \sum_{j=1}^n \eta_j(t)\sigma_{ij}(t)\, dt \qquad (2.3.61) \\
&= \sigma_{i\eta}(t)\, dt. \qquad (2.3.62)
\end{aligned}$$

Alternatively, by employing equations (2.2.126) and (2.3.53), the above expression can be rewritten as

$$\begin{aligned}
d\left\langle \log X_i, \log Z_\eta \right\rangle_t &= \sum_{\nu=1}^n \xi_{i\nu}(t) \left[\sum_{j=1}^n \eta_j(t)\xi_{j\nu}(t) \right] dt \\
&= \sum_{\nu=1}^n \xi_{i\nu}(t)\xi_{\eta\nu}(t)\, dt \\
&= \sigma_{i\eta}(t)\, dt.
\end{aligned}$$

Furthermore, by appealing to equation (2.2.131) and making use of equation (2.3.53), we can again show that

$$\begin{aligned}
d\left\langle \log X_i, \log Z_\eta \right\rangle_t &= d\left\langle \int_0^\cdot \sum_{\nu=1}^n \xi_{i\nu,s}\, dW_{\nu,s}, \int_0^\cdot \sum_{\upsilon=1}^n \xi_{\eta\upsilon,s}\, dW_{\upsilon,s} \right\rangle_t \\
&= d\left\langle \sum_{\nu=1}^n \int_0^\cdot \xi_{i\nu,s}\, dW_{\nu,s}, \sum_{\upsilon=1}^n \int_0^\cdot \xi_{\eta\upsilon,s}\, dW_{\upsilon,s} \right\rangle_t \\
&= \sum_{\nu,\upsilon=1}^n d\left\langle \int_0^\cdot \xi_{i\nu,s}\, dW_{\nu,s}, \int_0^\cdot \xi_{\eta\upsilon,s}\, dW_{\upsilon,s} \right\rangle_t \\
&= \sum_{\nu,\upsilon=1}^n \xi_{i\nu}(t)\xi_{\eta\upsilon}(t)\, d\left\langle W_\nu, W_\upsilon \right\rangle_t \\
&= \sum_{\nu,\upsilon=1}^n \xi_{i\nu}(t)\xi_{\eta\upsilon}(t)\, \rho_{\nu\upsilon}(t)\, dt \\
&= \sum_{\nu=1}^n \xi_{i\nu}(t)\xi_{\eta\nu}(t)\, dt \qquad (2.3.63) \\
&= \sigma_{i\eta}(t)\, dt. \qquad (2.3.64)
\end{aligned}$$

Again, this can be shown by appealing to equation (2.2.112) of Corollary 2.2.25. Thus, by employing equations (2.2.44) and (2.3.51), we obtain

$$\begin{aligned}
d\left\langle \log X_i, \log Z_\eta \right\rangle_t &= d\left\langle \int_0^\cdot d\log X_{i,s}, \int_0^\cdot \sum_{j=1}^n \eta_{j,s}\, d\log X_{j,s} \right\rangle_t \\
&= d\left\langle \int_0^\cdot d\log X_{i,s}, \sum_{j=1}^n \int_0^\cdot \eta_{j,s}\, d\log X_{j,s} \right\rangle_t \\
&= \sum_{j=1}^n d\left\langle \int_0^\cdot d\log X_{i,s}, \int_0^\cdot \eta_{j,s}\, d\log X_{j,s} \right\rangle_t \\
&= \sum_{j=1}^n \eta_j(t)\, d\left\langle \log X_i, \log X_j \right\rangle_t \qquad (2.3.65) \\
&= \sum_{j=1}^n \eta_j(t)\sigma_{ij}(t)\, dt \qquad (2.3.66) \\
&= \sigma_{i\eta}(t)\, dt. \qquad (2.3.67)
\end{aligned}$$

In conclusion, for all $i = 1, 2, \ldots, n$, we have

$$\begin{aligned}
d\left\langle \log X_i, \log Z_\eta \right\rangle_t &= \sigma_{i\eta}(t)\, dt, \qquad t \in [0,\infty), \qquad \text{a.s.,} \quad \text{or,} \qquad (2.3.68) \\
\left\langle \log X_i, \log Z_\eta \right\rangle_t &= \int_0^t \sigma_{i\eta}(s)\, ds, \qquad t \in [0,\infty), \qquad \text{a.s.} \qquad (2.3.69)
\end{aligned}$$

This is the differential of the cross-variation (covariation) of the ith stock and the portfolio η, which is comparable to the results obtained in (2.2.102) and (2.2.103). Consequently, for all $i, j = 1, \ldots, n$, we have the following definition.

Definition 2.3.7 (Covariance Process). *For stocks $X_i = \{X_i(t), t \in [0,\infty)\}$ and $X_j = \{X_j(t), t \in [0,\infty)\}$, for $i, j = 1, 2, \ldots, n$, and portfolio $\eta = \{\boldsymbol{\eta}(t) = (\eta_1(t), \ldots, \eta_n(t)), t \in [0,\infty)\}$, the processes*

$$\begin{aligned}
\sigma_{i\eta}(t) &\triangleq \frac{d}{dt}\left\langle \log X_i, \log Z_\eta \right\rangle_t, \qquad t \in [0,\infty), \qquad \textit{a.s., and,} \qquad (2.3.70) \\
\sigma_{j\eta}(t) &\triangleq \frac{d}{dt}\left\langle \log X_j, \log Z_\eta \right\rangle_t, \qquad t \in [0,\infty), \qquad \textit{a.s.,} \qquad (2.3.71)
\end{aligned}$$

*are referred to as the **covariance processes**. The former represents the covariance of the ith stock X_i and the portfolio η, for all $i = 1, 2, \ldots, n$, i.e., the covariance of the processes $\log X_i$ and $\log Z_\eta$, and indicates how the*

ith stock X_i varies relative to the portfolio η. Similarly, the latter represents the covariance of the jth stock X_j and the portfolio η, for all $j = 1, 2, \ldots, n$.

Setting $\pi := \eta$ in equation (2.2.105), together with equations (2.2.84) and (2.3.68), yields

$$d\left\langle X_i, Z_\eta \right\rangle_t = X_i(t) Z_\eta(t)\, d\left\langle \log X_i, \log Z_\eta \right\rangle_t = \sigma_{i\eta}(t)\, X_i(t) Z_\eta(t)\, dt. \tag{2.3.72}$$

Alternatively, from equations (2.2.62), (2.2.87) and (2.3.51), we obtain

$$\begin{aligned}
d\left\langle X_i, Z_\eta \right\rangle_t &= d\left\langle \int_0^{\cdot} dX_{i,s}, \int_0^{\cdot} Z_{\eta,s} \sum_{j=1}^n \eta_{j,s} \frac{dX_{j,s}}{X_{j,s}} \right\rangle_t \\
&= d\left\langle \int_0^{\cdot} dX_{i,s}, \sum_{j=1}^n \int_0^{\cdot} Z_{\eta,s} \eta_{j,s} \frac{dX_{j,s}}{X_{j,s}} \right\rangle_t \\
&= \sum_{j=1}^n d\left\langle \int_0^{\cdot} dX_{i,s}, \int_0^{\cdot} Z_{\eta,s} \eta_{j,s} \frac{dX_{j,s}}{X_{j,s}} \right\rangle_t \\
&= \sum_{j=1}^n Z_\eta(t)\, \eta_j(t) \frac{1}{X_j(t)}\, d\left\langle X_i, X_j \right\rangle_t && (2.3.73) \\
&= \sum_{j=1}^n X_i(t) Z_\eta(t)\, \eta_j(t)\, d\left\langle \log X_i, \log X_j \right\rangle_t && (2.3.74) \\
&= X_i(t) Z_\eta(t) \sum_{j=1}^n \eta_j(t) \sigma_{ij}(t)\, dt && (2.3.75) \\
&= \sigma_{i\eta}(t)\, X_i(t) Z_\eta(t)\, dt. && (2.3.76)
\end{aligned}$$

The preceding results can also be derived for the market which incorporates dividends. Thus, the cross-variation of the ith stock and the portfolio η, for the dividend case, can be obtained from equations (2.2.153) and (2.2.162). Consequently, for $i = 1, 2, \ldots, n$, we obtain the analogues to (2.3.59) and (2.3.62),

$$d\left\langle \log \widehat{X}_i, \log \widehat{Z}_\eta \right\rangle_t = d\left\langle \log X_i, \log Z_\eta \right\rangle_t = \sigma_{i\eta}(t)\, dt, \quad \text{or,} \tag{2.3.77}$$

$$\left\langle \log \widehat{X}_i, \log \widehat{Z}_\eta \right\rangle_t = \left\langle \log X_i, \log Z_\eta \right\rangle_t = \int_0^t \sigma_{i\eta}(s)\, ds. \tag{2.3.78}$$

Consequently, from equation (2.3.77), we have

$$d\left\langle \widehat{X}_i, \widehat{Z}_\eta \right\rangle_t = \widehat{X}_i(t) \widehat{Z}_\eta(t)\, d\left\langle \log \widehat{X}_i, \log \widehat{Z}_\eta \right\rangle_t = \sigma_{i\eta}(t)\, \widehat{X}_i(t) \widehat{Z}_\eta(t)\, dt. \tag{2.3.79}$$

Moreover, setting $\pi := \eta$ in equation (2.2.75), in conjunction with (2.3.55), we obtain

$$d\left\langle \log Z_\eta \right\rangle_t = \sigma_{\eta\eta}(t)\, dt, \qquad t \in [0, \infty), \qquad \text{a.s., or,} \tag{2.3.80}$$

$$\left\langle \log Z_\eta \right\rangle_t = \int_0^t \sigma_{\eta\eta}(s)\, ds, \qquad t \in [0, \infty), \qquad \text{a.s.} \tag{2.3.81}$$

2.3.1 The Relative Covariance Process

Recall the covariance process, given in equation (2.2.46) by

$$\sigma_{ij}(t) = \frac{d}{dt} \left\langle \log X_i, \log X_j \right\rangle_t.$$

Let us now consider the process of individual stocks' covariances relative to some portfolio η, namely, the **relative covariance process**.

Definition 2.3.8 (Relative Covariance). *For stocks $X_i = \{X_i(t), t \in [0,\infty)\}$ and $X_j = \{X_j(t), t \in [0,\infty)\}$, for all $i,j = 1,2,\ldots,n$, and any arbitrary portfolio $\eta = \{\boldsymbol{\eta}(t) = (\eta_1(t),\ldots,\eta_n(t)), t \in [0,\infty)\}$, the $(n \times n)-$matrix-valued process $\boldsymbol{\tau}^{\boldsymbol{\eta}}(t) = \left(\tau_{ij}^{\eta}(t)\right)_{1\le i,j\le n}$, defined by*

$$\tau_{ij}^{\eta}(t) \triangleq \frac{d}{dt}\left\langle \log\left(X_i/Z_\eta\right), \log\left(X_j/Z_\eta\right)\right\rangle_t = \frac{d}{dt}\left\langle R_i^\eta, R_j^\eta\right\rangle_t, \qquad t \in [0,\infty), \qquad a.s., \tag{2.3.82}$$

*is called the **relative covariance process**, i.e., the covariance of the ith stock X_i and the jth stock X_j, both relative to the portfolio η. More specifically, the covariance of the relative return processes $\log\left(X_i/Z_\eta\right)$ and $\log\left(X_j/Z_\eta\right)$.*

As a result, we shall also call the $(n \times n)-$matrix-valued process $\boldsymbol{\tau}^{\boldsymbol{\eta}}(t) = \left(\tau_{ij}^{\eta}(t)\right)_{1\le i,j\le n}$ above, the process of the individual stocks' covariances all relative to the portfolio η. Thus, from the preceding definition, we can recapture (2.3.2) of Definition 2.3.2 as

$$\tau_{ij}^{\chi_k}(t) = \frac{d}{dt}\left\langle \log\left(X_i/Z_{\chi_k}\right), \log\left(X_j/Z_{\chi_k}\right)\right\rangle_t, \qquad t \in [0,\infty), \qquad \text{a.s.}, \tag{2.3.83}$$

where χ_k represents the portfolio consisting of the single stock X_k and $Z_{\chi_k} = \{Z_{\chi_k}(t), t \in [0,\infty)\}$ represents the value of the portfolio χ_k. Thus, $\tau_{ij}^{\chi_k}$ can alternatively be interpreted as the covariance of the ith stock X_i versus the jth stock X_j, both relative to the portfolio χ_k. Since $k \in \{1,2,\ldots,n\}$, the relative covariance with respect to the portfolios $\chi_1, \chi_2, \ldots, \chi_n$, with corresponding value processes $Z_{\chi_1} = \{Z_{\chi_1}(t), t \in [0,\infty)\}, Z_{\chi_2} = \{Z_{\chi_2}(t), t \in [0,\infty)\}, \ldots, Z_{\chi_n} = \{Z_{\chi_n}(t), t \in [0,\infty)\}$, is given respectively by

$$\begin{aligned}
\tau_{ij}^{\chi_1}(t) &= \frac{d}{dt}\left\langle \log\left(X_i/X_1\right), \log\left(X_j/X_1\right)\right\rangle_t &&= \frac{d}{dt}\left\langle \log\left(X_i/Z_{\chi_1}\right), \log\left(X_j/Z_{\chi_1}\right)\right\rangle_t,\\
\tau_{ij}^{\chi_2}(t) &= \frac{d}{dt}\left\langle \log\left(X_i/X_2\right), \log\left(X_j/X_2\right)\right\rangle_t &&= \frac{d}{dt}\left\langle \log\left(X_i/Z_{\chi_2}\right), \log\left(X_j/Z_{\chi_2}\right)\right\rangle_t,\\
&\vdots && \vdots\\
\tau_{ij}^{\chi_n}(t) &= \frac{d}{dt}\left\langle \log\left(X_i/X_n\right), \log\left(X_j/X_n\right)\right\rangle_t &&= \frac{d}{dt}\left\langle \log\left(X_i/Z_{\chi_n}\right), \log\left(X_j/Z_{\chi_n}\right)\right\rangle_t.
\end{aligned}$$

Recall equations (2.3.50), (2.3.68) and (2.3.80), for $i,j = 1,2,\ldots,n$ and for $t \in [0,\infty)$, a.s., we obtain

$$\begin{aligned}
\tau_{ij}^{\eta}(t)\,dt &= d\left\langle \log\left(X_i/Z_\eta\right), \log\left(X_j/Z_\eta\right)\right\rangle_t \\
&= d\left\langle \log X_i, \log X_j\right\rangle_t - d\left\langle \log X_i, \log Z_\eta\right\rangle_t - d\left\langle \log X_j, \log Z_\eta\right\rangle_t + d\left\langle \log Z_\eta\right\rangle_t \\
&= \sigma_{ij}(t)\,dt - \sigma_{i\eta}(t)\,dt - \sigma_{j\eta}(t)\,dt + \sigma_{\eta\eta}(t)\,dt \\
&= \left(\sigma_{ij}(t) - \sigma_{i\eta}(t) - \sigma_{j\eta}(t) + \sigma_{\eta\eta}(t)\right)dt.
\end{aligned} \tag{2.3.84}$$

This implies, for $i,j = 1,2,\ldots,n$ and for $t \in [0,\infty)$, a.s., that

$$\tau_{ij}^{\eta}(t) = \sigma_{ij}(t) - \sigma_{i\eta}(t) - \sigma_{j\eta}(t) + \sigma_{\eta\eta}(t). \tag{2.3.85}$$

Again, for $i,j = 1,2,\ldots,n$ and for $t \in [0,\infty)$, a.s., using the portfolio representation and formulae established thus far, we can also recapture (2.3.9) as

$$\tau_{ij}^{\chi_k}(t) = \sigma_{ij}(t) - \sigma_{i\chi_k}(t) - \sigma_{j\chi_k}(t) + \sigma_{\chi_k\chi_k}(t), \tag{2.3.86}$$

where η in (2.3.85) is simply substituted by χ_k, a portfolio in the same respects as η, but comprising the single stock X_k.

Since the covariance matrix $\boldsymbol{\sigma}(t)$ is an $n \times n$ symmetric matrix, the relative covariance matrix is also an $n \times n$ symmetric matrix, and for all $t \in [0,\infty)$, we have $\boldsymbol{\tau}^{\boldsymbol{\eta}}(t) = \left(\boldsymbol{\tau}^{\boldsymbol{\eta}}\right)^{\mathsf{T}}(t)$ or $\tau_{ij}^{\eta}(t) = \tau_{ji}^{\eta}(t)$, for all $i,j = 1,2,\ldots,n$. From (2.3.82), for all $i,j = 1,\ldots,n$, we obtain

$$d\left\langle \log\left(X_i/Z_\eta\right), \log\left(X_j/Z_\eta\right)\right\rangle_t = \tau_{ij}^{\eta}(t)\,dt, \qquad t \in [0,\infty), \qquad \text{a.s., or,} \tag{2.3.87}$$

$$\left\langle \log\left(X_i/Z_\eta\right), \log\left(X_j/Z_\eta\right)\right\rangle_t = \int_0^t \tau_{ij}^{\eta}(s)\,ds, \qquad t \in [0,\infty), \qquad \text{a.s.} \tag{2.3.88}$$

Moreover, we have

$$\int_0^t \left|\tau_{ij}^{\eta}(s)\right| ds \quad < \quad \infty, \qquad t \in [0,\infty), \qquad \text{a.s.,} \tag{2.3.89}$$

which follows from equations (2.3.88) and (2.2.54), and the boundedness of the portfolio weight processes.

Consider a vector $\mathbf{e}_i$ which denotes the ith unit vector in $\mathbb{R}^n$, this is a vector in which the ith coordinate is 1 and the other coordinates are all 0, i.e., we have

$$\mathbf{e}_i \quad = \quad (0,\ldots,0,1,0,\ldots,0) \in \mathbb{R}^n. \tag{2.3.90}$$

Recall from (2.3.55), that

$$\sigma_{\eta\eta}(t) \quad = \quad \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t), \qquad t \in [0,\infty). \tag{2.3.91}$$

In addition, from (2.3.51) and employing the unit vector above, for $i = 1,2,\ldots,n$, we offer the following matrix representation

$$\sigma_{i\eta}(t) \quad = \quad \sum_{j=1}^{n} \eta_j(t)\sigma_{ij}(t) \quad = \quad \mathbf{e}_i\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t), \qquad t \in [0,\infty). \tag{2.3.92}$$

Moreover, for $j = 1,2,\ldots,n$, we have the analogue to (2.3.92),

$$\sigma_{j\eta}(t) \quad = \quad \sum_{i=1}^{n} \eta_i(t)\sigma_{ji}(t) \quad = \quad \mathbf{e}_j\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t), \qquad t \in [0,\infty). \tag{2.3.93}$$

It will be useful to note the following

$$\begin{aligned} \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t) \quad &= \quad \Big(\sigma_{1\eta}(t), \sigma_{2\eta}(t), \ldots, \sigma_{i\eta}(t), \ldots, \sigma_{n\eta}(t)\Big), \quad \text{and,} \\ \boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t) \quad &= \quad \Big(\sigma_{1\eta}(t), \sigma_{2\eta}(t), \ldots, \sigma_{i\eta}(t), \ldots, \sigma_{n\eta}(t)\Big)^{\mathsf{T}}. \end{aligned}$$

By the symmetry of $\boldsymbol{\sigma}(t)$ and the fact that $\mathbf{e}_i\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)$ is just a number, and is thus equal to its transpose, for $j = 1,2,\ldots,n$, we have

$$\begin{aligned} \mathbf{e}_i\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t) \quad &= \quad \Big(\mathbf{e}_i\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)\Big)^{\mathsf{T}} \\ &= \quad \boldsymbol{\eta}(t)\boldsymbol{\sigma}^{\mathsf{T}}(t)\mathbf{e}_i^{\mathsf{T}} \\ &= \quad \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\mathbf{e}_i^{\mathsf{T}}. \end{aligned}$$

Then, for $i = 1,2,\ldots,n$, it is evident that (2.3.92) can be expressed as

$$\sigma_{i\eta}(t) \quad = \quad \sum_{j=1}^{n} \eta_j(t)\sigma_{ji}(t) \quad = \quad \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\mathbf{e}_i^{\mathsf{T}}, \qquad t \in [0,\infty), \tag{2.3.94}$$

and, for $j = 1,2,\ldots,n$, (2.3.93) can be expressed as

$$\sigma_{j\eta}(t) \quad = \quad \sum_{i=1}^{n} \eta_i(t)\sigma_{ij}(t) \quad = \quad \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\mathbf{e}_j^{\mathsf{T}}, \qquad t \in [0,\infty). \tag{2.3.95}$$

Thus, in conclusion, for $i = 1,2,\ldots,n$, we have

$$\sigma_{i\eta}(t) \quad = \quad \sum_{k=1}^{n} \eta_k(t)\sigma_{ik}(t) \quad = \quad \sum_{k=1}^{n} \eta_k(t)\sigma_{ki}(t), \tag{2.3.96}$$

and, equivalently, in matrix form, we have

$$\sigma_{i\eta}(t) \quad = \quad \mathbf{e}_i\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t) \quad = \quad \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\mathbf{e}_i^{\mathsf{T}}. \tag{2.3.97}$$

For $j = 1, 2, \ldots, n$, we have

$$\sigma_{j\eta}(t) \quad = \quad \sum_{k=1}^{n} \eta_k(t)\sigma_{jk}(t) \quad = \quad \sum_{k=1}^{n} \eta_k(t)\sigma_{kj}(t), \tag{2.3.98}$$

and, equivalently, in matrix form, we have

$$\sigma_{j\eta}(t) \quad = \quad \mathbf{e}_j\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t) \quad = \quad \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\mathbf{e}_j^{\mathsf{T}}. \tag{2.3.99}$$

Thus, since $\sigma_{ij}(t) = \mathbf{e}_i\boldsymbol{\sigma}(t)\mathbf{e}_j^{\mathsf{T}}$, for $t \in [0, \infty)$, a.s., we have for $i, j = 1, 2, \ldots, n$,

$$\begin{aligned}
\tau_{ij}^{\eta}(t) &= \sigma_{ij}(t) - \sigma_{i\eta}(t) - \sigma_{j\eta}(t) + \sigma_{\eta\eta}(t) \\
&= \mathbf{e}_i\boldsymbol{\sigma}(t)\mathbf{e}_j^{\mathsf{T}} - \mathbf{e}_i\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t) - \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\mathbf{e}_j^{\mathsf{T}} + \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t) && (2.3.100)\\
&= \Big(\mathbf{e}_i - \boldsymbol{\eta}(t)\Big)\boldsymbol{\sigma}(t)\Big(\mathbf{e}_j^{\mathsf{T}} - \boldsymbol{\eta}^{\mathsf{T}}(t)\Big) && (2.3.101)\\
&= \Big(\mathbf{e}_i - \boldsymbol{\eta}(t)\Big)\boldsymbol{\sigma}(t)\Big(\mathbf{e}_j - \boldsymbol{\eta}(t)\Big)^{\mathsf{T}} && (2.3.102)\\
&= \Big(\boldsymbol{\eta}(t) - \mathbf{e}_i\Big)\boldsymbol{\sigma}(t)\Big(\boldsymbol{\eta}(t) - \mathbf{e}_j\Big)^{\mathsf{T}} && (2.3.103)\\
&:= \mathbf{e}_i\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\mathbf{e}_j^{\mathsf{T}}, && (2.3.104)
\end{aligned}$$

which follows from the results (2.3.91), (2.3.92) and (2.3.95). Furthermore, the matrix form is given by

$$\boldsymbol{\tau}^{\boldsymbol{\eta}}(t) \quad = \quad \boldsymbol{\sigma}(t) - \boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)\mathbf{1} - \mathbf{1}^{\mathsf{T}}\boldsymbol{\eta}(t)\boldsymbol{\sigma}(t) + \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t), \tag{2.3.105}$$

where $\mathbf{1}$ is the $(1 \times n)-$vector of 1s, i.e., $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$, the $n-$dimensional row vector with 1 in all entries. Again. The matrix representation $\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)\mathbf{1}$ in equation (2.3.105) indicates that the component, $\mathbf{e}_i\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)$, of the relative covariance is the same across columns and the matrix representation $\mathbf{1}^{\mathsf{T}}\boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)$ in equation (2.3.105) indicates that the component, $\boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\mathbf{e}_j^{\mathsf{T}}$, of the relative covariance is the same across rows, as shown below

$$\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)\mathbf{1} \quad = \quad \begin{bmatrix}
\sigma_{1\eta}(t) & \sigma_{1\eta}(t) & \cdots & \sigma_{1\eta}(t) & \cdots & \sigma_{1\eta}(t) \\
\sigma_{2\eta}(t) & \sigma_{2\eta}(t) & \cdots & \sigma_{2\eta}(t) & \cdots & \sigma_{2\eta}(t) \\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
\sigma_{j\eta}(t) & \sigma_{j\eta}(t) & \cdots & \sigma_{j\eta}(t) & \cdots & \sigma_{j\eta}(t) \\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
\sigma_{n\eta}(t) & \sigma_{n\eta}(t) & \cdots & \sigma_{n\eta}(t) & \cdots & \sigma_{n\eta}(t)
\end{bmatrix}, \tag{2.3.106}$$

and

$$\mathbf{1}^{\mathsf{T}}\boldsymbol{\eta}(t)\boldsymbol{\sigma}(t) \quad = \quad \begin{bmatrix}
\sigma_{1\eta}(t) & \sigma_{2\eta}(t) & \cdots & \sigma_{j\eta}(t) & \cdots & \sigma_{n\eta}(t) \\
\sigma_{1\eta}(t) & \sigma_{2\eta}(t) & \cdots & \sigma_{j\eta}(t) & \cdots & \sigma_{n\eta}(t) \\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
\sigma_{1\eta}(t) & \sigma_{2\eta}(t) & \cdots & \sigma_{j\eta}(t) & \cdots & \sigma_{n\eta}(t) \\
\vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
\sigma_{1\eta}(t) & \sigma_{2\eta}(t) & \cdots & \sigma_{j\eta}(t) & \cdots & \sigma_{n\eta}(t)
\end{bmatrix}. \tag{2.3.107}$$

Alternatively, using (2.3.98), we can write the above as

$$\tau_{ij}^{\eta}(t) \quad = \quad \sigma_{ij}(t) - \sum_{k=1}^{n} \eta_k(t)\sigma_{ik}(t) - \sum_{k=1}^{n} \eta_k(t)\sigma_{jk}(t) + \sigma_{\eta\eta}(t).$$

The relative covariance can also be expressed in terms of the volatility processes. By recalling equations (2.2.46),

(2.3.51) and (2.3.55), from equation (2.3.85), we deduce the following

$$
\begin{aligned}
\tau_{ij}^{\eta}(t) &= \sigma_{ij}(t) - \sum_{k=1}^{n} \eta_k(t)\sigma_{ik}(t) - \sum_{m=1}^{n} \eta_m(t)\sigma_{jm}(t) + \sum_{k,m=1}^{n} \eta_k(t)\sigma_{km}(t)\eta_m(t) \\
&= \sum_{\nu=1}^{n} \xi_{i\nu}(t)\xi_{j\nu}(t) - \sum_{k=1}^{n} \eta_k(t) \sum_{\nu=1}^{n} \xi_{i\nu}(t)\xi_{k\nu}(t) - \sum_{m=1}^{n} \eta_m(t) \sum_{\nu=1}^{n} \xi_{j\nu}(t)\xi_{m\nu}(t) \\
&\quad + \sum_{k,m=1}^{n} \eta_k(t)\eta_m(t) \sum_{\nu=1}^{n} \xi_{k\nu}(t)\xi_{m\nu}(t) \\
&= \sum_{\nu=1}^{n} \xi_{i\nu}(t)\xi_{j\nu}(t) - \sum_{\nu=1}^{n} \xi_{i\nu}(t) \left[\sum_{k=1}^{n} \eta_k(t)\xi_{k\nu}(t) \right] - \sum_{\nu=1}^{n} \xi_{j\nu}(t) \left[\sum_{m=1}^{n} \eta_m(t)\xi_{m\nu}(t) \right] \\
&\quad + \sum_{\nu=1}^{n} \left[\sum_{k=1}^{n} \eta_k(t)\xi_{k\nu}(t) \right] \left[\sum_{m=1}^{n} \eta_m(t)\xi_{m\nu}(t) \right] \\
&= \sum_{\nu=1}^{n} \xi_{i\nu}(t)\xi_{j\nu}(t) - \sum_{\nu=1}^{n} \xi_{i\nu}(t)\xi_{\eta\nu}(t) - \sum_{\nu=1}^{n} \xi_{j\nu}(t)\xi_{\eta\nu}(t) + \sum_{\nu=1}^{n} \xi_{\eta\nu}^2(t) \qquad (2.3.108) \\
&= \sum_{\nu=1}^{n} \Big(\xi_{i\nu}(t)\xi_{j\nu}(t) - \xi_{i\nu}(t)\xi_{\eta\nu}(t) - \xi_{j\nu}(t)\xi_{\eta\nu}(t) + \xi_{\eta\nu}^2(t) \Big) \\
&= \sum_{\nu=1}^{n} \Big(\xi_{i\nu}(t) - \xi_{\eta\nu}(t) \Big) \Big(\xi_{j\nu}(t) - \xi_{\eta\nu}(t) \Big), \qquad (2.3.109)
\end{aligned}
$$

where equation (2.3.108) follows by setting $\pi := \eta$ in equation (2.2.126). The last term in expression (2.3.108) also follows from equation (2.3.56). Alternatively, appealing to the cross-variation of the relative return process, in (2.3.43), yields the equivalent result

$$
\begin{aligned}
\big\langle \log\big(X_i/Z_\eta\big), \log\big(X_j/Z_\eta\big) \big\rangle_t &= \left\langle \int_0^{\cdot} \sum_{\nu=1}^{n} \big(\xi_{i\nu,s} - \xi_{\eta\nu,s}\big)\, dW_{\nu,s}, \int_0^{\cdot} \sum_{\upsilon=1}^{n} \big(\xi_{j\upsilon,s} - \xi_{\eta\upsilon,s}\big)\, dW_{\upsilon,s} \right\rangle_t \\
&= \left\langle \sum_{\nu=1}^{n} \int_0^{\cdot} \big(\xi_{i\nu,s} - \xi_{\eta\nu,s}\big)\, dW_{\nu,s}, \sum_{\upsilon=1}^{n} \int_0^{\cdot} \big(\xi_{j\upsilon,s} - \xi_{\eta\upsilon,s}\big)\, dW_{\upsilon,s} \right\rangle_t \\
&= \sum_{\nu,\upsilon=1}^{n} \left\langle \int_0^{\cdot} \big(\xi_{i\nu,s} - \xi_{\eta\nu,s}\big)\, dW_{\nu,s}, \int_0^{\cdot} \big(\xi_{j\upsilon,s} - \xi_{\eta\upsilon,s}\big)\, dW_{\upsilon,s} \right\rangle_t \\
&= \sum_{\nu,\upsilon=1}^{n} \int_0^{t} \Big(\xi_{i\nu}(s) - \xi_{\eta\nu}(s)\Big)\Big(\xi_{j\upsilon}(s) - \xi_{\eta\upsilon}(s)\Big)\, d\langle W_\nu, W_\upsilon \rangle_s \\
&= \int_0^{t} \sum_{\nu,\upsilon=1}^{n} \Big(\xi_{i\nu}(s) - \xi_{\eta\nu}(s)\Big)\Big(\xi_{j\upsilon}(s) - \xi_{\eta\upsilon}(s)\Big)\, \rho_{\nu\upsilon}(s)\, ds \\
&= \int_0^{t} \sum_{\nu=1}^{n} \Big(\xi_{i\nu}(s) - \xi_{\eta\nu}(s)\Big)\Big(\xi_{j\nu}(s) - \xi_{\eta\nu}(s)\Big)\, ds. \qquad (2.3.110)
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
d\big\langle \log\big(X_i/Z_\eta\big), \log\big(X_j/Z_\eta\big) \big\rangle_t &= d\left\langle \int_0^{\cdot} \sum_{\nu=1}^{n} \big(\xi_{i\nu,s} - \xi_{\eta\nu,s}\big)\, dW_{\nu,s}, \int_0^{\cdot} \sum_{\upsilon=1}^{n} \big(\xi_{j\upsilon,s} - \xi_{\eta\upsilon,s}\big)\, dW_{\upsilon,s} \right\rangle_t \\
&= d\left(\int_0^{t} \sum_{\nu=1}^{n} \Big(\xi_{i\nu}(s) - \xi_{\eta\nu}(s)\Big)\Big(\xi_{j\nu}(s) - \xi_{\eta\nu}(s)\Big)\, ds \right) \\
&= \sum_{\nu=1}^{n} \Big(\xi_{i\nu}(t) - \xi_{\eta\nu}(t)\Big)\Big(\xi_{j\nu}(t) - \xi_{\eta\nu}(t)\Big)\, dt. \qquad (2.3.111)
\end{aligned}
$$

Finally, by employing (2.3.82) in Definition 2.3.8, we achieve (2.3.109). Thus, the following identities are

obtained

$$\tau_{ij}^{\eta}(t) = \frac{d}{dt}\left\langle \log\left(X_i/Z_\eta\right), \log\left(X_j/Z_\eta\right)\right\rangle_t = \sum_{\nu=1}^{n}\Big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\Big)\Big(\xi_{j\nu}(t)-\xi_{\eta\nu}(t)\Big), \quad \text{and,} \tag{2.3.112}$$

$$\tau_{ij}^{\eta}(t)\,dt = d\left\langle \log\left(X_i/Z_\eta\right), \log\left(X_j/Z_\eta\right)\right\rangle_t = \sum_{\nu=1}^{n}\Big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\Big)\Big(\xi_{j\nu}(t)-\xi_{\eta\nu}(t)\Big)\,dt. \tag{2.3.113}$$

Again, by appealing to equation (2.3.45) of Lemma 2.3.6, the result of (2.3.82) is also derived as follows

$$\begin{aligned}
d\left\langle \log\left(X_i/Z_\eta\right), \log\left(X_j/Z_\eta\right)\right\rangle_t &= d\left\langle \int_0^{\cdot}\sum_{k=1}^{n}\eta_{k,s}\,d\log\left(X_{i,s}/X_{k,s}\right), \int_0^{\cdot}\sum_{m=1}^{n}\eta_{m,s}\,d\log\left(X_{j,s}/X_{m,s}\right)\right\rangle_t \\
&= d\left\langle \sum_{k=1}^{n}\int_0^{\cdot}\eta_{k,s}\,d\log\left(X_{i,s}/X_{k,s}\right), \sum_{m=1}^{n}\int_0^{\cdot}\eta_{m,s}\,d\log\left(X_{j,s}/X_{m,s}\right)\right\rangle_t \\
&= \sum_{k,m=1}^{n} d\left\langle \int_0^{\cdot}\eta_{k,s}\,d\log\left(X_{i,s}/X_{k,s}\right), \int_0^{\cdot}\eta_{m,s}\,d\log\left(X_{j,s}/X_{m,s}\right)\right\rangle_t \\
&= \sum_{k,m=1}^{n}\eta_k(t)\eta_m(t)\,d\left\langle \log\left(X_i/X_k\right), \log\left(X_j/X_m\right)\right\rangle_t && (2.3.114)\\
&= \sum_{k,m=1}^{n}\eta_k(t)\tau_{ij}^{\chi_k\chi_m}(t)\eta_m(t)\,dt && (2.3.115)\\
&= \sum_{k,m=1}^{n}\eta_k(t)\Big[\sigma_{ij}(t)-\sigma_{im}(t)-\sigma_{jk}(t)+\sigma_{km}(t)\Big]\eta_m(t)\,dt && (2.3.116)\\
&= \Bigg(\sigma_{ij}(t)\left[\sum_{k=1}^{n}\eta_k(t)\right]\left[\sum_{m=1}^{n}\eta_m(t)\right]-\left[\sum_{k=1}^{n}\eta_k(t)\right]\left[\sum_{m=1}^{n}\eta_m(t)\sigma_{im}(t)\right] \\
&\qquad -\left[\sum_{k=1}^{n}\eta_k(t)\sigma_{jk}(t)\right]\left[\sum_{m=1}^{n}\eta_m(t)\right]+\sum_{k,m=1}^{n}\eta_k(t)\sigma_{km}(t)\eta_m(t)\Bigg)\,dt \\
&= \big(\sigma_{ij}(t)-\sigma_{i\eta}(t)-\sigma_{j\eta}(t)+\sigma_{\eta\eta}(t)\big)\,dt && (2.3.117)\\
&= \tau_{ij}^{\eta}(t)\,dt, && (2.3.118)
\end{aligned}$$

where equation (2.3.115) follows from (2.3.18), equation (2.3.116) follows from (2.3.19), equation (2.3.117) follows from equations (2.3.51), (2.3.52) and (2.2.94), and lastly, equation (2.3.118) follows from (2.3.85). Therefore, we notice the following result akin to (2.2.94),

$$\tau_{ij}^{\eta}(t) = \sum_{k,m=1}^{n}\eta_k(t)\tau_{ij}^{\chi_k\chi_m}(t)\eta_m(t). \tag{2.3.119}$$

2.3.2 The Relative Variance Process

Recall, the variance process given in (2.2.50) by

$$\sigma_{ii}(t) = \frac{d}{dt}\left\langle \log X_i\right\rangle_t. \tag{2.3.120}$$

Let us now consider the process of individual stocks' variances relative to some portfolio η, namely, the **relative variance process**.

Definition 2.3.9 (Relative Variance). *For a stock $X_i = \{X_i(t), t\in[0,\infty)\}$, for $i = 1,2,\ldots,n$, and any arbitrary portfolio $\eta = \big\{\boldsymbol{\eta}(t) = \big(\eta_1(t),\ldots,\eta_n(t)\big), t\in[0,\infty)\big\}$, the process $\tau_{ii}^{\eta} = \big\{\tau_{ii}^{\eta}(t), t\in[0,\infty)\big\}$, defined by*

$$\tau_{ii}^{\eta}(t) \triangleq \frac{d}{dt}\left\langle \log\left(X_i/Z_\eta\right)\right\rangle_t = \frac{d}{dt}\left\langle R_i^{\eta}\right\rangle_t, \qquad t\in[0,\infty), \qquad a.s., \tag{2.3.121}$$

is called the ***relative variance process*** *of the ith stock X_i versus the portfolio η, i.e., the variance of the relative return process $\log\big(X_i/Z_\eta\big)$, or the variance of the ith stock X_i relative to the portfolio η, and are the diagonal elements of the relative covariance matrix.*

By employing the above definition and the portfolio representation adopted thus far, equation (2.3.21) of Definition 2.3.4 can be recaptured as follows

$$\tau_{ii}^{\chi_k}(t) \;=\; \frac{d}{dt}\big\langle \log\big(X_i/Z_{\chi_k}\big)\big\rangle_t, \qquad t\in[0,\infty), \qquad \text{a.s.} \tag{2.3.122}$$

In a similar fashion, for the portfolios $\chi_1,\chi_2,\ldots,\chi_n$, we have

$$\begin{aligned}
\tau_{ii}^{\chi_1}(t) &= \frac{d}{dt}\big\langle \log\big(X_i/X_1\big)\big\rangle_t &&= \frac{d}{dt}\big\langle \log\big(X_i/Z_{\chi_1}\big)\big\rangle_t,\\
\tau_{ii}^{\chi_2}(t) &= \frac{d}{dt}\big\langle \log\big(X_i/X_2\big)\big\rangle_t &&= \frac{d}{dt}\big\langle \log\big(X_i/Z_{\chi_2}\big)\big\rangle_t,\\
&\vdots && \vdots\\
\tau_{ii}^{\chi_n}(t) &= \frac{d}{dt}\big\langle \log\big(X_i/X_n\big)\big\rangle_t &&= \frac{d}{dt}\big\langle \log\big(X_i/Z_{\chi_n}\big)\big\rangle_t.
\end{aligned}$$

Thus, $\tau_{ii}^{\chi_k}$ can alternatively be interpreted as the variance of the ith stock relative to the portfolio χ_k. Now, from (2.3.121), we obtain

$$d\big\langle \log\big(X_i/Z_\eta\big)\big\rangle_t \;=\; \tau_{ii}^{\eta}(t)\,dt, \qquad t\in[0,\infty), \qquad \text{a.s., or,} \tag{2.3.123}$$

$$\big\langle \log\big(X_i/Z_\eta\big)\big\rangle_t \;=\; \int_0^t \tau_{ii}^{\eta}(s)\,ds, \qquad t\in[0,\infty), \qquad \text{a.s.} \tag{2.3.124}$$

Since $\big\langle \log\big(X_i/Z_\eta\big)\big\rangle$ is a.s. nondecreasing, this implies, for $t\in[0,\infty)$, a.s., that

$$\frac{d}{dt}\big\langle \log\big(X_i/Z_\eta\big)\big\rangle_t \;=\; \tau_{ii}^{\eta}(t) \;\geq\; 0. \tag{2.3.125}$$

Furthermore, for all $t\in[0,\infty)$, we get

$$\begin{aligned}
\big\langle \log\big(X_i/Z_\eta\big)\big\rangle_t &= \big\langle \log X_i - \log Z_\eta\big\rangle_t\\
&= \big\langle \log X_i\big\rangle_t - \big\langle \log X_i, \log Z_\eta\big\rangle_t - \big\langle \log Z_\eta, \log X_i\big\rangle_t + \big\langle \log Z_\eta\big\rangle_t\\
&= \big\langle \log X_i\big\rangle_t - 2\big\langle \log X_i, \log Z_\eta\big\rangle_t + \big\langle \log Z_\eta\big\rangle_t,
\end{aligned} \tag{2.3.126}$$

so that, we have

$$d\big\langle \log\big(X_i/Z_\eta\big)\big\rangle_t \;=\; d\big\langle \log X_i\big\rangle_t - 2\,d\big\langle \log X_i, \log Z_\eta\big\rangle_t + d\big\langle \log Z_\eta\big\rangle_t. \tag{2.3.127}$$

Therefore, by appealing to equations (2.2.45), (2.3.68) and (2.2.95), we equivalently obtain

$$\begin{aligned}
\tau_{ii}^{\eta}(t)\,dt &= d\big\langle \log X_i\big\rangle_t - 2\,d\big\langle \log X_i, \log Z_\eta\big\rangle_t + d\big\langle \log Z_\eta\big\rangle_t\\
&= \sigma_{ii}(t)\,dt - 2\,\sigma_{i\eta}(t)\,dt + \sigma_{\eta\eta}(t)\,dt\\
&= \big(\sigma_{ii}(t) - 2\,\sigma_{i\eta}(t) + \sigma_{\eta\eta}(t)\big)\,dt,
\end{aligned}$$

which also yields

$$\tau_{ii}^{\eta}(t) \;=\; \sigma_{ii}(t) - 2\,\sigma_{i\eta}(t) + \sigma_{\eta\eta}(t). \tag{2.3.128}$$

Thus, for $t\in[0,\infty)$, a.s., equation (2.3.27) can be rewritten as

$$\tau_{ii}^{\chi_k}(t) \;=\; \sigma_{ii}(t) - 2\,\sigma_{i\chi_k}(t) + \sigma_{\chi_k\chi_k}(t). \tag{2.3.129}$$

Analogous to (2.3.109), for $i=1,\ldots,n$ and for $t\in[0,\infty)$, we can also express the relative variance in terms of the volatility processes, and we have

$$\tau_{ii}^{\eta}(t) \;=\; \sum_{\nu=1}^{n}\Big(\xi_{i\nu}(t) - \xi_{\eta\nu}(t)\Big)^2. \tag{2.3.130}$$

For all $i = 1, 2, \ldots, n$ and for $t \in [0, \infty)$, this can be derived as follows

$$\begin{aligned}
\left\langle \log\left(X_i/Z_\eta\right)\right\rangle_t &= \left\langle \int_0^{\cdot} \sum_{\nu=1}^{n} \left(\xi_{i\nu,s} - \xi_{\eta\nu,s}\right) dW_{\nu,s} \right\rangle_t \\
&= \left\langle \sum_{\nu=1}^{n} \int_0^{\cdot} \left(\xi_{i\nu,s} - \xi_{\eta\nu,s}\right) dW_{\nu,s} \right\rangle_t \\
&= \sum_{\nu=1}^{n} \left\langle \int_0^{\cdot} \left(\xi_{i\nu,s} - \xi_{\eta\nu,s}\right) dW_{\nu,s} \right\rangle_t \\
&= \sum_{\nu=1}^{n} \int_0^{t} \Big(\xi_{i\nu}(s) - \xi_{\eta\nu}(s)\Big)^2 d\left\langle W_\nu\right\rangle_s \\
&= \int_0^{t} \sum_{\nu=1}^{n} \Big(\xi_{i\nu}(s) - \xi_{\eta\nu}(s)\Big)^2 ds. \qquad (2.3.131)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
d\left\langle \log\left(X_i/Z_\eta\right)\right\rangle_t &= d\left\langle \int_0^{\cdot} \sum_{\nu=1}^{n} \left(\xi_{i\nu,s} - \xi_{\eta\nu,s}\right) dW_{\nu,s} \right\rangle_t \\
&= d\left(\int_0^{t} \sum_{\nu=1}^{n} \Big(\xi_{i\nu}(s) - \xi_{\eta\nu}(s)\Big)^2 ds \right) \\
&= \sum_{\nu=1}^{n} \Big(\xi_{i\nu}(t) - \xi_{\eta\nu}(t)\Big)^2 dt. \qquad (2.3.132)
\end{aligned}$$

Thus, equation (2.3.130) follows from (2.3.121) in Definition 2.3.9. To summarise, we have

$$\tau_{ii}^{\eta}(t) = \frac{d}{dt}\left\langle \log\left(X_i/Z_\eta\right)\right\rangle_t = \sum_{\nu=1}^{n} \Big(\xi_{i\nu}(t) - \xi_{\eta\nu}(t)\Big)^2, \quad \text{and,} \qquad (2.3.133)$$

$$\tau_{ii}^{\eta}(t)\, dt = d\left\langle \log\left(X_i/Z_\eta\right)\right\rangle_t = \sum_{\nu=1}^{n} \Big(\xi_{i\nu}(t) - \xi_{\eta\nu}(t)\Big)^2 dt. \qquad (2.3.134)$$

Now, from (2.3.129), we have

$$\begin{aligned}
\tau_{kk}^{\chi_i}(t) &= \frac{d}{dt}\left\langle \log\left(X_k/Z_{\chi_i}\right)\right\rangle_t \\
&= \sigma_{kk}(t) - 2\,\sigma_{k\chi_i}(t) + \sigma_{\chi_i\chi_i}(t) \\
&= \sigma_{ii}(t) - 2\,\sigma_{i\chi_k}(t) + \sigma_{\chi_k\chi_k}(t) \\
&= \frac{d}{dt}\left\langle \log\left(X_i/Z_{\chi_k}\right)\right\rangle_t.
\end{aligned}$$

Consequently, we arrive at the following result

$$\tau_{kk}^{\chi_i}(t) = \tau_{ii}^{\chi_k}(t). \qquad (2.3.135)$$

Alternatively, by appealing to equation (2.3.45) of Lemma 2.3.6, we shall verify (2.3.123) as follows

$$
\begin{aligned}
d\left\langle \log\left(X_i/Z_\eta\right)\right\rangle_t &= d\left\langle \int_0^{\cdot} \sum_{k=1}^n \eta_{k,s}\, d\log\left(X_{i,s}/X_{k,s}\right), \int_0^{\cdot} \sum_{m=1}^n \eta_{m,s}\, d\log\left(X_{i,s}/X_{m,s}\right)\right\rangle_t \\
&= d\left\langle \sum_{k=1}^n \int_0^{\cdot} \eta_{k,s}\, d\log\left(X_{i,s}/X_{k,s}\right), \sum_{m=1}^n \int_0^{\cdot} \eta_{m,s}\, d\log\left(X_{i,s}/X_{m,s}\right)\right\rangle_t \\
&= \sum_{k,m=1}^n d\left\langle \int_0^{\cdot} \eta_{k,s}\, d\log\left(X_{i,s}/X_{k,s}\right), \int_0^{\cdot} \eta_{m,s}\, d\log\left(X_{i,s}/X_{m,s}\right)\right\rangle_t \\
&= \sum_{k,m=1}^n \eta_k(t)\eta_m(t)\, d\left\langle \log\left(X_i/X_k\right), \log\left(X_i/X_m\right)\right\rangle_t \qquad (2.3.136)\\
&= \sum_{k,m=1}^n \eta_k(t)\tau_{ii}^{\mathsf{X}_k\mathsf{X}_m}(t)\eta_m(t)\, dt \qquad (2.3.137)\\
&= \sum_{k,m=1}^n \eta_k(t)\Big[\sigma_{ii}(t) - \sigma_{im}(t) - \sigma_{ik}(t) + \sigma_{km}(t)\Big]\eta_m(t)\, dt \qquad (2.3.138)\\
&= \Bigg(\sigma_{ii}(t)\left[\sum_{k=1}^n \eta_k(t)\right]\left[\sum_{m=1}^n \eta_m(t)\right] - \left[\sum_{k=1}^n \eta_k(t)\right]\left[\sum_{m=1}^n \eta_m(t)\sigma_{im}(t)\right] \\
&\qquad - \left[\sum_{k=1}^n \eta_k(t)\sigma_{ik}(t)\right]\left[\sum_{m=1}^n \eta_m(t)\right] + \sum_{k,m=1}^n \eta_k(t)\sigma_{km}(t)\eta_m(t)\Bigg)\, dt \\
&= \left(\sigma_{ii}(t) - 2\,\sigma_{i\eta}(t) + \sigma_{\eta\eta}(t)\right) dt \qquad (2.3.139)\\
&= \tau_{ii}^{\eta}(t)\, dt, \qquad (2.3.140)
\end{aligned}
$$

where equation (2.3.137) follows from equation (2.3.38), equation (2.3.138) follows from equation (2.3.39), equation (2.3.139) follows from equations (2.2.94) and (2.3.51), and lastly, equation (2.3.140) follows from equation (2.3.128). Therefore, we notice the following result

$$
\tau_{ii}^{\eta}(t) = \sum_{k,m=1}^n \eta_k(t)\tau_{ii}^{\mathsf{X}_k\mathsf{X}_m}(t)\eta_m(t). \qquad (2.3.141)
$$

2.4 Some Fundamental Properties

2.4.1 Properties of the Relative Covariance Process

The relative covariance process exhibits some very useful properties, which will be of significance for prospective usage. We commence with the next lemma by establishing an elementary property for the relative covariance.

Lemma 2.4.1. *The $(n\times n)-$matrix-valued process, $\boldsymbol{\tau}^{\boldsymbol{\eta}}(t) = \left(\tau_{ij}^{\eta}(t)\right)_{1\le i,j\le n}$, of the individual stocks' covariances relative to the portfolio η, satisfies the elementary property for all $i = 1,\ldots,n$,*

$$
\sum_{j=1}^n \tau_{ij}^{\eta}(t)\eta_j(t) = 0, \qquad t\in[0,\infty). \qquad (2.4.1)
$$

Alternatively, expressed in matrix form, we have the equivalent property

$$
\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\boldsymbol{\eta}^{\mathsf{T}}(t) = \mathbf{0}^{\mathsf{T}}, \qquad t\in[0,\infty), \qquad (2.4.2)
$$

where $\mathbf{0}$ is the $1\times n$ zero vector, i.e., $\mathbf{0} = (0,\ldots,0)\in\mathbb{R}^n$.

Proof. For $t \in [0, \infty)$, we obtain

$$\begin{aligned}
\sum_{j=1}^{n} \tau_{ij}^{\eta}(t)\eta_j(t) &= \sum_{j=1}^{n} \Big[\sigma_{ij}(t) - \sigma_{i\eta}(t) - \sigma_{j\eta}(t) + \sigma_{\eta\eta}(t)\Big]\eta_j(t) && (2.4.3)\\
&= \sum_{j=1}^{n} \sigma_{ij}(t)\eta_j(t) - \sigma_{i\eta}(t)\sum_{j=1}^{n}\eta_j(t) - \sum_{j=1}^{n}\sigma_{j\eta}(t)\eta_j(t) + \sigma_{\eta\eta}(t)\sum_{j=1}^{n}\eta_j(t)\\
&= \sigma_{i\eta}(t) - \sigma_{i\eta}(t) - \sum_{j=1}^{n}\Big[\sum_{i=1}^{n}\sigma_{ji}(t)\eta_i(t)\Big]\eta_j(t) + \sigma_{\eta\eta}(t) && (2.4.4)\\
&= \sigma_{\eta\eta}(t) - \sum_{i,j=1}^{n}\eta_i(t)\sigma_{ij}(t)\eta_j(t) && (2.4.5)\\
&= \sigma_{\eta\eta}(t) - \sigma_{\eta\eta}(t) && (2.4.6)\\
&= 0,
\end{aligned}$$

where equation (2.4.3) follows from equation (2.3.85), equation (2.4.4) follows from equation (2.3.51), and finally, equation (2.4.6) follows from equation (2.2.94). ■

The result $\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\boldsymbol{\eta}^{\mathsf{T}}(t) = \mathbf{0}^{\mathsf{T}}$, for all $t \in [0, \infty)$, implies that

$$\boldsymbol{\eta}(t)\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\boldsymbol{\eta}^{\mathsf{T}}(t) = \sum_{i,j=1}^{n}\eta_i(t)\tau_{ij}^{\eta}(t)\eta_j(t) = 0, \qquad t \in [0, \infty). \tag{2.4.7}$$

Another property, utilising the results from the previous lemma, is illustrated in the following lemma.

Lemma 2.4.2 ([**Fernholz (2002)**]). *For an arbitrary portfolio η, the $(n \times n)-$matrix-valued process, $\boldsymbol{\tau}^{\boldsymbol{\eta}}(t) = \big(\tau_{ij}^{\eta}(t)\big)_{1\le i,j\le n}$, of the individual stocks' covariances relative to the portfolio η, i.e., the relative covariance matrix, is positive semidefinite with rank $n-1$, for $t \in [0, \infty)$, a.s., and the null space*[13] *of the relative covariance matrix $\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)$ is spanned by the vector $\boldsymbol{\eta}(t)$, for $t \in [0, \infty)$, a.s.*

Proof. Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\mathbf{x} \neq \mathbf{0}$, and let $t \in [0, \infty)$. Then it follows from equations (2.3.85), (2.3.51) and (2.3.55) that

$$\begin{aligned}
\mathbf{x}\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\mathbf{x}^{\mathsf{T}} &= \sum_{i,j=1}^{n} x_i\tau_{ij}^{\eta}(t)x_j\\
&= \sum_{i,j=1}^{n} x_i\Big[\sigma_{ij}(t) - \sigma_{i\eta}(t) - \sigma_{j\eta}(t) + \sigma_{\eta\eta}(t)\Big]x_j\\
&= \sum_{i,j=1}^{n} x_i\sigma_{ij}(t)x_j - \sum_{i,j=1}^{n} x_i\sigma_{i\eta}(t)x_j - \sum_{i,j=1}^{n} x_i\sigma_{j\eta}(t)x_j + \sum_{i,j=1}^{n} x_i\sigma_{\eta\eta}(t)x_j\\
&= \sum_{i,j=1}^{n} x_i\sigma_{ij}(t)x_j - 2\sum_{i,j=1}^{n} x_ix_j\sigma_{i\eta}(t) + \sum_{i,j=1}^{n} x_i\sigma_{\eta\eta}(t)x_j\\
&= \sum_{i,j=1}^{n} x_i\sigma_{ij}(t)x_j - 2\sum_{i,j=1}^{n} x_ix_j\Big[\sum_{k=1}^{n}\eta_k(t)\sigma_{ik}(t)\Big] + \sigma_{\eta\eta}(t)\sum_{i,j=1}^{n} x_ix_j\\
&= \sum_{i,j=1}^{n} x_i\sigma_{ij}(t)x_j - 2\Big[\sum_{i,k=1}^{n} x_i\sigma_{ik}(t)\eta_k(t)\Big]\sum_{j=1}^{n} x_j + \sigma_{\eta\eta}(t)\sum_{i=1}^{n} x_i\sum_{j=1}^{n} x_j\\
&= \mathbf{x}\boldsymbol{\sigma}(t)\mathbf{x}^{\mathsf{T}} - 2\,\mathbf{x}\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)\sum_{j=1}^{n} x_j + \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)\Big(\sum_{i=1}^{n} x_i\Big)\Big(\sum_{j=1}^{n} x_j\Big).
\end{aligned}$$

[13] In linear algebra, the *kernel* or *null space* of a matrix $\mathbf{A}$ is the set of all vectors $\mathbf{x}$ for which $\mathbf{A}\mathbf{x} = \mathbf{0}$, i.e., the set of vectors that map to the zero vector. More precisely, the null space of an $m \times n$ matrix $\mathbf{A}$ is the set $\mathrm{Null}(\mathbf{A}) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{0}\}$.

Consequently, we have

$$\mathbf{x}\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\mathbf{x}^{\mathsf{T}} = \mathbf{x}\boldsymbol{\sigma}(t)\mathbf{x}^{\mathsf{T}} - 2\,\mathbf{x}\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)\sum_{i=1}^{n} x_i + \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)\Big(\sum_{i=1}^{n} x_i\Big)^2. \tag{2.4.8}$$

Alternatively, by considering equation (2.3.105), we equivalently obtain

$$\begin{aligned}
\mathbf{x}\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\mathbf{x}^{\mathsf{T}} &= \mathbf{x}\Big(\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)\mathbf{1} - \mathbf{1}^{\mathsf{T}}\boldsymbol{\eta}(t)\boldsymbol{\sigma}(t) + \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)\Big)\mathbf{x}^{\mathsf{T}} \\
&= \mathbf{x}\Big(\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)\mathbf{1} - \mathbf{1}^{\mathsf{T}}\boldsymbol{\eta}(t)\boldsymbol{\sigma}(t) + \mathbf{1}^{\mathsf{T}}\boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)\mathbf{1}\Big)\mathbf{x}^{\mathsf{T}} \\
&= \mathbf{x}\boldsymbol{\sigma}(t)\mathbf{x}^{\mathsf{T}} - \mathbf{x}\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)\mathbf{1}\mathbf{x}^{\mathsf{T}} - \mathbf{x}\mathbf{1}^{\mathsf{T}}\boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\mathbf{x}^{\mathsf{T}} + \mathbf{x}\mathbf{1}^{\mathsf{T}}\boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)\mathbf{1}\mathbf{x}^{\mathsf{T}} \\
&= \mathbf{x}\boldsymbol{\sigma}(t)\mathbf{x}^{\mathsf{T}} - \mathbf{x}\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)\sum_{i=1}^{n} x_i - \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\mathbf{x}^{\mathsf{T}}\sum_{i=1}^{n} x_i + \Big(\sum_{i=1}^{n} x_i\Big)\boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)\Big(\sum_{i=1}^{n} x_i\Big) \\
&= \mathbf{x}\boldsymbol{\sigma}(t)\mathbf{x}^{\mathsf{T}} - \mathbf{x}\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)\sum_{i=1}^{n} x_i - \mathbf{x}\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)\sum_{i=1}^{n} x_i + \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)\Big(\sum_{i=1}^{n} x_i\Big)^2 \\
&= \mathbf{x}\boldsymbol{\sigma}(t)\mathbf{x}^{\mathsf{T}} - 2\,\mathbf{x}\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)\sum_{i=1}^{n} x_i + \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)\Big(\sum_{i=1}^{n} x_i\Big)^2.
\end{aligned}$$

There are two cases that we shall consider:

- First, suppose that $\sum_{i=1}^{n} x_i = 0$. Then it follows from equation (2.4.8) that

  $$\mathbf{x}\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\mathbf{x}^{\mathsf{T}} = \mathbf{x}\boldsymbol{\sigma}(t)\mathbf{x}^{\mathsf{T}}.$$

  Now, recall from Lemma 2.2.15 that $\boldsymbol{\sigma}(t)$ is a.s. positive definite for $t \in [0, \infty)$, which together with the assumption that $\mathbf{x} \neq \mathbf{0}$, we ascertain that

  $$\mathbf{x}\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\mathbf{x}^{\mathsf{T}} > 0.$$

- Now, consider the juxtaposed case where we suppose that on the other hand we have $\sum_{i=1}^{n} x_i = a \neq 0$, where $a \in \mathbb{R}$. So that, in this notation, equation (2.4.8) becomes

  $$\mathbf{x}\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\mathbf{x}^{\mathsf{T}} = \mathbf{x}\boldsymbol{\sigma}(t)\mathbf{x}^{\mathsf{T}} - 2a\mathbf{x}\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t) + a^2\boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t). \tag{2.4.9}$$

  Let us consider the vector $\mathbf{y} := \mathbf{x}/a = a^{-1}\mathbf{x}$, then it is obvious to see that this vector satisfies $\sum_{i=1}^{n} y_i = 1$. Thus, we have $\mathbf{x} = a\mathbf{y}$. By applying this simple substitution we obtain

  $$\begin{aligned}
  \mathbf{x}\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\mathbf{x}^{\mathsf{T}} &= a^2\mathbf{y}\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\mathbf{y}^{\mathsf{T}}, \quad \text{or,} \\
  \mathbf{y}\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\mathbf{y}^{\mathsf{T}} &= a^{-2}\mathbf{x}\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\mathbf{x}^{\mathsf{T}}.
  \end{aligned}$$

  Due to this result, and to this end, it suffices to consider the expression $\mathbf{y}\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\mathbf{y}^{\mathsf{T}}$. Consequently, using the fact that $\sum_{i=1}^{n} y_i = 1$ and setting $\mathbf{x} := a\mathbf{y}$ in equation (2.4.8), we observe that

  $$\begin{aligned}
  \mathbf{y}\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\mathbf{y}^{\mathsf{T}} &= \mathbf{y}\boldsymbol{\sigma}(t)\mathbf{y}^{\mathsf{T}} - 2\,\mathbf{y}\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)\sum_{i=1}^{n} y_i + \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)\Big(\sum_{i=1}^{n} y_i\Big)^2 && (2.4.10) \\
  &= \mathbf{y}\boldsymbol{\sigma}(t)\mathbf{y}^{\mathsf{T}} - 2\,\mathbf{y}\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t) + \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t) \\
  &= \mathbf{y}\boldsymbol{\sigma}(t)\mathbf{y}^{\mathsf{T}} - \mathbf{y}\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t) - \mathbf{y}\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t) + \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t) \\
  &= \mathbf{y}\boldsymbol{\sigma}(t)\mathbf{y}^{\mathsf{T}} - \mathbf{y}\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t) - \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\mathbf{y}^{\mathsf{T}} + \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t) \\
  &= \Big(\mathbf{y} - \boldsymbol{\eta}(t)\Big)\boldsymbol{\sigma}(t)\Big(\mathbf{y}^{\mathsf{T}} - \boldsymbol{\eta}^{\mathsf{T}}(t)\Big) \\
  &= \Big(\mathbf{y} - \boldsymbol{\eta}(t)\Big)\boldsymbol{\sigma}(t)\Big(\mathbf{y} - \boldsymbol{\eta}(t)\Big)^{\mathsf{T}} && (2.4.11) \\
  &\equiv \Big(\boldsymbol{\eta}(t) - \mathbf{y}\Big)\boldsymbol{\sigma}(t)\Big(\boldsymbol{\eta}(t) - \mathbf{y}\Big)^{\mathsf{T}}, && (2.4.12)
  \end{aligned}$$

since $\mathbf{y}\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)$ is just some number, and is thus equivalent to its transpose. The result then follows by appealing to the symmetry of the covariance matrix $\boldsymbol{\sigma}(t)$, where $\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}^{\mathsf{T}}(t)$. Again, using the result that $\boldsymbol{\sigma}(t)$ is a.s. positive definite by Lemma 2.2.15, we have that $\mathbf{y}\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\mathbf{y}^{\mathsf{T}} > 0$. In particular, notice that $\mathbf{y}\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\mathbf{y}^{\mathsf{T}} = 0$ if and only if $\mathbf{y} = \boldsymbol{\eta}(t)$, or equivalently if $\mathbf{x} = a\boldsymbol{\eta}(t)$. Thus, by using the fact that $\mathbf{y}\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\mathbf{y}^{\mathsf{T}} \geq 0$, we can conclude that

$$a^2\mathbf{y}\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\mathbf{y}^{\mathsf{T}} \quad = \quad \mathbf{x}\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\mathbf{x}^{\mathsf{T}} \quad \geq \quad 0, \qquad t \in [0,\infty), \qquad \text{a.s.},$$

which reveals that $\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)$ is positive semidefinite for $t \in [0,\infty)$, a.s.

Denote the kth element in the ith unit vector $\mathbf{e}_i$ by e_{ik}, such that $\mathbf{e}_i = \big(e_{i1},\ldots,e_{i(i-1)},e_{ii},e_{i(i+1)},\ldots,e_{in}\big) = (0,\ldots,0,1,0,\ldots,0)$. Likewise, denote the kth element in the jth unit vector $\mathbf{e}_j$ by e_{jk}, such that $\mathbf{e}_j = \big(e_{j1},\ldots,e_{j(j-1)},e_{jj},e_{j(j+1)},\ldots,e_{jn}\big) = (0,\ldots,0,1,0,\ldots,0)$. Thus, we have

$$\sum_{k=1}^{n} e_{ik} \quad = \quad 1, \qquad\qquad \text{and} \qquad\qquad \sum_{k=1}^{n} e_{jk} \quad = \quad 1,$$

which implies that

$$\Big(\sum_{k=1}^{n} e_{ik}\Big)\Big(\sum_{k=1}^{n} e_{jk}\Big) \quad = \quad 1.$$

Then the result (2.4.11) is also apparent from equations (2.3.102) and (2.3.104), where we have

$$\mathbf{e}_i\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\mathbf{e}_j^{\mathsf{T}} \quad = \quad \Big(\mathbf{e}_i - \boldsymbol{\eta}(t)\Big)\boldsymbol{\sigma}(t)\Big(\mathbf{e}_j - \boldsymbol{\eta}(t)\Big)^{\mathsf{T}}.$$

Now, since $\sum_{i=1}^{n}\pi_i(t) = 1$, by setting $\mathbf{y} := \boldsymbol{\pi}(t)$ in equation (2.4.10), it follows that for any portfolio π, for $t \in [0,\infty)$, we a.s. get

$$\begin{aligned}
\boldsymbol{\pi}(t)\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\boldsymbol{\pi}^{\mathsf{T}}(t) &= \boldsymbol{\pi}(t)\boldsymbol{\sigma}(t)\boldsymbol{\pi}^{\mathsf{T}}(t) - 2\,\boldsymbol{\pi}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)\sum_{i=1}^{n}\pi_i(t) + \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)\Big(\sum_{i=1}^{n}\pi_i(t)\Big)^2 && (2.4.13)\\
&= \boldsymbol{\pi}(t)\boldsymbol{\sigma}(t)\boldsymbol{\pi}^{\mathsf{T}}(t) - 2\,\boldsymbol{\pi}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t) + \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)\\
&= \boldsymbol{\pi}(t)\boldsymbol{\sigma}(t)\boldsymbol{\pi}^{\mathsf{T}}(t) - \boldsymbol{\pi}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t) - \boldsymbol{\pi}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t) + \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)\\
&= \boldsymbol{\pi}(t)\boldsymbol{\sigma}(t)\boldsymbol{\pi}^{\mathsf{T}}(t) - \boldsymbol{\pi}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t) - \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\boldsymbol{\pi}^{\mathsf{T}}(t) + \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t) && (2.4.14)\\
&= \Big(\boldsymbol{\pi}(t) - \boldsymbol{\eta}(t)\Big)\boldsymbol{\sigma}(t)\Big(\boldsymbol{\pi}^{\mathsf{T}}(t) - \boldsymbol{\eta}^{\mathsf{T}}(t)\Big)\\
&= \Big(\boldsymbol{\pi}(t) - \boldsymbol{\eta}(t)\Big)\boldsymbol{\sigma}(t)\Big(\boldsymbol{\pi}(t) - \boldsymbol{\eta}(t)\Big)^{\mathsf{T}}, && (2.4.15)
\end{aligned}$$

where equation (2.4.14) follows from the fact that $\boldsymbol{\pi}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t)$ is just some number and is thus equivalent to its transpose $\boldsymbol{\eta}(t)\boldsymbol{\sigma}^{\mathsf{T}}(t)\boldsymbol{\pi}^{\mathsf{T}}(t)$. The result then follows by appealing to the symmetry of the covariance matrix $\boldsymbol{\sigma}(t)$. Consequently, for the null space of $\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)$, for $t \in [0,\infty)$, a.s., we require that

$$\boldsymbol{\pi}(t)\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\boldsymbol{\pi}^{\mathsf{T}}(t) \quad = \quad \Big(\boldsymbol{\pi}(t) - \boldsymbol{\eta}(t)\Big)\boldsymbol{\sigma}(t)\Big(\boldsymbol{\pi}(t) - \boldsymbol{\eta}(t)\Big)^{\mathsf{T}} \quad = \quad 0. \tag{2.4.16}$$

This is achieved if and only if $\boldsymbol{\pi}(t) = \boldsymbol{\eta}(t)$ for $t \in [0,\infty)$. Thus, $\boldsymbol{\eta}(t)$ spans the null space of $\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)$, a.s., and the rank of $\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)$ is $n-1$, which holds a.s. for any $t \in [0,\infty)$. This result is also apparent from the result obtained in the previous lemma, in which it was shown that $\boldsymbol{\eta}(t)\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\boldsymbol{\eta}^{\mathsf{T}}(t) = 0$. ■

Thus far, we have defined and discussed the covariance of single individual stocks relative to some portfolio. We would like to expand this notion to portfolios. In doing so, we shall define the relative variance of portfolios, which is presented next.

2.4.2 The Relative Portfolio Variance Process

The preceding lemma introduced the expression $\boldsymbol{\pi}(t)\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\boldsymbol{\pi}^{\mathsf{T}}(t)$, which we shall now formally define as the **relative variance of portfolios**.

Definition 2.4.3 (Relative Variance of Portfolios). *For portfolios* $\pi = \big\{\boldsymbol{\pi}(t) = \big(\pi_1(t), \ldots, \pi_n(t)\big), t \in [0,\infty)\big\}$ *and* $\eta = \big\{\boldsymbol{\eta}(t) = \big(\eta_1(t), \ldots, \eta_n(t)\big), t \in [0,\infty)\big\}$, *the process* $\tau^{\eta}_{\pi\pi} = \big\{\tau^{\eta}_{\pi\pi}(t), t \in [0,\infty)\big\}$ *defined by*

$$\tau^{\eta}_{\pi\pi}(t) \triangleq \boldsymbol{\pi}(t)\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\boldsymbol{\pi}^{\mathsf{T}}(t) = \sum_{i,j=1}^{n} \pi_i(t)\tau^{\eta}_{ij}(t)\pi_j(t), \qquad t \in [0,\infty), \tag{2.4.17}$$

is called the ***relative variance process*** *of portfolio* π *versus* η, *and the process* $\tau^{\pi}_{\eta\eta} = \big\{\tau^{\pi}_{\eta\eta}(t), t \in [0,\infty)\big\}$ *defined by*

$$\tau^{\pi}_{\eta\eta}(t) \triangleq \boldsymbol{\eta}(t)\boldsymbol{\tau}^{\boldsymbol{\pi}}(t)\boldsymbol{\eta}^{\mathsf{T}}(t) = \sum_{i,j=1}^{n} \eta_i(t)\tau^{\pi}_{ij}(t)\eta_j(t), \qquad t \in [0,\infty), \tag{2.4.18}$$

is called the ***relative variance process*** *of portfolio* η *versus* π.

The relative variance, just as the relative covariance, exhibits some rather interesting properties, one of which we shall explore in the following lemma.

Lemma 2.4.4. *For portfolios* $\pi = \big\{\boldsymbol{\pi}(t) = \big(\pi_1(t), \ldots, \pi_n(t)\big), t \in [0,\infty)\big\}$ *and* $\eta = \big\{\boldsymbol{\eta}(t) = \big(\eta_1(t), \ldots, \eta_n(t)\big), t \in [0,\infty)\big\}$, *the relative variance in* (2.4.17), *satisfies the following property*

$$\tau^{\eta}_{\pi\pi}(t) = \tau^{\pi}_{\eta\eta}(t), \qquad t \in [0,\infty), \qquad a.s. \tag{2.4.19}$$

Proof. Since $\sum_{i=1}^{n} \pi_i(t) = 1$ and $\sum_{i=1}^{n} \eta_i(t) = 1$, using the same procedure employed in arriving at (2.4.15), we can show for $t \in [0,\infty)$, a.s., that

$$\begin{aligned}
\boldsymbol{\eta}(t)\boldsymbol{\tau}^{\boldsymbol{\pi}}(t)\boldsymbol{\eta}^{\mathsf{T}}(t) &= \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t) - 2\,\boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\boldsymbol{\pi}^{\mathsf{T}}(t)\sum_{i=1}^{n}\eta_i(t) + \boldsymbol{\pi}(t)\boldsymbol{\sigma}(t)\boldsymbol{\pi}^{\mathsf{T}}(t)\Big(\sum_{i=1}^{n}\eta_i(t)\Big)^2 \\
&= \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^{\mathsf{T}}(t) - 2\,\boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\boldsymbol{\pi}^{\mathsf{T}}(t) + \boldsymbol{\pi}(t)\boldsymbol{\sigma}(t)\boldsymbol{\pi}^{\mathsf{T}}(t) \\
&= \Big(\boldsymbol{\eta}(t) - \boldsymbol{\pi}(t)\Big)\boldsymbol{\sigma}(t)\Big(\boldsymbol{\eta}(t) - \boldsymbol{\pi}(t)\Big)^{\mathsf{T}} && (2.4.20) \\
&= \Big(\boldsymbol{\pi}(t) - \boldsymbol{\eta}(t)\Big)\boldsymbol{\sigma}(t)\Big(\boldsymbol{\pi}(t) - \boldsymbol{\eta}(t)\Big)^{\mathsf{T}} && (2.4.21) \\
&= \boldsymbol{\pi}(t)\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\boldsymbol{\pi}^{\mathsf{T}}(t). && (2.4.22)
\end{aligned}$$

Therefore, by definition, for $t \in [0,\infty)$, we have the result

$$\tau^{\pi}_{\eta\eta}(t) = \tau^{\eta}_{\pi\pi}(t).$$

■

The result presented in the lemma above is akin to the derived result of (2.3.36). The results are similar since each stock in the market represents a portfolio in itself, i.e., we have

$$\begin{aligned}
\tau^{\chi_i}_{\chi_k\chi_k}(t) &= \frac{d}{dt}\big\langle \log\big(Z_{\chi_k}/Z_{\chi_i}\big)\big\rangle_t \\
&= \sigma_{\chi_k\chi_k}(t) - 2\,\sigma_{\chi_k\chi_i}(t) + \sigma_{\chi_i\chi_i}(t) \\
&= \sigma_{\chi_i\chi_i}(t) - 2\,\sigma_{\chi_i\chi_k}(t) + \sigma_{\chi_k\chi_k}(t) \\
&= \frac{d}{dt}\big\langle \log\big(Z_{\chi_i}/Z_{\chi_k}\big)\big\rangle_t.
\end{aligned}$$

Consequently, for $t \in [0,\infty)$, we get

$$\tau^{\chi_i}_{\chi_k\chi_k}(t) = \tau^{\chi_k}_{\chi_i\chi_i}(t). \tag{2.4.23}$$

Obviously, the relative variance process of a portfolio versus itself is zero. This follows easily from equation (2.4.7) (which is a consequence of Lemma 2.4.1) in conjunction with the matrix representation of the relative variance process of portfolios provided in Definition 2.4.3. Thus, we have

$$\boldsymbol{\eta}(t)\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\boldsymbol{\eta}^{\mathsf{T}}(t) = \tau^{\eta}_{\eta\eta}(t) = 0, \quad \text{and,} \tag{2.4.24}$$

$$\boldsymbol{\pi}(t)\boldsymbol{\tau}^{\boldsymbol{\pi}}(t)\boldsymbol{\pi}^{\mathsf{T}}(t) = \tau^{\pi}_{\pi\pi}(t) = 0. \tag{2.4.25}$$

2.4.3 Properties of the Excess Growth Rate Process

Recall from equation (2.2.66) in Proposition 2.2.20 that the portfolio growth rate is the weighted average of the growth rates of the individual component stocks plus the excess growth rate of the portfolio. Since the excess growth rate plays such a critical role in the construction of the portfolio growth rate, it is necessary to develop tools that will facilitate its calculation. This directs us to the following lemma, which establishes the so-called "numéraire invariance" property of the excess growth rate of a portfolio π. This property, namely that the excess growth rate of a portfolio π is **numéraire invariant**, is of particular interest when the numéraire is the market portfolio, which will be of focus in Section 2.12.

Lemma 2.4.5 (Numéraire Invariance Property). *Let π and η be any two arbitrary portfolios. Then a.s., for $t \in [0, \infty)$, we have the* ***numéraire invariance property***

$$\gamma^*_{\pi}(t) = \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\tau^{\eta}_{ii}(t) - \sum_{i,j=1}^{n} \pi_i(t)\tau^{\eta}_{ij}(t)\pi_j(t)\right). \tag{2.4.26}$$

Proof. We want to show that equation (2.4.26) is equivalent to equation (2.2.108). By making use of the relations (2.3.85), (2.3.128) and the fact that $\sum_{i=1}^{n} \pi_i(t) = 1$ for all $t \in [0, \infty)$, we obtain

$$\begin{aligned}
\sum_{i=1}^{n} \pi_i(t)\tau^{\eta}_{ii}(t) &= \sum_{i=1}^{n} \pi_i(t)\Big[\sigma_{ii}(t) - 2\,\sigma_{i\eta}(t) + \sigma_{\eta\eta}(t)\Big] \\
&= \sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - 2\sum_{i=1}^{n} \pi_i(t)\sigma_{i\eta}(t) + \sum_{i=1}^{n} \pi_i(t)\sigma_{\eta\eta}(t) \\
&= \sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - 2\sum_{i=1}^{n} \pi_i(t)\sigma_{i\eta}(t) + \sigma_{\eta\eta}(t)\sum_{i=1}^{n} \pi_i(t) \\
&= \sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - 2\sum_{i=1}^{n} \pi_i(t)\sigma_{i\eta}(t) + \sigma_{\eta\eta}(t),
\end{aligned} \tag{2.4.27}$$

and,

$$\begin{aligned}
\sum_{i,j=1}^{n} \pi_i(t)\tau_{ij}^{\eta}(t)\pi_j(t) &= \sum_{i,j=1}^{n} \pi_i(t)\Big[\sigma_{ij}(t) - \sigma_{i\eta}(t) - \sigma_{j\eta}(t) + \sigma_{\eta\eta}(t)\Big]\pi_j(t) \\
&= \sum_{i,j=1}^{n} \pi_i(t)\sigma_{ij}(t)\pi_j(t) - \sum_{i,j=1}^{n} \pi_i(t)\sigma_{i\eta}(t)\pi_j(t) - \sum_{i,j=1}^{n} \pi_i(t)\sigma_{j\eta}(t)\pi_j(t) \\
&\quad + \sum_{i,j=1}^{n} \pi_i(t)\sigma_{\eta\eta}(t)\pi_j(t) \\
&= \sum_{i,j=1}^{n} \pi_i(t)\sigma_{ij}(t)\pi_j(t) - \sum_{j=1}^{n} \pi_j(t)\left[\sum_{i=1}^{n} \pi_i(t)\sigma_{i\eta}(t)\right] - \sum_{i=1}^{n} \pi_i(t)\left[\sum_{j=1}^{n} \pi_j(t)\sigma_{j\eta}(t)\right] \\
&\quad + \sigma_{\eta\eta}(t)\left[\sum_{i=1}^{n} \pi_i(t)\right]\left[\sum_{j=1}^{n} \pi_j(t)\right] \\
&= \sum_{i,j=1}^{n} \pi_i(t)\sigma_{ij}(t)\pi_j(t) - \sum_{i=1}^{n} \pi_i(t)\sigma_{i\eta}(t) - \sum_{j=1}^{n} \pi_j(t)\sigma_{j\eta}(t) + \sigma_{\eta\eta}(t) \\
&= \sum_{i,j=1}^{n} \pi_i(t)\sigma_{ij}(t)\pi_j(t) - 2\sum_{i=1}^{n} \pi_i(t)\sigma_{i\eta}(t) + \sigma_{\eta\eta}(t). \qquad (2.4.28)
\end{aligned}$$

Therefore, upon comparison of the preceding expressions (2.4.27) and (2.4.28), we confirm the result as follows

$$\begin{aligned}
\frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\tau_{ii}^{\eta}(t) - \sum_{i,j=1}^{n} \pi_i(t)\tau_{ij}^{\eta}(t)\pi_j(t)\right) &= \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - 2\sum_{i=1}^{n} \pi_i(t)\sigma_{i\eta}(t) + \sigma_{\eta\eta}(t)\right. \\
&\qquad \left. - \sum_{i,j=1}^{n} \pi_i(t)\sigma_{ij}(t)\pi_j(t) + 2\sum_{i=1}^{n} \pi_i(t)\sigma_{i\eta}(t) - \sigma_{\eta\eta}(t)\right) \\
&= \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^{n} \pi_i(t)\sigma_{ij}(t)\pi_j(t)\right) \\
&\equiv \gamma_{\pi}^{*}(t),
\end{aligned}$$

which follows from equation (2.2.108). So, for $t \in [0, \infty)$, we have the result

$$\gamma_{\pi}^{*}(t) = \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\tau_{ii}^{\eta}(t) - \sum_{i,j=1}^{n} \pi_i(t)\tau_{ij}^{\eta}(t)\pi_j(t)\right).$$

■

This proposition demonstrates the fact that the excess growth rate of the portfolio π does not depend on the choice of the benchmark portfolio η, the portfolio with which the performance of the portfolio π is to be contrasted. The expression given in equation (2.4.26) offers a more useful approach to calculating the excess growth rate, since it requires an estimation of the relative variances as opposed to an estimation of the variances themselves. Thus, by employing a change of numéraire technique we can essentially replace the variances in equation (2.2.109) by the relative variances without changing the value of the excess growth rate. The portfolio growth rate does not depend on the choice of the numéraire asset from which variances are calculated.

Remark 2.4.6. Notice, by applying equation (2.4.17), we can express equation (2.4.26) as

$$\gamma_{\pi}^{*}(t) = \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\tau_{ii}^{\eta}(t) - \tau_{\pi\pi}^{\eta}(t)\right), \qquad (2.4.29)$$

which is comparable to the expression given in equation (2.2.109),

$$\gamma_{\pi}^{*}(t) = \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - \sigma_{\pi\pi}(t)\right). \qquad (2.4.30)$$

Therefore, from equation (2.4.29), we have

$$\sum_{i=1}^{n} \pi_i(t)\tau_{ii}^{\eta}(t) \quad = \quad \tau_{\pi\pi}^{\eta}(t) + 2\,\gamma_{\pi}^{*}(t). \tag{2.4.31}$$

The next lemma simplifies the expression for the excess growth rate of a portfolio π, by affording the portfolio π the opportunity of being the numéraire portfolio.

Lemma 2.4.7 ([**Fernholz (2002)**]). *Let π be any arbitrary portfolio. Then a.s., for $t \in [0, \infty)$, we obtain the representation*

$$\gamma_{\pi}^{*}(t) \quad = \quad \frac{1}{2}\sum_{i=1}^{n} \pi_i(t)\tau_{ii}^{\pi}(t), \tag{2.4.32}$$

for the excess growth rate, as a weighted average of the individual stocks' variances τ_{ii}^{π} relative to the portfolio π.

Proof. Let $\eta := \pi$ in Lemma 2.4.5. Then from equation (2.4.26) we have

$$\gamma_{\pi}^{*}(t) \quad = \quad \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\tau_{ii}^{\pi}(t) - \sum_{i,j=1}^{n} \pi_i(t)\tau_{ij}^{\pi}(t)\pi_j(t)\right), \tag{2.4.33}$$

or from equation (2.4.29), we have

$$\gamma_{\pi}^{*}(t) \quad = \quad \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\tau_{ii}^{\pi}(t) - \tau_{\pi\pi}^{\pi}(t)\right). \tag{2.4.34}$$

Now, from equation (2.4.11) (setting both $\mathbf{y}$ and $\boldsymbol{\eta}(t)$ to $\boldsymbol{\pi}(t)$) together with equation (2.4.17) of Definition 2.4.3, we obtain

$$\tau_{\pi\pi}^{\pi}(t) \quad = \quad \boldsymbol{\pi}(t)\boldsymbol{\tau}^{\boldsymbol{\pi}}(t)\boldsymbol{\pi}^{\mathsf{T}}(t) \quad = \quad \Big(\boldsymbol{\pi}(t) - \boldsymbol{\pi}(t)\Big)\boldsymbol{\sigma}(t)\Big(\boldsymbol{\pi}(t) - \boldsymbol{\pi}(t)\Big)^{\mathsf{T}} \quad = \quad 0.$$

Alternatively, refer to the comment made after the proof of Lemma 2.4.4, i.e., consider equations (2.4.24) and (2.4.25). So, from Definition 2.4.3 and the result above, we have

$$\tau_{\pi\pi}^{\pi}(t) \quad = \quad \boldsymbol{\pi}(t)\boldsymbol{\tau}^{\boldsymbol{\pi}}(t)\boldsymbol{\pi}^{\mathsf{T}}(t) \quad = \quad \sum_{i,j=1}^{n} \pi_i(t)\tau_{ij}^{\pi}(t)\pi_j(t) \quad = \quad 0, \tag{2.4.35}$$

which, when substituted into either equation (2.4.33) or equation (2.4.34), yields the desired result. An alternative proof is provided in Fernholz & Karatzas (2009), which we shall formally proffer here. Recalling the result (2.4.1) and putting this to use by setting $\eta := \pi$ in equation (2.4.26), we obtain the following representation

$$\begin{aligned}\gamma_{\pi}^{*}(t) \quad &= \quad \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\tau_{ii}^{\pi}(t) - \sum_{i=1}^{n} \pi_i(t)\left[\sum_{j=1}^{n} \tau_{ij}^{\pi}(t)\pi_j(t)\right]\right) \\ &= \quad \frac{1}{2}\sum_{i=1}^{n} \pi_i(t)\tau_{ii}^{\pi}(t).\end{aligned}$$

■

Therefore, we have

$$\sum_{i=1}^{n} \pi_i(t)\tau_{ii}^{\pi}(t) \quad = \quad 2\,\gamma_{\pi}^{*}(t). \tag{2.4.36}$$

Here we discover that the excess growth rate of the portfolio π can be expressed purely in terms of the relative variances of the stocks versus the portfolio π. This lemma allows us to draw some conclusions regarding the positivity of the excess growth rate. We mentioned earlier that the excess growth rate of a portfolio will be strictly positive if the portfolio holds at least two or more stocks with no short sales; and will be nonnegative for strictly long-only portfolios. In the next proposition, the reasoning for this will be revealed.

Proposition 2.4.8 ([**Fernholz (2002)**]). *Let π be a portfolio with nonnegative weights (i.e., $\pi_i(t) \geq 0$ for all $i = 1, \ldots, n$, and $t \in [0, \infty)$).*[14]

(*i*) *Then, we get for any strictly long-only portfolio π the following property*

$$\gamma_\pi^*(t) \quad \geq \quad 0, \qquad t \in [0, \infty), \qquad a.s., \tag{2.4.37}$$

(*i.e., for a strictly long-only portfolio π that does not permit short sales, and in which* ***all the weights*** *of the portfolio are nonnegative, the excess growth rate of π will be nonnegative*);

(*ii*) *If there exists both an i, j, $i \neq j$, for $i, j = 1, \ldots, n$, such that $0 < \pi_i(t), \pi_j(t) < 1$, for all $t \in [0, \infty)$, then*

$$\gamma_\pi^*(t) \quad > \quad 0, \qquad t \in [0, \infty), \qquad a.s., \tag{2.4.38}$$

(*i.e., for a portfolio π in which* ***at least two stocks*** *are held with no short sales, the excess growth rate of π will be strictly positive*).

Proof. There are two cases that we shall consider:

- First, suppose that π is a portfolio with nonnegative weights. This implies that $\pi_i(t) \geq 0$, for all $i = 1, \ldots, n$. Recall, from the inequality (2.3.125), that the individual stocks' variances τ_{ii}^π relative to the portfolio π are nonnegative, i.e., $\tau_{ii}^\pi(t) \geq 0$ for $t \in [0, \infty)$, a.s. Thus, all of the terms on the right-hand side of the expression (2.4.32) are a.s. nonnegative. Therefore, the excess growth rate is nonnegative

$$\gamma_\pi^*(t) \quad \geq \quad 0, \qquad t \in [0, \infty), \qquad \text{a.s.}$$

- The condition (ii), stated in the second part of the proposition, that there exists $i, j = 1, \ldots, n$, $i \neq j$ such that $0 < \pi_i(t), \pi_j(t) < 1$, for all $t \in [0, \infty)$, implies that at least two of the weights $\pi_1, \pi_2, \ldots, \pi_n$ are positive. Lemma 2.4.2 implies that $\boldsymbol{\tau}^{\boldsymbol{\pi}}(t)$ is positive semidefinite with rank $n - 1$, for $t \in [0, \infty)$, a.s. Using (2.3.125), for $i = 1, \ldots, n$ and $t \in [0, \infty)$, we have

$$\tau_{ii}^\pi(t) \quad = \quad \mathbf{e}_i \boldsymbol{\tau}^{\boldsymbol{\pi}}(t) \mathbf{e}_i^\mathsf{T} \quad \geq \quad 0. \tag{2.4.39}$$

  Since $\boldsymbol{\tau}^{\boldsymbol{\pi}}(t)$ has rank $n - 1$, equality can hold in (2.4.39) for at most one value of i. Moreover, since at least two of the weights $\pi_1, \pi_2, \ldots, \pi_n$ are positive for any $t \in [0, \infty)$, the right-hand side of the expression (2.4.32) is a.s. positive. Therefore, the excess growth rate will be positive if the portfolio holds at least two or more stocks with no short sales, and we have

$$\gamma_\pi^*(t) \quad > \quad 0, \qquad t \in [0, \infty), \qquad \text{a.s.}$$

■

Therefore, the above proposition declares that for a strictly long-only portfolio this excess growth rate is always nonnegative and is strictly positive for such portfolios that do not concentrate their holdings in just one stock. Note that when the portfolio comprises a single stock, it emerges that the excess growth rate of the portfolio satisfies $\gamma_\pi^*(t) = 0$. Thus, the excess growth rate occurs only when the portfolio is diversified into more than one stock. The aforementioned proposition suggests that for multistock portfolios with no short sales, the excess growth rate is nonnegative, and consequently, the portfolio growth rate always exceeds the weighted average of the growth rates of the individual component stocks. This follows from equation (2.2.111), in which the portfolio growth rate is expressed as the weighted average of the growth rates of the individual component stocks plus an additional element, the excess growth rate. Hence, for $t \in [0, \infty)$, a.s., we observe

$$\gamma_\pi^*(t) \quad = \quad \gamma_\pi(t) - \sum_{i=1}^n \pi_i(t) \gamma_i(t) \quad \geq \quad 0,$$

[14] Clearly, if the portfolio π exhibits all nonnegative weights, then there are no short sales allowed. This implies that $0 \leq \pi_i(t) < 1$, since $\sum_{i=1}^n \pi_i(t) = 1$. Thus, as a consequence of this condition, all the weights must be restricted to the interval $[0, 1)$.

which implies a.s., for $t \in [0, \infty)$, that

$$\gamma_\pi(t) \quad \geq \quad \sum_{i=1}^{n} \pi_i(t)\gamma_i(t).$$

Consider a market where all the stock growth rates are equal, i.e., $\gamma_i(t) \equiv \gamma(t)$ for all $i = 1, \ldots, n$ and for all $t \in [0, \infty)$. In accordance with the expressions above, for $t \in [0, \infty)$, a.s., we obtain

$$\begin{aligned} \gamma_\pi(t) &\geq \sum_{i=1}^{n} \pi_i(t)\gamma(t) \\ &= \gamma(t) \sum_{i=1}^{n} \pi_i(t) \\ &= \gamma(t). \end{aligned}$$

In such a market, all multistock portfolios without short sales have a higher growth rate than the common growth rate of the component stocks, i.e., $\gamma_\pi(t) \geq \gamma(t)$. This nature will be of particular importance later when we come to examine the concept of diversity in equity markets, this will be divulged in Chapter 4.

In order for us to proceed at certain junctures in the following chapters, it is imperative that we develop lemmas that will assist us in this regard. To progress we need to prove several lemmas that relate the size of the weights of the stocks in a portfolio to the excess growth rate of the portfolio. The first lemma establishes a lower bound on the relative variances of the stocks versus the portfolio π.

Lemma 2.4.9 ([**Fernholz (2002)**]). *Let π be a portfolio with nonnegative weights (i.e., a long-only portfolio) in a nondegenerate market (i.e., assume that the covariance process $\boldsymbol{\sigma}(t)$ satisfies the strong nondegeneracy condition in* (2.2.55)*). Then, for every long-only portfolio π, there exists an $\varepsilon > 0$ such that for $i = 1, \ldots, n$, we have the following inequality*

$$\tau_{ii}^{\pi}(t) \quad \geq \quad \varepsilon \left(1 - \pi_i(t)\right)^2, \qquad t \in [0, \infty), \qquad a.s. \tag{2.4.40}$$

Proof. Let $\mathbf{e}_i$ be the ith unit vector in $\mathbb{R}^n$. For $1 \leq i \leq n$ and $t \in [0, \infty)$, consider the vector $\mathbf{x}(t) \triangleq \boldsymbol{\pi}(t) - \mathbf{e}_i$, where $\mathbf{x}(t) \in \mathbb{R}^n$. Also, let e_{ik} denote the kth element in the ith unit vector $\mathbf{e}_i$, and let e_{jk} denote the kth element in the jth unit vector $\mathbf{e}_j$. The kth element in the vector $\mathbf{x}(t)$ is then given by $x_k(t) = \pi_k(t) - e_{ik}$. Hence, for $i, k = 1, 2, \ldots, n$, we have

$$e_{ik} \quad = \quad \begin{cases} 1 & \text{if} \quad k = i, \\ 0 & \text{if} \quad k \neq i, \end{cases} \tag{2.4.41}$$

which resembles the well-known *Kronecker delta function*. Hence, for $i = 1, 2, \ldots, n$, consider the vector

$$\begin{aligned} \mathbf{x}(t) &= \Big(\pi_1(t) - e_{i1}, \ldots, \pi_{i-1}(t) - e_{i(i-1)}, \pi_i(t) - e_{ii}, \pi_{i+1}(t) - e_{i(i+1)}, \ldots, \pi_n(t) - e_{in}\Big) \\ &= \Big(\pi_1(t) - 0, \ldots, \pi_{i-1}(t) - 0, \pi_i(t) - 1, \pi_{i+1}(t) - 0, \ldots, \pi_n(t) - 0\Big) \\ &= \Big(\pi_1(t), \ldots, \pi_{i-1}(t), \pi_i(t) - 1, \pi_{i+1}(t), \ldots, \pi_n(t)\Big). \end{aligned}$$

By setting $\eta := \pi$ in equation (2.3.85), for $t \in [0, \infty)$, we a.s. have

$$\tau_{ii}^{\pi}(t) \quad = \quad \sigma_{ii}(t) - 2\,\sigma_{i\pi}(t) + \sigma_{\pi\pi}(t).$$

From equation (2.4.41), we notice the following

$$\sigma_{jk}(t)e_{ik} \quad = \quad \begin{cases} \sigma_{ji}(t) & \text{if} \quad k = i, \\ 0 & \text{if} \quad k \neq i, \end{cases} \tag{2.4.42}$$

$$\sigma_{jk}(t)e_{ij} \quad = \quad \begin{cases} \sigma_{ik}(t) & \text{if} \quad j = i, \\ 0 & \text{if} \quad j \neq i. \end{cases} \tag{2.4.43}$$

The foregoing results imply the well-known *sifting property*, which is a familiar property of the *Kronecker delta function*, and is given in the following equations by

$$\sigma_{ji}(t) = \sum_{k=1}^{n} \sigma_{jk}(t) e_{ik}, \quad \text{and,} \tag{2.4.44}$$

$$\sigma_{ik}(t) = \sum_{j=1}^{n} \sigma_{jk}(t) e_{ij}. \tag{2.4.45}$$

By setting $\eta := \pi$ in the expressions (2.3.51) and (2.3.55), employing equations (2.4.44) and (2.4.45), and using the fact that $\boldsymbol{\sigma}(t)$ is a symmetric matrix, we arrive at the following

$$\begin{aligned}
\tau_{ii}^{\pi}(t) &= \sigma_{ii}(t) - 2\,\sigma_{i\pi}(t) + \sigma_{\pi\pi}(t) \\
&= \sigma_{ii}(t) - \sum_{j=1}^{n} \pi_j(t)\sigma_{ij}(t) - \sum_{k=1}^{n} \pi_k(t)\sigma_{ik}(t) + \sum_{j,k=1}^{n} \pi_j(t)\sigma_{jk}(t)\pi_k(t) \\
&= \sigma_{ii}(t) - \sum_{j=1}^{n} \pi_j(t)\sigma_{ji}(t) - \sum_{k=1}^{n} \pi_k(t)\sigma_{ik}(t) + \sum_{j,k=1}^{n} \pi_j(t)\sigma_{jk}(t)\pi_k(t) \\
&= \sum_{j=1}^{n} e_{ij}\Big[\sum_{k=1}^{n} \sigma_{jk}(t)e_{ik}\Big] - \sum_{j=1}^{n} \pi_j(t)\Big[\sum_{k=1}^{n} \sigma_{jk}(t)e_{ik}\Big] - \sum_{k=1}^{n} \pi_k(t)\Big[\sum_{j=1}^{n} \sigma_{jk}(t)e_{ij}\Big] + \sum_{j,k=1}^{n} \pi_j(t)\sigma_{jk}(t)\pi_k(t) \\
&= \sum_{j,k=1}^{n} e_{ij}\sigma_{jk}(t)e_{ik} - \sum_{j,k=1}^{n} \pi_j(t)\sigma_{jk}(t)e_{ik} - \sum_{j,k=1}^{n} \pi_k(t)\sigma_{jk}(t)e_{ij} + \sum_{j,k=1}^{n} \pi_j(t)\sigma_{jk}(t)\pi_k(t) \qquad (2.4.46) \\
&= \sum_{j,k=1}^{n} \sigma_{jk}(t)\Big[e_{ij}e_{ik} - \pi_j(t)e_{ik} - \pi_k(t)e_{ij} + \pi_j(t)\pi_k(t)\Big] \\
&= \sum_{j,k=1}^{n} \Big(e_{ij} - \pi_j(t)\Big)\sigma_{jk}(t)\Big(e_{ik} - \pi_k(t)\Big) = \Big(\mathbf{e}_i - \boldsymbol{\pi}(t)\Big)\boldsymbol{\sigma}(t)\Big(\mathbf{e}_i - \boldsymbol{\pi}(t)\Big)^{\mathsf{T}} \\
&= \sum_{j,k=1}^{n} \Big(\pi_j(t) - e_{ij}\Big)\sigma_{jk}(t)\Big(\pi_k(t) - e_{ik}\Big) = \Big(\boldsymbol{\pi}(t) - \mathbf{e}_i\Big)\boldsymbol{\sigma}(t)\Big(\boldsymbol{\pi}(t) - \mathbf{e}_i\Big)^{\mathsf{T}} \qquad (2.4.47) \\
&= \sum_{j,k=1}^{n} x_j(t)\sigma_{jk}(t)x_k(t) = \mathbf{x}(t)\boldsymbol{\sigma}(t)\mathbf{x}^{\mathsf{T}}(t). \qquad (2.4.48)
\end{aligned}$$

Notice that equation (2.4.46) is equivalent to $\mathbf{e}_i\boldsymbol{\sigma}(t)\mathbf{e}_i^{\mathsf{T}} - \boldsymbol{\pi}(t)\boldsymbol{\sigma}(t)\mathbf{e}_i^{\mathsf{T}} - \mathbf{e}_i\boldsymbol{\sigma}(t)\boldsymbol{\pi}^{\mathsf{T}}(t) + \boldsymbol{\pi}(t)\boldsymbol{\sigma}(t)\boldsymbol{\pi}^{\mathsf{T}}(t)$, which is analogous to the result obtained in equation (2.3.100). An alternative approach to the derivation above, is to set $\mathbf{x} := \mathbf{e}_i$ in the proof of Lemma 2.4.2. Then $\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} e_{ij} = 1$. Implementing this in equation (2.4.8) gives

$$\begin{aligned}
\tau_{ii}^{\pi}(t) = \mathbf{e}_i\boldsymbol{\tau}^{\boldsymbol{\pi}}(t)\mathbf{e}_i^{\mathsf{T}} &= \mathbf{e}_i\boldsymbol{\sigma}(t)\mathbf{e}_i^{\mathsf{T}} - 2\,\mathbf{e}_i\boldsymbol{\sigma}(t)\boldsymbol{\pi}^{\mathsf{T}}(t) + \boldsymbol{\pi}(t)\boldsymbol{\sigma}(t)\boldsymbol{\pi}^{\mathsf{T}}(t) \\
&= \Big(\boldsymbol{\pi}(t) - \mathbf{e}_i\Big)\boldsymbol{\sigma}(t)\Big(\boldsymbol{\pi}(t) - \mathbf{e}_i\Big)^{\mathsf{T}} \\
&= \mathbf{x}(t)\boldsymbol{\sigma}(t)\mathbf{x}^{\mathsf{T}}(t),
\end{aligned}$$

which is what we obtained above. Recall the strong nondegeneracy condition in Definition 2.2.12. Let $\varepsilon > 0$ be chosen as in (2.2.55), so that

$$\mathbf{x}\boldsymbol{\sigma}(t)\mathbf{x}^{\mathsf{T}} \geq \varepsilon\,\|\mathbf{x}\|^2, \qquad \mathbf{x} \in \mathbb{R}^n, \qquad t \in [0,\infty), \qquad \text{a.s.} \tag{2.4.49}$$

Thus, from equations (2.4.47) and (2.4.48) in conjunction with the strong nondegeneracy condition (2.4.49) and with $\mathbf{x}(t) = \boldsymbol{\pi}(t) - \mathbf{e}_i$, we have a.s., for $t \in [0,\infty)$,

$$\tau_{ii}^{\pi}(t) = \Big(\boldsymbol{\pi}(t) - \mathbf{e}_i\Big)\boldsymbol{\sigma}(t)\Big(\boldsymbol{\pi}(t) - \mathbf{e}_i\Big)^{\mathsf{T}} = \mathbf{x}(t)\boldsymbol{\sigma}(t)\mathbf{x}^{\mathsf{T}}(t) \geq \varepsilon\,\big\|\mathbf{x}(t)\big\|^2.$$

Since,

$$\begin{aligned}\left\|\mathbf{x}(t)\right\|^2 &= \left\|\boldsymbol{\pi}(t)-\mathbf{e}_i\right\|^2 = \pi_1^2(t)+\cdots+\pi_{i-1}^2(t)+\big(\pi_i(t)-1\big)^2+\pi_{i+1}^2(t)+\cdots+\pi_n^2(t)\\ &= \big(\pi_i(t)-1\big)^2+\sum_{j\neq i}^{n}\pi_j^2(t)\\ &= \big(1-\pi_i(t)\big)^2+\sum_{j\neq i}^{n}\pi_j^2(t),\end{aligned}$$

it follows from the fact that $\pi_i^2(t)\geq 0$ for all $i=1,\ldots,n$, that

$$\left\|\mathbf{x}(t)\right\|^2 = \left\|\boldsymbol{\pi}(t)-\mathbf{e}_i\right\|^2 \geq \big(1-\pi_i(t)\big)^2, \qquad t\in[0,\infty), \qquad \text{a.s.},$$

and the lemma follows,

$$\tau_{ii}^{\pi}(t) \geq \varepsilon\left\|\mathbf{x}(t)\right\|^2 = \varepsilon\left\|\boldsymbol{\pi}(t)-\mathbf{e}_i\right\|^2 \geq \varepsilon\big(1-\pi_i(t)\big)^2. \tag{2.4.50}$$

■

In order to set the stage for notions and developments that follow, let us introduce the reverse-order-statistics notation for the weight processes.

Definition 2.4.10 (Reverse-order Statistics)**.** *For a portfolio π, it is convenient to introduce the following notation*

$$\pi_{\max}(t) \triangleq \max_{1\leq i\leq n}\pi_i(t), \qquad t\in[0,\infty), \tag{2.4.51}$$

and

$$\pi_{\min}(t) \triangleq \min_{1\leq i\leq n}\pi_i(t), \qquad t\in[0,\infty). \tag{2.4.52}$$

Also, we shall use the ***reverse-order-statistics notation*** *for the weights of a portfolio $\pi(t)$, ranked at time $t\in[0,\infty)$ in decreasing order, from the largest portfolio weight $\pi_{(1)}(t)$ to the smallest portfolio weight $\pi_{(n)}(t)$,*

$$\max_{1\leq i\leq n}\pi_i(t) \triangleq \pi_{(1)}(t) \geq \pi_{(2)}(t) \geq \cdots \geq \pi_{(n-1)}(t) \geq \pi_{(n)}(t) \triangleq \min_{1\leq i\leq n}\pi_i(t). \tag{2.4.53}$$

These portfolio weights, defined in this fashion, are referred to as the ***ranked portfolio weight processes****, given by*

$$\pi_{(\cdot)} = \Big\{\boldsymbol{\pi}_{(\cdot)}(t)=\big(\pi_{(1)}(t),\ldots,\pi_{(n)}(t)\big), t\in[0,\infty)\Big\}.$$

With this notation, we can restate Lemma 2.4.9 in its most useful form.

Lemma 2.4.11 ([**Fernholz (2002)**])**.** *Let π be a portfolio in a nondegenerate market. Then, for every portfolio π, there exists an $\varepsilon>0$ such that for $i=1,\ldots,n$, we have, in the notation of* (2.4.51) *of Definition* 2.4.10*, the following inequality*

$$\tau_{ii}^{\pi}(t) \geq \varepsilon\big(1-\pi_{\max}(t)\big)^2, \qquad t\in[0,\infty), \qquad a.s. \tag{2.4.54}$$

Proof. This follows immediately from Lemma 2.4.9, since $\pi_{\max}(t)\geq\pi_i(t)$ for all $i=1,\ldots,n$, so that

$$\begin{aligned}1-\pi_i(t) &\geq 1-\pi_{\max}(t), \quad \text{and,}\\ \big(1-\pi_i(t)\big)^2 &\geq \big(1-\pi_{\max}(t)\big)^2.\end{aligned}$$

Consequently, for all $i=1,2,\ldots,n$, we have

$$\tau_{ii}^{\pi}(t) \geq \varepsilon\big(1-\pi_i(t)\big)^2 \geq \varepsilon\big(1-\pi_{\max}(t)\big)^2.$$

■

The next lemma strengthens Proposition 2.4.8 for nondegenerate markets.

Lemma 2.4.12 ([**Fernholz (2002)**]). *Let π be a portfolio with nonnegative weights (i.e., a long-only portfolio) in a nondegenerate market. Then, for every long-only portfolio π, there exists an $\varepsilon > 0$ such that we also have, in the notation of* (2.4.51) *of Definition* 2.4.10*, the following inequality*

$$\gamma_\pi^*(t) \quad \geq \quad \frac{\varepsilon}{2}\big(1-\pi_{\max}(t)\big)^2, \qquad t \in [0,\infty), \qquad a.s. \tag{2.4.55}$$

Proof. By Lemma 2.4.7, for $t \in [0,\infty)$, a.s., we have

$$\gamma_\pi^*(t) \quad = \quad \frac{1}{2}\sum_{i=1}^n \pi_i(t)\tau_{ii}^\pi(t),$$

which, together with the strong nondegeneracy condition and the previous lemma, gives

$$\gamma_\pi^*(t) \quad \geq \quad \frac{1}{2}\sum_{i=1}^n \pi_i(t)\Big[\varepsilon\big(1-\pi_{\max}(t)\big)^2\Big] \quad = \quad \frac{\varepsilon}{2}\big(1-\pi_{\max}(t)\big)^2\sum_{i=1}^n \pi_i(t) \quad = \quad \frac{\varepsilon}{2}\big(1-\pi_{\max}(t)\big)^2,$$

where ε is chosen as in Lemma 2.4.11, since the π_i are nonnegative. ■

Remark 2.4.13. Fernholz & Karatzas (2009) offer an alternative expression for the lower bound of the excess growth rate, which we shall disclose in the next lemma. This lemma is analogous to Lemma 2.4.12, however, we shall find it convenient to cast the inequality (2.4.55) in Lemma 2.4.12 in an equivalent form, given in (2.4.56) in the lemma below. The archetypal formulation for the lower bound of the excess growth rate, after which the preceding and following lemmas are fashioned, is in fact attributable to Fernholz (1999a), a detailed account of this can be found in Appendix C.

Lemma 2.4.14 ([**Fernholz & Karatzas (2009)**]). *Let π be a portfolio with nonnegative weights (i.e., a long-only portfolio) in a nondegenerate market. Then, for every long-only portfolio π, there exists an $\varepsilon > 0$ such that we also have, in the notation of* (2.4.51) *of Definition* 2.4.10*, the following inequality*

$$\gamma_\pi^*(t) \quad \geq \quad \frac{\varepsilon}{2}\big(1-\pi_{\max}(t)\big), \qquad t \in [0,\infty), \qquad a.s. \tag{2.4.56}$$

Proof. The result follows immediately from Lemma 2.4.12, since $\big(1-\pi_{\max}(t)\big) \geq \big(1-\pi_{\max}(t)\big)^2$, which is a consequence of the fact that the weights are assumed to be nonnegative, i.e., $0 \leq \pi_i(t) \leq 1$ for all $i = 1,\ldots,n$. The proof offered by Fernholz & Karatzas (2009) is also presented here. Recall, from Lemma 2.4.9, that

$$\begin{aligned}
\tau_{ii}^\pi(t) \quad \geq \quad \varepsilon\left\|\boldsymbol{\pi}(t)-\mathbf{e}_i\right\|^2 \quad &= \quad \varepsilon\Big(\pi_1^2(t)+\cdots+\pi_{i-1}^2(t)+\big(\pi_i(t)-1\big)^2+\pi_{i+1}^2(t)+\cdots+\pi_n^2(t)\Big) \\
&= \quad \varepsilon\left(\big(\pi_i(t)-1\big)^2+\sum_{j\neq i}^n \pi_j^2(t)\right) \\
&= \quad \varepsilon\left(\big(1-\pi_i(t)\big)^2+\sum_{j\neq i}^n \pi_j^2(t)\right).
\end{aligned}$$

Now, refer back to equation (2.4.32). Substituting the inequality above into equation (2.4.32), along with the assumption that the weights are all nonnegative (i.e., with $\pi_i(t) \geq 0$ valid for all $i = 1,2,\ldots,n$), this lower

estimate gives

$$\begin{aligned}
\gamma_\pi^*(t) &\geq \frac{\varepsilon}{2}\sum_{i=1}^n \pi_i(t)\Big[\big(1-\pi_i(t)\big)^2 + \sum_{j\neq i}^n \pi_j^2(t)\Big] \\
&= \frac{\varepsilon}{2}\left(\sum_{i=1}^n \pi_i(t)\big(1-\pi_i(t)\big)^2 + \sum_{i=1}^n \pi_i(t)\sum_{j\neq i}^n \pi_j^2(t)\right) \\
&= \frac{\varepsilon}{2}\left(\sum_{i=1}^n \pi_i(t)\big(1-\pi_i(t)\big)^2 + \sum_{i=1}^n \pi_i(t)\Big[\sum_{j=1}^n \pi_j^2(t) - \pi_i^2(t)\Big]\right) \\
&= \frac{\varepsilon}{2}\left(\sum_{i=1}^n \pi_i(t)\big(1-\pi_i(t)\big)^2 + \sum_{i=1}^n \pi_i(t)\sum_{j=1}^n \pi_j^2(t) - \sum_{i=1}^n \pi_i^3(t)\right) \\
&= \frac{\varepsilon}{2}\left(\sum_{i=1}^n \pi_i(t)\big(1-\pi_i(t)\big)^2 + \sum_{i=1}^n \pi_i^2(t) - \sum_{i=1}^n \pi_i^3(t)\right) \\
&= \frac{\varepsilon}{2}\left(\sum_{i=1}^n \pi_i(t)\big(1-\pi_i(t)\big)^2 + \sum_{i=1}^n \pi_i^2(t)\big(1-\pi_i(t)\big)\right) \\
&= \frac{\varepsilon}{2}\sum_{i=1}^n \Big[\pi_i(t)\big(1-\pi_i(t)\big)^2 + \pi_i^2(t)\big(1-\pi_i(t)\big)\Big] \\
&= \frac{\varepsilon}{2}\sum_{i=1}^n \pi_i(t)\big(1-\pi_i(t)\big)\Big[\big(1-\pi_i(t)\big) + \pi_i(t)\Big] \\
&= \frac{\varepsilon}{2}\sum_{i=1}^n \pi_i(t)\big(1-\pi_i(t)\big) \\
&\geq \frac{\varepsilon}{2}\sum_{i=1}^n \pi_i(t)\big(1-\pi_{\max}(t)\big) \\
&= \frac{\varepsilon}{2}\big(1-\pi_{\max}(t)\big),
\end{aligned}$$

since $\sum_{i=1}^n \pi_i(t) = 1$ and $1-\pi_i(t) \geq 1-\pi_{\max}(t)$ or $1-\pi_{\max}(t) \leq 1-\pi_i(t)$ from the fact that $\pi_i(t) \leq \pi_{\max}(t)$ for all $i = 1, 2, \ldots, n$, and we have the desired result. ■

The previous lemmas propose lower bound conditions on both the relative variances of the stocks in the portfolio π and the excess growth rate of the portfolio π, in a *nondegenerate market*. These lemmas prove to be especially useful in later examples of portfolio dominance when $\gamma_\pi^*(t)$ is required to be bounded below by a positive constant. In particular, both Lemma 2.4.12 and Lemma 2.4.14 demonstrate that, in such a market, if $\pi_{\max}$ is bounded away from 1 then the excess growth rate γ_π^* is bounded away from 0. In the following lemma, we deduce an upper bound estimate for the excess growth rate of the portfolio π. The lemma shows that, in a market with *bounded variance*, if the excess growth rate is bounded away from 0 then $\pi_{\max}$ is bounded away from 1. This establishes the converse to the preceding lemmas.

Lemma 2.4.15 ([**Fernholz (2002)**]). *Let π be a portfolio with nonnegative weights (i.e., a long-only portfolio) in a market with bounded variance (i.e., assume that the market, via the covariance process $\boldsymbol{\sigma}(t)$, satisfies the uniform boundedness condition in* (2.2.57)*). Then, for every long-only portfolio π, there exists a number $\varepsilon > 0$ such that we also have, in the notation of* (2.4.51) *of Definition* 2.4.10*, the following inequality*

$$\pi_{\max}(t) \leq 1 - \varepsilon\gamma_\pi^*(t), \qquad t \in [0,\infty), \qquad a.s. \tag{2.4.57}$$

Proof. Firstly, since we assume that the weights of the portfolio π are nonnegative (i.e., we exclusively consider long-only portfolios), for all $i = 1, \ldots, n$ and $t \in [0,\infty)$, we have $0 \leq \pi_i(t) < 1$. Furthermore, the market is assumed to exhibit bounded variance and consequently satisfies the expression (2.2.57) provided in Definition 2.2.14. Thus, we can choose $K > 0$ such that

$$\mathbf{x}\boldsymbol{\sigma}(t)\mathbf{x}^\mathsf{T} \leq K\,\|\mathbf{x}\|^2, \qquad \mathbf{x} \in \mathbb{R}^n, \qquad t \in [0,\infty), \qquad \text{a.s.} \tag{2.4.58}$$

Let $\mathbf{e}_i$ be the ith unit vector in $\mathbb{R}^n$. Then, setting $\mathbf{x} := \mathbf{e}_i$ in the inequality (2.4.58), gives, for all $i = 1, \ldots, n$,

$$\mathbf{e}_i \boldsymbol{\sigma}(t) \mathbf{e}_i^{\mathsf{T}} \quad \leq \quad K \left\| \mathbf{e}_i \right\|^2, \qquad \mathbf{e}_i \in \mathbb{R}^n, \qquad t \in [0, \infty), \qquad \text{a.s.}$$

Hence, from equation (2.2.48), for all $i = 1, \ldots, n$ and $t \in [0, \infty)$, we have

$$\sigma_{ii}(t) \quad \leq \quad K \left\| \mathbf{e}_i \right\|^2.$$

By the definition of the Euclidean norm and that of the unit vector $\mathbf{e}_i$, it is easily verified that

$$\sigma_{ii}(t) \quad \leq \quad K, \qquad t \in [0, \infty), \qquad \text{a.s.}, \tag{2.4.59}$$

since $\left\| \mathbf{e}_i \right\|^2 = 1$. For any integer k, $k = 1, \ldots, n$, the long-only condition on the portfolio suggests that $\pi_k(t) < 1$ (more specifically, $0 \leq \pi_k(t) < 1$). Consequently, let us define

$$\eta_i(t) \quad \triangleq \quad \begin{cases} \dfrac{\pi_i(t)}{\big(1 - \pi_k(t)\big)} & \text{if} \quad i \neq k, \\ 0 & \text{if} \quad i = k, \end{cases} \tag{2.4.60}$$

for $i = 1, \ldots, n$ and for $t \in [0, \infty)$. Since, by assumption we have $\pi_k(t) < 1$, this implies that $1 - \pi_k(t) > 0$. This result together with the fact that $\pi_i(t) \geq 0$, suggests that $\eta = \big\{ \boldsymbol{\eta}(t) = \big(\eta_1(t), \ldots, \eta_n(t)\big), t \in [0, \infty) \big\}$ defines a portfolio with nonnegative weights (i.e., $0 \leq \eta_i(t) < 1$) and with $\eta_k(t) = 0$, for all $t \in [0, \infty)$. We can show that η does indeed satisfy the properties of a long-only portfolio as outlined in Definition 2.2.17 as follows:

- Since $\eta_i(t) \geq 0$ for $i = 1, 2, \ldots, n$, the weights $\eta_1(t), \ldots, \eta_n(t)$ are bounded on $[0, \infty)$. In addition, from the properties of the portfolio π, $\sum_{i=1}^n \pi_i(t) = 1$ and $0 \leq \pi_i(t) < 1$, for $t \in [0, \infty)$ and for $i = 1, 2, \ldots, n$, we have $\pi_i(t) + \pi_k(t) \leq 1$, which implies $\pi_i(t) \leq 1 - \pi_k(t)$ for $i = 1, 2, \ldots, n$. Consequently, for $i = 1, 2, \ldots, n$, we obtain

$$\eta_i(t) \quad = \quad \frac{\pi_i(t)}{1 - \pi_k(t)} \quad \leq \quad 1.$$

- Using equation (2.4.60), we obtain

$$\sum_{i=1}^n \eta_i(t) \quad = \quad \sum_{i \neq k}^n \frac{\pi_i(t)}{\big(1 - \pi_k(t)\big)} + \eta_k(t) \quad = \quad \frac{1}{\big(1 - \pi_k(t)\big)} \sum_{i \neq k}^n \pi_i(t) \quad = \quad \frac{1}{\big(1 - \pi_k(t)\big)} \big(1 - \pi_k(t)\big) \quad = \quad 1,$$

  since $\eta_k(t) = 0$. Therefore, the portfolio η satisfies $\eta_1(t) + \cdots + \eta_n(t) = 1$, for all $t \in [0, \infty)$.

Recall that $\boldsymbol{\sigma}(t)$ is positive definite for $t \in [0, \infty)$, which, by equation (2.2.94) of Definition 2.2.22, implies that $\sigma_{\eta\eta}(t) = \boldsymbol{\eta}(t) \boldsymbol{\sigma}(t) \boldsymbol{\eta}^{\mathsf{T}}(t) \geq 0$. This fact, together with equation (2.2.109) and the inequality (2.4.59), yields a.s., for $t \in [0, \infty)$,

$$2\, \gamma_\eta^*(t) \quad = \quad \sum_{i=1}^n \eta_i(t) \sigma_{ii}(t) - \sigma_{\eta\eta}(t) \quad \leq \quad \sum_{i=1}^n \eta_i(t) \sigma_{ii}(t) \quad \leq \quad \sum_{i=1}^n \eta_i(t) K \quad = \quad K \sum_{i=1}^n \eta_i(t) \quad = \quad K. \tag{2.4.61}$$

Let $\mathbf{e}_k$ be the kth unit vector in $\mathbb{R}^n$. For $1 \leq k \leq n$ and $t \in [0, \infty)$, consider the vector $\mathbf{x}(t) \triangleq \boldsymbol{\eta}(t) - \mathbf{e}_k$, where $\mathbf{x}(t) \in \mathbb{R}^n$. Also, let e_{ki} denote the ith element in the kth unit vector $\mathbf{e}_k$. The ith element in the vector $\mathbf{x}(t)$ is then given by $x_i(t) = \eta_i(t) - e_{ki}$. Hence, for $i, k = 1, 2, \ldots, n$, we have

$$e_{ki} \quad = \quad \begin{cases} 1 & \text{if} \quad i = k, \\ 0 & \text{if} \quad i \neq k. \end{cases} \tag{2.4.62}$$

Hence, for $k = 1, 2, \ldots, n$, consider the vector

$$\begin{aligned} \mathbf{x}(t) \quad &= \quad \Big(\eta_1(t) - e_{k1}, \ldots, \eta_{k-1}(t) - e_{k(k-1)}, \eta_k(t) - e_{kk}, \eta_{k+1}(t) - e_{k(k+1)}, \ldots, \eta_n(t) - e_{kn} \Big) \\ &= \quad \Big(\eta_1(t) - 0, \ldots, \eta_{k-1}(t) - 0, \eta_k(t) - 1, \eta_{k+1}(t) - 0, \ldots, \eta_n(t) - 0 \Big) \\ &= \quad \Big(\eta_1(t), \ldots, \eta_{k-1}(t), \eta_k(t) - 1, \eta_{k+1}(t), \ldots, \eta_n(t) \Big) \\ &= \quad \Big(\eta_1(t), \ldots, \eta_{k-1}(t), -1, \eta_{k+1}(t), \ldots, \eta_n(t) \Big). \end{aligned}$$

For $t \in [0, \infty)$, by equation (2.3.128), we have a.s.,

$$\tau_{kk}^{\eta}(t) \quad = \quad \sigma_{kk}(t) - 2\,\sigma_{k\eta}(t) + \sigma_{\eta\eta}(t).$$

From equation (2.4.62), we notice the following

$$\sigma_{ij}(t)e_{kj} \quad = \quad \begin{cases} \sigma_{ik}(t) & \text{if} \quad j = k, \\ 0 & \text{if} \quad j \neq k, \end{cases} \tag{2.4.63}$$

$$\sigma_{ij}(t)e_{ki} \quad = \quad \begin{cases} \sigma_{kj}(t) & \text{if} \quad i = k, \\ 0 & \text{if} \quad i \neq k. \end{cases} \tag{2.4.64}$$

The above expressions imply

$$\sigma_{ik}(t) \quad = \quad \sum_{j=1}^{n} \sigma_{ij}(t)e_{kj}, \quad \text{and,} \tag{2.4.65}$$

$$\sigma_{kj}(t) \quad = \quad \sum_{i=1}^{n} \sigma_{ij}(t)e_{ki}. \tag{2.4.66}$$

From the expressions (2.3.51) and (2.3.55), in conjunction with equations (2.4.65) and (2.4.66), and using the fact that $\boldsymbol{\sigma}(t)$ is a symmetric matrix, we can apply the same procedure as in the proof of Lemma 2.4.9, to obtain the result

$$\begin{aligned}
\tau_{kk}^{\eta}(t) &= \sigma_{kk}(t) - 2\,\sigma_{k\eta}(t) + \sigma_{\eta\eta}(t) \\
&= \sigma_{kk}(t) - \sum_{i=1}^{n} \eta_i(t)\sigma_{ki}(t) - \sum_{j=1}^{n} \eta_j(t)\sigma_{kj}(t) + \sum_{i,j=1}^{n} \eta_i(t)\sigma_{ij}(t)\eta_j(t) \\
&= \sigma_{kk}(t) - \sum_{i=1}^{n} \eta_i(t)\sigma_{ik}(t) - \sum_{j=1}^{n} \eta_j(t)\sigma_{kj}(t) + \sum_{i,j=1}^{n} \eta_i(t)\sigma_{ij}(t)\eta_j(t) \\
&= \sum_{i=1}^{n} e_{ki} \sum_{j=1}^{n} \sigma_{ij}(t)e_{kj} - \sum_{i=1}^{n} \eta_i(t)\Big[\sum_{j=1}^{n} \sigma_{ij}(t)e_{kj}\Big] - \sum_{j=1}^{n} \eta_j(t)\Big[\sum_{i=1}^{n} \sigma_{ij}(t)e_{ki}\Big] + \sum_{i,j=1}^{n} \eta_i(t)\sigma_{ij}(t)\eta_j(t) \\
&= \sum_{i,j=1}^{n} e_{ki}\sigma_{ij}(t)e_{kj} - \sum_{i,j=1}^{n} \eta_i(t)\sigma_{ij}(t)e_{kj} - \sum_{i,j=1}^{n} \eta_j(t)\sigma_{ij}(t)e_{ki} + \sum_{i,j=1}^{n} \eta_i(t)\sigma_{ij}(t)\eta_j(t) \\
&= \sum_{i,j=1}^{n} \sigma_{ij}(t)\Big[e_{ki}e_{kj} - \eta_i(t)e_{kj} - \eta_j(t)e_{ki} + \eta_i(t)\eta_j(t)\Big] \\
&= \sum_{i,j=1}^{n} \Big(e_{ki} - \eta_i(t)\Big)\sigma_{ij}(t)\Big(e_{kj} - \eta_j(t)\Big) \quad = \quad \Big(\mathbf{e}_k - \boldsymbol{\eta}(t)\Big)\boldsymbol{\sigma}(t)\Big(\mathbf{e}_k - \boldsymbol{\eta}(t)\Big)^{\mathsf{T}} \\
&= \sum_{i,j=1}^{n} \Big(\eta_i(t) - e_{ki}\Big)\sigma_{ij}(t)\Big(\eta_j(t) - e_{kj}\Big) \quad = \quad \Big(\boldsymbol{\eta}(t) - \mathbf{e}_k\Big)\boldsymbol{\sigma}(t)\Big(\boldsymbol{\eta}(t) - \mathbf{e}_k\Big)^{\mathsf{T}} \qquad (2.4.67) \\
&= \sum_{i,j=1}^{n} x_i(t)\sigma_{ij}(t)x_j(t) \quad = \quad \mathbf{x}(t)\boldsymbol{\sigma}(t)\mathbf{x}^{\mathsf{T}}(t). \qquad (2.4.68)
\end{aligned}$$

Thus, from equations (2.4.67) and (2.4.68) in conjunction with the uniform boundedness condition (2.4.58) and with $\mathbf{x}(t) = \boldsymbol{\eta}(t) - \mathbf{e}_k$, we have a.s., for $t \in [0, \infty)$,

$$\tau_{kk}^{\eta}(t) \quad = \quad \Big(\boldsymbol{\eta}(t) - \mathbf{e}_k\Big)\boldsymbol{\sigma}(t)\Big(\boldsymbol{\eta}(t) - \mathbf{e}_k\Big)^{\mathsf{T}} \quad = \quad \mathbf{x}(t)\boldsymbol{\sigma}(t)\mathbf{x}^{\mathsf{T}}(t) \quad \leq \quad K\,\big\|\mathbf{x}(t)\big\|^2.$$

Now from equation (2.4.60), for the Euclidean norm in the above inequality, we obtain[15]

$$\begin{aligned}
\big\|\mathbf{x}(t)\big\|^2 &= \big\|\boldsymbol{\eta}(t)-\mathbf{e}_k\big\|^2 = \eta_1^2(t)+\cdots+\eta_{k-1}^2(t)+\big(\eta_k(t)-1\big)^2+\eta_{k+1}^2(t)+\cdots+\eta_n^2(t)\\
&= \sum_{i\neq k}^{n}\eta_i^2(t)+\big(\eta_k(t)-1\big)^2\\
&= \sum_{i\neq k}^{n}\left(\frac{\pi_i(t)}{1-\pi_k(t)}\right)^2+1\\
&= \frac{1}{\big(1-\pi_k(t)\big)^2}\left(\sum_{i\neq k}^{n}\pi_i^2(t)\right)+1\\
&\leq \frac{1}{\big(1-\pi_k(t)\big)^2}\left(\sum_{i\neq k}^{n}\pi_i(t)\right)^2+1\\
&= \frac{1}{\big(1-\pi_k(t)\big)^2}\big(1-\pi_k(t)\big)^2+1\\
&= 2.
\end{aligned}$$

Hence, we have the inequality

$$\big\|\mathbf{x}(t)\big\|^2 = \big\|\boldsymbol{\eta}(t)-\mathbf{e}_k\big\|^2 \leq 2, \qquad t\in[0,\infty), \qquad \text{a.s.} \tag{2.4.69}$$

Thus, for $k=1,2,\ldots,n$, $t\in[0,\infty)$, a.s., we have

$$\tau_{kk}^{\eta}(t) \leq K\big\|\mathbf{x}(t)\big\|^2 = K\big\|\boldsymbol{\eta}(t)-\mathbf{e}_k\big\|^2 \leq 2K. \tag{2.4.70}$$

By equation (2.2.108), for $t\in[0,\infty)$, a.s., we obtain

$$\begin{aligned}
2\,\gamma_\pi^*(t) &= \sum_{i=1}^{n}\pi_i(t)\sigma_{ii}(t)-\sum_{i,j=1}^{n}\pi_i(t)\sigma_{ij}(t)\pi_j(t)\\
&= \pi_k(t)\sigma_{kk}(t)+\sum_{i\neq k}^{n}\pi_i(t)\sigma_{ii}(t)-\pi_k^2(t)\sigma_{kk}(t)\\
&\quad -\sum_{i\neq k}^{n}\pi_i(t)\sigma_{ik}(t)\pi_k(t)-\sum_{j\neq k}^{n}\pi_k(t)\sigma_{kj}(t)\pi_j(t)-\sum_{i,j\neq k}^{n}\pi_i(t)\sigma_{ij}(t)\pi_j(t)\\
&= \pi_k(t)\sigma_{kk}(t)+\sum_{i\neq k}^{n}\pi_i(t)\sigma_{ii}(t)-\pi_k^2(t)\sigma_{kk}(t)\\
&\quad -\pi_k(t)\left(\sum_{i\neq k}^{n}\sigma_{ik}(t)\pi_i(t)\right)-\pi_k(t)\left(\sum_{j\neq k}^{n}\sigma_{jk}(t)\pi_j(t)\right)-\sum_{i,j\neq k}^{n}\pi_i(t)\sigma_{ij}(t)\pi_j(t)\\
&= \pi_k(t)\sigma_{kk}(t)+\sum_{i\neq k}^{n}\pi_i(t)\sigma_{ii}(t)-\pi_k^2(t)\sigma_{kk}(t)\\
&\quad -2\,\pi_k(t)\left(\sum_{i\neq k}^{n}\sigma_{ik}(t)\pi_i(t)\right)-\sum_{i,j\neq k}^{n}\pi_i(t)\sigma_{ij}(t)\pi_j(t).
\end{aligned}$$

From equation (2.4.60), we have $\pi_i(t)=\eta_i(t)\big(1-\pi_k(t)\big)$ for all $i\neq k$, and for $i=k$, we have $\eta_k(t)=0$. By

[15] By the result: $\big(\sum_{i\neq k}\pi_i(t)\big)^2=\sum_{i\neq k}\pi_i^2(t)+\sum_{i,j\neq k}\pi_i(t)\pi_j(t)\geq\sum_{i\neq k}\pi_i^2(t)$, for $k=1,\ldots,n$, since the weights of the portfolio π are all nonnegative, $0\leq\pi_i(t)<1$.

employing this result together with equations (2.3.51) and (2.3.55), we can simplify the above as follows

$$
\begin{aligned}
2\gamma_\pi^*(t) &= \pi_k(t)\sigma_{kk}(t) + \sum_{i=1}^n \Big[\eta_i(t)\big(1-\pi_k(t)\big)\Big]\sigma_{ii}(t) - \pi_k^2(t)\sigma_{kk}(t) \\
&\quad - 2\,\pi_k(t)\Big(\sum_{i=1}^n \sigma_{ik}(t)\Big[\eta_i(t)\big(1-\pi_k(t)\big)\Big]\Big) - \sum_{i,j=1}^n \sigma_{ij}(t)\Big[\eta_i(t)\big(1-\pi_k(t)\big)\Big]\Big[\eta_j(t)\big(1-\pi_k(t)\big)\Big] \\
&= \pi_k(t)\sigma_{kk}(t) + \big(1-\pi_k(t)\big)\sum_{i=1}^n \eta_i(t)\sigma_{ii}(t) - \pi_k^2(t)\sigma_{kk}(t) \\
&\quad - 2\,\pi_k(t)\big(1-\pi_k(t)\big)\sum_{i=1}^n \eta_i(t)\sigma_{ik}(t) - \big(1-\pi_k(t)\big)^2 \sum_{i,j=1}^n \eta_i(t)\sigma_{ij}(t)\eta_j(t) \\
&= \big(\pi_k(t) - \pi_k^2(t)\big)\sigma_{kk}(t) + \big(1-\pi_k(t)\big)\sum_{i=1}^n \eta_i(t)\sigma_{ii}(t) \\
&\quad - 2\,\pi_k(t)\big(1-\pi_k(t)\big)\sigma_{k\eta}(t) - \big(1-\pi_k(t)\big)^2\sigma_{\eta\eta}(t) \\
&= \pi_k(t)\big(1-\pi_k(t)\big)\sigma_{kk}(t) + \big(1-\pi_k(t)\big)\sum_{i=1}^n \eta_i(t)\sigma_{ii}(t) \\
&\quad - 2\,\pi_k(t)\big(1-\pi_k(t)\big)\sigma_{k\eta}(t) - \Big(\big(1-\pi_k(t)\big) - \pi_k(t)\big(1-\pi_k(t)\big)\Big)\sigma_{\eta\eta}(t) \\
&= \pi_k(t)\big(1-\pi_k(t)\big)\sigma_{kk}(t) + \big(1-\pi_k(t)\big)\sum_{i=1}^n \eta_i(t)\sigma_{ii}(t) \\
&\quad - 2\,\pi_k(t)\big(1-\pi_k(t)\big)\sigma_{k\eta}(t) - \big(1-\pi_k(t)\big)\sigma_{\eta\eta}(t) + \pi_k(t)\big(1-\pi_k(t)\big)\sigma_{\eta\eta}(t) \\
&= \pi_k(t)\big(1-\pi_k(t)\big)\Big(\sigma_{kk}(t) - 2\,\sigma_{k\eta}(t) + \sigma_{\eta\eta}(t)\Big) + \big(1-\pi_k(t)\big)\left(\sum_{i=1}^n \eta_i(t)\sigma_{ii}(t) - \sigma_{\eta\eta}(t)\right). \qquad (2.4.71)
\end{aligned}
$$

Consequently, from equations (2.3.128) and (2.2.109), we get

$$
\begin{aligned}
2\,\gamma_\pi^*(t) &= \pi_k(t)\big(1-\pi_k(t)\big)\big(\tau_{kk}^\eta(t)\big) + \big(1-\pi_k(t)\big)\big(2\,\gamma_\eta^*(t)\big) \\
&\le \pi_k(t)\big(1-\pi_k(t)\big)2K + \big(1-\pi_k(t)\big)K \qquad (2.4.72) \\
&\le \big(1-\pi_k(t)\big)2K + \big(1-\pi_k(t)\big)K \qquad (2.4.73) \\
&= \big(1-\pi_k(t)\big)\big(2K+K\big) \qquad (2.4.74) \\
&= 3K\,\big(1-\pi_k(t)\big), \qquad (2.4.75)
\end{aligned}
$$

where the inequality (2.4.72) is implied by the inequalities (2.4.61) and (2.4.70), and the inequality (2.4.73) holds since $\pi_k(t) < 1$. Therefore, we have

$$
\gamma_\pi^*(t) \le \frac{3K}{2}\,\big(1-\pi_k(t)\big), \qquad (2.4.76)
$$

which holds for all $k = 1, 2, \ldots, n$. Since the preceding result also holds for the largest stock with corresponding weight given by $\pi_{\max}$, we have

$$
\gamma_\pi^*(t) \le \frac{3K}{2}\,\big(1-\pi_{\max}(t)\big). \qquad (2.4.77)
$$

Now, recall the inequality (2.4.57), which can be recaptured as follows

$$
\begin{aligned}
\pi_{\max}(t) &\le 1 - \varepsilon\gamma_\pi^*(t), \\
\varepsilon\gamma_\pi^*(t) &\le 1 - \pi_{\max}(t), \quad \text{and,} \\
\gamma_\pi^*(t) &\le \frac{1}{\varepsilon}\,\big(1-\pi_{\max}(t)\big). \qquad (2.4.78)
\end{aligned}
$$

Thus, the inequality (2.4.57) follows from the inequality (2.4.77) by taking $\varepsilon := \frac{2}{3K}$ in (2.4.78). ■

Fernholz & Karatzas (2009) provide an alternative formulation of the previous lemma, which we shall present in the following lemma. The reason for including it here, is that the proof is vastly simplified and is a more succinct view of the upper bound placed on the excess growth rate. The results are akin to those obtained in the previous lemma.

Lemma 2.4.16 ([**Fernholz & Karatzas (2009)**]). *Let π be a portfolio with nonnegative weights (i.e., a long-only portfolio) in a market with bounded variance (i.e., assume that, via the covariance process $\boldsymbol{\sigma}(t)$, the market uniform boundedness condition holds in* (2.2.57)*). Then, for every long-only portfolio π, there exists a number $\varepsilon > 0$ such that for $i = 1, \ldots, n$, we also have, in the notation of* (2.4.51) *of Definition* 2.4.10*, the following inequalities*

$$\tau_{ii}^{\pi}(t) \leq \varepsilon\big(1-\pi_i(t)\big)\big(2-\pi_i(t)\big), \qquad t \in [0,\infty), \qquad a.s., \quad and, \tag{2.4.79}$$

$$\gamma_{\pi}^{*}(t) \leq \varepsilon\big(1-\pi_{\max}(t)\big), \qquad t \in [0,\infty), \qquad a.s. \tag{2.4.80}$$

Proof. From equations (2.4.47) and (2.4.48), in conjunction with the uniform boundedness condition (2.2.57) and with $\mathbf{x}(t) = \boldsymbol{\pi}(t) - \mathbf{e}_i$, where $\mathbf{e}_i$ represents the ith unit vector in $\mathbb{R}^n$, for $t \in [0,\infty)$, a.s., we have

$$\tau_{ii}^{\pi}(t) = \Big(\boldsymbol{\pi}(t) - \mathbf{e}_i\Big)\boldsymbol{\sigma}(t)\Big(\boldsymbol{\pi}(t) - \mathbf{e}_i\Big)^{\mathsf{T}} = \mathbf{x}(t)\boldsymbol{\sigma}(t)\mathbf{x}^{\mathsf{T}}(t) \leq K\left\|\mathbf{x}(t)\right\|^2.$$

Now, for the Euclidean norm in the above inequality, we obtain

$$\begin{aligned}
\left\|\mathbf{x}(t)\right\|^2 = \left\|\boldsymbol{\pi}(t) - \mathbf{e}_i\right\|^2 &= \pi_1^2(t) + \cdots + \pi_{i-1}^2(t) + \big(\pi_i(t) - 1\big)^2 + \pi_{i+1}^2(t) + \cdots + \pi_n^2(t) \\
&= \big(\pi_i(t) - 1\big)^2 + \sum_{j \neq i}^{n} \pi_j^2(t) \\
&= \big(1 - \pi_i(t)\big)^2 + \sum_{j \neq i}^{n} \pi_j^2(t).
\end{aligned}$$

Thus, for $i = 1, 2, \ldots, n$, $t \in [0,\infty)$, a.s., we have

$$\tau_{ii}^{\pi}(t) \leq K\left\|\mathbf{x}(t)\right\|^2 = K\left\|\boldsymbol{\pi}(t) - \mathbf{e}_i\right\|^2 = K\left(\big(1-\pi_i(t)\big)^2 + \sum_{j \neq i}^{n} \pi_j^2(t)\right),$$

this is analogous to the proofs of Lemma 2.4.9 and Lemma 2.4.14. Moreover, with the knowledge that $0 \leq \pi_i(t) < 1$, we arrive at the following[16]

$$\begin{aligned}
\tau_{ii}^{\pi}(t) &\leq K\left(\big(1-\pi_i(t)\big)^2 + \sum_{j \neq i}^{n} \pi_j(t)\right) \\
&= K\left(\big(1-\pi_i(t)\big)^2 + \big(1-\pi_i(t)\big)\right) \\
&= K\left(\big(1-\pi_i(t)\big)\big(1-\pi_i(t)+1\big)\right) \\
&= K\big(1-\pi_i(t)\big)\big(2-\pi_i(t)\big).
\end{aligned}$$

Thus, the result as claimed in (2.4.79) follows by taking $\varepsilon := K$. Since $2 - \pi_i(t) \geq 1$ for all $i = 1, \ldots, n$, we have

$$\tau_{ii}^{\pi}(t) \leq K\big(1-\pi_i(t)\big). \tag{2.4.81}$$

Now, by incorporating this upper bound estimate (2.4.81) for the relative variances of the stocks in the portfolio π, into equation (2.4.32) of Lemma 2.4.7, we obtain the following upper bound on the excess growth rate of the

[16] Since, with $0 \leq \pi_i(t) < 1$, it follows that $\pi_i^2(t) \leq \pi_i(t)$, and that $\sum \pi_i^2(t) \leq \sum \pi_i(t)$.

portfolio π,

$$\begin{aligned}
\gamma_\pi^*(t) &= \frac{1}{2}\sum_{i=1}^n \pi_i(t)\tau_{ii}^\pi(t) \leq \sum_{i=1}^n \pi_i(t)\tau_{ii}^\pi(t) \\
&\leq \sum_{i=1}^n \pi_i(t)\Big[K\big(1-\pi_i(t)\big)\Big] \\
&= K\sum_{i=1}^n \pi_i(t)\big(1-\pi_i(t)\big) \\
&= K\left(\pi_{(1)}(t)\big(1-\pi_{(1)}(t)\big) + \sum_{k=2}^n \pi_{(k)}(t)\big(1-\pi_{(k)}(t)\big)\right) \\
&\leq K\left(\big(1-\pi_{(1)}(t)\big) + \sum_{k=2}^n \pi_{(k)}(t)\right) \\
&= K\Big(\big(1-\pi_{(1)}(t)\big) + \big(1-\pi_{(1)}(t)\big)\Big) \\
&= 2K\big(1-\pi_{(1)}(t)\big).
\end{aligned}$$

By taking $\varepsilon := 2K$ and recalling the reverse-order-statistics notation of Definition 2.4.10 where $\pi_{\max}(t) = \pi_{(1)}(t)$, we obtain (2.4.80). This result is equivalent to that obtained in Lemma 2.4.15. The parallel is obtained by taking $\varepsilon := \frac{1}{2K}$ in (2.4.57) or (2.4.78). ■

Thus, in summary, the strong nondegeneracy condition and the uniform boundedness condition, encapsulated in the following compact form

$$\varepsilon\,\|\mathbf{x}\|^2 \leq \mathbf{x}\boldsymbol{\sigma}(t)\mathbf{x}^\mathsf{T} \leq K\,\|\mathbf{x}\|^2, \qquad \mathbf{x}\in\mathbb{R}^n, \qquad t\in[0,\infty), \qquad \text{a.s.,} \tag{2.4.82}$$

lead to the following inequalities for the relative covariance for $i = 1,\ldots,n$,

$$\varepsilon\big(1-\pi_i(t)\big)^2 \leq \tau_{ii}^\pi(t) \leq K\big(1-\pi_i(t)\big)\big(2-\pi_i(t)\big), \qquad t\in[0,\infty), \qquad \text{a.s.} \tag{2.4.83}$$

Furthermore, we revealed that these same conditions imply the following bounds on the excess growth rate[17]

$$\frac{\varepsilon}{2}\big(1-\pi_{\max}(t)\big) \leq \gamma_\pi^*(t) \leq 2K\big(1-\pi_{\max}(t)\big), \qquad t\in[0,\infty), \qquad \text{a.s.,} \tag{2.4.84}$$

which, in the reverse-order-statistics notation (2.4.53) of Definition 2.4.10, can equivalently be expressed as

$$\frac{\varepsilon}{2}\big(1-\pi_{(1)}(t)\big) \leq \gamma_\pi^*(t) \leq 2K\big(1-\pi_{(1)}(t)\big), \qquad t\in[0,\infty), \qquad \text{a.s.} \tag{2.4.85}$$

2.5 Quotient Process

We shall first introduce the quotient process of stocks, and then proceed to that of a stock and a portfolio.

Definition 2.5.1 (Quotient Process of Stocks). *For stock $X_i = \{X_i(t), t\in[0,\infty)\}$ for $i = 1,2,\ldots,n$, and stock $X_j = \big\{X_j(t), t\in[0,\infty)\big\}$, the process $X_i/X_j = \big\{X_i(t)/X_j(t), t\in[0,\infty)\big\}$, is called the* ***quotient process*** *of X_i versus X_j.*

[17] The bounds on the excess growth rate can also be expressed as $\frac{\varepsilon}{2}\big(1-\pi_{\max}(t)\big) \leq \gamma_\pi^*(t) \leq \varepsilon_1\big(1-\pi_{\max}(t)\big)$, where $\varepsilon_1 = 2K$. A tighter bound on the excess growth rate, than the one provided in (2.4.84), is given by $\frac{\varepsilon}{2}\big(1-\pi_{\max}(t)\big) \leq \gamma_\pi^*(t) \leq \frac{3K}{2}\big(1-\pi_{\max}(t)\big)$ (if K is the same for both bounds). This bound is the same as $\frac{\varepsilon}{2}\big(1-\pi_{\max}(t)\big) \leq \gamma_\pi^*(t) \leq \frac{1}{\varepsilon_2}\big(1-\pi_{\max}(t)\big)$, where $\varepsilon_2 = \frac{2}{3K}$. This gives a tighter bound than (2.4.84), since $\varepsilon_1 > \frac{1}{\varepsilon_2}$. However, if $\varepsilon_1 = \varepsilon_2 = \varepsilon > 0$, then the bound $\frac{\varepsilon}{2}\big(1-\pi_{\max}(t)\big) \leq \gamma_\pi^*(t) \leq \frac{3K_2}{2}\big(1-\pi_{\max}(t)\big) \equiv \frac{1}{\varepsilon}\big(1-\pi_{\max}(t)\big)$ is tighter than the bound $\frac{\varepsilon}{2}\big(1-\pi_{\max}(t)\big) \leq \gamma_\pi^*(t) \leq 2K_1\big(1-\pi_{\max}(t)\big) \equiv \varepsilon\big(1-\pi_{\max}(t)\big)$, when $\varepsilon > 1$.

By applying Itô's formula to $\exp\big(\log\big(X_i(t)/X_j(t)\big)\big)$, the quotient process of X_i versus X_j can be described by the following dynamics for $t \in [0,\infty)$, a.s.,

$$d\left(\frac{X_i(t)}{X_j(t)}\right) = \frac{X_i(t)}{X_j(t)}\, d\log\Big(X_i(t)/X_j(t)\Big) + \frac{1}{2}\,\frac{X_i(t)}{X_j(t)}\, d\left\langle \log\big(X_i/X_j\big)\right\rangle_t, \tag{2.5.1}$$

which can be restated as follows

$$d\left(\frac{X_i(t)}{X_j(t)}\right) = \frac{X_i(t)}{X_j(t)}\, d\log\Big(X_i(t)/X_j(t)\Big) + \frac{1}{2}\,\frac{X_i(t)}{X_j(t)}\, d\left\langle \log\big(X_i/Z_{\chi_j}\big)\right\rangle_t. \tag{2.5.2}$$

Thus, from equation (2.3.22) or equation (2.3.122), we obtain

$$d\left(\frac{X_i(t)}{X_j(t)}\right) = \frac{X_i(t)}{X_j(t)}\, d\log\Big(X_i(t)/X_j(t)\Big) + \frac{1}{2}\,\frac{X_i(t)}{X_j(t)}\, \tau_{ii}^{\chi_j}(t)\, dt, \tag{2.5.3}$$

which, together with equation (2.3.1), gives

$$d\left(\frac{X_i(t)}{X_j(t)}\right) = \frac{X_i(t)}{X_j(t)}\left[\big(\gamma_i(t)-\gamma_j(t)\big)\, dt + \sum_{\nu=1}^{n}\big(\xi_{i\nu}(t)-\xi_{j\nu}(t)\big)\, dW_\nu(t)\right] + \frac{1}{2}\,\frac{X_i(t)}{X_j(t)}\,\tau_{ii}^{\chi_j}(t)\, dt \tag{2.5.4}$$

$$= \left(\gamma_i(t)-\gamma_j(t)+\frac{1}{2}\,\tau_{ii}^{\chi_j}(t)\right)\frac{X_i(t)}{X_j(t)}\, dt + \frac{X_i(t)}{X_j(t)}\sum_{\nu=1}^{n}\big(\xi_{i\nu}(t)-\xi_{j\nu}(t)\big)\, dW_\nu(t) \tag{2.5.5}$$

$$= \frac{X_i(t)}{X_j(t)}\left[\left(\gamma_i(t)-\gamma_j(t)+\frac{1}{2}\,\tau_{ii}^{\chi_j}(t)\right) dt + \sum_{\nu=1}^{n}\big(\xi_{i\nu}(t)-\xi_{j\nu}(t)\big)\, dW_\nu(t)\right]. \tag{2.5.6}$$

We can rewrite equation (2.5.3) in the following beneficial form

$$\frac{d\big(X_i(t)/X_j(t)\big)}{X_i(t)/X_j(t)} = d\log\Big(X_i(t)/X_j(t)\Big) + \frac{1}{2}\,\tau_{ii}^{\chi_j}(t)\, dt. \tag{2.5.7}$$

Therefore, the differential of the quadratic variation of the aforementioned quotient process of X_i versus a fixed stock X_k, $k \in \{1,2,\ldots,n\}$, given above in equation (2.5.3), can be derived from (2.3.122) as follows for $t \in [0,\infty)$, and for all $i = 1,2,\ldots,n$,

$$d\left\langle \frac{X_i}{X_k}\right\rangle_t = \left(\frac{X_i(t)}{X_k(t)}\right)^2 d\left\langle \log\big(X_i/X_k\big)\right\rangle_t \tag{2.5.8}$$

$$= \tau_{ii}^{\chi_k}(t)\left(\frac{X_i(t)}{X_k(t)}\right)^2 dt. \tag{2.5.9}$$

A derivation of the cross-variation of the quotient processes can also be acquired by making use of equation (2.3.3) as follows for $t \in [0,\infty)$, and for all $i = 1,2,\ldots,n$,

$$d\left\langle \frac{X_i}{X_k}, \frac{X_j}{X_k}\right\rangle_t = \left(\frac{X_i(t)X_j(t)}{X_k^2(t)}\right) d\left\langle \log\big(X_i/X_k\big), \log\big(X_j/X_k\big)\right\rangle_t \tag{2.5.10}$$

$$= \tau_{ij}^{\chi_k}(t)\left(\frac{X_i(t)X_j(t)}{X_k^2(t)}\right) dt. \tag{2.5.11}$$

Akin to the notation adopted in equations (2.3.38) and (2.3.18), we shall require the following notation, for all $i,j = 1,2,\ldots,n$ and for fixed $k,m \in \{1,2,\ldots,n\}$,

$$d\left\langle \frac{X_i}{X_k}, \frac{X_i}{X_m}\right\rangle_t = \left(\frac{X_i^2(t)}{X_k(t)X_m(t)}\right) d\left\langle \log\big(X_i/X_k\big), \log\big(X_i/X_m\big)\right\rangle_t \tag{2.5.12}$$

$$= \tau_{ii}^{\chi_k\chi_m}(t)\left(\frac{X_i^2(t)}{X_k(t)X_m(t)}\right) dt, \tag{2.5.13}$$

and,

$$d\left\langle \frac{X_i}{X_k}, \frac{X_j}{X_m} \right\rangle_t = \left(\frac{X_i(t)X_j(t)}{X_k(t)X_m(t)} \right) d\left\langle \log\left(X_i/X_k\right), \log\left(X_j/X_m\right) \right\rangle_t \tag{2.5.14}$$

$$= \tau_{ij}^{\chi_k \chi_m}(t) \left(\frac{X_i(t)X_j(t)}{X_k(t)X_m(t)} \right) dt. \tag{2.5.15}$$

In what follows, from equation (2.2.109) of Definition 2.2.24, we shall also require the following summation equality

$$\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) = 2\,\gamma_\pi^*(t) + \sigma_{\pi\pi}(t). \tag{2.5.16}$$

We shall express the relative return process $d\log\left(X_i/Z_\eta\right)$ in terms of the aforementioned quotient processes. This is done by substituting the adjusted form of (2.5.7), given by

$$d\log\Big(X_i(t)/X_j(t)\Big) = \frac{d\left(X_i(t)/X_j(t)\right)}{X_i(t)/X_j(t)} - \frac{1}{2}\,\tau_{ii}^{\chi_j}(t)\,dt, \tag{2.5.17}$$

into equation (2.3.45) of Lemma 2.3.6, as follows

$$\begin{aligned}
&d\log\Big(X_i(t)/Z_\eta(t)\Big) \\
&= \sum_{j=1}^{n} \eta_j(t) \left[\frac{d\left(X_i(t)/X_j(t)\right)}{X_i(t)/X_j(t)} - \frac{1}{2}\,\tau_{ii}^{\chi_j}(t)\,dt \right] - \gamma_\eta^*(t)\,dt \\
&= \sum_{j=1}^{n} \eta_j(t) \frac{X_j(t)}{X_i(t)}\, d\left(\frac{X_i(t)}{X_j(t)}\right) - \left(\gamma_\eta^*(t) + \frac{1}{2}\sum_{j=1}^{n} \eta_j(t)\tau_{ii}^{\chi_j}(t) \right) dt \\
&= \sum_{j=1}^{n} \eta_j(t) \frac{X_j(t)}{X_i(t)}\, d\left(\frac{X_i(t)}{X_j(t)}\right) - \left(\gamma_\eta^*(t) + \frac{1}{2}\sum_{j=1}^{n} \eta_j(t)\Big[\sigma_{ii}(t) - 2\,\sigma_{ij}(t) + \sigma_{jj}(t)\Big] \right) dt \qquad (2.5.18) \\
&= \sum_{j=1}^{n} \eta_j(t) \frac{X_j(t)}{X_i(t)}\, d\left(\frac{X_i(t)}{X_j(t)}\right) - \left(\gamma_\eta^*(t) + \frac{1}{2}\left[\sigma_{ii}(t) - 2\sum_{j=1}^{n} \eta_j(t)\sigma_{ij}(t) + \sum_{j=1}^{n} \eta_j(t)\sigma_{jj}(t)\right] \right) dt \\
&= \sum_{j=1}^{n} \eta_j(t) \frac{X_j(t)}{X_i(t)}\, d\left(\frac{X_i(t)}{X_j(t)}\right) - \left(\gamma_\eta^*(t) + \frac{1}{2}\Big[\sigma_{ii}(t) - 2\,\sigma_{i\eta}(t) + \sigma_{\eta\eta}(t) + 2\,\gamma_\eta^*(t)\Big] \right) dt \qquad (2.5.19) \\
&= \sum_{j=1}^{n} \eta_j(t) \frac{X_j(t)}{X_i(t)}\, d\left(\frac{X_i(t)}{X_j(t)}\right) - \left(\gamma_\eta^*(t) + \frac{1}{2}\Big[\tau_{ii}^{\eta}(t) + 2\,\gamma_\eta^*(t)\Big] \right) dt \qquad (2.5.20) \\
&= \sum_{j=1}^{n} \eta_j(t) \frac{X_j(t)}{X_i(t)}\, d\left(\frac{X_i(t)}{X_j(t)}\right) - \left(2\,\gamma_\eta^*(t) + \frac{1}{2}\,\tau_{ii}^{\eta}(t) \right) dt, \qquad (2.5.21)
\end{aligned}$$

where equation (2.5.18) follows from equation (2.3.27), equation (2.5.19) follows from equations (2.3.51) and (2.5.16), and lastly, equation (2.5.20) follows from equation (2.3.128). Hence, the following excess growth rate consequence emerges

$$\gamma_\eta^*(t) = \frac{1}{2}\left(\sum_{j=1}^{n} \eta_j(t)\tau_{ii}^{\chi_j}(t) - \tau_{ii}^{\eta}(t) \right), \tag{2.5.22}$$

which is akin to the expression (2.4.29) of the numéraire invariance property of Lemma 2.4.5. In fact, setting the numéraire portfolio to χ_i in the expression (2.4.29) of the numéraire invariance property, we equivalently obtain

$$\gamma_\eta^*(t) = \frac{1}{2}\left(\sum_{j=1}^{n} \eta_j(t)\tau_{jj}^{\chi_i}(t) - \tau_{\eta\eta}^{\chi_i}(t) \right) = \frac{1}{2}\left(\sum_{j=1}^{n} \eta_j(t)\tau_{ii}^{\chi_j}(t) - \tau_{ii}^{\eta}(t) \right),$$

since $\tau_{jj}^{\chi_i}(t) \equiv \tau_{ii}^{\chi_j}(t)$ by (2.3.36), and $\tau_{\eta\eta}^{\chi_i}(t) \equiv \tau_{\chi_i\chi_i}^{\eta}(t) \equiv \tau_{ii}^{\eta}(t)$ by equation (2.4.23) and equation (2.4.19) of Lemma 2.4.4. Moreover, equation (2.5.22) can be recaptured as

$$\sum_{j=1}^{n} \eta_j(t)\tau_{ii}^{\chi_j}(t) \quad = \quad 2\,\gamma_\eta^*(t) + \tau_{ii}^{\eta}(t). \tag{2.5.23}$$

The interested reader can verify that substituting equation (2.5.6) into equation (2.5.21), in conjunction with equation (2.5.22), yields equation (2.3.43).

Definition 2.5.2 (Quotient Process). *For a stock $X_i = \{X_i(t), t \in [0,\infty)\}$ with initial capital $X_i(0) = x_i$, for $i = 1, 2, \ldots, n$, portfolio $\eta = \big\{\boldsymbol{\eta}(t) = \big(\eta_1(t), \ldots, \eta_n(t)\big), t \in [0,\infty)\big\}$ and portfolio value process $Z_{w,\eta} = \big\{Z_{w,\eta}(t), t \in [0,\infty)\big\}$ with initial capital $Z_{w,\eta}(0) = w > 0$, with $w := X_i(0)$, the process $X_i/Z_\eta = \big\{X_i(t)/Z_\eta(t), t \in [0,\infty)\big\}$, i.e.,*

$$\frac{X_i(t)}{Z_\eta(t)} \quad = \quad \left(\frac{X_i(t)}{Z_{w,\eta}(t)}\right)\Bigg|_{w=X_i(0)}, \qquad t \in [0,\infty), \tag{2.5.24}$$

*is called the **quotient process** of X_i versus η, for all $i = 1, 2, \ldots, n$.*

An application of Itô's formula to $\exp\big(\log\big(X_i(t)/Z_\eta(t)\big)\big)$ leads to the following

$$d\left(\frac{X_i(t)}{Z_\eta(t)}\right) \quad = \quad \frac{X_i(t)}{Z_\eta(t)}\, d\log\Big(X_i(t)/Z_\eta(t)\Big) + \frac{1}{2}\,\frac{X_i(t)}{Z_\eta(t)}\, d\left\langle \log\big(X_i/Z_\eta\big)\right\rangle_t. \tag{2.5.25}$$

Substituting equation (2.3.123) into equation (2.5.25), gives

$$d\left(\frac{X_i(t)}{Z_\eta(t)}\right) \quad = \quad \frac{X_i(t)}{Z_\eta(t)}\, d\log\Big(X_i(t)/Z_\eta(t)\Big) + \frac{1}{2}\,\frac{X_i(t)}{Z_\eta(t)}\,\tau_{ii}^{\eta}(t)\, dt. \tag{2.5.26}$$

The expression above can be rewritten in the more useful form, which will be enlisted at a later stage, as follows

$$\frac{d\big(X_i(t)/Z_\eta(t)\big)}{X_i(t)/Z_\eta(t)} \quad = \quad d\log\Big(X_i(t)/Z_\eta(t)\Big) + \frac{1}{2}\,\tau_{ii}^{\eta}(t)\, dt. \tag{2.5.27}$$

Substituting equation (2.3.42) into equation (2.5.26), offers the following representation for the dynamics of the quotient process

$$\begin{aligned}
d\left(\frac{X_i(t)}{Z_\eta(t)}\right) &= \frac{X_i(t)}{Z_\eta(t)}\left[\big(\gamma_i(t) - \gamma_\eta(t)\big)\, dt + \sum_{j,\nu=1}^{n} \eta_j(t)\big(\xi_{i\nu}(t) - \xi_{j\nu}(t)\big)\, dW_\nu(t)\right] + \frac{1}{2}\,\frac{X_i(t)}{Z_\eta(t)}\,\tau_{ii}^{\eta}(t)\, dt && (2.5.28)\\
&= \left(\gamma_i(t) - \gamma_\eta(t) + \frac{1}{2}\,\tau_{ii}^{\eta}(t)\right)\frac{X_i(t)}{Z_\eta(t)}\, dt + \frac{X_i(t)}{Z_\eta(t)}\sum_{j,\nu=1}^{n} \eta_j(t)\big(\xi_{i\nu}(t) - \xi_{j\nu}(t)\big)\, dW_\nu(t) && (2.5.29)\\
&= \frac{X_i(t)}{Z_\eta(t)}\left[\left(\gamma_i(t) - \gamma_\eta(t) + \frac{1}{2}\,\tau_{ii}^{\eta}(t)\right) dt + \sum_{j,\nu=1}^{n} \eta_j(t)\big(\xi_{i\nu}(t) - \xi_{j\nu}(t)\big)\, dW_\nu(t)\right]. && (2.5.30)
\end{aligned}$$

In addition, substituting equation (2.3.43) into equation (2.5.26), offers the following representation for the dynamics of the quotient process

$$\begin{aligned}
d\left(\frac{X_i(t)}{Z_\eta(t)}\right) &= \frac{X_i(t)}{Z_\eta(t)}\left[\big(\gamma_i(t) - \gamma_\eta(t)\big)\, dt + \sum_{\nu=1}^{n} \big(\xi_{i\nu}(t) - \xi_{\eta\nu}(t)\big)\, dW_\nu(t)\right] + \frac{1}{2}\,\frac{X_i(t)}{Z_\eta(t)}\,\tau_{ii}^{\eta}(t)\, dt && (2.5.31)\\
&= \left(\gamma_i(t) - \gamma_\eta(t) + \frac{1}{2}\,\tau_{ii}^{\eta}(t)\right)\frac{X_i(t)}{Z_\eta(t)}\, dt + \frac{X_i(t)}{Z_\eta(t)}\sum_{\nu=1}^{n} \big(\xi_{i\nu}(t) - \xi_{\eta\nu}(t)\big)\, dW_\nu(t) && (2.5.32)\\
&= \frac{X_i(t)}{Z_\eta(t)}\left[\left(\gamma_i(t) - \gamma_\eta(t) + \frac{1}{2}\,\tau_{ii}^{\eta}(t)\right) dt + \sum_{\nu=1}^{n} \big(\xi_{i\nu}(t) - \xi_{\eta\nu}(t)\big)\, dW_\nu(t)\right]. && (2.5.33)
\end{aligned}$$

Thus, equation (2.5.6) can be reformulated in much the same way as above, as follows

$$d\left(\frac{X_i(t)}{Z_{\chi_j}(t)}\right) = \frac{X_i(t)}{Z_{\chi_j}(t)}\left[\left(\gamma_i(t)-\gamma_{\chi_j}(t)+\frac{1}{2}\,\tau_{ii}^{\chi_j}(t)\right)dt+\sum_{\nu=1}^{n}\left(\xi_{i\nu}(t)-\xi_{\chi_j\nu}(t)\right)dW_\nu(t)\right]. \tag{2.5.34}$$

Therefore, the differential of the quadratic variation, of the aforementioned quotient process of X_i versus the portfolio η, can be derived from equation (2.5.25) as follows, for $t\in[0,\infty)$, and for all $i=1,2,\ldots,n$,

$$d\left\langle\frac{X_i}{Z_\eta}\right\rangle_t = \left(\frac{X_i(t)}{Z_\eta(t)}\right)^2 d\left\langle\log\left(X_i/Z_\eta\right)\right\rangle_t \tag{2.5.35}$$

$$= \tau_{ii}^{\eta}(t)\left(\frac{X_i(t)}{Z_\eta(t)}\right)^2 dt. \tag{2.5.36}$$

A derivation of the cross-variation of the quotient processes can also be acquired by making use of equation (2.3.84), as follows for $t\in[0,\infty)$, and for all $i=1,2,\ldots,n$,

$$d\left\langle\frac{X_i}{Z_\eta},\frac{X_j}{Z_\eta}\right\rangle_t = \left(\frac{X_i(t)X_j(t)}{Z_\eta^2(t)}\right) d\left\langle\log\left(X_i/Z_\eta\right),\log\left(X_j/Z_\eta\right)\right\rangle_t \tag{2.5.37}$$

$$= \tau_{ij}^{\eta}(t)\left(\frac{X_i(t)X_j(t)}{Z_\eta^2(t)}\right) dt. \tag{2.5.38}$$

Lemma 2.5.3. *Let X_i denote the ith stock in the market $\mathcal{M}$ and let η be any arbitrary portfolio in the market $\mathcal{M}$. Then, a.s., for $t\in[0,\infty)$, we obtain the dynamics*

$$d\left(\frac{X_i(t)}{Z_\eta(t)}\right) = \frac{X_i(t)}{Z_\eta(t)}\left[\left(\frac{1}{2}\,\tau_{ii}^{\eta}(t)-\gamma_\eta^*(t)\right)dt+\sum_{j=1}^{n}\eta_j(t)\,d\log\left(X_i(t)/X_j(t)\right)\right], \tag{2.5.39}$$

for the relative arithmetic return of stock X_i versus portfolio η.

Proof. This easily follows by substituting equation (2.3.45) of Lemma 2.3.6 into equation (2.5.26), to obtain the result

$$\begin{aligned} d\left(\frac{X_i(t)}{Z_\eta(t)}\right) &= \frac{1}{2}\,\frac{X_i(t)}{Z_\eta(t)}\,\tau_{ii}^{\eta}(t)\,dt+\frac{X_i(t)}{Z_\eta(t)}\left[\sum_{j=1}^{n}\eta_j(t)\,d\log\left(X_i(t)/X_j(t)\right)-\gamma_\eta^*(t)\,dt\right] \\ &= \frac{X_i(t)}{Z_\eta(t)}\left[\left(\frac{1}{2}\,\tau_{ii}^{\eta}(t)-\gamma_\eta^*(t)\right)dt+\sum_{j=1}^{n}\eta_j(t)\,d\log\left(X_i(t)/X_j(t)\right)\right]. \end{aligned}$$

■

Equation (2.5.30) can be verified by substituting equation (2.3.1) into equation (2.5.39) of the above lemma

$$\begin{aligned} d\left(\frac{X_i(t)}{Z_\eta(t)}\right) &= \frac{X_i(t)}{Z_\eta(t)}\left[\left(\frac{1}{2}\,\tau_{ii}^{\eta}(t)-\gamma_\eta^*(t)\right)dt+\sum_{j=1}^{n}\eta_j(t)\left[\left(\gamma_i(t)-\gamma_j(t)\right)dt+\sum_{\nu=1}^{n}\left(\xi_{i\nu}(t)-\xi_{j\nu}(t)\right)dW_\nu(t)\right]\right] \\ &= \frac{X_i(t)}{Z_\eta(t)}\left[\left(\gamma_i(t)-\sum_{j=1}^{n}\eta_j(t)\gamma_j(t)-\gamma_\eta^*(t)+\frac{1}{2}\,\tau_{ii}^{\eta}(t)\right)dt+\sum_{j,\nu=1}^{n}\eta_j(t)\left(\xi_{i\nu}(t)-\xi_{j\nu}(t)\right)dW_\nu(t)\right] \\ &= \frac{X_i(t)}{Z_\eta(t)}\left[\left(\gamma_i(t)-\gamma_\eta(t)+\frac{1}{2}\,\tau_{ii}^{\eta}(t)\right)dt+\sum_{j,\nu=1}^{n}\eta_j(t)\left(\xi_{i\nu}(t)-\xi_{j\nu}(t)\right)dW_\nu(t)\right], \end{aligned}$$

where the last equality follows from equation (2.2.111). In addition, equation (2.5.33) can also be verified by substituting equation (2.3.1) into equation (2.5.39) of the above lemma

$$\begin{aligned} d\left(\frac{X_i(t)}{Z_\eta(t)}\right) &= \frac{X_i(t)}{Z_\eta(t)}\left[\left(\gamma_i(t)-\sum_{j=1}^{n}\eta_j(t)\gamma_j(t)-\gamma_\eta^*(t)+\frac{1}{2}\,\tau_{ii}^{\eta}(t)\right)dt+\sum_{\nu=1}^{n}\left[\xi_{i\nu}(t)-\sum_{j=1}^{n}\eta_j(t)\xi_{j\nu}(t)\right]dW_\nu(t)\right] \\ &= \frac{X_i(t)}{Z_\eta(t)}\left[\left(\gamma_i(t)-\gamma_\eta(t)+\frac{1}{2}\,\tau_{ii}^{\eta}(t)\right)dt+\sum_{\nu=1}^{n}\left(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\right)dW_\nu(t)\right], \end{aligned}$$

where the last equality follows from equations (2.2.111) and (2.2.126).

Lemma 2.5.4. *Let X_i denote the ith stock in the market $\mathcal{M}$ and let η be any arbitrary portfolio in the market $\mathcal{M}$. Then, a.s., for $t \in [0, \infty)$, we obtain the dynamics*

$$d\left(\frac{X_i(t)}{Z_\eta(t)}\right) = \frac{X_i(t)}{Z_\eta(t)}\left[\left(\frac{1}{2}\,\tau_{ii}^{\eta}(t) - \frac{1}{2}\sum_{j=1}^{n}\eta_j(t)\tau_{ii}^{\chi_j}(t) - \gamma_\eta^*(t)\right)dt + \sum_{j=1}^{n}\eta_j(t)\frac{X_j(t)}{X_i(t)}\,d\left(\frac{X_i(t)}{X_j(t)}\right)\right] \tag{2.5.40}$$

$$= \frac{X_i(t)}{Z_\eta(t)}\left[-2\,\gamma_\eta^*(t)\,dt + \sum_{j=1}^{n}\eta_j(t)\frac{X_j(t)}{X_i(t)}\,d\left(\frac{X_i(t)}{X_j(t)}\right)\right], \tag{2.5.41}$$

for the relative arithmetic return of stock X_i versus portfolio η.

Proof. For this proof, we shall require a modification of equation (2.5.7), given by

$$d\log\Big(X_i(t)/X_j(t)\Big) = \frac{d\left(X_i(t)/X_j(t)\right)}{X_i(t)/X_j(t)} - \frac{1}{2}\,\tau_{ii}^{\chi_j}(t)\,dt. \tag{2.5.42}$$

Thus, utilising equations (2.5.39) and (2.5.42), we obtain

$$d\left(\frac{X_i(t)}{Z_\eta(t)}\right) = \frac{X_i(t)}{Z_\eta(t)}\left[\left(\frac{1}{2}\,\tau_{ii}^{\eta}(t) - \gamma_\eta^*(t)\right)dt + \sum_{j=1}^{n}\eta_j(t)\left[\frac{d\left(X_i(t)/X_j(t)\right)}{X_i(t)/X_j(t)} - \frac{1}{2}\,\tau_{ii}^{\chi_j}(t)\,dt\right]\right].$$

Simplifying completes the proof of the first part (2.5.40). To prove the second part (2.5.41), it suffices to show that the drift component in (2.5.40) reduces to $-2\,\gamma_\eta^*(t)$. This follows by employing equation (2.5.23), as follows

$$\frac{1}{2}\,\tau_{ii}^{\eta}(t) - \frac{1}{2}\sum_{j=1}^{n}\eta_j(t)\tau_{ii}^{\chi_j}(t) - \gamma_\eta^*(t) = \frac{1}{2}\,\tau_{ii}^{\eta}(t) - \frac{1}{2}\Big(2\,\gamma_\eta^*(t) + \tau_{ii}^{\eta}(t)\Big) - \gamma_\eta^*(t) = -2\,\gamma_\eta^*(t).$$

Thus, the equality above provides the substitution required to obtain equation (2.5.41). Since equation (2.5.21) expresses $d\log\big(X_i/Z_\eta\big)$ directly in terms of the quotient processes, an alternative means of obtaining the above result is to substitute equation (2.5.21) into equation (2.5.26), as follows

$$d\left(\frac{X_i(t)}{Z_\eta(t)}\right) = \frac{X_i(t)}{Z_\eta(t)}\left[\frac{1}{2}\,\tau_{ii}^{\eta}(t)\,dt + \left(\sum_{j=1}^{n}\eta_j(t)\frac{X_j(t)}{X_i(t)}\,d\left(\frac{X_i(t)}{X_j(t)}\right) - \left(2\,\gamma_\eta^*(t) + \frac{1}{2}\,\tau_{ii}^{\eta}(t)\right)dt\right)\right]$$

$$= \frac{X_i(t)}{Z_\eta(t)}\left[-2\,\gamma_\eta^*(t)\,dt + \sum_{j=1}^{n}\eta_j(t)\frac{X_j(t)}{X_i(t)}\,d\left(\frac{X_i(t)}{X_j(t)}\right)\right].$$

■

The result of equation (2.5.33) can be reaffirmed by substituting equation (2.5.6) into equation (2.5.40), as follows

$$d\left(\frac{X_i(t)}{Z_\eta(t)}\right) = \frac{X_i(t)}{Z_\eta(t)}\left[\left(\frac{1}{2}\,\tau_{ii}^{\eta}(t) - \frac{1}{2}\sum_{j=1}^{n}\eta_j(t)\tau_{ii}^{\chi_j}(t) - \gamma_\eta^*(t)\right)dt + \sum_{j=1}^{n}\eta_j(t)\left[\left(\gamma_i(t) - \gamma_j(t) + \frac{1}{2}\,\tau_{ii}^{\chi_j}(t)\right)dt\right.\right.$$
$$\left.\left. + \sum_{\nu=1}^{n}\big(\xi_{i\nu}(t) - \xi_{j\nu}(t)\big)\,dW_\nu(t)\right]\right]$$

$$= \frac{X_i(t)}{Z_\eta(t)}\left[\left(\frac{1}{2}\,\tau_{ii}^{\eta}(t) - \frac{1}{2}\sum_{j=1}^{n}\eta_j(t)\tau_{ii}^{\chi_j}(t) + \gamma_i(t) - \sum_{j=1}^{n}\eta_j(t)\gamma_j(t) - \gamma_\eta^*(t) + \frac{1}{2}\sum_{j=1}^{n}\eta_j(t)\tau_{ii}^{\chi_j}(t)\right)dt\right.$$
$$\left. + \sum_{\nu=1}^{n}\left[\xi_{i\nu}(t) - \sum_{j=1}^{n}\eta_j(t)\xi_{j\nu}(t)\right]dW_\nu(t)\right]$$

$$= \frac{X_i(t)}{Z_\eta(t)}\left[\left(\gamma_i(t) - \gamma_\eta(t) + \frac{1}{2}\,\tau_{ii}^{\eta}(t)\right)dt + \sum_{\nu=1}^{n}\big(\xi_{i\nu}(t) - \xi_{\eta\nu}(t)\big)\,dW_\nu(t)\right],$$

where the last expression follows from equations (2.2.111) and (2.2.126). Furthermore, by appealing to equation (2.5.41) of Lemma 2.5.4, we can alternatively derive the quadratic variation of the quotient process as follows for $t \in [0, \infty)$, and for all $i = 1, 2, \ldots, n$,

$$
\begin{aligned}
d\left\langle \frac{X_i}{Z_\eta} \right\rangle_t &= d\left\langle \int_0^\cdot \frac{X_{i,s}}{Z_{\eta,s}} \sum_{k=1}^n \eta_{k,s} \frac{X_{k,s}}{X_{i,s}}\, d\left(\frac{X_{i,s}}{X_{k,s}}\right), \int_0^\cdot \frac{X_{i,s}}{Z_{\eta,s}} \sum_{m=1}^n \eta_{m,s} \frac{X_{m,s}}{X_{i,s}}\, d\left(\frac{X_{i,s}}{X_{m,s}}\right) \right\rangle_t \\
&= d\left\langle \sum_{k=1}^n \int_0^\cdot \eta_{k,s} \frac{X_{k,s}}{Z_{\eta,s}}\, d\left(\frac{X_{i,s}}{X_{k,s}}\right), \sum_{m=1}^n \int_0^\cdot \eta_{m,s} \frac{X_{m,s}}{Z_{\eta,s}}\, d\left(\frac{X_{i,s}}{X_{m,s}}\right) \right\rangle_t \\
&= \sum_{k,m=1}^n d\left\langle \int_0^\cdot \eta_{k,s} \frac{X_{k,s}}{Z_{\eta,s}}\, d\left(\frac{X_{i,s}}{X_{k,s}}\right), \int_0^\cdot \eta_{m,s} \frac{X_{m,s}}{Z_{\eta,s}}\, d\left(\frac{X_{i,s}}{X_{m,s}}\right) \right\rangle_t \\
&= \sum_{k,m=1}^n \eta_k(t)\eta_m(t) \frac{X_k(t)X_m(t)}{Z_\eta^2(t)}\, d\left\langle \frac{X_i}{X_k}, \frac{X_i}{X_m} \right\rangle_t && (2.5.43) \\
&= \sum_{k,m=1}^n \eta_k(t)\eta_m(t) \left(\frac{X_i(t)}{Z_\eta(t)}\right)^2 d\left\langle \log\left(X_i/X_k\right), \log\left(X_i/X_m\right) \right\rangle_t && (2.5.44) \\
&= \left(\frac{X_i(t)}{Z_\eta(t)}\right)^2 \sum_{k,m=1}^n \eta_k(t) \tau_{ii}^{\mathsf{X}_k \mathsf{X}_m}(t) \eta_m(t)\, dt && (2.5.45) \\
&= \tau_{ii}^\eta(t) \left(\frac{X_i(t)}{Z_\eta(t)}\right)^2 dt, && (2.5.46)
\end{aligned}
$$

where (2.5.44) follows from equation (2.5.12), equation (2.5.45) follows from equation (2.3.38), and finally, equation (2.5.46) follows from equation (2.3.141). Moreover, we can also derive the covariation of the quotient process as follows for $t \in [0, \infty)$, and for all $i = 1, 2, \ldots, n$,

$$
\begin{aligned}
d\left\langle \frac{X_i}{Z_\eta}, \frac{X_j}{Z_\eta} \right\rangle_t &= d\left\langle \int_0^\cdot \frac{X_{i,s}}{Z_{\eta,s}} \sum_{k=1}^n \eta_{k,s} \frac{X_{k,s}}{X_{i,s}}\, d\left(\frac{X_{i,s}}{X_{k,s}}\right), \int_0^\cdot \frac{X_{j,s}}{Z_{\eta,s}} \sum_{m=1}^n \eta_{m,s} \frac{X_{m,s}}{X_{j,s}}\, d\left(\frac{X_{j,s}}{X_{m,s}}\right) \right\rangle_t \\
&= d\left\langle \sum_{k=1}^n \int_0^\cdot \eta_{k,s} \frac{X_{k,s}}{Z_{\eta,s}}\, d\left(\frac{X_{i,s}}{X_{k,s}}\right), \sum_{m=1}^n \int_0^\cdot \eta_{m,s} \frac{X_{m,s}}{Z_{\eta,s}}\, d\left(\frac{X_{j,s}}{X_{m,s}}\right) \right\rangle_t \\
&= \sum_{k,m=1}^n d\left\langle \int_0^\cdot \eta_{k,s} \frac{X_{k,s}}{Z_{\eta,s}}\, d\left(\frac{X_{i,s}}{X_{k,s}}\right), \int_0^\cdot \eta_{m,s} \frac{X_{m,s}}{Z_{\eta,s}}\, d\left(\frac{X_{j,s}}{X_{m,s}}\right) \right\rangle_t \\
&= \sum_{k,m=1}^n \eta_k(t)\eta_m(t) \frac{X_k(t)X_m(t)}{Z_\eta^2(t)}\, d\left\langle \frac{X_i}{X_k}, \frac{X_j}{X_m} \right\rangle_t && (2.5.47) \\
&= \sum_{k,m=1}^n \eta_k(t)\eta_m(t) \left(\frac{X_i(t)X_j(t)}{Z_\eta^2(t)}\right) d\left\langle \log\left(X_i/X_k\right), \log\left(X_j/X_m\right) \right\rangle_t && (2.5.48) \\
&= \left(\frac{X_i(t)X_j(t)}{Z_\eta^2(t)}\right) \sum_{k,m=1}^n \eta_k(t) \tau_{ij}^{\mathsf{X}_k \mathsf{X}_m}(t) \eta_m(t)\, dt && (2.5.49) \\
&= \tau_{ij}^\eta(t) \left(\frac{X_i(t)X_j(t)}{Z_\eta^2(t)}\right) dt, && (2.5.50)
\end{aligned}
$$

where (2.5.48) follows from equation (2.5.14), equation (2.5.49) follows from equation (2.3.18), and finally, equation (2.5.50) follows from equation (2.3.119).

2.6 Relative Return of Portfolios

Definition 2.6.1 (Relative Return of Portfolios). *For portfolios* $\pi = \left\{\boldsymbol{\pi}(t) = \left(\pi_1(t), \ldots, \pi_n(t)\right), t \in [0, \infty)\right\}$ *and* $\eta = \left\{\boldsymbol{\eta}(t) = \left(\eta_1(t), \ldots, \eta_n(t)\right), t \in [0, \infty)\right\}$, *the process* $\log\left(Z_\pi/Z_\eta\right) = \left\{\log\left(Z_\pi(t)/Z_\eta(t)\right), t \in\right.$

$[0,\infty)\}$, *i.e.,*

$$\log\Big(Z_\pi(t)/Z_\eta(t)\Big) \;=\; \log\Big(Z_{w,\pi}(t)/Z_{w,\eta}(t)\Big)\Big|_{w=Z_{w,\pi}(0)=Z_{w,\eta}(0)}, \qquad t\in[0,\infty), \tag{2.6.1}$$

is called the ***relative return process*** *of π versus η.*

Let us consider the structure of the relative return process. Suppose that π and η are portfolios, then recalling equation (2.2.112) in Corollary 2.2.25, for $t\in[0,\infty)$, a.s., we have the following

$$d\log Z_\pi(t) \;=\; \sum_{i=1}^n \pi_i(t)\,d\log X_i(t) + \gamma_\pi^*(t)\,dt, \quad \text{and,} \tag{2.6.2}$$

$$d\log Z_\eta(t) \;=\; \sum_{i=1}^n \eta_i(t)\,d\log X_i(t) + \gamma_\eta^*(t)\,dt. \tag{2.6.3}$$

Consequently, a.s., for $t\in[0,\infty)$, we obtain

$$\begin{aligned}
d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) &= d\log Z_\pi(t) - d\log Z_\eta(t)\\
&= \Big(\sum_{i=1}^n \pi_i(t)\,d\log X_i(t) + \gamma_\pi^*(t)\,dt\Big) - d\log Z_\eta(t)\Big(\sum_{i=1}^n \pi_i(t)\Big)\\
&= \sum_{i=1}^n \pi_i(t)\,d\log X_i(t) - \sum_{i=1}^n \pi_i(t)\,d\log Z_\eta(t) + \gamma_\pi^*(t)\,dt\\
&= \sum_{i=1}^n \pi_i(t)\Big[d\log X_i(t) - d\log Z_\eta(t)\Big] + \gamma_\pi^*(t)\,dt\\
&= \sum_{i=1}^n \pi_i(t)\,d\log\Big(X_i(t)/Z_\eta(t)\Big) + \gamma_\pi^*(t)\,dt.
\end{aligned} \tag{2.6.4}$$

In a similar fashion, we have

$$d\log\Big(Z_\eta(t)/Z_\pi(t)\Big) \;=\; \sum_{i=1}^n \eta_i(t)\,d\log\Big(X_i(t)/Z_\pi(t)\Big) + \gamma_\eta^*(t)\,dt, \quad \text{and,} \tag{2.6.5}$$

$$d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) \;=\; \sum_{i=1}^n \eta_i(t)\,d\log\Big(Z_\pi(t)/X_i(t)\Big) - \gamma_\eta^*(t)\,dt. \tag{2.6.6}$$

Lemma 2.6.2. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t\in[0,\infty)$, we obtain the dynamics*

$$d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) \;=\; \big(\gamma_\pi(t)-\gamma_\eta(t)\big)\,dt + \sum_{i,\nu=1}^n \big(\pi_i(t)-\eta_i(t)\big)\xi_{i\nu}(t)\,dW_\nu(t), \tag{2.6.7}$$

for the relative return of portfolio π versus portfolio η.

Proof. This result is recovered easily by enlisting the use of equation (2.6.4) in conjunction with equation

(2.3.42), we thus obtain

$$\begin{aligned}
d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) &= \gamma_\pi^*(t)\,dt + \sum_{i=1}^n \pi_i(t)\,d\log\Big(X_i(t)/Z_\eta(t)\Big) \\
&= \gamma_\pi^*(t)\,dt + \sum_{i=1}^n \pi_i(t)\Big[\big(\gamma_i(t)-\gamma_\eta(t)\big)\,dt + \sum_{j,\nu=1}^n \eta_j(t)\big(\xi_{i\nu}(t)-\xi_{j\nu}(t)\big)\,dW_\nu(t)\Big] \\
&= \Big(\gamma_\pi^*(t) + \sum_{i=1}^n \pi_i(t)\gamma_i(t) - \gamma_\eta(t)\Big)\,dt + \sum_{\nu=1}^n\Big[\sum_{i,j=1}^n \pi_i(t)\eta_j(t)\big(\xi_{i\nu}(t)-\xi_{j\nu}(t)\big)\,dW_\nu(t)\Big] \\
&= \big(\gamma_\pi(t)-\gamma_\eta(t)\big)\,dt + \sum_{\nu=1}^n\Big[\sum_{i=1}^n \pi_i(t)\xi_{i\nu}(t) - \sum_{j=1}^n \eta_j(t)\xi_{j\nu}(t)\Big]\,dW_\nu(t) \\
&= \big(\gamma_\pi(t)-\gamma_\eta(t)\big)\,dt + \sum_{\nu=1}^n\Big[\sum_{i=1}^n \pi_i(t)\xi_{i\nu}(t) - \sum_{i=1}^n \eta_i(t)\xi_{i\nu}(t)\Big]\,dW_\nu(t) \\
&= \big(\gamma_\pi(t)-\gamma_\eta(t)\big)\,dt + \sum_{i,\nu=1}^n \big(\pi_i(t)-\eta_i(t)\big)\xi_{i\nu}(t)\,dW_\nu(t), \qquad (2.6.8)
\end{aligned}$$

where equation (2.6.8) follows from equation (2.2.111). Alternatively, employing equation (2.2.65) of Proposition 2.2.20 yields

$$\begin{aligned}
d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) &= d\log Z_\pi(t) - d\log Z_\eta(t) \\
&= \Big(\gamma_\pi(t)\,dt + \sum_{i,\nu=1}^n \pi_i(t)\xi_{i\nu}(t)\,dW_\nu(t)\Big) - \Big(\gamma_\eta(t)\,dt + \sum_{i,\nu=1}^n \eta_i(t)\xi_{i\nu}(t)\,dW_\nu(t)\Big) \\
&= \big(\gamma_\pi(t)-\gamma_\eta(t)\big)\,dt + \sum_{i,\nu=1}^n \big(\pi_i(t)-\eta_i(t)\big)\xi_{i\nu}(t)\,dW_\nu(t).
\end{aligned}$$

■

Lemma 2.6.3. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t \in [0,\infty)$, we obtain the dynamics*

$$d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) = \big(\gamma_\pi(t)-\gamma_\eta(t)\big)\,dt + \sum_{i,\nu=1}^n \pi_i(t)\big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t), \qquad (2.6.9)$$

for the relative return of portfolio π versus portfolio η.

Proof. This result is recovered easily by enlisting the use of equation (2.6.4) in conjunction with equation (2.3.43), we thus obtain

$$\begin{aligned}
d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) &= \gamma_\pi^*(t)\,dt + \sum_{i=1}^n \pi_i(t)\,d\log\Big(X_i(t)/Z_\eta(t)\Big) \\
&= \gamma_\pi^*(t)\,dt + \sum_{i=1}^n \pi_i(t)\Big[\big(\gamma_i(t)-\gamma_\eta(t)\big)\,dt + \sum_{\nu=1}^n \big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t)\Big] \\
&= \Big(\gamma_\pi^*(t) + \sum_{i=1}^n \pi_i(t)\gamma_i(t) - \gamma_\eta(t)\Big)\,dt + \sum_{i,\nu=1}^n \pi_i(t)\big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t) \\
&= \big(\gamma_\pi(t)-\gamma_\eta(t)\big)\,dt + \sum_{i,\nu=1}^n \pi_i(t)\big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t), \qquad (2.6.10)
\end{aligned}$$

where equation (2.6.10) follows from equation (2.2.111). Alternatively, simplifying equation (2.6.7) of Lemma

2.6.2 using equation (2.2.126) of Definition 2.2.28, gives

$$\begin{aligned}
d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) &= \big(\gamma_\pi(t)-\gamma_\eta(t)\big)\,dt+\sum_{i,\nu=1}^{n}\big(\pi_i(t)-\eta_i(t)\big)\xi_{i\nu}(t)\,dW_\nu(t)\\
&= \big(\gamma_\pi(t)-\gamma_\eta(t)\big)\,dt+\sum_{\nu=1}^{n}\Bigg[\sum_{i=1}^{n}\pi_i(t)\xi_{i\nu}(t)-\sum_{i=1}^{n}\eta_i(t)\xi_{i\nu}(t)\Bigg]\,dW_\nu(t)\\
&= \big(\gamma_\pi(t)-\gamma_\eta(t)\big)\,dt+\sum_{\nu=1}^{n}\Bigg[\sum_{i=1}^{n}\pi_i(t)\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\Bigg]\,dW_\nu(t)\\
&= \big(\gamma_\pi(t)-\gamma_\eta(t)\big)\,dt+\sum_{i,\nu=1}^{n}\pi_i(t)\big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t).
\end{aligned}$$

Moreover, employing equation (2.2.65) along with equation (2.2.131), yields the same result

$$\begin{aligned}
d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) &= d\log Z_\pi(t)-d\log Z_\eta(t)\\
&= \Bigg(\gamma_\pi(t)\,dt+\sum_{i,\nu=1}^{n}\pi_i(t)\xi_{i\nu}(t)\,dW_\nu(t)\Bigg)-\Bigg(\gamma_\eta(t)\,dt+\sum_{\nu=1}^{n}\xi_{\eta\nu}(t)\,dW_\nu(t)\Bigg)\\
&= \big(\gamma_\pi(t)-\gamma_\eta(t)\big)\,dt+\sum_{i,\nu=1}^{n}\pi_i(t)\big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t).
\end{aligned}$$

■

Lemma 2.6.4. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t\in[0,\infty)$, we obtain the dynamics*

$$d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) = \big(\gamma_\pi(t)-\gamma_\eta(t)\big)\,dt+\sum_{\nu=1}^{n}\big(\xi_{\pi\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t), \tag{2.6.11}$$

for the relative return of portfolio π versus portfolio η.

Proof. This result follows easily from equation (2.6.7) of Lemma 2.6.2, which, when allied with (2.2.126) of Definition 2.2.28, yields the desired result

$$\begin{aligned}
d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) &= \big(\gamma_\pi(t)-\gamma_\eta(t)\big)\,dt+\sum_{i,\nu=1}^{n}\big(\pi_i(t)-\eta_i(t)\big)\xi_{i\nu}(t)\,dW_\nu(t)\\
&= \big(\gamma_\pi(t)-\gamma_\eta(t)\big)\,dt+\sum_{\nu=1}^{n}\Bigg[\sum_{i=1}^{n}\pi_i(t)\xi_{i\nu}(t)-\sum_{i=1}^{n}\eta_i(t)\xi_{i\nu}(t)\Bigg]\,dW_\nu(t)\\
&= \big(\gamma_\pi(t)-\gamma_\eta(t)\big)\,dt+\sum_{\nu=1}^{n}\big(\xi_{\pi\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t).
\end{aligned}$$

Alternatively, simplifying equation (2.6.9) of Lemma 2.6.3 using equation (2.2.126) of Definition 2.2.28, gives

$$\begin{aligned}
d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) &= \big(\gamma_\pi(t)-\gamma_\eta(t)\big)\,dt+\sum_{i,\nu=1}^{n}\pi_i(t)\big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t)\\
&= \big(\gamma_\pi(t)-\gamma_\eta(t)\big)\,dt+\sum_{\nu=1}^{n}\Bigg[\sum_{i=1}^{n}\pi_i(t)\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\Bigg]\,dW_\nu(t)\\
&= \big(\gamma_\pi(t)-\gamma_\eta(t)\big)\,dt+\sum_{\nu=1}^{n}\big(\xi_{\pi\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t).
\end{aligned}$$

Moreover, employing equation (2.2.131) yields

$$\begin{aligned} d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) &= d\log Z_\pi(t) - d\log Z_\eta(t) \\ &= \Big(\gamma_\pi(t)\,dt + \sum_{\nu=1}^n \xi_{\pi\nu}(t)\,dW_\nu(t)\Big) - \Big(\gamma_\eta(t)\,dt + \sum_{\nu=1}^n \xi_{\eta\nu}(t)\,dW_\nu(t)\Big) \\ &= \big(\gamma_\pi(t) - \gamma_\eta(t)\big)\,dt + \sum_{\nu=1}^n \big(\xi_{\pi\nu}(t) - \xi_{\eta\nu}(t)\big)\,dW_\nu(t). \end{aligned}$$

■

Lemma 2.6.5. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t \in [0,\infty)$, we obtain the dynamics*

$$d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) = \big(\gamma_\pi^*(t) - \gamma_\eta^*(t)\big)\,dt + \sum_{i,j=1}^n \pi_i(t)\eta_j(t)\,d\log\Big(X_i(t)/X_j(t)\Big), \tag{2.6.12}$$

for the relative return of portfolio π versus portfolio η.

Proof. From equation (2.6.4) and equation (2.3.45) of Lemma 2.3.6, we get the result

$$\begin{aligned} d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) &= \sum_{i=1}^n \pi_i(t)\Big[\sum_{j=1}^n \eta_j(t)\,d\log\Big(X_i(t)/X_j(t)\Big) - \gamma_\eta^*(t)\,dt\Big] + \gamma_\pi^*(t)\,dt \\ &= \big(\gamma_\pi^*(t) - \gamma_\eta^*(t)\big)\,dt + \sum_{i,j=1}^n \pi_i(t)\eta_j(t)\,d\log\Big(X_i(t)/X_j(t)\Big). \end{aligned}$$

Alternatively, by employing equation (2.2.112) of Corollary 2.2.25, we obtain the equivalent result

$$\begin{aligned} d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) &= d\log Z_\pi(t) - d\log Z_\eta(t) \\ &= \Big(\sum_{i=1}^n \pi_i(t)\,d\log X_i(t) + \gamma_\pi^*(t)\,dt\Big) - \Big(\sum_{j=1}^n \eta_j(t)\,d\log X_j(t) + \gamma_\eta^*(t)\,dt\Big) \\ &= \big(\gamma_\pi^*(t) - \gamma_\eta^*(t)\big)\,dt + \sum_{i=1}^n \pi_i(t)\,d\log X_i(t)\Big[\sum_{j=1}^n \eta_j(t)\Big] - \sum_{j=1}^n \eta_j(t)\,d\log X_j(t)\Big[\sum_{i=1}^n \pi_i(t)\Big] \\ &= \big(\gamma_\pi^*(t) - \gamma_\eta^*(t)\big)\,dt + \sum_{i,j=1}^n \pi_i(t)\eta_j(t)\,d\log X_i(t) - \sum_{i,j=1}^n \pi_i(t)\eta_j(t)\,d\log X_j(t) \\ &= \big(\gamma_\pi^*(t) - \gamma_\eta^*(t)\big)\,dt + \sum_{i,j=1}^n \pi_i(t)\eta_j(t)\Big[d\log X_i(t) - d\log X_j(t)\Big] \\ &= \big(\gamma_\pi^*(t) - \gamma_\eta^*(t)\big)\,dt + \sum_{i,j=1}^n \pi_i(t)\eta_j(t)\,d\log\Big(X_i(t)/X_j(t)\Big). \end{aligned}$$

■

We can reexpress equation (2.6.12) as

$$d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) = \big(\gamma_\pi^*(t) - \gamma_\eta^*(t)\big)\,dt + \sum_{i,j=1}^n \pi_i(t)\eta_j(t)\,d\log\Big(Z_{\chi_i}(t)/Z_{\chi_j}(t)\Big). \tag{2.6.13}$$

We can reaffirm equation (2.6.11) of Lemma 2.6.4, by substituting equation (2.3.1) into equation (2.6.12). Thus,

we have

$$
\begin{aligned}
& d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) \\
&\quad = \big(\gamma_\pi^*(t)-\gamma_\eta^*(t)\big)\,dt + \sum_{i,j=1}^n \pi_i(t)\eta_j(t)\Big[\big(\gamma_i(t)-\gamma_j(t)\big)\,dt + \sum_{\nu=1}^n \big(\xi_{i\nu}(t)-\xi_{j\nu}(t)\big)\,dW_\nu(t)\Big] \\
&\quad = \Big(\gamma_\pi^*(t) + \sum_{i=1}^n \pi_i(t)\gamma_i(t) - \gamma_\eta^*(t) - \sum_{j=1}^n \eta_j(t)\gamma_j(t)\Big)\,dt + \sum_{\nu=1}^n \Big[\sum_{i=1}^n \pi_i(t)\xi_{i\nu}(t) - \sum_{j=1}^n \eta_j(t)\xi_{j\nu}(t)\Big]\,dW_\nu(t) \\
&\quad = \big(\gamma_\pi(t)-\gamma_\eta(t)\big)\,dt + \sum_{\nu=1}^n \big(\xi_{\pi\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t),
\end{aligned}
$$

where the last expression follows from equations (2.2.111) and (2.2.126).

Lemma 2.6.6. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t \in [0,\infty)$, we obtain the dynamics*

$$
d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) = \Big(-\frac{1}{2}\,\tau_{\pi\pi}^\eta(t) - 2\,\gamma_\eta^*(t)\Big)\,dt + \sum_{i,j=1}^n \pi_i(t)\eta_j(t)\,\frac{X_j(t)}{X_i(t)}\,d\Big(\frac{X_i(t)}{X_j(t)}\Big), \tag{2.6.14}
$$

for the relative return of portfolio π versus portfolio η.

Proof. From equation (2.5.42) and equation (2.6.12) of Lemma 2.6.5, we obtain

$$
d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) = \big(\gamma_\pi^*(t)-\gamma_\eta^*(t)\big)\,dt + \sum_{i,j=1}^n \pi_i(t)\eta_j(t)\Big[\frac{d\big(X_i(t)/X_j(t)\big)}{X_i(t)/X_j(t)} - \frac{1}{2}\,\tau_{ii}^{\chi_j}(t)\,dt\Big]. \tag{2.6.15}
$$

From equations (2.4.31) and (2.4.19) of Lemma 2.4.4, in the above expression, we have

$$
\begin{aligned}
\sum_{i,j=1}^n \pi_i(t)\tau_{ii}^{\chi_j}(t)\eta_j(t) &= \sum_{j=1}^n \eta_j(t)\Big[\sum_{i=1}^n \pi_i(t)\tau_{ii}^{\chi_j}(t)\Big] \\
&= \sum_{j=1}^n \eta_j(t)\tau_{\pi\pi}^{\chi_j}(t) + 2\,\gamma_\pi^*(t) \\
&= \sum_{j=1}^n \eta_j(t)\tau_{jj}^{\pi}(t) + 2\,\gamma_\pi^*(t) \\
&= \tau_{\eta\eta}^{\pi}(t) + 2\,\gamma_\pi^*(t) + 2\,\gamma_\eta^*(t) \\
&= \tau_{\pi\pi}^{\eta}(t) + 2\,\gamma_\pi^*(t) + 2\,\gamma_\eta^*(t).
\end{aligned}
$$

Inserting the above result into equation (2.6.15) yields the desired result. ■

Lemma 2.6.7. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t \in [0,\infty)$, we obtain the dynamics*

$$
d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) = \big(\gamma_\pi^*(t)-\gamma_\eta^*(t)\big)\,dt + \sum_{i=1}^n \big(\pi_i(t)-\eta_i(t)\big)\,d\log\Big(X_i(t)/Z_\eta(t)\Big), \tag{2.6.16}
$$

for the relative return of portfolio π versus portfolio η.

Proof. By employing equations (2.3.43), (2.2.111) and (2.2.126), we determine the following

$$
\begin{aligned}
\sum_{i=1}^n \eta_i(t)\,d\log\Big(X_i(t)/Z_\eta(t)\Big) &= \sum_{i=1}^n \eta_i(t)\Big[\big(\gamma_i(t)-\gamma_\eta(t)\big)\,dt + \sum_{\nu=1}^n \big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t)\Big] \\
&= \Big(\sum_{i=1}^n \eta_i(t)\gamma_i(t) - \gamma_\eta(t)\Big)\,dt + \sum_{\nu=1}^n \Big[\sum_{i=1}^n \eta_i(t)\xi_{i\nu}(t) - \xi_{\eta\nu}(t)\Big]\,dW_\nu(t) \\
&= -\gamma_\eta^*(t)\,dt + \sum_{\nu=1}^n \big(\xi_{\eta\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t) \\
&= -\gamma_\eta^*(t)\,dt.
\end{aligned}
$$

Therefore, a.s., for $t \in [0, \infty)$, we have

$$\gamma_\eta^*(t)\,dt + \sum_{i=1}^n \eta_i(t)\,d\log\Big(X_i(t)/Z_\eta(t)\Big) \quad = \quad 0. \tag{2.6.17}$$

Consequently, subtracting the left-hand side of the above result from the right-hand side of equation (2.6.4), establishes the required result, since the above result in no way alters equation (2.6.4),

$$\begin{aligned} d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) &= \sum_{i=1}^n \pi_i(t)\,d\log\Big(X_i(t)/Z_\eta(t)\Big) + \gamma_\pi^*(t)\,dt - \Big(\gamma_\eta^*(t)\,dt + \sum_{i=1}^n \eta_i(t)\,d\log\Big(X_i(t)/Z_\eta(t)\Big)\Big) \\ &= \big(\gamma_\pi^*(t) - \gamma_\eta^*(t)\big)\,dt + \sum_{i=1}^n \big(\pi_i(t) - \eta_i(t)\big)\,d\log\Big(X_i(t)/Z_\eta(t)\Big). \end{aligned}$$

■

We shall again verify that equation (2.6.16) of Lemma 2.6.7 results in equation (2.6.12) of Lemma 2.6.5. Firstly, we shall require the following relation obtained by using equation (2.3.1),

$$\begin{aligned} &\sum_{i,j=1}^n \eta_i(t)\eta_j(t)\,d\log\Big(X_i(t)/X_j(t)\Big) \\ &\quad= \sum_{i,j=1}^n \eta_i(t)\eta_j(t)\Big[\big(\gamma_i(t) - \gamma_j(t)\big)\,dt + \sum_{\nu=1}^n \big(\xi_{i\nu}(t) - \xi_{j\nu}(t)\big)\,dW_\nu(t)\Big] \\ &\quad= \Big(\sum_{i=1}^n \eta_i(t)\gamma_i(t) - \sum_{j=1}^n \eta_j(t)\gamma_j(t)\Big)\,dt + \sum_{\nu=1}^n \Big[\sum_{i=1}^n \eta_i(t)\xi_{i\nu}(t) - \sum_{j=1}^n \eta_j(t)\xi_{j\nu}(t)\Big]\,dW_\nu(t) \\ &\quad= 0. \end{aligned}$$

Consequently, employing the above result and enlisting equation (2.3.45), we have

$$\begin{aligned} d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) &= \big(\gamma_\pi^*(t) - \gamma_\eta^*(t)\big)\,dt + \sum_{i=1}^n \big(\pi_i(t) - \eta_i(t)\big)\Big[\sum_{j=1}^n \eta_j(t)\,d\log\Big(X_i(t)/X_j(t)\Big) - \gamma_\eta^*(t)\,dt\Big] \\ &= \big(\gamma_\pi^*(t) - \gamma_\eta^*(t)\big)\,dt + \sum_{i,j=1}^n \pi_i(t)\eta_j(t)\,d\log\Big(X_i(t)/X_j(t)\Big) \\ &\quad - \sum_{i,j=1}^n \eta_i(t)\eta_j(t)\,d\log\Big(X_i(t)/X_j(t)\Big) - \gamma_\eta^*(t)\,dt + \gamma_\eta^*(t)\,dt \\ &= \big(\gamma_\pi^*(t) - \gamma_\eta^*(t)\big)\,dt + \sum_{i,j=1}^n \pi_i(t)\eta_j(t)\,d\log\Big(X_i(t)/X_j(t)\Big). \end{aligned}$$

We shall do the same as above, but now verifying equation (2.6.11) of Lemma 2.6.4 using equation (2.3.43), as follows

$$\begin{aligned} &d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) \\ &\quad= \big(\gamma_\pi^*(t) - \gamma_\eta^*(t)\big)\,dt + \sum_{i=1}^n \big(\pi_i(t) - \eta_i(t)\big)\Big[\big(\gamma_i(t) - \gamma_\eta(t)\big)\,dt + \sum_{\nu=1}^n \big(\xi_{i\nu}(t) - \xi_{\eta\nu}(t)\big)dW_\nu(t)\Big] \\ &\quad= \Big(\gamma_\pi^*(t) + \sum_{i=1}^n \pi_i(t)\gamma_i(t) - \gamma_\eta^*(t) - \sum_{i=1}^n \eta_i(t)\gamma_i(t) - \gamma_\eta(t) + \gamma_\eta(t)\Big)\,dt \\ &\qquad + \sum_{\nu=1}^n \Big[\sum_{i=1}^n \pi_i(t)\xi_{i\nu}(t) - \xi_{\eta\nu}(t) - \sum_{i=1}^n \eta_i(t)\xi_{i\nu}(t) + \xi_{\eta\nu}(t)\Big]\,dW_\nu(t) \\ &\quad= \big(\gamma_\pi(t) - \gamma_\eta(t)\big)\,dt + \sum_{\nu=1}^n \big(\xi_{\pi\nu}(t) - \xi_{\eta\nu}(t)\big)\,dW_\nu(t). \end{aligned}$$

Lemma 2.6.8. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t \in [0, \infty)$ we obtain the dynamics*

$$d \log \Big(Z_\pi(t)/Z_\eta(t) \Big) = \sum_{i=1}^n \pi_i(t) \frac{Z_\eta(t)}{X_i(t)} \, d \left(\frac{X_i(t)}{Z_\eta(t)} \right) - \frac{1}{2} \, \tau_{\pi\pi}^\eta(t) \, dt, \tag{2.6.18}$$

for the relative return of portfolio π versus portfolio η.

Proof. From equation (2.5.27), the relative logarithmic return can be expressed as follows

$$d \log \Big(X_i(t)/Z_\eta(t) \Big) = \frac{d \left(X_i(t)/Z_\eta(t) \right)}{X_i(t)/Z_\eta(t)} - \frac{1}{2} \, \tau_{ii}^\eta(t) \, dt, \tag{2.6.19}$$

which, together with the numéraire invariance property (2.4.29), yields the following representation for the relative return of π versus η

$$\begin{aligned} d \log \Big(Z_\pi(t)/Z_\eta(t) \Big) &= \gamma_\pi^*(t) \, dt + \sum_{i=1}^n \pi_i(t) \, d \log \Big(X_i(t)/Z_\eta(t) \Big) \\ &= \frac{1}{2} \left(\sum_{i=1}^n \pi_i(t) \tau_{ii}^\eta(t) - \tau_{\pi\pi}^\eta(t) \right) dt + \sum_{i=1}^n \pi_i(t) \left[\frac{d \left(X_i(t)/Z_\eta(t) \right)}{X_i(t)/Z_\eta(t)} - \frac{1}{2} \, \tau_{ii}^\eta(t) \, dt \right] \\ &= \frac{1}{2} \sum_{i=1}^n \pi_i(t) \tau_{ii}^\eta(t) \, dt - \frac{1}{2} \, \tau_{\pi\pi}^\eta(t) \, dt + \sum_{i=1}^n \pi_i(t) \frac{Z_\eta(t)}{X_i(t)} \, d \left(\frac{X_i(t)}{Z_\eta(t)} \right) - \frac{1}{2} \sum_{i=1}^n \pi_i(t) \tau_{ii}^\eta(t) \, dt. \end{aligned}$$

The first and last terms cancel, establishing the result. ■

Equation (2.6.18) can be recaptured in the following more insightful form, which will adopted at a later stage

$$d \log \Big(Z_\pi(t)/Z_\eta(t) \Big) = \sum_{i=1}^n \left(\frac{\pi_i(t)}{X_i(t)/Z_\eta(t)} \right) d \left(\frac{X_i(t)}{Z_\eta(t)} \right) - \frac{1}{2} \sum_{i,j=1}^n \pi_i(t) \tau_{ij}^\eta(t) \pi_j(t) \, dt. \tag{2.6.20}$$

We shall verify the result (2.6.18) of Lemma 2.6.8, by appealing to (2.6.16) of Lemma 2.6.7. From equation (2.6.16) together with equations (2.6.19), (2.4.31) and (2.4.32) of Lemma 2.4.7, we obtain

$$\begin{aligned} & d \log \Big(Z_\pi(t)/Z_\eta(t) \Big) \\ &= \big(\gamma_\pi^*(t) - \gamma_\eta^*(t) \big) \, dt + \sum_{i=1}^n \big(\pi_i(t) - \eta_i(t) \big) \left[\frac{d \left(X_i(t)/Z_\eta(t) \right)}{X_i(t)/Z_\eta(t)} - \frac{1}{2} \, \tau_{ii}^\eta(t) \, dt \right] \\ &= \big(\gamma_\pi^*(t) - \gamma_\eta^*(t) \big) \, dt + \sum_{i=1}^n \big(\pi_i(t) - \eta_i(t) \big) \frac{Z_\eta(t)}{X_i(t)} \, d \left(\frac{X_i(t)}{Z_\eta(t)} \right) - \frac{1}{2} \left(\sum_{i=1}^n \pi_i(t) \tau_{ii}^\eta(t) - \sum_{i=1}^n \eta_i(t) \tau_{ii}^\eta(t) \right) dt \\ &= \big(\gamma_\pi^*(t) - \gamma_\eta^*(t) \big) \, dt + \sum_{i=1}^n \big(\pi_i(t) - \eta_i(t) \big) \frac{Z_\eta(t)}{X_i(t)} \, d \left(\frac{X_i(t)}{Z_\eta(t)} \right) - \frac{1}{2} \Big(2 \, \gamma_\pi^*(t) + \tau_{\pi\pi}^\eta(t) - 2 \, \gamma_\eta^*(t) \Big) \, dt \\ &= \sum_{i=1}^n \big(\pi_i(t) - \eta_i(t) \big) \frac{Z_\eta(t)}{X_i(t)} \, d \left(\frac{X_i(t)}{Z_\eta(t)} \right) - \frac{1}{2} \, \tau_{\pi\pi}^\eta(t) \, dt \qquad (2.6.21) \\ &= \sum_{i=1}^n \pi_i(t) \frac{Z_\eta(t)}{X_i(t)} \, d \left(\frac{X_i(t)}{Z_\eta(t)} \right) - \sum_{i=1}^n \eta_i(t) \frac{Z_\eta(t)}{X_i(t)} \, d \left(\frac{X_i(t)}{Z_\eta(t)} \right) - \frac{1}{2} \, \tau_{\pi\pi}^\eta(t) \, dt. \qquad (2.6.22) \end{aligned}$$

The second term in the expression above can be simplified by making use of equation (2.5.26) and equation (2.4.32) of Lemma 2.4.7

$$\begin{aligned} \sum_{i=1}^n \eta_i(t) \frac{Z_\eta(t)}{X_i(t)} \, d \left(\frac{X_i(t)}{Z_\eta(t)} \right) &= \sum_{i=1}^n \eta_i(t) \frac{Z_\eta(t)}{X_i(t)} \left[\frac{1}{2} \, \frac{X_i(t)}{Z_\eta(t)} \, \tau_{ii}^\eta(t) \, dt + \frac{X_i(t)}{Z_\eta(t)} \, d \log \Big(X_i(t)/Z_\eta(t) \Big) \right] \\ &= \frac{1}{2} \sum_{i=1}^n \eta_i(t) \tau_{ii}^\eta(t) \, dt + \sum_{i=1}^n \eta_i(t) \, d \log \Big(X_i(t)/Z_\eta(t) \Big) \\ &= \gamma_\eta^*(t) \, dt + \sum_{i=1}^n \eta_i(t) \, d \log \Big(X_i(t)/Z_\eta(t) \Big). \end{aligned}$$

Therefore, by recalling (2.6.17), we get

$$\sum_{i=1}^{n} \eta_i(t) \frac{Z_\eta(t)}{X_i(t)}\, d\left(\frac{X_i(t)}{Z_\eta(t)}\right) \quad = \quad 0. \tag{2.6.23}$$

Thus, equation (2.6.23) allied with equation (2.4.24), results in the reduction of equation (2.6.22) to equation (2.6.18), which is precisely what we set out to show. Now, using (2.5.32), we shall show that (2.6.18) of Lemma 2.6.8 reduces to (2.6.11) of Lemma 2.6.4.

$$\begin{aligned}
&d \log\Big(Z_\pi(t)/Z_\eta(t)\Big) \\
&= \sum_{i=1}^{n} \pi_i(t) \frac{Z_\eta(t)}{X_i(t)} \left[\frac{X_i(t)}{Z_\eta(t)} \left(\gamma_i(t) - \gamma_\eta(t) + \frac{1}{2}\, \tau_{ii}^{\eta}(t) \right) dt + \frac{X_i(t)}{Z_\eta(t)} \sum_{\nu=1}^{n} \big(\xi_{i\nu}(t) - \xi_{\eta\nu}(t)\big)\, dW_\nu(t) \right] - \frac{1}{2}\, \tau_{\pi\pi}^{\eta}(t)\, dt \\
&= \left(\sum_{i=1}^{n} \pi_i(t)\gamma_i(t) - \gamma_\eta(t) + \frac{1}{2} \sum_{i=1}^{n} \pi_i(t)\tau_{ii}^{\eta}(t) - \frac{1}{2}\, \tau_{\pi\pi}^{\eta}(t) \right) dt + \sum_{\nu=1}^{n} \left[\sum_{i=1}^{n} \pi_i(t)\xi_{i\nu}(t) - \xi_{\eta\nu}(t) \right] dW_\nu(t) \\
&= \left(\sum_{i=1}^{n} \pi_i(t)\gamma_i(t) - \gamma_\eta(t) + \frac{1}{2} \Big(2\, \gamma_\pi^*(t) + \tau_{\pi\pi}^{\eta}(t) \Big) - \frac{1}{2}\, \tau_{\pi\pi}^{\eta}(t) \right) dt + \sum_{\nu=1}^{n} \big(\xi_{\pi\nu}(t) - \xi_{\eta\nu}(t)\big)\, dW_\nu(t) \qquad (2.6.24) \\
&= \big(\gamma_\pi(t) - \gamma_\eta(t)\big)\, dt + \sum_{\nu=1}^{n} \big(\xi_{\pi\nu}(t) - \xi_{\eta\nu}(t)\big)\, dW_\nu(t), \qquad (2.6.25)
\end{aligned}$$

where equation (2.6.24) follows from equations (2.4.31) and (2.2.126), and equation (2.6.25) follows from equation (2.2.111).

Now, the quadratic variation of the relative return process can be determined a.s., for $t \in [0, \infty)$, as follows

$$\begin{aligned}
\big\langle \log\big(Z_\pi/Z_\eta\big) \big\rangle_t &= \big\langle \log Z_\pi - \log Z_\eta \big\rangle_t \\
&= \big\langle \log Z_\pi \big\rangle_t - \big\langle \log Z_\pi, \log Z_\eta \big\rangle_t - \big\langle \log Z_\eta, \log Z_\pi \big\rangle_t + \big\langle \log Z_\eta \big\rangle_t \qquad (2.6.26) \\
&= \big\langle \log Z_\pi \big\rangle_t - 2 \big\langle \log Z_\pi, \log Z_\eta \big\rangle_t + \big\langle \log Z_\eta \big\rangle_t. \qquad (2.6.27)
\end{aligned}$$

Therefore, for $t \in [0, \infty)$, a.s., we have

$$\begin{aligned}
d \big\langle \log\big(Z_\pi/Z_\eta\big) \big\rangle_t &= d \big\langle \log Z_\pi \big\rangle_t - d \big\langle \log Z_\pi, \log Z_\eta \big\rangle_t - d \big\langle \log Z_\eta, \log Z_\pi \big\rangle_t + d \big\langle \log Z_\eta \big\rangle_t \qquad (2.6.28) \\
&= d \big\langle \log Z_\pi \big\rangle_t - 2\, d \big\langle \log Z_\pi, \log Z_\eta \big\rangle_t + d \big\langle \log Z_\eta \big\rangle_t. \qquad (2.6.29)
\end{aligned}$$

Thus, we require $\big\langle \log Z_\pi, \log Z_\eta \big\rangle$, this is calculated by employing equation (2.2.65) together with equations

(2.2.37) and (2.3.51), as follows

$$
\begin{aligned}
\left\langle \log Z_\pi, \log Z_\eta \right\rangle_t &= \left\langle \int_0^\cdot \sum_{i,\nu=1}^n \pi_{i,s}\xi_{i\nu,s}\, dW_{\nu,s}, \int_0^\cdot \sum_{j,\upsilon=1}^n \eta_{j,s}\xi_{j\upsilon,s}\, dW_{\upsilon,s} \right\rangle_t \\
&= \left\langle \sum_{\nu=1}^n \int_0^\cdot \sum_{i=1}^n \pi_{i,s}\xi_{i\nu,s}\, dW_{\nu,s}, \sum_{\upsilon=1}^n \int_0^\cdot \sum_{j=1}^n \eta_{j,s}\xi_{j\upsilon,s}\, dW_{\upsilon,s} \right\rangle_t \\
&= \sum_{\nu,\upsilon=1}^n \left\langle \int_0^\cdot \sum_{i=1}^n \pi_{i,s}\xi_{i\nu,s}\, dW_{\nu,s}, \int_0^\cdot \sum_{j=1}^n \eta_{j,s}\xi_{j\upsilon,s}\, dW_{\upsilon,s} \right\rangle_t \\
&= \sum_{\nu,\upsilon=1}^n \int_0^t \sum_{i,j=1}^n \pi_i(s)\eta_j(s)\xi_{i\nu}(s)\xi_{j\upsilon}(s)\, d\left\langle W_\nu, W_\upsilon \right\rangle_s \\
&= \int_0^t \sum_{i,j=1}^n \pi_i(s)\eta_j(s) \sum_{\nu,\upsilon=1}^n \xi_{i\nu}(s)\xi_{j\upsilon}(s)\, \rho_{\nu\upsilon}(s)\, ds \\
&= \int_0^t \sum_{i,j=1}^n \pi_i(s)\eta_j(s) \sum_{\nu=1}^n \xi_{i\nu}(s)\xi_{j\nu}(s)\, ds \qquad (2.6.30) \\
&= \int_0^t \sum_{i,j=1}^n \pi_i(s)\sigma_{ij}(s)\eta_j(s)\, ds \qquad (2.6.31) \\
&= \int_0^t \sum_{i=1}^n \pi_i(s) \left[\sum_{j=1}^n \eta_j(s)\sigma_{ij}(s) \right] ds \qquad (2.6.32) \\
&= \int_0^t \sum_{i=1}^n \pi_i(s)\sigma_{i\eta}(s)\, ds. \qquad (2.6.33)
\end{aligned}
$$

Alternatively, we have

$$
\begin{aligned}
\left\langle \log Z_\pi, \log Z_\eta \right\rangle_t &= \int_0^t \sum_{j=1}^n \eta_j(s) \left[\sum_{i=1}^n \pi_i(s)\sigma_{ji}(s) \right] ds \qquad (2.6.34) \\
&= \int_0^t \sum_{j=1}^n \eta_j(s)\sigma_{j\pi}(s)\, ds. \qquad (2.6.35)
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
d\left\langle \log Z_\pi, \log Z_\eta \right\rangle_t &= d\left\langle \int_0^\cdot \sum_{i,\nu=1}^n \pi_{i,s}\xi_{i\nu,s}\, dW_{\nu,s}, \int_0^\cdot \sum_{j,\upsilon=1}^n \eta_{j,s}\xi_{j\upsilon,s}\, dW_{\upsilon,s} \right\rangle_t \\
&= d\left(\int_0^t \sum_{i,j=1}^n \pi_i(s)\eta_j(s) \sum_{\nu=1}^n \xi_{i\nu}(s)\xi_{j\nu}(s)\, ds \right) \\
&= \sum_{i,j=1}^n \pi_i(t)\eta_j(t) \sum_{\nu=1}^n \xi_{i\nu}(t)\xi_{j\nu}(t)\, dt \qquad (2.6.36) \\
&= \sum_{i,j=1}^n \pi_i(t)\sigma_{ij}(t)\eta_j(t)\, dt \qquad (2.6.37) \\
&= \sum_{i=1}^n \pi_i(t) \left[\sum_{j=1}^n \eta_j(t)\sigma_{ij}(t) \right] dt \qquad (2.6.38) \\
&= \sum_{i=1}^n \pi_i(t)\sigma_{i\eta}(t)\, dt. \qquad (2.6.39)
\end{aligned}
$$

Alternatively, we have

$$d\left\langle \log Z_\pi, \log Z_\eta \right\rangle_t = \sum_{j=1}^n \eta_j(t) \left[\sum_{i=1}^n \pi_i(t)\sigma_{ji}(t) \right] dt \tag{2.6.40}$$

$$= \sum_{j=1}^n \eta_j(t)\sigma_{j\pi}(t)\, dt. \tag{2.6.41}$$

Hence, a.s., for all $t \in [0, \infty)$, we have

$$d\left\langle \log Z_\pi, \log Z_\eta \right\rangle_t = \sum_{i=1}^n \pi_i(t)\sigma_{i\eta}(t)\, dt = \sum_{j=1}^n \eta_j(t)\sigma_{j\pi}(t)\, dt. \tag{2.6.42}$$

In addition, we have

$$d\left\langle \log Z_\pi, \log Z_\eta \right\rangle_t = \sum_{i,j=1}^n \pi_i(t)\sigma_{ij}(t)\eta_j(t)\, dt = \boldsymbol{\pi}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^\mathsf{T}(t)\, dt. \tag{2.6.43}$$

Let us now introduce the following definition.

Definition 2.6.9 (Portfolio Covariance Process). *For portfolios* $\pi = \big\{\boldsymbol{\pi}(t) = \big(\pi_1(t), \ldots, \pi_n(t)\big), t \in [0, \infty)\big\}$ *and* $\eta = \big\{\boldsymbol{\eta}(t) = \big(\eta_1(t), \ldots, \eta_n(t)\big), t \in [0, \infty)\big\}$, *the process* $\sigma_{\pi\eta} = \big\{\sigma_{\pi\eta}(t), t \in [0, \infty)\big\}$, *defined by*

$$\sigma_{\pi\eta}(t) \triangleq \frac{d}{dt}\left\langle \log Z_\pi, \log Z_\eta \right\rangle_t, \qquad t \in [0, \infty), \qquad a.s., \tag{2.6.44}$$

is referred to as the ***portfolio covariance process****, which represents the covariance process of portfolio* π *and portfolio* η*, i.e., the covariance of the processes* $\log Z_\pi$ *and* $\log Z_\eta$ *and indicates how portfolio* π *varies relative to portfolio* η*.*

Therefore, from equation (2.6.43), for $t \in [0, \infty)$, we have

$$\sigma_{\pi\eta}(t) = \sum_{i,j=1}^n \pi_i(t)\sigma_{ij}(t)\eta_j(t) = \boldsymbol{\pi}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^\mathsf{T}(t). \tag{2.6.45}$$

The preceding equation of the covariance process of portfolio π and portfolio η is comparable to that of the variance process of portfolio η, provided in equation (2.3.55). Therefore, for all $t \in [0, \infty)$, a.s., we have the following

$$d\left\langle \log Z_\pi, \log Z_\eta \right\rangle_t = \sigma_{\pi\eta}(t)\, dt, \quad \text{or,} \tag{2.6.46}$$

$$\left\langle \log Z_\pi, \log Z_\eta \right\rangle_t = \int_0^t \sigma_{\pi\eta}(s)\, ds. \tag{2.6.47}$$

Furthermore, we can express $\sigma_{\pi\eta}$ in terms of the volatility processes. Considering equations (2.6.42) and (2.6.44), in conjunction with equations (2.3.51), (2.3.53) and (2.2.126), we get

$$\sigma_{\pi\eta}(t) = \sum_{i=1}^n \pi_i(t)\sigma_{i\eta}(t) \tag{2.6.48}$$

$$= \sum_{i=1}^n \pi_i(t) \left[\sum_{\nu=1}^n \xi_{i\nu}(t)\xi_{\eta\nu}(t) \right]$$

$$= \sum_{\nu=1}^n \xi_{\eta\nu}(t) \left[\sum_{i=1}^n \pi_i(t)\xi_{i\nu}(t) \right]$$

$$= \sum_{\nu=1}^n \xi_{\pi\nu}(t)\xi_{\eta\nu}(t). \tag{2.6.49}$$

Alternatively, by making use of equations (2.3.52), (2.3.54) and (2.2.126), we similarly obtain

$$\begin{aligned}
\sigma_{\pi\eta}(t) &= \sum_{j=1}^{n} \eta_j(t)\sigma_{j\pi}(t) \\
&= \sum_{j=1}^{n} \eta_j(t)\left[\sum_{\nu=1}^{n} \xi_{j\nu}(t)\xi_{\pi\nu}(t)\right] \\
&= \sum_{\nu=1}^{n} \xi_{\pi\nu}(t)\left[\sum_{j=1}^{n} \eta_j(t)\xi_{j\nu}(t)\right] \\
&= \sum_{\nu=1}^{n} \xi_{\pi\nu}(t)\xi_{\eta\nu}(t).
\end{aligned}$$

Thus, in summary, and for the purposes of comparison, for all $i, j = 1, \ldots, n$, and for $t \in [0, \infty)$, we have the following covariance processes

$$\begin{aligned}
\sigma_{ij}(t) &= \sum_{\nu=1}^{n} \xi_{i\nu}(t)\xi_{j\nu}(t), \quad \text{and,} \\
\sigma_{i\eta}(t) &= \sum_{\nu=1}^{n} \xi_{i\nu}(t)\xi_{\eta\nu}(t), \quad \text{and,} \\
\sigma_{\pi\eta}(t) &= \sum_{\nu=1}^{n} \xi_{\pi\nu}(t)\xi_{\eta\nu}(t), \quad \text{and,} \\
\sigma_{\eta\eta}(t) &= \sum_{\nu=1}^{n} \xi_{\eta\nu}^2(t).
\end{aligned}$$

Equation (2.6.46) can be verified in a variety of ways, which we shall demonstrate below. Firstly, by appealing to equations (2.2.65) and (2.2.131), we determine the following

$$\begin{aligned}
d\left\langle \log Z_\pi, \log Z_\eta \right\rangle_t &= d\left\langle \int_0^{\cdot} \sum_{i,\nu=1}^{n} \pi_{i,s}\xi_{i\nu,s}\, dW_{\nu,s}, \int_0^{\cdot} \sum_{\upsilon=1}^{n} \xi_{\eta\upsilon,s}\, dW_{\upsilon,s} \right\rangle_t \\
&= d\left\langle \sum_{\nu=1}^{n} \int_0^{\cdot} \sum_{i=1}^{n} \pi_{i,s}\xi_{i\nu,s}\, dW_{\nu,s}, \sum_{\upsilon=1}^{n} \int_0^{\cdot} \xi_{\eta\upsilon,s}\, dW_{\upsilon,s} \right\rangle_t \\
&= \sum_{\nu,\upsilon=1}^{n} d\left\langle \int_0^{\cdot} \sum_{i=1}^{n} \pi_{i,s}\xi_{i\nu,s}\, dW_{\nu,s}, \int_0^{\cdot} \xi_{\eta\upsilon,s}\, dW_{\upsilon,s} \right\rangle_t \\
&= \sum_{\nu,\upsilon=1}^{n} \sum_{i=1}^{n} \pi_i(t)\xi_{i\nu}(t)\xi_{\eta\upsilon}(t)\, d\left\langle W_\nu, W_\upsilon \right\rangle_t \\
&= \sum_{i=1}^{n} \pi_i(t) \sum_{\nu,\upsilon=1}^{n} \xi_{i\nu}(t)\xi_{\eta\upsilon}(t)\, \rho_{\nu\upsilon}(t)\, dt \\
&= \sum_{i=1}^{n} \pi_i(t) \sum_{\nu=1}^{n} \xi_{i\nu}(t)\xi_{\eta\nu}(t)\, dt \qquad (2.6.50) \\
&= \sum_{i=1}^{n} \pi_i(t)\sigma_{i\eta}(t)\, dt \qquad (2.6.51) \\
&= \sigma_{\pi\eta}(t)\, dt, \qquad (2.6.52)
\end{aligned}$$

where equation (2.6.51) follows from equation (2.3.53) and equation (2.6.52) follows from equation (2.6.48).

Secondly, by appealing to equation (2.2.131), we determine the following

$$
\begin{aligned}
d\left\langle \log Z_\pi, \log Z_\eta \right\rangle_t &= d\left\langle \int_0^\cdot \sum_{\nu=1}^n \xi_{\pi\nu,s}\, dW_{\nu,s}, \int_0^\cdot \sum_{\upsilon=1}^n \xi_{\eta\upsilon,s}\, dW_{\upsilon,s} \right\rangle_t \\
&= d\left\langle \sum_{\nu=1}^n \int_0^\cdot \xi_{\pi\nu,s}\, dW_{\nu,s}, \sum_{\upsilon=1}^n \int_0^\cdot \xi_{\eta\upsilon,s}\, dW_{\upsilon,s} \right\rangle_t \\
&= \sum_{\nu,\upsilon=1}^n d\left\langle \int_0^\cdot \xi_{\pi\nu,s}\, dW_{\nu,s}, \int_0^\cdot \xi_{\eta\upsilon,s}\, dW_{\upsilon,s} \right\rangle_t \\
&= \sum_{\nu,\upsilon=1}^n \xi_{\pi\nu}(t)\xi_{\eta\upsilon}(t)\, d\left\langle W_\nu, W_\upsilon \right\rangle_t \\
&= \sum_{\nu,\upsilon=1}^n \xi_{\pi\nu}(t)\xi_{\eta\upsilon}(t)\, \rho_{\nu\upsilon}(t)\, dt \\
&= \sum_{\nu=1}^n \xi_{\pi\nu}(t)\xi_{\eta\nu}(t)\, dt \qquad (2.6.53) \\
&= \sigma_{\pi\eta}(t)\, dt, \qquad (2.6.54)
\end{aligned}
$$

where equation (2.6.54) follows from equation (2.6.49). For the next method, we shall employ equation (2.2.112) of Corollary 2.2.25, to obtain

$$
\begin{aligned}
d\left\langle \log Z_\pi, \log Z_\eta \right\rangle_t &= d\left\langle \int_0^\cdot \sum_{i=1}^n \pi_{i,s}\, d\log X_{i,s}, \int_0^\cdot \sum_{j=1}^n \eta_{j,s}\, d\log X_{j,s} \right\rangle_t \\
&= d\left\langle \sum_{i=1}^n \int_0^\cdot \pi_{i,s}\, d\log X_{i,s}, \sum_{j=1}^n \int_0^\cdot \eta_{j,s}\, d\log X_{j,s} \right\rangle_t \\
&= \sum_{i,j=1}^n d\left\langle \int_0^\cdot \pi_{i,s}\, d\log X_{i,s}, \int_0^\cdot \eta_{j,s}\, d\log X_{j,s} \right\rangle_t \\
&= \sum_{i,j=1}^n \pi_i(t)\eta_j(t)\, d\left\langle \log X_i, \log X_j \right\rangle_t \qquad (2.6.55) \\
&= \sum_{i,j=1}^n \pi_i(t)\sigma_{ij}(t)\eta_j(t)\, dt \qquad (2.6.56) \\
&= \sigma_{\pi\eta}(t)\, dt, \qquad (2.6.57)
\end{aligned}
$$

where equation (2.6.56) follows from equation (2.2.44), and equation (2.6.57) follows from equation (2.6.45). Finally, another application of equation (2.2.112) of Corollary 2.2.25, yields the following

$$
\begin{aligned}
d\left\langle \log Z_\pi, \log Z_\eta \right\rangle_t &= d\left\langle \int_0^\cdot \sum_{i=1}^n \pi_{i,s}\, d\log X_{i,s}, \int_0^\cdot d\log Z_{\eta,s} \right\rangle_t \\
&= d\left\langle \sum_{i=1}^n \int_0^\cdot \pi_{i,s}\, d\log X_{i,s}, \int_0^\cdot d\log Z_{\eta,s} \right\rangle_t \\
&= \sum_{i=1}^n d\left\langle \int_0^\cdot \pi_{i,s}\, d\log X_{i,s}, \int_0^\cdot d\log Z_{\eta,s} \right\rangle_t \\
&= \sum_{i=1}^n \pi_i(t)\, d\left\langle \log X_i, \log Z_\eta \right\rangle_t \qquad (2.6.58) \\
&= \sum_{i=1}^n \pi_i(t)\sigma_{i\eta}(t)\, dt \qquad (2.6.59) \\
&= \sigma_{\pi\eta}(t)\, dt, \qquad (2.6.60)
\end{aligned}
$$

where equation (2.6.59) follows from equation (2.3.68), and equation (2.6.60) follows from equation (2.6.48). Employing equation (2.2.105) together with equation (2.6.46), yields

$$d\left\langle Z_\pi, Z_\eta\right\rangle_t = Z_\pi(t)Z_\eta(t)\, d\left\langle \log Z_\pi, \log Z_\eta\right\rangle_t = \sigma_{\pi\eta}(t)\, Z_\pi(t)Z_\eta(t)\, dt. \tag{2.6.61}$$

Alternatively, from equations (2.2.62), (2.2.87) and (2.6.45), we obtain

$$\begin{aligned}
d\left\langle Z_\pi, Z_\eta\right\rangle_t &= d\left\langle \int_0^\cdot Z_{\pi,s}\sum_{i=1}^n \pi_{i,s}\,\frac{dX_{i,s}}{X_{i,s}}, \int_0^\cdot Z_{\eta,s}\sum_{j=1}^n \eta_{j,s}\,\frac{dX_{j,s}}{X_{j,s}}\right\rangle_t \\
&= d\left\langle \sum_{i=1}^n\int_0^\cdot Z_{\pi,s}\pi_{i,s}\,\frac{dX_{i,s}}{X_{i,s}}, \sum_{j=1}^n\int_0^\cdot Z_{\eta,s}\eta_{j,s}\,\frac{dX_{j,s}}{X_{j,s}}\right\rangle_t \\
&= \sum_{i,j=1}^n d\left\langle \int_0^\cdot Z_{\pi,s}\pi_{i,s}\,\frac{dX_{i,s}}{X_{i,s}}, \int_0^\cdot Z_{\eta,s}\eta_{j,s}\,\frac{dX_{j,s}}{X_{j,s}}\right\rangle_t \\
&= \sum_{i,j=1}^n Z_\pi(t)Z_\eta(t)\,\pi_i(t)\eta_j(t)\frac{1}{X_i(t)X_j(t)}\, d\left\langle X_i, X_j\right\rangle_t && (2.6.62)\\
&= \sum_{i,j=1}^n Z_\pi(t)Z_\eta(t)\,\pi_i(t)\eta_j(t)\, d\left\langle \log X_i, \log X_j\right\rangle_t && (2.6.63)\\
&= Z_\pi(t)Z_\eta(t)\sum_{i,j=1}^n \pi_i(t)\sigma_{ij}(t)\eta_j(t)\, dt && (2.6.64)\\
&= \sigma_{\pi\eta}(t)\, Z_\pi(t)Z_\eta(t)\, dt. && (2.6.65)
\end{aligned}$$

In addition, we can also use equations (2.2.62), (2.3.72) and (2.6.48), to yield the same result

$$\begin{aligned}
d\left\langle Z_\pi, Z_\eta\right\rangle_t &= d\left\langle \int_0^\cdot Z_{\pi,s}\sum_{i=1}^n \pi_{i,s}\,\frac{dX_{i,s}}{X_{i,s}}, \int_0^\cdot dZ_{\eta,s}\right\rangle_t \\
&= d\left\langle \sum_{i=1}^n\int_0^\cdot Z_{\pi,s}\pi_{i,s}\,\frac{dX_{i,s}}{X_{i,s}}, \int_0^\cdot dZ_{\eta,s}\right\rangle_t \\
&= \sum_{i=1}^n d\left\langle \int_0^\cdot Z_{\pi,s}\pi_{i,s}\,\frac{dX_{i,s}}{X_{i,s}}, \int_0^\cdot dZ_{\eta,s}\right\rangle_t \\
&= \sum_{i=1}^n Z_\pi(t)\,\pi_i(t)\frac{1}{X_i(t)}\, d\left\langle X_i, Z_\eta\right\rangle_t && (2.6.66)\\
&= \sum_{i=1}^n Z_\pi(t)Z_\eta(t)\,\pi_i(t)\, d\left\langle \log X_i, \log Z_\eta\right\rangle_t && (2.6.67)\\
&= Z_\pi(t)Z_\eta(t)\sum_{i=1}^n \pi_i(t)\sigma_{i\eta}(t)\, dt && (2.6.68)\\
&= \sigma_{\pi\eta}(t)\, Z_\pi(t)Z_\eta(t)\, dt. && (2.6.69)
\end{aligned}$$

The preceding results can also be derived for the market which incorporates dividends. Thus, the cross-variation of portfolio π and the portfolio η, for the dividend case, can be obtained from equation (2.2.162). Consequently, for $i = 1, 2, \ldots, n$, we obtain the analogues to (2.6.46) and (2.6.47),

$$d\left\langle \log \widehat{Z}_\pi, \log \widehat{Z}_\eta\right\rangle_t = d\left\langle \log Z_\pi, \log Z_\eta\right\rangle_t = \sigma_{\pi\eta}(t)\, dt, \quad \text{or,} \tag{2.6.70}$$

$$\left\langle \log \widehat{Z}_\pi, \log \widehat{Z}_\eta\right\rangle_t = \left\langle \log Z_\pi, \log Z_\eta\right\rangle_t = \int_0^t \sigma_{\pi\eta}(s)\, ds. \tag{2.6.71}$$

Consequently, from equation (2.6.70), we have

$$d\left\langle \widehat{Z}_\pi, \widehat{Z}_\eta\right\rangle_t = \widehat{Z}_\pi(t)\widehat{Z}_\eta(t)\, d\left\langle \log \widehat{Z}_\pi, \log \widehat{Z}_\eta\right\rangle_t = \sigma_{\pi\eta}(t)\, \widehat{Z}_\pi(t)\widehat{Z}_\eta(t)\, dt. \tag{2.6.72}$$

Now, from equation (2.6.28) together with equations (2.2.75), (2.6.42), and (2.2.95), we arrive at the following

$$\begin{aligned}
d\left\langle \log\left(Z_\pi/Z_\eta\right)\right\rangle_t &= d\left\langle \log Z_\pi\right\rangle_t - d\left\langle \log Z_\pi, \log Z_\eta\right\rangle_t - d\left\langle \log Z_\eta, \log Z_\pi\right\rangle_t + d\left\langle \log Z_\eta\right\rangle_t \\
&= \sum_{i,j=1}^n \pi_i(t)\sigma_{ij}(t)\pi_j(t)\,dt - \sum_{i=1}^n \pi_i(t)\sigma_{i\eta}(t)\,dt - \sum_{j=1}^n \pi_j(t)\sigma_{j\eta}(t)\,dt + \sigma_{\eta\eta}(t)\,dt \\
&= \sum_{i,j=1}^n \pi_i(t)\sigma_{ij}(t)\pi_j(t)\,dt - \left[\sum_{j=1}^n \pi_j(t)\right]\sum_{i=1}^n \pi_i(t)\sigma_{i\eta}(t)\,dt - \left[\sum_{i=1}^n \pi_i(t)\right]\sum_{j=1}^n \pi_j(t)\sigma_{j\eta}(t)\,dt \\
&\quad + \left[\sum_{i=1}^n \pi_i(t)\right]\left[\sum_{j=1}^n \pi_j(t)\right]\sigma_{\eta\eta}(t)\,dt \\
&= \sum_{i,j=1}^n \pi_i(t)\sigma_{ij}(t)\pi_j(t)\,dt - \sum_{i,j=1}^n \pi_i(t)\sigma_{i\eta}(t)\pi_j(t)\,dt - \sum_{i,j=1}^n \pi_i(t)\sigma_{j\eta}(t)\pi_j(t)\,dt \\
&\quad + \sum_{i,j=1}^n \pi_i(t)\sigma_{\eta\eta}(t)\pi_j(t)\,dt \\
&= \sum_{i,j=1}^n \pi_i(t)\Big[\sigma_{ij}(t) - \sigma_{i\eta}(t) - \sigma_{j\eta}(t) + \sigma_{\eta\eta}(t)\Big]\pi_j(t)\,dt,
\end{aligned}$$

which, when combined with equation (2.3.85), reduces to the following

$$d\left\langle \log\left(Z_\pi/Z_\eta\right)\right\rangle_t = \sum_{i,j=1}^n \pi_i(t)\tau_{ij}^\eta(t)\pi_j(t)\,dt. \tag{2.6.73}$$

Moreover, employing equations (2.2.95), (2.6.46), and (2.3.80) in equation (2.6.29), gives a.s., for $t \in [0,\infty)$,

$$d\left\langle \log\left(Z_\pi/Z_\eta\right)\right\rangle_t = \sigma_{\pi\pi}(t)\,dt - 2\,\sigma_{\pi\eta}(t)\,dt + \sigma_{\eta\eta}(t)\,dt \tag{2.6.74}$$

$$= \left(\sigma_{\pi\pi}(t) - 2\,\sigma_{\pi\eta}(t) + \sigma_{\eta\eta}(t)\right)dt, \tag{2.6.75}$$

$$\frac{d}{dt}\left\langle \log\left(Z_\pi/Z_\eta\right)\right\rangle_t = \sigma_{\pi\pi}(t) - 2\,\sigma_{\pi\eta}(t) + \sigma_{\eta\eta}(t). \tag{2.6.76}$$

Consequently, by employing equation (2.2.94) of Definition 2.2.22, equation (2.6.45), and equation (2.3.55), we obtain the following matrix representation

$$\begin{aligned}
&\frac{d}{dt}\left\langle \log\left(Z_\pi/Z_\eta\right)\right\rangle_t \\
&= \sum_{i,j=1}^n \pi_i(t)\sigma_{ij}(t)\pi_j(t) - \sum_{i,j=1}^n \pi_i(t)\sigma_{ij}(t)\eta_j(t) - \sum_{i,j=1}^n \eta_i(t)\sigma_{ij}(t)\pi_j(t) + \sum_{i,j=1}^n \eta_i(t)\sigma_{ij}(t)\eta_j(t) \\
&= \boldsymbol{\pi}(t)\boldsymbol{\sigma}(t)\boldsymbol{\pi}^\mathsf{T}(t) - \boldsymbol{\pi}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^\mathsf{T}(t) - \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\boldsymbol{\pi}^\mathsf{T}(t) + \boldsymbol{\eta}(t)\boldsymbol{\sigma}(t)\boldsymbol{\eta}^\mathsf{T}(t) \\
&= \Big(\boldsymbol{\pi}(t) - \boldsymbol{\eta}(t)\Big)\boldsymbol{\sigma}(t)\Big(\boldsymbol{\pi}^\mathsf{T}(t) - \boldsymbol{\eta}^\mathsf{T}(t)\Big) \\
&= \Big(\boldsymbol{\pi}(t) - \boldsymbol{\eta}(t)\Big)\boldsymbol{\sigma}(t)\Big(\boldsymbol{\pi}(t) - \boldsymbol{\eta}(t)\Big)^\mathsf{T}.
\end{aligned} \tag{2.6.77}$$

Thus, by equation (2.4.15), we arrive at

$$\frac{d}{dt}\left\langle \log\left(Z_\pi/Z_\eta\right)\right\rangle_t = \boldsymbol{\pi}(t)\boldsymbol{\tau}^{\boldsymbol{\eta}}(t)\boldsymbol{\pi}^\mathsf{T}(t) = \sum_{i,j=1}^n \pi_i(t)\tau_{ij}^\eta(t)\pi_j(t). \tag{2.6.78}$$

Finally, recalling equation (2.4.17) of Definition 2.4.3, the preceding expression is simply the relative variance process of π versus η, i.e., for $t \in [0,\infty)$, a.s., we have

$$d\left\langle \log\left(Z_\pi/Z_\eta\right)\right\rangle_t = \tau_{\pi\pi}^\eta(t)\,dt, \quad \text{or,} \tag{2.6.79}$$

$$\left\langle \log\left(Z_\pi/Z_\eta\right)\right\rangle_t = \int_0^t \tau_{\pi\pi}^\eta(s)\,ds. \tag{2.6.80}$$

This leads us to the following alternative definition to Definition 2.4.3, of the relative variance of portfolios.

Definition 2.6.10 (Relative Variance of Portfolios). *For portfolios* $\pi = \big\{\boldsymbol{\pi}(t) = \big(\pi_1(t), \ldots, \pi_n(t)\big), t \in [0,\infty)\big\}$ *and* $\eta = \big\{\boldsymbol{\eta}(t) = \big(\eta_1(t), \ldots, \eta_n(t)\big), t \in [0,\infty)\big\}$, *the process* $\tau^{\eta}_{\pi\pi} = \big\{\tau^{\eta}_{\pi\pi}(t), t \in [0,\infty)\big\}$, *defined by*

$$\tau^{\eta}_{\pi\pi}(t) \triangleq \frac{d}{dt}\big\langle \log\big(Z_\pi/Z_\eta\big)\big\rangle_t, \qquad t \in [0,\infty), \qquad a.s., \tag{2.6.81}$$

is called the ***relative variance process*** *of* π *versus* η*; and the process* $\tau^{\pi}_{\eta\eta} = \big\{\tau^{\pi}_{\eta\eta}(t), t \in [0,\infty)\big\}$, *defined by*

$$\tau^{\pi}_{\eta\eta}(t) \triangleq \frac{d}{dt}\big\langle \log\big(Z_\eta/Z_\pi\big)\big\rangle_t, \qquad t \in [0,\infty), \qquad a.s., \tag{2.6.82}$$

is called the ***relative variance process*** *of* η *versus* π.

From equation (2.4.19) of Lemma 2.4.4, it unfolds a.s., for $t \in [0,\infty)$, that

$$\frac{d}{dt}\big\langle \log\big(Z_\pi/Z_\eta\big)\big\rangle_t = \frac{d}{dt}\big\langle \log\big(Z_\eta/Z_\pi\big)\big\rangle_t, \quad \text{or,} \tag{2.6.83}$$

$$d\big\langle \log\big(Z_\pi/Z_\eta\big)\big\rangle_t = d\big\langle \log\big(Z_\eta/Z_\pi\big)\big\rangle_t. \tag{2.6.84}$$

Consequently, from equations (2.6.76) and (2.6.81), for the relative variance of portfolio π versus portfolio η for $t \in [0,\infty)$, a.s., we have the following form

$$\tau^{\eta}_{\pi\pi}(t) = \sigma_{\pi\pi}(t) - 2\,\sigma_{\pi\eta}(t) + \sigma_{\eta\eta}(t), \tag{2.6.85}$$

which is akin to the relative variance of the ith stock versus the portfolio η, derived in equation (2.3.128). By putting equations (2.2.136) and (2.6.49) to use, the portfolio relative variance above (2.6.85), can be expressed in terms of the volatility processes as follows

$$\begin{aligned}
\tau^{\eta}_{\pi\pi}(t) &= \sum_{\nu=1}^{n} \xi^2_{\pi\nu}(t) - 2\sum_{\nu=1}^{n} \xi_{\pi\nu}(t)\xi_{\eta\nu}(t) + \sum_{\nu=1}^{n} \xi^2_{\eta\nu}(t) \\
&= \sum_{\nu=1}^{n} \Big(\xi^2_{\pi\nu}(t) - 2\,\xi_{\pi\nu}(t)\xi_{\eta\nu}(t) + \xi^2_{\eta\nu}(t)\Big) \\
&= \sum_{\nu=1}^{n} \Big(\xi_{\pi\nu}(t) - \xi_{\eta\nu}(t)\Big)^2,
\end{aligned} \tag{2.6.86}$$

which, again, bears a resemblance to the individual stock relative variances in equation (2.3.130). An alternative approach would be to consider the quadratic variation of the relative portfolio return process. Thus, by appealing to equation (2.6.7) of Lemma 2.6.2, we shall verify the expression (2.6.79) as follows

$$\begin{aligned}
\big\langle \log\big(Z_\pi/Z_\eta\big)\big\rangle_t &= \Big\langle \int_0^{\cdot} \sum_{i,\nu=1}^{n} \big(\pi_{i,s} - \eta_{i,s}\big)\xi_{i\nu,s}\, dW_{\nu,s}, \int_0^{\cdot} \sum_{j,\upsilon=1}^{n} \big(\pi_{j,s} - \eta_{j,s}\big)\xi_{j\upsilon,s}\, dW_{\upsilon,s} \Big\rangle_t \\
&= \Big\langle \sum_{\nu=1}^{n} \int_0^{\cdot} \sum_{i=1}^{n} \big(\pi_{i,s} - \eta_{i,s}\big)\xi_{i\nu,s}\, dW_{\nu,s}, \sum_{\upsilon=1}^{n} \int_0^{\cdot} \sum_{j=1}^{n} \big(\pi_{j,s} - \eta_{j,s}\big)\xi_{j\upsilon,s}\, dW_{\upsilon,s} \Big\rangle_t \\
&= \sum_{\nu,\upsilon=1}^{n} \Big\langle \int_0^{\cdot} \sum_{i=1}^{n} \big(\pi_{i,s} - \eta_{i,s}\big)\xi_{i\nu,s}\, dW_{\nu,s}, \int_0^{\cdot} \sum_{j=1}^{n} \big(\pi_{j,s} - \eta_{j,s}\big)\xi_{j\upsilon,s}\, dW_{\upsilon,s} \Big\rangle_t \\
&= \sum_{\nu,\upsilon=1}^{n} \int_0^{t} \sum_{i,j=1}^{n} \big(\pi_i(s) - \eta_i(s)\big)\big(\pi_j(s) - \eta_j(s)\big)\xi_{i\nu}(s)\xi_{j\upsilon}(s)\, d\,\langle W_\nu, W_\upsilon\rangle_s \\
&= \int_0^{t} \sum_{i,j=1}^{n} \big(\pi_i(s) - \eta_i(s)\big)\big(\pi_j(s) - \eta_j(s)\big) \sum_{\nu,\upsilon=1}^{n} \xi_{i\nu}(s)\xi_{j\upsilon}(s)\, \rho_{\nu\upsilon}(s)\, ds \\
&= \int_0^{t} \sum_{i,j=1}^{n} \big(\pi_i(s) - \eta_i(s)\big)\big(\pi_j(s) - \eta_j(s)\big) \sum_{\nu=1}^{n} \xi_{i\nu}(s)\xi_{j\nu}(s)\, ds \qquad (2.6.87) \\
&= \int_0^{t} \sum_{i,j=1}^{n} \big(\pi_i(s) - \eta_i(s)\big)\sigma_{ij}(s)\big(\pi_j(s) - \eta_j(s)\big)\, ds \qquad (2.6.88) \\
&= \int_0^{t} \tau^{\eta}_{\pi\pi}(s)\, ds, \qquad (2.6.89)
\end{aligned}$$

which is as a result of a combination of (2.2.37) and (2.6.77). Therefore, we have

$$\begin{aligned}
d\left\langle \log\left(Z_\pi/Z_\eta\right)\right\rangle_t &= d\left\langle \int_0^\cdot \sum_{i,\nu=1}^n \left(\pi_{i,s}-\eta_{i,s}\right)\xi_{i\nu,s}\,dW_{\nu,s}, \int_0^\cdot \sum_{j,\upsilon=1}^n \left(\pi_{j,s}-\eta_{j,s}\right)\xi_{j\upsilon,s}\,dW_{\upsilon,s}\right\rangle_t \\
&= d\left(\int_0^t \sum_{i,j=1}^n \left(\pi_i(s)-\eta_i(s)\right)\left(\pi_j(s)-\eta_j(s)\right)\sum_{\nu=1}^n \xi_{i\nu}(s)\xi_{j\nu}(s)\,ds\right) \\
&= \sum_{i,j=1}^n \left(\pi_i(t)-\eta_i(t)\right)\left(\pi_j(t)-\eta_j(t)\right)\sum_{\nu=1}^n \xi_{i\nu}(t)\xi_{j\nu}(t)\,dt \qquad (2.6.90)\\
&= \sum_{i,j=1}^n \left(\pi_i(t)-\eta_i(t)\right)\sigma_{ij}(t)\left(\pi_j(t)-\eta_j(t)\right)dt \qquad (2.6.91)\\
&= \tau^\eta_{\pi\pi}(t)\,dt. \qquad (2.6.92)
\end{aligned}$$

Moreover, by appealing to equation (2.6.9) of Lemma 2.6.3, we shall also verify the expression (2.6.79) as follows

$$\begin{aligned}
d\left\langle \log\left(Z_\pi/Z_\eta\right)\right\rangle_t &= d\left\langle \int_0^\cdot \sum_{i,\nu=1}^n \pi_{i,s}\left(\xi_{i\nu,s}-\xi_{\eta\nu,s}\right)dW_{\nu,s}, \int_0^\cdot \sum_{j,\upsilon=1}^n \pi_{j,s}\left(\xi_{j\upsilon,s}-\xi_{\eta\upsilon,s}\right)dW_{\upsilon,s}\right\rangle_t \\
&= d\left\langle \sum_{\nu=1}^n \int_0^\cdot \sum_{i=1}^n \pi_{i,s}\left(\xi_{i\nu,s}-\xi_{\eta\nu,s}\right)dW_{\nu,s}, \sum_{\upsilon=1}^n \int_0^\cdot \sum_{j=1}^n \pi_{j,s}\left(\xi_{j\upsilon,s}-\xi_{\eta\upsilon,s}\right)dW_{\upsilon,s}\right\rangle_t \\
&= \sum_{\nu,\upsilon=1}^n d\left\langle \int_0^\cdot \sum_{i=1}^n \pi_{i,s}\left(\xi_{i\nu,s}-\xi_{\eta\nu,s}\right)dW_{\nu,s}, \int_0^\cdot \sum_{j=1}^n \pi_{j,s}\left(\xi_{j\upsilon,s}-\xi_{\eta\upsilon,s}\right)dW_{\upsilon,s}\right\rangle_t \\
&= \sum_{\nu,\upsilon=1}^n \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\Big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\Big)\Big(\xi_{j\upsilon}(t)-\xi_{\eta\upsilon}(t)\Big)\,d\left\langle W_\nu, W_\upsilon\right\rangle_t \\
&= \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\sum_{\nu,\upsilon=1}^n \Big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\Big)\Big(\xi_{j\upsilon}(t)-\xi_{\eta\upsilon}(t)\Big)\,\rho_{\nu\upsilon}(t)\,dt \\
&= \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\sum_{\nu=1}^n \Big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\Big)\Big(\xi_{j\nu}(t)-\xi_{\eta\nu}(t)\Big)\,dt. \qquad (2.6.93)
\end{aligned}$$

Thus, from equation (2.3.109), we have

$$d\left\langle \log\left(Z_\pi/Z_\eta\right)\right\rangle_t = \sum_{i,j=1}^n \pi_i(t)\tau^\eta_{ij}(t)\pi_j(t)\,dt,$$

which by Definition 2.4.3, equation (2.4.17) gives us the desired result

$$d\left\langle \log\left(Z_\pi/Z_\eta\right)\right\rangle_t = \tau^\eta_{\pi\pi}(t)\,dt.$$

Furthermore, simplifying (2.6.93), using equation (2.2.126), yields

$$\begin{aligned}
d\left\langle \log\left(Z_\pi/Z_\eta\right)\right\rangle_t &= \sum_{\nu=1}^n \left[\sum_{i=1}^n \pi_i(t)\Big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\Big)\right]\left[\sum_{j=1}^n \pi_j(t)\Big(\xi_{j\nu}(t)-\xi_{\eta\nu}(t)\Big)\right]dt \\
&= \sum_{\nu=1}^n \left[\sum_{i=1}^n \pi_i(t)\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\right]\left[\sum_{j=1}^n \pi_j(t)\xi_{j\nu}(t)-\xi_{\eta\nu}(t)\right]dt \\
&= \sum_{\nu=1}^n \Big(\xi_{\pi\nu}(t)-\xi_{\eta\nu}(t)\Big)\Big(\xi_{\pi\nu}(t)-\xi_{\eta\nu}(t)\Big)\,dt \\
&= \sum_{\nu=1}^n \Big(\xi_{\pi\nu}(t)-\xi_{\eta\nu}(t)\Big)^2\,dt, \qquad (2.6.94)
\end{aligned}$$

which is comparable to equation (2.3.131). Therefore,

$$d\left\langle \log\left(Z_\pi/Z_\eta\right)\right\rangle_t = \sum_{\nu=1}^n \Big(\xi_{\pi\nu}(t)-\xi_{\eta\nu}(t)\Big)^2\,dt. \qquad (2.6.95)$$

Hence, we have the analogues to (2.3.133) and (2.3.134), respectively,

$$\tau_{\pi\pi}^{\eta}(t) = \frac{d}{dt}\left\langle \log\left(Z_\pi/Z_\eta\right)\right\rangle_t = \sum_{\nu=1}^{n}\Big(\xi_{\pi\nu}(t)-\xi_{\eta\nu}(t)\Big)^2, \quad \text{and,} \tag{2.6.96}$$

$$\tau_{\pi\pi}^{\eta}(t)\,dt = d\left\langle \log\left(Z_\pi/Z_\eta\right)\right\rangle_t = \sum_{\nu=1}^{n}\Big(\xi_{\pi\nu}(t)-\xi_{\eta\nu}(t)\Big)^2 dt. \tag{2.6.97}$$

Employing equation (2.6.11) of Lemma 2.6.4, also gives

$$\begin{aligned}
d\left\langle \log\left(Z_\pi/Z_\eta\right)\right\rangle_t &= d\left\langle \int_0^{\cdot}\sum_{\nu=1}^{n}\left(\xi_{\pi\nu,s}-\xi_{\eta\nu,s}\right)dW_{\nu,s}, \int_0^{\cdot}\sum_{\upsilon=1}^{n}\left(\xi_{\pi\upsilon,s}-\xi_{\eta\upsilon,s}\right)dW_{\upsilon,s}\right\rangle_t \\
&= d\left\langle \sum_{\nu=1}^{n}\int_0^{\cdot}\left(\xi_{\pi\nu,s}-\xi_{\eta\nu,s}\right)dW_{\nu,s}, \sum_{\upsilon=1}^{n}\int_0^{\cdot}\left(\xi_{\pi\upsilon,s}-\xi_{\eta\upsilon,s}\right)dW_{\upsilon,s}\right\rangle_t \\
&= \sum_{\nu,\upsilon=1}^{n} d\left\langle \int_0^{\cdot}\left(\xi_{\pi\nu,s}-\xi_{\eta\nu,s}\right)dW_{\nu,s}, \int_0^{\cdot}\left(\xi_{\pi\upsilon,s}-\xi_{\eta\upsilon,s}\right)dW_{\upsilon,s}\right\rangle_t \\
&= \sum_{\nu,\upsilon=1}^{n}\Big(\xi_{\pi\nu}(t)-\xi_{\eta\nu}(t)\Big)\Big(\xi_{\pi\upsilon}(t)-\xi_{\eta\upsilon}(t)\Big)\,d\left\langle W_\nu, W_\upsilon\right\rangle_t \\
&= \sum_{\nu,\upsilon=1}^{n}\Big(\xi_{\pi\nu}(t)-\xi_{\eta\nu}(t)\Big)\Big(\xi_{\pi\upsilon}(t)-\xi_{\eta\upsilon}(t)\Big)\,\rho_{\nu\upsilon}(t)\,dt \\
&= \sum_{\nu=1}^{n}\Big(\xi_{\pi\nu}(t)-\xi_{\eta\nu}(t)\Big)^2 dt \qquad (2.6.98)\\
&= \tau_{\pi\pi}^{\eta}(t)\,dt, \qquad (2.6.99)
\end{aligned}$$

which follows from equation (2.6.86). Lastly, from equation (2.6.12) of Lemma 2.6.5, we obtain the following analogous result

$$\begin{aligned}
d\left\langle \log\left(Z_\pi/Z_\eta\right)\right\rangle_t &= d\left\langle \int_0^{\cdot}\sum_{i,k=1}^{n}\pi_{i,s}\eta_{k,s}\,d\log\left(X_{i,s}/X_{k,s}\right), \int_0^{\cdot}\sum_{j,m=1}^{n}\pi_{j,s}\eta_{m,s}\,d\log\left(X_{j,s}/X_{m,s}\right)\right\rangle_t \\
&= d\left\langle \sum_{i=1}^{n}\int_0^{\cdot}\sum_{k=1}^{n}\pi_{i,s}\eta_{k,s}\,d\log\left(X_{i,s}/X_{k,s}\right), \sum_{j=1}^{n}\int_0^{\cdot}\sum_{m=1}^{n}\pi_{j,s}\eta_{m,s}\,d\log\left(X_{j,s}/X_{m,s}\right)\right\rangle_t \\
&= \sum_{i,j=1}^{n} d\left\langle \int_0^{\cdot}\sum_{k=1}^{n}\pi_{i,s}\eta_{k,s}\,d\log\left(X_{i,s}/X_{k,s}\right), \int_0^{\cdot}\sum_{m=1}^{n}\pi_{j,s}\eta_{m,s}\,d\log\left(X_{j,s}/X_{m,s}\right)\right\rangle_t \\
&= \sum_{i,j=1}^{n}\sum_{k,m=1}^{n}\pi_i(t)\pi_j(t)\eta_k(t)\eta_m(t)\,d\left\langle \log\left(X_i/X_k\right), \log\left(X_j/X_m\right)\right\rangle_t \qquad (2.6.100)\\
&= \sum_{i,j=1}^{n}\pi_i(t)\pi_j(t)\sum_{k,m=1}^{n}\eta_k(t)\tau_{ij}^{X_kX_m}(t)\eta_m(t)\,dt \qquad (2.6.101)\\
&= \sum_{i,j=1}^{n}\pi_i(t)\tau_{ij}^{\eta}(t)\pi_j(t)\,dt \qquad (2.6.102)\\
&= \tau_{\pi\pi}^{\eta}(t)\,dt, \qquad (2.6.103)
\end{aligned}$$

where equation (2.6.101) follows from equation (2.3.18), equation (2.6.102) follows from equation (2.3.119) and equation (2.6.103) follows from (2.6.78).

2.7 Quotient Process of Portfolios

Definition 2.7.1 (Quotient Process of Portfolios). *For portfolios π and η, the process $Z_\pi/Z_\eta = \big\{Z_\pi(t)/Z_\eta(t), t \in [0,\infty)\big\}$, i.e.,*

$$\frac{Z_\pi(t)}{Z_\eta(t)} = \left(\frac{Z_{w,\pi}(t)}{Z_{w,\eta}(t)}\right)\Bigg|_{w=Z_{w,\pi}(0)=Z_{w,\eta}(0)}, \qquad t \in [0,\infty), \tag{2.7.1}$$

*is called the **quotient process** of π versus η.*

Lemma 2.7.2. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t \in [0,\infty)$, we obtain the dynamics*

$$d\left(\frac{Z_\pi(t)}{Z_\eta(t)}\right) = \frac{Z_\pi(t)}{Z_\eta(t)}\left[\left(\gamma_\pi^*(t) + \frac{1}{2}\,\tau_{\pi\pi}^\eta(t)\right)dt + \sum_{i=1}^n \pi_i(t)\,d\log\Big(X_i(t)/Z_\eta(t)\Big)\right], \tag{2.7.2}$$

for the relative arithmetic return of portfolio π versus portfolio η.

Proof. Itô's formula, applied to $\exp\big(\log\big(Z_\pi(t)/Z_\eta(t)\big)\big)$, yields the following for all $t \in [0,\infty)$, a.s.,

$$d\left(\frac{Z_\pi(t)}{Z_\eta(t)}\right) = \frac{Z_\pi(t)}{Z_\eta(t)}\,d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) + \frac{1}{2}\,\frac{Z_\pi(t)}{Z_\eta(t)}\,d\big\langle\log\big(Z_\pi/Z_\eta\big)\big\rangle_t. \tag{2.7.3}$$

By appealing to equation (2.6.4), the dynamics of the relative return of portfolio π versus portfolio η, we shall confirm the result of (2.6.79), as follows

$$\begin{aligned}
d\big\langle\log\big(Z_\pi/Z_\eta\big)\big\rangle_t &= d\left\langle \int_0^\cdot \sum_{i=1}^n \pi_{i,s}\,d\log\big(X_{i,s}/Z_{\eta,s}\big), \int_0^\cdot \sum_{j=1}^n \pi_{j,s}\,d\log\big(X_{j,s}/Z_{\eta,s}\big)\right\rangle_t \\
&= d\left\langle \sum_{i=1}^n \int_0^\cdot \pi_{i,s}\,d\log\big(X_{i,s}/Z_{\eta,s}\big), \sum_{j=1}^n \int_0^\cdot \pi_{j,s}\,d\log\big(X_{j,s}/Z_{\eta,s}\big)\right\rangle_t \\
&= \sum_{i,j=1}^n d\left\langle \int_0^\cdot \pi_{i,s}\,d\log\big(X_{i,s}/Z_{\eta,s}\big), \int_0^\cdot \pi_{j,s}\,d\log\big(X_{j,s}/Z_{\eta,s}\big)\right\rangle_t \\
&= \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\,d\big\langle\log\big(X_i/Z_\eta\big), \log\big(X_j/Z_\eta\big)\big\rangle_t \qquad (2.7.4)\\
&= \sum_{i,j=1}^n \pi_i(t)\tau_{ij}^\eta(t)\pi_j(t)\,dt \qquad (2.7.5)\\
&= \tau_{\pi\pi}^\eta(t)\,dt, \qquad (2.7.6)
\end{aligned}$$

where equation (2.7.5) follows from equation (2.3.82) of Definition 2.3.8, and equation (2.7.6) follows from equation (2.4.17) of Definition 2.4.3. Then, substituting the above result into equation (2.7.3), yields

$$d\left(\frac{Z_\pi(t)}{Z_\eta(t)}\right) = \frac{Z_\pi(t)}{Z_\eta(t)}\,d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) + \frac{1}{2}\,\frac{Z_\pi(t)}{Z_\eta(t)}\,\tau_{\pi\pi}^\eta(t)\,dt, \tag{2.7.7}$$

which, in conjunction with equation (2.6.4), gives the desired result

$$\begin{aligned}
d\left(\frac{Z_\pi(t)}{Z_\eta(t)}\right) &= \frac{Z_\pi(t)}{Z_\eta(t)}\left[\gamma_\pi^*(t)\,dt + \sum_{i=1}^n \pi_i(t)\,d\log\Big(X_i(t)/Z_\eta(t)\Big)\right] + \frac{1}{2}\,\frac{Z_\pi(t)}{Z_\eta(t)}\,\tau_{\pi\pi}^\eta(t)\,dt \\
&= \frac{Z_\pi(t)}{Z_\eta(t)}\left[\left(\gamma_\pi^*(t) + \frac{1}{2}\,\tau_{\pi\pi}^\eta(t)\right)dt + \sum_{i=1}^n \pi_i(t)\,d\log\Big(X_i(t)/Z_\eta(t)\Big)\right].
\end{aligned}$$

■

We can rewrite equation (2.7.7) in the following useful forms

$$\frac{d\left(Z_\pi(t)/Z_\eta(t)\right)}{Z_\pi(t)/Z_\eta(t)} = d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) + \frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\,dt, \tag{2.7.8}$$

and

$$d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) = \frac{d\left(Z_\pi(t)/Z_\eta(t)\right)}{Z_\pi(t)/Z_\eta(t)} - \frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\,dt. \tag{2.7.9}$$

Lemma 2.7.3. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t \in [0,\infty)$, we obtain the dynamics*

$$d\left(\frac{Z_\pi(t)}{Z_\eta(t)}\right) = \frac{Z_\pi(t)}{Z_\eta(t)}\left[\left(\gamma_\pi(t) - \gamma_\eta(t) + \frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)dt + \sum_{i,\nu=1}^{n}\big(\pi_i(t) - \eta_i(t)\big)\xi_{i\nu}(t)\,dW_\nu(t)\right], \tag{2.7.10}$$

for the relative arithmetic return of portfolio π versus portfolio η.

Proof. Upon substitution of equation (2.6.7) into equation (2.7.7), we obtain

$$\begin{aligned}
d\left(\frac{Z_\pi(t)}{Z_\eta(t)}\right) &= \frac{Z_\pi(t)}{Z_\eta(t)}\left[\big(\gamma_\pi(t) - \gamma_\eta(t)\big)\,dt + \sum_{i,\nu=1}^{n}\big(\pi_i(t) - \eta_i(t)\big)\xi_{i\nu}(t)\,dW_\nu(t)\right] + \frac{1}{2}\,\frac{Z_\pi(t)}{Z_\eta(t)}\,\tau^\eta_{\pi\pi}(t)\,dt \\
&= \left(\gamma_\pi(t) - \gamma_\eta(t) + \frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)\frac{Z_\pi(t)}{Z_\eta(t)}\,dt + \frac{Z_\pi(t)}{Z_\eta(t)}\sum_{i,\nu=1}^{n}\big(\pi_i(t) - \eta_i(t)\big)\xi_{i\nu}(t)\,dW_\nu(t) \\
&= \frac{Z_\pi(t)}{Z_\eta(t)}\left[\left(\gamma_\pi(t) - \gamma_\eta(t) + \frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)dt + \sum_{i,\nu=1}^{n}\big(\pi_i(t) - \eta_i(t)\big)\xi_{i\nu}(t)\,dW_\nu(t)\right],
\end{aligned}$$

which completes the proof. ■

Lemma 2.7.4. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t \in [0,\infty)$, we obtain the dynamics*

$$d\left(\frac{Z_\pi(t)}{Z_\eta(t)}\right) = \frac{Z_\pi(t)}{Z_\eta(t)}\left[\left(\gamma_\pi(t) - \gamma_\eta(t) + \frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)dt + \sum_{i,\nu=1}^{n}\pi_i(t)\big(\xi_{i\nu}(t) - \xi_{\eta\nu}(t)\big)\,dW_\nu(t)\right], \tag{2.7.11}$$

for the relative arithmetic return of portfolio π versus portfolio η.

Proof. Upon substitution of equation (2.6.9) into equation (2.7.7), we obtain

$$\begin{aligned}
d\left(\frac{Z_\pi(t)}{Z_\eta(t)}\right) &= \frac{Z_\pi(t)}{Z_\eta(t)}\left[\big(\gamma_\pi(t) - \gamma_\eta(t)\big)\,dt + \sum_{i,\nu=1}^{n}\pi_i(t)\big(\xi_{i\nu}(t) - \xi_{\eta\nu}(t)\big)\,dW_\nu(t)\right] + \frac{1}{2}\,\frac{Z_\pi(t)}{Z_\eta(t)}\,\tau^\eta_{\pi\pi}(t)\,dt \\
&= \left(\gamma_\pi(t) - \gamma_\eta(t) + \frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)\frac{Z_\pi(t)}{Z_\eta(t)}\,dt + \frac{Z_\pi(t)}{Z_\eta(t)}\sum_{i,\nu=1}^{n}\pi_i(t)\big(\xi_{i\nu}(t) - \xi_{\eta\nu}(t)\big)\,dW_\nu(t) \\
&= \frac{Z_\pi(t)}{Z_\eta(t)}\left[\left(\gamma_\pi(t) - \gamma_\eta(t) + \frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)dt + \sum_{i,\nu=1}^{n}\pi_i(t)\big(\xi_{i\nu}(t) - \xi_{\eta\nu}(t)\big)\,dW_\nu(t)\right],
\end{aligned}$$

which completes the proof. ■

The result put forth in the preceding lemma is akin to the result established in equation (2.5.33), for the relative arithmetic return of the ith stock versus portfolio η.

Lemma 2.7.5. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t \in [0, \infty)$, we obtain the dynamics*

$$d\left(\frac{Z_\pi(t)}{Z_\eta(t)}\right) = \frac{Z_\pi(t)}{Z_\eta(t)}\left[\left(\gamma_\pi(t) - \gamma_\eta(t) + \frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)dt + \sum_{\nu=1}^n \big(\xi_{\pi\nu}(t) - \xi_{\eta\nu}(t)\big)\,dW_\nu(t)\right], \tag{2.7.12}$$

for the relative arithmetic return of portfolio π versus portfolio η.

Proof. This result follows easily by a simple modification of the previous lemma and employing equation (2.2.126), much in the same way as was done for Lemma 2.6.4. Hence, we deduce

$$\begin{aligned} d\left(\frac{Z_\pi(t)}{Z_\eta(t)}\right) &= \frac{Z_\pi(t)}{Z_\eta(t)}\left[\left(\gamma_\pi(t) - \gamma_\eta(t) + \frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)dt + \sum_{\nu=1}^n \left[\sum_{i=1}^n \pi_i(t)\xi_{i\nu}(t) - \xi_{\eta\nu}(t)\right]dW_\nu(t)\right] \\ &= \frac{Z_\pi(t)}{Z_\eta(t)}\left[\left(\gamma_\pi(t) - \gamma_\eta(t) + \frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)dt + \sum_{\nu=1}^n \big(\xi_{\pi\nu}(t) - \xi_{\eta\nu}(t)\big)\,dW_\nu(t)\right]. \end{aligned}$$

Alternatively, upon substitution of equation (2.6.11) of Lemma 2.6.4 into equation (2.7.7), we further obtain

$$\begin{aligned} d\left(\frac{Z_\pi(t)}{Z_\eta(t)}\right) &= \frac{Z_\pi(t)}{Z_\eta(t)}\left[\big(\gamma_\pi(t) - \gamma_\eta(t)\big)\,dt + \sum_{\nu=1}^n \big(\xi_{\pi\nu}(t) - \xi_{\eta\nu}(t)\big)\,dW_\nu(t)\right] + \frac{1}{2}\,\frac{Z_\pi(t)}{Z_\eta(t)}\,\tau^\eta_{\pi\pi}(t)\,dt \\ &= \left(\gamma_\pi(t) - \gamma_\eta(t) + \frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)\frac{Z_\pi(t)}{Z_\eta(t)}\,dt + \frac{Z_\pi(t)}{Z_\eta(t)}\sum_{\nu=1}^n \big(\xi_{\pi\nu}(t) - \xi_{\eta\nu}(t)\big)\,dW_\nu(t) \\ &= \frac{Z_\pi(t)}{Z_\eta(t)}\left[\left(\gamma_\pi(t) - \gamma_\eta(t) + \frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)dt + \sum_{\nu=1}^n \big(\xi_{\pi\nu}(t) - \xi_{\eta\nu}(t)\big)\,dW_\nu(t)\right]. \end{aligned}$$

■

Lemma 2.7.6. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t \in [0, \infty)$, we obtain the dynamics*

$$d\left(\frac{Z_\pi(t)}{Z_\eta(t)}\right) = \frac{Z_\pi(t)}{Z_\eta(t)}\left[\left(\gamma^*_\pi(t) - \gamma^*_\eta(t) + \frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)dt + \sum_{i,j=1}^n \pi_i(t)\eta_j(t)\,d\log\Big(X_i(t)/X_j(t)\Big)\right], \tag{2.7.13}$$

for the relative arithmetic return of portfolio π versus portfolio η.

Proof. This is easily derived by inserting the result (2.3.45) derived in Lemma 2.3.6, into (2.7.2) of Lemma 2.7.2. Alternatively, by substituting equation (2.6.12) of Lemma 2.6.5 into equation (2.7.7), we obtain the required result

$$\begin{aligned} d\left(\frac{Z_\pi(t)}{Z_\eta(t)}\right) &= \frac{Z_\pi(t)}{Z_\eta(t)}\left[\big(\gamma^*_\pi(t) - \gamma^*_\eta(t)\big)\,dt + \sum_{i,j=1}^n \pi_i(t)\eta_j(t)\,d\log\Big(X_i(t)/X_j(t)\Big)\right] + \frac{1}{2}\,\frac{Z_\pi(t)}{Z_\eta(t)}\,\tau^\eta_{\pi\pi}(t)\,dt \\ &= \frac{Z_\pi(t)}{Z_\eta(t)}\left[\left(\gamma^*_\pi(t) - \gamma^*_\eta(t) + \frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)dt + \sum_{i,j=1}^n \pi_i(t)\eta_j(t)\,d\log\Big(X_i(t)/X_j(t)\Big)\right]. \end{aligned}$$

■

Lemma 2.7.7. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t \in [0, \infty)$, we obtain the dynamics*

$$d\left(\frac{Z_\pi(t)}{Z_\eta(t)}\right) = \frac{Z_\pi(t)}{Z_\eta(t)}\left[-\,2\,\gamma^*_\eta(t)\,dt + \sum_{i,j=1}^n \pi_i(t)\eta_j(t)\frac{X_j(t)}{X_i(t)}\,d\left(\frac{X_i(t)}{X_j(t)}\right)\right], \tag{2.7.14}$$

for the relative arithmetic return of portfolio π versus portfolio η.

Proof. Upon substitution of equation (2.6.14) of Lemma 2.6.6 into equation (2.7.7), we obtain the required result

$$\begin{aligned} d\left(\frac{Z_\pi(t)}{Z_\eta(t)}\right) &= \frac{Z_\pi(t)}{Z_\eta(t)}\left[\left(-\frac{1}{2}\,\tau^\eta_{\pi\pi}(t) - 2\,\gamma^*_\eta(t)\right)dt + \sum_{i,j=1}^n \pi_i(t)\eta_j(t)\,\frac{X_j(t)}{X_i(t)}\,d\left(\frac{X_i(t)}{X_j(t)}\right)\right] + \frac{1}{2}\,\frac{Z_\pi(t)}{Z_\eta(t)}\,\tau^\eta_{\pi\pi}(t)\,dt \\ &= \frac{Z_\pi(t)}{Z_\eta(t)}\left[-2\,\gamma^*_\eta(t)\,dt + \sum_{i,j=1}^n \pi_i(t)\eta_j(t)\frac{X_j(t)}{X_i(t)}\,d\left(\frac{X_i(t)}{X_j(t)}\right)\right]. \end{aligned}$$

■

Lemma 2.7.8. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t \in [0,\infty)$, we obtain the dynamics*

$$d\left(\frac{Z_\pi(t)}{Z_\eta(t)}\right) = \frac{Z_\pi(t)}{Z_\eta(t)}\left[\left(\gamma^*_\pi(t) - \gamma^*_\eta(t) + \frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)dt + \sum_{i=1}^n \big(\pi_i(t) - \eta_i(t)\big)\,d\log\Big(X_i(t)/Z_\eta(t)\Big)\right], \tag{2.7.15}$$

for the relative arithmetic return of portfolio π versus portfolio η.

Proof. Adopting equation (2.6.16) of Lemma 2.6.7, we obtain

$$\begin{aligned} d\left\langle \log\big(Z_\pi/Z_\eta\big)\right\rangle_t &= d\left\langle \int_0^\cdot \sum_{i=1}^n \big(\pi_{i,s} - \eta_{i,s}\big)\,d\log\big(X_{i,s}/Z_{\eta,s}\big), \int_0^\cdot \sum_{j=1}^n \big(\pi_{j,s} - \eta_{j,s}\big)\,d\log\big(X_{j,s}/Z_{\eta,s}\big)\right\rangle_t \\ &= d\left\langle \sum_{i=1}^n \int_0^\cdot \big(\pi_{i,s} - \eta_{i,s}\big)\,d\log\big(X_{i,s}/Z_{\eta,s}\big), \sum_{j=1}^n \int_0^\cdot \big(\pi_{j,s} - \eta_{j,s}\big)\,d\log\big(X_{j,s}/Z_{\eta,s}\big)\right\rangle_t \\ &= \sum_{i,j=1}^n d\left\langle \int_0^\cdot \big(\pi_{i,s} - \eta_{i,s}\big)\,d\log\big(X_{i,s}/Z_{\eta,s}\big), \int_0^\cdot \big(\pi_{j,s} - \eta_{j,s}\big)\,d\log\big(X_{j,s}/Z_{\eta,s}\big)\right\rangle_t \\ &= \sum_{i,j=1}^n \big(\pi_i(t) - \eta_i(t)\big)\big(\pi_j(t) - \eta_j(t)\big)\,d\left\langle \log\big(X_i/Z_\eta\big), \log\big(X_j/Z_\eta\big)\right\rangle_t \\ &= \sum_{i,j=1}^n \big(\pi_i(t) - \eta_i(t)\big)\tau^\eta_{ij}(t)\big(\pi_j(t) - \eta_j(t)\big)\,dt \\ &= \sum_{i,j=1}^n \pi_i(t)\tau^\eta_{ij}(t)\pi_j(t)\,dt - \sum_{i,j=1}^n \pi_i(t)\tau^\eta_{ij}(t)\eta_j(t)\,dt - \sum_{i,j=1}^n \eta_i(t)\tau^\eta_{ij}(t)\pi_j(t)\,dt \\ &\quad + \sum_{i,j=1}^n \eta_i(t)\tau^\eta_{ij}(t)\eta_j(t)\,dt \\ &= \sum_{i,j=1}^n \pi_i(t)\tau^\eta_{ij}(t)\pi_j(t)\,dt - \sum_{i=1}^n \pi_i(t)\left[\sum_{j=1}^n \eta_j(t)\tau^\eta_{ij}(t)\right]dt - \sum_{j=1}^n \pi_j(t)\left[\sum_{i=1}^n \eta_i(t)\tau^\eta_{ij}(t)\right]dt \\ &\quad + \sum_{i,j=1}^n \eta_i(t)\tau^\eta_{ij}(t)\eta_j(t)\,dt, \end{aligned}$$

which, in conjunction with equation (2.4.1) of Lemma 2.4.1, equation (2.4.7) and recalling the symmetry property of the relative covariance matrix, i.e., $\tau^\eta_{ij}(t) = \tau^\eta_{ji}(t)$ for all $t \in [0,\infty)$, becomes

$$d\left\langle \log\big(Z_\pi/Z_\eta\big)\right\rangle_t = \sum_{i,j=1}^n \pi_i(t)\tau^\eta_{ij}(t)\pi_j(t)\,dt = \tau^\eta_{\pi\pi}(t)\,dt,$$

which agrees with the result (2.6.79). Thus, inserting equation (2.6.16) into equation (2.7.7), yields

$$\begin{aligned} d\left(\frac{Z_\pi(t)}{Z_\eta(t)}\right) &= \frac{Z_\pi(t)}{Z_\eta(t)}\left[\big(\gamma^*_\pi(t) - \gamma^*_\eta(t)\big)\,dt + \sum_{i=1}^n \big(\pi_i(t) - \eta_i(t)\big)\,d\log\Big(X_i(t)/Z_\eta(t)\Big)\right] + \frac{1}{2}\,\frac{Z_\pi(t)}{Z_\eta(t)}\,\tau^\eta_{\pi\pi}(t)\,dt \\ &= \frac{Z_\pi(t)}{Z_\eta(t)}\left[\left(\gamma^*_\pi(t) - \gamma^*_\eta(t) + \frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)dt + \sum_{i=1}^n \big(\pi_i(t) - \eta_i(t)\big)\,d\log\Big(X_i(t)/Z_\eta(t)\Big)\right]. \end{aligned}$$

■

Lemma 2.7.9. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t \in [0, \infty)$, we obtain the dynamics*

$$d\left(\frac{Z_\pi(t)}{Z_\eta(t)}\right) = \sum_{i=1}^{n} \pi_i(t) \frac{Z_\pi(t)}{X_i(t)}\, d\left(\frac{X_i(t)}{Z_\eta(t)}\right), \tag{2.7.16}$$

for the relative arithmetic return of portfolio π versus portfolio η.

Proof. By substituting equation (2.6.18) into equation (2.7.7), we obtain the required result

$$\begin{aligned} d\left(\frac{Z_\pi(t)}{Z_\eta(t)}\right) &= \frac{Z_\pi(t)}{Z_\eta(t)}\left[\sum_{i=1}^{n} \pi_i(t) \frac{Z_\eta(t)}{X_i(t)}\, d\left(\frac{X_i(t)}{Z_\eta(t)}\right) - \frac{1}{2}\,\tau^{\eta}_{\pi\pi}(t)\, dt\right] + \frac{1}{2}\,\frac{Z_\pi(t)}{Z_\eta(t)}\,\tau^{\eta}_{\pi\pi}(t)\, dt \\ &= \sum_{i=1}^{n} \pi_i(t) \frac{Z_\pi(t)}{X_i(t)}\, d\left(\frac{X_i(t)}{Z_\eta(t)}\right). \end{aligned}$$

Notice that, by substituting (2.5.32) into (2.7.16), combined with (2.2.111) and (2.4.29), we obtain (2.7.11). ■

From the above results, it can be easily verified that the quadratic variation of the relative arithmetic return process satisfies the following for all $t \in [0, \infty)$, a.s.,

$$d\left\langle \frac{Z_\pi}{Z_\eta} \right\rangle_t = \left(\frac{Z_\pi(t)}{Z_\eta(t)}\right)^2 d\left\langle \log\left(Z_\pi/Z_\eta\right)\right\rangle_t \tag{2.7.17}$$

$$= \tau^{\eta}_{\pi\pi}(t) \left(\frac{Z_\pi(t)}{Z_\eta(t)}\right)^2 dt. \tag{2.7.18}$$

Alternatively, from equation (2.7.14) of Lemma 2.7.7, we equivalently obtain

$$\begin{aligned} d\left\langle \frac{Z_\pi}{Z_\eta} \right\rangle_t &= d\left\langle \int_0^{\cdot} \frac{Z_{\pi,s}}{Z_{\eta,s}} \sum_{i,k=1}^{n} \pi_{i,s}\eta_{k,s} \frac{X_{k,s}}{X_{i,s}}\, d\left(\frac{X_{i,s}}{X_{k,s}}\right), \int_0^{\cdot} \frac{Z_{\pi,s}}{Z_{\eta,s}} \sum_{j,m=1}^{n} \pi_{j,s}\eta_{m,s} \frac{X_{m,s}}{X_{j,s}}\, d\left(\frac{X_{j,s}}{X_{m,s}}\right) \right\rangle_t \\ &= d\left\langle \sum_{i=1}^{n} \int_0^{\cdot} \sum_{k=1}^{n} \pi_{i,s}\eta_{k,s} \frac{Z_{\pi,s}}{Z_{\eta,s}} \frac{X_{k,s}}{X_{i,s}}\, d\left(\frac{X_{i,s}}{X_{k,s}}\right), \sum_{j=1}^{n} \int_0^{\cdot} \sum_{m=1}^{n} \pi_{j,s}\eta_{m,s} \frac{Z_{\pi,s}}{Z_{\eta,s}} \frac{X_{m,s}}{X_{j,s}}\, d\left(\frac{X_{j,s}}{X_{m,s}}\right) \right\rangle_t \\ &= \sum_{i,j=1}^{n} d\left\langle \int_0^{\cdot} \sum_{k=1}^{n} \pi_{i,s}\eta_{k,s} \frac{Z_{\pi,s}}{Z_{\eta,s}} \frac{X_{k,s}}{X_{i,s}}\, d\left(\frac{X_{i,s}}{X_{k,s}}\right), \int_0^{\cdot} \sum_{m=1}^{n} \pi_{j,s}\eta_{m,s} \frac{Z_{\pi,s}}{Z_{\eta,s}} \frac{X_{m,s}}{X_{j,s}}\, d\left(\frac{X_{j,s}}{X_{m,s}}\right) \right\rangle_t \\ &= \sum_{i,j=1}^{n} \sum_{k,m=1}^{n} \pi_i(t)\pi_j(t)\eta_k(t)\eta_m(t) \left(\frac{Z_\pi(t)}{Z_\eta(t)}\right)^2 \frac{X_k(t)X_m(t)}{X_i(t)X_j(t)}\, d\left\langle \frac{X_i}{X_k}, \frac{X_j}{X_m} \right\rangle_t \qquad (2.7.19) \\ &= \sum_{i,j=1}^{n} \sum_{k,m=1}^{n} \pi_i(t)\pi_j(t)\eta_k(t)\eta_m(t) \left(\frac{Z_\pi(t)}{Z_\eta(t)}\right)^2 d\left\langle \log\left(X_i/X_k\right), \log\left(X_j/X_m\right)\right\rangle_t \qquad (2.7.20) \\ &= \left(\frac{Z_\pi(t)}{Z_\eta(t)}\right)^2 \sum_{i,j=1}^{n} \pi_i(t)\pi_j(t) \sum_{k,m=1}^{n} \eta_k(t)\tau^{X_kX_m}_{ij}(t)\eta_m(t)\, dt \qquad (2.7.21) \\ &= \left(\frac{Z_\pi(t)}{Z_\eta(t)}\right)^2 \sum_{i,j=1}^{n} \pi_i(t)\tau^{\eta}_{ij}(t)\pi_j(t)\, dt \qquad (2.7.22) \\ &= \tau^{\eta}_{\pi\pi}(t) \left(\frac{Z_\pi(t)}{Z_\eta(t)}\right)^2 dt, \qquad (2.7.23) \end{aligned}$$

where equation (2.7.20) follows from equation (2.5.14), equation (2.7.21) follows from equation (2.3.18), equation (2.7.22) follows from equation (2.3.119), and finally, equation (2.7.23) follows from equation (2.4.17) of Definition

2.4.3. In addition, we can also use equation (2.7.16) of Lemma 2.7.9, to yield the same result

$$\begin{aligned}
d\left\langle \frac{Z_\pi}{Z_\eta} \right\rangle_t &= d\left\langle \int_0^\cdot \sum_{i=1}^n \pi_{i,s} \frac{Z_{\pi,s}}{X_{i,s}}\, d\left(\frac{X_{i,s}}{Z_{\eta,s}}\right), \int_0^\cdot \sum_{j=1}^n \pi_{j,s} \frac{Z_{\pi,s}}{X_{j,s}}\, d\left(\frac{X_{j,s}}{Z_{\eta,s}}\right) \right\rangle_t \\
&= d\left\langle \sum_{i=1}^n \int_0^\cdot \pi_{i,s} \frac{Z_{\pi,s}}{X_{i,s}}\, d\left(\frac{X_{i,s}}{Z_{\eta,s}}\right), \sum_{j=1}^n \int_0^\cdot \pi_{j,s} \frac{Z_{\pi,s}}{X_{j,s}}\, d\left(\frac{X_{j,s}}{Z_{\eta,s}}\right) \right\rangle_t \\
&= \sum_{i,j=1}^n d\left\langle \int_0^\cdot \pi_{i,s} \frac{Z_{\pi,s}}{X_{i,s}}\, d\left(\frac{X_{i,s}}{Z_{\eta,s}}\right), \int_0^\cdot \pi_{j,s} \frac{Z_{\pi,s}}{X_{j,s}}\, d\left(\frac{X_{j,s}}{Z_{\eta,s}}\right) \right\rangle_t \\
&= \sum_{i,j=1}^n \pi_i(t)\pi_j(t) \frac{Z_\pi^2(t)}{X_i(t)X_j(t)}\, d\left\langle \frac{X_i}{Z_\eta}, \frac{X_j}{Z_\eta} \right\rangle_t && (2.7.24) \\
&= \sum_{i,j=1}^n \pi_i(t)\pi_j(t) \left(\frac{Z_\pi(t)}{Z_\eta(t)}\right)^2 d\left\langle \log\left(X_i/Z_\eta\right), \log\left(X_j/Z_\eta\right) \right\rangle_t && (2.7.25) \\
&= \left(\frac{Z_\pi(t)}{Z_\eta(t)}\right)^2 \sum_{i,j=1}^n \pi_i(t)\tau_{ij}^\eta(t)\pi_j(t)\, dt && (2.7.26) \\
&= \tau_{\pi\pi}^\eta(t) \left(\frac{Z_\pi(t)}{Z_\eta(t)}\right)^2 dt, && (2.7.27)
\end{aligned}$$

where equation (2.7.25) follows from equation (2.5.37), equation (2.7.26) follows from either of equations (2.3.84), (2.3.113) or (2.3.118), and finally, equation (2.7.27) follows from equation (2.4.17) of Definition 2.4.3.

2.8 Relative Total Return

Definition 2.8.1 (Relative Total Return of Stocks). *For a stock $X_i = \{X_i(t), t \in [0,\infty)\}$, for $i = 1, 2, \ldots, n$, and stock $X_j = \{X_j(t), t \in [0,\infty)\}$, the process $\log\left(\widehat{X}_i/\widehat{X}_j\right) = \left\{\log\left(\widehat{X}_i(t)/\widehat{X}_j(t)\right), t \in [0,\infty)\right\}$, is called the* ***relative augmented return process*** *(or, the* ***relative total return process****) of X_i versus X_j.*

By equation (2.2.151), the total return of the ith stock relative to the jth stock can be described by the following dynamics

$$\begin{aligned}
d\log\left(\widehat{X}_i(t)/\widehat{X}_j(t)\right) &= d\log\widehat{X}_i(t) - d\log\widehat{X}_j(t) \\
&= \Big(d\log X_i(t) + \delta_i(t)\, dt\Big) - \Big(d\log X_j(t) + \delta_j(t)\, dt\Big) \\
&= d\log\Big(X_i(t)/X_j(t)\Big) + \big(\delta_i(t) - \delta_j(t)\big)\, dt. && (2.8.1)
\end{aligned}$$

Hence, from equations (2.3.1) and (2.2.152) of Definition 2.2.35, we have

$$\begin{aligned}
d\log\left(\widehat{X}_i(t)/\widehat{X}_j(t)\right) &= \left(\big(\gamma_i(t) - \gamma_j(t)\big)\, dt + \sum_{\nu=1}^n \big(\xi_{i\nu}(t) - \xi_{j\nu}(t)\big)\, dW_\nu(t)\right) + \big(\delta_i(t) - \delta_j(t)\big)\, dt \\
&= \big(\vartheta_i(t) - \vartheta_j(t)\big)\, dt + \sum_{\nu=1}^n \big(\xi_{i\nu}(t) - \xi_{j\nu}(t)\big)\, dW_\nu(t). && (2.8.2)
\end{aligned}$$

Alternatively, by equation (2.2.153), the total return of the ith stock relative to the jth stock can be described by the following dynamics

$$\begin{aligned}
d\log\left(\widehat{X}_i(t)/\widehat{X}_j(t)\right) &= d\log\widehat{X}_i(t) - d\log\widehat{X}_j(t) \\
&= \left(\vartheta_i(t)\, dt + \sum_{\nu=1}^n \xi_{i\nu}(t)\, dW_\nu(t)\right) - \left(\vartheta_j(t)\, dt + \sum_{\nu=1}^n \xi_{j\nu}(t)\, dW_\nu(t)\right) \\
&= \big(\vartheta_i(t) - \vartheta_j(t)\big)\, dt + \sum_{\nu=1}^n \big(\xi_{i\nu}(t) - \xi_{j\nu}(t)\big)\, dW_\nu(t). && (2.8.3)
\end{aligned}$$

Furthermore, the cross-variation for the relative total return process, for $t \in [0, \infty)$, is a.s. given by

$$\begin{aligned}\left\langle \log\left(\widehat{X}_i/\widehat{X}_k\right), \log\left(\widehat{X}_j/\widehat{X}_k\right)\right\rangle_t &= \left\langle \log \widehat{X}_i - \log \widehat{X}_k, \log \widehat{X}_j - \log \widehat{X}_k\right\rangle_t \\ &= \left\langle \log \widehat{X}_i, \log \widehat{X}_j\right\rangle_t - \left\langle \log \widehat{X}_i, \log \widehat{X}_k\right\rangle_t - \left\langle \log \widehat{X}_k, \log \widehat{X}_j\right\rangle_t + \left\langle \log \widehat{X}_k\right\rangle_t \\ &= \left\langle \log \widehat{X}_i, \log \widehat{X}_j\right\rangle_t - \left\langle \log \widehat{X}_i, \log \widehat{X}_k\right\rangle_t - \left\langle \log \widehat{X}_j, \log \widehat{X}_k\right\rangle_t + \left\langle \log \widehat{X}_k\right\rangle_t. \end{aligned} \tag{2.8.4}$$

The above expression, together with equations (2.2.179) and (2.2.175), yields the following result

$$\left\langle \log\left(\widehat{X}_i/\widehat{X}_k\right), \log\left(\widehat{X}_j/\widehat{X}_k\right)\right\rangle_t = \left\langle \log X_i, \log X_j\right\rangle_t - \left\langle \log X_i, \log X_k\right\rangle_t - \left\langle \log X_j, \log X_k\right\rangle_t + \left\langle \log X_k\right\rangle_t, \tag{2.8.5}$$

which, from equation (2.3.10), is equivalent to the cross-variation for the relative return process. Consequently, we have

$$\left\langle \log\left(\widehat{X}_i/\widehat{X}_k\right), \log\left(\widehat{X}_j/\widehat{X}_k\right)\right\rangle_t \equiv \left\langle \log\left(X_i/X_k\right), \log\left(X_j/X_k\right)\right\rangle_t. \tag{2.8.6}$$

This is the cross-variation process for continuous semimartingales, $\log\left(\widehat{X}_i/\widehat{X}_k\right)$ and $\log\left(\widehat{X}_j/\widehat{X}_k\right)$. The above expression reveals that the cross-variation process is unaltered when dividends are introduced into the model, this is what we would expect since dividends are only captured in the drift. Therefore, we have

$$\begin{aligned} d\left\langle \log\left(\widehat{X}_i/\widehat{X}_k\right), \log\left(\widehat{X}_j/\widehat{X}_k\right)\right\rangle_t &= d\left\langle \log \widehat{X}_i, \log \widehat{X}_j\right\rangle_t - d\left\langle \log \widehat{X}_i, \log \widehat{X}_k\right\rangle_t - d\left\langle \log \widehat{X}_j, \log \widehat{X}_k\right\rangle_t + d\left\langle \log \widehat{X}_k\right\rangle_t \\ &= d\left\langle \log X_i, \log X_j\right\rangle_t - d\left\langle \log X_i, \log X_k\right\rangle_t - d\left\langle \log X_j, \log X_k\right\rangle_t + d\left\langle \log X_k\right\rangle_t \\ &\equiv d\left\langle \log\left(X_i/X_k\right), \log\left(X_j/X_k\right)\right\rangle_t. \end{aligned} \tag{2.8.7}$$

Consequently, from equation (2.8.1) and equation (2.3.2) of Definition 2.3.2 (or, equation (2.3.3)), of the relative covariance process, it is apparent that

$$d\left\langle \log\left(\widehat{X}_i/\widehat{X}_k\right), \log\left(\widehat{X}_j/\widehat{X}_k\right)\right\rangle_t = d\left\langle \log\left(X_i/X_k\right), \log\left(X_j/X_k\right)\right\rangle_t = \tau_{ij}^{\chi_k}(t)\,dt, \quad \text{or,} \tag{2.8.8}$$

$$\left\langle \log\left(\widehat{X}_i/\widehat{X}_k\right), \log\left(\widehat{X}_j/\widehat{X}_k\right)\right\rangle_t = \left\langle \log\left(X_i/X_k\right), \log\left(X_j/X_k\right)\right\rangle_t = \int_0^t \tau_{ij}^{\chi_k}(s)\,ds. \tag{2.8.9}$$

Analogously, for the quadratic variation process of $\log\left(\widehat{X}_i/\widehat{X}_k\right)$, we obtain the following expressions

$$\begin{aligned}\left\langle \log\left(\widehat{X}_i/\widehat{X}_k\right)\right\rangle_t &= \left\langle \log \widehat{X}_i - \log \widehat{X}_k\right\rangle_t \\ &= \left\langle \log \widehat{X}_i\right\rangle_t - \left\langle \log \widehat{X}_i, \log \widehat{X}_k\right\rangle_t - \left\langle \log \widehat{X}_k, \log \widehat{X}_i\right\rangle_t + \left\langle \log \widehat{X}_k\right\rangle_t \\ &= \left\langle \log \widehat{X}_i\right\rangle_t - 2\left\langle \log \widehat{X}_i, \log \widehat{X}_k\right\rangle_t + \left\langle \log \widehat{X}_k\right\rangle_t. \end{aligned} \tag{2.8.10}$$

The above expression, together with equations (2.2.175) and (2.2.179), yields the following result

$$\left\langle \log\left(\widehat{X}_i/\widehat{X}_k\right)\right\rangle_t = \left\langle \log X_i\right\rangle_t - 2\left\langle \log X_i, \log X_k\right\rangle_t + \left\langle \log X_k\right\rangle_t. \tag{2.8.11}$$

Consequently, from (2.3.28), we have

$$\left\langle \log\left(\widehat{X}_i/\widehat{X}_k\right)\right\rangle_t \equiv \left\langle \log\left(X_i/X_k\right)\right\rangle_t. \tag{2.8.12}$$

Therefore, we have

$$\begin{aligned} d\left\langle \log\left(\widehat{X}_i/\widehat{X}_k\right)\right\rangle_t &= d\left\langle \log \widehat{X}_i\right\rangle_t - 2\,d\left\langle \log \widehat{X}_i, \log \widehat{X}_k\right\rangle_t + d\left\langle \log \widehat{X}_k\right\rangle_t \\ &= d\left\langle \log X_i\right\rangle_t - 2\,d\left\langle \log X_i, \log X_k\right\rangle_t + d\left\langle \log X_k\right\rangle_t \\ &\equiv d\left\langle \log\left(X_i/X_k\right)\right\rangle_t. \end{aligned} \tag{2.8.13}$$

In addition, from equation (2.8.1) as well as from the foregoing arguments, in conjunction with equation (2.3.22), for all $t \in [0, \infty)$, the quadratic variation of $\log\left(\widehat{X}_i/\widehat{X}_k\right)$ is given by

$$d\left\langle \log\left(\widehat{X}_i/\widehat{X}_k\right)\right\rangle_t = d\left\langle \log\left(X_i/X_k\right)\right\rangle_t = \tau_{ii}^{\chi_k}(t)\,dt, \quad \text{or,} \tag{2.8.14}$$

$$\left\langle \log\left(\widehat{X}_i/\widehat{X}_k\right)\right\rangle_t = \left\langle \log\left(X_i/X_k\right)\right\rangle_t = \int_0^t \tau_{ii}^{\chi_k}(s)\,ds. \tag{2.8.15}$$

In addition, from equations (2.3.18) and (2.3.38), we have the following

$$d\left\langle \log\left(\widehat{X}_i/\widehat{X}_k\right), \log\left(\widehat{X}_j/\widehat{X}_m\right)\right\rangle_t = d\left\langle \log\left(X_i/X_k\right), \log\left(X_j/X_m\right)\right\rangle_t = \tau_{ij}^{\mathsf{X}_k\mathsf{X}_m}(t)\,dt, \quad \text{and,} \tag{2.8.16}$$

$$d\left\langle \log\left(\widehat{X}_i/\widehat{X}_k\right), \log\left(\widehat{X}_i/\widehat{X}_m\right)\right\rangle_t = d\left\langle \log\left(X_i/X_k\right), \log\left(X_i/X_m\right)\right\rangle_t = \tau_{ii}^{\mathsf{X}_k\mathsf{X}_m}(t)\,dt. \tag{2.8.17}$$

Definition 2.8.2 (Relative Total Return). *For a stock $X_i = \{X_i(t), t \in [0,\infty)\}$ with initial capital $X_i(0) = x_i$, for $i = 1, 2, \ldots, n$, portfolio $\eta = \left\{\boldsymbol{\eta}(t) = \left(\eta_1(t), \ldots, \eta_n(t)\right), t \in [0,\infty)\right\}$ and portfolio value process $Z_{w,\eta} = \left\{Z_{w,\eta}(t), t \in [0,\infty)\right\}$ with initial capital $Z_{w,\eta}(0) = w > 0$, with $w := X_i(0)$, the process $\log\left(\widehat{X}_i/\widehat{Z}_\eta\right) = \left\{\log\left(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\right), t \in [0,\infty)\right\}$, i.e.,*

$$\log\left(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\right) = \log\left(\widehat{X}_i(t)/\widehat{Z}_{w,\eta}(t)\right)\Big|_{w=X_i(0)}, \qquad t \in [0,\infty), \tag{2.8.18}$$

*is called the **relative augmented return process** (or, the **relative total return process**) of X_i versus η, for all $i = 1, 2, \ldots, n$.*

The relative total return process is denoted by $\widehat{R}_i^\eta(t) := \log\left(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\right) = \log \widehat{X}_i(t) - \log \widehat{Z}_\eta(t)$. By adopting this notation, and employing equations (2.2.151) and (2.2.157), we obtain

$$\begin{aligned} d\widehat{R}_i^\eta(t) &= d\log\left(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\right) \\ &= d\log \widehat{X}_i(t) - d\log \widehat{Z}_\eta(t) \\ &= \left(d\log X_i(t) + \delta_i(t)\,dt\right) - \left(d\log Z_\eta(t) + \delta_\eta(t)\,dt\right) \\ &= d\log\left(X_i(t)/Z_\eta(t)\right) + \left(\delta_i(t) - \delta_\eta(t)\right)dt. \end{aligned} \tag{2.8.19}$$

Consequently, from equation (2.3.43), we have

$$\begin{aligned} d\log\left(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\right) &= \left(\left(\gamma_i(t) - \gamma_\eta(t)\right)dt + \sum_{\nu=1}^n \left(\xi_{i\nu}(t) - \xi_{\eta\nu}(t)\right)dW_\nu(t)\right) + \left(\delta_i(t) - \delta_\eta(t)\right)dt \\ &= \left(\vartheta_i(t) - \vartheta_\eta(t)\right)dt + \sum_{\nu=1}^n \left(\xi_{i\nu}(t) - \xi_{\eta\nu}(t)\right)dW_\nu(t), \end{aligned} \tag{2.8.20}$$

where the last equation follows from equation (2.2.152) of Definition 2.2.35 and equation (2.2.161) of Definition 2.2.38. By comparing equations (2.2.153) and (2.2.162) (or, (2.2.163)), we arrive at the following

$$\begin{aligned} d\widehat{R}_i^\eta(t) &= d\log\left(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\right) \\ &= d\log \widehat{X}_i(t) - d\log \widehat{Z}_\eta(t) \\ &= \left(\vartheta_i(t)\,dt + \sum_{\nu=1}^n \xi_{i\nu}(t)\,dW_\nu(t)\right) - \left(\vartheta_\eta(t)\,dt + \sum_{j,\nu=1}^n \eta_j(t)\xi_{j\nu}(t)\,dW_\nu(t)\right) \\ &= \left(\vartheta_i(t)\,dt + \sum_{j,\nu=1}^n \eta_j(t)\xi_{i\nu}(t)\,dW_\nu(t)\right) - \left(\vartheta_\eta(t)\,dt + \sum_{j,\nu=1}^n \eta_j(t)\xi_{j\nu}(t)\,dW_\nu(t)\right) \\ &= \left(\vartheta_i(t) - \vartheta_\eta(t)\right)dt + \sum_{j,\nu=1}^n \eta_j(t)\left(\xi_{i\nu}(t) - \xi_{j\nu}(t)\right)dW_\nu(t). \end{aligned} \tag{2.8.21}$$

Moreover, by appealing to equations (2.2.153), (2.2.162) (or, (2.2.163)), and (2.2.126), we can also arrive at the

following

$$\begin{aligned}
d\widehat{R}_i^{\eta}(t) &= d\log\Big(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\Big) \\
&= d\log\widehat{X}_i(t) - d\log\widehat{Z}_\eta(t) \\
&= \Big(\vartheta_i(t)\,dt + \sum_{\nu=1}^{n}\xi_{i\nu}(t)\,dW_\nu(t)\Big) - \Big(\vartheta_\eta(t)\,dt + \sum_{j,\nu=1}^{n}\eta_j(t)\xi_{j\nu}(t)\,dW_\nu(t)\Big) \\
&= \Big(\vartheta_i(t)\,dt + \sum_{\nu=1}^{n}\xi_{i\nu}(t)\,dW_\nu(t)\Big) - \Big(\vartheta_\eta(t)\,dt + \sum_{\nu=1}^{n}\Big[\sum_{j=1}^{n}\eta_j(t)\xi_{j\nu}(t)\Big]\,dW_\nu(t)\Big) \\
&= \Big(\vartheta_i(t)\,dt + \sum_{\nu=1}^{n}\xi_{i\nu}(t)\,dW_\nu(t)\Big) - \Big(\vartheta_\eta(t)\,dt + \sum_{\nu=1}^{n}\xi_{\eta\nu}(t)\,dW_\nu(t)\Big) \\
&= \big(\vartheta_i(t) - \vartheta_\eta(t)\big)\,dt + \sum_{\nu=1}^{n}\big(\xi_{i\nu}(t) - \xi_{\eta\nu}(t)\big)\,dW_\nu(t). \qquad (2.8.22)
\end{aligned}$$

It is evident that the relative total return process (2.8.18) is a continuous semimartingale, since both $\log\widehat{X}_i$ and $\log\widehat{Z}_\eta$ are continuous semimartingales provided that the requisite integrability conditions are fulfilled.

Lemma 2.8.3. *Let X_i denote the ith stock in the market $\mathcal{M}$ and let η be any arbitrary portfolio in the market $\mathcal{M}$. Then, a.s., for $t \in [0,\infty)$ and for $i = 1, 2, \ldots, n$, we obtain the dynamics*

$$d\log\Big(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\Big) = \sum_{j=1}^{n}\eta_j(t)\,d\log\Big(\widehat{X}_i(t)/\widehat{X}_j(t)\Big) - \vartheta_\eta^*(t)\,dt, \qquad (2.8.23)$$

for the relative total return of stock X_i versus portfolio η.

Proof. From equation (2.2.170), we have the equivalent result for the portfolio η, given by

$$d\log\widehat{Z}_\eta(t) = \sum_{i=1}^{n}\eta_i(t)\,d\log\widehat{X}_i(t) + \vartheta_\eta^*(t)\,dt.$$

The above result allows us to capture the relative total return process of the ith stock X_i versus the portfolio η, as follows

$$\begin{aligned}
d\log\Big(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\Big) &= d\log\widehat{X}_i(t) - d\log\widehat{Z}_\eta(t) \\
&= d\log\widehat{X}_i(t)\Big(\sum_{j=1}^{n}\eta_j(t)\Big) - \Big(\sum_{j=1}^{n}\eta_j(t)\,d\log\widehat{X}_j(t) + \vartheta_\eta^*(t)\,dt\Big) \\
&= \sum_{j=1}^{n}\eta_j(t)\,d\log\widehat{X}_i(t) - \sum_{j=1}^{n}\eta_j(t)\,d\log\widehat{X}_j(t) - \vartheta_\eta^*(t)\,dt \\
&= \sum_{j=1}^{n}\eta_j(t)\Big[d\log\widehat{X}_i(t) - d\log\widehat{X}_j(t)\Big] - \vartheta_\eta^*(t)\,dt \\
&= \sum_{j=1}^{n}\eta_j(t)\,d\log\Big(\widehat{X}_i(t)/\widehat{X}_j(t)\Big) - \vartheta_\eta^*(t)\,dt,
\end{aligned}$$

giving the required result. ■

The result of (2.8.21) can also be derived by substituting (2.8.3) into equation (2.8.23), obtained as follows

$$\begin{aligned}
d\log\left(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\right) &= \sum_{j=1}^{n}\eta_j(t)\,d\log\left(\widehat{X}_i(t)/\widehat{X}_j(t)\right) - \vartheta_\eta^*(t)\,dt \\
&= \sum_{j=1}^{n}\eta_j(t)\left[\big(\vartheta_i(t)-\vartheta_j(t)\big)\,dt + \sum_{\nu=1}^{n}\big(\xi_{i\nu}(t)-\xi_{j\nu}(t)\big)\,dW_\nu(t)\right] - \vartheta_\eta^*(t)\,dt \\
&= \left(\vartheta_i(t) - \sum_{j=1}^{n}\eta_j(t)\vartheta_j(t) - \vartheta_\eta^*(t)\right)dt + \sum_{j,\nu=1}^{n}\eta_j(t)\big(\xi_{i\nu}(t)-\xi_{j\nu}(t)\big)\,dW_\nu(t) \\
&= \big(\vartheta_i(t)-\vartheta_\eta(t)\big)\,dt + \sum_{j,\nu=1}^{n}\eta_j(t)\big(\xi_{i\nu}(t)-\xi_{j\nu}(t)\big)\,dW_\nu(t), \qquad (2.8.24)
\end{aligned}$$

where equation (2.8.24) is obtained by utilising equation (2.2.168). Moreover, the result of (2.8.22) can also be derived by substituting (2.8.3) into equation (2.8.23), obtained as follows

$$\begin{aligned}
d\log\left(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\right) &= \sum_{j=1}^{n}\eta_j(t)\,d\log\left(\widehat{X}_i(t)/\widehat{X}_j(t)\right) - \vartheta_\eta^*(t)\,dt \\
&= \sum_{j=1}^{n}\eta_j(t)\left[\big(\vartheta_i(t)-\vartheta_j(t)\big)\,dt + \sum_{\nu=1}^{n}\big(\xi_{i\nu}(t)-\xi_{j\nu}(t)\big)\,dW_\nu(t)\right] - \vartheta_\eta^*(t)\,dt \\
&= \left(\vartheta_i(t) - \sum_{j=1}^{n}\eta_j(t)\vartheta_j(t) - \vartheta_\eta^*(t)\right)dt + \sum_{\nu=1}^{n}\left[\xi_{i\nu}(t) - \sum_{j=1}^{n}\eta_j(t)\xi_{j\nu}(t)\right]dW_\nu(t) \\
&= \big(\vartheta_i(t)-\vartheta_\eta(t)\big)\,dt + \sum_{\nu=1}^{n}\big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t), \qquad (2.8.25)
\end{aligned}$$

where equation (2.8.25) is obtained by recalling equations (2.2.168) and (2.2.126).

Furthermore, the cross-variation for the relative total return process, for $t \in [0,\infty)$, is a.s. given by

$$\begin{aligned}
\left\langle \log\left(\widehat{X}_i/\widehat{Z}_\eta\right), \log\left(\widehat{X}_j/\widehat{Z}_\eta\right)\right\rangle_t &= \left\langle \log\widehat{X}_i - \log\widehat{Z}_\eta, \log\widehat{X}_j - \log\widehat{Z}_\eta\right\rangle_t \\
&= \left\langle \log\widehat{X}_i, \log\widehat{X}_j\right\rangle_t - \left\langle \log\widehat{X}_i, \log\widehat{Z}_\eta\right\rangle_t - \left\langle \log\widehat{Z}_\eta, \log\widehat{X}_j\right\rangle_t + \left\langle \log\widehat{Z}_\eta\right\rangle_t \\
&= \left\langle \log\widehat{X}_i, \log\widehat{X}_j\right\rangle_t - \left\langle \log\widehat{X}_i, \log\widehat{Z}_\eta\right\rangle_t - \left\langle \log\widehat{X}_j, \log\widehat{Z}_\eta\right\rangle_t + \left\langle \log\widehat{Z}_\eta\right\rangle_t. \qquad (2.8.26)
\end{aligned}$$

The above expression, together with equations (2.2.179), (2.3.78), and (2.2.176), yields the following result

$$\left\langle \log\left(\widehat{X}_i/\widehat{Z}_\eta\right), \log\left(\widehat{X}_j/\widehat{Z}_\eta\right)\right\rangle_t = \left\langle \log X_i, \log X_j\right\rangle_t - \left\langle \log X_i, \log Z_\eta\right\rangle_t - \left\langle \log X_j, \log Z_\eta\right\rangle_t + \left\langle \log Z_\eta\right\rangle_t, \qquad (2.8.27)$$

which, from equation (2.3.49), is equivalent to the cross-variation for the relative return process. Consequently, we have

$$\left\langle \log\left(\widehat{X}_i/\widehat{Z}_\eta\right), \log\left(\widehat{X}_j/\widehat{Z}_\eta\right)\right\rangle_t \equiv \left\langle \log\left(X_i/Z_\eta\right), \log\left(X_j/Z_\eta\right)\right\rangle_t. \qquad (2.8.28)$$

This is the cross-variation process for continuous semimartingales, $\log\left(\widehat{X}_i/\widehat{Z}_\eta\right)$ and $\log\left(\widehat{X}_j/\widehat{Z}_\eta\right)$. The above expression reveals that the cross-variation process is unaltered when dividends are introduced into the model, this is what we would expect since dividends are only captured in the drift. Therefore, we have

$$\begin{aligned}
d\left\langle \log\left(\widehat{X}_i/\widehat{Z}_\eta\right), \log\left(\widehat{X}_j/\widehat{Z}_\eta\right)\right\rangle_t &= d\left\langle \log\widehat{X}_i, \log\widehat{X}_j\right\rangle_t - d\left\langle \log\widehat{X}_i, \log\widehat{Z}_\eta\right\rangle_t - d\left\langle \log\widehat{X}_j, \log\widehat{Z}_\eta\right\rangle_t + d\left\langle \log\widehat{Z}_\eta\right\rangle_t \\
&= d\left\langle \log X_i, \log X_j\right\rangle_t - d\left\langle \log X_i, \log Z_\eta\right\rangle_t - d\left\langle \log X_j, \log Z_\eta\right\rangle_t + d\left\langle \log Z_\eta\right\rangle_t \\
&\equiv d\left\langle \log\left(X_i/Z_\eta\right), \log\left(X_j/Z_\eta\right)\right\rangle_t. \qquad (2.8.29)
\end{aligned}$$

Consequently, from equation (2.8.19) and equation (2.3.82) of Definition 2.3.8, of the relative covariance process, it is apparent that

$$d\left\langle \log\left(\widehat{X}_i/\widehat{Z}_\eta\right), \log\left(\widehat{X}_j/\widehat{Z}_\eta\right)\right\rangle_t = d\left\langle \log\left(X_i/Z_\eta\right), \log\left(X_j/Z_\eta\right)\right\rangle_t = \tau_{ij}^\eta(t)\,dt, \quad \text{or,} \tag{2.8.30}$$

$$\left\langle \log\left(\widehat{X}_i/\widehat{Z}_\eta\right), \log\left(\widehat{X}_j/\widehat{Z}_\eta\right)\right\rangle_t = \left\langle \log\left(X_i/Z_\eta\right), \log\left(X_j/Z_\eta\right)\right\rangle_t = \int_0^t \tau_{ij}^\eta(s)\,ds. \tag{2.8.31}$$

Analogously, for the quadratic variation process of $\log\left(\widehat{X}_i/\widehat{Z}_\eta\right)$, we obtain the following expressions

$$\begin{aligned}
\left\langle \log\left(\widehat{X}_i/\widehat{Z}_\eta\right)\right\rangle_t &= \left\langle \log \widehat{X}_i - \log \widehat{Z}_\eta\right\rangle_t \\
&= \left\langle \log \widehat{X}_i\right\rangle_t - \left\langle \log \widehat{X}_i, \log \widehat{Z}_\eta\right\rangle_t - \left\langle \log \widehat{Z}_\eta, \log \widehat{X}_i\right\rangle_t + \left\langle \log \widehat{Z}_\eta\right\rangle_t \\
&= \left\langle \log \widehat{X}_i\right\rangle_t - 2\left\langle \log \widehat{X}_i, \log \widehat{Z}_\eta\right\rangle_t + \left\langle \log \widehat{Z}_\eta\right\rangle_t.
\end{aligned} \tag{2.8.32}$$

The above expression, together with equations (2.2.175), (2.8.26), and (2.2.176), yields the following result

$$\left\langle \log\left(\widehat{X}_i/\widehat{Z}_\eta\right)\right\rangle_t = \left\langle \log X_i\right\rangle_t - 2\left\langle \log X_i, \log Z_\eta\right\rangle_t + \left\langle \log Z_\eta\right\rangle_t. \tag{2.8.33}$$

Consequently, from (2.3.126), we have

$$\left\langle \log\left(\widehat{X}_i/\widehat{Z}_\eta\right)\right\rangle_t \equiv \left\langle \log\left(X_i/Z_\eta\right)\right\rangle_t. \tag{2.8.34}$$

Therefore, we have

$$\begin{aligned}
d\left\langle \log\left(\widehat{X}_i/\widehat{Z}_\eta\right)\right\rangle_t &= d\left\langle \log \widehat{X}_i\right\rangle_t - 2\,d\left\langle \log \widehat{X}_i, \log \widehat{Z}_\eta\right\rangle_t + d\left\langle \log \widehat{Z}_\eta\right\rangle_t \\
&= d\left\langle \log X_i\right\rangle_t - 2\,d\left\langle \log X_i, \log Z_\eta\right\rangle_t + d\left\langle \log Z_\eta\right\rangle_t \\
&\equiv d\left\langle \log\left(X_i/Z_\eta\right)\right\rangle_t.
\end{aligned} \tag{2.8.35}$$

In addition, from equation (2.8.19) as well as from the foregoing arguments, in conjunction with equation (2.3.123), for all $t \in [0, \infty)$, we have

$$d\left\langle \log\left(\widehat{X}_i/\widehat{Z}_\eta\right)\right\rangle_t = d\left\langle \log\left(X_i/Z_\eta\right)\right\rangle_t = \tau_{ii}^\eta(t)\,dt, \quad \text{or,} \tag{2.8.36}$$

$$\left\langle \log\left(\widehat{X}_i/\widehat{Z}_\eta\right)\right\rangle_t = \left\langle \log\left(X_i/Z_\eta\right)\right\rangle_t = \int_0^t \tau_{ii}^\eta(s)\,ds. \tag{2.8.37}$$

2.9 Total Quotient Process

Definition 2.9.1 (Total Quotient Process of Stocks). *For stock $X_i = \{X_i(t), t \in [0,\infty)\}$ for $i = 1, 2, \ldots, n$, and stock $X_j = \left\{X_j(t), t \in [0,\infty)\right\}$, the process $\widehat{X}_i/\widehat{X}_j = \left\{\widehat{X}_i(t)/\widehat{X}_j(t), t \in [0,\infty)\right\}$, is called the* ***total quotient process*** *of X_i versus X_j.*

An application of Itô's formula to $\exp\left(\log\left(\widehat{X}_i(t)/\widehat{X}_j(t)\right)\right)$, along with equations (2.8.14) and (2.8.2), results in the following

$$d\left(\frac{\widehat{X}_i(t)}{\widehat{X}_j(t)}\right) = \frac{\widehat{X}_i(t)}{\widehat{X}_j(t)}\,d\log\left(\widehat{X}_i(t)/\widehat{X}_j(t)\right) + \frac{1}{2}\,\frac{\widehat{X}_i(t)}{\widehat{X}_j(t)}\,d\left\langle \log\left(\widehat{X}_i/\widehat{X}_j\right)\right\rangle_t \tag{2.9.1}$$

$$= \frac{\widehat{X}_i(t)}{\widehat{X}_j(t)}\,d\log\left(\widehat{X}_i(t)/\widehat{X}_j(t)\right) + \frac{1}{2}\,\frac{\widehat{X}_i(t)}{\widehat{X}_j(t)}\,\tau_{ii}^{\chi_j}(t)\,dt \tag{2.9.2}$$

$$= \frac{\widehat{X}_i(t)}{\widehat{X}_j(t)}\left[\left(\vartheta_i(t) - \vartheta_j(t)\right)dt + \sum_{\nu=1}^n \left(\xi_{i\nu}(t) - \xi_{j\nu}(t)\right)dW_\nu(t)\right] + \frac{1}{2}\,\frac{\widehat{X}_i(t)}{\widehat{X}_j(t)}\,\tau_{ii}^{\chi_j}(t)\,dt \tag{2.9.3}$$

$$= \left(\vartheta_i(t) - \vartheta_j(t) + \frac{1}{2}\,\tau_{ii}^{\chi_j}(t)\right)\frac{\widehat{X}_i(t)}{\widehat{X}_j(t)}\,dt + \frac{\widehat{X}_i(t)}{\widehat{X}_j(t)}\sum_{\nu=1}^n \left(\xi_{i\nu}(t) - \xi_{j\nu}(t)\right)dW_\nu(t) \tag{2.9.4}$$

$$= \frac{\widehat{X}_i(t)}{\widehat{X}_j(t)}\left[\left(\vartheta_i(t) - \vartheta_j(t) + \frac{1}{2}\,\tau_{ii}^{\chi_j}(t)\right)dt + \sum_{\nu=1}^n \left(\xi_{i\nu}(t) - \xi_{j\nu}(t)\right)dW_\nu(t)\right]. \tag{2.9.5}$$

We can rewrite equation (2.9.2) in the following beneficial forms

$$\frac{d\left(\widehat{X}_i(t)/\widehat{X}_j(t)\right)}{\widehat{X}_i(t)/\widehat{X}_j(t)} = d\log\left(\widehat{X}_i(t)/\widehat{X}_j(t)\right) + \frac{1}{2}\,\tau_{ii}^{\mathsf{X}_j}(t)\,dt, \tag{2.9.6}$$

and

$$d\log\left(\widehat{X}_i(t)/\widehat{X}_j(t)\right) = \frac{d\left(\widehat{X}_i(t)/\widehat{X}_j(t)\right)}{\widehat{X}_i(t)/\widehat{X}_j(t)} - \frac{1}{2}\,\tau_{ii}^{\mathsf{X}_j}(t)\,dt. \tag{2.9.7}$$

The quadratic variation of the aforementioned process can be obtained from equations (2.9.2) and (2.8.14), as follows

$$d\left\langle \frac{\widehat{X}_i}{\widehat{X}_k}\right\rangle_t = \left(\frac{\widehat{X}_i(t)}{\widehat{X}_k(t)}\right)^2 d\left\langle \log\left(\widehat{X}_i/\widehat{X}_k\right)\right\rangle_t \tag{2.9.8}$$

$$= \tau_{ii}^{\mathsf{X}_k}(t)\left(\frac{\widehat{X}_i(t)}{\widehat{X}_k(t)}\right)^2 dt. \tag{2.9.9}$$

For the covariation, from equations (2.9.2) and (2.8.8), we have

$$d\left\langle \frac{\widehat{X}_i}{\widehat{X}_k}, \frac{\widehat{X}_j}{\widehat{X}_k}\right\rangle_t = \left(\frac{\widehat{X}_i(t)\widehat{X}_j(t)}{\widehat{X}_k^2(t)}\right) d\left\langle \log\left(\widehat{X}_i/\widehat{X}_k\right), \log\left(\widehat{X}_j/\widehat{X}_k\right)\right\rangle_t \tag{2.9.10}$$

$$= \tau_{ij}^{\mathsf{X}_k}(t)\left(\frac{\widehat{X}_i(t)\widehat{X}_j(t)}{\widehat{X}_k^2(t)}\right) dt. \tag{2.9.11}$$

In addition, from equations (2.8.16) and (2.8.17), we have the following

$$d\left\langle \frac{\widehat{X}_i}{\widehat{X}_k}, \frac{\widehat{X}_i}{\widehat{X}_m}\right\rangle_t = \left(\frac{\widehat{X}_i^2(t)}{\widehat{X}_k(t)\widehat{X}_m(t)}\right) d\left\langle \log\left(\widehat{X}_i/\widehat{X}_k\right), \log\left(\widehat{X}_i/\widehat{X}_m\right)\right\rangle_t \tag{2.9.12}$$

$$= \tau_{ii}^{\mathsf{X}_k\mathsf{X}_m}(t)\left(\frac{\widehat{X}_i^2(t)}{\widehat{X}_k(t)\widehat{X}_m(t)}\right) dt, \tag{2.9.13}$$

and,

$$d\left\langle \frac{\widehat{X}_i}{\widehat{X}_k}, \frac{\widehat{X}_j}{\widehat{X}_m}\right\rangle_t = \left(\frac{\widehat{X}_i(t)\widehat{X}_j(t)}{\widehat{X}_k(t)\widehat{X}_m(t)}\right) d\left\langle \log\left(\widehat{X}_i/\widehat{X}_k\right), \log\left(\widehat{X}_j/\widehat{X}_m\right)\right\rangle_t \tag{2.9.14}$$

$$= \tau_{ij}^{\mathsf{X}_k\mathsf{X}_m}(t)\left(\frac{\widehat{X}_i(t)\widehat{X}_j(t)}{\widehat{X}_k(t)\widehat{X}_m(t)}\right) dt. \tag{2.9.15}$$

We shall express the relative total return process $d\log\left(\widehat{X}_i/\widehat{Z}_\eta\right)$ in terms of the aforementioned quotient processes.

This is done by substituting (2.9.7) into equation (2.8.23) of Lemma 2.8.3, as follows

$$
\begin{aligned}
&d\log\Big(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\Big)\\
&\quad= \sum_{j=1}^n \eta_j(t)\left[\frac{d\left(\widehat{X}_i(t)/\widehat{X}_j(t)\right)}{\widehat{X}_i(t)/\widehat{X}_j(t)} - \frac{1}{2}\,\tau_{ii}^{\chi_j}(t)\,dt\right] - \vartheta_\eta^*(t)\,dt\\
&\quad= \sum_{j=1}^n \eta_j(t)\frac{\widehat{X}_j(t)}{\widehat{X}_i(t)}\,d\left(\frac{\widehat{X}_i(t)}{\widehat{X}_j(t)}\right) - \left(\vartheta_\eta^*(t) + \frac{1}{2}\sum_{j=1}^n \eta_j(t)\tau_{ii}^{\chi_j}(t)\right)dt\\
&\quad= \sum_{j=1}^n \eta_j(t)\frac{\widehat{X}_j(t)}{\widehat{X}_i(t)}\,d\left(\frac{\widehat{X}_i(t)}{\widehat{X}_j(t)}\right) - \left(\vartheta_\eta^*(t) + \frac{1}{2}\sum_{j=1}^n \eta_j(t)\Big[\sigma_{ii}(t) - 2\,\sigma_{ij}(t) + \sigma_{jj}(t)\Big]\right)dt \qquad (2.9.16)\\
&\quad= \sum_{j=1}^n \eta_j(t)\frac{\widehat{X}_j(t)}{\widehat{X}_i(t)}\,d\left(\frac{\widehat{X}_i(t)}{\widehat{X}_j(t)}\right) - \left(\vartheta_\eta^*(t) + \frac{1}{2}\left[\sigma_{ii}(t) - 2\sum_{j=1}^n \eta_j(t)\sigma_{ij}(t) + \sum_{j=1}^n \eta_j(t)\sigma_{jj}(t)\right]\right)dt\\
&\quad= \sum_{j=1}^n \eta_j(t)\frac{\widehat{X}_j(t)}{\widehat{X}_i(t)}\,d\left(\frac{\widehat{X}_i(t)}{\widehat{X}_j(t)}\right) - \left(\vartheta_\eta^*(t) + \frac{1}{2}\Big[\sigma_{ii}(t) - 2\,\sigma_{i\eta}(t) + \sigma_{\eta\eta}(t) + 2\,\vartheta_\eta^*(t)\Big]\right)dt \qquad (2.9.17)\\
&\quad= \sum_{j=1}^n \eta_j(t)\frac{\widehat{X}_j(t)}{\widehat{X}_i(t)}\,d\left(\frac{\widehat{X}_i(t)}{\widehat{X}_j(t)}\right) - \left(\vartheta_\eta^*(t) + \frac{1}{2}\Big[\tau_{ii}^\eta(t) + 2\,\vartheta_\eta^*(t)\Big]\right)dt \qquad (2.9.18)\\
&\quad= \sum_{j=1}^n \eta_j(t)\frac{\widehat{X}_j(t)}{\widehat{X}_i(t)}\,d\left(\frac{\widehat{X}_i(t)}{\widehat{X}_j(t)}\right) - \left(2\,\vartheta_\eta^*(t) + \frac{1}{2}\,\tau_{ii}^\eta(t)\right)dt, \qquad (2.9.19)
\end{aligned}
$$

where equation (2.9.16) follows from equation (2.3.27), equation (2.9.17) follows from equations (2.3.51) and (2.5.16) along with the fact that $\gamma_\eta^*(t) \equiv \vartheta_\eta^*(t)$, and lastly, equation (2.9.18) follows from equation (2.3.128).

Definition 2.9.2 (Total Quotient Process). *For a stock $X_i = \{X_i(t), t \in [0,\infty)\}$ with initial capital $X_i(0) = x_i$, for $i = 1, 2, \ldots, n$, portfolio $\eta = \big\{\boldsymbol{\eta}(t) = \big(\eta_1(t), \ldots, \eta_n(t)\big), t \in [0,\infty)\big\}$ and portfolio value process $Z_{w,\eta} = \big\{Z_{w,\eta}(t), t \in [0,\infty)\big\}$ with initial capital $Z_{w,\eta}(0) = w > 0$, with $w := X_i(0)$, the process $\widehat{X}_i/\widehat{Z}_\eta = \big\{\widehat{X}_i(t)/\widehat{Z}_\eta(t), t \in [0,\infty)\big\}$, i.e.,*

$$
\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)} = \left(\frac{\widehat{X}_i(t)}{\widehat{Z}_{w,\eta}(t)}\right)\Bigg|_{w=X_i(0)}, \qquad t \in [0,\infty), \qquad (2.9.20)
$$

*is called the **total quotient process** of X_i versus η, for all $i = 1, 2, \ldots, n$.*

An application of Itô's formula to $\exp\big(\log\big(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\big)\big)$ along with equation (2.8.36), results in the following

$$
\begin{aligned}
d\left(\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\right) &= \frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\,d\log\Big(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\Big) + \frac{1}{2}\,\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\,d\left\langle\log\big(\widehat{X}_i/\widehat{Z}_\eta\big)\right\rangle_t \qquad (2.9.21)\\
&= \frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\,d\log\Big(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\Big) + \frac{1}{2}\,\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\,\tau_{ii}^\eta(t)\,dt \qquad (2.9.22)\\
&= \frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\left[\big(\vartheta_i(t) - \vartheta_\eta(t)\big)\,dt + \sum_{j,\nu=1}^n \eta_j(t)\big(\xi_{i\nu}(t) - \xi_{j\nu}(t)\big)\,dW_\nu(t)\right] + \frac{1}{2}\,\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\,\tau_{ii}^\eta(t)\,dt \qquad (2.9.23)\\
&= \left(\vartheta_i(t) - \vartheta_\eta(t) + \frac{1}{2}\,\tau_{ii}^\eta(t)\right)\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\,dt + \frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\sum_{j,\nu=1}^n \eta_j(t)\big(\xi_{i\nu}(t) - \xi_{j\nu}(t)\big)\,dW_\nu(t) \qquad (2.9.24)\\
&= \frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\left[\left(\vartheta_i(t) - \vartheta_\eta(t) + \frac{1}{2}\,\tau_{ii}^\eta(t)\right)dt + \sum_{j,\nu=1}^n \eta_j(t)\big(\xi_{i\nu}(t) - \xi_{j\nu}(t)\big)\,dW_\nu(t)\right], \qquad (2.9.25)
\end{aligned}
$$

where equation (2.9.24) follows from equation (2.8.21). Substituting equation (2.8.22) into equation (2.9.22),

offers the following additional representation for the dynamics of the total quotient process

$$d\left(\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\right) = \frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\left[\big(\vartheta_i(t)-\vartheta_\eta(t)\big)\,dt+\sum_{\nu=1}^{n}\big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t)\right]+\frac{1}{2}\,\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\,\tau_{ii}^{\eta}(t)\,dt \tag{2.9.26}$$

$$= \left(\vartheta_i(t)-\vartheta_\eta(t)+\frac{1}{2}\,\tau_{ii}^{\eta}(t)\right)\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\,dt+\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\sum_{\nu=1}^{n}\big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t) \tag{2.9.27}$$

$$= \frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\left[\left(\vartheta_i(t)-\vartheta_\eta(t)+\frac{1}{2}\,\tau_{ii}^{\eta}(t)\right)dt+\sum_{\nu=1}^{n}\big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t)\right]. \tag{2.9.28}$$

The expression above, (2.9.22), can be rewritten in the more useful form, which will be enlisted at a later stage, as follows

$$\frac{d\left(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\right)}{\widehat{X}_i(t)/\widehat{Z}_\eta(t)} = d\log\left(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\right)+\frac{1}{2}\,\tau_{ii}^{\eta}(t)\,dt. \tag{2.9.29}$$

The quadratic variation of the aforementioned process can be derived from equation (2.9.22) as follows

$$d\left\langle\frac{\widehat{X}_i}{\widehat{Z}_\eta}\right\rangle_t = \left(\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\right)^2 d\left\langle\log\left(\widehat{X}_i/\widehat{Z}_\eta\right)\right\rangle_t \tag{2.9.30}$$

$$= \tau_{ii}^{\eta}(t)\left(\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\right)^2 dt. \tag{2.9.31}$$

A derivation of the cross-variation of the relative return processes can also be obtained from equation (2.9.22), as follows

$$d\left\langle\frac{\widehat{X}_i}{\widehat{Z}_\eta},\frac{\widehat{X}_j}{\widehat{Z}_\eta}\right\rangle_t = \left(\frac{\widehat{X}_i(t)\widehat{X}_j(t)}{\widehat{Z}_\eta^2(t)}\right) d\left\langle\log\left(\widehat{X}_i/\widehat{Z}_\eta\right),\log\left(\widehat{X}_j/\widehat{Z}_\eta\right)\right\rangle_t \tag{2.9.32}$$

$$= \tau_{ij}^{\eta}(t)\left(\frac{\widehat{X}_i(t)\widehat{X}_j(t)}{\widehat{Z}_\eta^2(t)}\right) dt. \tag{2.9.33}$$

Lemma 2.9.3. *Let X_i denote the ith stock in the market $\mathcal{M}$ and let η be any arbitrary portfolio in the market $\mathcal{M}$. Then, a.s., for $t\in[0,\infty)$, we obtain the dynamics*

$$d\left(\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\right) = \frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\left[\left(\frac{1}{2}\,\tau_{ii}^{\eta}(t)-\vartheta_\eta^*(t)\right)dt+\sum_{j=1}^{n}\eta_j(t)\,d\log\left(\widehat{X}_i(t)/\widehat{X}_j(t)\right)\right], \tag{2.9.34}$$

for the relative arithmetic total return of stock X_i versus portfolio η.

Proof. This easily follows by substituting equation (2.8.23) of Lemma 2.8.3 into equation (2.9.22), to obtain the result

$$d\left(\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\right) = \frac{1}{2}\,\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\,\tau_{ii}^{\eta}(t)\,dt+\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\left[\sum_{j=1}^{n}\eta_j(t)\,d\log\left(\widehat{X}_i(t)/\widehat{X}_j(t)\right)-\vartheta_\eta^*(t)\,dt\right]$$

$$= \frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\left[\left(\frac{1}{2}\,\tau_{ii}^{\eta}(t)-\vartheta_\eta^*(t)\right)dt+\sum_{j=1}^{n}\eta_j(t)\,d\log\left(\widehat{X}_i(t)/\widehat{X}_j(t)\right)\right].$$

■

Equation (2.9.25) can be verified by substituting equation (2.8.3) into equation (2.9.34) of the above lemma

$$d\left(\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\right) = \frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\left[\left(\frac{1}{2}\,\tau_{ii}^{\eta}(t)-\vartheta_\eta^*(t)\right)dt+\sum_{j=1}^{n}\eta_j(t)\left[\big(\vartheta_i(t)-\vartheta_j(t)\big)\,dt+\sum_{\nu=1}^{n}\big(\xi_{i\nu}(t)-\xi_{j\nu}(t)\big)\,dW_\nu(t)\right]\right]$$

$$= \frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\left[\left(\vartheta_i(t)-\sum_{j=1}^{n}\eta_j(t)\vartheta_j(t)-\vartheta_\eta^*(t)+\frac{1}{2}\,\tau_{ii}^{\eta}(t)\right)dt+\sum_{j,\nu=1}^{n}\eta_j(t)\big(\xi_{i\nu}(t)-\xi_{j\nu}(t)\big)\,dW_\nu(t)\right]$$

$$= \frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\left[\left(\vartheta_i(t)-\vartheta_\eta(t)+\frac{1}{2}\,\tau_{ii}^{\eta}(t)\right)dt+\sum_{j,\nu=1}^{n}\eta_j(t)\big(\xi_{i\nu}(t)-\xi_{j\nu}(t)\big)\,dW_\nu(t)\right],$$

where the last equality follows from equation (2.2.168). In addition, equation (2.9.28) can also be verified by substituting equation (2.8.3) into equation (2.9.34) of the above lemma

$$\begin{aligned} d\left(\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\right) &= \frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\left[\left(\vartheta_i(t) - \sum_{j=1}^n \eta_j(t)\vartheta_j(t) - \vartheta^*_\eta(t) + \frac{1}{2}\,\tau^\eta_{ii}(t)\right)dt + \sum_{\nu=1}^n\left[\xi_{i\nu}(t) - \sum_{j=1}^n \eta_j(t)\xi_{j\nu}(t)\right]dW_\nu(t)\right] \\ &= \frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\left[\left(\vartheta_i(t) - \vartheta_\eta(t) + \frac{1}{2}\,\tau^\eta_{ii}(t)\right)dt + \sum_{\nu=1}^n\big(\xi_{i\nu}(t) - \xi_{\eta\nu}(t)\big)\,dW_\nu(t)\right], \end{aligned}$$

where the last equality follows from equations (2.2.126) and (2.2.168).

Lemma 2.9.4. *Let X_i denote the ith stock in the market $\mathcal{M}$ and let η be any arbitrary portfolio in the market $\mathcal{M}$. Then, a.s., for $t \in [0,\infty)$, we obtain the dynamics*

$$d\left(\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\right) = \frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\left[\left(\frac{1}{2}\,\tau^\eta_{ii}(t) - \frac{1}{2}\sum_{j=1}^n \eta_j(t)\tau^{\chi_j}_{ii}(t) - \vartheta^*_\eta(t)\right)dt + \sum_{j=1}^n \eta_j(t)\frac{\widehat{X}_j(t)}{\widehat{X}_i(t)}\,d\left(\frac{\widehat{X}_i(t)}{\widehat{X}_j(t)}\right)\right] \tag{2.9.35}$$

$$= \frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\left[-\,2\,\vartheta^*_\eta(t)\,dt + \sum_{j=1}^n \eta_j(t)\frac{\widehat{X}_j(t)}{\widehat{X}_i(t)}\,d\left(\frac{\widehat{X}_i(t)}{\widehat{X}_j(t)}\right)\right], \tag{2.9.36}$$

for the relative arithmetic total return of stock X_i versus portfolio η.

Proof. For this proof, we shall require a modification of equation (2.9.6), given by

$$d\log\left(\widehat{X}_i(t)/\widehat{X}_j(t)\right) = \frac{d\left(\widehat{X}_i(t)/\widehat{X}_j(t)\right)}{\widehat{X}_i(t)/\widehat{X}_j(t)} - \frac{1}{2}\,\tau^{\chi_j}_{ii}(t)\,dt. \tag{2.9.37}$$

Thus, utilising equations (2.9.34) and (2.9.37), we obtain

$$d\left(\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\right) = \frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\left[\left(\frac{1}{2}\,\tau^\eta_{ii}(t) - \vartheta^*_\eta(t)\right)dt + \sum_{j=1}^n \eta_j(t)\left[\frac{d\left(\widehat{X}_i(t)/\widehat{X}_j(t)\right)}{\widehat{X}_i(t)/\widehat{X}_j(t)} - \frac{1}{2}\,\tau^{\chi_j}_{ii}(t)\,dt\right]\right].$$

Simplifying completes the proof of the first part (2.9.35). To prove the second part (2.9.36), it suffices to show that the drift component in (2.9.35) reduces to $-2\,\vartheta^*_\eta(t)$. This follows by employing equation (2.5.23) and recalling that $\gamma^*_\eta(t) \equiv \vartheta^*_\eta(t)$, as follows

$$\frac{1}{2}\,\tau^\eta_{ii}(t) - \frac{1}{2}\sum_{j=1}^n \eta_j(t)\tau^{\chi_j}_{ii}(t) - \vartheta^*_\eta(t) = \frac{1}{2}\,\tau^\eta_{ii}(t) - \frac{1}{2}\left(2\,\vartheta^*_\eta(t) + \tau^\eta_{ii}(t)\right) - \vartheta^*_\eta(t) = -2\,\vartheta^*_\eta(t).$$

Thus, the equality above provides the substitution required to obtain equation (2.9.36). Since equation (2.9.19) expresses $d\log\left(\widehat{X}_i/\widehat{Z}_\eta\right)$ directly in terms of the quotient processes, an alternative means of obtaining the above result is to substitute equation (2.9.19) into equation (2.9.22), as follows

$$\begin{aligned} d\left(\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\right) &= \frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\left[\frac{1}{2}\,\tau^\eta_{ii}(t)\,dt + \left(\sum_{j=1}^n \eta_j(t)\frac{\widehat{X}_j(t)}{\widehat{X}_i(t)}\,d\left(\frac{\widehat{X}_i(t)}{\widehat{X}_j(t)}\right) - \left(2\,\vartheta^*_\eta(t) + \frac{1}{2}\,\tau^\eta_{ii}(t)\right)dt\right)\right] \\ &= \frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\left[-\,2\,\vartheta^*_\eta(t)\,dt + \sum_{j=1}^n \eta_j(t)\frac{\widehat{X}_j(t)}{\widehat{X}_i(t)}\,d\left(\frac{\widehat{X}_i(t)}{\widehat{X}_j(t)}\right)\right]. \end{aligned}$$

■

2.10 Relative Total Return of Portfolios

Definition 2.10.1 (Relative Total Return of Portfolios). *For $\pi = \big\{\boldsymbol{\pi}(t) = \big(\pi_1(t),\ldots,\pi_n(t)\big), t \in [0,\infty)\big\}$, and $\eta = \big\{\boldsymbol{\eta}(t) = \big(\eta_1(t),\ldots,\eta_n(t)\big), t \in [0,\infty)\big\}$, the process $\log\left(\widehat{Z}_\pi/\widehat{Z}_\eta\right) = \big\{\log\left(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\right), t \in [0,\infty)\big\}$, i.e.,*

$$\log\left(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\right) = \log\left(\widehat{Z}_{w,\pi}(t)/\widehat{Z}_{w,\eta}(t)\right)\Big|_{w=Z_{w,\pi}(0)=Z_{w,\eta}(0)}, \qquad t \in [0,\infty), \tag{2.10.1}$$

*is called the **relative total return process** of π versus η [refer to Fernholz (2002, Definition 1.2.4)].*

Now, let us consider the dynamics of the relative total return process of portfolio π versus portfolio η. By equation (2.2.157), the total return of portfolio π relative to portfolio η can be described by the following dynamics

$$\begin{aligned} d\log\Big(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\Big) &= d\log\widehat{Z}_\pi(t) - d\log\widehat{Z}_\eta(t) \\ &= \Big(d\log Z_\pi(t) + \delta_\pi(t)\,dt\Big) - \Big(d\log Z_\eta(t) + \delta_\eta(t)\,dt\Big) \\ &= d\log\Big(Z_\pi(t)/Z_\eta(t)\Big) + \big(\delta_\pi(t) - \delta_\eta(t)\big)\,dt. \end{aligned} \tag{2.10.2}$$

Consequently, from equation (2.6.4), we have

$$\begin{aligned} d\log\Big(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\Big) &= \Big(\sum_{i=1}^n \pi_i(t)\,d\log\Big(X_i(t)/Z_\eta(t)\Big) + \gamma_\pi^*(t)\,dt\Big) + \big(\delta_\pi(t) - \delta_\eta(t)\big)\,dt \\ &= \Big(\sum_{i=1}^n \pi_i(t)\,d\log\Big(X_i(t)/Z_\eta(t)\Big) + \gamma_\pi^*(t)\,dt\Big) + \Big(\sum_{i=1}^n \pi_i(t)\delta_i(t) - \delta_\eta(t)\Big)\,dt \\ &= \sum_{i=1}^n \pi_i(t)\Big[d\log\Big(X_i(t)/Z_\eta(t)\Big) + \big(\delta_i(t) - \delta_\eta(t)\big)\,dt\Big] + \gamma_\pi^*(t)\,dt \\ &= \sum_{i=1}^n \pi_i(t)\,d\log\Big(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\Big) + \gamma_\pi^*(t)\,dt, \end{aligned} \tag{2.10.3}$$

which follows from equation (2.2.154) of Definition 2.2.36 and equation (2.8.19). Using equation (2.2.170), for the relative total return process of π versus η, for $t \in [0,\infty)$, a.s., we obtain

$$\begin{aligned} d\log\Big(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\Big) &= d\log\widehat{Z}_\pi(t) - d\log\widehat{Z}_\eta(t) \\ &= \Big(\sum_{i=1}^n \pi_i(t)\,d\log\widehat{X}_i(t) + \vartheta_\pi^*(t)\,dt\Big) - d\log\widehat{Z}_\eta(t)\Big(\sum_{i=1}^n \pi_i(t)\Big) \\ &= \sum_{i=1}^n \pi_i(t)\,d\log\widehat{X}_i(t) - \sum_{i=1}^n \pi_i(t)\,d\log\widehat{Z}_\eta(t) + \vartheta_\pi^*(t)\,dt \\ &= \sum_{i=1}^n \pi_i(t)\Big[d\log\widehat{X}_i(t) - d\log\widehat{Z}_\eta(t)\Big] + \vartheta_\pi^*(t)\,dt \\ &= \sum_{i=1}^n \pi_i(t)\,d\log\Big(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\Big) + \vartheta_\pi^*(t)\,dt \qquad (2.10.4) \\ &= \sum_{i=1}^n \pi_i(t)\,d\log\Big(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\Big) + \gamma_\pi^*(t)\,dt. \qquad (2.10.5) \end{aligned}$$

This shows that we can express the relative total return process of π versus η in terms of the individual stock relative total returns against the portfolio η.

Lemma 2.10.2. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t \in [0,\infty)$, we obtain the dynamics*

$$d\log\Big(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\Big) = \big(\vartheta_\pi(t) - \vartheta_\eta(t)\big)\,dt + \sum_{i,\nu=1}^n \big(\pi_i(t) - \eta_i(t)\big)\xi_{i\nu}(t)\,dW_\nu(t), \tag{2.10.6}$$

for the relative total return of portfolio π versus portfolio η.

Proof. This result is recovered easily by enlisting the use of equation (2.10.4) in conjunction with equation

(2.8.21), we thus obtain

$$\begin{aligned}
d\log\Big(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\Big) &= \vartheta_\pi^*(t)\,dt + \sum_{i=1}^n \pi_i(t)\,d\log\Big(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\Big) \\
&= \vartheta_\pi^*(t)\,dt + \sum_{i=1}^n \pi_i(t)\Big[\big(\vartheta_i(t)-\vartheta_\eta(t)\big)\,dt + \sum_{j,\nu=1}^n \eta_j(t)\big(\xi_{i\nu}(t)-\xi_{j\nu}(t)\big)\,dW_\nu(t)\Big] \\
&= \Big(\vartheta_\pi^*(t) + \sum_{i=1}^n \pi_i(t)\vartheta_i(t) - \vartheta_\eta(t)\Big)\,dt + \sum_{\nu=1}^n\Big[\sum_{i,j=1}^n \pi_i(t)\eta_j(t)\big(\xi_{i\nu}(t)-\xi_{j\nu}(t)\big)\,dW_\nu(t)\Big] \\
&= \big(\vartheta_\pi(t)-\vartheta_\eta(t)\big)\,dt + \sum_{\nu=1}^n\Big[\sum_{i=1}^n \pi_i(t)\xi_{i\nu}(t) - \sum_{j=1}^n \eta_j(t)\xi_{j\nu}(t)\Big]\,dW_\nu(t) \\
&= \big(\vartheta_\pi(t)-\vartheta_\eta(t)\big)\,dt + \sum_{\nu=1}^n\Big[\sum_{i=1}^n \pi_i(t)\xi_{i\nu}(t) - \sum_{i=1}^n \eta_i(t)\xi_{i\nu}(t)\Big]\,dW_\nu(t) \\
&= \big(\vartheta_\pi(t)-\vartheta_\eta(t)\big)\,dt + \sum_{i,\nu=1}^n \big(\pi_i(t)-\eta_i(t)\big)\xi_{i\nu}(t)\,dW_\nu(t), \qquad (2.10.7)
\end{aligned}$$

where equation (2.10.7) follows from equation (2.2.168). Alternatively, employing equation (2.2.162) yields

$$\begin{aligned}
d\log\Big(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\Big) &= d\log\widehat{Z}_\pi(t) - d\log\widehat{Z}_\eta(t) \\
&= \Big(\vartheta_\pi(t)\,dt + \sum_{i,\nu=1}^n \pi_i(t)\xi_{i\nu}(t)\,dW_\nu(t)\Big) - \Big(\vartheta_\eta(t)\,dt + \sum_{i,\nu=1}^n \eta_i(t)\xi_{i\nu}(t)\,dW_\nu(t)\Big) \\
&= \big(\vartheta_\pi(t)-\vartheta_\eta(t)\big)\,dt + \sum_{i,\nu=1}^n \big(\pi_i(t)-\eta_i(t)\big)\xi_{i\nu}(t)\,dW_\nu(t).
\end{aligned}$$

■

Lemma 2.10.3. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t \in [0,\infty)$, we obtain the dynamics*

$$d\log\Big(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\Big) = \big(\vartheta_\pi(t)-\vartheta_\eta(t)\big)\,dt + \sum_{i,\nu=1}^n \pi_i(t)\big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t), \qquad (2.10.8)$$

for the relative total return of portfolio π versus portfolio η.

Proof. This result is recovered easily by enlisting the use of either equation (2.10.4) or equation (2.10.5), in conjunction with equation (2.8.22). Thus, we obtain

$$\begin{aligned}
d\log\Big(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\Big) &= \vartheta_\pi^*(t)\,dt + \sum_{i=1}^n \pi_i(t)\,d\log\Big(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\Big) \\
&= \vartheta_\pi^*(t)\,dt + \sum_{i=1}^n \pi_i(t)\Big[\big(\vartheta_i(t)-\vartheta_\eta(t)\big)\,dt + \sum_{\nu=1}^n \big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t)\Big] \\
&= \Big(\vartheta_\pi^*(t) + \sum_{i=1}^n \pi_i(t)\vartheta_i(t) - \vartheta_\eta(t)\Big)\,dt + \sum_{i,\nu=1}^n \pi_i(t)\big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t) \\
&= \big(\vartheta_\pi(t)-\vartheta_\eta(t)\big)\,dt + \sum_{i,\nu=1}^n \pi_i(t)\big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t), \qquad (2.10.9)
\end{aligned}$$

where the last expression follows from (2.2.168). Alternatively, simplifying equation (2.10.6) of Lemma 2.10.2

using equation (2.2.126) of Definition 2.2.28, gives

$$\begin{aligned}
d\log\Big(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\Big) &= \big(\vartheta_\pi(t)-\vartheta_\eta(t)\big)\,dt+\sum_{i,\nu=1}^{n}\big(\pi_i(t)-\eta_i(t)\big)\xi_{i\nu}(t)\,dW_\nu(t)\\
&= \big(\vartheta_\pi(t)-\vartheta_\eta(t)\big)\,dt+\sum_{\nu=1}^{n}\Bigg[\sum_{i=1}^{n}\pi_i(t)\xi_{i\nu}(t)-\sum_{i=1}^{n}\eta_i(t)\xi_{i\nu}(t)\Bigg]\,dW_\nu(t)\\
&= \big(\vartheta_\pi(t)-\vartheta_\eta(t)\big)\,dt+\sum_{\nu=1}^{n}\Bigg[\sum_{i=1}^{n}\pi_i(t)\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\Bigg]\,dW_\nu(t)\\
&= \big(\vartheta_\pi(t)-\vartheta_\eta(t)\big)\,dt+\sum_{i,\nu=1}^{n}\pi_i(t)\big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t).
\end{aligned}$$

Moreover, employing equations (2.2.162) and (2.2.163), gives the same result

$$\begin{aligned}
d\log\Big(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\Big) &= d\log\widehat{Z}_\pi(t)-d\log\widehat{Z}_\eta(t)\\
&= \Bigg(\vartheta_\pi(t)\,dt+\sum_{i,\nu=1}^{n}\pi_i(t)\xi_{i\nu}(t)\,dW_\nu(t)\Bigg)-\Bigg(\vartheta_\eta(t)\,dt+\sum_{\nu=1}^{n}\xi_{\eta\nu}(t)\,dW_\nu(t)\Bigg)\\
&= \big(\vartheta_\pi(t)-\vartheta_\eta(t)\big)\,dt+\sum_{i,\nu=1}^{n}\pi_i(t)\big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t).
\end{aligned}$$

■

Lemma 2.10.4. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t\in[0,\infty)$, we obtain the dynamics*

$$d\log\Big(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\Big) = \big(\vartheta_\pi(t)-\vartheta_\eta(t)\big)\,dt+\sum_{\nu=1}^{n}\big(\xi_{\pi\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t), \tag{2.10.10}$$

for the relative total return of portfolio π versus portfolio η.

Proof. This result follows easily from equation (2.10.6) of Lemma 2.10.2, which, when allied with (2.2.126) of Definition 2.2.28, yields the desired result

$$\begin{aligned}
d\log\Big(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\Big) &= \big(\vartheta_\pi(t)-\vartheta_\eta(t)\big)\,dt+\sum_{i,\nu=1}^{n}\big(\pi_i(t)-\eta_i(t)\big)\xi_{i\nu}(t)\,dW_\nu(t)\\
&= \big(\vartheta_\pi(t)-\vartheta_\eta(t)\big)\,dt+\sum_{\nu=1}^{n}\Bigg[\sum_{i=1}^{n}\pi_i(t)\xi_{i\nu}(t)-\sum_{i=1}^{n}\eta_i(t)\xi_{i\nu}(t)\Bigg]\,dW_\nu(t)\\
&= \big(\vartheta_\pi(t)-\vartheta_\eta(t)\big)\,dt+\sum_{\nu=1}^{n}\big(\xi_{\pi\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t).
\end{aligned}$$

Alternatively, simplifying equation (2.10.8) of Lemma 2.6.3 using equation (2.2.126) of Definition 2.2.28, gives

$$\begin{aligned}
d\log\Big(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\Big) &= \big(\vartheta_\pi(t)-\vartheta_\eta(t)\big)\,dt+\sum_{i,\nu=1}^{n}\pi_i(t)\big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t)\\
&= \big(\vartheta_\pi(t)-\vartheta_\eta(t)\big)\,dt+\sum_{\nu=1}^{n}\Bigg[\sum_{i=1}^{n}\pi_i(t)\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\Bigg]\,dW_\nu(t)\\
&= \big(\vartheta_\pi(t)-\vartheta_\eta(t)\big)\,dt+\sum_{\nu=1}^{n}\big(\xi_{\pi\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t).
\end{aligned}$$

Moreover, employing equation (2.2.163) yields

$$\begin{aligned} d\log\Big(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\Big) &= d\log\widehat{Z}_\pi(t) - d\log\widehat{Z}_\eta(t) \\ &= \Big(\vartheta_\pi(t)\,dt + \sum_{\nu=1}^n \xi_{\pi\nu}(t)\,dW_\nu(t)\Big) - \Big(\vartheta_\eta(t)\,dt + \sum_{\nu=1}^n \xi_{\eta\nu}(t)\,dW_\nu(t)\Big) \\ &= \big(\vartheta_\pi(t) - \vartheta_\eta(t)\big)\,dt + \sum_{\nu=1}^n \big(\xi_{\pi\nu}(t) - \xi_{\eta\nu}(t)\big)\,dW_\nu(t). \end{aligned}$$

■

Lemma 2.10.5. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t \in [0,\infty)$, we obtain the dynamics*

$$d\log\Big(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\Big) = \big(\vartheta^*_\pi(t) - \vartheta^*_\eta(t)\big)\,dt + \sum_{i,j=1}^n \pi_i(t)\eta_j(t)\,d\log\Big(\widehat{X}_i(t)/\widehat{X}_j(t)\Big), \tag{2.10.11}$$

for the relative total return of portfolio π versus portfolio η.

Proof. Inserting equation (2.8.23) into equation (2.10.4), we easily obtain the result

$$\begin{aligned} d\log\Big(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\Big) &= \sum_{i=1}^n \pi_i(t)\Big[\sum_{j=1}^n \eta_j(t)\,d\log\Big(\widehat{X}_i(t)/\widehat{X}_j(t)\Big) - \vartheta^*_\eta(t)\,dt\Big] + \vartheta^*_\pi(t)\,dt \\ &= \big(\vartheta^*_\pi(t) - \vartheta^*_\eta(t)\big)\,dt + \sum_{i,j=1}^n \pi_i(t)\eta_j(t)\,d\log\Big(\widehat{X}_i(t)/\widehat{X}_j(t)\Big). \end{aligned}$$

Alternatively, by employing equation (2.2.170), we obtain the equivalent result

$$\begin{aligned} d\log\Big(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\Big) &= d\log\widehat{Z}_\pi(t) - d\log\widehat{Z}_\eta(t) \\ &= \Big(\sum_{i=1}^n \pi_i(t)\,d\log\widehat{X}_i(t) + \vartheta^*_\pi(t)\,dt\Big) - \Big(\sum_{j=1}^n \eta_j(t)\,d\log\widehat{X}_j(t) + \vartheta^*_\eta(t)\,dt\Big) \\ &= \big(\vartheta^*_\pi(t) - \vartheta^*_\eta(t)\big)\,dt + \sum_{i=1}^n \pi_i(t)\,d\log\widehat{X}_i(t)\Big[\sum_{j=1}^n \eta_j(t)\Big] - \sum_{j=1}^n \eta_j(t)\,d\log\widehat{X}_j(t)\Big[\sum_{i=1}^n \pi_i(t)\Big] \\ &= \big(\vartheta^*_\pi(t) - \vartheta^*_\eta(t)\big)\,dt + \sum_{i,j=1}^n \pi_i(t)\eta_j(t)\,d\log\widehat{X}_i(t) - \sum_{i,j=1}^n \pi_i(t)\eta_j(t)\,d\log\widehat{X}_j(t) \\ &= \big(\vartheta^*_\pi(t) - \vartheta^*_\eta(t)\big)\,dt + \sum_{i,j=1}^n \pi_i(t)\eta_j(t)\Big[d\log\widehat{X}_i(t) - d\log\widehat{X}_j(t)\Big] \\ &= \big(\vartheta^*_\pi(t) - \vartheta^*_\eta(t)\big)\,dt + \sum_{i,j=1}^n \pi_i(t)\eta_j(t)\,d\log\Big(\widehat{X}_i(t)/\widehat{X}_j(t)\Big). \end{aligned}$$

■

Lemma 2.10.6. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t \in [0,\infty)$, we obtain the dynamics*

$$d\log\Big(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\Big) = \Big(-\frac{1}{2}\,\tau^\eta_{\pi\pi}(t) - 2\,\vartheta^*_\eta(t)\Big)\,dt + \sum_{i,j=1}^n \pi_i(t)\eta_j(t)\,\frac{\widehat{X}_j(t)}{\widehat{X}_i(t)}\,d\Big(\frac{\widehat{X}_i(t)}{\widehat{X}_j(t)}\Big), \tag{2.10.12}$$

for the relative total return of portfolio π versus portfolio η.

Proof. From equation (2.9.7) and equation (2.10.11) of Lemma 2.10.5, we obtain

$$d\log\Big(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\Big) = \big(\vartheta^*_\pi(t)-\vartheta^*_\eta(t)\big)\,dt + \sum_{i,j=1}^n \pi_i(t)\eta_j(t)\bigg[\frac{d\,\big(\widehat{X}_i(t)/\widehat{X}_j(t)\big)}{\widehat{X}_i(t)/\widehat{X}_j(t)} - \frac{1}{2}\,\tau^{\chi_j}_{ii}(t)\,dt\bigg]. \tag{2.10.13}$$

From equations (2.3.27) and (2.5.16), along with the fact that $\gamma^*_\pi(t)\equiv\vartheta^*_\pi(t)$, in the above expression, we have

$$\begin{aligned}
\sum_{i,j=1}^n \pi_i(t)\tau^{\chi_j}_{ii}(t)\eta_j(t) &= \sum_{i,j=1}^n \pi_i(t)\Big[\sigma_{ii}(t)-2\,\sigma_{ij}(t)+\sigma_{jj}(t)\Big]\eta_j(t)\\
&= \sum_{i=1}^n \pi_i(t)\sigma_{ii}(t) - 2\sum_{i,j=1}^n \pi_i(t)\sigma_{ij}(t)\eta_j(t) + \sum_{j=1}^n \eta_j(t)\sigma_{jj}(t)\\
&= \sigma_{\pi\pi}(t)+2\,\vartheta^*_\pi(t)-2\,\sigma_{\pi\eta}(t)+\sigma_{\eta\eta}(t)+2\,\vartheta^*_\eta(t)\\
&= \tau^\eta_{\pi\pi}(t)+2\,\vartheta^*_\pi(t)+2\,\vartheta^*_\eta(t).
\end{aligned}$$

Inserting the above result into equation (2.10.13) yields the desired result. ■

Lemma 2.10.7. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t\in[0,\infty)$, we obtain the dynamics*

$$d\log\Big(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\Big) = \big(\vartheta^*_\pi(t)-\vartheta^*_\eta(t)\big)\,dt + \sum_{i=1}^n \big(\pi_i(t)-\eta_i(t)\big)\,d\log\Big(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\Big), \tag{2.10.14}$$

for the relative total return of portfolio π versus portfolio η.

Proof. By following the same procedure as was adopted in the proof of Lemma 2.6.7, and employing equation (2.8.22), equation (2.2.164) of Lemma 2.2.39 and equation (2.2.126) of Definition 2.2.28, we obtain

$$\begin{aligned}
\sum_{i=1}^n \eta_i(t)\,d\log\Big(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\Big) &= \sum_{i=1}^n \eta_i(t)\bigg[\big(\vartheta_i(t)-\vartheta_\eta(t)\big)\,dt + \sum_{\nu=1}^n \big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t)\bigg]\\
&= \bigg(\sum_{i=1}^n \eta_i(t)\vartheta_i(t)-\vartheta_\eta(t)\bigg)\,dt + \sum_{\nu=1}^n\bigg[\sum_{i=1}^n \eta_i(t)\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\bigg]\,dW_\nu(t)\\
&= -\vartheta^*_\eta(t)\,dt + \sum_{\nu=1}^n \big(\xi_{\eta\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t)\\
&= -\vartheta^*_\eta(t)\,dt.
\end{aligned}$$

Therefore, for all $t\in[0,\infty)$, a.s., we have

$$\vartheta^*_\eta(t)\,dt + \sum_{i=1}^n \eta_i(t)\,d\log\Big(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\Big) = 0. \tag{2.10.15}$$

Consequently, subtracting the left-hand side of equation (2.10.15) from the right-hand side of equation (2.10.5), establishes the required result

$$\begin{aligned}
d\log\Big(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\Big) &= \sum_{i=1}^n \pi_i(t)\,d\log\Big(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\Big) + \vartheta^*_\pi(t)\,dt - \bigg(\vartheta^*_\eta(t)\,dt + \sum_{i=1}^n \eta_i(t)\,d\log\Big(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\Big)\bigg)\\
&= \big(\vartheta^*_\pi(t)-\vartheta^*_\eta(t)\big)\,dt + \sum_{i=1}^n \big(\pi_i(t)-\eta_i(t)\big)\,d\log\Big(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\Big).
\end{aligned}$$

■

Lemma 2.10.8. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t \in [0, \infty)$, we obtain the dynamics*

$$d \log \left(\widehat{Z}_\pi(t) / \widehat{Z}_\eta(t) \right) = \sum_{i=1}^n \pi_i(t) \frac{\widehat{Z}_\eta(t)}{\widehat{X}_i(t)} \, d \left(\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)} \right) - \frac{1}{2} \tau_{\pi\pi}^\eta(t) \, dt, \tag{2.10.16}$$

for the relative total return of portfolio π versus portfolio η.

Proof. From equation (2.9.22), we can express the relative logarithmic total return of X_i versus η as follows

$$d \log \left(\widehat{X}_i(t) / \widehat{Z}_\eta(t) \right) = \frac{d \left(\widehat{X}_i(t) / \widehat{Z}_\eta(t) \right)}{\widehat{X}_i(t) / \widehat{Z}_\eta(t)} - \frac{1}{2} \tau_{ii}^\eta(t) \, dt. \tag{2.10.17}$$

The combination of equations (2.10.5) and (2.10.17) with the numéraire invariance property (2.4.29), yields the following representation for the relative total return of π versus η

$$\begin{aligned} d \log \left(\widehat{Z}_\pi(t) / \widehat{Z}_\eta(t) \right) &= \vartheta_\pi^*(t) \, dt + \sum_{i=1}^n \pi_i(t) \, d \log \left(\widehat{X}_i(t) / \widehat{Z}_\eta(t) \right) \\ &= \frac{1}{2} \left(\sum_{i=1}^n \pi_i(t) \tau_{ii}^\eta(t) - \tau_{\pi\pi}^\eta(t) \right) dt + \sum_{i=1}^n \pi_i(t) \left[\frac{d \left(\widehat{X}_i(t) / \widehat{Z}_\eta(t) \right)}{\widehat{X}_i(t) / \widehat{Z}_\eta(t)} - \frac{1}{2} \tau_{ii}^\eta(t) \, dt \right] \\ &= \frac{1}{2} \sum_{i=1}^n \pi_i(t) \tau_{ii}^\eta(t) \, dt - \frac{1}{2} \tau_{\pi\pi}^\eta(t) \, dt + \sum_{i=1}^n \pi_i(t) \frac{\widehat{Z}_\eta(t)}{\widehat{X}_i(t)} \, d \left(\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)} \right) - \frac{1}{2} \sum_{i=1}^n \pi_i(t) \tau_{ii}^\eta(t) \, dt. \end{aligned}$$

The first and last terms cancel, establishing the result. ■

We shall verify the result (2.10.16) of Lemma 2.10.8, by appealing to (2.10.14) of Lemma 2.10.7. From equation (2.10.14) together with equations (2.10.17), (2.4.31) and (2.4.32) of Lemma 2.4.7 (along with the fact that $\gamma_\pi^*(t) \equiv \vartheta_\pi^*(t)$), we obtain

$$\begin{aligned} & d \log \left(\widehat{Z}_\pi(t) / \widehat{Z}_\eta(t) \right) \\ &= \left(\vartheta_\pi^*(t) - \vartheta_\eta^*(t) \right) dt + \sum_{i=1}^n \left(\pi_i(t) - \eta_i(t) \right) \left[\frac{d \left(\widehat{X}_i(t) / \widehat{Z}_\eta(t) \right)}{\widehat{X}_i(t) / \widehat{Z}_\eta(t)} - \frac{1}{2} \tau_{ii}^\eta(t) \, dt \right] \\ &= \left(\vartheta_\pi^*(t) - \vartheta_\eta^*(t) \right) dt + \sum_{i=1}^n \left(\pi_i(t) - \eta_i(t) \right) \frac{\widehat{Z}_\eta(t)}{\widehat{X}_i(t)} \, d \left(\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)} \right) - \frac{1}{2} \left(\sum_{i=1}^n \pi_i(t) \tau_{ii}^\eta(t) - \sum_{i=1}^n \eta_i(t) \tau_{ii}^\eta(t) \right) dt \\ &= \left(\vartheta_\pi^*(t) - \vartheta_\eta^*(t) \right) dt + \sum_{i=1}^n \left(\pi_i(t) - \eta_i(t) \right) \frac{\widehat{Z}_\eta(t)}{\widehat{X}_i(t)} \, d \left(\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)} \right) - \frac{1}{2} \left(2 \, \vartheta_\pi^*(t) + \tau_{\pi\pi}^\eta(t) - 2 \, \vartheta_\eta^*(t) \right) dt \\ &= \sum_{i=1}^n \left(\pi_i(t) - \eta_i(t) \right) \frac{\widehat{Z}_\eta(t)}{\widehat{X}_i(t)} \, d \left(\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)} \right) - \frac{1}{2} \tau_{\pi\pi}^\eta(t) \, dt \qquad (2.10.18) \\ &= \sum_{i=1}^n \pi_i(t) \frac{\widehat{Z}_\eta(t)}{\widehat{X}_i(t)} \, d \left(\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)} \right) - \sum_{i=1}^n \eta_i(t) \frac{\widehat{Z}_\eta(t)}{\widehat{X}_i(t)} \, d \left(\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)} \right) - \frac{1}{2} \tau_{\pi\pi}^\eta(t) \, dt. \qquad (2.10.19) \end{aligned}$$

The second term in the expression above can be simplified by making use of equation (2.9.22) and equation (2.4.32) of Lemma 2.4.7,

$$\begin{aligned} \sum_{i=1}^n \eta_i(t) \frac{\widehat{Z}_\eta(t)}{\widehat{X}_i(t)} \, d \left(\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)} \right) &= \sum_{i=1}^n \eta_i(t) \frac{\widehat{Z}_\eta(t)}{\widehat{X}_i(t)} \left[\frac{1}{2} \frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)} \tau_{ii}^\eta(t) \, dt + \frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)} \, d \log \left(\widehat{X}_i(t) / \widehat{Z}_\eta(t) \right) \right] \\ &= \frac{1}{2} \sum_{i=1}^n \eta_i(t) \tau_{ii}^\eta(t) \, dt + \sum_{i=1}^n \eta_i(t) \, d \log \left(\widehat{X}_i(t) / \widehat{Z}_\eta(t) \right) \\ &= \vartheta_\eta^*(t) \, dt + \sum_{i=1}^n \eta_i(t) \, d \log \left(\widehat{X}_i(t) / \widehat{Z}_\eta(t) \right). \end{aligned}$$

Therefore, by recalling (2.10.15), we get

$$\sum_{i=1}^{n} \eta_i(t) \frac{\widehat{Z}_\eta(t)}{\widehat{X}_i(t)}\, d\left(\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\right) \quad = \quad 0. \tag{2.10.20}$$

Thus, equation (2.10.20) results in the reduction of equation (2.10.19) to equation (2.10.16), which is precisely what we set out to show. Furthermore, for the quadratic variation process of $\log\left(\widehat{Z}_\pi/\widehat{Z}_\eta\right)$, we obtain the following expressions

$$\begin{aligned}
\left\langle \log\left(\widehat{Z}_\pi/\widehat{Z}_\eta\right)\right\rangle_t &= \left\langle \log \widehat{Z}_\pi - \log \widehat{Z}_\eta\right\rangle_t \\
&= \left\langle \log \widehat{Z}_\pi\right\rangle_t - \left\langle \log \widehat{Z}_\pi, \log \widehat{Z}_\eta\right\rangle_t - \left\langle \log \widehat{Z}_\eta, \log \widehat{Z}_\pi\right\rangle_t + \left\langle \log \widehat{Z}_\eta\right\rangle_t && (2.10.21)\\
&= \left\langle \log \widehat{Z}_\pi\right\rangle_t - 2\left\langle \log \widehat{Z}_\pi, \log \widehat{Z}_\eta\right\rangle_t + \left\langle \log \widehat{Z}_\eta\right\rangle_t. && (2.10.22)
\end{aligned}$$

The above expression, together with equations (2.2.176) and (2.6.71), yields the following result

$$\left\langle \log\left(\widehat{Z}_\pi/\widehat{Z}_\eta\right)\right\rangle_t \quad = \quad \left\langle \log Z_\pi\right\rangle_t - 2\left\langle \log Z_\pi, \log Z_\eta\right\rangle_t + \left\langle \log Z_\eta\right\rangle_t. \tag{2.10.23}$$

Consequently, from (2.6.27), we have

$$\left\langle \log\left(\widehat{Z}_\pi/\widehat{Z}_\eta\right)\right\rangle_t \quad \equiv \quad \left\langle \log\left(Z_\pi/Z_\eta\right)\right\rangle_t. \tag{2.10.24}$$

This is the quadratic variation process for the continuous semimartingale $\log\left(\widehat{Z}_\pi/\widehat{Z}_\eta\right)$. The above expression reveals that the quadratic variation process is unaltered when dividends are introduced into the model, this is what we would expect since dividends are only captured in the drift. Therefore, we have

$$\begin{aligned}
d\left\langle \log\left(\widehat{Z}_\pi/\widehat{Z}_\eta\right)\right\rangle_t &= d\left\langle \log \widehat{Z}_\pi\right\rangle_t - d\left\langle \log \widehat{Z}_\pi, \log \widehat{Z}_\eta\right\rangle_t - d\left\langle \log \widehat{Z}_\eta, \log \widehat{Z}_\pi\right\rangle_t + d\left\langle \log \widehat{Z}_\eta\right\rangle_t \\
&= d\left\langle \log \widehat{Z}_\pi\right\rangle_t - 2\, d\left\langle \log \widehat{Z}_\pi, \log \widehat{Z}_\eta\right\rangle_t + d\left\langle \log \widehat{Z}_\eta\right\rangle_t \\
&= d\left\langle \log Z_\pi\right\rangle_t - 2\, d\left\langle \log Z_\pi, \log Z_\eta\right\rangle_t + d\left\langle \log Z_\eta\right\rangle_t \\
&\equiv d\left\langle \log\left(Z_\pi/Z_\eta\right)\right\rangle_t. && (2.10.25)
\end{aligned}$$

Consequently, from equation (2.10.2) as well as from the foregoing arguments, in conjunction with equation (2.6.79), for all $t \in [0, \infty)$, we have

$$d\left\langle \log\left(\widehat{Z}_\pi/\widehat{Z}_\eta\right)\right\rangle_t \quad = \quad d\left\langle \log\left(Z_\pi/Z_\eta\right)\right\rangle_t \quad = \quad \tau^\eta_{\pi\pi}(t)\, dt, \quad \text{or}, \tag{2.10.26}$$

$$\left\langle \log\left(\widehat{Z}_\pi/\widehat{Z}_\eta\right)\right\rangle_t \quad = \quad \left\langle \log\left(Z_\pi/Z_\eta\right)\right\rangle_t \quad = \quad \int_0^t \tau^\eta_{\pi\pi}(s)\, ds. \tag{2.10.27}$$

2.11 Total Quotient Process of Portfolios

Definition 2.11.1 (Total Quotient Process of Portfolios). *For portfolios π and η, the process $\widehat{Z}_\pi/\widehat{Z}_\eta = \left\{\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t), t \in [0, \infty)\right\}$, i.e.,*

$$\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)} \quad = \quad \left(\frac{\widehat{Z}_{w,\pi}(t)}{\widehat{Z}_{w,\eta}(t)}\right)\Bigg|_{w=Z_{w,\pi}(0)=Z_{w,\eta}(0)}, \qquad t \in [0, \infty), \tag{2.11.1}$$

*is called the **total quotient process** of π versus η.*

Lemma 2.11.2. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t \in [0, \infty)$, we obtain the dynamics*

$$d\left(\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\right) \quad = \quad \frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\left[\left(\vartheta^*_\pi(t) + \frac{1}{2}\tau^\eta_{\pi\pi}(t)\right) dt + \sum_{i=1}^{n} \pi_i(t)\, d\log\left(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\right)\right], \tag{2.11.2}$$

for the relative arithmetic total return of portfolio π versus portfolio η.

Proof. Itô's formula, applied to $\exp\big(\log\big(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\big)\big)$, yields the following for all $t \in [0,\infty)$, a.s.,

$$d\left(\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\right) = \frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\, d\log\Big(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\Big) + \frac{1}{2}\,\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\, d\big\langle \log\big(\widehat{Z}_\pi/\widehat{Z}_\eta\big)\big\rangle_t. \tag{2.11.3}$$

Then, substituting the result (2.10.26) into equation (2.11.3), yields

$$d\left(\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\right) = \frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\, d\log\Big(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\Big) + \frac{1}{2}\,\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\,\tau^\eta_{\pi\pi}(t)\,dt, \tag{2.11.4}$$

which, in conjunction with equation (2.10.4), gives the desired result

$$\begin{aligned} d\left(\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\right) &= \frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\left[\vartheta^*_\pi(t)\,dt + \sum_{i=1}^n \pi_i(t)\, d\log\Big(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\Big)\right] + \frac{1}{2}\,\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\,\tau^\eta_{\pi\pi}(t)\,dt \\ &= \frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\left[\left(\vartheta^*_\pi(t) + \frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)dt + \sum_{i=1}^n \pi_i(t)\, d\log\Big(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\Big)\right]. \end{aligned}$$

■

We can rewrite equation (2.11.4) in the following useful forms

$$\frac{d\big(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\big)}{\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)} = d\log\Big(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\Big) + \frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\,dt, \tag{2.11.5}$$

and

$$d\log\Big(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\Big) = \frac{d\big(\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)\big)}{\widehat{Z}_\pi(t)/\widehat{Z}_\eta(t)} - \frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\,dt. \tag{2.11.6}$$

Lemma 2.11.3. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t \in [0,\infty)$, we obtain the dynamics*

$$d\left(\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\right) = \frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\left[\left(\vartheta_\pi(t) - \vartheta_\eta(t) + \frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)dt + \sum_{i,\nu=1}^n \big(\pi_i(t) - \eta_i(t)\big)\xi_{i\nu}(t)\,dW_\nu(t)\right], \tag{2.11.7}$$

for the relative arithmetic total return of portfolio π versus portfolio η.

Proof. Upon substitution of equation (2.10.6) into equation (2.11.4), we obtain

$$\begin{aligned} d\left(\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\right) &= \frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\left[\big(\vartheta_\pi(t) - \vartheta_\eta(t)\big)\,dt + \sum_{i,\nu=1}^n \big(\pi_i(t) - \eta_i(t)\big)\xi_{i\nu}(t)\,dW_\nu(t)\right] + \frac{1}{2}\,\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\,\tau^\eta_{\pi\pi}(t)\,dt \\ &= \left(\vartheta_\pi(t) - \vartheta_\eta(t) + \frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\,dt + \frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\sum_{i,\nu=1}^n \big(\pi_i(t) - \eta_i(t)\big)\xi_{i\nu}(t)\,dW_\nu(t) \\ &= \frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\left[\left(\vartheta_\pi(t) - \vartheta_\eta(t) + \frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)dt + \sum_{i,\nu=1}^n \big(\pi_i(t) - \eta_i(t)\big)\xi_{i\nu}(t)\,dW_\nu(t)\right], \end{aligned}$$

which completes the proof. ■

Lemma 2.11.4. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t \in [0,\infty)$, we obtain the dynamics*

$$d\left(\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\right) = \frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\left[\left(\vartheta_\pi(t) - \vartheta_\eta(t) + \frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)dt + \sum_{i,\nu=1}^n \pi_i(t)\big(\xi_{i\nu}(t) - \xi_{\eta\nu}(t)\big)\,dW_\nu(t)\right], \tag{2.11.8}$$

for the relative arithmetic total return of portfolio π versus portfolio η.

Proof. Upon substitution of equation (2.10.8) into equation (2.11.4), we obtain

$$\begin{aligned} d\left(\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\right) &= \frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\left[\big(\vartheta_\pi(t)-\vartheta_\eta(t)\big)\,dt+\sum_{i,\nu=1}^n \pi_i(t)\big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t)\right]+\frac{1}{2}\,\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\,\tau^\eta_{\pi\pi}(t)\,dt \\ &= \left(\vartheta_\pi(t)-\vartheta_\eta(t)+\frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\,dt+\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\sum_{i,\nu=1}^n \pi_i(t)\big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t) \\ &= \frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\left[\left(\vartheta_\pi(t)-\vartheta_\eta(t)+\frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)dt+\sum_{i,\nu=1}^n \pi_i(t)\big(\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t)\right], \end{aligned}$$

which completes the proof. ■

The result put forth in the preceding lemma is akin to the result established in equation (2.9.28), for the relative total arithmetic return of the ith stock versus portfolio η.

Lemma 2.11.5. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t \in [0,\infty)$, we obtain the dynamics*

$$d\left(\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\right) = \frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\left[\left(\vartheta_\pi(t)-\vartheta_\eta(t)+\frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)dt+\sum_{\nu=1}^n \big(\xi_{\pi\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t)\right], \tag{2.11.9}$$

for the relative arithmetic total return of portfolio π versus portfolio η.

Proof. This result follows easily by a simple modification of the previous lemma and employing equation (2.2.126), much in the same way as was done for Lemma 2.10.4. Hence, we deduce

$$\begin{aligned} d\left(\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\right) &= \frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\left[\left(\vartheta_\pi(t)-\vartheta_\eta(t)+\frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)dt+\sum_{\nu=1}^n\left[\sum_{i=1}^n \pi_i(t)\xi_{i\nu}(t)-\xi_{\eta\nu}(t)\right]dW_\nu(t)\right] \\ &= \frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\left[\left(\vartheta_\pi(t)-\vartheta_\eta(t)+\frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)dt+\sum_{\nu=1}^n \big(\xi_{\pi\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t)\right]. \end{aligned}$$

Alternatively, upon substitution of equation (2.10.10) of Lemma 2.10.4 into equation (2.11.4), we further obtain

$$\begin{aligned} d\left(\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\right) &= \frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\left[\big(\vartheta_\pi(t)-\vartheta_\eta(t)\big)\,dt+\sum_{\nu=1}^n \big(\xi_{\pi\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t)\right]+\frac{1}{2}\,\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\,\tau^\eta_{\pi\pi}(t)\,dt \\ &= \left(\vartheta_\pi(t)-\vartheta_\eta(t)+\frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\,dt+\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\sum_{\nu=1}^n \big(\xi_{\pi\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t) \\ &= \frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\left[\left(\vartheta_\pi(t)-\vartheta_\eta(t)+\frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)dt+\sum_{\nu=1}^n \big(\xi_{\pi\nu}(t)-\xi_{\eta\nu}(t)\big)\,dW_\nu(t)\right]. \end{aligned}$$

■

Lemma 2.11.6. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t \in [0,\infty)$, we obtain the dynamics*

$$d\left(\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\right) = \frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\left[\left(\vartheta^*_\pi(t)-\vartheta^*_\eta(t)+\frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)dt+\sum_{i,j=1}^n \pi_i(t)\eta_j(t)\,d\log\left(\widehat{X}_i(t)/\widehat{X}_j(t)\right)\right], \tag{2.11.10}$$

for the relative arithmetic total return of portfolio π versus portfolio η.

Proof. This is easily derived by inserting the result (2.8.23) derived in Lemma 2.8.3, into (2.11.2) of Lemma 2.11.2. Alternatively, by substituting equation (2.10.11) of Lemma 2.10.5 into equation (2.11.4), we obtain the

required result

$$\begin{aligned} d\left(\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\right) &= \frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\left[\left(\vartheta^*_\pi(t)-\vartheta^*_\eta(t)\right)dt+\sum_{i,j=1}^n \pi_i(t)\eta_j(t)\,d\log\left(\widehat{X}_i(t)/\widehat{X}_j(t)\right)\right]+\frac{1}{2}\,\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\,\tau^\eta_{\pi\pi}(t)\,dt \\ &= \frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\left[\left(\vartheta^*_\pi(t)-\vartheta^*_\eta(t)+\frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)dt+\sum_{i,j=1}^n \pi_i(t)\eta_j(t)\,d\log\left(\widehat{X}_i(t)/\widehat{X}_j(t)\right)\right]. \end{aligned}$$

■

Lemma 2.11.7. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t\in[0,\infty)$, we obtain the dynamics*

$$d\left(\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\right) = \frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\left[-2\,\vartheta^*_\eta(t)\,dt+\sum_{i,j=1}^n \pi_i(t)\eta_j(t)\frac{\widehat{X}_j(t)}{\widehat{X}_i(t)}\,d\left(\frac{\widehat{X}_i(t)}{\widehat{X}_j(t)}\right)\right], \tag{2.11.11}$$

for the relative arithmetic total return of portfolio π versus portfolio η.

Proof. Upon substitution of equation (2.10.12) of Lemma 2.10.6 into equation (2.11.4), we obtain the required result

$$\begin{aligned} d\left(\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\right) &= \frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\left[\left(-\frac{1}{2}\,\tau^\eta_{\pi\pi}(t)-2\,\vartheta^*_\eta(t)\right)dt+\sum_{i,j=1}^n \pi_i(t)\eta_j(t)\,\frac{\widehat{X}_j(t)}{\widehat{X}_i(t)}\,d\left(\frac{\widehat{X}_i(t)}{\widehat{X}_j(t)}\right)\right]+\frac{1}{2}\,\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\,\tau^\eta_{\pi\pi}(t)\,dt \\ &= \frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\left[-2\,\vartheta^*_\eta(t)\,dt+\sum_{i,j=1}^n \pi_i(t)\eta_j(t)\frac{\widehat{X}_j(t)}{\widehat{X}_i(t)}\,d\left(\frac{\widehat{X}_i(t)}{\widehat{X}_j(t)}\right)\right]. \end{aligned}$$

■

Lemma 2.11.8. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t\in[0,\infty)$, we obtain the dynamics*

$$d\left(\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\right) = \frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\left[\left(\vartheta^*_\pi(t)-\vartheta^*_\eta(t)+\frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)dt+\sum_{i=1}^n \left(\pi_i(t)-\eta_i(t)\right)d\log\left(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\right)\right], \tag{2.11.12}$$

for the relative arithmetic total return of portfolio π versus portfolio η.

Proof. By inserting equation (2.10.14) into equation (2.11.4), the following is obtained

$$\begin{aligned} d\left(\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\right) &= \frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\left[\left(\vartheta^*_\pi(t)-\vartheta^*_\eta(t)\right)dt+\sum_{i=1}^n \left(\pi_i(t)-\eta_i(t)\right)d\log\left(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\right)\right]+\frac{1}{2}\,\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\,\tau^\eta_{\pi\pi}(t)\,dt \\ &= \frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\left[\left(\vartheta^*_\pi(t)-\vartheta^*_\eta(t)+\frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\right)dt+\sum_{i=1}^n \left(\pi_i(t)-\eta_i(t)\right)d\log\left(\widehat{X}_i(t)/\widehat{Z}_\eta(t)\right)\right]. \end{aligned}$$

■

Lemma 2.11.9. *Let π and η be any arbitrary portfolios in the market $\mathcal{M}$. Then, a.s., for $t\in[0,\infty)$, we obtain the dynamics*

$$d\left(\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\right) = \sum_{i=1}^n \pi_i(t)\frac{\widehat{Z}_\pi(t)}{\widehat{X}_i(t)}\,d\left(\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\right), \tag{2.11.13}$$

for the relative arithmetic total return of portfolio π versus portfolio η.

Proof. By substituting equation (2.10.16) into equation (2.11.4), we obtain the required result

$$\begin{aligned} d\left(\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\right) &= \frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\left[\sum_{i=1}^n \pi_i(t)\frac{\widehat{Z}_\eta(t)}{\widehat{X}_i(t)}\, d\left(\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\right) - \frac{1}{2}\,\tau^\eta_{\pi\pi}(t)\,dt\right] + \frac{1}{2}\,\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\,\tau^\eta_{\pi\pi}(t)\,dt \\ &= \sum_{i=1}^n \pi_i(t)\frac{\widehat{Z}_\pi(t)}{\widehat{X}_i(t)}\, d\left(\frac{\widehat{X}_i(t)}{\widehat{Z}_\eta(t)}\right). \end{aligned}$$

Notice that, by substituting (2.9.27) into (2.11.13), combined with (2.2.168) and (2.4.29), we obtain (2.11.8). ■

From the above results, it can be easily verified that the quadratic variation of the relative arithmetic total return process satisfies the following for all $t \in [0,\infty)$, a.s.,

$$d\left\langle \frac{\widehat{Z}_\pi}{\widehat{Z}_\eta}\right\rangle_t = \left(\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\right)^2 d\left\langle \log\left(\widehat{Z}_\pi/\widehat{Z}_\eta\right)\right\rangle_t \tag{2.11.14}$$

$$= \tau^\eta_{\pi\pi}(t)\left(\frac{\widehat{Z}_\pi(t)}{\widehat{Z}_\eta(t)}\right)^2 dt. \tag{2.11.15}$$

2.12 The Market Portfolio

The portfolio we now introduce is perhaps the most important portfolio we shall consider. Recall that our main objective is to collate the performance of two different portfolios, of particular focus is a comparable benchmark portfolio. Such a portfolio that we shall pay particular heed to is the reverential market portfolio. The market portfolio plays a fundamental role in stochastic portfolio theory, since if one invests according to the market portfolio, one will achieve the "entire market" by means of a constant multiplier difference.

Definition 2.12.1 (The Market Portfolio). *The portfolio* $\mu = \{\boldsymbol{\mu}(t) = (\mu_1(t),\ldots,\mu_n(t)), t \in [0,\infty)\}$ *defined by*

$$\mu_i(t) \triangleq \frac{X_i(t)}{X_1(t)+\cdots+X_n(t)}, \qquad t \in [0,\infty), \tag{2.12.1}$$

for $i = 1,\ldots,n$, *is called the* ***market portfolio (process)*** *and the weights* $\mu_i(t)$ *are called the* ***market weight processes****, or simply, the* ***market weights*** *(we can use the terms, market weights, capitalisation weights or cap-weights, interchangeably).*

From the definition of the market portfolio, it is apparent that $0 < \mu_i(t) < 1$, for $t \in [0,\infty)$ and $i = 1,\ldots,n$, and thus exhibits positive weights for all the stocks. The importance of the market portfolio is derived from the status as the canonical benchmark for equity portfolio performance. It can be easily verified that μ satisfies the requirements of Definition 2.2.16, since clearly the weights are bounded on $[0,\infty)$, and

$$\sum_{i=1}^n \mu_i(t) = \sum_{i=1}^n \frac{X_i(t)}{X_1(t)+\cdots+X_n(t)} = \frac{1}{X_1(t)+\cdots+X_n(t)}\sum_{i=1}^n X_i(t) = \frac{X_1(t)+\cdots+X_n(t)}{X_1(t)+\cdots+X_n(t)} = 1.$$

So, we may think of the vector process $\boldsymbol{\mu}(t) = (\mu_1(t),\ldots,\mu_n(t))$ as a portfolio that invests the proportion $\mu_i(t)$ of current wealth in the ith stock at all times. Equivalently, this portfolio holds the same constant number of shares in all stocks at all times [Fernholz & Karatzas (2009)]. Thus, the market portfolio process $\boldsymbol{\mu}(t) = (\mu_1(t),\ldots,\mu_n(t))$ always lies in the *open unit* $(n-1)-$*simplex*, Δ^{n-1}, $\boldsymbol{\mu}(t) \in \Delta^{n-1}$, defined as in Definition 2.2.16, by

$$\Delta^{n-1} \triangleq \left\{\boldsymbol{\mu}(t) = (\mu_1(t),\ldots,\mu_n(t)) \in \mathbb{R}^n \;\middle|\; \mu_1(t) > 0,\ldots,\mu_n(t) > 0,\ \mu_1(t)+\cdots+\mu_n(t) = 1\right\},$$

or more precisely by (2.2.59),

$$\Delta^{n-1} = \left\{\boldsymbol{\mu}(t) = (\mu_1(t),\ldots,\mu_n(t)) \in \mathbb{R}^n \;\middle|\; \mu_1(t)+\cdots+\mu_n(t) = 1,\ 0 < \mu_i(t) < 1,\ i = 1,\ldots,n\right\},$$

and thus admits no short sales. Again, suppose we normalise so that each stock has always just a single share outstanding, so that the ith stock price at time t, $X_i(t)$, can also be interpreted as the capitalisation of the ith company at time t. Let $Z_{w,\mu} = \{Z_{w,\mu}(t), t \in [0,\infty)\}$, where the initial capital invested is given by $Z_{w,\mu}(0) = w > 0$ for all $t \in [0,\infty)$, denote the value of an investment in the prestigious market portfolio $\mu = \big\{\boldsymbol{\mu}(t) = \big(\mu_1(t), \ldots, \mu_n(t)\big), t \in [0,\infty)\big\}$. We shall refer to $Z_{w,\mu} = \big\{Z_{w,\mu}(t), t \in [0,\infty)\big\}$ as the *market portfolio value process* with initial capital $Z_{w,\mu}(0) = w > 0$. Thus the interpretation is that $\mu_i(t)$ represents the proportion of wealth $Z_{w,\mu}(t)$ invested at time t in the ith stock. Also, the quantity $\mu_i(t)$, can be interpreted as the *relative capitalisation* of the individual companies. With this in mind, let us officially define the *total market capitalisation process*, $Z = \{Z(t), t \in [0,\infty)\}$ with initial capital given by $Z(0) = z > 0$, as follows

$$Z(t) \triangleq X_1(t) + \cdots + X_n(t), \qquad t \in [0,\infty). \tag{2.12.2}$$

Thus, the quantity $Z(t)$, can be interpreted as the total capitalisation of the market. So that, for $i = 1, \ldots, n$, (2.12.1) becomes

$$\mu_i(t) \triangleq \frac{X_i(t)}{Z(t)}, \qquad t \in [0,\infty). \tag{2.12.3}$$

Moreover, we have

$$dZ(t) = d\big(X_1(t) + \cdots + X_n(t)\big) = d\left(\sum_{i=1}^n X_i(t)\right), \qquad t \in [0,\infty), \tag{2.12.4}$$

$$dZ(t) = dX_1(t) + \cdots + dX_n(t) = \sum_{i=1}^n dX_i(t), \qquad t \in [0,\infty). \tag{2.12.5}$$

We may think of the $(1 \times n)-$vector-valued process $\boldsymbol{\mu}(t) = \big(\mu_1(t), \mu_2(t), \ldots, \mu_n(t)\big)$ as a portfolio rule that invests a proportion $\mu_i(t)$ of current wealth in the ith stock, for all $t \in [0,\infty)$. Then, according to Corollary 2.2.19, (2.2.61), the resulting value process $Z_{w,\mu}(t)$, that corresponds to an initial capital of $Z_{w,\mu}(0) = w > 0$, satisfies

$$\frac{dZ_{w,\mu}(t)}{Z_{w,\mu}(t)} = \sum_{i=1}^n \mu_i(t)\,\frac{dX_i(t)}{X_i(t)}. \tag{2.12.6}$$

Thus, we obtain

$$\frac{dZ_{w,\mu}(t)}{Z_{w,\mu}(t)} = \sum_{i=1}^n \frac{X_i(t)}{Z(t)}\,\frac{dX_i(t)}{X_i(t)} = \sum_{i=1}^n \frac{dX_i(t)}{Z(t)} = \frac{dZ(t)}{Z(t)}, \tag{2.12.7}$$

as postulated by (2.12.3). In other words, from (2.12.7) we deduce the following for all $t \in [0,\infty)$,

$$\frac{dZ_{w,\mu}(t)}{Z_{w,\mu}(t)} = \frac{dZ(t)}{Z(t)} \tag{2.12.8}$$

$$\frac{Z_{w,\mu}(t) - Z_{w,\mu}(0)}{Z_{w,\mu}(t)} = \frac{Z(t) - Z(0)}{Z(t)}$$

$$1 - \frac{Z_{w,\mu}(0)}{Z_{w,\mu}(t)} = 1 - \frac{Z(0)}{Z(t)}$$

$$\frac{Z_{w,\mu}(0)}{Z_{w,\mu}(t)} = \frac{Z(0)}{Z(t)}$$

$$\frac{Z_{w,\mu}(t)}{Z_{w,\mu}(0)} = \frac{Z(t)}{Z(0)} \tag{2.12.9}$$

$$Z_{w,\mu}(t) = \frac{Z_{w,\mu}(0)}{Z(0)}\,Z(t) \tag{2.12.10}$$

$$= \frac{w}{Z(0)}\,Z(t) \tag{2.12.11}$$

$$= \frac{w}{z}\,Z(t) \tag{2.12.12}$$

$$= \frac{w}{z}\,\big(X_1(t) + \cdots + X_n(t)\big), \tag{2.12.13}$$

with $z \triangleq Z(0) = X_1(0) + \cdots + X_n(0)$. So, by letting k be

$$k \triangleq \frac{Z_{w,\mu}(0)}{Z(0)} = \frac{w}{Z(0)} = \frac{w}{z},$$

where k is a constant, we have

$$Z_{w,\mu}(t) = k\,Z(t). \tag{2.12.14}$$

Thus, with the appropriate initial conditions, i.e., if we start with initial capital $w = Z_{w,\mu}(0) = Z(0) = X_1(0)+\cdots+X_n(0) = z$, we get $k = 1$ and hence $Z_{w,\mu}(t) \equiv Z(t)$ for all $t \in [0,\infty)$, the total market capitalisation. In other words, investing according to the portfolio process $\boldsymbol{\mu}(t)$ amounts to ownership of the entire market in proportion to the original initial investment, $Z_{w,\mu}(0) = Z(0) = w$ [Fernholz, Karatzas & Kardaras (2005), Fernholz & Karatzas (2009)]. The portfolio μ essentially "mirrors the market", in the sense that the ratio of its value to the total market capitalisation, $Z(t)$, is a.s. constant over time [Banner & Fernholz (2008)]. This consequence is what gives us reason to refer to μ as the market portfolio in $\mathcal{M}$. Thus, if at time $t = 0$, $Z_{w,\mu}(0) \equiv X_1(0) + \cdots + X_n(0)$, we have

$$Z_{w,\mu}(t) \triangleq X_1(t) + \cdots + X_n(t), \qquad t \in [0,\infty). \tag{2.12.15}$$

Hence, the value of the market portfolio represents the combined capitalisation of all the stocks in the market. Furthermore, $Z_{w,\mu}(t)$ satisfies the equivalence to (2.2.64),

$$\frac{dZ_{w,\mu}(t)}{Z_{w,\mu}(t)} = \sum_{i=1}^{n} \mu_i(t)\,\frac{dX_i(t)}{X_i(t)}. \tag{2.12.16}$$

Remark 2.12.2. In recognition of the special status of the market portfolio, henceforth, we shall reserve the notation μ to exclusively represent this portfolio, and $Z_w = Z_{w,\mu}(t)$ in (2.12.15) to represent its value. In light of this prominence, we shall use the notation

$$\boldsymbol{\tau} = \Big\{\boldsymbol{\tau}(t) = \big(\tau_{ij}(t)\big)_{1\le i,j\le n}, t \in [0,\infty)\Big\}, \tag{2.12.17}$$

to represent

$$\boldsymbol{\tau}^{\boldsymbol{\mu}} = \Big\{\boldsymbol{\tau}^{\boldsymbol{\mu}}(t) = \big(\tau^{\mu}_{ij}(t)\big)_{1\le i,j\le n}, t \in [0,\infty)\Big\}, \tag{2.12.18}$$

and $\tau_{ij} = \big\{\tau_{ij}(t), t \in [0,\infty)\big\}$ to represent $\tau^{\mu}_{ij} = \big\{\tau^{\mu}_{ij}(t), t \in [0,\infty)\big\}$, namely, the covariances of the individual stocks relative to the entire market, and $\tau_{ii} = \big\{\tau_{ii}(t), t \in [0,\infty)\big\}$ to represent $\tau^{\mu}_{ii} = \big\{\tau^{\mu}_{ii}(t), t \in [0,\infty)\big\}$, namely, the variances of the individual stocks relative to the entire market. Further to these, for a portfolio π, we shall use $\tau_{\pi\pi} = \big\{\tau_{\pi\pi}(t), t \in [0,\infty)\big\}$ to represent $\tau^{\mu}_{\pi\pi} = \big\{\tau^{\mu}_{\pi\pi}(t), t \in [0,\infty)\big\}$. This is in keeping with and following the convention that Fernholz (2002) appoints. Furthermore, we observe from (2.12.1), that the logarithm of the ith market weight can be interpreted as the relative return of the ith stock versus the market portfolio μ.

Definition 2.12.1 and (2.12.15) imply that the market weight processes $\mu_i(t)$, for $i = 1,\ldots,n$, are *quotient processes* and can be expressed as

$$\mu_i(t) = \frac{X_i(t)}{Z_{w,\mu}(t)}, \qquad t \in [0,\infty). \tag{2.12.19}$$

Much in the same way as was done for the portfolio π, we shall set the initial wealth, $Z_{w,\mu}(0) = w$, invested in the market portfolio μ to $w = 1$ unit of currency. Henceforward, we shall write $Z_\mu(t) \equiv Z_{1,\mu}(t)$, which represents the portfolio value process or wealth process that corresponds to the market portfolio μ and initial capital given by $Z_\mu(0) = w = 1$ unit of currency, e.g. $w = \$1$. Thus, let $Z_\mu = \big\{Z_\mu(t), t \in [0,\infty)\big\}$ where $Z_\mu(t) > 0$ for all $t \in [0,\infty)$, denote the value of an investment in the market portfolio μ with initial capital $w = 1$. Therefore, all the results derived for the portfolio value process $Z_{w,\mu}$ with initial capital $Z_{w,\mu}(0) = w > 0$ can be similarly derived for the portfolio value process Z_μ with initial capital $Z_\mu(0) = w = 1$.

Now, let us consider the following process $\log \mu = \big\{\log \boldsymbol{\mu}(t) = \big(\log \mu_1(t), \ldots, \log \mu_n(t)\big), t \in [0,\infty)\big\}$. According to the expression given in (2.12.19), we obtain, for $i = 1, \ldots, n$,

$$\log \mu_i(t) \;=\; \log\Big(X_i(t)/Z_\mu(t)\Big) \;=\; \log\Big(X_i(t)/Z_{w,\mu}(t)\Big)\Big|_{w=X_i(0)}, \qquad t \in [0,\infty), \tag{2.12.20}$$

which, as dictated by Definition 2.3.5, represents the relative return of the ith stock versus the market portfolio μ. Moreover, (2.12.20) provides a measure of the relative performance of each of the individual stocks in the market versus the canonical benchmark market portfolio.

By analogy with (2.2.65), Proposition 2.2.20, we have for $Z_\mu(0) = w > 0$,

$$d\log Z_\mu(t) \;=\; \gamma_\mu(t)\,dt + \sum_{i,\nu=1}^{n} \mu_i(t)\xi_{i\nu}(t)\,dW_\nu(t), \qquad t \in [0,\infty), \qquad \text{a.s.} \tag{2.12.21}$$

Further to the above expression, by analogy with (2.2.131), we have for $Z_\mu(0) = w > 0$,

$$d\log Z_\mu(t) \;=\; \gamma_\mu(t)\,dt + \sum_{\nu=1}^{n} \xi_{\mu\nu}(t)\,dW_\nu(t), \qquad t \in [0,\infty), \qquad \text{a.s.}, \tag{2.12.22}$$

consult expression (2.12.24) below for details. Now, by combining (2.2.20) and (2.12.21) (or, (2.12.22)), into (2.12.20), we obtain the dynamics of the market weights in (2.12.19) for all stocks $i = 1, \ldots, n$, and $t \in [0,\infty)$, a.s.,

$$\begin{aligned}
d\log \mu_i(t) &= d\log X_i(t) - d\log Z_\mu(t) \\
&= \Big(\gamma_i(t)\,dt + \sum_{\nu=1}^{n} \xi_{i\nu}(t)\,dW_\nu(t)\Big) - \Big(\gamma_\mu(t)\,dt + \sum_{j,\nu=1}^{n} \mu_j(t)\xi_{j\nu}(t)\,dW_\nu(t)\Big) \\
&= \Big(\gamma_i(t)\,dt + \sum_{\nu=1}^{n} \xi_{i\nu}(t)\,dW_\nu(t)\Big) - \Big(\gamma_\mu(t)\,dt + \sum_{\nu=1}^{n}\Big[\sum_{j=1}^{n} \mu_j(t)\xi_{j\nu}(t)\Big]\,dW_\nu(t)\Big) \\
&= \Big(\gamma_i(t)\,dt + \sum_{\nu=1}^{n} \xi_{i\nu}(t)\,dW_\nu(t)\Big) - \Big(\gamma_\mu(t)\,dt + \sum_{\nu=1}^{n} \xi_{\mu\nu}(t)\,dW_\nu(t)\Big) \\
&= \big(\gamma_i(t) - \gamma_\mu(t)\big)\,dt + \sum_{\nu=1}^{n} \big(\xi_{i\nu}(t) - \xi_{\mu\nu}(t)\big)\,dW_\nu(t).
\end{aligned} \tag{2.12.23}$$

We see that the logarithmic change in the market weight μ_i is equal to the log-return of X_i relative to the market. We obtain (2.12.23) by setting $\pi := \mu$ in (2.2.126), we thus have the analogue for $\nu = 1, \ldots, n$,

$$\xi_{\mu\nu}(t) \;=\; \sum_{i=1}^{n} \mu_i(t)\xi_{i\nu}(t), \tag{2.12.24}$$

which, when substituted above, gives the analogue to (2.3.43). Hence, from (2.3.87) and (2.3.88), and by setting $\eta := \mu$ for the market portfolio, for $i, j = 1, 2, \ldots, n$ it follows that

$$d\big\langle \log \mu_i, \log \mu_j \big\rangle_t \;=\; d\big\langle \log\big(X_i/Z_\mu\big), \log\big(X_j/Z_\mu\big)\big\rangle_t \;=\; \tau_{ij}(t)\,dt, \quad \text{or,} \tag{2.12.25}$$

$$\big\langle \log \mu_i, \log \mu_j \big\rangle_t \;=\; \big\langle \log\big(X_i/Z_\mu\big), \log\big(X_j/Z_\mu\big)\big\rangle_t \;=\; \int_0^t \tau_{ij}(s)\,ds, \tag{2.12.26}$$

and from (2.3.123) and (2.3.124), and by setting $\eta := \mu$ for the market portfolio, for $i = 1, 2, \ldots, n$ it follows that

$$d\big\langle \log \mu_i \big\rangle_t \;=\; d\big\langle \log\big(X_i/Z_\mu\big)\big\rangle_t \;=\; \tau_{ii}(t)\,dt, \quad \text{or,} \tag{2.12.27}$$

$$\big\langle \log \mu_i \big\rangle_t \;=\; \big\langle \log\big(X_i/Z_\mu\big)\big\rangle_t \;=\; \int_0^t \tau_{ii}(s)\,ds. \tag{2.12.28}$$

Note the following quantities analogous to equations (2.3.112) and (2.3.113), for the market portfolio $\eta := \mu$, for $i, j = 1, 2, \ldots, n$,

$$\tau_{ij}(t) = \frac{d}{dt}\left\langle \log \mu_i, \log \mu_j \right\rangle_t = \sum_{\nu=1}^{n} \Big(\xi_{i\nu}(t) - \xi_{\mu\nu}(t)\Big)\Big(\xi_{j\nu}(t) - \xi_{\mu\nu}(t)\Big), \quad \text{and,} \tag{2.12.29}$$

$$\tau_{ij}(t)\,dt = d\left\langle \log \mu_i, \log \mu_j \right\rangle_t = \sum_{\nu=1}^{n} \Big(\xi_{i\nu}(t) - \xi_{\mu\nu}(t)\Big)\Big(\xi_{j\nu}(t) - \xi_{\mu\nu}(t)\Big)\,dt, \tag{2.12.30}$$

namely, the covariances of the individual stocks relative to the entire market, as well as those analogous to equations (2.3.133) and (2.3.134), for the market portfolio $\eta := \mu$, for $i = 1, 2, \ldots, n$,

$$\tau_{ii}(t) = \frac{d}{dt}\left\langle \log \mu_i \right\rangle_t = \sum_{\nu=1}^{n} \Big(\xi_{i\nu}(t) - \xi_{\mu\nu}(t)\Big)^2, \quad \text{and,} \tag{2.12.31}$$

$$\tau_{ii}(t)\,dt = d\left\langle \log \mu_i \right\rangle_t = \sum_{\nu=1}^{n} \Big(\xi_{i\nu}(t) - \xi_{\mu\nu}(t)\Big)^2\,dt, \tag{2.12.32}$$

namely, the variances of the individual stocks relative to the entire market.

Consider the following form for the market portfolio process $\mu_i(t) = \exp\big(\log \mu_i(t)\big)$, we can apply Itô's formula to $\exp\big(\log \mu_i(t)\big)$. So, by letting $\log \mu_i(t) = Y(t)$, the form of the function to be used in Itô's formula is given by $F\big(t, Y(t)\big) = \exp\big(Y(t)\big)$, and the following are easily obtained

$$\begin{aligned}
\frac{\partial F}{\partial t}(t, y) &= 0, \\
\frac{\partial F}{\partial y}(t, y) &= \exp\big(Y(t)\big), \\
\frac{\partial^2 F}{\partial y^2}(t, y) &= \exp\big(Y(t)\big).
\end{aligned}$$

We then arrive at the following

$$dF\big(t, Y(t)\big) = \exp\big(Y(t)\big)\,dY(t) + \frac{1}{2}\exp\big(Y(t)\big)\,d\left\langle Y \right\rangle_t, \qquad t \in [0, \infty), \qquad \text{a.s.} \tag{2.12.33}$$

Since, $Y(t) = \log \mu_i(t)$, the following formula for $d\mu_i$ is obtained for $t \in [0, \infty)$, a.s.,

$$d\mu_i(t) = \mu_i(t)\,d\log \mu_i(t) + \frac{1}{2}\,\mu_i(t)\,d\left\langle \log \mu_i \right\rangle_t \tag{2.12.34}$$

$$= \mu_i(t)\,d\log \mu_i(t) + \frac{1}{2}\,\mu_i(t)\,\tau_{ii}(t)\,dt. \tag{2.12.35}$$

The expression above can be rewritten in the following more useful forms

$$\frac{d\mu_i(t)}{\mu_i(t)} = d\log \mu_i(t) + \frac{1}{2}\,\tau_{ii}(t)\,dt, \tag{2.12.36}$$

and,

$$d\log \mu_i(t) = \frac{d\mu_i(t)}{\mu_i(t)} - \frac{1}{2}\,\tau_{ii}(t)\,dt. \tag{2.12.37}$$

By substituting (2.12.23) into (2.12.35), we obtain

$$d\mu_i(t) = \mu_i(t)\left[\big(\gamma_i(t) - \gamma_\mu(t)\big)\,dt + \sum_{\nu=1}^{n}\big(\xi_{i\nu}(t) - \xi_{\mu\nu}(t)\big)\,dW_\nu(t)\right] + \frac{1}{2}\,\mu_i(t)\,\tau_{ii}(t)\,dt \tag{2.12.38}$$

$$= \left(\gamma_i(t) - \gamma_\mu(t) + \frac{1}{2}\,\tau_{ii}(t)\right)\mu_i(t)\,dt + \mu_i(t)\sum_{\nu=1}^{n}\big(\xi_{i\nu}(t) - \xi_{\mu\nu}(t)\big)\,dW_\nu(t) \tag{2.12.39}$$

$$= \mu_i(t)\left[\left(\gamma_i(t) - \gamma_\mu(t) + \frac{1}{2}\,\tau_{ii}(t)\right)dt + \sum_{\nu=1}^{n}\big(\xi_{i\nu}(t) - \xi_{\mu\nu}(t)\big)\,dW_\nu(t)\right]. \tag{2.12.40}$$

Thus, we have

$$\frac{d\mu_i(t)}{\mu_i(t)} = \left(\gamma_i(t) - \gamma_\mu(t) + \frac{1}{2}\,\tau_{ii}(t)\right) dt + \sum_{\nu=1}^{n} \left(\xi_{i\nu}(t) - \xi_{\mu\nu}(t)\right) dW_\nu(t). \tag{2.12.41}$$

The quadratic variation of the market weights, for $i = 1, 2, \ldots, n$, can be achieved from (2.5.35) or by considering (2.12.35) along with (2.12.27), as follows

$$\begin{aligned} d\left\langle \mu_i \right\rangle_t = d\left\langle \frac{X_i}{Z_\mu} \right\rangle_t &= \left(\frac{X_i(t)}{Z_\mu(t)}\right)^2 d\left\langle \log\left(X_i/Z_\mu\right)\right\rangle_t && (2.12.42)\\ &= \mu_i^2(t)\, d\left\langle \log \mu_i \right\rangle_t && (2.12.43)\\ &= \mu_i^2(t)\, \tau_{ii}(t)\, dt && (2.12.44)\\ &= \tau_{ii}(t)\, \mu_i^2(t)\, dt. && (2.12.45) \end{aligned}$$

From (2.5.37) or by considering (2.12.35) along with (2.12.25), we arrive at the following for the cross-variation of the market weights,

$$\begin{aligned} d\left\langle \mu_i, \mu_j \right\rangle_t = d\left\langle \frac{X_i}{Z_\mu}, \frac{X_j}{Z_\mu} \right\rangle_t &= \left(\frac{X_i(t)X_j(t)}{Z_\mu^2(t)}\right) d\left\langle \log\left(X_i/Z_\mu\right), \log\left(X_j/Z_\mu\right)\right\rangle_t && (2.12.46)\\ &= \mu_i(t)\mu_j(t)\, d\left\langle \log \mu_i, \log \mu_j \right\rangle_t && (2.12.47)\\ &= \mu_i(t)\mu_j(t)\, \tau_{ij}(t)\, dt && (2.12.48)\\ &= \tau_{ij}(t)\, \mu_i(t)\mu_j(t)\, dt. && (2.12.49) \end{aligned}$$

Consequently, we deduce the following for $t \in [0, \infty)$, a.s.,

$$\frac{d\left\langle \mu_i \right\rangle_t}{\mu_i^2(t)} = d\left\langle \log \mu_i \right\rangle_t = \tau_{ii}(t)\, dt, \tag{2.12.50}$$

and,

$$\frac{d\left\langle \mu_i, \mu_j \right\rangle_t}{\mu_i(t)\mu_j(t)} = d\left\langle \log \mu_i, \log \mu_j \right\rangle_t = \tau_{ij}(t)\, dt. \tag{2.12.51}$$

Therefore, for $i = 1, 2, \ldots, n$, $t \in [0, \infty)$, a.s., we also have for the market portfolio $\eta := \mu$,

$$\tau_{ii}(t) = \frac{d}{dt}\left\langle \log \mu_i \right\rangle_t = \frac{d\left\langle \mu_i \right\rangle_t}{\mu_i^2(t)\, dt} = \frac{1}{\mu_i^2(t)}\, \frac{d}{dt}\left\langle \mu_i \right\rangle_t, \tag{2.12.52}$$

namely, the variances of the individual stocks relative to the entire market, and, for $i, j = 1, 2, \ldots, n$, $t \in [0, \infty)$, a.s., we also have for the market portfolio $\eta := \mu$,

$$\tau_{ij}(t) = \frac{d}{dt}\left\langle \log \mu_i, \log \mu_j \right\rangle_t = \frac{d\left\langle \mu_i, \mu_j \right\rangle_t}{\mu_i(t)\mu_j(t)\, dt} = \frac{1}{\mu_i(t)\mu_j(t)}\, \frac{d}{dt}\left\langle \mu_i, \mu_j \right\rangle_t, \tag{2.12.53}$$

namely, the covariances of the individual stocks relative to the entire market.

Remark 2.12.3. The following relations, for $t \in [0, \infty)$, a.s.,

$$\begin{aligned} d\left\langle \log \mu_i, \log \mu_j \right\rangle_t &= \tau_{ij}^{\mu}(t)\, dt = \tau_{ij}(t)\, dt, \quad \text{and,}\\ d\left\langle \log \mu_i \right\rangle_t &= \tau_{ii}^{\mu}(t)\, dt = \tau_{ii}(t)\, dt, \end{aligned}$$

also,

$$\begin{aligned} d\left\langle \mu_i, \mu_j \right\rangle_t &= \mu_i(t)\mu_j(t)\, \tau_{ij}^{\mu}(t)\, dt = \mu_i(t)\mu_j(t)\, \tau_{ij}(t)\, dt, \quad \text{and,}\\ d\left\langle \mu_i \right\rangle_t &= \mu_i^2(t)\, \tau_{ii}^{\mu}(t)\, dt = \mu_i^2(t)\, \tau_{ii}(t)\, dt, \end{aligned}$$

are unique to the market weights, this uniqueness stems from the fact that the cross-variation of any two market weights can be written as the product of these two market weights and their corresponding relative covariance. Similar results cannot be expected to hold for arbitrary portfolio weights. The following lemma is the analogous version of Lemma 2.4.1, for the market portfolio. We include it here to show how the new results concerning the market portfolio can be used to prove the same result.

Lemma 2.12.4. *The $(n \times n)-$matrix-valued process, $\boldsymbol{\tau}^{\boldsymbol{\mu}}(t) = \big(\tau^{\mu}_{ij}(t)\big)_{1 \le i,j \le n}$, of the individual stocks' covariances relative to the market portfolio μ satisfies the elementary property for all $i = 1, \ldots, n$,*

$$\sum_{j=1}^{n} \tau^{\mu}_{ij}(t)\mu_j(t) \quad = \quad \sum_{j=1}^{n} \tau_{ij}(t)\mu_j(t) \quad = \quad 0, \qquad t \in [0, \infty). \tag{2.12.54}$$

Alternatively, expressed in matrix form, we have the equivalent property

$$\boldsymbol{\tau}^{\boldsymbol{\mu}}(t)\boldsymbol{\mu}^{\mathsf{T}}(t) \quad = \quad \mathbf{0}^{\mathsf{T}}, \qquad t \in [0, \infty), \tag{2.12.55}$$

where $\mathbf{0}$ is the $(1 \times n)-$zero vector, i.e., $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^n$.

Proof. By using (2.12.48) and by appealing to the bilinearity of the quadratic (cross-variation) covariation process, we obtain for $i = 1, 2, \ldots, n$, $t \in [0, \infty)$, a.s.,

$$\sum_{j=1}^{n} \mu_i(t)\tau_{ij}(t)\mu_j(t)\, dt \quad = \quad \sum_{j=1}^{n} d\left\langle \mu_i, \mu_j \right\rangle_t \quad = \quad d\left\langle \mu_i, \sum_{j=1}^{n} \mu_j \right\rangle_t \quad = \quad d\left\langle \mu_i, 1 \right\rangle_t \quad = \quad 0.$$

Therewith, we also have the following

$$\int_0^t \sum_{j=1}^{n} \mu_i(s)\tau_{ij}(s)\mu_j(s)\, ds \quad = \quad 0.$$

Therefore,

$$\sum_{j=1}^{n} \mu_i(t)\tau_{ij}(t)\mu_j(t) \quad = \quad \mu_i(t)\sum_{j=1}^{n} \tau_{ij}(t)\mu_j(t) \quad = \quad 0, \tag{2.12.56}$$

and since $\mu_i(t) \neq 0$ for all $t \in [0, \infty)$, for $i = 1, 2, \ldots, n$, i.e., $\boldsymbol{\mu}(t) \neq \mathbf{0}$, the result follows. ■

Recall the strong nondegeneracy condition (2.2.55) in Definition 2.2.12, Fernholz (1998b) provides an alternative definition for this market condition on the volatility structure, in terms of the relative variances.

Definition 2.12.5 (Nondegeneracy)**.** *The market $\mathcal{M}$ is **nondegenerate** if there exists an $\varepsilon > 0$ such that for all $i = 1, \ldots, n$,*

$$\tau_{ii}(t) \quad \ge \quad \varepsilon, \qquad t \in [0, \infty), \qquad a.s. \tag{2.12.57}$$

Remark 2.12.6. Definition 2.12.5 is compatible with condition (2.2.55), but neither implies the other. The values of the relative variances in (2.12.57) should be fairly simple to estimate in actual equity markets, while condition (2.2.55) could be more difficult to verify because it depends on the values of the eigenvalues of the covariance matrix $\boldsymbol{\sigma}(t)$. Both of these conditions are probably consistent with the behaviour of actual equity markets. There is one significant difference, however: condition (2.2.55) is helpful in proving the absence of arbitrage, while condition (2.12.57) implies its presence, as will be shown later [Fernholz (1998b)].

Lemma 2.12.7 (**[Fernholz (1998b)]**)**.** *For the market portfolio μ, $\boldsymbol{\tau}^{\boldsymbol{\mu}}(t)$ is positive semidefinite for $t \in [0, \infty)$, a.s., and $\boldsymbol{\mu}(t)$ is in the null space of $\boldsymbol{\tau}(t) = \boldsymbol{\tau}^{\boldsymbol{\mu}}(t)$.*

Proof. This follows directly from Lemma 2.4.2, by setting $\eta := \mu$ in (2.4.16),

$$\boldsymbol{\pi}(t)\boldsymbol{\tau}^{\boldsymbol{\mu}}(t)\boldsymbol{\pi}^{\mathsf{T}}(t) \quad = \quad \Big(\boldsymbol{\pi}(t) - \boldsymbol{\mu}(t)\Big)\boldsymbol{\sigma}(t)\Big(\boldsymbol{\pi}(t) - \boldsymbol{\mu}(t)\Big)^{\mathsf{T}} \quad = \quad 0, \qquad t \in [0, \infty), \qquad \text{a.s.}, \tag{2.12.58}$$

if and only if $\boldsymbol{\pi}(t) = \boldsymbol{\mu}(t)$ for all $t \in [0, \infty)$. Thus, $\boldsymbol{\mu}(t)$ spans the null space of $\boldsymbol{\tau}(t)$. ■

From equation (2.4.17) of Definition 2.4.3 and setting $\eta := \mu$ therein, we have the variance of portfolio π relative to the market portfolio,

$$\tau_{\pi\pi}(t) \quad = \quad \tau^{\mu}_{\pi\pi}(t) \quad = \quad \sum_{i,j=1}^{n} \pi_i(t)\tau^{\mu}_{ij}(t)\pi_j(t) \quad = \quad \sum_{i,j=1}^{n} \pi_i(t)\tau_{ij}(t)\pi_j(t). \tag{2.12.59}$$

Moreover, recall from the numéraire invariance property, and by setting $\eta := \mu$ in (2.4.26), we have

$$\begin{aligned}\gamma_\pi^*(t) &= \frac{1}{2}\left(\sum_{i=1}^n \pi_i(t)\tau_{ii}^\mu(t) - \sum_{i,j=1}^n \pi_i(t)\tau_{ij}^\mu(t)\pi_j(t)\right)\\ &= \frac{1}{2}\left(\sum_{i=1}^n \pi_i(t)\tau_{ii}(t) - \sum_{i,j=1}^n \pi_i(t)\tau_{ij}(t)\pi_j(t)\right). \end{aligned} \tag{2.12.60}$$

By applying equation (2.12.59) or from equation (2.4.29), we can express the above as

$$\gamma_\pi^*(t) = \frac{1}{2}\left(\sum_{i=1}^n \pi_i(t)\tau_{ii}(t) - \tau_{\pi\pi}^\mu(t)\right) \tag{2.12.61}$$

$$= \frac{1}{2}\left(\sum_{i=1}^n \pi_i(t)\tau_{ii}(t) - \tau_{\pi\pi}(t)\right). \tag{2.12.62}$$

In the particular case, where $\eta = \mu$, the market portfolio, (2.6.4) can be expressed in a notably useful form. This we encapsulate in the following proposition.

Proposition 2.12.8 ([**Fernholz (2002)**]). *Let π be any arbitrary portfolio in the market $\mathcal{M}$. Then, a.s., for $t \in [0, \infty)$, we get the dynamics*

$$d\log\Big(Z_\pi(t)/Z_\mu(t)\Big) = \sum_{i=1}^n \pi_i(t)\, d\log\mu_i(t) + \gamma_\pi^*(t)\, dt, \tag{2.12.63}$$

for the relative return of an arbitrary portfolio π with respect to the market.

Proof. The above equality (2.12.63) is the special case of (2.6.4) with $\eta := \mu$. Recall, (2.12.20), and by implementing the result formulated in (2.6.4), we then arrive at,

$$\begin{aligned} d\log\Big(Z_\pi(t)/Z_\mu(t)\Big) &= \sum_{i=1}^n \pi_i(t)\, d\log\Big(X_i(t)/Z_\mu(t)\Big) + \gamma_\pi^*(t)\, dt\\ &= \sum_{i=1}^n \pi_i(t)\, d\log\mu_i(t) + \gamma_\pi^*(t)\, dt. \end{aligned}$$

■

The crucial consequence that ensues from this proposition is that we can represent the relative return of a portfolio versus the market portfolio in terms of the changes in the market weights. Representations of the relative log-return of a portfolio π can be useful in highlighting certain characteristics of portfolio behaviour. Particularly, the representation in (2.12.63) shows that the relative log-return of π is the weighted average of the changes in the logarithms of the market weights, plus the excess growth rate of the portfolio π. Alternatively, in a more notable fashion, the *relative log-return of* π is the weighted average of the *relative log-returns of the stocks* in the portfolio, plus the excess growth rate of the portfolio, this is akin to (2.2.112), in which the log-return of π is the weighted average of the log-returns of the stocks in the portfolio, plus the excess growth rate of the portfolio. This relationship is of extreme importance since it serves as the cornerstone to the development of the theory of portfolio generating functions, which we shall explore in Chapters 5 and 6. Thus, it will be necessary to develop an understanding of the behaviour of the components just mentioned.

In the particular case that $\pi = \mu$, the left-hand side of the representation vanishes, exposing the fact that for the market portfolio the weighted average of the changes in the logarithms of the market weights will exactly nullify the (possibly positive) excess growth rate term. In this sense, the market portfolio is unable to capture the excess growth it generates [Fernholz (2003b)].

Another useful expression of (2.6.4) is presented in the following lemma.

Lemma 2.12.9 ([**Fernholz & Karatzas (2009)**]). *Let π be any arbitrary portfolio in the market $\mathcal{M}$. Then, a.s., for $t \in [0, \infty)$, we get the dynamics*

$$d\log\Big(Z_\pi(t)/Z_\mu(t)\Big) = \big(\gamma_\pi^*(t) - \gamma_\mu^*(t)\big)\,dt + \sum_{i=1}^n \big(\pi_i(t) - \mu_i(t)\big)\,d\log\mu_i(t), \tag{2.12.64}$$

for the relative return of an arbitrary portfolio π with respect to the market.

Proof. Recall (2.12.23), the dynamics for the market weights, from this we obtain

$$\begin{aligned}\sum_{i=1}^n \mu_i(t)\,d\log\mu_i(t) &= \sum_{i=1}^n \mu_i(t)\Big[\big(\gamma_i(t) - \gamma_\mu(t)\big)\,dt + \sum_{\nu=1}^n \big(\xi_{i\nu}(t) - \xi_{\mu\nu}(t)\big)\,dW_\nu(t)\Big] \\ &= \sum_{i=1}^n \mu_i(t)\big(\gamma_i(t) - \gamma_\mu(t)\big)\,dt + \sum_{i,\nu=1}^n \mu_i(t)\big(\xi_{i\nu}(t) - \xi_{\mu\nu}(t)\big)\,dW_\nu(t) \\ &= \sum_{i=1}^n \mu_i(t)\big(\gamma_i(t) - \gamma_\mu(t)\big)\,dt + \sum_{\nu=1}^n \Big[\sum_{i=1}^n \mu_i(t)\xi_{i\nu}(t) - \xi_{\mu\nu}(t)\Big]\,dW_\nu(t),\end{aligned} \tag{2.12.65}$$

by applying (2.12.24) to (2.12.65), we establish the following

$$\begin{aligned}\sum_{i=1}^n \mu_i(t)\,d\log\mu_i(t) &= \sum_{i=1}^n \mu_i(t)\big(\gamma_i(t) - \gamma_\mu(t)\big)\,dt + \sum_{\nu=1}^n \big(\xi_{\mu\nu}(t) - \xi_{\mu\nu}(t)\big)\,dW_\nu(t) \\ &= \sum_{i=1}^n \mu_i(t)\big(\gamma_i(t) - \gamma_\mu(t)\big)\,dt.\end{aligned} \tag{2.12.66}$$

Moreover, by setting $\pi := \mu$ in (2.2.111), we have the analogue given by

$$\gamma_\mu(t) = \sum_{i=1}^n \mu_i(t)\gamma_i(t) + \gamma_\mu^*(t). \tag{2.12.67}$$

By using (2.12.67), we ascertain

$$\begin{aligned}\sum_{i=1}^n \mu_i(t)\,d\log\mu_i(t) &= \sum_{i=1}^n \mu_i(t)\big(\gamma_i(t) - \gamma_\mu(t)\big)\,dt \\ &= \Big(\sum_{i=1}^n \mu_i(t)\gamma_i(t) - \gamma_\mu(t)\Big)\,dt \\ &= -\gamma_\mu^*(t)\,dt.\end{aligned} \tag{2.12.68}$$

Therefore, a.s., for $t \in [0, \infty)$, we have

$$\gamma_\mu^*(t)\,dt + \sum_{i=1}^n \mu_i(t)\,d\log\mu_i(t) = 0. \tag{2.12.69}$$

Consequently, subtracting the left-hand side of the above result from the right-hand side of equation (2.12.63), establishes the desired result, since the above result in no way alters equation (2.12.63),

$$d\log\Big(Z_\pi(t)/Z_\mu(t)\Big) = \sum_{i=1}^n \pi_i(t)\,d\log\mu_i(t) + \gamma_\pi^*(t)\,dt - \Big(\gamma_\mu^*(t)\,dt + \sum_{i=1}^n \mu_i(t)\,d\log\mu_i(t)\Big) \tag{2.12.70}$$

$$= \big(\gamma_\pi^*(t) - \gamma_\mu^*(t)\big)\,dt + \sum_{i=1}^n \big(\pi_i(t) - \mu_i(t)\big)\,d\log\mu_i(t). \tag{2.12.71}$$

An alternative proof is offered in Appendix C. ■

The next results have been fashioned and adapted from Fernholz & Karatzas (2009, Remark 3.1), which we have formulated in the next lemmas. We also expound upon the results by presenting formal proofs.

Lemma 2.12.10 ([**Fernholz & Karatzas (2009)**]). *Let π be a portfolio in the market $\mathcal{M}$. Then, a.s., for $t \in [0, \infty)$, we obtain the dynamics*

$$d \log \Big(Z_\pi(t)/Z_\mu(t) \Big) = \sum_{i=1}^n \frac{\pi_i(t)}{\mu_i(t)} \, d\mu_i(t) - \frac{1}{2} \sum_{i,j=1}^n \pi_i(t) \tau_{ij}(t) \pi_j(t) \, dt \tag{2.12.72}$$

$$= \sum_{i=1}^n \frac{\pi_i(t)}{\mu_i(t)} \, d\mu_i(t) - \frac{1}{2} \, \tau_{\pi\pi}(t) \, dt, \tag{2.12.73}$$

for the relative return of an arbitrary portfolio π with respect to the market.

Proof. To prove the relative logarithmic return expression from equations (2.12.63), (2.12.60) (or (2.12.62)), (2.12.37), and (2.12.59) we carry out the following

$$\begin{aligned} d \log \Big(Z_\pi(t)/Z_\mu(t) \Big) &= \gamma_\pi^*(t) \, dt + \sum_{i=1}^n \pi_i(t) \, d \log \mu_i(t) \\ &= \frac{1}{2} \left(\sum_{i=1}^n \pi_i(t) \tau_{ii}(t) - \sum_{i,j=1}^n \pi_i(t) \tau_{ij}(t) \pi_j(t) \right) dt + \sum_{i=1}^n \pi_i(t) \left[\frac{d\mu_i(t)}{\mu_i(t)} - \frac{1}{2} \, \tau_{ii}(t) \, dt \right] \\ &= \frac{1}{2} \left(\sum_{i=1}^n \pi_i(t) \tau_{ii}(t) - \tau_{\pi\pi}(t) \right) dt + \sum_{i=1}^n \frac{\pi_i(t)}{\mu_i(t)} \, d\mu_i(t) - \frac{1}{2} \sum_{i=1}^n \pi_i(t) \tau_{ii}(t) \, dt \\ &= \frac{1}{2} \sum_{i=1}^n \pi_i(t) \tau_{ii}(t) \, dt - \frac{1}{2} \, \tau_{\pi\pi}(t) \, dt + \sum_{i=1}^n \frac{\pi_i(t)}{\mu_i(t)} \, d\mu_i(t) - \frac{1}{2} \sum_{i=1}^n \pi_i(t) \tau_{ii}(t) \, dt. \end{aligned}$$

The first and last terms cancel, establishing the result. Thus, (2.12.63), in conjunction with (2.12.37), (2.12.59) and the numéraire invariance property (2.12.60) (or (2.12.62)), implies that for any arbitrary portfolio π, we have the above relative return formula. ■

By simply setting $\eta := \mu$ in equations (2.6.79) and (2.6.80), we obtain the relative variance process of π versus the market portfolio, i.e., for $t \in [0, \infty)$, a.s., we have

$$d \left\langle \log \left(Z_\pi / Z_\mu \right) \right\rangle_t = \tau_{\pi\pi}^\mu(t) \, dt = \tau_{\pi\pi}(t) \, dt, \quad \text{or,} \tag{2.12.74}$$

$$\left\langle \log \left(Z_\pi / Z_\mu \right) \right\rangle_t = \int_0^t \tau_{\pi\pi}^\mu(s) \, ds = \int_0^t \tau_{\pi\pi}(s) \, ds. \tag{2.12.75}$$

In addition, setting $\eta := \mu$ in equations (2.6.96) and (2.6.97), gives the following expressions

$$\tau_{\pi\pi}^\mu(t) = \tau_{\pi\pi}(t) = \frac{d}{dt} \left\langle \log \left(Z_\pi / Z_\mu \right) \right\rangle_t = \sum_{\nu=1}^n \Big(\xi_{\pi\nu}(t) - \xi_{\mu\nu}(t) \Big)^2, \quad \text{and,} \tag{2.12.76}$$

$$\tau_{\pi\pi}^\mu(t) \, dt = \tau_{\pi\pi}(t) \, dt = d \left\langle \log \left(Z_\pi / Z_\mu \right) \right\rangle_t = \sum_{\nu=1}^n \Big(\xi_{\pi\nu}(t) - \xi_{\mu\nu}(t) \Big)^2 \, dt. \tag{2.12.77}$$

Lemma 2.12.11. *Let π be a portfolio in the market $\mathcal{M}$. Then, a.s., for $t \in [0, \infty)$, we obtain the dynamics*

$$d \left(\frac{Z_\pi(t)}{Z_\mu(t)} \right) = \frac{Z_\pi(t)}{Z_\mu(t)} \left[\left(\gamma_\pi^*(t) + \frac{1}{2} \, \tau_{\pi\pi}(t) \right) dt + \sum_{i=1}^n \pi_i(t) \, d \log \mu_i(t) \right], \tag{2.12.78}$$

for the relative arithmetic return of an arbitrary portfolio π with respect to the market.

Proof. Let us apply Itô's formula to $\exp \big(\log \big(Z_\pi(t)/Z_\mu(t) \big) \big)$. Thus, we ascertain the following

$$d \left(\frac{Z_\pi(t)}{Z_\mu(t)} \right) = \frac{Z_\pi(t)}{Z_\mu(t)} \, d \log \Big(Z_\pi(t)/Z_\mu(t) \Big) + \frac{1}{2} \, \frac{Z_\pi(t)}{Z_\mu(t)} \, d \left\langle \log \left(Z_\pi / Z_\mu \right) \right\rangle_t. \tag{2.12.79}$$

By appealing to equation (2.12.63) of Proposition 2.12.8, the dynamics of the relative return of portfolio π versus the market portfolio μ, we shall confirm the result of (2.12.74), as follows

$$\begin{aligned}
d\left\langle \log\left(Z_\pi/Z_\mu\right)\right\rangle_t &= d\left\langle \int_0^\cdot \sum_{i=1}^n \pi_{i,s}\, d\log\mu_{i,s}, \int_0^\cdot \sum_{j=1}^n \pi_{j,s}\, d\log\mu_{j,s}\right\rangle_t \\
&= d\left\langle \sum_{i=1}^n \int_0^\cdot \pi_{i,s}\, d\log\mu_{i,s}, \sum_{j=1}^n \int_0^\cdot \pi_{j,s}\, d\log\mu_{j,s}\right\rangle_t \\
&= \sum_{i,j=1}^n d\left\langle \int_0^\cdot \pi_{i,s}\, d\log\mu_{i,s}, \int_0^\cdot \pi_{j,s}\, d\log\mu_{j,s}\right\rangle_t \\
&= \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\, d\left\langle \log\mu_i, \log\mu_j\right\rangle_t \qquad (2.12.80)\\
&= \sum_{i,j=1}^n \pi_i(t)\tau_{ij}(t)\pi_j(t)\, dt \qquad (2.12.81)\\
&= \tau_{\pi\pi}(t)\, dt, \qquad (2.12.82)
\end{aligned}$$

where equation (2.12.81) follows from equation (2.12.25), and equation (2.12.82) follows from equation (2.12.59). Then, substituting the above result or result (2.12.74) into equation (2.12.79), yields

$$d\left(\frac{Z_\pi(t)}{Z_\mu(t)}\right) = \frac{Z_\pi(t)}{Z_\mu(t)}\, d\log\left(Z_\pi(t)/Z_\mu(t)\right) + \frac{1}{2}\,\frac{Z_\pi(t)}{Z_\mu(t)}\,\tau_{\pi\pi}(t)\, dt, \qquad (2.12.83)$$

which, in conjunction with equation (2.12.63) of Proposition 2.12.8, gives the desired result

$$\begin{aligned}
d\left(\frac{Z_\pi(t)}{Z_\mu(t)}\right) &= \frac{Z_\pi(t)}{Z_\mu(t)}\left[\gamma_\pi^*(t)\, dt + \sum_{i=1}^n \pi_i(t)\, d\log\mu_i(t)\right] + \frac{1}{2}\,\frac{Z_\pi(t)}{Z_\mu(t)}\,\tau_{\pi\pi}(t)\, dt \\
&= \frac{Z_\pi(t)}{Z_\mu(t)}\left[\left(\gamma_\pi^*(t) + \frac{1}{2}\,\tau_{\pi\pi}(t)\right) dt + \sum_{i=1}^n \pi_i(t)\, d\log\mu_i(t)\right].
\end{aligned}$$

■

We can rewrite equation (2.12.83) in the following useful forms

$$\frac{d\left(Z_\pi(t)/Z_\mu(t)\right)}{Z_\pi(t)/Z_\mu(t)} = d\log\left(Z_\pi(t)/Z_\mu(t)\right) + \frac{1}{2}\,\tau_{\pi\pi}(t)\, dt, \qquad (2.12.84)$$

and

$$d\log\left(Z_\pi(t)/Z_\mu(t)\right) = \frac{d\left(Z_\pi(t)/Z_\mu(t)\right)}{Z_\pi(t)/Z_\mu(t)} - \frac{1}{2}\,\tau_{\pi\pi}(t)\, dt. \qquad (2.12.85)$$

Lemma 2.12.12. *Let π be a portfolio in the market $\mathcal{M}$. Then, a.s., for $t \in [0,\infty)$, we obtain the dynamics*

$$d\left(\frac{Z_\pi(t)}{Z_\mu(t)}\right) = \frac{Z_\pi(t)}{Z_\mu(t)}\left[\left(\gamma_\pi^*(t) - \gamma_\mu^*(t) + \frac{1}{2}\,\tau_{\pi\pi}(t)\right) dt + \sum_{i=1}^n \left(\pi_i(t) - \mu_i(t)\right) d\log\mu_i(t)\right], \qquad (2.12.86)$$

for the relative arithmetic return of an arbitrary portfolio π with respect to the market.

Proof. Adopting equation (2.12.64) of Lemma 2.12.9 and applying equation (2.12.25), we obtain

$$\begin{aligned}
d\left\langle \log\left(Z_\pi/Z_\mu\right)\right\rangle_t &= d\left\langle \int_0^\cdot \sum_{i=1}^n \left(\pi_{i,s}-\mu_{i,s}\right) d\log\mu_{i,s}, \int_0^\cdot \sum_{j=1}^n \left(\pi_{j,s}-\mu_{j,s}\right) d\log\mu_{j,s}\right\rangle_t \\
&= d\left\langle \sum_{i=1}^n \int_0^\cdot \left(\pi_{i,s}-\mu_{i,s}\right) d\log\mu_{i,s}, \sum_{j=1}^n \int_0^\cdot \left(\pi_{j,s}-\mu_{j,s}\right) d\log\mu_{j,s}\right\rangle_t \\
&= \sum_{i,j=1}^n d\left\langle \int_0^\cdot \left(\pi_{i,s}-\mu_{i,s}\right) d\log\mu_{i,s}, \int_0^\cdot \left(\pi_{j,s}-\mu_{j,s}\right) d\log\mu_{j,s}\right\rangle_t \\
&= \sum_{i,j=1}^n \left(\pi_i(t)-\mu_i(t)\right)\left(\pi_j(t)-\mu_j(t)\right) d\left\langle \log\mu_i, \log\mu_j\right\rangle_t \\
&= \sum_{i,j=1}^n \left(\pi_i(t)-\mu_i(t)\right)\tau_{ij}(t)\left(\pi_j(t)-\mu_j(t)\right) dt \\
&= \sum_{i,j=1}^n \pi_i(t)\tau_{ij}(t)\pi_j(t)\,dt - \sum_{i,j=1}^n \pi_i(t)\tau_{ij}(t)\mu_j(t)\,dt - \sum_{i,j=1}^n \mu_i(t)\tau_{ij}(t)\pi_j(t)\,dt \\
&\quad + \sum_{i,j=1}^n \mu_i(t)\tau_{ij}(t)\mu_j(t)\,dt \\
&= \sum_{i,j=1}^n \pi_i(t)\tau_{ij}(t)\pi_j(t)\,dt - \sum_{i=1}^n \pi_i(t)\left[\sum_{j=1}^n \mu_j(t)\tau_{ij}(t)\right] dt - \sum_{j=1}^n \pi_j(t)\left[\sum_{i=1}^n \mu_i(t)\tau_{ij}(t)\right] dt \\
&\quad + \sum_{i,j=1}^n \mu_i(t)\tau_{ij}(t)\mu_j(t)\,dt,
\end{aligned}$$

which, in conjunction with equation (2.12.54) of Lemma 2.12.4, equation (2.12.56) and recalling the symmetry property of the relative covariance matrix, i.e., $\tau_{ij}(t) = \tau_{ji}(t)$ for all $t \in [0,\infty)$, becomes

$$d\left\langle \log\left(Z_\pi/Z_\mu\right)\right\rangle_t = \sum_{i,j=1}^n \pi_i(t)\tau_{ij}(t)\pi_j(t)\,dt = \tau_{\pi\pi}(t)\,dt,$$

which follows from (2.12.59) and therefore agrees with the result (2.12.74). Thus, by inserting equation (2.12.64) of Lemma 2.12.9 into equation (2.12.83), the following is obtained

$$\begin{aligned}
d\left(\frac{Z_\pi(t)}{Z_\mu(t)}\right) &= \frac{Z_\pi(t)}{Z_\mu(t)}\left[\left(\gamma_\pi^*(t)-\gamma_\mu^*(t)\right) dt + \sum_{i=1}^n \left(\pi_i(t)-\mu_i(t)\right) d\log\mu_i(t)\right] + \frac{1}{2}\,\frac{Z_\pi(t)}{Z_\mu(t)}\,\tau_{\pi\pi}(t)\,dt \\
&= \frac{Z_\pi(t)}{Z_\mu(t)}\left[\left(\gamma_\pi^*(t)-\gamma_\mu^*(t)+\frac{1}{2}\,\tau_{\pi\pi}(t)\right) dt + \sum_{i=1}^n \left(\pi_i(t)-\mu_i(t)\right) d\log\mu_i(t)\right].
\end{aligned}$$

■

Lemma 2.12.13 ([**Fernholz & Karatzas (2009)**]). *Let π be a portfolio in the market $\mathcal{M}$. Then, a.s., for $t \in [0,\infty)$, we obtain the dynamics*

$$d\left(\frac{Z_\pi(t)}{Z_\mu(t)}\right) = \frac{Z_\pi(t)}{Z_\mu(t)} \sum_{i=1}^n \frac{\pi_i(t)}{\mu_i(t)}\,d\mu_i(t), \tag{2.12.87}$$

for the relative arithmetic return of an arbitrary portfolio π with respect to the market.

Proof. By substituting equation (2.12.73) of Lemma 2.12.10 into equation (2.12.83), we confirm the required

result

$$\begin{aligned} d\left(\frac{Z_\pi(t)}{Z_\mu(t)}\right) &= \frac{Z_\pi(t)}{Z_\mu(t)}\left[\sum_{i=1}^n \frac{\pi_i(t)}{\mu_i(t)}\,d\mu_i(t) - \frac{1}{2}\,\tau_{\pi\pi}(t)\,dt\right] + \frac{1}{2}\,\frac{Z_\pi(t)}{Z_\mu(t)}\,\tau_{\pi\pi}(t)\,dt \\ &= \frac{Z_\pi(t)}{Z_\mu(t)}\sum_{i=1}^n \frac{\pi_i(t)}{\mu_i(t)}\,d\mu_i(t). \end{aligned}$$

■

From (2.7.17) or by considering (2.12.83) along with (2.12.74), it can be easily verified that the quadratic variation of the relative arithmetic return process satisfies the following for all $t \in [0,\infty)$, a.s.,

$$d\left\langle \frac{Z_\pi}{Z_\mu}\right\rangle_t = \left(\frac{Z_\pi(t)}{Z_\mu(t)}\right)^2 d\left\langle \log\left(Z_\pi/Z_\mu\right)\right\rangle_t \tag{2.12.88}$$

$$= \tau^\mu_{\pi\pi}(t)\left(\frac{Z_\pi(t)}{Z_\mu(t)}\right)^2 dt \tag{2.12.89}$$

$$= \tau_{\pi\pi}(t)\left(\frac{Z_\pi(t)}{Z_\mu(t)}\right)^2 dt. \tag{2.12.90}$$

Alternatively, from equation (2.12.87) of Lemma 2.12.13, we equivalently obtain

$$\begin{aligned} d\left\langle \frac{Z_\pi}{Z_\mu}\right\rangle_t &= d\left\langle \int_0^\cdot \frac{Z_{\pi,s}}{Z_{\mu,s}}\sum_{i=1}^n \frac{\pi_{i,s}}{\mu_{i,s}}\,d\mu_{i,s}, \int_0^\cdot \frac{Z_{\pi,s}}{Z_{\mu,s}}\sum_{j=1}^n \frac{\pi_{j,s}}{\mu_{j,s}}\,d\mu_{j,s}\right\rangle_t \\ &= d\left\langle \sum_{i=1}^n \int_0^\cdot \frac{Z_{\pi,s}}{Z_{\mu,s}}\frac{\pi_{i,s}}{\mu_{i,s}}\,d\mu_{i,s}, \sum_{j=1}^n \int_0^\cdot \frac{Z_{\pi,s}}{Z_{\mu,s}}\frac{\pi_{j,s}}{\mu_{j,s}}\,d\mu_{j,s}\right\rangle_t \\ &= \sum_{i,j=1}^n d\left\langle \int_0^\cdot \frac{Z_{\pi,s}}{Z_{\mu,s}}\frac{\pi_{i,s}}{\mu_{i,s}}\,d\mu_{i,s}, \int_0^\cdot \frac{Z_{\pi,s}}{Z_{\mu,s}}\frac{\pi_{j,s}}{\mu_{j,s}}\,d\mu_{j,s}\right\rangle_t \\ &= \sum_{i,j=1}^n \left(\frac{Z_\pi(t)}{Z_\mu(t)}\right)^2 \frac{\pi_i(t)\pi_j(t)}{\mu_i(t)\mu_j(t)}\,d\left\langle \mu_i,\mu_j\right\rangle_t \qquad (2.12.91) \\ &= \sum_{i,j=1}^n \left(\frac{Z_\pi(t)}{Z_\mu(t)}\right)^2 \pi_i(t)\pi_j(t)\,d\left\langle \log\mu_i, \log\mu_j\right\rangle_t \qquad (2.12.92) \\ &= \left(\frac{Z_\pi(t)}{Z_\mu(t)}\right)^2 \sum_{i,j=1}^n \pi_i(t)\tau_{ij}(t)\pi_j(t)\,dt \qquad (2.12.93) \\ &= \tau_{\pi\pi}(t)\left(\frac{Z_\pi(t)}{Z_\mu(t)}\right)^2 dt, \qquad (2.12.94) \end{aligned}$$

where equation (2.12.92) follows from equation (2.12.47), equation (2.12.93) follows from equation (2.12.25), and finally, equation (2.12.94) follows from equation (2.12.59).

In conclusion, all of the market-related results derived in Sections 2.3–2.11 and not shown here, can also be applied and extended to the market portfolio by simply setting $\eta := \mu$ in the expressions therein.

2.12.1 The Market's Intrinsic Volatility

Recall, (2.4.32), for the market portfolio μ, this expression becomes

$$\gamma^*_\mu(t) = \frac{1}{2}\sum_{i=1}^n \mu_i(t)\tau^\mu_{ii}(t) = \frac{1}{2}\sum_{i=1}^n \mu_i(t)\tau_{ii}(t), \tag{2.12.95}$$

the summation on the right-hand side is the average, according to the market weights of individual stocks, of these stocks' variances relative to the market. Thus, the excess growth rate of the market portfolio μ measures,

at any time $t \in [0, \infty)$, the amount of available volatility in the market, namely, the relative variation of the stocks in the market. Consequently, (2.12.95) gives an interpretation of the excess growth rate of the market portfolio, as a *measure of the market's available "intrinsic" volatility*, i.e., the intrinsic volatility available in the market at any given time.

2.13 Alternative Total Return

Definition 2.13.1 (Alternative Total Return of a Portfolio). *Consider a market $\mathcal{M}$ comprising n stocks $X_1, X_2, \ldots, X_n$. Let $\delta_1, \delta_2, \ldots, \delta_n$ be the respective dividend rate processes and let $\widehat{X}_1, \widehat{X}_2, \ldots, \widehat{X}_n$ be the total return processes of the stocks in the market (see Definition 2.2.34, equation (2.2.149)). Then the process $\widetilde{Z} = \left\{\widetilde{Z}(t), t \in [0, \infty)\right\}$ defined by*

$$\widetilde{Z}(t) \triangleq X_1(t) \exp\left(\int_0^t \delta_1(s)\, ds\right) + \cdots + X_n(t) \exp\left(\int_0^t \delta_n(s)\, ds\right), \qquad t \in [0, \infty), \tag{2.13.1}$$

$$= \widehat{X}_1(t) + \cdots + \widehat{X}_n(t), \qquad t \in [0, \infty), \tag{2.13.2}$$

signifies the return process of a portfolio in which dividends of each stock are reinvested in the exact same stock.

Therefore, the return process, $\widetilde{Z}$, represents the value of a portfolio with the same weights as the portfolio and in which the dividends of each stock are reinvested in the same stock. Thus, with this definition, we have the following initial condition

$$\widetilde{Z}(0) = \widehat{X}_1(0) + \cdots + \widehat{X}_n(0) \tag{2.13.3}$$

$$= X_1(0) + \cdots + X_n(0) \tag{2.13.4}$$

$$= Z_{w,\mu}(0) = w. \tag{2.13.5}$$

For a market without dividends, i.e., $\delta_i = 0$ for all $i = 1, \ldots, n$, it is clear that $\widetilde{Z}(t) = Z_{w,\mu}(t)$ for all $t \in [0, \infty)$ and with initial capital $\widetilde{Z}(0) = Z_{w,\mu}(0) = w$.

2.14 Portfolio Optimisation

At this stage, we can already formulate some fairly interesting optimisation techniques. In this section we shall formulate and investigate portfolio optimisation techniques using the stochastic portfolio theory developed. Portfolio optimisation lies at the core of modern equity management, and is thus an essential facet to this dissertation. Here we shall contrast two approaches: the classical portfolio optimisation methodology pioneered by Markowitz (1952) to the portfolio optimisation methodology exploiting the use of stochastic portfolio theory. The majority of this section is attributed to the examples formulated by Fernholz (2002). Moreover, the reader may consult Fernholz & Karatzas (2009) for a more detailed treatment of this subject, in which these and other additional techniques are presented as optimisation problems. Refer to Fernholz & Karatzas (2009) for those optimisation problems that are not mentioned here in this section. We commence with the approach forged by Markowitz (1952).

2.14.1 Classical Portfolio Optimisation

2.14.1.1 Portfolio Variance Minimisation: Quadratic Criterion, Linear Constraint

Classical Markowitz [Markowitz (1952)] portfolio optimisation is accomplished by minimising the objective function given by the variance of the portfolio π, (2.2.94)

$$\sigma_{\pi\pi}(t) = \sum_{i,j=1}^{n} \pi_i(t)\sigma_{ij}(t)\pi_j(t), \tag{2.14.1}$$

among all portfolios π, subject to the linear constraints

$$\alpha_\pi(t) = \sum_{i=1}^{n} \pi_i(t)\alpha_i(t) \geq \alpha_0, \tag{2.14.2}$$

$$\sum_{i=1}^{n} \pi_i(t) = 1, \qquad \text{with} \qquad \pi_1(t), \ldots, \pi_n(t) \geq 0, \tag{2.14.3}$$

that is, with a rate of return greater than or equal to a given contant $\alpha_0 \in \mathbb{R}$ and over all vectors $\boldsymbol{\pi}(t) \in \overline{\Delta}^{n-1}$ (i.e., we restrict ourselves to long-only portfolios). Thus, the classical Markowitz approach appoints the use of the *portfolio rate of return* within its set of constraints, the portfolio rate of return was initially formulated in (2.2.120). Hence, the optimisation finds the portfolio with no short sales that has the minimum variance for a given portfolio rate of return $\alpha_0 \in \mathbb{R}$, a portfolio determined in this manner is typically termed an efficient portfolio. Such an optimisation technique can be easily performed using conventional *quadratic programming* techniques [see Wolfe (1959)].

2.14.1.2 Portfolio Relative Variance Minimisation: Quadratic Criterion, Linear Constraint

Alternatively, if equipped with the knowledge of the relative variances of the individual component stocks versus some benchmark portfolio η, we can redesign this optimisation problem by considering the slightly modified variation of the previously stated problem. The approach taken here is to minimise the relative variance of the portfolio π versus η, (2.4.17),

$$\tau^{\eta}_{\pi\pi}(t) = \sum_{i,j=1}^{n} \pi_i(t)\tau^{\eta}_{ij}(t)\pi_j(t), \tag{2.14.4}$$

subject to the linear constraints

$$\alpha_\pi(t) = \sum_{i=1}^{n} \pi_i(t)\alpha_i(t) \geq \alpha_0, \tag{2.14.5}$$

$$\sum_{i=1}^{n} \pi_i(t) = 1, \qquad \text{with} \qquad \pi_1(t), \ldots, \pi_n(t) \geq 0, \tag{2.14.6}$$

that is, over all vectors $\boldsymbol{\pi}(t) \in \overline{\Delta}^{n-1}$ (i.e., we restrict ourselves to long-only portfolios).

2.14.2 Stochastic Portfolio Optimisation

2.14.2.1 Portfolio Variance Minimisation: Quadratic Criterion, Quadratic Constraint

However, as will be revealed later, the growth rate of a portfolio π is a more adequate take on the analysis of long-term portfolio behaviour. Thus, a minimisation of the portfolio variance under a constraint on the *portfolio growth rate* as opposed to that on the portfolio rate of return should be considered. In this particular instance, (2.14.2) is replaced by (2.14.7), consequently the optimisation problem can be reformulated as follows: minimise the portfolio variance

$$\sigma_{\pi\pi}(t) = \sum_{i,j=1}^{n} \pi_i(t)\sigma_{ij}(t)\pi_j(t),$$

among all portfolios π, subject to the following constraints

$$\gamma_\pi(t) = \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^{n} \pi_i(t)\sigma_{ij}(t)\pi_j(t)\right) \geq \gamma_0, \tag{2.14.7}$$

$$\sum_{i=1}^{n} \pi_i(t) = 1, \qquad \text{with} \qquad \pi_1(t), \ldots, \pi_n(t) \geq 0, \tag{2.14.8}$$

that is, with growth rate at least equal to a given constant γ_0 and over all vectors $\boldsymbol{\pi}(t) \in \overline{\Delta}^{n-1}$ (i.e., we restrict ourselves to long-only portfolios). Thus, in a similar fashion to the previous optimisation scheme (apart from the fact that we appoint the use of the portfolio growth rate rather than the portfolio rate of return), we are interested in finding the portfolio with no short sales that has the minimum variance for a given fixed portfolio growth rate, γ_0. The constraint given by (2.14.7), is nonlinear, and the conventional quadratic programmes are rendered useless. However, by implementing a slight alteration to (2.14.7), we obtain the equivalent expression

$$\sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \frac{1}{2}\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) \quad \geq \quad \gamma_0 + \frac{1}{2}\sum_{i,j=1}^{n} \pi_i(t)\sigma_{ij}(t)\pi_j(t), \tag{2.14.9}$$

or, also equivalently

$$\sum_{i=1}^{n} \pi_i(t)\left(\gamma_i(t) + \frac{1}{2}\,\sigma_{ii}(t)\right) \quad \geq \quad \gamma_0 + \frac{1}{2}\sum_{i,j=1}^{n} \pi_i(t)\sigma_{ij}(t)\pi_j(t), \tag{2.14.10}$$

and we shall see later that in certain cases the quadratic term in (2.14.9) can be controlled, and this permits the use of the quadratic programming technique.

2.14.2.2 Portfolio Relative Variance Minimisation: Quadratic Criterion, Quadratic Constraint

Certain stock portfolios, variously, called "risk-controlled" portfolios or "enhanced index" portfolios are constructed to maintain a low relative variance with a particular index or benchmark portfolio [Fernholz (2002)]. The "tracking error", $\mathsf{TE}_\pi(t)$, of such a portfolio is the square root of the relative variance, and is typically held to be about 2% a year. Hence, optimisation in this case should minimise the tracking error, or equivalently, the relative variance of the portfolio versus the benchmark portfolio, subject to certain constraints. Suppose that η is a benchmark portfolio, and we employ (2.4.26) to represent the excess growth rate of the portfolio π, $\gamma^*_\pi(t)$, we can then optimise the portfolio by minimising the variance of the portfolio relative to the benchmark portfolio,

$$\tau^\eta_{\pi\pi}(t) \quad = \quad \sum_{i,j=1}^{n} \pi_i(t)\tau^\eta_{ij}(t)\pi_j(t) \quad = \quad \big(\mathsf{TE}_\pi(t)\big)^2, \tag{2.14.11}$$

subject to the following constraints

$$\sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t)\tau^\eta_{ii}(t) - \sum_{i,j=1}^{n} \pi_i(t)\tau^\eta_{ij}(t)\pi_j(t)\right) \quad \geq \quad \gamma_0,$$

or, equivalently,

$$\sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \frac{1}{2}\sum_{i=1}^{n} \pi_i(t)\tau^\eta_{ii}(t) \quad \geq \quad \gamma_0 + \frac{1}{2}\sum_{i,j=1}^{n} \pi_i(t)\tau^\eta_{ij}(t)\pi_j(t), \tag{2.14.12}$$

$$= \quad \gamma_0 + \frac{1}{2}\big(\mathsf{TE}_\pi(t)\big)^2, \tag{2.14.13}$$

or, again, equivalently,

$$\sum_{i=1}^{n} \pi_i(t)\left(\gamma_i(t) + \frac{1}{2}\,\tau^\eta_{ii}(t)\right) \quad \geq \quad \gamma_0 + \frac{1}{2}\sum_{i,j=1}^{n} \pi_i(t)\tau^\eta_{ij}(t)\pi_j(t), \tag{2.14.14}$$

$$= \quad \gamma_0 + \frac{1}{2}\big(\mathsf{TE}_\pi(t)\big)^2, \tag{2.14.15}$$

and,

$$\sum_{i=1}^{n} \pi_i(t) \quad = \quad 1, \qquad \text{with} \qquad \pi_1(t), \ldots, \pi_n(t) \quad \geq \quad 0, \tag{2.14.16}$$

that is, over all vectors $\boldsymbol{\pi}(t) \in \overline{\Delta}^{n-1}$ (i.e., we restrict ourselves to long-only portfolios). The constraint in (2.14.12) is obtained by following the same reasoning to that used when optimising the portfolio variance. It is clear that this constraint is quadratic, which is nonlinear in nature, as was the case in the previous optimisation problem. Recall that in such an event the usual quadratic programming techniques are no longer useful. However, by applying a simple adjustment, the optimisation problem can be transformed into one that consists only of linear constraints, which admits the use of the conventional techniques.

Lemma 2.4.5, the "numéraire invariance" property, implies that (2.14.12) is equivalent to (2.14.9), thus these two optimisation procedures are essentially one and the same. The quadratic term in (2.14.12) (i.e., the last term) is half the square of the tracking error (as represented in (2.14.13)), so this term will be about $\frac{1}{2}\left(\mathsf{TE}_\pi(t)\right)^2 = 0.02\%$ a year if the tracking error is held to about 2%. In practice, the inaccuracy induced by ignoring the 0.02% a year in the estimated portfolio growth rate is likely to be inconsequential [Fernholz (2002)]. Hence, we can remove the quadratic component in (2.14.12) without any loss of accuracy, we can thus replace the constraint given in (2.14.12) with the following modified constraint

$$\sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \frac{1}{2}\sum_{i=1}^{n} \pi_i(t)\tau_{ii}^{\eta}(t) \quad \geq \quad \gamma_0, \tag{2.14.17}$$

or, equivalently

$$\sum_{i=1}^{n} \pi_i(t)\left(\gamma_i(t) + \frac{1}{2}\,\tau_{ii}^{\eta}(t)\right) \quad \geq \quad \gamma_0. \tag{2.14.18}$$

Since this is linear in π, conventional quadratic programming can be used for optimisation with this constraint.

Although the constraint (2.14.17) is compatible with conventional techniques, it suffers the same drawbacks as the original Markowitz model since it is necessary to accurately forecast the growth rates, $\gamma_i(t)$ for all $i = 1, \ldots, n$ [Fernholz (2003b)]. In practice, it is quite difficult, if not impossible, to accurately forecast these growth rates. However, if we consider a large-stock index, it may not be entirely unreasonable to assume that over the long term most stocks will have about the same average growth rate, at least for the stocks in this large-stock index η. If this, however, is not the case, and one of the stocks had a significantly higher growth rate compared to that of its large-stock counterparts, then it would dominate the market. This notion of the stock dominating the market is directly related to the concept of market diversity, which will be disclosed in the next chapter. Market diversity lends itself to many interesting applications in the area of stochastic portfolio theory, consequently a market not satisfying the diversity market condition will go by the wayside for the moment. Moreover, if a stock had a significantly lower growth rate, then it would not be a large stock. Let us assume that all the stocks share a common growth rate[18], $\gamma_i(t) \equiv \gamma(t)$, for all $i = 1, \ldots, n$, then Lemma 2.4.7 implies that the growth rate of the benchmark portfolio η is

$$\begin{aligned} \gamma_\eta(t) &= \sum_{i=1}^{n} \eta_i(t)\gamma(t) + \gamma_\eta^*(t) \\ &= \gamma(t) + \frac{1}{2}\sum_{i=1}^{n} \eta_i(t)\tau_{ii}^{\eta}(t), \qquad t \in [0, \infty), \qquad \text{a.s.} \end{aligned} \tag{2.14.19}$$

By applying the same reasoning as that in the previous optimisation problem, the "tracking error" of π relative to η, $\mathsf{TE}_\pi(t)$, can again be assumed to be small enough. Thus the inaccuracy induced by this small amount is, in a similar vein, inconsequential. So the effect of removing such a term is negligible. Consequently, we can surmise that

$$\frac{1}{2}\sum_{i,j=1}^{n} \pi_i(t)\tau_{ij}^{\eta}(t)\pi_j(t) \quad = \quad \frac{1}{2}\left(\mathsf{TE}_\pi(t)\right)^2 \quad \approx \quad 0,$$

and thus,

$$\tau_{\pi\pi}^{\eta}(t) \quad \approx \quad 0.$$

[18] Then, since $\gamma_i(t) \equiv \gamma(t)$ for all $i = 1, \ldots, n$, we have $\sum_{i=1}^{n} \pi_i(t)\gamma_i(t) = \sum_{i=1}^{n} \pi_i(t)\gamma(t) = \gamma(t)$.

This suggests that the expression, given by (2.14.17), will be of particular focus. The common growth rate assumption of all the stocks in the market, results in the following expression

$$\gamma(t) + \frac{1}{2}\sum_{i=1}^{n} \pi_i(t)\tau_{ii}^{\eta}(t) \quad \geq \quad \gamma_0.$$

Now, suppose that $\gamma_0 > 0$ is some given fixed constant, and we wish to optimise π such that

$$\gamma_\pi(t) \quad \geq \quad \gamma_\eta(t) + \gamma_0. \tag{2.14.20}$$

So, the optimisation is performed subject to the constraint given by (2.14.20), which states that the growth rate of the portfolio π must be greater than the growth rate of the benchmark portfolio η, by an amount γ_0. Since, for both portfolios, π and η, the weighted average of the growth rates of the individual component stocks is precisely the common growth rate of all the stocks in the portfolio $\gamma(t)$, it follows that

$$\gamma_\pi^*(t) \quad \geq \quad \gamma_\eta^*(t) + \gamma_0 \tag{2.14.21}$$

$$\frac{1}{2}\sum_{i=1}^{n} \pi_i(t)\tau_{ii}^{\eta}(t) \quad \geq \quad \gamma_0 + \frac{1}{2}\sum_{i=1}^{n} \eta_i(t)\tau_{ii}^{\eta}(t). \tag{2.14.22}$$

The left-hand side of is a consequence of the negligible tracking error, where the excess growth rate can be re-expressed by eliminating the $\tau_{\pi\pi}^{\eta}(t)$ component, whereas the right-hand side is a consequence of Lemma 2.4.7. Thus in the case of a market in which all the stocks evolve by a common growth rate, the optimisation problem can be formulated as follows: minimise the relative variance

$$\tau_{\pi\pi}^{\eta}(t) \quad = \quad \sum_{i,j=1}^{n} \pi_i(t)\tau_{ij}^{\eta}(t)\pi_j(t), \tag{2.14.23}$$

subject to the following constraints

$$\frac{1}{2}\sum_{i=1}^{n} \pi_i(t)\tau_{ii}^{\eta}(t) \quad \geq \quad \gamma_0 + \frac{1}{2}\sum_{i=1}^{n} \eta_i(t)\tau_{ii}^{\eta}(t), \tag{2.14.24}$$

$$\sum_{i=1}^{n} \pi_i(t) \quad = \quad 1, \qquad \text{with} \qquad \pi_1(t), \ldots, \pi_n(t) \quad \geq \quad 0, \tag{2.14.25}$$

that is, over all vectors $\boldsymbol{\pi}(t) \in \overline{\Delta}^{n-1}$ (i.e., we restrict ourselves to long-only portfolios). This minimisation procedure produces a portfolio with growth rate about γ_0 greater than the benchmark portfolio's growth rate. The constraint in (2.14.24) is linear, thus admitting the implementation of conventional quadratic programming techniques. This optimisation involves only the covariance process $\boldsymbol{\tau^{\eta}}$, and thus no predictions regarding any future growth rates or indeed rates of returns is necessary.

2.14.2.3 Portfolio Growth Maximisation

Thus far, we have considered a portfolio optimisation scheme that involves the minimisation of the portfolio variance, however, it will often be of interest to perform an optimisation that encompasses the maximisation of the portfolio growth rate. Suppose that we wish to find the portfolio π that maximises the expected value of $\log Z_\pi(t)$ for $t \in [0, \infty)$. Since for $t \in [0, \infty)$,

$$d \log Z_\pi(t) \quad = \quad \gamma_\pi(t)\, dt + \sum_{i,\nu=1}^{n} \pi_i(t)\xi_{i\nu}(t)\, dW_\nu(t). \tag{2.14.26}$$

The last term in the expression (2.14.26), is the martingale component of $d \log Z_\pi$, and as such the expected value of this martingale component is zero. Thus, maximising the expected value of $\log Z_\pi$ amounts to maximising the portfolio growth rate, γ_π. Thus, we shall maximise the drift in the equation for $\log Z_\pi$. Consequently, the

optimisation problem can be formulated as follows: maximise the portfolio growth rate

$$\gamma_\pi(t) = \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \frac{1}{2}\sum_{i=1}^{n} \pi_i(t)\sigma_{ii}(t) - \frac{1}{2}\sum_{i,j=1}^{n} \pi_i(t)\sigma_{ij}(t)\pi_j(t) \tag{2.14.27}$$

$$= \sum_{i=1}^{n} \pi_i(t)\left(\gamma_i(t) + \frac{1}{2}\,\sigma_{ii}(t)\right) - \frac{1}{2}\sum_{i,j=1}^{n} \pi_i(t)\sigma_{ij}(t)\pi_j(t), \tag{2.14.28}$$

among all portfolios π, subject to the following constraint

$$\sum_{i=1}^{n} \pi_i(t) = 1, \qquad \text{with} \qquad \pi_1(t), \ldots, \pi_n(t) \geq 0, \tag{2.14.29}$$

that is, over all vectors $\boldsymbol{\pi}(t) \in \overline{\Delta}^{n-1}$ (i.e., we restrict ourselves to long-only portfolios). Fernholz (2002) argues that such portfolios carry too high a level of risk for most investors. This is largely due to the fact that maximising the expected value of $\log Z_\pi$ produces the portfolio with the greatest asymptotic value.

Thus far, in our presentation of the theory, we have concentrated on the fundamental components of stochastic portfolio theory, of which we have provided an extensive analysis. In this chapter, there have been several hints at the notion of long-term portfolio performance and diversity. In what follows we examine these concepts in greater detail and discuss their relevance within the context of this theory. In particular, the central notion of stochastic portfolio theory will be featured, namely that it is the growth rate of a portfolio that determines its long-term behaviour. Furthermore, an extraordinary aspect of this framework is divulged, specifically that diversity enforces an implicit structural stability on the equity market.

2.15 Summary and Conclusion

This chapter was ultimately devoted to presenting the foundations upon which the entire stochastic portfolio theory is built. Fundamental concepts from stocks and portfolios, through to the notion of the excess growth rate and relative return were encapsulated in this chapter. This theory is rich and prevalent in the available literature, the majority of which is found in the Fernholz literature and its associated collaborations. At the heart of stochastic portfolio theory lies the epitomical financial equity market model which is based on the geometric rate of return (alternatively, the logarithmic rate of return) of stocks and portfolios. This is unlike the arithmetic rate of return which is employed by the usual traditional equity market models. We also introduced the covariance process (and explored some of its key characteristics), the variance process, the portfolio variance process and the fundamental market conditions. The relative covariance process and the relative variance process were also given mention here. We established several results for the relative return process which included descriptions of the dynamics of the relative return process. More precisely, the relative return process of a stock versus an arbitrary benchmark portfolio can be expressed as a weighted average of the relative return processes of the individual stocks and an additional component, the excess growth rate. The quadratic variance and the covariance of these relative return processes was also derived. We developed certain fundamental and useful properties of the relative covariance process and of the excess growth rate process that are essential to our analysis, moreover crucial upper and lower bounds on both the excess growth rate as well as the relative covariance and relative variance are imposed. We then discussed several different processes that were of interest to us, these include: total return, total return of portfolios, quotient process, relative return of portfolios, quotient process of portfolios, relative total return, total quotient process, relative total return of portfolios and the total quotient process of portfolios. The market portfolio is also introduced and results concerning it are divulged. A brief mention of portfolio optimisation techniques is given some consideration within the scope of stochastic portfolio theory. In particular, the classical portfolio optimisation approach inspired by Markowitz (1952) is compared to the portfolio optimisation approach adopting stochastic portfolio theory, i.e., the stochastic portfolio optimisation approach.

Chapter 3

Stock Market and Portfolio Behaviour

3.1 Introduction

Historically, portfolio theory, as instigated by Markowitz (1952), has placed specific emphasis on the expected rate of return and on the variance of a portfolio of stocks. The primary focus of this chapter is to illustrate the significance of the growth rate in influencing long-term behaviour. We are concerned with long-term investing since it is of obvious interest to long-term investors, so, we are concerned with the notion of long-term behaviour, which consequently directs the attention to the growth rate. In this chapter, we shall demonstrate that the growth rate rather than the rate of return governs the long-term behaviour of a portfolio of stocks. Thus, for long-term investments, it would seem reasonable to consider growth rates rather than the rates of return. The theory in this chapter has been largely developed by Fernholz (2002) and Fernholz & Karatzas (2009), who, to our knowledge, pioneered the notion of growth rates as the determinants of long-term stock and portfolio behaviour.

To analyse long-term behaviour of stocks, portfolios, or the market itself, it is appropriate that we consider the time-average values rather than the expected values of the processes under consideration. The reason for this, is that in practice, we are able to observe the time-average value, whereas the expected value is merely a theoretical construct. Hence, for the growth rate γ_i of the ith stock, X_i, we shall consider

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \gamma_i(t)\, dt,$$

as opposed to $\mathbb{E}\big[\gamma_i(t)\big]$. Likewise, for a market weight μ_i, we shall study

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \log \mu_i(t)\, dt,$$

as opposed to $\mathbb{E}\big[\log \mu_i(t)\big]$.

In Section 3.2, the affirmation that the long-term behaviour of stocks and portfolios is driven by the corresponding growth rates is confirmed. We confirm this affirmation by demonstrating that it is indeed the growth rate and not the rate of return that determines the long-term behaviour and performance of stocks and of a portfolio of stocks in an equity market. Thus, for the sake of the investor who is contemplating a long time horizon in his potential investment plan, this result provides us with the necessary nudge in the growth rate direction that we need to rather inspect the growth rates of stocks and portfolios instead of the rates of return of stocks and portfolios when implementing long-term investment policies and strategies. This is in stark contrast to the traditional theories of portfolio behaviour, in which the rates of return played the primary role. As a result, the rate of return is of no consequence to us within the logarithmic setting, and all of our focus will be shifted to the growth rate. The gravity of such a result must be duly emphasised and it must be made absolutely crystal clear, that this result is of the utmost magnitude and crucial significance within the entire stochastic portfolio theory

universe, as it sets the tone and scene for the rest of the theory that hinges upon and is impacted by this one specific result. In particular, a discussion of the long-term behaviour of stocks and portfolios is addressed here. In this section, the connection between the justification of the portfolio growth rate as the determinant of long-term portfolio behaviour and the uniform boundedness condition, in which all the eigenvalues of the covariance matrix process are uniformly bounded away from infinity, also becomes apparent. The long-term relative behaviour of stocks and portfolios is given in Section 3.3, in which we shall simply consider the long-term behaviour of the relative return of portfolios. It shall be shown that the long-term behaviour of the relative return process of a portfolio versus another portfolio is determined by the difference in the growth rates of the two portfolios. Furthermore, since a stock can be considered to be a portfolio holding only a single stock, this result can also be applied to stocks. Section 3.4 concerns the long-term relative behaviour of the stocks in the market, and as such a minimal structural stability market condition shall be introduced and imposed, namely the coherence of a market. We shall provide a formal definition of the concept of coherence in the equity market, which states that the equity market model encountered in the previous chapter is coherent if the relative capitalisations, i.e., if the market weights, satisfy a particular requirement put forth in this definition. The requirement is essentially that none of the stocks in the equity market can decline too rapidly with respect to the market as a whole, over some future time horizon. We shall refer to this as the market coherence condition that is placed on an equity market. Furthermore, this market coherence condition conjectures that the long-term relative log-return of the stocks versus the market portfolio approaches zero. We shall also present conditions that are equivalent to the condition of coherence of an equity market, referred to as the equivalence coherence conditions. In particular, these equivalence conditions that are comparable to the coherence of the equity market, are established under a requisite condition on the covariance structure of the equity market. These equivalence conditions reveal that in a coherent equity market, the time-average difference between the market portfolio and a stock in the market portfolio will be zero. Roughly speaking, this insinuates that the market's long-term realised growth rate does not exceed that of its component stocks. Moreover, the market coherence condition, is justified when the time-average difference between the growth rates of any two stocks is zero. Note that this pertains only to the differences, the time average of the growth rate of an individual stock may not necessarily exist. The effect that the long-term behaviour of the individual stocks has on the entire equity market shall also be illustrated in this section, by studying certain simple portfolios. This will also enable us to derive the long-term behaviour of these simple portfolios. In addition, the simple portfolios that we shall consider are contained within markets with constant parameters, examples included here are the market with equal growth rates, the market with constant growth rates, the market with both equal and constant growth rates and the market with constant weights. The market with equal growth rates is referred to as the equal-growth-rate market, and comprises a portfolio that exhibits the same growth rate process for all of the stocks in the equity market. The market with constant growth rates is referred to as the constant-growth-rate market, and comprises a portfolio that exhibits a constant growth rate process for all of the stocks in the equity market. The market with both equal and constant growth rates is referred to as the equal-constant-growth-rate market, and comprises a portfolio that exhibits the same constant growth rate process for all of the stocks in the equity market. The market with constant weights is referred to as the constant-weighted market, and is comprised of a portfolio in which all the stocks in the market have a constant weighting. This portfolio is called the constant-weighted portfolio. We shall prove that these simplified portfolios, i.e. the equal-growth-rate market and the equal-constant-growth-rate market, possess the market coherence quality. In particular, if all the stocks in the equity market have the same growth rate process, then it shall be shown that the equity market is coherent. The converse to this will also be shown, i.e., for the case in which the growth rates of all the stocks are constant. In particular, if all the stocks in the market have constant growth rate processes, then it will be shown that the equity market is coherent if and only if the growth rates of all the stocks are all equal. Thus, if all the stocks in the equity market have constant equal growth rate processes, then the equity market is coherent. However, for the case in which all the stocks in the equity market have constant yet different growth rate processes, the market will not be coherent. Thus, a portfolio that exhibits constant growth rates for all its stocks is not necessarily coherent, information regarding the equality of the growth rates is required. We shall also establish that in a market in which all the stocks have equal growth rates, over the long term, the average excess growth rate of this market is asymptotically negligible. This implies that, asymptotically, the growth rate of the market will be the same as the common equal growth rate of the stocks. Thus, over the long term, the contribution of the average excess growth rate of

this market to the growth rate of the market must be minimal, as a result the market excess growth rate term vanishes from all formulations relating to long-term relative portfolio behaviour. We shall also examine and gain some insight into some key features of these markets under the condition of coherence, a case in point is the long-term relative behaviour of the constant-weighted portfolio. In this regard, we shall consider the impact that the condition of coherence in the market has on the performance of the constant-weighted portfolio relative to the market portfolio, by demonstrating that in a nondegenerate and coherent market, the constant-weighted portfolio will asymptotically outperform the market portfolio. This provides us with a sneak peek into and our first encounter of the notion of arbitrage in equity markets, more specifically, the notion of relative arbitrage in equity markets. The long-term behaviour of dividend-paying stocks and portfolios is discussed in Section 3.5, in which we integrate the concept of dividends with the long-term behaviour of stocks and portfolios. We do so by exchanging the return processes with the total return processes. We shall expose the fact that the long-term performance of a portfolio in a market which includes dividends into the mix, is determined by its augmented or total portfolio growth rate process. Furthermore, for the market in which the stocks pay dividends, we shall reformulate the coherence condition of the market and we shall establish that the same equivalence coherence conditions hold. In addition, in this section, we shall consider an example of a market that exhibits equal augmented growth rates. The market with equal augmented growth rates is referred to as the equal-augmented-growth-rate market, and comprises a portfolio that exhibits the same augmented growth rate process for all of the dividend-paying stocks in the equity market. We shall prove that the equal-augmented-growth-rate market also possesses the market coherence quality. In particular, if all the stocks in the equity market have the same augmented growth rate process and the same nonnegative dividend rate process, then it shall be shown that the equity market is coherent. Thus, an equal-augmented-growth-rate market is coherent when the dividends for each stock are equal. This chapter finishes with Section 3.6 in which we provide a brief summary and conclusion on the results presented here in this chapter.

3.2 The Long-Term Behaviour of Stocks and Portfolios

In this section, we shall corroborate the rationale behind the assertion that the portfolio growth rate determines long-term portfolio behaviour. This advances us to Proposition 3.2.2, we provide expanded versions of the proofs given in Fernholz (2002), filling in details where necessary. But, firstly, we obtain some results that will be useful in the main part of the proof of this proposition. Thus, we proceed with the presentation of the following lemma.

Lemma 3.2.1 ([**Fernholz (2002)**]). *Let M be a continuous local martingale such that*

$$\lim_{t\to\infty} \frac{1}{t^2} \langle M \rangle_t \log\log t = 0, \qquad a.s. \tag{3.2.1}$$

Then

$$\lim_{t\to\infty} \frac{1}{t} M(t) = 0, \qquad a.s. \tag{3.2.2}$$

Proof. This proof is taken directly from Fernholz (2002, Lemma 1.3.2).

By extending the measure space Ω, if necessary, we can construct a standard one-dimensional Brownian motion $W_0 = \{W_0(t), \mathcal{F}_t, t \in [0,\infty)\}$ independent of the continuous local martingale $M = \{M(t), \mathcal{F}_t, t \in [0,\infty)\}$. Then define the following process $M_0 = \{M_0(t), \mathcal{F}_t, t \in [0,\infty)\}$, by

$$M_0(t) \triangleq M(t) + W_0(t), \qquad t \in [0,\infty). \tag{3.2.3}$$

Thus, M_0 is a continuous local martingale, and we obtain

$$\langle M_0 \rangle_t = \langle M + W_0 \rangle_t = \langle M + W_0, M + W_0 \rangle_t \tag{3.2.4}$$

$$= \langle M, M \rangle_t + \langle W_0, M \rangle_t + \langle M, W_0 \rangle_t + \langle W_0, W_0 \rangle_t. \tag{3.2.5}$$

Consequently, the independence of M and W_0, implies that $\langle M, W_0\rangle_t = \langle W_0, M\rangle_t = 0$, which yields[1]

$$\begin{aligned} \langle M_0\rangle_t &= \langle M\rangle_t + \langle W_0\rangle_t && (3.2.6)\\ &= \langle M\rangle_t + t, \qquad t\in[0,\infty), \qquad \text{a.s.} && (3.2.7)\end{aligned}$$

Moreover, (3.2.7) implies the following

$$\begin{aligned} \lim_{t\to\infty}\frac{1}{t^2}\langle M_0\rangle_t \log\log t &= \lim_{t\to\infty}\frac{1}{t^2}\big(\langle M\rangle_t + t\big)\log\log t && (3.2.8)\\ &= \lim_{t\to\infty}\frac{1}{t^2}\langle M\rangle_t \log\log t + \lim_{t\to\infty}\frac{1}{t^2}\, t\,\log\log t. && (3.2.9)\end{aligned}$$

Thus, (3.2.1), gives

$$\begin{aligned} \lim_{t\to\infty}\frac{1}{t^2}\langle M_0\rangle_t \log\log t &= \lim_{t\to\infty}\frac{1}{t^2}\, t\,\log\log t && (3.2.10)\\ &= \lim_{t\to\infty}\frac{1}{t}\,\log\log t && (3.2.11)\\ &= 0, && (3.2.12)\end{aligned}$$

The above result suggests that the process $\langle M_0\rangle_t$ increases at a slower rate than t^2. We shall put this finding to use shortly. Furthermore, from (3.2.7) we see that

$$\lim_{t\to\infty}\langle M_0\rangle_t = \lim_{t\to\infty}\big(\langle M\rangle_t + t\big) = \lim_{t\to\infty}\langle M\rangle_t + \lim_{t\to\infty} t = \infty, \qquad \text{a.s.} \qquad (3.2.13)$$

So, the time-change theorem for local martingales [Karatzas & Shreve (1991, Theorem 3.4.6)], can be applied to show that there exists a Brownian motion $B = \{B(t), \mathcal{F}_t, t\in[0,\infty)\}$ such that

$$B\big(\langle M_0\rangle_t\big) = M_0(t), \qquad t\in[0,\infty), \qquad \text{a.s.} \qquad (3.2.14)$$

Due to (3.2.13), we can apply the law of the iterated logarithm for Brownian motion [Karatzas & Shreve (1991, Theorem 2.9.23)], to $B\big(\langle M_0\rangle_t\big)$, to obtain

$$\overline{\lim_{\langle M_0\rangle_t\to\infty}}\frac{B\big(\langle M_0\rangle_t\big)}{\sqrt{2\langle M_0\rangle_t \log\log\langle M_0\rangle_t}} = 1, \qquad \text{a.s.,} \qquad (3.2.15)$$

which, along with (3.2.14), implies that

$$\overline{\lim_{t\to\infty}}\frac{\big|M_0(t)\big|}{\sqrt{2\langle M_0\rangle_t \log\log\langle M_0\rangle_t}} = \limsup_{t\to\infty}\frac{\big|M_0(t)\big|}{\sqrt{2\langle M_0\rangle_t \log\log\langle M_0\rangle_t}} = 1, \qquad \text{a.s.} \qquad (3.2.16)$$

From (3.2.12) it follows that $\langle M_0\rangle_t$ grows more slowly than t^2, so we can replace $\log t$ by $\log\langle M_0\rangle_t$ in (3.2.12), and we have

$$\lim_{t\to\infty}\frac{1}{t^2}\langle M_0\rangle_t \log\log\langle M_0\rangle_t = 0, \qquad \text{a.s.}$$

Hence,

$$\lim_{t\to\infty}\frac{1}{t}\sqrt{\langle M_0\rangle_t \log\log\langle M_0\rangle_t} = 0, \qquad \text{a.s.,}$$

and this and (3.2.16) imply that

$$\lim_{t\to\infty}\frac{1}{t}M_0(t) = 0, \qquad \text{a.s.}$$

Since the strong law of large numbers for Brownian motion [Karatzas & Shreve (1991, Problem 2.9.3)] implies that

$$\lim_{t\to\infty}\frac{1}{t}W_0(t) = 0, \qquad \text{a.s.,}$$

[1] Recall that the quadratic variation of the Brownian motion process is $\langle W\rangle_t = t$.

the lemma follows from (3.2.3)

$$\lim_{t\to\infty}\frac{1}{t}\,M(t) \;=\; \lim_{t\to\infty}\frac{1}{t}\,M_0(t) - \lim_{t\to\infty}\frac{1}{t}\,W_0(t) \;=\; 0, \qquad \text{a.s.} \tag{3.2.17}$$

■

Proposition 3.2.2 ([**Fernholz (2002)**]). *For any portfolio π in $\mathcal{M}$ and corresponding portfolio value process $Z_{w,\pi}$ with initial capital $Z_{w,\pi}(0) = w > 0$, we have*

$$\lim_{T\to\infty}\frac{1}{T}\left(\log Z_{w,\pi}(T) - \int_0^T \gamma_\pi(t)\,dt\right) \;=\; 0, \qquad a.s. \tag{3.2.18}$$

For initial capital $Z_{w,\pi}(0) = w = 1$, we have the portfolio value process $Z_\pi = Z_{1,\pi}$ that satisfies

$$\lim_{T\to\infty}\frac{1}{T}\left(\log Z_\pi(T) - \int_0^T \gamma_\pi(t)\,dt\right) \;=\; 0, \qquad a.s. \tag{3.2.19}$$

Proof. According to Proposition 2.2.20, we have a.s., for $t \in [0,\infty)$, the canonical decomposition of the semimartingale $\log Z_{w,\pi}(t)$,

$$d\log Z_{w,\pi}(t) \;=\; \gamma_\pi(t)\,dt + \sum_{i,\nu=1}^{n}\pi_i(t)\xi_{i\nu}(t)\,dW_\nu(t),$$

which is directly integrable to obtain the following

$$\begin{aligned}\log Z_{w,\pi}(T) - \log Z_{w,\pi}(0) &= \log\Big(Z_{w,\pi}(T)/Z_{w,\pi}(0)\Big)\\ &= \log\Big(Z_{w,\pi}(T)/w\Big)\\ &= \int_0^T \gamma_\pi(t)\,dt + \int_0^T \sum_{i,\nu=1}^{n}\pi_i(t)\xi_{i\nu}(t)\,dW_\nu(t).\end{aligned}$$

Let us define the process $V = \{V(t), \mathcal{F}_t, t \in [0,\infty)\}$, by

$$V(T) \;\triangleq\; \log\Big(Z_{w,\pi}(T)/Z_{w,\pi}(0)\Big) - \int_0^T \gamma_\pi(t)\,dt, \qquad T \in [0,\infty), \tag{3.2.20}$$

$$= \log\Big(Z_{w,\pi}(T)/w\Big) - \int_0^T \gamma_\pi(t)\,dt \tag{3.2.21}$$

$$= \int_0^T \sum_{i,\nu=1}^{n}\pi_i(t)\xi_{i\nu}(t)\,dW_\nu(t) \tag{3.2.22}$$

$$= \int_0^T\left(\sum_{i=1}^{n}\pi_i(t)\xi_{i1}(t)\right)dW_1(t) + \int_0^T\left(\sum_{i=1}^{n}\pi_i(t)\xi_{i2}(t)\right)dW_2(t) + \cdots + \int_0^T\left(\sum_{i=1}^{n}\pi_i(t)\xi_{in}(t)\right)dW_n(t).$$

We can infer from the formulation of the process $V(T)$ that it is a continuous martingale, this is due to the fact the $V(T)$ can be expressed as a summation of Itô integrals, which are martingales. Consequently, by considering the differential of the quadratic variational process of V, we can affirm the following

$$d\langle V\rangle_t \;=\; d\left\langle \int_0^{\cdot}\sum_{i,\nu=1}^{n}\pi_{i,s}\xi_{i\nu,s}\,dW_{\nu,s}\right\rangle_t.$$

Upon closer investigation into this equation, we observe from the derivation leading to (2.2.75), that $d\left\langle \log Z_{w,\pi}\right\rangle_t$ is equivalent to $d\langle V\rangle_t$, this along with (2.2.94) permits us to establish the following

$$\begin{aligned} d\langle V\rangle_t \;=\; d\left\langle \log Z_{w,\pi}\right\rangle_t &= \sum_{i,j=1}^{n}\pi_i(t)\sigma_{ij}(t)\pi_j(t)\,dt\\ &= \sigma_{\pi\pi}(t)\,dt\\ \langle V\rangle_t \;=\; \left\langle \log Z_{w,\pi}\right\rangle_t &= \int_0^t \sigma_{\pi\pi}(s)\,ds, \qquad t\in[0,\infty), \qquad \text{a.s.}\end{aligned} \tag{3.2.23}$$

At this point in the proof, we briefly digress, so as to recall a condition inherent in Definition 2.2.3. In particular, condition (ii) is given by

$$\lim_{t\to\infty} \frac{1}{t}\left(\xi_1^2(t) + \cdots + \xi_n^2(t)\right)\log\log t \quad = \quad 0, \qquad \text{a.s.} \tag{3.2.24}$$

Thus, for a financial market, $\mathcal{M}$, consisting of n stocks, we can reformulate (3.2.24) to obtain the following for $i = 1, \ldots, n$,

$$\lim_{t\to\infty} \frac{1}{t}\left(\xi_{i1}^2(t) + \cdots + \xi_{in}^2(t)\right)\log\log t \quad = \quad \lim_{t\to\infty} \frac{1}{t}\left(\sum_{\nu=1}^{n} \xi_{i\nu}^2(t)\right)\log\log t \quad = \quad 0, \qquad \text{a.s.} \tag{3.2.25}$$

Therefore, by applying (2.2.47), for all $i = 1, \ldots, n$, the preceding equation becomes

$$\lim_{t\to\infty} \frac{1}{t}\,\sigma_{ii}(t)\log\log t \quad = \quad 0, \qquad \text{a.s.} \tag{3.2.26}$$

Now, recall from (2.2.46), that for $i, j = 1, \ldots, n$,

$$\sigma_{ij}(t) \quad = \quad \sum_{\nu=1}^{n} \xi_{i\nu}(t)\xi_{j\nu}(t). \tag{3.2.27}$$

Consequently (2.2.94) becomes

$$\sigma_{\pi\pi}(t) \quad = \quad \sum_{i,j=1}^{n} \pi_i(t)\sigma_{ij}(t)\pi_j(t) \quad = \quad \sum_{i,j=1}^{n} \pi_i(t)\pi_j(t)\sum_{\nu=1}^{n} \xi_{i\nu}(t)\xi_{j\nu}(t) \tag{3.2.28}$$

$$= \quad \sum_{\nu=1}^{n}\left[\sum_{i=1}^{n} \pi_i(t)\xi_{i\nu}(t)\right]\left[\sum_{j=1}^{n} \pi_j(t)\xi_{j\nu}(t)\right] \tag{3.2.29}$$

$$= \quad \sum_{\nu=1}^{n}\left(\sum_{i=1}^{n} \pi_i(t)\xi_{i\nu}(t)\right)^2. \tag{3.2.30}$$

So, by the Cauchy-Schwarz inequality,[2] we have

$$\sigma_{\pi\pi}(t) \quad \leq \quad \sum_{\nu=1}^{n}\left(\sum_{i=1}^{n} \pi_i^2(t)\right)\left(\sum_{i=1}^{n} \xi_{i\nu}^2(t)\right) \tag{3.2.31}$$

$$= \quad \left(\sum_{i=1}^{n} \pi_i^2(t)\right)\left(\sum_{i,\nu=1}^{n} \xi_{i\nu}^2(t)\right) \tag{3.2.32}$$

$$= \quad \left(\sum_{i=1}^{n} \pi_i^2(t)\right)\left(\sum_{i=1}^{n} \sigma_{ii}(t)\right). \tag{3.2.33}$$

Hence, (3.2.26) together with the fact that the proportions in π are a.s. bounded, i.e.,

$$\lim_{t\to\infty} \frac{1}{t}\,\pi_i(t)\log\log t \quad = \quad 0, \qquad \text{a.s.,} \tag{3.2.34}$$

imply that

$$\lim_{t\to\infty} \frac{1}{t}\,\sigma_{\pi\pi}(t)\log\log t \quad = \quad 0, \qquad \text{a.s.} \tag{3.2.35}$$

Now, recall that $d\langle V\rangle_t = \sigma_{\pi\pi}(t)\,dt$. By substituting this expression into that for $\sigma_{\pi\pi}(t)$ in (3.2.35), we can deduce that

$$\lim_{t\to\infty} \frac{1}{t}\left(\frac{d\langle V\rangle_t}{dt}\right)\log\log t \quad = \quad 0, \qquad \text{a.s.} \tag{3.2.36}$$

[2] Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then a version of the Cauchy-Schwarz inequality is given by

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \quad \leq \quad \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right).$$

Let us consider that $d\langle V\rangle_t = \langle V\rangle_t - \langle V\rangle_0$ and $dt = t - 0$. It is evident from (3.2.23), that $\langle V\rangle_0 = 0$, which alludes to the following

$$\lim_{t\to\infty} \frac{1}{t}\left(\frac{\langle V\rangle_t}{t}\right)\log\log t = 0, \qquad \text{a.s.,} \tag{3.2.37}$$

$$\lim_{t\to\infty} \frac{1}{t^2}\langle V\rangle_t \log\log t = 0, \qquad \text{a.s.} \tag{3.2.38}$$

Consequently, by setting $\langle M\rangle_t := \langle V\rangle_t$, we can apply Lemma 3.2.1 to obtain

$$\lim_{t\to\infty} \frac{1}{t} V(t) = 0, \qquad \text{a.s.} \tag{3.2.39}$$

Likewise,

$$\lim_{T\to\infty} \frac{1}{T} V(T) = 0, \qquad \text{a.s.} \tag{3.2.40}$$

Substituting (3.2.20) into the equation above, yields

$$\lim_{T\to\infty} \frac{1}{T}\left(\log\left(Z_{w,\pi}(T)/Z_{w,\pi}(0)\right) - \int_0^T \gamma_\pi(t)\,dt\right) = \lim_{T\to\infty} \frac{1}{T}\left(\log\left(Z_{w,\pi}(T)/w\right) - \int_0^T \gamma_\pi(t)\,dt\right) = 0.$$

Thus,

$$\lim_{T\to\infty} \frac{1}{T}\left(\log Z_{w,\pi}(T) - \log Z_{w,\pi}(0) - \int_0^T \gamma_\pi(t)\,dt\right) = \lim_{T\to\infty} \frac{1}{T}\left(\log Z_{w,\pi}(T) - \log w - \int_0^T \gamma_\pi(t)\,dt\right) = 0,$$

and by noticing that $Z_{w,\pi}(0) = w > 0$, $w \in \mathbb{R}$, results in

$$\lim_{T\to\infty} \frac{1}{T}\log Z_{w,\pi}(0) = \lim_{T\to\infty} \frac{1}{T}\log w = 0, \qquad \text{a.s.,}$$

we eventually obtain the required result a.s.,

$$\lim_{T\to\infty} \frac{1}{T}\left(\log Z_{w,\pi}(T) - \int_0^T \gamma_\pi(t)\,dt\right) = 0.$$

■

The aforementioned proposition was originally formulated in Fernholz (1999a), and the proof of this result is somewhat different, the reader is referred to this text for a detailed account. Since a stock can be considered a portfolio holding a single stock, Proposition 3.2.2 can also be applied to stocks, and thus a relation of the form (3.2.19) also holds for stocks. This will be substantiated in the following corollary.

Corollary 3.2.3. *Let X be a stock with growth rate γ. Then*

$$\lim_{T\to\infty} \frac{1}{T}\left(\log X(T) - \int_0^T \gamma(t)\,dt\right) = 0, \qquad \textit{a.s.} \tag{3.2.41}$$

Proof. Apply Proposition 3.2.2 to a portfolio in which the weight corresponding to X is 1 and all the other weights are 0. The required result follows. ■

Thus, to fortify the result above, we can state that for a financial market $\mathcal{M}$, comprising n stocks, we have for $i = 1, \ldots, n$,

$$\lim_{T\to\infty} \frac{1}{T}\left(\log X_i(T) - \int_0^T \gamma_i(t)\,dt\right) = 0, \qquad \text{a.s.,} \tag{3.2.42}$$

which is formalised in the corollary below.

Corollary 3.2.4. *Let X_i be the ith stock with growth rate γ_i, for all $i = 1, 2, \ldots, n$. Then*

$$\lim_{T\to\infty} \frac{1}{T}\left(\log X_i(T) - \int_0^T \gamma_i(t)\,dt\right) = 0, \qquad a.s. \tag{3.2.43}$$

This a.s. relationship is valid when the individual stock variances $\sigma_{ii}(t)$ do not increase too quickly, in particular, such that

$$\lim_{T\to\infty} \frac{1}{T^2} \log\log T \int_0^T \sigma_{ii}(t)\,dt = 0, \qquad \text{a.s.} \tag{3.2.44}$$

From (3.2.38), we obtain a relation equivalent to (3.2.44), under which (3.2.19) is valid,

$$\lim_{T\to\infty} \frac{1}{T^2} \log\log T \int_0^T \sigma_{\pi\pi}(t)\,dt = 0, \qquad \text{a.s.} \tag{3.2.45}$$

Fernholz & Karatzas (2009) propose an alternative form to that in (3.2.45), which makes apparent the connection between the justification of the portfolio growth rate as the determinant of long-term portfolio behaviour and the uniform boundedness condition as presented in Definition 2.2.14,

$$\lim_{T\to\infty} \frac{1}{T^2} \log\log T \int_0^T \left\|\boldsymbol{\sigma}(t)\right\|\,dt = 0, \qquad \text{a.s.} \tag{3.2.46}$$

In fact, as Fernholz & Karatzas (2009) point out, this condition is satisfied when all the eigenvalues of the covariance matrix process, $\boldsymbol{\sigma} = \{\boldsymbol{\sigma}(t), t \in [0, \infty)\}$ are uniformly bounded away from infinity.

Let us now briefly summarise the impact and significance of such a result. The preceding proposition postulated that it is the growth rate of a stock or portfolio that characterises the long-term stock or portfolio behaviour. Specifically, (3.2.19), shows that the portfolio growth rate is an essential determinant of portfolio performance, especially over the long time horizon. Its subsequent proof substantiated this claim, revealing that in the interests of the long-term investor, we should be analysing the growth rate of a stock or portfolio rather than the rates of return. This is in stark contrast to the traditional theories of portfolio behaviour, in which the rates of return played the primary role. Hereafter, the rate of return is of no concern to us within the logarithmic setting, and all of our focus will be shifted to the growth rate. The consequences of such a result cannot be stressed enough.

3.3 The Long-Term Relative Behaviour of Stocks and Portfolios

In this section we shall consider the long-term behaviour of the relative return of portfolios.

Corollary 3.3.1. *Let π and η be two portfolios in the market $\mathcal{M}$. Then,*

$$\lim_{T\to\infty} \frac{1}{T}\left(\log\left(Z_\pi(T)/Z_\eta(T)\right) - \int_0^T \left(\gamma_\pi(t) - \gamma_\eta(t)\right)dt\right) = 0, \qquad a.s. \tag{3.3.1}$$

Proof. By equation (3.2.19) of Proposition 3.2.2, we have

$$\lim_{T\to\infty} \frac{1}{T}\left(\log Z_\pi(T) - \int_0^T \gamma_\pi(t)\,dt\right) = 0, \qquad \text{a.s.},$$

and

$$\lim_{T\to\infty} \frac{1}{T}\left(\log Z_\eta(T) - \int_0^T \gamma_\eta(t)\,dt\right) = 0, \qquad \text{a.s.}$$

It then follows that

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\left(\log\Big(Z_\pi(T)/Z_\eta(T)\Big)-\int_0^T\big(\gamma_\pi(t)-\gamma_\eta(t)\big)\,dt\right)\\ &= \lim_{T\to\infty}\frac{1}{T}\left(\log Z_\pi(T)-\int_0^T\gamma_\pi(t)\,dt\right)-\lim_{T\to\infty}\frac{1}{T}\left(\log Z_\eta(T)-\int_0^T\gamma_\eta(t)\,dt\right)\\ &= 0.\end{aligned}$$

■

Thus, the long-term behaviour of the relative return process of π versus η is determined by the difference in the growth rates of the two portfolios, i.e., the long-term relative behaviour of the portfolios is determined by $\gamma_\pi(t)-\gamma_\eta(t)$. Once again, since a stock can be considered to be a portfolio holding a single stock, Corollary 3.3.1 can also be applied to stocks, and thus a relation of the form (3.3.1) also holds for stocks. This will be substantiated in the following corollary.

Corollary 3.3.2. *Let X_i, for $i=1,2,\ldots,n$, be stocks in the market $\mathcal{M}$ and let η be a portfolio in the market $\mathcal{M}$. Then, for $i=1,2,\ldots,n$,*

$$\lim_{T\to\infty}\frac{1}{T}\left(\log\Big(X_i(T)/Z_\eta(T)\Big)-\int_0^T\big(\gamma_i(t)-\gamma_\eta(t)\big)\,dt\right) = 0, \qquad a.s. \tag{3.3.2}$$

3.4 The Long-Term Relative Behaviour of the Market

Thus far, we have considered the long-term behaviour of stocks and portfolios. In this section we shall examine the long-term relative behaviour of the stocks in the market. We thus illustrate through various simplified portfolios, the effect that the long-term behaviour of the individual stocks has on the market. This will consequently permit us to establish the long-term behaviour of certain simple portfolios. To begin with, it is necessary for some of the subsequent results here, to impose a structural condition on the market, namely the coherence of the market. This we shall introduce in the next subsection in Definition 3.4.1.

3.4.1 Coherence

Let us now define the concept of coherence in the market. We shall say that the market model $\mathcal{M}$ encountered in the previous chapter is coherent if the relative capitalisations of (2.12.19), i.e., if the market weights, satisfy the requirement in the following definition.

Definition 3.4.1 (Coherence)**.** *The market $\mathcal{M}$ is **coherent** if for each $i=1,\ldots,n$, we have*

$$\lim_{t\to\infty}\frac{1}{t}\log\mu_i(t) = 0, \qquad a.s., \tag{3.4.1}$$

equivalently, if for each $i=1,\ldots,n$ and $T\in[0,\infty)$, we have

$$\lim_{T\to\infty}\frac{1}{T}\log\mu_i(T) = 0, \qquad a.s. \tag{3.4.2}$$

Recall that $\mu_i(t)$ signify the market weights, and in line with the fact that the market portfolio satisfies the requirements of the definition of a portfolio, it is evident that $0<\mu_i(t)<1$, for $t\in[0,\infty)$ and $i=1,\ldots,n$. Consequently, $\log\mu_i(t)<0$, for $t\in[0,\infty)$ and $i=1,\ldots,n$. With this in mind, it is apparent that the market coherence condition is valid if none of the stocks in the market declines too rapidly with respect to the market as a whole, over some future time horizon. Furthermore, the coherence condition conjectures that the long-term relative log-return of the stocks versus the market portfolio approaches zero.

Recall, following Definition 2.12.1, that the market weight processes can be expressed as

$$\mu_i(t) \quad = \quad \frac{X_i(t)}{Z_\mu(t)}, \qquad t \in [0, \infty).$$

As such, the condition (3.4.1), can be reformulated as

$$\lim_{t\to\infty} \frac{1}{t}\Big(\log X_i(t) - \log Z_\mu(t)\Big) \quad = \quad 0, \qquad \text{a.s.} \tag{3.4.3}$$

This expression should be kept in mind, as it is of vital importance in substantiating the next proposition which presents some rather illuminating equivalence conditions. Thus, the following proposition establishes, under the condition (2.2.57) (equivalently, conditions (3.2.45) or (3.2.46)) on the covariance structure, certain conditions to which the coherence of the market is comparable. That is, under the aforementioned condition on the covariance structure, it can be shown that the coherence of the market is equivalent to each of the following two conditions provided in the following proposition.

Proposition 3.4.2 ([**Fernholz (2002)**]). *Let $\mathcal{M}$ denote the market with stocks $X_1, \ldots, X_n$. Then the following statements are equivalent:*

(*i*) *The market, $\mathcal{M}$, is coherent;*

(*ii*) *for each $i = 1, \ldots, n$,*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \big(\gamma_i(t) - \gamma_\mu(t)\big)\, dt \quad = \quad 0, \qquad \textit{a.s.;} \tag{3.4.4}$$

(*iii*) *for each pair $i, j = 1, \ldots, n$,*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \big(\gamma_i(t) - \gamma_j(t)\big)\, dt \quad = \quad 0, \qquad \textit{a.s.} \tag{3.4.5}$$

Proof. We expound upon the proof given by Fernholz (2002).

In order to prove the equivalence conditions, we shall prove that condition (i) implies condition (ii), which then implies condition (iii) which in turn implies condition (i).

- (i) $\Rightarrow$ (ii):

  Suppose that the market $\mathcal{M}$ is coherent, as per condition (i). Then, in accordance with Definition 3.4.1, the market is coherent if for $i = 1, \ldots, n$, the following is satisfied

  $$\lim_{T\to\infty} \frac{1}{T}\Big(\log X_i(T) - \log Z_\mu(T)\Big) \quad = \quad 0, \qquad \text{a.s.}$$

  Recall Proposition 3.2.2, where it was shown that for any portfolio π in the market $\mathcal{M}$, the growth rate of the portfolio π determines the long-term performance of π. Thus, the proposition holds for the market portfolio μ. By setting $\pi := \mu$ in this case, we obtain the following result

  $$\lim_{T\to\infty} \frac{1}{T}\left(\log Z_\mu(T) - \int_0^T \gamma_\mu(t)\, dt\right) \quad = \quad 0, \qquad \text{a.s.,} \tag{3.4.6}$$

  moreover, by Corollary 3.2.3, and (3.2.20), we have for $i = 1, \ldots, n$,

  $$\lim_{T\to\infty} \frac{1}{T}\left(\log X_i(T) - \int_0^T \gamma_i(t)\, dt\right) \quad = \quad 0, \qquad \text{a.s.} \tag{3.4.7}$$

  By equating the left-hand side part of (3.4.6) and (3.4.7), then rearranging we obtain the required result, namely condition (ii).

- (ii) $\Rightarrow$ (iii):

  Now, suppose that condition (ii) holds, we thus have for $i = 1, \ldots, n$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \big(\gamma_i(t) - \gamma_\mu(t)\big)\, dt \quad = \quad 0, \qquad \text{a.s.}$$

  However, just as this condition holds for some $i = 1, \ldots, n$, so does it hold for some $j = 1, \ldots, n$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \big(\gamma_j(t) - \gamma_\mu(t)\big)\, dt \quad = \quad 0, \qquad \text{a.s.}$$

  Again, by equating the left-hand side of these expressions, we obtain the required result, namely condition (iii).

- (iii) $\Rightarrow$ (i):

  Suppose that condition (iii) holds. By consulting the intimation advocated by Fernholz (2002, Proposition 2.1.2), we shall explicitly show the dependence of all random variables and processes on $\omega \in \Omega$. Thus, Corollary 3.2.3 and condition (iii) imply that there is a subset $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$ such that for $\omega \in \Omega'$, and for $i, j = 1, \ldots, n$,

$$\lim_{T\to\infty} \frac{1}{T} \left(\log X_i(T,\omega) - \int_0^T \gamma_i(t,\omega)\, dt \right) \quad = \quad 0, \tag{3.4.8}$$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \big(\gamma_i(t,\omega) - \gamma_j(t,\omega)\big)\, dt \quad = \quad 0. \tag{3.4.9}$$

  Equipped with this knowledge, we need to show that the market $\mathcal{M}$ is coherent, and accordingly, satisfies the conditions put forth in Definition 3.4.1. For our purposes it will be satisfactory to show that (3.4.3) holds.

  Let us choose $\omega \in \Omega'$. Then (3.4.8) and (3.4.9) with $j = 1$ fixed, imply that for $i = 1, \ldots, n$,

$$\lim_{T\to\infty} \frac{1}{T} \left(\log X_i(T,\omega) - \int_0^T \gamma_1(t,\omega)\, dt \right) \quad = \quad 0. \tag{3.4.10}$$

  In particular,

$$\lim_{T\to\infty} \frac{1}{T} \left(\log X_1(T,\omega) - \int_0^T \gamma_1(t,\omega)\, dt \right) \quad = \quad 0. \tag{3.4.11}$$

  Hence, if this expression (3.4.10) goes to 0 for all $i = 1, \ldots, n$, then it will also do so for the maximum over $i = 1, \ldots, n$,

$$\lim_{T\to\infty} \frac{1}{T} \left(\max_{1\le i\le n} \big(\log X_i(T,\omega)\big) - \int_0^T \gamma_1(t,\omega)\, dt \right) \quad = \quad 0.$$

  Due to the fact that the logarithmic function is a strictly increasing function, we have $\max\big(\log X_i(T,\omega)\big) = \log\big(\max X_i(T,\omega)\big)$, which yields the following expression, equivalent to that above,

$$\lim_{T\to\infty} \frac{1}{T} \left(\log\left(\max_{1\le i\le n} X_i(T,\omega) \right) - \int_0^T \gamma_1(t,\omega)\, dt \right) \quad = \quad 0. \tag{3.4.12}$$

  Now, since $X_i(t,\omega)$ is nonnegative for all $t \in [0,\infty)$ and $i = 1, \ldots, n$, and

$$X_k(t,\omega) \quad \le \quad \max_{1\le i\le n} X_i(t,\omega), \qquad \text{for all } \ k = 1, \ldots, n,$$

  we can infer that for $t \in [0,\infty)$,

$$X_1(t,\omega) \quad \le \quad X_1(t,\omega) + X_2(t,\omega) + \cdots + X_n(t,\omega) \quad \le \quad n \max_{1\le i\le n} X_i(t,\omega),$$

so that we also have for $t \in [0, \infty)$,

$$X_1(t,\omega) \quad \leq \quad Z_\mu(t,\omega) \quad \leq \quad n \max_{1 \leq i \leq n} X_i(t,\omega).$$

Once more, the fact that the logarithmic function is strictly increasing, and recalling that the total capitalisation of the market can be represented in terms of the individual stock capitalisations as $Z_\mu(t) = X_1(t) + X_2(t) + \cdots + X_n(t)$, permits us to form the following

$$\log X_1(t,\omega) \quad \leq \quad \log Z_\mu(t,\omega) \quad \leq \quad \log n + \log\left(\max_{1 \leq i \leq n} X_i(t,\omega)\right). \tag{3.4.13}$$

Let us modify (3.4.13) slightly, by subtracting $\int_0^T \gamma_1(t,\omega)\,dt$ and by taking the long-term average,

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T}\left(\log X_1(T,\omega) - \int_0^T \gamma_1(t,\omega)\,dt\right) \\
&\qquad \leq \quad \lim_{T\to\infty} \frac{1}{T}\left(\log Z_\mu(T,\omega) - \int_0^T \gamma_1(t,\omega)\,dt\right) \\
&\qquad \leq \quad \lim_{T\to\infty} \frac{1}{T}\log n + \lim_{T\to\infty} \frac{1}{T}\left(\log\left(\max_{1 \leq i \leq n} X_i(T,\omega)\right) - \int_0^T \gamma_1(t,\omega)\,dt\right).
\end{aligned}$$

In addition, because n is simply a constant, it is apparent then that

$$\lim_{T\to\infty} \frac{1}{T}\log n \quad = \quad 0. \tag{3.4.14}$$

It thus follows from (3.4.11), (3.4.12) and the result above (3.4.14), that

$$0 \quad \leq \quad \lim_{T\to\infty} \frac{1}{T}\left(\log Z_\mu(T,\omega) - \int_0^T \gamma_1(t,\omega)\,dt\right) \quad \leq \quad 0,$$

which implies that

$$\lim_{T\to\infty} \frac{1}{T}\left(\log Z_\mu(T,\omega) - \int_0^T \gamma_1(t,\omega)\,dt\right) \quad = \quad 0. \tag{3.4.15}$$

We finally have from (3.4.10) and (3.4.15), that

$$\lim_{T\to\infty} \frac{1}{T}\Big(\log X_i(T,\omega) - \log Z_\mu(T,\omega)\Big) \quad = \quad 0.$$

This is precisely the coherence condition provided in (3.4.3), and since this holds for any $\omega \in \Omega'$, we can conclude that the market $\mathcal{M}$ is indeed coherent, demonstrating that the equivalence conditions are justified. Note the argument presented can similarly make use of the min $\big(\log X_i(t,\omega)\big)$ as demonstrated in Fernholz (1999a, Proposition 3.2).

■

This proposition suggests that in a coherent market, the time-average difference between the market portfolio and a stock in the market portfolio will be zero. Roughly speaking, this insinuates that the market's long-term realised growth rate does not exceed that of its component stocks. Moreover, the coherence condition, is justified when the time-average difference between the growth rates of any two stocks is zero. Note that this pertains only to the differences, the time average of the growth rate of an individual stock may not necessarily exist.

3.4.2 Markets with Constant Parameters

The notion of a coherent market allows us now to reveal certain simplified portfolios that satisfy this criterion, a case in point, is the portfolio which exhibits the *same growth rate* process for all of the stocks. We shall substantiate this claim in the corollary following the next. First, we show a result for the equal-growth-rate market.

3.4.2.1 Market with Equal Growth Rates

Corollary 3.4.3. *Suppose that all the stocks in the market $\mathcal{M}$ have the* ***same growth rate process****. Then for the market portfolio, μ, in $\mathcal{M}$, we have*

$$\lim_{T\to\infty} \frac{1}{T}\left(\log Z_\mu(T) - \int_0^T \gamma(t)\,dt\right) = 0, \qquad a.s., \tag{3.4.16}$$

where $Z_\mu(t) = X_1(t) + \cdots + X_n(t)$.

Proof. The proof presented here is taken from Fernholz (1999a, Proof of Proposition 3.2).

Since all the stocks in the market share the same common growth rate, $\gamma(t)$, so that $\gamma_i(t) \equiv \gamma(t)$ for all $i = 1, \ldots, n$ and $t \in [0, \infty)$, (3.2.42) becomes

$$\lim_{T\to\infty} \frac{1}{T}\left(\log X_i(T) - \int_0^T \gamma(t)\,dt\right) = 0, \qquad \text{a.s.}, \tag{3.4.17}$$

and by Proposition 3.2.2, we have for the market portfolio in $\mathcal{M}$,

$$\lim_{T\to\infty} \frac{1}{T}\left(\log Z_\mu(T) - \int_0^T \gamma_\mu(t)\,dt\right) = 0, \qquad \text{a.s.}, \tag{3.4.18}$$

where $Z_\mu(t) = X_1(t) + \cdots + X_n(t)$ represents the total capitalisation of the market portfolio.

Once again, if (3.4.17) goes to 0 for all $i = 1, \ldots, n$, then it will also do so for the maximum over $i = 1, \ldots, n$, i.e., for the biggest stock we have

$$\lim_{T\to\infty} \frac{1}{T}\left(\max_{1\le i\le n}\big(\log X_i(T)\big) - \int_0^T \gamma(t)\,dt\right) = 0, \qquad \text{a.s.},$$

and also for the minimum over $i = 1, \ldots, n$, i.e., for the smallest stock we have

$$\lim_{T\to\infty} \frac{1}{T}\left(\min_{1\le i\le n}\big(\log X_i(T)\big) - \int_0^T \gamma(t)\,dt\right) = 0, \qquad \text{a.s.}$$

Due to the fact that the logarithmic function is a strictly increasing function, we have that $\max\big(\log X_i(T)\big) = \log\big(\max X_i(T)\big)$, which yields the following expressions, equivalent to those above,

$$\lim_{T\to\infty} \frac{1}{T}\left(\log\left(\max_{1\le i\le n} X_i(T)\right) - \int_0^T \gamma(t)\,dt\right) = 0, \qquad \text{a.s.}, \tag{3.4.19}$$

and

$$\lim_{T\to\infty} \frac{1}{T}\left(\log\left(\min_{1\le i\le n} X_i(T)\right) - \int_0^T \gamma(t)\,dt\right) = 0, \qquad \text{a.s.} \tag{3.4.20}$$

But then, employing the reverse-order-statistics notation concept of Definition 2.4.10, we also have

$$\lim_{T\to\infty} \frac{1}{T}\left(\log X_{(1)}(T) - \int_0^T \gamma(t)\,dt\right) = 0, \qquad \text{a.s.}, \tag{3.4.21}$$

for the biggest stock $X_{(1)}(t) \triangleq \max_{1\le i\le n} X_i(t)$, and

$$\lim_{T\to\infty} \frac{1}{T}\left(\log X_{(n)}(T) - \int_0^T \gamma(t)\,dt\right) = 0, \qquad \text{a.s.}, \tag{3.4.22}$$

for the smallest stock $X_{(n)}(t) \triangleq \min_{1\le i\le n} X_i(t)$. Now, note since $X_i(t)$ is nonnegative for all $t \in [0,\infty)$ and $i = 1,\ldots,n$, and

$$\min_{1\le i\le n} X_i(t) \;\le\; X_k(t) \;\le\; \max_{1\le i\le n} X_i(t), \qquad \text{for all } k = 1,\ldots,n,$$

$$n \min_{1\le i\le n} X_i(t) \;=\; \sum_{k=1}^{n} \min_{1\le i\le n} X_i(t) \;\le\; Z_\mu(t) \;=\; \sum_{k=1}^{n} X_k(t) \;\le\; \sum_{k=1}^{n} \max_{1\le i\le n} X_i(t) \;=\; n \max_{1\le i\le n} X_i(t),$$

we can infer the following inequalities for $t \in [0,\infty)$,

$$n \min_{1\le i\le n} X_i(t) \;\le\; X_1(t) + X_2(t) + \cdots + X_n(t) \;\le\; n \max_{1\le i\le n} X_i(t),$$

so that we also have for $t \in [0,\infty)$,

$$n \min_{1\le i\le n} X_i(t) \;\le\; Z_\mu(t) \;\le\; n \max_{1\le i\le n} X_i(t).$$

Once more, the fact that the logarithmic function is strictly increasing, and recalling that the total capitalisation of the market can be represented in terms of the individual stock capitalisations as $Z_\mu(t) = X_1(t) + X_2(t) + \cdots + X_n(t)$, permits us to form the following

$$\log n + \log\left(\min_{1\le i\le n} X_i(t)\right) \;\le\; \log Z_\mu(t) \;\le\; \log n + \log\left(\max_{1\le i\le n} X_i(t)\right). \tag{3.4.23}$$

We now modify (3.4.23) slightly, by subtracting $\int_0^T \gamma(t)\,dt$ and by taking the long-term average,

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T} \log n + \lim_{T\to\infty} \frac{1}{T}\left(\log\left(\min_{1\le i\le n} X_i(T)\right) - \int_0^T \gamma(t)\,dt\right) \\
&\qquad \le \lim_{T\to\infty} \frac{1}{T}\left(\log Z_\mu(T) - \int_0^T \gamma(t)\,dt\right) \\
&\qquad \le \lim_{T\to\infty} \frac{1}{T} \log n + \lim_{T\to\infty} \frac{1}{T}\left(\log\left(\max_{1\le i\le n} X_i(T)\right) - \int_0^T \gamma(t)\,dt\right).
\end{aligned}$$

In addition, because n is simply a constant, it is apparent then that

$$\lim_{T\to\infty} \frac{1}{T} \log n \;=\; 0.$$

It thus follows from (3.4.19) and the result above, that

$$0 \;\le\; \lim_{T\to\infty} \frac{1}{T}\left(\log Z_\mu(T) - \int_0^T \gamma(t)\,dt\right) \;\le\; 0,$$

which implies the result

$$\lim_{T\to\infty} \frac{1}{T}\left(\log Z_\mu(T) - \int_0^T \gamma(t)\,dt\right) \;=\; 0. \tag{3.4.24}$$

■

Therefore, from (3.4.17) combined with the result (3.4.16) or (3.4.24) of Corollary 3.4.3, we have that

$$\lim_{T\to\infty} \frac{1}{T}\Big(\log X_i(T) - \log Z_\mu(T)\Big) \;=\; 0, \qquad \text{a.s.}, \tag{3.4.25}$$

which is exactly the market coherence condition given in (3.4.3). Furthermore, (3.4.19) and (3.4.21) in conjunction again with the result (3.4.16) or (3.4.24) of Corollary 3.4.3, gives the following

$$\lim_{T\to\infty} \frac{1}{T}\left(\log\left(\max_{1\le i\le n} X_i(T)\right) - \log Z_\mu(T)\right) \;=\; \lim_{T\to\infty} \frac{1}{T}\Big(\log X_{(1)}(T) - \log Z_\mu(T)\Big) \;=\; 0, \qquad \text{a.s.}, \tag{3.4.26}$$

this is also a version of the market coherence condition given in (3.4.3) but for the biggest stock $X_{(1)}(t) \triangleq \max_{1\le i\le n} X_i(t)$. Since, from the coherence definition 3.4.1, the coherence condition (3.4.2) is also valid for the largest stock, so that we have

$$\lim_{T\to\infty} \frac{1}{T} \log\left(\max_{1\le i\le n} \mu_i(T)\right) \;=\; \lim_{T\to\infty} \frac{1}{T} \log \mu_{(1)}(T) \;=\; 0, \qquad \text{a.s.,} \tag{3.4.27}$$

where, from the reverse-order-statistics notation, we have

$$\max_{1\le i\le n} \mu_i(t) \;\triangleq\; \mu_{(1)}(t) \;=\; \frac{X_{(1)}(t)}{Z_\mu(t)}, \qquad t\in[0,\infty).$$

Further to the above result, (3.4.20) and (3.4.22) in conjunction again with the result (3.4.16) or (3.4.24) of Corollary 3.4.3, gives the following

$$\lim_{T\to\infty} \frac{1}{T}\left(\log\left(\min_{1\le i\le n} X_i(T)\right) - \log Z_\mu(T)\right) \;=\; \lim_{T\to\infty} \frac{1}{T}\Big(\log X_{(n)}(T) - \log Z_\mu(T)\Big) \;=\; 0, \qquad \text{a.s.,} \tag{3.4.28}$$

this is also a version of the market coherence condition given in (3.4.3) but now for the smallest stock $X_{(n)}(t) \triangleq \min_{1\le i\le n} X_i(t)$. Since, from the coherence definition 3.4.1, the coherence condition (3.4.2) is also valid for the smallest stock, so that we have

$$\lim_{T\to\infty} \frac{1}{T} \log\left(\min_{1\le i\le n} \mu_i(T)\right) \;=\; \lim_{T\to\infty} \frac{1}{T} \log \mu_{(n)}(T) \;=\; 0, \qquad \text{a.s.,} \tag{3.4.29}$$

where, from the reverse-order-statistics notation, we have

$$\min_{1\le i\le n} \mu_i(t) \;\triangleq\; \mu_{(n)}(t) \;=\; \frac{X_{(n)}(t)}{Z_\mu(t)}, \qquad t\in[0,\infty).$$

Finally, from (3.4.18) together with the result (3.4.16) or (3.4.24) of Corollary 3.4.3, we thus have that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \big(\gamma(t) - \gamma_\mu(t)\big)\,dt \;=\; 0, \qquad \text{a.s.,} \tag{3.4.30}$$

which is the coherence-equivalent condition (ii), (3.4.4), of Proposition 3.4.2, but for the equal-growth-rate market, i.e., for which $\gamma_i(t) \equiv \gamma(t)$ for all $i = 1, 2, \ldots, n$.

Corollary 3.4.4. *Suppose that all the stocks in the market $\mathcal{M}$ have the* ***same growth rate process****. Then the market $\mathcal{M}$ is coherent.*

Proof. Suppose that all the stocks in the market have the same growth rate process, then $\gamma_i(t) \equiv \gamma(t)$ for all $t \in [0,\infty)$ and $i = 1,\ldots,n$. Which further implies that $\gamma_i(t) = \gamma_j(t)$ for all $i,j = 1,\ldots,n$. With this information, we can conclude that condition (iii) of Proposition 3.4.2 holds, and that due to the equivalence conditions, the market $\mathcal{M}$ is coherent. Furthermore, by appealing to Corollary 3.4.3, we can also verify condition (ii) of Proposition 3.4.2. Therefore, (3.4.17) in conjunction with (3.4.16) yields for all $i = 1,\ldots,n$,

$$\lim_{T\to\infty} \frac{1}{T}\Big(\log X_i(T) - \log Z_\mu(T)\Big) \;=\; 0.$$

This is precisely the coherence condition provided in (3.4.3), we thus conclude that the equal-growth-rate market $\mathcal{M}$ is indeed coherent. More specifically, the following expression holds

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \big(\gamma(t) - \gamma_\mu(t)\big)\,dt \;=\; 0, \qquad \text{a.s.,} \tag{3.4.31}$$

which is just the coherent-equivalence condition (ii) of Proposition 3.4.2. ■

The next corollary establishes the converse to Corollary 3.4.4, for the case that the growth rates of all the stocks are constant.

Corollary 3.4.5. *Suppose that all the stocks in the market $\mathcal{M}$ have constant growth rates. Then $\mathcal{M}$ is coherent if and only if the growth rates are all equal.*

Proof. If the growth rates are all equal, i.e., $\gamma_i(t) = \gamma_j(t) \equiv \gamma$, for all $i, j = 1, \ldots, n$, then the market $\mathcal{M}$ is coherent by Corollary 3.4.4. If, however, $X_i(t)$ and $X_j(t)$ have different constant growth rates, then condition (iii) of Proposition 3.4.2 will fail, and so the market $\mathcal{M}$ is not coherent. ■

This corollary reveals that for a portfolio in which all the stocks have constant equal growth rates, the market is coherent. However, for a portfolio in which all the stocks have constant yet different growth rates, the market is not coherent. In conclusion, a portfolio that exhibits constant growth rates for all its stocks is not necessarily coherent, information regarding the equality of the growth rates is required.

Once again, suppose that all the stocks in the market have the same growth rate (i.e., let $\gamma_i(t) \equiv \gamma(t)$, for all $i = 1, \ldots, n$), then (2.2.111) implies, by setting $\pi := \mu$, that the growth rate of the market portfolio, μ, is

$$\gamma_\mu(t) \quad = \quad \sum_{i=1}^{n} \mu_i(t)\gamma(t) + \gamma_\mu^*(t) \quad = \quad \gamma(t) + \gamma_\mu^*(t), \qquad t \in [0, \infty), \qquad \text{a.s.} \tag{3.4.32}$$

Recall Corollary 3.4.4, since all the stocks in the market have the same growth rate, the market will be coherent. Condition (ii) of Proposition 3.4.2, is now interpreted as

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \big(\gamma(t) - \gamma_\mu(t)\big)\, dt \quad = \quad 0, \qquad \text{a.s.},$$

which implies that, asymptotically, the growth rate of the market will be the same as the common growth rate of the stocks. This is intuitively obvious, since according to Corollary 3.2.3, the long-term behaviour of a stock is characterised by its growth rate, in addition, since $Z_\mu(t)$ is the simple summation of $X_i(t)$ for $i = 1, \ldots, n$, all of which exhibit the same growth rate $\gamma(t)$, it is apparent that $Z_\mu(t)$ must in some sense share this common growth rate. It follows that, over the long term, the contribution of $\gamma_\mu^*(t)$ to $\gamma_\mu(t)$ must be minimal. This statement will be justified in the following proposition.

Proposition 3.4.6 ([**Fernholz (2002)**]). *Suppose that all the stocks in the market $\mathcal{M}$ have the same growth rate ($\gamma_i(t) \equiv \gamma(t)$, for all $i = 1, 2, \ldots, n$). Then we have*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \gamma_\mu^*(t)\, dt \quad = \quad 0, \qquad \textit{a.s.} \tag{3.4.33}$$

Proof. Since all the growth rates of the stocks are equal, Corollary 3.4.4 states that the market $\mathcal{M}$ is coherent. By (ii) of the equivalence conditions of Proposition 3.4.2,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \big(\gamma(t) - \gamma_\mu(t)\big)\, dt \quad = \quad 0, \qquad \text{a.s.}$$

In addition, (3.4.32) translates to $\gamma_\mu^*(t) = \gamma_\mu(t) - \gamma(t)$ or, that prior, to $\gamma(t) - \gamma_\mu(t) = -\gamma_\mu^*(t)$, because of the assumption of equal growth rates. Combining this with the expression above yields the required result:

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \big(\gamma(t) - \gamma_\mu(t)\big)\, dt \quad = \quad \lim_{T\to\infty} \frac{1}{T} \int_0^T \big(\gamma(t) - \gamma(t) - \gamma_\mu^*(t)\big)\, dt \quad = \quad -\lim_{T\to\infty} \frac{1}{T} \int_0^T \gamma_\mu^*(t)\, dt \quad = \quad 0,$$

and (3.4.33) follows as a result. ■

Proposition 3.4.6 encapsulates the fact that in a market in which all the stocks have equal growth rates, over the long term, the average excess growth rate of this market is asymptotically negligible. Effectively, the $\gamma_\mu^*(t)$ term vanishes from all formulations relating to long-term portfolio performance evaluation. This consequence will assist in establishing a rather crucial result, namely that of the diverse nature of the equal-growth-rate market, which will be revealed shortly. An alternative proof to Proposition 3.4.6 is offered in Fernholz (1999a, Proposition 3.2).

3.4.2.2 The Constant-Weighted Portfolio

Definition 3.4.7 (The Constant-Weighted Portfolio). *A portfolio π is **constant-weighted** if the weight processes π_i are all constant in t, such that for all $i = 1, \ldots, n$,*

$$\pi_i(t) \quad \triangleq \quad p_i, \qquad t \in [0, \infty), \tag{3.4.34}$$

where the p_i are nonnegative constants such that $p_1 + p_2 + \cdots + p_n = 1$.

3.4.2.3 The Long-Term Relative Behaviour of the Constant-Weighted Portfolio

The next proposition gives some insight into the long-term behaviour of constant-weighted portfolios.

Proposition 3.4.8 ([**Fernholz (2002)**]). *Suppose that the market $\mathcal{M}$ is nondegenerate and coherent, and that π is a constant-weighted portfolio with at least two positive weights and no negative weights. Then*

$$\liminf_{T\to\infty} \frac{1}{T} \log\Big(Z_\pi(T)/Z_\mu(T)\Big) \quad > \quad 0, \qquad a.s. \tag{3.4.35}$$

Proof. Suppose that π is constant-weighted and thus satisfies the criteria laid out in Definition 3.4.7. Recall the reverse-order-statistics notation of Definition 2.4.10,

$$\pi_{\max}(t) \quad \triangleq \quad \max_{1\le i\le n} \pi_i(t), \qquad t \in [0, \infty).$$

Furthermore, let us define a constant $p_{\max}$, in accordance with the notation above, as follows

$$p_{\max} \quad \triangleq \quad \max_{1\le i\le n} p_i \quad = \quad \max_{1\le i\le n} \pi_i(t) \quad = \quad \pi_{\max}(t) \quad < \quad 1.$$

Since the $\mathcal{M}$ is **nondegenerate**, (2.4.56) of Lemma 2.4.14 and (2.4.84) implies that there exists an $\varepsilon > 0$ such that

$$\gamma^*_\pi(t) \quad \ge \quad \frac{\varepsilon}{2}\big(1 - \pi_{\max}(t)\big), \qquad t \in [0, \infty), \qquad \text{a.s.},$$

thus, following the adopted notation, we have

$$\gamma^*_\pi(t) \quad \ge \quad \frac{\varepsilon}{2}\big(1 - p_{\max}\big), \qquad t \in [0, \infty), \qquad \text{a.s.}$$

Therefore, the time-average of the excess growth rate is given by

$$\frac{1}{T}\int_0^T \gamma^*_\pi(t)\,dt \quad \ge \quad \frac{1}{T}\int_0^T \frac{\varepsilon}{2}\big(1 - p_{\max}\big)\,dt \quad = \quad \frac{\varepsilon}{2}\big(1 - p_{\max}\big), \qquad T \in [0, \infty), \qquad \text{a.s.} \tag{3.4.36}$$

So, the long-term average is given by[3]

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T \gamma^*_\pi(t)\,dt \quad \ge \quad \liminf_{T\to\infty} \frac{\varepsilon}{2}\big(1 - p_{\max}\big) \quad = \quad \frac{\varepsilon}{2}\big(1 - p_{\max}\big). \tag{3.4.37}$$

Now, recall (2.12.63) of Proposition 2.12.8, we obtain from (3.4.34)

$$d\log\Big(Z_{w,\pi}(t)/Z_{w,\mu}(t)\Big) \quad = \quad \sum_{i=1}^n p_i\, d\log\mu_i(t) + \gamma^*_\pi(t)\,dt, \qquad t \in [0, \infty), \qquad \text{a.s.}$$

From this we arrive at the following

$$\begin{aligned}
\log\Big(Z_{w,\pi}(T)/Z_{w,\mu}(T)\Big) - \log\Big(Z_{w,\pi}(0)/Z_{w,\mu}(0)\Big) \quad &= \quad \sum_{i=1}^n p_i\Big[\log\mu_i(T) - \log\mu_i(0)\Big] + \int_0^T \gamma^*_\pi(t)\,dt \\
&= \quad \sum_{i=1}^n p_i\Big[\log\mu_i(T) - \log\Big(X_i(0)/Z_{w,\mu}(0)\Big)\Big] + \int_0^T \gamma^*_\pi(t)\,dt.
\end{aligned}$$

[3]Consider any two functions $f(t)$ and $g(t)$, and suppose that $f(t) \ge g(t)$ for all $t \in [0, \infty)$, then $\limsup_{t\to\infty} f(t) \ge \limsup_{t\to\infty} g(t)$ and $\liminf_{t\to\infty} f(t) \ge \liminf_{t\to\infty} g(t)$.

Hence,

$$\begin{aligned}\log\Big(Z_{w,\pi}(T)/Z_{w,\mu}(T)\Big) - \log\Big(Z_{w,\pi}(0)/Z_{w,\mu}(0)\Big) - \int_0^T \gamma_\pi^*(t)\,dt &= \sum_{i=1}^n p_i\Big[\log\mu_i(T) - \log\Big(X_i(0)/Z_{w,\mu}(0)\Big)\Big] \\ &= \sum_{i=1}^n p_i\Big[\log\mu_i(T) - \log\big(x_i/w\big)\Big].\end{aligned}$$

Recall the assumptions listed at the inception of Chapter 2, which mentioned that each company has a single share of stock outstanding. In accordance with this assumption, the stock price at time t, $X_i(t)$, for $i = 1, \ldots, n$ also represents the total capitalisation of the company at time t. Thus, the initial value of the ith stock is also the initial value of the total capitalisation of the company. With this in mind, and taking the long-term average we obtain

$$\begin{aligned}&\lim_{T\to\infty}\frac{1}{T}\left(\log\Big(Z_{w,\pi}(T)/Z_{w,\mu}(T)\Big) - \log\Big(Z_{w,\pi}(0)/Z_{w,\mu}(0)\Big) - \int_0^T \gamma_\pi^*(t)\,dt\right) \\ &\qquad = \lim_{T\to\infty}\frac{1}{T}\left(\sum_{i=1}^n p_i\log\mu_i(T) - \sum_{i=1}^n p_i\log\Big(X_i(0)/Z_{w,\mu}(0)\Big)\right) \\ &\qquad = \lim_{T\to\infty}\frac{1}{T}\left(\sum_{i=1}^n p_i\log\mu_i(T) - \sum_{i=1}^n p_i\log\big(x_i/w\big)\right).\end{aligned}$$

Since p_i, x_i and w are all fixed positive constants, i.e., $p_i \in \mathbb{R}^+$, $x_i \in \mathbb{R}^+$ and $w \in \mathbb{R}^+$, let $K \in \mathbb{R}$ and $M \in \mathbb{R}$ be fixed constants defined as follows

$$K \triangleq \log\Big(Z_{w,\pi}(0)/Z_{w,\mu}(0)\Big) = 0, \tag{3.4.38}$$

$$M \triangleq \sum_{i=1}^n p_i\log\Big(X_i(0)/Z_{w,\mu}(0)\Big) = \sum_{i=1}^n p_i\log\big(x_i/w\big). \tag{3.4.39}$$

Then, we have

$$\lim_{T\to\infty}\frac{1}{T}\left(\log\Big(Z_{w,\pi}(T)/Z_{w,\mu}(T)\Big) - K - \int_0^T \gamma_\pi^*(t)\,dt\right) = \lim_{T\to\infty}\frac{1}{T}\left(\sum_{i=1}^n p_i\log\mu_i(T) - M\right).$$

Thus,

$$\lim_{T\to\infty}\frac{1}{T}\left(\log\Big(Z_{w,\pi}(T)/Z_{w,\mu}(T)\Big) - \int_0^T \gamma_\pi^*(t)\,dt\right) - \lim_{T\to\infty}\frac{1}{T}K = \lim_{T\to\infty}\frac{1}{T}\left(\sum_{i=1}^n p_i\log\mu_i(T)\right) - \lim_{T\to\infty}\frac{1}{T}M.$$

Note that since K and M are constants, we have

$$\lim_{T\to\infty}\frac{1}{T}K = 0, \qquad \text{a.s.}, \tag{3.4.40}$$

$$\lim_{T\to\infty}\frac{1}{T}M = 0, \qquad \text{a.s.} \tag{3.4.41}$$

Hence, we infer the following

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\left(\log\Big(Z_{w,\pi}(T)/Z_{w,\mu}(T)\Big) - \int_0^T \gamma_\pi^*(t)\,dt\right) &= \lim_{T\to\infty}\frac{1}{T}\left(\sum_{i=1}^n p_i\log\mu_i(T)\right) \\ &= \sum_{i=1}^n p_i\left(\lim_{T\to\infty}\frac{1}{T}\log\mu_i(T)\right) \\ &= 0,\end{aligned}$$

since the $\mathcal{M}$ is **coherent** and thus condition (3.4.1) holds. Moreover, we assert that

$$\lim_{T\to\infty}\frac{1}{T}\log\Big(Z_{w,\pi}(T)/Z_{w,\mu}(T)\Big) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \gamma_\pi^*(t)\,dt.$$

We thus ascertain from the result that ensued from the nondegeneracy condition, (3.4.37), that

$$\liminf_{T\to\infty} \frac{1}{T} \log\Big(Z_{w,\pi}(T)/Z_{w,\mu}(T)\Big) \quad = \quad \liminf_{T\to\infty} \frac{1}{T}\int_0^T \gamma_\pi^*(t)\,dt \quad \geq \quad \frac{\varepsilon}{2}\big(1-p_{\max}\big) \quad > \quad 0,$$

since $\varepsilon > 0$ and all the weights of the portfolio π are assumed to be nonnegative, so that $0 \leq p_i < 1$ for all $i = 1, \ldots, n$. Subsequently, we obtain the required result

$$\liminf_{T\to\infty} \frac{1}{T} \log\Big(Z_\pi(T)/Z_\mu(T)\Big) \quad > \quad 0, \qquad \text{a.s.}$$

■

The preceding proposition is quite enlightening since it relates the notion of relative arbitrage in the market. In particular, Proposition 3.4.8, explores the consequences that a coherent and nondegenerate market has on the performance of the constant-weighted portfolio relative to the market portfolio. Specifically, the proposition states that in a market that exhibits nondegeneracy and coherence, constant-weighted portfolios of more than a single stock will asymptotically outperform the market a.s. The notion of relative arbitrage in the market, under certain market conditions, is a unique and novel feature of stochastic portfolio theory. We shall build on this notion, by studying those specifications that allow us to generate relative arbitrage in the market. This will equip us with the necessary tools to investigate the relative performance of various portfolios. A case in point, is the concept of diversity in the market, a discussion of which will be deferred to the next chapter. Relative arbitrage will be examined closely in Chapter 7.

3.5 The Long-Term Behaviour of Dividend-Paying Stocks and Portfolios

Proposition 3.5.1. *For any portfolio π in $\mathcal{M}$,*

$$\lim_{T\to\infty} \frac{1}{T}\left(\log \widehat{Z}_\pi(T) - \int_0^T \vartheta_\pi(t)\,dt\right) \quad = \quad 0, \qquad a.s. \tag{3.5.1}$$

Proof. Recall from Proposition 3.2.2, equation (3.2.19), the analogous result for the case without dividends

$$\lim_{T\to\infty} \frac{1}{T}\left(\log Z_{w,\pi}(T) - \int_0^T \gamma_\pi(t)\,dt\right) \quad = \quad 0, \qquad \text{a.s.} \tag{3.5.2}$$

Thus, by making use of (2.2.156) and (2.2.161) in Definition 2.2.38, we obtain

$$\log Z_{w,\pi}(T) - \int_0^T \gamma_\pi(t)\,dt \quad \equiv \quad \log Z_{w,\pi}(T) - \int_0^T \big(\vartheta_\pi(t) - \delta_\pi(t)\big)\,dt, \qquad \text{a.s.,} \tag{3.5.3}$$

$$= \quad \log Z_{w,\pi}(T) + \int_0^T \delta_\pi(t)\,dt - \int_0^T \vartheta_\pi(t)\,dt \tag{3.5.4}$$

$$= \quad \log \widehat{Z}_{w,\pi}(T) - \int_0^T \vartheta_\pi(t)\,dt. \tag{3.5.5}$$

Therefore, we have

$$\lim_{T\to\infty} \frac{1}{T}\left(\log Z_{w,\pi}(T) - \int_0^T \gamma_\pi(t)\,dt\right) \quad \equiv \quad \lim_{T\to\infty} \frac{1}{T}\left(\log \widehat{Z}_{w,\pi}(T) - \int_0^T \vartheta_\pi(t)\,dt\right), \tag{3.5.6}$$

which together with (3.5.2) gives us the required result,

$$\lim_{T\to\infty} \frac{1}{T}\left(\log \widehat{Z}_{w,\pi}(T) - \int_0^T \vartheta_\pi(t)\,dt\right) \quad = \quad 0, \qquad \text{a.s.} \tag{3.5.7}$$

Alternatively, integrating (2.2.162) directly yields

$$\begin{aligned}
\log \widehat{Z}_{w,\pi}(T) - \log \widehat{Z}_{w,\pi}(0) &= \log\left(\widehat{Z}_{w,\pi}(T)/\widehat{Z}_{w,\pi}(0)\right) && (3.5.8)\\
&= \log\left(\widehat{Z}_{w,\pi}(T)/w\right) && (3.5.9)\\
&= \int_0^T \vartheta_\pi(t)\,dt + \int_0^T \sum_{i,\nu=1}^n \pi_i(t)\xi_{i\nu}(t)\,dW_\nu(t). && (3.5.10)
\end{aligned}$$

Let us define the process $V = \{V(t), \mathcal{F}_t, t \in [0,\infty)\}$, much in the same manner as was done in the proof of Proposition 3.2.2

$$\begin{aligned}
V(T) &\triangleq \log\left(\widehat{Z}_{w,\pi}(T)/\widehat{Z}_{w,\pi}(0)\right) - \int_0^T \vartheta_\pi(t)\,dt, \qquad T \in [0,\infty), && (3.5.11)\\
&= \log\left(\widehat{Z}_{w,\pi}(T)/w\right) - \int_0^T \vartheta_\pi(t)\,dt && (3.5.12)\\
&= \int_0^T \sum_{i,\nu=1}^n \pi_i(t)\xi_{i\nu}(t)\,dW_\nu(t). && (3.5.13)
\end{aligned}$$

Following the same steps as in the proof of Proposition 3.2.2, we obtain

$$\lim_{T\to\infty} \frac{1}{T} V(T) = 0, \qquad \text{a.s.}, \tag{3.5.14}$$

therefore, we have a.s.,

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{T}\left(\log\left(\widehat{Z}_{w,\pi}(T)/\widehat{Z}_{w,\pi}(0)\right) - \int_0^T \vartheta_\pi(t)\,dt\right) &= \lim_{T\to\infty} \frac{1}{T}\left(\log\left(\widehat{Z}_{w,\pi}(T)/w\right) - \int_0^T \vartheta_\pi(t)\,dt\right) = 0,\\
\lim_{T\to\infty} \frac{1}{T}\left(\log \widehat{Z}_{w,\pi}(T) - \log \widehat{Z}_{w,\pi}(0) - \int_0^T \vartheta_\pi(t)\,dt\right) &= \lim_{T\to\infty} \frac{1}{T}\left(\log \widehat{Z}_{w,\pi}(T) - \log w - \int_0^T \vartheta_\pi(t)\,dt\right) = 0,
\end{aligned}$$

and by noticing that $\widehat{Z}_{w,\pi}(0) = Z_{w,\pi}(0) = w > 0$, $w \in \mathbb{R}$, results in

$$\lim_{T\to\infty} \frac{1}{T} \log \widehat{Z}_{w,\pi}(0) = \lim_{T\to\infty} \frac{1}{T} \log w = 0, \qquad \text{a.s.},$$

we subsequently obtain

$$\lim_{T\to\infty} \frac{1}{T}\left(\log \widehat{Z}_{w,\pi}(T) - \int_0^T \vartheta_\pi(t)\,dt\right) = 0,$$

concluding the proof. ■

This reveals that the long-term performance of a portfolio π in a market which incorporates dividends is determined by its augmented or total portfolio growth rate. This was to be expected, since it is in analogy with the result recovered for a market without dividends.

Corollary 3.5.2. *Let $\mathcal{M}$ be a market comprising n stocks, $X_1, \ldots, X_n$, with respective augmented growth rates $\vartheta_1, \ldots, \vartheta_n$. Then for the ith stock, X_i, with augmented growth rate ϑ_i, we have for $i = 1, 2, \ldots, n$,*

$$\lim_{T\to\infty} \frac{1}{T}\left(\log \widehat{X}_i(T) - \int_0^T \vartheta_i(t)\,dt\right) = 0, \qquad \textit{a.s.} \tag{3.5.15}$$

Proof. Recall from Corollary 3.2.3, equation (3.2.42) the analogous result without dividends

$$\lim_{T\to\infty} \frac{1}{T}\left(\log X_i(T) - \int_0^T \gamma_i(t)\,dt\right) = 0, \qquad \text{a.s.} \tag{3.5.16}$$

Employing equation (2.2.152) of Definition 2.2.35 and (2.2.150), yields the following

$$\log X_i(T) - \int_0^T \gamma_i(t)\,dt \equiv \log X_i(T) - \int_0^T \big(\vartheta_i(t) - \delta_i(t)\big)\,dt, \qquad \text{a.s.,} \tag{3.5.17}$$

$$= \log X_i(T) + \int_0^T \delta_i(t)\,dt - \int_0^T \vartheta_i(t)\,dt \tag{3.5.18}$$

$$= \log \widehat{X}_i(T) - \int_0^T \vartheta_i(t)\,dt. \tag{3.5.19}$$

Therefore, we have

$$\lim_{T\to\infty} \frac{1}{T}\left(\log X_i(T) - \int_0^T \gamma_i(t)\,dt\right) = \lim_{T\to\infty} \frac{1}{T}\left(\log \widehat{X}_i(T) - \int_0^T \vartheta_i(t)\,dt\right). \tag{3.5.20}$$

Thus, using (3.5.16), we get the result

$$\lim_{T\to\infty} \frac{1}{T}\left(\log \widehat{X}_i(T) - \int_0^T \vartheta_i(t)\,dt\right) = 0, \qquad \text{a.s.} \tag{3.5.21}$$

■

3.5.1 Coherence

Recall the coherence market condition given by (3.4.1) and subsequently by (3.4.3). For the market in which the stocks pay dividends, we can reformulate the condition of coherence on the market. Thus, by making use of (2.2.150) and (2.2.156) for the market portfolio, we have for $i = 1, \ldots, n$,

$$\lim_{t\to\infty} \frac{1}{t}\left(\log \widehat{X}_i(t) - \log \widehat{Z}_\mu(t) - \int_0^t \big(\delta_i(s) - \delta_\mu(s)\big)\,ds\right) = 0, \qquad \text{a.s.,} \tag{3.5.22}$$

or, equivalently, for $T \in [0, \infty)$, for each $i = 1, 2, \ldots, n$, we have

$$\lim_{T\to\infty} \frac{1}{T}\left(\log \widehat{X}_i(T) - \log \widehat{Z}_\mu(T) - \int_0^T \big(\delta_i(t) - \delta_\mu(t)\big)\,dt\right) = 0, \qquad \text{a.s.} \tag{3.5.23}$$

Therefore, a market $\mathcal{M}$ is coherent if (3.5.22) or (3.5.23) is satisfied. By Corollary 3.5.2 and Proposition 3.5.1, which for the market portfolio is given by,

$$\lim_{T\to\infty} \frac{1}{T}\left(\log \widehat{Z}_\mu(T) - \int_0^T \vartheta_\mu(t)\,dt\right) = 0, \qquad \text{a.s.,} \tag{3.5.24}$$

we have for $i = 1, \ldots, n$,

$$\lim_{T\to\infty} \frac{1}{T}\left(\int_0^T \big(\vartheta_i(t) - \vartheta_\mu(t)\big)\,dt - \int_0^T \big(\delta_i(t) - \delta_\mu(t)\big)\,dt\right) = 0, \qquad \text{a.s.,} \tag{3.5.25}$$

$$\lim_{T\to\infty} \frac{1}{T}\left(\int_0^T \big(\vartheta_i(t) - \delta_i(t)\big)\,dt - \int_0^T \big(\vartheta_\mu(t) - \delta_\mu(t)\big)\,dt\right) = 0, \qquad \text{a.s.,} \tag{3.5.26}$$

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \big(\gamma_i(t) - \gamma_\mu(t)\big)\,dt = 0, \qquad \text{a.s.,} \tag{3.5.27}$$

which is precisely condition (ii) of the equivalent statements provided in Proposition 3.4.2. Thus, for a market which pays dividends, the same coherence equivalence conditions (conditions (ii) and (iii) of Proposition 3.4.2) apply. We can simply restate it as (3.5.26) where the inclusion of the dividend rate process is clearly visible.

3.5.2 Market with Equal Augmented Growth Rates

Recall that if all the stocks in the market have the same growth rate process, then by Corollary 3.4.3, we have

$$\lim_{T\to\infty} \frac{1}{T}\left(\log Z_\mu(T) - \int_0^T \gamma(t)\,dt\right) = 0, \qquad \text{a.s.}$$

Thus, all the stocks in the market exhibit the same potential for capital growth. Then, the expression above, suggests that the market portfolio, specified by $Z_\mu(t) = \sum_{i=1}^n X_i(t)$, yields the same growth as all the stocks in the market and consequently possesses the same growth potential as the other stocks. Now, if we consider dividend-paying stocks that have the same augmented growth rate process then again all the stocks have the same potential for capital growth. So, in a similar, intuitive fashion, $\widetilde{Z}(t) = \sum_{i=1}^n \widehat{X}_i(t)$, grows at the same rate and has the same growth potential as all the stocks in the market. Therefore, for the equal-augmented-growth-rate market we have the following similar result

$$\lim_{T\to\infty} \frac{1}{T}\left(\log \widetilde{Z}(T) - \int_0^T \vartheta(t)\,dt\right) = 0, \qquad \text{a.s.}$$

We formulate this result more precisely in the following proposition as well as providing a formal proof.

Proposition 3.5.3. *Suppose that all the stocks in the market $\mathcal{M}$ have nonnegative dividend rates and the* ***same augmented growth rate process****. Then for any portfolio in $\mathcal{M}$, we have*

$$\lim_{T\to\infty} \frac{1}{T}\left(\log \widetilde{Z}(T) - \int_0^T \vartheta(t)\,dt\right) = 0, \qquad \textit{a.s.}, \tag{3.5.28}$$

where $\widetilde{Z}$ is the alternative total return process defined in Definition 2.13.1 *as $\widetilde{Z}(t) = \widehat{X}_1(t) + \cdots + \widehat{X}_n(t)$.*

Proof. An equal-augmented-growth-rate market implies that $\vartheta_i(t) = \vartheta(t)$ for all $i = 1, 2, \ldots, n$, and for all $t \in [0, \infty)$. Thus (3.5.15) of Corollary 3.5.2 becomes

$$\lim_{T\to\infty} \frac{1}{T}\left(\log \widehat{X}_i(T) - \int_0^T \vartheta(t)\,dt\right) = 0, \qquad \text{a.s.} \tag{3.5.29}$$

We shall now follow the same steps as was employed in the proof of Corollary 3.4.3, but now taking dividends into account, to arrive at the result. We start off by noticing that if (3.5.29) goes to 0 for all stocks, it will obviously, also do so for the maximum over $i = 1, \ldots, n$. Thus, for the largest stock in the market we have

$$\lim_{T\to\infty} \frac{1}{T}\left(\max_{1\le i\le n}\left(\log \widehat{X}_i(T)\right) - \int_0^T \vartheta(t)\,dt\right) = 0, \qquad \text{a.s.} \tag{3.5.30}$$

Analogously, for the smallest stock in the market we have

$$\lim_{T\to\infty} \frac{1}{T}\left(\min_{1\le i\le n}\left(\log \widehat{X}_i(T)\right) - \int_0^T \vartheta(t)\,dt\right) = 0, \qquad \text{a.s.} \tag{3.5.31}$$

Using the strictly increasing property of the logarithmic function, we obtain the following equivalent expressions to those above

$$\lim_{T\to\infty} \frac{1}{T}\left(\log\left(\max_{1\le i\le n} \widehat{X}_i(T)\right) - \int_0^T \vartheta(t)\,dt\right) = 0, \qquad \text{a.s.}, \tag{3.5.32}$$

$$\lim_{T\to\infty} \frac{1}{T}\left(\log\left(\min_{1\le i\le n} \widehat{X}_i(T)\right) - \int_0^T \vartheta(t)\,dt\right) = 0, \qquad \text{a.s.} \tag{3.5.33}$$

Recall that $X_i(t)$ and $\delta_i(t)$ are nonnegative for all $t \in [0, \infty)$ and $i = 1, \ldots, n$, therefore $\widehat{X}_i(t)$ is nonnegative for all $i = 1, \ldots, n$, $t \in [0, \infty)$. Notice that for all $k = 1, \ldots, n$, we have

$$\min_{1\le i\le n} \widehat{X}_i(t) \le \widehat{X}_k(t) \le \max_{1\le i\le n} \widehat{X}_i(t). \tag{3.5.34}$$

By taking the summation, we get

$$\sum_{k=1}^{n} \min_{1\le i\le n} \widehat{X}_i(t) \quad \le \quad \sum_{k=1}^{n} \widehat{X}_k(t) \quad \le \quad \sum_{k=1}^{n} \max_{1\le i\le n} \widehat{X}_i(t), \tag{3.5.35}$$

$$n \min_{1\le i\le n} \widehat{X}_i(t) \quad \le \quad \widehat{X}_1(t) + \widehat{X}_2(t) + \cdots + \widehat{X}_n(t) \quad \le \quad n \max_{1\le i\le n} \widehat{X}_i(t). \tag{3.5.36}$$

From Definition 2.13.1 we have $\widetilde{Z}(t) = \widehat{X}_1(t) + \widehat{X}_n(t) + \cdots + \widehat{X}_n(t)$, which we use to obtain[4]

$$n \min_{1\le i\le n} \widehat{X}_i(t) \quad \le \quad \widetilde{Z}(t) \quad \le \quad n \max_{1\le i\le n} \widehat{X}_i(t). \tag{3.5.37}$$

The strictly increasing nature of the logarithmic function is once again employed to obtain

$$\log n + \log\left(\min_{1\le i\le n} \widehat{X}_i(t)\right) \quad \le \quad \log \widetilde{Z}(t) \quad \le \quad \log n + \log\left(\max_{1\le i\le n} \widehat{X}_i(t)\right). \tag{3.5.38}$$

Subtracting $\int_0^T \vartheta(t)\,dt$ and taking the long-term average gives

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{T} \log n + \lim_{T\to\infty} \frac{1}{T} &\left(\log\left(\min_{1\le i\le n} \widehat{X}_i(T)\right) - \int_0^T \vartheta(t)\,dt\right) \\
&\le \lim_{T\to\infty} \frac{1}{T}\left(\log \widetilde{Z}(T) - \int_0^T \vartheta(t)\,dt\right) \\
&\le \lim_{T\to\infty} \frac{1}{T} \log n + \lim_{T\to\infty} \frac{1}{T}\left(\log\left(\max_{1\le i\le n} \widehat{X}_i(T)\right) - \int_0^T \vartheta(t)\,dt\right).
\end{aligned}$$

Thus, using (3.5.32) in conjunction with (3.5.33), it can be shown that

$$\lim_{T\to\infty} \frac{1}{T}\left(\log \widetilde{Z}(T) - \int_0^T \vartheta(t)\,dt\right) \quad = \quad 0, \qquad \text{a.s.}, \tag{3.5.39}$$

which concludes the proof. ■

Proposition 3.5.4. *Suppose that all the stocks in the market $\mathcal{M}$ have nonnegative dividend rates and the* ***same augmented growth rate process****, $\vartheta(t)$. Then for the* ***market portfolio****, μ, we have*

$$\lim_{T\to\infty} \frac{1}{T}\left(\log Z_\mu(T) - \int_0^T \vartheta(t)\,dt\right) \quad \le \quad 0, \qquad \textit{a.s.} \tag{3.5.40}$$

Proof. Firstly, recall Definition 2.2.34, equation (2.2.149), the total return process for the ith stock X_i, $\widehat{X}_i = \left\{\widehat{X}_i(t), t \in [0,\infty)\right\}$ for $i = 1, 2, \ldots, n$, which is given by

$$\widehat{X}_i(t) \quad = \quad X_i(t) \exp\left(\int_0^t \delta_i(s)\,ds\right), \qquad t \in [0,\infty). \tag{3.5.41}$$

Furthermore, recall the alternative total return process $\widetilde{Z}$ given by equation (2.13.2), Definition 2.13.1, as

$$\widetilde{Z}(t) \quad = \quad \widehat{X}_1(t) + \cdots + \widehat{X}_n(t), \qquad t \in [0,\infty), \tag{3.5.42}$$

which represents the value of a portfolio in $\mathcal{M}$ with $\widetilde{Z}(0) = Z_\mu(0)$ in which all the dividends of each stock are reinvested in the same stock. Since all the dividend-paying stocks, $\widehat{X}_i$, have the same common augmented growth rate ($\vartheta_i(t) \equiv \vartheta(t)$ for all $i = 1, \ldots, n$), Proposition 3.5.3 establishes that

$$\lim_{T\to\infty} \frac{1}{T}\left(\log \widetilde{Z}(T) - \int_0^T \vartheta(t)\,dt\right) \quad = \quad 0, \qquad \text{a.s.} \tag{3.5.43}$$

[4]Note that $\widetilde{Z}(t) = \widehat{X}_1(t) + \cdots + \widehat{X}_n(t)$, but that $\widehat{Z}_\mu(t) \neq \widehat{X}_1(t) + \cdots + \widehat{X}_n(t)$, since $\widehat{Z}_\mu(t) = Z_\mu(t) \exp\left(\int_0^t \delta_\mu(s)\,ds\right)$ which represents the portfolio value in which dividends are reinvested proportionally across all stocks in the market according to the market weights of the stocks, whereas $\widetilde{Z}(t) = X_1(t) \exp\left(\int_0^t \delta_1(s)\,ds\right) + \cdots + X_n(t) \exp\left(\int_0^t \delta_n(s)\,ds\right)$ which represents the value of a portfolio in which the dividends of each stock are reinvested back into each respective stock.

Alternatively, the same steps as in the proof of Proposition 3.2.2 can be used to establish the same result [see Fernholz (2002)]. Since we only consider nonnegative dividend rates, i.e., $\delta_1(t) \geq 0, \ldots, \delta_n(t) \geq 0$, for all $t \in [0, \infty)$, we gather from (3.5.41) that $X_1(t) \leq \widehat{X}_1(t), X_2(t) \leq \widehat{X}_2(t)$, and so on, a.s., for $t \in [0, \infty)$. Hence, for all $i = 1, \ldots, n$, $X_i(t) \leq \widehat{X}_i(t)$, a.s., for $t \in [0, \infty)$, it thus follows from (2.12.15) and (3.5.42) that

$$\begin{aligned} X_1(t) + \cdots + X_n(t) &\leq \widehat{X}_1(t) + \cdots + \widehat{X}_n(t), \qquad t \in [0, \infty), \qquad \text{a.s.} \\ Z_\mu(t) &\leq \widetilde{Z}(t), \qquad t \in [0, \infty), \qquad \text{a.s.} \end{aligned}$$

Hence, we have

$$\log Z_\mu(t) \leq \log \widetilde{Z}(t), \qquad t \in [0, \infty), \qquad \text{a.s.}$$

Now, consider $T \in [0, \infty)$, let us adjust the above inequality by subtracting $\int_0^T \vartheta(t)\, dt$ from both sides as follows

$$\log Z_\mu(T) - \int_0^T \vartheta(t)\, dt \leq \log \widetilde{Z}(T) - \int_0^T \vartheta(t)\, dt, \qquad T \in [0, \infty), \qquad \text{a.s.}$$

Hence, taking the long-range average and applying (3.5.43), we obtain

$$\lim_{T \to \infty} \frac{1}{T} \left(\log Z_\mu(T) - \int_0^T \vartheta(t)\, dt \right) \leq \lim_{T \to \infty} \frac{1}{T} \left(\log \widetilde{Z}(T) - \int_0^T \vartheta(t)\, dt \right) = 0,$$

we subsequently deduce the desired result

$$\lim_{T \to \infty} \frac{1}{T} \left(\log Z_\mu(T) - \int_0^T \vartheta(t)\, dt \right) \leq 0. \tag{3.5.44}$$

■

From Lemma 2.2.39 and Definition 2.2.40, the augmented growth rate of the market is given by

$$\vartheta_\mu(t) = \sum_{i=1}^n \mu_i(t)\vartheta_i(t) + \gamma_\mu^*(t) \tag{3.5.45}$$

$$= \sum_{i=1}^n \mu_i(t)\vartheta_i(t) + \vartheta_\mu^*(t). \tag{3.5.46}$$

Hence, if all the stocks in the market share a common augmented growth rate, then we have

$$\vartheta_\mu(t) = \vartheta(t) + \gamma_\mu^*(t), \tag{3.5.47}$$

which is the analogue of (3.4.32), for the growth rate of the market, where the stocks have the same growth rate process.

Proposition 3.5.5. *Suppose that all the stocks in the market $\mathcal{M}$ have nonnegative dividend rates and the* ***same augmented growth rate process****, $\vartheta(t)$. Then for the* ***market portfolio****, μ, we have*

$$\lim_{T \to \infty} \frac{1}{T} \left(\log \widehat{Z}_\mu(T) - \int_0^T \vartheta(t)\, dt \right) \geq 0, \qquad \textit{a.s.} \tag{3.5.48}$$

Proof. From Proposition 3.5.1 for the market, we have

$$\lim_{T \to \infty} \frac{1}{T} \left(\log \widehat{Z}_\mu(T) - \int_0^T \vartheta_\mu(t)\, dt \right) = 0, \qquad \text{a.s.} \tag{3.5.49}$$

Now, for an equal-augmented-growth-rate market, using (3.5.47), (3.5.49) a.s. becomes

$$\lim_{T \to \infty} \frac{1}{T} \left(\log \widehat{Z}_\mu(T) - \int_0^T \left(\vartheta(t) + \gamma_\mu^*(t)\right) dt \right) = \lim_{T \to \infty} \frac{1}{T} \left(\log \widehat{Z}_\mu(T) - \int_0^T \vartheta(t)\, dt - \int_0^T \gamma_\mu^*(t)\, dt \right) = 0, \tag{3.5.50}$$

Since $0 < \mu_i(t) < 1$ for all $i = 1, \ldots, n$, $t \in [0, \infty)$, Proposition 2.4.8 implies that for the market we have $\gamma_\mu^*(t) \geq 0$, for all $t \in [0, \infty)$, a.s. Consequently, we have the following inequality

$$\log \widehat{Z}_\mu(T) - \int_0^T \vartheta(t)\, dt \quad \geq \quad \log \widehat{Z}_\mu(T) - \int_0^T \vartheta(t)\, dt - \int_0^T \gamma_\mu^*(t)\, dt. \tag{3.5.51}$$

Therefore, we obtain

$$\lim_{T\to\infty} \frac{1}{T} \left(\log \widehat{Z}_\mu(T) - \int_0^T \vartheta(t)\, dt \right) \quad \geq \quad \lim_{T\to\infty} \frac{1}{T} \left(\log \widehat{Z}_\mu(T) - \int_0^T \vartheta(t)\, dt - \int_0^T \gamma_\mu^*(t)\, dt \right) \quad = \quad 0, \tag{3.5.52}$$

and the result is derived. ■

For the coherence of a market we require condition (iii) of Proposition 3.4.2 to hold. Thus, for $i, j = 1, \ldots, n$, we have a.s.

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \big(\gamma_i(t) - \gamma_j(t)\big)\, dt \quad = \quad \lim_{T\to\infty} \frac{1}{T} \left(\int_0^T \big(\vartheta_i(t) - \vartheta_j(t)\big)\, dt - \int_0^T \big(\delta_i(t) - \delta_j(t)\big)\, dt \right) \quad = \quad 0. \tag{3.5.53}$$

Furthermore, condition (ii) of Proposition 3.4.2 is equivalent to (3.5.25),

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \big(\gamma_i(t) - \gamma_\mu(t)\big)\, dt \quad = \quad \lim_{T\to\infty} \frac{1}{T} \left(\int_0^T \big(\vartheta_i(t) - \vartheta_\mu(t)\big)\, dt - \int_0^T \big(\delta_i(t) - \delta_\mu(t)\big)\, dt \right) \quad = \quad 0. \tag{3.5.54}$$

Now, let's place ourselves in the *equal-augmented-growth-rate market*, $\vartheta_i(t) \equiv \vartheta(t)$, then the above coherence condition (3.5.53) reduces to

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \big(\delta_i(t) - \delta_j(t)\big)\, dt \quad = \quad 0, \qquad \text{a.s.}, \tag{3.5.55}$$

and using (3.5.47), (3.5.54) reduces to

$$\lim_{T\to\infty} \frac{1}{T} \left(\int_0^T -\gamma_\mu^*(t)\, dt - \int_0^T \big(\delta_i(t) - \delta_\mu(t)\big)\, dt \right) \quad = \quad 0, \qquad \text{a.s.} \tag{3.5.56}$$

Thus, we can restate Proposition 3.4.2, for the equal-augmented-growth-rate market as follows.

Proposition 3.5.6. *Let $\mathcal{M}$ denote the market with stocks $X_1, \ldots, X_n$, that have the* ***same augmented growth rate process****. Then the following statements are equivalent:*

(*i*) *The market, $\mathcal{M}$, is coherent;*

(*ii*) *for each $i = 1, \ldots, n$,*

$$\lim_{T\to\infty} \frac{1}{T} \left(\int_0^T -\gamma_\mu^*(t)\, dt - \int_0^T \big(\delta_i(t) - \delta_\mu(t)\big)\, dt \right) \quad = \quad 0, \qquad \textit{a.s.;} \tag{3.5.57}$$

(*iii*) *for each pair $i, j = 1, \ldots, n$,*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \big(\delta_i(t) - \delta_j(t)\big)\, dt \quad = \quad 0, \qquad \textit{a.s.} \tag{3.5.58}$$

Corollary 3.5.7. *Suppose that all the stocks in the market $\mathcal{M}$ have the* ***same augmented growth rate process*** *and the* ***same nonnegative dividend rate process****. Then the market $\mathcal{M}$ is coherent.*

Proof. Thus the conditions, $\vartheta_i(t) \equiv \vartheta(t)$ and $\delta_i(t) \equiv \delta(t)$ for all $i = 1, \ldots, n$, $t \in [0, \infty)$, correspond to the equal-growth-rate market, since for all $i = 1, \ldots, n$, $t \in [0, \infty)$, we have

$$\gamma_i(t) \quad = \quad \vartheta(t) - \delta(t) \quad \triangleq \quad \gamma(t).$$

Thus the result follows from Corollary 3.4.4, since all the stocks in the market $\mathcal{M}$ have the same growth rate process. Alternatively, since the dividend rates for each stock are the same, condition (iii) of Proposition 3.5.6 implies that the market is coherent. Moreover, $\delta_i(t) = \delta_\mu(t) = \delta(t)$ for all $i = 1, \ldots, n$, which is achieved by $\delta_\mu(t) = \sum_{i=1}^n \mu_i(t)\delta(t) = \delta(t)$. Since the conditions imposed amount to the equal-growth-rate market, we also have the result from Proposition 3.4.6. Hence,

$$\lim_{T\to\infty} \left(\int_0^T -\gamma_\mu^*(t)\, dt - \int_0^T \big(\delta_i(t) - \delta_\mu(t)\big)\, dt \right) \quad = \quad \lim_{T\to\infty} \left(\int_0^T -\gamma_\mu^*(t)\, dt - \int_0^T \big(\delta(t) - \delta(t)\big)\, dt \right) \tag{3.5.59}$$

$$= \quad -\lim_{T\to\infty} \int_0^T \gamma_\mu^*(t)\, dt \tag{3.5.60}$$

$$= \quad 0. \tag{3.5.61}$$

Thus, by verifying condition (ii), (3.5.57), of Proposition 3.5.6, we show coherence and fortify the result. ■

Thus, an equal-augmented-growth-rate market is coherent when the dividends for each stock are equal. However, we cannot draw any conclusive results regarding the coherence of a market in which all stocks have the same augmented growth rate. Thus, we need to impose an additional requirement on the market, that of "equal dividends".

Proposition 3.5.8. *Suppose that all the stocks in the market $\mathcal{M}$ have the same augmented growth rate and the same nonnegative dividend rate. Then*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \gamma_\mu^*(t)\, dt \quad = \quad 0, \qquad a.s. \tag{3.5.62}$$

Proof. Again, this is just the equal-growth-rate market, thus in accordance with Proposition 3.4.6 we have the required result. Alternatively, if all the stocks in the market share a common dividend rate, $\delta_i(t) = \delta(t)$ for all $i = 1, \ldots, n$, $t \in [0, \infty)$, then it is obvious that the market-weighted average will also be equal to this common rate,

$$\delta_\mu(t) \quad = \quad \sum_{i=1}^n \mu_i(t)\delta_i(t) \quad = \quad \delta(t).$$

Thus, $\delta_i(t) = \delta_\mu(t) = \delta(t)$, for all $i = 1, \ldots, n$. Moreover, Corollary 3.5.7 states that the market $\mathcal{M}$ is coherent, thus by (ii) of the equivalence conditions of Proposition 3.5.6, we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T -\gamma_\mu^*(t)\, dt \quad = \quad 0. \tag{3.5.63}$$

■

Remark 3.5.9. It should be clear that an *equal-growth-rate market* need not necessarily imply an *equal-augmented-growth-rate market*, and vice versa. If growth rates are equal, then $\vartheta_i(t) = \gamma(t) + \delta_i(t)$ for all $i = 1, \ldots, n$. Therefore, the augmented growth rates are not necessarily equal. Conversely, if the augmented growth rates are equal, then $\gamma_i(t) = \vartheta(t) - \delta_i(t)$ for all $i = 1, \ldots, n$, and again the growth rates are not necessarily equal to each other. In fact, one such scenario in which this equality does indeed occur, is when each stock exhibits the same dividend rate and so the difference between the growth rate and the augmented growth rate is accounted for by dividends. Thus, Proposition 3.5.8 is somewhat misleading, in that the result (3.5.62) occurs purely for an equal-growth-rate market. Of course, the result also holds for an equal-augmented-growth-rate market, but in which all stocks share the same dividend rate.

3.6 Summary and Conclusion

In this chapter we introduced the concept of long-term stock, portfolio and market behaviour. Thus far, we have established that the growth rate plays the lead role in the analysis of long-term stock and portfolio performance. Recall that it will be of interest to evaluate the performance of portfolios relative to some benchmark portfolio, in that regard a discussion of long-term relative behaviour was provided. In particular, we discovered that within a market satisfying the nondegeneracy and coherence conditions, the constant-weighted portfolio dominates the canonical benchmark portfolio. This is one such example of relative arbitrage, which we touched on in this chapter and which we shall further develop in coming chapters. In the next chapter, we shall introduce another market condition, namely that of diversity.

Chapter 4

Stock Market Diversity of Capital Distributions

4.1 Introduction

The concept of stock market diversity is the inspiration of E. Robert Fernholz, and was developed through research conducted at INTECH during the first few years of the origination of stochastic portfolio theory. The fundamental notion of stochastic portfolio theory was germinated in Fernholz & Shay (1982), but it was in Fernholz (1999a) that the notion of stock market diversity, as we currently think of it, was conceptualised and formulated. However, Sharpe's work [Sharpe (1964)] on market equilibrium in conjunction with Markowitz's work [Markowitz (1952)] on portfolio theory, established that the diversification of portfolios, commonly employed by risk averse investors to alleviate the effect of volatility, will lead to a *diverse equity market*. Consequently, it is of no surprise that diversity of the distribution of capital is a prominent feature of equity markets [Fernholz (1999a)].

In this chapter we shall study the diversity of the distribution of capital in an equity market composed of stocks.[1] Heuristically speaking, a market is said to be "diverse" if the capital is spread among a reasonably large number of stocks or in a similar sense if the market is one that avoids concentration of all its capital into a few large stocks (possibly, a single stock). Consequently, to assist in its formulation, the notion of diversity for a financial equity market essentially postulates that no single stock should ever be allowed to dominate the entire market in terms of relative capitalisation. This assertion is motivated by the *antitrust law* or more commonly known as *competition law*, in fact, diversity could be considered a consequence of antitrust law. Antitrust law is universal in modern economics, because at least since Smith (1776), it has been generally accepted that excessive concentration of either production or capital is likely to interfere with competition, and be detrimental to a national economy [Fernholz (2002)]. The notion of antitrust law arose out of the need to combat the development of trusts and monopolies and hence any anticompetitive practices, that were dominating the free market economy.[2] The genesis of such monopolies, presented those within the monopoly, with the power and control over the economy. The antitrust laws were thus intended to prohibit the use of power to control the marketplace. Indeed, economists believe that the possession of such control injures both individuals and the public because it leads to anticompetitive practices in an effort to obtain or maintain total control, which in turn, is a hindrance to the competitive structure of the economy. This ultimately results in the stagnation of markets and the depression of economic growth. Consequently, antitrust laws were introduced to reduce market domination by individual corporations, and thus prevent the excessive concentration of capital and economic power in the hands of a few large corporations, as well as prohibiting anticompetitive conduct and preserving

[1] Diversity is a concept that is meaningful for equity markets, but probably not for more general asset classes. Nevertheless, some of the results in this chapter may be relevant for passive portfolios comprising more general types of assets [Fernholz & Karatzas (2009)].

[2] In the late 1800s in America.

the quintessential nature of competitive markets. These laws were referred to as the Sherman Antitrust Act, which forms the basis for most antitrust legislation and remains the core of antitrust policy. In fact, since the implementation of this act, all economically developed nations have adopted some form of antitrust legislation, that parallel the Sherman Antitrust Act. Within the context of developing nations, it is uncertain whether some form (mild or otherwise) of competition or antitrust policy is a reasonable consideration. In these developing, particularly low-income countries, the markets are usually very small and fragmented so that developing scale sufficient to raise competitiveness and engage in international markets is a major challenge. There, however, exists a larger problem, namely that of poor governance. In societies with widespread corruption, inadequate public finances, and weak judiciary and oversight institutions, competition policy may become another tool for capture by vested interests, becoming in itself a barrier to entry. A discussion of the nonexistence of antitrust law within developing nations is noteworthy since it relates to the discussion of diversity within such nations. The concept of diversity is in some manner implied by the existence of antitrust law, thus without its enforcement, as is the case in developing nations, the diversity requirement is not necessarily satisfied. We won't elaborate on this issue any further, since we are more concerned with the effect that antitrust legislation may have on the distribution of capital in the equity market, rather than with the economic rationale for such legislation. For any further information on antitrust laws, the reader should refer to the substantial amount of available literature. We won't consider this any further, beyond noting an earlier comment that any credible antitrust law should prevent prolonged concentration of practically all the market capital into a single company [Fernholz (2002)], and consequently that the imposition and enactment of such a law implies the diversity of these markets.

This chapter shall proceed with Section 4.2, in which we explore the notion of diversity in equity markets. Diversity of a stock market is a measure of how uniformly the capital is apportioned among the stocks in the market. This is tantamount to saying that diversity is a measure of the level of the concentration of capital into each stock in the market. With this in mind, a market is diverse if it avoids the extreme concentration of capital into single stocks and if at no time the largest stock in the market accounts for almost the entire market capitalisation. Within the context of stochastic portfolio theory, the concept of diversity in equity markets is of integral consideration. We shall proffer a formal definition of the notion of diversity in financial equity markets, together with the formal definitions of the allied, successively weaker notions of weak diversity, uniform weak diversity and asymptotic weak diversity. This section will also contain some consequences of stock market diversity. In this regard, we shall show that diversity can be characterised in terms of the excess growth rate of the market portfolio (the market excess growth rate). Thus, there exists a relationship between the excess growth rate of the market and diversity. This relationship amounts to imposing a condition on the excess growth rate of the market in order for the equity market to be diverse. This will then allow us to take a look at diversity (or rather the lack thereof) in both the equal-growth-rate equity market and the constant-growth-rate equity market. In Section 4.3, we shall delve into the idea of maintaining stock market diversity in equity markets by showing that dividends can be employed as a means to maintain stock market diversity. Section 4.4 investigates the measurement of diversity in equity markets, as it is crucial that we not only understand the concept of diversity in equity markets but also be able to measure it in some manner. Recall that diversity is seen as a measure of how the capital is distributed among all the stocks in an equity market, we thus require some means by which to effectively measure this quantity. The measurement tool adopted by Fernholz for this purpose, are the aptly named measures of diversity. Measures of diversity are of interest primarily for two reasons: diversity is an observable characteristic of equity markets that is amenable to stochastic analysis, and measures of diversity can be used to construct portfolios with desirable investment characteristics. Quantifying the effect of stock market diversity shall be discussed in Section 4.5. It is here that we shall review the means by which we are able to quantify the degree of stock market diversity of an equity market, i.e. by which the degree of stock market diversity of an equity market can be measured quantitatively. To achieve this, we first need to understand the concept of the distribution of equity capital of the market as well as the related capital distribution curve. The distribution of equity capital in the market, also the spread of equity capital in the market, is just a representation of the allocation of the available capital in the equity market to each of the stocks in the equity market. To enable us to acquire a greater grasp of the structure of the equity capital distribution, we shall formally define the equity capital distribution and the capital distribution curve in terms of the market weights. This essentially entails positioning the ranking of these market weights in decreasing order from the largest market weight through to the smallest market weight. The capital distribution curve of

the market is then the graph representation of the family of the ranked market weights. The capital distribution curve graph most commonly considered and produced in the literature is the log-log plot of the market weights arranged in descending rank, i.e., the logarithms of the market weights versus the logarithms of their respective ranks. Thus, the capital distribution curve graph, that is obtained by plotting ordered logarithmic relative capitalisations against the logarithms of their respective ranks, is a decreasing curve. The reason that we shall consider the capital distribution curve graph on the log-log scale is that it brings to the fore the most useful and easily interpretive information than what is revealed with a plot on a normal standard scale. Next, we shall characterise stock market diversity in terms of the distribution of equity capital, and consequently, in terms of the capital distribution curve. So that changes in stock market diversity occur in conjunction with changes in the distribution of equity capital. Since the movement of capital between the larger and the smaller stocks in an equity market impacts on the distribution of equity capital, this shall also have an effect on the degree of stock market diversity and cause it to vary over time. Thus, we shall consider these changes in stock market diversity. These changes can be attributed to the fact that when capital flows from the smaller stocks into the larger stocks, the resulting equity capital distribution is less concentrated into the smaller stocks with most of the capital accumulated into just a few of the largest stocks. As a consequence, the level of diversity in the market declines, since most of the capital has shifted into larger stocks which causes the equity capital to have an uneven distribution. Whereas, when the capital ebbs back into the smaller stocks which occurs by injecting and redistributing capital from the larger stocks into and among the smaller stocks, the capital that the largest stocks had amassed is now more evenly distributed among all the stocks in the equity market. Thus, as the concentration of capital reduces, the level of diversity in the equity market surges. Therefore, diversity of the distribution is lower when capital is concentrated mostly into a few large stocks and is higher when capital is more uniformly spread across all the stocks in the equity market. Evidently, minimum diversity would occur if the capital in the market were entirely invested in a single (largest) stock, all the capital from the remaining smaller stocks in the market aggregates into the largest stock. Moreover, maximum diversity would be attained if all the stocks had the exact same capitalisation, so capital flows from the larger stocks into the smaller stocks in a uniform fashion. We shall also supply a definition of functions that are measures of diversity in this section, which initially requires us to establish and formulate the requisite fundamental properties that a sufficient measure of diversity should possess, namely the symmetry property and the concavity property. Whereas, the definition of diversity provides a criterion for determining whether or not a market is diverse; a measure of diversity provides a mathematical measure of the degree of diversity in the market (i.e., how diverse is the market). In essence, a positive twice continuously differentiable function of the market weights is said to be a measure of diversity if it is both symmetric and concave. We shall introduce and define the two concepts of symmetric functions and concave functions, as well as the related concepts of symmetry and concavity. A real-valued function of multiple variables is said to be symmetric if its value is independent of the order of the variables, i.e. is invariant under any permutations of the variables. Thus, in what remains in this section, we shall exposit the attributes ascribed to these such functions and discuss thereafter the line of reasoning in establishing these attributes. The symmetry property ensures that all the stocks in the market receive equal treatment so that the degree of diversity in the market is not affected by the ordering of the stocks. The concavity property ensures that diversity increases by mixing the equity capital distribution, as it is required that transferring capital from a larger stock to a smaller stock increases the value of the diversity measure, and analogously transferring capital from a smaller stock to a larger stock decreases the value of that measure. The criterion that encapsulates and fulfills this characteristic behaviour of diversity is that of concavity. Furthermore, since maximum diversity is associated with equal weights, we require a measure of diversity to be maximised by equal weights. Likewise, minimum diversity must be attained when the weight of the largest stock is 1 and the weights of the remaining smaller stocks are 0. Thus, a measure of diversity must be minimised when all the capital is concentrated into a single (largest) stock. It is very important to note that measures of diversity have already previously been encountered and utilised in probability and information theory, as well as in mathematical ecology. Here, we are merely adapting the concept of measures of diversity to the financial domain. In Section 4.6, we shall present and study several examples of potential measures of diversity. These include, but are not limited to, the following examples of diversity measures: entropy, modified entropy, the $\mathbf{D}_p$ index, the normalised version of the $\mathbf{D}_p$ index, Rényi entropy, the quadratic Gini coefficient, the quartic Gini coefficient, the Gini-Simpson index, an admissible market-dominating diversity measure, the geometric mean and the modified geometric mean. We

also put forth the buy-and-hold function in this section, however, it is not a valid measure of diversity as it fails to satisfy the required symmetry condition, which is one of the sufficient diversity measure requirements. In each case, we shall present and define the (diversity measure) function of the market weights under consideration, establish maximum and minimum diversity bounds imposed on the (diversity measure) function through the diversity constraints on the market weights, and then we shall demonstrate that these functions are indeed appropriate measures of diversity for our purposes, by appealing to the definition of diversity measures that requires a positive twice continuously differentiable real-valued function of the market weights to satisfy both the symmetry and concavity conditions. Showing that these functions are concave will invoke a calculation of the corresponding Hessian matrices and a determination that these Hessian matrices are negative semidefinite, or negative definite for strictly concave functions. Entropy is the prototypal measure of diversity that is most frequently encountered in the Fernholz literature and is the most prevalent in its use. Entropy is a standard measure of the uniformity of a probability distribution that was first used in thermodynamics and statistical mechanics, and more recently has been used in probability theory and information theory. The notion of entropy was first introduced by Shannon (1948) in his mathematical formulation of information theory and as a measure of randomness in probability theory. However, even though the entropy function displays a ubiquity in the literature, it is not the only suitable measure of diversity that is referenced and utilised. Thus, it is necessary to explore all avenues of available diversity measures that satisfy the necessary constraints. Another measure of diversity other than the entropy function that we shall be particularly concerned with is the $\mathbf{D}_p$-function, as Fernholz, Garvy & Hannon (1998) employ it to construct an institutional equity investment product of popular status. The $\mathbf{D}_p$-function exhibits advantages over the entropy function, in that the built-in inherent parameter p can be tweaked to accommodate specific instances that we may come across in practice. A characterisation of stock market diversity in terms of the market entropy process is elucidated in the penultimate section, Section 4.7. Here we establish the common thread between the entropy function and diverse equity markets. The link is essentially that the equity market is diverse if and only if the market entropy process is sufficiently bounded below by a positive constant, i.e. it is sufficiently bounded away from zero. This stems from the fact (i.e., from the bounds of the market entropy function) that a value of zero for the market entropy process signifies an equity market that is not diverse, whereas a positive value other than zero for the market entropy process is indicative of an equity market displaying some form of diversity at varying levels. We end this chapter with a summary and conclusion in Section 4.8.

4.2 Diversity of Equity Markets

The *diversity* of a stock market is a measure of how uniformly the capital is apportioned among the stocks in the market. This is tantamount to saying that diversity is a measure of the level of the concentration of capital into each stock in the market. With this in mind, a market is diverse if it avoids the extreme concentration of capital into single stocks. The notion of stock market diversity was initially considered and formulated in Fernholz (1999a), and has been further expounded upon in Fernholz (2002, 2005), Fernholz, Karatzas & Kardaras (2005) and Fernholz & Karatzas (2009). Diversity was, historically, among the first phenomena to be analysed exploiting the application of stochastic portfolio theory. At present, diversity is one of the key components that pervades stochastic portfolio theory and its consequences are essential to the study of portfolio generating functions, which will be explored in the following chapter. Much of the theory presented in this section is standard and the reader may consult any of the aforementioned texts for a comparative account and for clarity of exposition. We have, for the most part, enlisted the contributions made by Fernholz (1999a, 2002).

To make this notion of stock market diversity in a financial equity market precise, we shall now formally define the concept of diversity in an equity market, along with the allied, successively weaker notions of weak diversity, uniform weak diversity and asymptotic weak diversity, all in precise terms.

Definition 4.2.1 (Diversity). *The market $\mathcal{M}$ is **diverse** if there exists a number $\delta > 0$, $\delta \in (0,1)$ such that*

$$\mu_{(1)}(t) \quad \leq \quad 1-\delta, \qquad t \in [0,\infty), \qquad a.s. \tag{4.2.1}$$

*The market $\mathcal{M}$ is **diverse** on the time horizon $[0,T]$, with $T > 0$ a given real number, if there exists a number*

$\delta > 0$, $\delta \in (0,1)$ such that

$$\mu_{(1)}(t) \quad \leq \quad 1-\delta, \qquad t \in [0,T], \qquad a.s., \tag{4.2.2}$$

in the reverse-order-statistics notation (2.4.53) *of Definition* 2.4.10*, where* $\mu_{\max}(t) \triangleq \max_{1\leq i\leq n} \mu_i(t) \triangleq \mu_{(1)}(t)$*, i.e.,* $\mu_{(1)}$ *represents the largest market weight. Hereinafter, we shall adopt this notation.*

In a similar vein, we have the following definition.

Definition 4.2.2 (Weak Diversity, Uniform Weak Diversity, Asymptotic Weak Diversity)**.** *The market $\mathcal{M}$ is **weakly diverse** on the time horizon $[0,T]$, with $T > 0$ a given real number, if there exists a number $\delta > 0$, $\delta \in (0,1)$ such that we have*

$$\frac{1}{T}\int_0^T \mu_{(1)}(t)\,dt \quad \leq \quad 1-\delta, \qquad a.s. \tag{4.2.3}$$

*We say that $\mathcal{M}$ is **uniformly weakly diverse** on $[T_0,\infty)$, for some real number $T_0 > 0$, if there exists a number $\delta \in (0,1)$ such that* (4.2.3) *holds a.s. for every $T \in [T_0,\infty)$. Moreover, the market $\mathcal{M}$ is called **asymptotically weakly diverse** if, for some $\delta \in (0,1)$, we have*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mu_{(1)}(t)\,dt \quad \leq \quad 1-\delta, \qquad a.s., \tag{4.2.4}$$

alternatively, if

$$\overline{\lim_{T\to\infty}} \frac{1}{T}\int_0^T \mu_{(1)}(t)\,dt \quad = \quad \limsup_{T\to\infty} \frac{1}{T}\int_0^T \mu_{(1)}(t)\,dt \quad \leq \quad 1-\delta, \qquad a.s. \tag{4.2.5}$$

The notion of diversity was first introduced in Fernholz (1999a), whereas the allied weaker notion of a weakly diverse market was formulated in Fernholz (2002), and thereafter the notions of a uniformly weakly diverse market and an asymptotically weakly diverse market were conceptualised in Fernholz, Karatzas & Kardaras (2005). Recall that $\mu_{(1)}(t)$ corresponds to the market weight of the largest stock in the market $\mathcal{M}$ at time $t \in [0,\infty)$. Consequently, it is apparent from the foregoing definitions, that a market is diverse if at no time the largest stock in the market accounts for almost the entire market capitalisation. In essence, (4.2.1), amounts to saying that the market is diverse if the market weight for the largest stock is bounded away from 1, i.e., all the capital is not concentrated into the largest stock. Since, this property requires that the largest stock at any time $t \in [0,\infty)$ not dominate the entire market, it stands to reason that this condition applies to all the stocks in the market. Since, if the largest market weight is bounded away from 1, as is requisite for a market satisfying the stronger diversity condition, it is apparent then that the market weights of the remaining ("smaller") stocks should also be bounded away from 1. Thus, a market is diverse if at no time any single stock accounts for almost the entire market capitalisation [refer to, Fernholz (1999a)], and similarly if the cap-weights of all the stocks in the market are bounded away from 1. This is intuitively obvious, since if the largest cap-weight is bounded away from 1 so will all the other cap-weights of the stocks in the market. However, since it is not likely that the smallest stock will dominate the market in terms of relative capitalisation, we formulate the notions of diversity[3] and weak diversity[4] in terms of the largest market weight, which has been the convention

[3]In Fernholz (1999a, Definition 3.2), the formulation of diversity is somewhat different to that provided here, in that (4.2.1) is replaced with $\mu_i(t) \leq 1-\delta$, $t \in [0,\infty)$, a.s., for $i = 1,\ldots,n$. Since, $\mu_i(t) \leq \max_{1\leq i\leq n} \mu_i(t) = \mu_{(1)}(t) \leq 1-\delta$. The formulation provided here is compact and makes intuitive sense in line with the reverse-order-statistics notation.

[4]Thus, the alternative formulation of weak diversity is as follows: the market $\mathcal{M}$ is **weakly diverse** on the time horizon $[0,T]$, with $T > 0$ a given real number, if there exists a number $\delta > 0$, $\delta \in (0,1)$ such that for $i = 1,2,\ldots,n$,

$$\frac{1}{T}\int_0^T \mu_i(t)\,dt \quad \leq \quad 1-\delta, \qquad \text{a.s.}$$

Since,

$$\frac{1}{T}\int_0^T \mu_i(t)\,dt \quad \leq \quad \frac{1}{T}\int_0^T \mu_{(1)}(t)\,dt \quad \leq \quad \frac{1}{T}\int_0^T (1-\delta)\,dt \quad = \quad 1-\delta.$$

adopted herein and in Fernholz (2002). Moreover, a market is said to be weakly diverse if the aforesaid notion of diversity holds on average over the time horizon $[0, T]$. Hence, a weakly diverse market is one in which no single stock holds the entire market capital on average over time. Clearly, if a market is diverse then it is also weakly diverse, however, if a market satisfies the weak diversity requirement it need not necessarily satisfy the stronger diversity requirement.

It is noted by Fernholz (2002), that the notions of diversity and weak diversity are fairly weak empirical requirements, and that actual equity markets of any importance will indeed satisfy both of these properties. Furthermore, as we have already alluded to, the imposition of the aforementioned diversity conditions are a weak consequence of actual antitrust laws, and any market that possesses some form of these laws would ensure the validity of either of the diversity requirements. So, the concept of a diverse equity market is in some respect guaranteed by the existence of any semblance of antitrust laws. Despite the fact that diversity is not very restrictive, many simple hypothetical markets fail to satisfy this condition. An example of such a market is one with equal or constant growth rates, however since diversity is a subtle property, it would seem that equal growth rates for all stocks would ensure market diversity. In fact, the opposite is true, in a nondegenerate market, if all the stocks have the same growth rate, then it can be shown that the market is not diverse.

Fernholz (2002) further remarks that in an economy such as that of the U.S., it is unlikely that a single stock or company could account for even half of the total market capitalisation, this is due to the fact that the requisite measures have been put into place by the U.S. judicial system to combat dominating stocks in the market and thus the enforcement of the antitrust laws is implied. The enactment of these laws in the U.S. equity market suggests that the diversity conditions are consistent with the structure of the U.S. equity market. Consequently, any market model bearing even a remote resemblance to the real U.S. equity market can safely be assumed to satisfy the diversity conditions. Fernholz (1999a) points out that actual or real equity markets are diverse according to Definition 4.2.1. The requirement that real financial equity markets are diverse seems to be reasonable from a regulatory point of view, otherwise an entirely different society would be encountered altogether. Thus, diversity appears to be a natural occurrence in real equity markets. So, to assume that some moderate form of diversity exists in real financial equity markets is not unfounded. But what of hypothetical equity markets with deceptively desirable properties, such as a common growth rate among the stocks in the market? It would naïvely seem that in such hypothetical markets, diversity would naturally be maintained, but (as we have already hinted at) we shall see that this is not the case. So, even if all the stocks have the same growth rate, some technique is still required to maintain diversity in such a market circumstance, since without it the market is non-diverse. Consequently, in the next section, we shall explore the techniques that are needed to ensure the diversity of such hypothetical markets.

Even though the concept of diversity is a delicate consideration, we shall reveal that it has strong mathematical consequences. These and some other ideas will be of focus in the following subsection.

4.2.1 Consequences of Stock Market Diversity

4.2.1.1 Diversity and the Market Excess Growth Rate

We shall now show, using the results of the lemmas derived in Chapter 2, that market diversity can be characterised in terms of the excess growth rate of the market portfolio (the market excess growth rate).

Proposition 4.2.3 ([**Fernholz (2002), Fernholz & Karatzas (2009)**]). *If the market $\mathcal{M}$ is* ***nondegenerate*** *and* ***diverse*** *(resp.,* ***weakly diverse*** *on the time horizon $[0, T]$), then there is a number $\zeta > 0$ such that the following is satisfied*

$$\gamma^*_\mu(t) \quad \geq \quad \zeta, \qquad t \in [0, \infty), \qquad a.s., \tag{4.2.6}$$

$$\left(resp., \quad \frac{1}{T}\int_0^T \gamma^*_\mu(t)\, dt \quad \geq \quad \zeta, \qquad a.s. \right). \tag{4.2.7}$$

Proof. Suppose that the market $\mathcal{M}$ is nondegenerate and diverse, so the diversity condition implies that there exists a $\delta \in (0,1)$ such that

$$\begin{aligned} \mu_{(1)}(t) &\leq 1-\delta, \qquad t \in [0,\infty), \qquad \text{a.s.,} \\ 1-\mu_{(1)}(t) &\geq \delta, \qquad t \in [0,\infty), \qquad \text{a.s.,} \end{aligned}$$

in accordance with Definition 4.2.1 of diversity. Also, since the market is nondegenerate, we have from Lemma 2.4.12 that there exists an $\varepsilon > 0$ such that

$$\gamma_\mu^*(t) \geq \frac{\varepsilon}{2}\big(1-\mu_{(1)}(t)\big)^2, \qquad t \in [0,\infty), \qquad \text{a.s.}$$

The previous expression leads us to deduce that $\big(1-\mu_{(1)}(t)\big)^2 \geq \delta^2$, for $t \in [0,\infty)$, a.s., which when allied with the expression above, gives

$$\gamma_\mu^*(t) \geq \frac{\varepsilon}{2}\delta^2, \qquad t \in [0,\infty), \qquad \text{a.s.,}$$

and (4.2.6) follows with $\zeta := \frac{\varepsilon}{2}\delta^2$. See also, Fernholz (1999a, Proposition 3.1) for an alternative proof. ■

This proposition establishes the crucial link between the excess growth rate of the market and the diversity of the market. We shall see later that it is precisely this link that will equip us with the means to maintain diversity in hypothetical equity markets, satisfying the criteria outlined above.

Corollary 4.2.4. *If the market $\mathcal{M}$ has **bounded variance**, and if there exists a number $\zeta > 0$ such that the first inequality* (4.2.6) *in Proposition* 4.2.3 (*resp., the second inequality* (4.2.7) *in Proposition* 4.2.3) *holds almost surely, then the basic market model $\mathcal{M}$ of* (2.2.20), (2.2.22) *is **diverse** for all $t \in [0,\infty)$* (*resp., **weakly diverse** on the time horizon* $[0,T]$).

Proof. Suppose that the market $\mathcal{M}$ has bounded variance and there exists a $\zeta > 0$ such that $\gamma_\mu^*(t) \geq \zeta$, $t \in [0,\infty)$, a.s. Since the market exhibits bounded variance, Lemma 2.4.15 suggests that we can choose an $\varepsilon > 0$ such that, a.s., for $t \in [0,\infty)$,

$$\mu_{(1)}(t) \leq 1-\varepsilon\gamma_\mu^*(t) \leq 1-\varepsilon\zeta,$$

therefore, the diversity requirement of Definition 4.2.1 is satisfied with $\delta := \varepsilon\zeta$, and hence the market $\mathcal{M}$ is diverse. ■

These results, in particular via (2.4.55) of Lemma 2.4.12 or (2.4.56) of Lemma 2.4.14, reveal that under the *strong nondegeneracy condition* (2.2.55), the first inequality (4.2.6) in Proposition 4.2.3 (respectively, the second inequality (4.2.7) in Proposition 4.2.3) is satisfied if diversity holds for all $t \in [0,\infty)$ (respectively, weak diversity holds on the time interval $[0,T]$). Furthermore, it follows directly from (2.4.57) of Lemma 2.4.15 or (2.4.80) of Lemma 2.4.16, that under the *uniform boundedness condition* (2.2.57), the basic market model $\mathcal{M}$ of (2.2.20), (2.2.22) is diverse for all $t \in [0,\infty)$ (respectively, weakly diverse on the time horizon $[0,T]$) if there exists a number $\zeta > 0$ such that the first inequality (4.2.6) in Proposition 4.2.3 (respectively, the second inequality (4.2.7) in Proposition 4.2.3) holds almost surely. Hence, for a market that is *nondegenerate* and exhibits a *bounded volatility structure*, we have the following

(i) **Diversity criterion:**

$$\mu_{(1)}(t) \leq 1-\delta \qquad \Longleftrightarrow \qquad \gamma_\mu^*(t) \geq \zeta; \tag{4.2.8}$$

(ii) **Weak diversity criterion:**

$$\frac{1}{T}\int_0^T \mu_{(1)}(t)\,dt \leq 1-\delta \qquad \Longleftrightarrow \qquad \frac{1}{T}\int_0^T \gamma_\mu^*(t)\,dt \geq \zeta; \tag{4.2.9}$$

(iii) **Asymptotic weak diversity criterion:**

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \mu_{(1)}(t)\,dt \;\leq\; 1-\delta \qquad \Longleftrightarrow \qquad \lim_{T\to\infty}\frac{1}{T}\int_0^T \gamma_\mu^*(t)\,dt \;\geq\; \zeta. \tag{4.2.10}$$

Thus, we have shown that there exists a relationship between the excess growth rate of the market and the diversity of the capital distribution. This relationship amounts to imposing a condition on the excess growth rate of the market when the market satisfies the *diversity criterion*, this will be referred to as the *excess-growth-based criterion* for the market. Indeed, in the case of a bounded and nondegenerate volatility structure (i.e., the eigenvalues of the covariance matrix are bounded away from both zero and infinity), the excess-growth-based criterion is equivalent to the diversity criterion, as stipulated by the preceding results. In summary, we have shown that diversity can be characterised in terms of the excess growth rate of the market and that this relationship determines market conditions that are compatible with market diversity.

4.2.1.2 Diversity of the Equal-Growth-Rate Market

Let us now investigate the diversity of the equal-growth-rate market. At the outset, the asymptotic negligibility property of the average excess growth rate of the equal-growth-rate market over the long term (see Proposition 3.4.6), could lead one to believe that such a hypothetical market is diverse. It is natural to make such an assumption, since even Fernholz (2002) remarks that such a property of a market (equal growth rates for all stocks), appears to ensure market diversity. In fact, it is quite the opposite as shall be demonstrated in the following corollary.

Corollary 4.2.5 ([**Fernholz (2002)**]). *Suppose that the market $\mathcal{M}$ is* ***nondegenerate****. If all the stocks in the market $\mathcal{M}$ have the* ***same growth rate*** *($\gamma_i(t) \equiv \gamma(t)$, for all $i = 1, 2, \ldots, n$), then $\mathcal{M}$ is* ***not diverse****. That is, such an equal-growth-rate market cannot be diverse, even weakly diverse, over long time horizons, provided that the strong nondegeneracy condition is also satisfied.*

Proof. This result follows from contradiction. If the market $\mathcal{M}$ is diverse, Proposition 4.2.3 implies that there exists a $\zeta > 0$ such that

$$\gamma_\mu^*(t) \;\geq\; \zeta, \qquad t \in [0,\infty), \qquad \text{a.s.}$$

Since a diverse market is also *weakly diverse*, we have in the respective case of weak diversity, from Proposition 4.2.3,

$$\frac{1}{T}\int_0^T \gamma_\mu^*(t)\,dt \;\geq\; \zeta, \qquad T \in [0,\infty), \qquad \text{a.s.},$$

moreover, if a market is diverse then it is also *asymptotically weakly diverse*, in which case the following expression is satisfied

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \gamma_\mu^*(t)\,dt \;\geq\; \zeta, \qquad T \in [0,\infty), \qquad \text{a.s.} \tag{4.2.11}$$

But, we know from Proposition 3.4.6, that for a market in which all the stocks have the same growth rate ($\gamma_i(t) \equiv \gamma(t)$ for $i = 1, \ldots, n$), we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \gamma_\mu^*(t)\,dt \;=\; 0, \qquad \text{a.s.}, \tag{4.2.12}$$

this, however, contradicts the notion of asymptotic weak diversity, (4.2.11). So, such an *equal-growth-rate* market cannot be diverse, since it does not satisfy the diversity criterion.

An alternative approach, is to prove that the diversity condition does not hold as per Definition 4.2.1 (i.e., all the capital is concentrated into the largest stock) and that we have

$$\mu_{(1)}(t) \;=\; 1, \qquad t \in [0,\infty), \qquad \text{a.s.}, \tag{4.2.13}$$

or, that this holds asymptotically on average over a time horizon $[0,T]$,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \mu_{(1)}(t)\,dt \;=\; 1, \qquad T\in[0,\infty), \qquad \text{a.s.} \tag{4.2.14}$$

To prove (4.2.14), we observe the following

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\int_0^T \gamma_\mu^*(t)\,dt &\geq \lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{\varepsilon}{2}\big(1-\mu_{(1)}(t)\big)^2\,dt\\ &= \frac{\varepsilon}{2}\lim_{T\to\infty}\frac{1}{T}\int_0^T \big(1-\mu_{(1)}(t)\big)^2\,dt\\ &\geq \frac{\varepsilon}{2}\left(\lim_{T\to\infty}\frac{1}{T}\int_0^T \big(1-\mu_{(1)}(t)\big)\,dt\right)^2,\end{aligned} \tag{4.2.15}$$

where the first inequality follows from Lemma 2.4.12, using the fact that the market is nondegenerate (a simplified approach would be to appeal to Lemma 2.4.14) and the second inequality follows from Jensen's inequality,[5] since $\big(1-\mu_{(1)}(t)\big)^2$ is a convex function ($\varphi(x)=x^2$ is convex), or from the Cauchy-Schwarz inequality.[6] Furthermore, since all the stocks in the market $\mathcal{M}$ have equal growth rates, we know the following result for equal-growth-rate markets

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \gamma_\mu^*(t)\,dt \;=\; 0.$$

We can thus conclude the following from (4.2.15),

$$\frac{\varepsilon}{2}\left(\lim_{T\to\infty}\frac{1}{T}\int_0^T \big(1-\mu_{(1)}(t)\big)\,dt\right)^2 \;\leq\; 0.$$

This expression, however, is not valid for values less than 0, and only makes sense if equated to 0. We thus have

$$\frac{\varepsilon}{2}\left(\lim_{T\to\infty}\frac{1}{T}\int_0^T \big(1-\mu_{(1)}(t)\big)\,dt\right)^2 \;=\; 0$$

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \big(1-\mu_{(1)}(t)\big)\,dt \;=\; 0 \tag{4.2.16}$$

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \mu_{(1)}(t)\,dt \;=\; \lim_{T\to\infty}\frac{1}{T}\int_0^T dt \;=\; 1. \tag{4.2.17}$$

This proves the desired result, thus if the market satisfies the *strong nondegeneracy condition*, then (2.4.55) and (3.4.33) imply (4.2.17). ■

Thus, we are left with the following equivalent conditions:

(i) **Asymptotic weak diversity fails:**

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \gamma_\mu^*(t)\,dt \;=\; 0 \quad\Longleftrightarrow\quad \lim_{T\to\infty}\frac{1}{T}\int_0^T \mu_{(1)}(t)\,dt \;=\; 1; \tag{4.2.18}$$

(ii) **Weak diversity fails:**

$$\frac{1}{T}\int_0^T \gamma_\mu^*(t)\,dt \;=\; 0 \quad\Longleftrightarrow\quad \frac{1}{T}\int_0^T \mu_{(1)}(t)\,dt \;=\; 1; \tag{4.2.19}$$

[5] Let X be a random variable defined on a finite probability space such that $\mathbb{E}\left[|X|\right]<\infty$ and let $\varphi:\mathbb{R}\to\mathbb{R}$ be a real-valued convex function of X such that $\mathbb{E}[|\varphi(X)|]<\infty$, then Jensen's inequality states that $\mathbb{E}\left[\varphi(X)\right]\geq\varphi\left(\mathbb{E}[X]\right)$.

[6] If $\mathcal{A}$ is a measurable subset of $\mathbb{R}^n$, and f and g are measurable real-valued functions on $\mathcal{A}$, then another version of the Cauchy-Schwarz inequality is given by $\left(\int_{\mathcal{A}} f(x)g(x)\,dx\right)^2 \leq \left(\int_{\mathcal{A}} f^2(x)\,dx\right)\left(\int_{\mathcal{A}} g^2(x)\,dx\right)$ or $\left(\int_{\mathcal{A}} f(x)\,dx\right)^2 \leq \left(\int_{\mathcal{A}} f^2(x)\,dx\right)$, stated differently as $\big(\mathbb{E}[XY]\big)^2 \leq \mathbb{E}\big[X^2\big]\mathbb{E}\big[Y^2\big]$. A generalisation of this result is given by Hölder's inequality, $\int_{\mathcal{A}}\left|f(x)g(x)\right|\,dx \leq \left(\int_{\mathcal{A}}\left|f(x)\right|^p\,dx\right)^{\frac{1}{p}}\left(\int_{\mathcal{A}}\left|g(x)\right|^q\,dx\right)^{\frac{1}{q}}$, where $p,q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, stated differently as $\mathbb{E}\big[XY\big]\leq\big(\mathbb{E}\big[|X|^p\big]\big)^{\frac{1}{p}}\big(\mathbb{E}\big[|Y|^q\big]\big)^{\frac{1}{q}}$.

(iii) **Diversity fails:**

$$\gamma_\mu^*(t) \;=\; 0 \qquad \Longleftrightarrow \qquad \mu_{(1)}(t) \;=\; 1. \tag{4.2.20}$$

We have thus considered the diversity of the *equal-growth-rate* market, or rather lack thereof. We have shown in Corollary 4.2.5 that for a *nondegenerate, equal-growth-rate market* that *asymptotic weak diversity* fails on long time horizons. It is obvious that if a market fails to satisfy the asymptotic weak diversity requirement then it also fails to satisfy the weak diversity requirement and consequently, the foremost requirement, that of diversity, also fails. In fact, if the weak diversity requirement is not satisfied, i.e., when

$$\frac{1}{T}\int_0^T \mu_{(1)}(t)\,dt \;=\; 1, \qquad \text{a.s.}, \tag{4.2.21}$$

then the only way in which (4.2.21) can hold is if $\mu_{(1)}(t) = 1$ for all $t \in [0, \infty)$, since $0 < \mu_i(t) \leq 1$ for $i = 1, \ldots, n$. Thus, if the market is *not weakly diverse* then it is *not diverse*. However, a market that is *not diverse* might possibly be weakly diverse, i.e., the lack of diversity of a market does not necessarily imply the lack of weak diversity of the market. Thus, for a *nondegenerate, equal-growth-rate market*, *weak diversity* fails on long time horizons. Furthermore, for a *nondegenerate, equal-growth-rate market*, *diversity* fails. In particular, such an equal-growth-rate market cannot be diverse, and is not even weakly diverse over long time horizons, provided that the strong nondegeneracy condition is also satisfied. This suggests that, every so often, at some time $t \in [0, \infty)$, a single stock (the largest stock) will dominate the entire market; eventually it will recede, only to be ousted, and subsequently superseded, by another stock which reprises the role of the absolutely dominant market leader, and so on and so forth.

4.2.1.3 Diversity of the Constant-Growth-Rate Market

The previous corollary showed that equal-growth-rate markets, under the assumption of strong nondegeneracy, are not diverse. Again, under the strong nondegeneracy assumption, the diversity of a market with constant, but not necessarily equal, growth rates is addressed next.

Corollary 4.2.6 ([**Fernholz (2002)**]). *Suppose that the market $\mathcal{M}$ is* ***nondegenerate****. If all the stocks in the market $\mathcal{M}$ have* ***constant growth rates*** *(though not necessarily equal), then $\mathcal{M}$ is* ***not diverse****. That is, such a constant-growth-rate market cannot be diverse, even weakly diverse, over long time horizons, provided that the strong nondegeneracy condition is also satisfied.*

Proof. Let us consider a market with constant parameters, in particular, *constant growth rates*, in which there are one or more stocks that share the highest growth rate, γ. Let $\mathfrak{M}$ denote the family of k stocks that share the highest growth rate, $\mathfrak{M} = \{X_1, X_2, \ldots, X_k\}$, where $\mathfrak{M} \subseteq \mathcal{M}$, i.e., $\mathfrak{M}$ represents the submarket of the financial market $\mathcal{M}$ composed of the stocks that share the highest growth rate. We can ignore all the stocks except those that share the highest growth rate, γ, since Corollary 3.2.3 implies that all stocks except those that share the highest growth rate will represent a negligible part of the market value in the long term. In passing, we note in a similar fashion to that derived for the equal-growth-rate market, that the contribution of $\gamma_\mu^*(t)$ to $\gamma_\mu(t)$ is minimal when only considering the stocks with the highest growth rate. Thus, using the following expressions:

$$\lim_{T\to\infty} \frac{1}{T}\left(\log Z_\mu(T) - \int_0^T \gamma_\mu(t)\,dt\right) \;=\; 0, \qquad \text{a.s.},$$

$$\gamma_\mu(t) \;=\; \sum_{i=1}^n \mu_i(t)\gamma_i + \gamma_\mu^*(t) \;=\; \gamma + \gamma_\mu^*(t), \tag{4.2.22}$$

where (4.2.22) is a consequence of the *constant-growth-rate market*, hence the γ_i, and the fact that we ignore all the stocks that do not lie in $\mathfrak{M}$, thus the only growth rates under consideration are those of the stocks that

have a common growth rate γ. Then, we have

$$\begin{aligned}
\lim_{T\to\infty}\frac{1}{T}\log Z_\mu(T) &= \lim_{T\to\infty}\frac{1}{T}\int_0^T\big(\gamma+\gamma_\mu^*(t)\big)\,dt\\
&= \lim_{T\to\infty}\frac{1}{T}\int_0^T\gamma\,dt+\lim_{T\to\infty}\frac{1}{T}\int_0^T\gamma_\mu^*(t)\,dt\\
&= \gamma+\lim_{T\to\infty}\frac{1}{T}\int_0^T\gamma_\mu^*(t)\,dt,
\end{aligned}$$

and since Z_μ cannot grow faster than all the stocks, in particular the stocks that lie in $\mathfrak{M}$, i.e., that share the highest growth rate, the last term must vanish. We thus have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\gamma_\mu^*(t)\,dt = 0,$$

which parallels the result for the equal-growth-rate market.

Continuing with this proof, we recall the following for the long-term behaviour of the stocks in the market, $\mathcal{M}$, for $i=1,\ldots,n$,

$$\lim_{T\to\infty}\frac{1}{T}\left(\log X_i(T)-\int_0^T\gamma_i(t)\,dt\right) = 0, \qquad \text{a.s.}$$

Since the growth rates are constants, we have for all $t\in[0,\infty)$, $\gamma_i(t)\equiv\gamma_i$. Then, we obtain for $i=1,\ldots,n$,

$$\begin{aligned}
\lim_{T\to\infty}\frac{1}{T}\Big(\log X_i(T)-\gamma_iT\Big) &= 0, \qquad \text{a.s.}\\
\lim_{T\to\infty}\frac{1}{T}\log X_i(T) &= \gamma_i, \qquad \text{a.s.}
\end{aligned}$$

Thus, for large enough T, we can deduce for $i=1,\ldots,n$,

$$\begin{aligned}
\log X_i(T) &\cong \gamma_iT, \qquad \text{a.s.}\\
X_i(T) &\cong \exp\big(\gamma_iT\big), \qquad \text{a.s.}
\end{aligned}$$

For all the stocks that do not lie in $\mathfrak{M}$, i.e., $\mathcal{M}\backslash\mathfrak{M}=\{X_{k+1},X_{k+2},\ldots,X_n\}$, we have for $i=k+1,\ldots,n$,

$$\begin{aligned}
\mu_i(T) = \frac{X_i(T)}{X_1(T)+\cdots+X_n(T)} &\cong \frac{\exp\big(\gamma_iT\big)}{\sum_{j=1}^n\exp\big(\gamma_jT\big)}\\
&= \frac{\exp\big((\gamma_i-\gamma)T\big)}{\sum_{j=1}^n\exp\big((\gamma_j-\gamma)T\big)}\\
&= \frac{\exp\big((\gamma_i-\gamma)T\big)}{\sum_{j=1,j\in\mathfrak{M}}^k\exp\big((\gamma_j-\gamma)T\big)+\sum_{j=k+1,j\in\mathcal{M}\backslash\mathfrak{M}}^n\exp\big((\gamma_j-\gamma)T\big)}\\
&= \frac{\exp\big((\gamma_i-\gamma)T\big)}{\sum_{j=1,j\in\mathfrak{M}}^k\exp\big((\gamma-\gamma)T\big)+\sum_{j=k+1,j\in\mathcal{M}\backslash\mathfrak{M}}^n\exp\big((\gamma_j-\gamma)T\big)}\\
&= \frac{\exp\big((\gamma_i-\gamma)T\big)}{k+\sum_{j=k+1,j\in\mathcal{M}\backslash\mathfrak{M}}^n\exp\big((\gamma_j-\gamma)T\big)},
\end{aligned}$$

so as $T\to\infty$, $\mu_i(T)\to 0$ for $i=k+1,\ldots,n$. This justifies the aforementioned claim that all the stocks that do not lie in $\mathfrak{M}$ are negligible over the long term, i.e., as $T\to\infty$. Thus the market under consideration over the long term is the submarket $\mathfrak{M}$. Thus, to show that the constant-growth-rate market $\mathcal{M}$ is not diverse, it suffices to show that the market $\mathfrak{M}$ is not diverse. Now, if the number of stocks in $\mathfrak{M}$ that share the highest growth rate is $k=1$, then all the capital is concentrated into this single stock and by definition, the market $\mathfrak{M}$ is not diverse. In addition, if $k\neq 1$, then the submarket $\mathfrak{M}$ consisting of all stocks that have the same highest growth rate is composed of more than a single stock. Since all the stocks have equal growth rates within $\mathfrak{M}$,

we conclude that $\mathfrak{M}$ is indeed an equal-growth-rate market satisfying the hypotheses of the previous corollary, Corollary 4.2.5. Hence, the submarket $\mathfrak{M}$ is not diverse; and so the constant-growth-rate market $\mathcal{M}$ is likewise not diverse. ■

The preceding corollary shows that the diversity requirement is not satisfied even for a hypothetical market in which all the stocks possess *constant growth rates*. This is because the *constant-growth-rate market* basically reduces to either (1) a market comprising a negligible subset of stocks and a single stock with the highest growth rate or (2) a market also comprising a negligible subset of stocks but now with a subset of k stocks that share the highest growth rate. In the former case, the equity market concentrates essentially into a single stock that accounts for all market value and the failure of diversity is implied. The latter case equates to the recently derived *non-diverse equal-growth-rate market*. Thus, constant growth rates are an impossible notion in a nondegenerate, diverse market. In addition, such an equity market with constant growth rates and covariances exhibits an implicit *instability*, i.e., such markets are *unstable* in the sense that they tend to concentrate into a single stock. In fact, the behaviour of diversity implies a certain stability in the market over the long term and we thus consider the diversity condition to be a *weak stability condition* on the market. Thus, hypothetical equity markets not satisfying the condition of diversity are in some weak sense, unstable. The aforementioned corollaries reveal that the deceivingly "nice" properties of equal or constant growth rates assigned to all the stocks in the market are not sufficient to maintain diversity.

We have seen that certain hypothetical nondegenerate equity markets, which we would have expected to display at least some weak form of stability, do not even fulfill some weak form of diversity. Therefore, diversity is clearly not a natural state for hypothetical equity markets fashioned from our general equity market model of stock price processes represented by continuous semimartingales. So, some mechanism is required to maintain it in hypothetical markets. In the next section we shall discuss the structure of this mechanism and in so doing present the conditions under which market diversity is maintained. At a later stage we shall tackle the stronger notion of how diversity can be guaranteed in equity markets.

4.3 Maintaining Diversity

An equity market fails to satisfy the diversity requirement when the entire capital of the market is concentrated into a single, usually largest, stock. As a result, we have seen that the simple hypothetical markets of equal or constant growth rates fail to satisfy this fundamental requirement. Thus, we would like to demonstrate whether it is possible to maintain diversity in hypothetical equity markets of this type, that are prone to high concentrations of capital into a single stock. Consequently, to maintain diversity in an equity market, one theory, proposed by Fernholz (1999a) and Fernholz (2002), is to somehow redistribute the capital of the market, so that it doesn't have the propensity to accumulate into a single stock. One such method of accomplishing this is to allow for the payment of dividends in markets of this type. This is because dividend payments are a natural means of redistributing capital and, consequently, of potentially maintaining diversity. In fact, any such redistribution can be considered to be a dividend in a generalised sense. In an effort to entertain this plausible theory, we shall incorporate nonnegative dividend rates into our general equity market model.

In some sense, we would like to show under what circumstances a market, in which all the stocks have the same growth rate and thus the same growth potential, maintains diversity. Thus, we need to consider a market in which all the stocks not only pay dividends but also display the same capacity for growth as with the equal-growth-rate market. Consequently, we shall consider dividend-paying stocks in an *equal-augmented-growth-rate market*. The combination of equal total growth rates and dividend rates models a situation in which all the stocks possess the same potential for capital growth, but some of the companies elect to distribute part of their capital in the form of dividends rather than investing in themselves [Fernholz (1999a)]. A market of this type appears to be similar to the equal-growth-rate market, in that all the stocks share the same growth prospects, and so one might come to the conclusion that this too is a non-diverse market. However, the inclusion of dividends is what makes the difference and results in a more even distribution of capital over the long term, which, in turn, results in the diversity of the market. Of course, merely throwing dividends into the mix

does not necessarily guarantee that diversity will be maintained over the long term. The following corollary demonstrates this point for the equal-augmented-growth-rate market under consideration. For a case in point, we shall consider the market in which all stocks pay a nonnegative, *but equal*, dividend rate.

Corollary 4.3.1. *Suppose that the $\mathcal{M}$ is nondegenerate. If all the stocks in $\mathcal{M}$ have the* ***same augmented growth rate*** *and the* ***same dividend rate****, then $\mathcal{M}$ is* ***not diverse****.*

Proof. The assumption of equal total growth rates, $\vartheta_i(t) \equiv \vartheta(t)$, for all $i = 1, \ldots, n$, and equal dividend rates, $\delta_i(t) \equiv \delta(t)$, for all $i = 1, \ldots, n$, yields $\gamma_i(t) = \vartheta_i(t) - \delta_i(t) = \vartheta(t) - \delta(t) \equiv \gamma(t)$, so that $\gamma_i(t) \equiv \gamma(t)$ for all $i = 1, \ldots, n$. Consequently, the conditions of equal total growth rates and dividends coincides with the equal-growth-rate market, which from Corollary 4.2.5 is not diverse. Alternatively, appealing to Corollary 3.5.7 along with Proposition 3.5.8, which states that for such an equal-augmented-growth-rate market with equal dividends for all stocks, we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \gamma_\mu^*(t)\, dt \quad = \quad 0, \qquad \text{a.s.}$$

Thus, due to the parity of this market with its equal-growth-rate counterpart, applying and following the same steps as in the proof of Corollary 4.2.5 for the equal-growth-rate market, establishes the non-diversity result for the equal-augmented-growth-rate market with equal dividends. ■

The corollary above shows that a common dividend rate among the stocks in an equal-augmented-growth-rate market is not sufficient to maintain diversity. Thus, "equal dividends" results in a non-diverse market. Hence, incorporating dividends in this fashion, where all stocks share the same dividend rate, is one such instance in which dividends do not meet the requirements to maintain diversity. We, thus, need to establish, and subsequently impose, the specific conditions on dividends that will achieve and maintain market diversity.

4.3.1 Dividends as a Means to Maintain Diversity

Proposition 4.3.2 ([**Fernholz (2002)**])**.** *Suppose that all the stocks in the market $\mathcal{M}$ have nonnegative dividend rates and the* ***same augmented growth rate****, ϑ. Then*

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T \big(\gamma_\mu(t) - \vartheta(t)\big)\, dt \quad \leq \quad 0, \qquad \textit{a.s.} \tag{4.3.1}$$

Proof. From Proposition 3.5.4, we have

$$\limsup_{T\to\infty} \frac{1}{T} \left(\log Z_\mu(T) - \int_0^T \vartheta(t)\, dt \right) \quad \leq \quad 0, \qquad \text{a.s.,} \tag{4.3.2}$$

and from Proposition 3.2.2, we have

$$\limsup_{T\to\infty} \frac{1}{T} \left(\log Z_\mu(T) - \int_0^T \gamma_\mu(t)\, dt \right) \quad = \quad 0, \qquad \text{a.s.} \tag{4.3.3}$$

Now, subtracting the left-hand side of (4.3.3) from the left-hand side of the inequality (4.3.2), implies the desired result

$$\limsup_{T\to\infty} \frac{1}{T} \left(\log Z_\mu(T) - \int_0^T \vartheta(t)\, dt \right) - \limsup_{T\to\infty} \frac{1}{T} \left(\log Z_\mu(T) - \int_0^T \gamma_\mu(t)\, dt \right) \quad \leq \quad 0, \qquad \text{a.s.,} \tag{4.3.4}$$

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T \big(\gamma_\mu(t) - \vartheta(t)\big)\, dt \quad \leq \quad 0, \qquad \text{a.s.} \tag{4.3.5}$$

Analogously, we have[7]

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T -\big(\vartheta(t)-\gamma_\mu(t)\big)\,dt \;=\; -\liminf_{T\to\infty} \frac{1}{T}\int_0^T \big(\vartheta(t)-\gamma_\mu(t)\big)\,dt \;\le\; 0, \qquad \text{a.s.} \tag{4.3.6}$$

Hence,

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T \big(\vartheta(t)-\gamma_\mu(t)\big)\,dt \;\ge\; 0, \qquad \text{a.s.} \tag{4.3.7}$$

■

Thus, the proposition above shows that for an equal-augmented-growth-rate market in which all the stocks have nonnegative dividend rates, the growth rate of the market cannot exceed the common augmented growth rate of the stocks.

Corollary 4.3.3 ([**Fernholz (2002)**]). *Suppose that all the stocks in the market $\mathcal{M}$ have nonnegative dividend rates and the* ***same augmented growth rate****. Then*

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \big(\gamma_\mu^*(t)-\delta_\mu(t)\big)\,dt \;\le\; 0, \qquad \textit{a.s.} \tag{4.3.8}$$

Proof. This proof follows on from the proof of Proposition 4.3.2. From Definition 2.2.38, by setting $\pi := \mu$ in (2.2.161), the augmented growth rate of the market portfolio is given by

$$\vartheta_\mu(t) \;=\; \gamma_\mu(t)+\delta_\mu(t), \qquad t\in[0,\infty). \tag{4.3.9}$$

Furthermore, recall Lemma 2.2.39, by setting $\pi := \mu$ in (2.2.164), we have

$$\vartheta_\mu(t) \;=\; \sum_{i=1}^n \mu_i(t)\vartheta_i(t)+\gamma_\mu^*(t), \qquad t\in[0,\infty). \tag{4.3.10}$$

Since all the stocks share the same augmented growth rate, $\vartheta_i(t)\equiv\vartheta(t)$, for $i=1,\dots,n$, we obtain the following from (4.3.10), for the *equal-augmented-growth-rate* market,

$$\vartheta_\mu(t) \;=\; \sum_{i=1}^n \mu_i(t)\vartheta(t)+\gamma_\mu^*(t) \tag{4.3.11}$$

$$\vartheta_\mu(t) \;=\; \vartheta(t)+\gamma_\mu^*(t), \qquad t\in[0,\infty), \qquad \text{a.s.} \tag{4.3.12}$$

Thus, equating the right-hand side of (4.3.9) to the right-hand side of (4.3.12), yields the following

$$\begin{aligned} \gamma_\mu(t)+\delta_\mu(t) &= \vartheta(t)+\gamma_\mu^*(t), \qquad t\in[0,\infty), \qquad \text{a.s.,}\\ \gamma_\mu(t)-\vartheta(t) &\equiv \gamma_\mu^*(t)-\delta_\mu(t), \qquad t\in[0,\infty), \qquad \text{a.s.} \end{aligned}$$

Consequently, we obtain

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \big(\gamma_\mu(t)-\vartheta(t)\big)\,dt \;\equiv\; \limsup_{T\to\infty} \frac{1}{T}\int_0^T \big(\gamma_\mu^*(t)-\delta_\mu(t)\big)\,dt, \qquad \text{a.s.} \tag{4.3.13}$$

Therefore (4.3.8) is equivalent to (4.3.1), and we have derived the required result:

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \big(\gamma_\mu^*(t)-\delta_\mu(t)\big)\,dt \;=\; \limsup_{T\to\infty} \frac{1}{T}\int_0^T \big(\gamma_\mu(t)-\vartheta(t)\big)\,dt \;\le\; 0, \qquad \text{a.s.}$$

We also have the analogous result,

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T \big(\delta_\mu(t)-\gamma_\mu^*(t)\big)\,dt \;\ge\; 0, \qquad \text{a.s.} \tag{4.3.14}$$

[7] The relationship of limit infimum and limit supremum for sequences of real numbers is as follows: $\limsup_{t\to\infty}\big(-f(t)\big) = -\liminf_{t\to\infty} f(t)$, or $\limsup_{t\to\infty} f(t) = -\liminf_{t\to\infty}\big(-f(t)\big)$.

■

Hence, for an equal-augmented-growth-rate market, the average dividend rate must at least equal the average excess growth rate of the market over the long term. Thus, in an attempt to show how to maintain diversity in an equal-augmented-growth-rate market, we need to determine the link between the result of Corollary 4.3.3 and the notion of diversity. This, in turn, leads us to the all-important property of the excess growth rate in diverse markets. Recall that the excess growth rate of the market is related to market diversity by Proposition 4.2.3. Thus, from the above corollary, for an equal-augmented-growth-rate market, the excess-growth-rate characterisation of diversity suggests that at least some dividends must be paid in order to maintain diversity in a market of this type. Be that as it may, simply allowing for the payment of dividends in any manner is not adequate enough to guarantee diversity. More specifically, this corollary fails to mention that if the stocks pay the same dividend rate then diversity fails. Consequently, it does not indicate how and by which stocks such dividend payments must be made. This we subtly address in the following corollary.

Corollary 4.3.4 ([**Fernholz (2002)**]). *Suppose that the market $\mathcal{M}$ is **nondegenerate**, and that all the stocks have nonnegative dividend rates and the **same augmented growth rate**. If $\mathcal{M}$ is **diverse**, then there exists a $\zeta > 0$ such that*

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T \delta_\mu(t)\,dt \quad \geq \quad \zeta, \qquad a.s. \tag{4.3.15}$$

Proof. We shall require the following analogous result, (4.5.3), of the previous corollary

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T \big(\delta_\mu(t) - \gamma_\mu^*(t)\big)\,dt \quad \geq \quad 0, \qquad \text{a.s.} \tag{4.3.16}$$

Moreover, if the market $\mathcal{M}$ is diverse then according to Proposition 4.2.3 there exists a $\zeta > 0$ such that $\gamma_\mu^*(t) = \vartheta_\mu^*(t)$ (from Definition 2.2.40), satisfies for all $t \in [0,\infty)$, a.s.

$$\gamma_\mu^*(t) \quad \geq \quad \zeta. \tag{4.3.17}$$

Thus, we have

$$-\gamma_\mu^*(t) \quad \leq \quad -\zeta.$$

By adding $\delta_\mu(t)$ to both sides of the inequality we obtain

$$\delta_\mu(t) - \gamma_\mu^*(t) \quad \leq \quad \delta_\mu(t) - \zeta.$$

Hence, using (4.3.16), we have a.s.[8]

$$0 \quad \leq \quad \liminf_{T\to\infty} \frac{1}{T}\int_0^T \big(\delta_\mu(t) - \gamma_\mu^*(t)\big)\,dt \quad \leq \quad \liminf_{T\to\infty} \frac{1}{T}\int_0^T \big(\delta_\mu(t) - \zeta\big)\,dt \tag{4.3.18}$$

$$= \quad \liminf_{T\to\infty} \left(\frac{1}{T}\int_0^T \delta_\mu(t)\,dt - \zeta\right) \tag{4.3.19}$$

$$= \quad \liminf_{T\to\infty} \frac{1}{T}\int_0^T \delta_\mu(t)\,dt - \zeta. \tag{4.3.20}$$

Therefore,

$$\liminf_{T\to\infty} \frac{1}{T}\int_0^T \delta_\mu(t)\,dt - \zeta \quad \geq \quad 0, \tag{4.3.21}$$

and the corollary follows. ■

[8] Let $f(t)$ be any function, and let $h(t)$ be a function with a well-defined limit, so that $\lim_{t\to\infty} h(t) = h^*$ for some (possibly infinite) value h^*. Then, $\limsup_{t\to\infty}\big(f(t)+h(t)\big) \equiv \limsup_{t\to\infty} f(t) + h^*$ and $\liminf_{t\to\infty}\big(f(t)+h(t)\big) \equiv \liminf_{t\to\infty} f(t) + h^*$. Alternatively, let $c \in \mathbb{R}$, then $\liminf_{t\to\infty}\big(c+f(t)\big) = c + \liminf_{t\to\infty} f(t)$ and $\limsup_{t\to\infty}\big(c+f(t)\big) = c + \limsup_{t\to\infty} f(t)$.

Clearly, we also have the following result

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \delta_\mu(t)\,dt \quad \geq \quad \liminf_{T\to\infty} \frac{1}{T}\int_0^T \delta_\mu(t)\,dt \quad \geq \quad \zeta. \tag{4.3.22}$$

Thus,

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \delta_\mu(t)\,dt \quad \geq \quad \zeta. \tag{4.3.23}$$

Intuitively, it makes sense that if an equal-growth-rate market is not diverse due to the largest stock dominating the entire market, then reducing the growth rate of the largest stock and increasing the growth rates of the smaller stocks might result in a diverse market. In order to reduce the growth potential of the largest stock, it needs to avoid reinvesting in itself and rather distribute part of its capital in the form of dividend payments. This is why incorporating dividends into the model is so important in maintaining the diversity of the market. Hence, if all the stocks have nonnegative dividend rates (which implies that all stocks redistribute part of their capital in the form of dividends), then in order for the market to remain diverse the larger stocks must pay a higher dividend than the smaller stocks. This has the effect of balancing out the capital in the market so that diversity is maintained. This broaches the discussion of the compatibility of market diversity with capital market equilibrium.

To follow up on the idea presented in Corollary 4.3.3, Corollary 4.3.4 states that the average dividend rate of the equal-augmented-growth-rate market must at least equal some strictly positive constant ζ, over the long term. Again, this implies that the dividend payment is a requisite condition in maintaining diversity. In particular, both Corollary 4.3.3 and 4.3.4, reinforce the fact that the excess growth rate-diversity relationship determines explicit conditions on the dividend structure of the market that are consistent with the notion of diversity. Moreover, Corollary 4.3.4 alludes to what was mentioned previously, namely that under certain circumstances the dividend rates of the larger stocks must be greater than those of the smaller stocks in order to maintain diversity. In particular, we illustrated that not all stocks can share the same dividend rate, since equal dividends do not serve to maintain diversity. Thus, we have shown how to maintain diversity in "seemingly diverse" markets, by allowing companies to pay dividends, but not all the same dividend. Furthermore, we investigated how dividends have the potential to maintain market diversity, but under the (rather restrictive) condition of equal augmented growth rates. The requirement of equal growth rates or equal augmented growth rates will be relaxed in the forthcoming chapters, where we shall study in greater detail, this condition on the dividend structure, as a means to maintain diversity.

4.4 The Measurement of Diversity

Within the context of stochastic portfolio theory, the concept of a diverse market is an integral consideration. Recall that diversity is seen as a measure of how the capital is distributed among all the stocks in an equity market, we thus require some means by which to effectively measure this quantity. The measurement tool adopted by Fernholz for this purpose, are the aptly named *measures of diversity*. We are interested in measures of market diversity for two reasons. The first is that market diversity is an observable characteristic of equity markets that is amenable to stochastic analysis. The second reason is that measures of diversity can be used to construct portfolios with desirable investment characteristics, as we shall see in the forthcoming chapters [Fernholz (2002)]. In this section we shall review the measurement of diversity and collect together some useful properties that are requisite of such measures. Consequently, we shall investigate the different measures of diversity, and explore one in particular that lends itself to and forms the foundations for what follows in the next chapter, where portfolio generating functions will be of primary concern.

Whereas, the definition of diversity provides a criterion for determining whether or not a market is diverse; a measure of diversity [see Rao (1982a,b,c, 1984) and Simpson (1949)] provides a mathematical measure of the *degree* of diversity in the market (i.e., how diverse the market is). There exist many measures of diversity,

of which *entropy* [see Shannon (1948), Shannon & Weaver (1964) and Rényi (1961)] is perhaps the one most frequently encountered and best known. As a result, we shall pay particular attention to the discussion of entropy and its appropriateness as a measure of diversity. There are, however, other measures of diversity which are more pertinent to the discussion. Hence, it is more useful to consider more general measures of diversity than to consider the entropy function alone. To this end, other functions are proposed as measures of diversity and we thus pursue and survey these different avenues in greater detail. However, we shall first proceed with a discussion on how stock market diversity can be quantified. This allows us to establish the properties that a sufficient measure of diversity should possess. Thereafter, we shall commence with the investigation of the relevant measures of diversity, and demonstrate that these functions do indeed satisfy the requirements as embodied by a definitive measure of diversity.

4.5 Quantifying the Effect of Diversity

We shall now discuss methods by which the level of diversity of an equity market can be measured quantitatively. In order to do so, we need to clarify the concept of the distribution of equity capital (or, the spread of equity capital) of the market. As we already know, the distribution of equity capital is simply the amount of capital that is allocated to each of the stocks in an equity market. We would like to present a formal definition of this concept in terms of the market weights. Since at a later stage in this dissertation we would like to investigate the equity capital distribution in greater detail, than will be done presently, it will be necessary to introduce the capital distribution curve[9] as a means to study this feature. This will be imperative, since we would like to achieve a deeper understanding of the structure of the capital distribution.

4.5.1 The Distribution of Capital and the Capital Distribution Curve

Recall that $\mu = \big\{\boldsymbol{\mu}(t) = \big(\mu_1(t), \mu_2(t), \ldots, \mu_n(t)\big), t \in [0, \infty)\big\}$ signifies the market portfolio, or more specifically, the vector of market weights (or, capitalisation weights, the so-called cap-weights) at time $t \in [0, \infty)$, as defined in Definition 2.12.1. Moreover, from Definition 2.4.10 and by ranking these market weights from the largest market weight $\mu_{(1)}$ to the smallest market weight $\mu_{(n)}$, we have the *ranked market portfolio weight processes*, denoted by

$$\mu_{(\cdot)} \quad = \quad \Big\{\boldsymbol{\mu}_{(\cdot)}(t) = \big(\mu_{(1)}(t), \ldots, \mu_{(n)}(t)\big), t \in [0, \infty)\Big\}, \tag{4.5.1}$$

where we adopt the reverse-order-statistics notation to represent the vector of ranked market weights, (4.5.1), at time $t \in [0, \infty)$ by

$$\max_{1 \le i \le n} \mu_i(t) \quad \triangleq \quad \mu_{(1)}(t) \quad \ge \quad \mu_{(2)}(t) \quad \ge \quad \cdots \quad \ge \quad \mu_{(n-1)}(t) \quad \ge \quad \mu_{(n)}(t) \quad \triangleq \quad \min_{1 \le i \le n} \mu_i(t). \tag{4.5.2}$$

The $\mu_{(k)}(t)$ are the rank processes associated with the market weights $\mu_k(t)$, for $k = 1, \ldots, n$.

Definition 4.5.1 (The Capital Distribution). *We shall refer to the arrangement of the market weights in the form* (4.5.2)*, i.e., in decreasing order from the largest market weight $\mu_{(1)}(t)$ to the smallest market weight $\mu_{(n)}(t)$, as the* ***capital distribution*** *of the market at time $t \in [0, \infty)$. In addition, the graph of the family of the ranked market weights, $\big(\mu_{(1)}(t), \mu_{(2)}(t), \ldots, \mu_{(n)}(t)\big)$, is regarded as the* ***capital distribution curve*** *of the market, or of a cap-weighted index, at time $t \in [0, \infty)$. More explicitly, the capital distribution curve is the log-log plot of the market weights arranged in descending rank, i.e., the logarithms of the market weights versus the logarithms of their respective ranks, $\log \mu_i(t)$ versus $\log r_i(t)$, for $i = 1, \ldots, n$, where $r_k(t)$ represents the rank of the kth stock at time $t \in [0, \infty)$, $\log r_k(t) \mapsto \log \mu_{(k)}(t)$.*

Thus, the capital distribution is the family of market weights arranged in descending order. Moreover, for a market $\mathcal{M}$ with weight processes $\mu_1, \mu_2, \ldots, \mu_n$, the vector $\big(\mu_{(1)}, \mu_{(2)}, \ldots, \mu_{(n)}\big)$ is called the capital distribution

[9] Often abbreviated to CDC, as is commonplace in the related literature. Henceforward, we shall occasionally adopt the use of CDC to represent the capital distribution curve.

of the market. In addition, the capital distribution for a subset of the market can also be defined, and this is of particular interest when the subset is an index (a cap-weighted index) such as the S&P 500 Index, which is composed of 500 of the largest stocks traded on the U.S. Stock Exchange. Although the S&P 500 Index is not the market, it usually holds about 70% of the total market capitalisation including all of the largest stocks. Thus, capital distribution curves of the S&P 500 Index give a reasonable idea of the capital distribution of the entire market. The capital distribution curve of the market or of a cap-weighted index, that is obtained by plotting ordered logarithmic relative capitalisations against the logarithms of their respective ranks, is a *decreasing* curve. For each point on the capital distribution curve, the vertical coordinate represents the logarithmic capitalisation weight of a particular stock, and the horizontal coordinate represents the logarithmic rank of that stock. An alternative to the log-log plot of the capital distribution curve, is the simple plot of the market weights versus their respective ranks, $r_k(t) \mapsto \mu_{(k)}(t)$. However, we shall consider the plot of the capital distribution on the log-log axes since it reveals the most useful and intuitive information than is revealed with the normal plot. Since the capital distribution of the market is equivalent to the size distribution of firms, it has been studied extensively. We shall defer a more discerning discussion of the distribution of capital and the capital distribution curve to a forthcoming chapter.

Thus, having introduced the notions above, we shall continue with the discussion on how to quantify the degree of diversity in an equity market. This involves the characterisation of market diversity in terms of the distribution of equity capital, and consequently in terms of the capital distribution curve. Another avenue of approach, is to consider fluctuations in the capital distribution, and from this glean the effect that any changes in the concentration of capital among the stocks in an equity market will have on diversity. This enables us to derive the key characteristics of stock market diversity which then contributes to the procurement of the fundamental properties of a measure of diversity. Below we provide a plot of the capital distribution curve for the South African JSE ALSI Top40 Index on the 2nd January 2002 and on the 23rd August 2007.

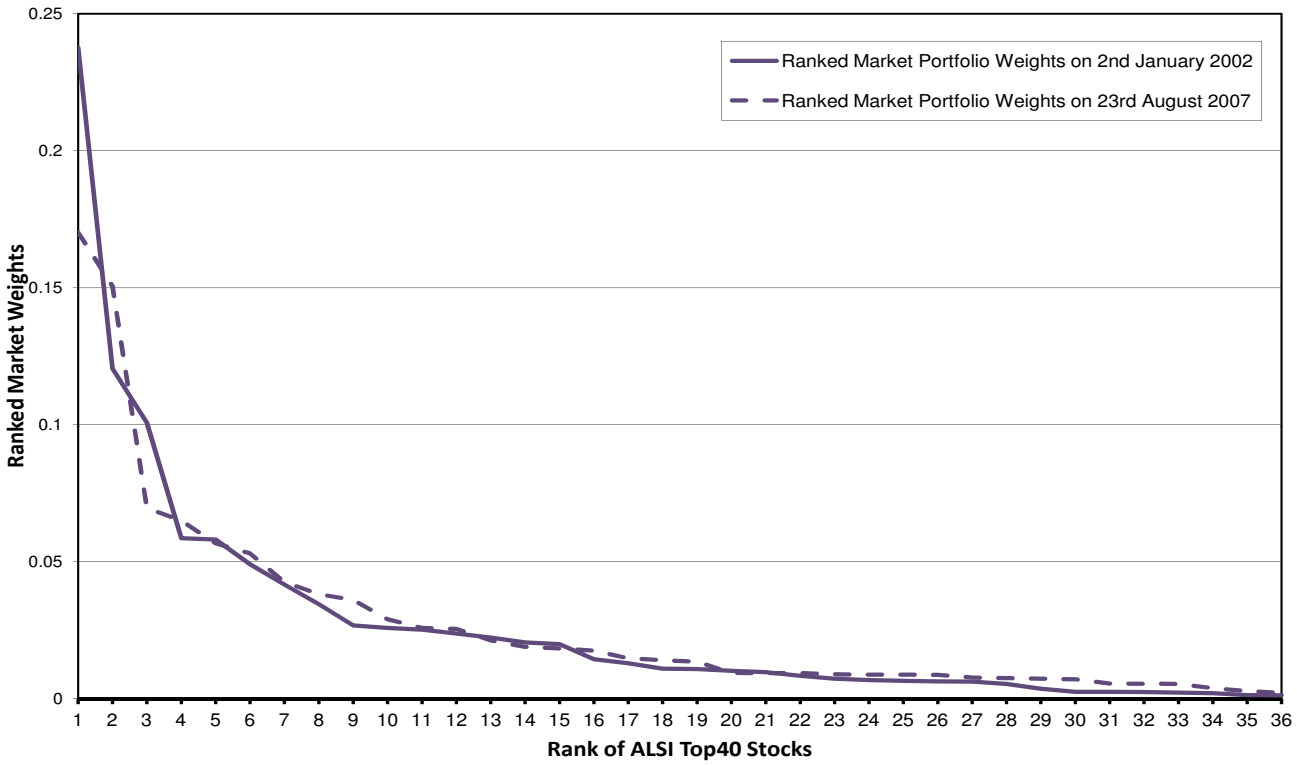

Figure 4.1: Capital Distribution Curve for the JSE ALSI Top40 Index on 2nd January 2002 (solid line) and on the 23rd August 2007 (dashed line).

4.5.1.1 Diversity and the Capital Distribution Curve

Firstly, let us consider a market that is not diverse, then a market of this nature essentially comprises a single stock, that being the largest stock with market weight $\mu_{(1)}$. In such a market, all of the capital is concentrated into this largest stock and we have $\mu_{(1)} = 1$ and $\mu_{(k)} = 0$ for $k = 2, \ldots, n$. Thus, a single point constitutes the capital distribution curve, and this point is positioned above rank 1 on the horizontal axis and to the right of 1 (i.e., 100%) on the vertical axis. This illustrates one of the extreme ends of the scale of diversity, namely the "completely undiversified (or, undistributed) market". On the other extreme end of the scale, we have the so-called "completely diverse market". A market is at its "most diverse" when the capital is equally distributed among all the stocks in an equity market. This corresponds to the notion that no stock is more important than any other, and thus such a market is akin to the equal-weighted market, i.e., $\mu_{(1)} = \mu_{(2)} = \cdots = \mu_{(n-1)} = \mu_{(n)} = \frac{1}{n}$, for a market consisting of n stocks. In this case, the corresponding capital distribution curve is a horizontal line at a height of $\frac{1}{n}$, on the vertical axis.

4.5.1.2 Changes in Diversity

It is evident from Definition 4.2.1 that diversity varies over time, and this variation in diversity is a result of the movement of capital between the larger and smaller stocks in an equity market. More specifically, the ebb and flow of capital between larger and smaller capitalisation stocks manifests itself in the variation in diversity. Thus, changes in diversity occur in conjunction with changes in the distribution of capital. Let us consider this continuous ebb and flow of capital in the market and the effect this will have on the level of diversity. When capital flows from the smaller stocks into the larger stocks, the resulting capital distribution is less concentrated into the smaller stocks with most of the capital accumulated into just a few of the largest stocks. As a consequence, the level of diversity in the market declines, since most of the capital has shifted into larger stocks which causes the equity capital to have an uneven distribution. Whereas, when the capital ebbs back into the smaller stocks which occurs by injecting and redistributing capital from the larger stocks into and among the smaller stocks, the capital that the largest stocks had amassed is now more evenly distributed among all the stocks in the market. Thus, as the concentration of capital reduces, the level of diversity in the market surges. To summarise, diversity of the distribution is lower when capital is concentrated mostly into a few large stocks and is higher when capital is more uniformly spread across all the stocks in the market. Evidently, minimum diversity would occur if the capital in the market were entirely invested in a single stock, i.e., $\mu_j(t) = 1$ for some $j \in \{1, 2, \ldots, n\}$, all the capital from the remaining smaller stocks in the market aggregates into the largest stock. This coincides with the aforementioned "completely undiversified market". Moreover, maximum diversity would be attained if all the stocks had the same capitalisation, i.e., $\mu_1(t) = \mu_2(t) = \cdots = \mu_n(t) = \frac{1}{n}$, so capital flows from the larger stocks into the smaller stocks in a uniform fashion, and this corresponds to the "completely diverse market".

4.5.2 Properties of a Measure of Diversity

In order for a function to be a measure of diversity it needs to possess certain characteristics exhibited by diversity or rather by the variation in diversity, as outlined in the preceding section and in Definition 4.2.1. Mathematically, the variation in diversity is the logarithmic change of a measure of diversity, and so the results obtained heretofore will enable us to establish these characteristics. Ultimately, what we require is a measure of diversity that is unequivocal in nature and that captures and retains the intuitive essence of diversity. The main characteristic of all distribution-based diversity measures is to comprise some "preference" for evenness. A market is said to be maximally diversified whenever all the market weights share the same elementary quantity and there should be minimal diversity whenever the overall quantity is accumulated in a single stock. Formally, let $\widetilde{\boldsymbol{\mu}}(t) = (1, 0, \ldots, 0)$ and $\overline{\boldsymbol{\mu}}(t) = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$, then a distribution-based diversity measure, $\mathbf{S}\big(\boldsymbol{\mu}(t)\big)$, should satisfy

$$\mathbf{S}\big(\widetilde{\boldsymbol{\mu}}(t)\big) \;\leq\; \mathbf{S}\big(\boldsymbol{\mu}(t)\big) \;\leq\; \mathbf{S}\big(\overline{\boldsymbol{\mu}}(t)\big).$$

From the theory of majorisation, it is well-known that the bounds above are satisfied when $\mathbf{S}\big(\boldsymbol{\mu}(t)\big)$ is *symmetric* and *concave* [refer to Marshall & Olkin (2009)]. In what follows, we shall exposit the attributes ascribed to these such functions and discuss thereafter the line of reasoning in establishing these attributes.

4.5.2.1 The Fundamental Properties of a Measure of Diversity

In the preceding subsection we established the essential characteristics displayed by diversity and reinforced the fact that changes in diversity are directly related to changes in the capital distribution. Armed with these facts, we are now able to formulate the three cardinal properties that a sufficient measure of diversity should satisfy, namely twice continuous differentiability, symmetry and concavity.

1. C^2 **Property:**

In the above, we have defined higher order derivatives. Here we focus on the second-order derivatives. To this end, it would be illuminating to recast its definition in the following equivalent form.

Definition 4.5.2 (Twice Continuously Differentiable Function). *A real-valued function of n real variables, $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$, defined on an open subset U of $\mathbb{R}^n$, i.e., $f : \mathbb{R}^n \to \mathbb{R}$ or $f : U \to \mathbb{R}$ for $U \subset \mathbb{R}^n$, is said to be* ***twice continuously differentiable*** *at $\mathbf{x} \in \mathbb{R}^n$, if the first-order partial derivatives with respect to the ith variable, i.e., $\frac{\partial f(\mathbf{x})}{\partial x_i}$, and the second-order partial derivatives with respect to the ith and jth variables, i.e., $\frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j}$, for $i, j = 1, \ldots, n$, exist and are continuous.*[10] *Moreover, a function f is said to be twice continuously differentiable in an open region $U \subset \mathbb{R}^n$, denoted by $f \in C^2(U)$, if it is twice continuously differentiable at every point $\mathbf{x}$ in U. Thus, a real-valued function defined on an open subset U of $\mathbb{R}^n$ is of class C^2, $f \in C^2$, if it is twice continuously differentiable in all n variables.*[11]

2. **Symmetry Property:**

We now introduce the concept of a symmetric function, and the related concept of symmetry.

Definition 4.5.3 (Symmetric Function). *A real-valued function of multiple variables, $f(\mathbf{x}) = f(x_1, x_2)$, defined on an open subset U of $\mathbb{R}^2$, $U \subset \mathbb{R}^2$, i.e., $f : \mathbb{R}^2 \to \mathbb{R}$ or $f : U \to \mathbb{R}$, is said to be* ***symmetric*** *if its value is independent of the order of the variables, i.e., $f(x_1, x_2) \equiv f(x_2, x_1)$, for all $x_1, x_2 \in U$. More precisely, a* ***symmetric function****, is a real-valued function of n variables, $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$, defined on an open subset U of $\mathbb{R}^n$, $U \subset \mathbb{R}^n$, i.e., $f : \mathbb{R}^n \to \mathbb{R}$ or $f : U \to \mathbb{R}$, that is invariant under any permutations of the variables x_i, $i = 1, \ldots, n$.*[12]

All stocks in the market should be treated in the same manner, which implies that a measure of diversity must be unchanged by any permutation of a given set of market weights. Thus, the degree of diversity in the market is not affected by the ordering of the stocks, but instead by the capital that is apportioned to each of the stocks. The condition that ensures the equal treatment of the stocks is symmetry and consequently, diversity, or rather the diversity measure should possess the symmetry property.

3. **Concavity Property:**

We now introduce the concept of a concave function, and the related concept of concavity.

[10] Alternatively, in accordance with Definition D.0.1 of Appendix D, a real-valued *continuously differentiable* function defined on an open subset U of $\mathbb{R}^n$, i.e., $f : \mathbb{R}^n \to \mathbb{R}$ or $f : U \to \mathbb{R}$ for $U \subset \mathbb{R}^n$, is said to be *twice continuously differentiable* at $\mathbf{x} \in \mathbb{R}^n$, if the second-order partial derivatives with respect to the ith and jth variables, i.e., $\frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j}$, for $i, j = 1, \ldots, n$, exist and are continuous.

[11] The notation $C^0(U)$ is sometimes used to denote the set of all continuous functions with domain U. Similarly, $C^1(U)$ is used to denote the set of differentiable functions whose derivative is continuous, $C^2(U)$ for the twice-differentiable functions whose second derivative is continuous, and so on.

[12] Alternatively, a *symmetric function* of n variables is one whose value at any $n-$tuple of arguments is the same as its value at any and every permutation of that $n-$tuple, i.e., the value of the function does not depend on the order of the $n-$tuple arguments.

Definition 4.5.4 (Concave Function, Strictly Concave Function). *Let $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$ be a real-valued function of n real variables defined on a convex*[13] *(not necessarily open) subset U of $\mathbb{R}^n$, $U \subset \mathbb{R}^n$, i.e., $f : \mathbb{R}^n \to \mathbb{R}$ or $f : U \to \mathbb{R}$. The function f is called a* ***concave function*** *if for all $0 \leq \theta \leq 1$ and $\mathbf{x}, \mathbf{y} \in U \subset \mathbb{R}^n$,*

$$f\big(\theta\mathbf{x} + (1-\theta)\mathbf{y}\big) \quad \geq \quad \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y}). \tag{4.5.3}$$

If the inequalities are strict for all $\mathbf{x}, \mathbf{y} \in U$, with $\mathbf{x} \neq \mathbf{y}$, and $0 < \theta < 1$, i.e.,

$$f\big(\theta\mathbf{x} + (1-\theta)\mathbf{y}\big) \quad > \quad \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y}), \tag{4.5.4}$$

then f is called a ***strictly concave function****.*

An alternative definition relating to the differentiability of the function f of a ***single variable*** *is as follows: Let $f(x)$, $f : \mathbb{R} \to \mathbb{R}$ or $f : U \to \mathbb{R}$ be a real-valued continuously differentiable function of a single variable $x \in U \subset \mathbb{R}$, defined on an open convex subset U in $\mathbb{R}$ (i.e., U is an open interval in $\mathbb{R}$). Then the function f is called a* ***concave function*** *on U if the function f' is a* ***decreasing function*** *on U.*

Intuitively, the geometrical interpretation is that f is a concave function if the chord joining any two points on the function lies on or below the function, or if the differentiable function f lies below all of its tangent lines. Clearly, if a function is strictly concave, it is automatically concave.

Recall, that diversity decreases when capital flows into the larger stocks and diversity increases when capital is transferred from the larger stocks to the smaller stocks in the market. Thus, it is required that transferring capital from a larger stock to a smaller stock increases the value of the diversity measure, and analogously transferring capital from a smaller stock to a larger stock decreases the value of that measure. The criterion that encapsulates and fulfills this characteristic behaviour of diversity is that of concavity. In concrete terms, this property means that diversity increases by mixing (the capital distribution). So, the requirement of concavity is essential when considering diversity measures. We shall see further on that the condition of concavity requires the diversity measure function to be twice continuously differentiable. Furthermore, since maximum diversity is associated with equal weights, we require a measure of diversity to be maximised by equal weights. Likewise, minimum diversity must be attained when the weight of the largest stock is 1 and the weights of all the other stocks are 0. Thus, a measure of diversity must be minimised when all the capital is concentrated into a single stock.

4.5.3 Measures of Diversity

Thus, having established the requisite properties of a diversity measure we are now in a position to provide a formal definition of measures of diversity.

Definition 4.5.5 (Diversity Measure). *Let U be an open neighbourhood in $\mathbb{R}^n$ of the open positive unit simplex Δ^{n-1}, where*

$$\Delta^{n-1} \quad = \quad \Big\{\boldsymbol{\mu}(t) = \big(\mu_1(t), \ldots, \mu_n(t)\big) \in \mathbb{R}^n \;\Big|\; \mu_1(t) + \cdots + \mu_n(t) = 1,\ 0 < \mu_i(t) < 1,\ i = 1, \ldots, n\Big\}.$$

Then, a ***positive twice continuously differentiable*** *function, i.e., $f \in C^2$ or $\mathbf{S} \in C^2$, of the market weights, $f(\boldsymbol{\mu}) \in C^2$ or $\mathbf{S}(\boldsymbol{\mu}) \in C^2$, defined on $U \subset \mathbb{R}^n$ of Δ^{n-1}, $f : \Delta^{n-1} \to \mathbb{R}^+$ or $f : U \to \mathbb{R}^+$, alternatively $\mathbf{S} : \Delta^{n-1} \to \mathbb{R}^+$ or $\mathbf{S} : U \to \mathbb{R}^+$, is a* ***measure of diversity*** *if it is* ***symmetric*** *and* ***concave****.*

[13] Given $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$, then $\mathbf{z}$ defined by $\mathbf{z} = \theta\mathbf{x} + (1-\theta)\mathbf{y}$, $0 \leq \theta \leq 1$ is called a *convex combination* of $\mathbf{x}$ and $\mathbf{y}$. Similarly, given k points, $\mathbf{x}_1, \ldots, \mathbf{x}_k$ in $\mathbb{R}^n$, then $\mathbf{z}$ defined by $\mathbf{z} = \theta_1\mathbf{x}_1 + \theta_2\mathbf{x}_2 + \cdots + \theta_k\mathbf{x}_k$, $0 \leq \theta_i \leq 1$, for $i = 1, \ldots, k$, and with $\sum_{i=1}^k \theta_i = 1$ is called a convex combination of these k points. Let U be a subset of $\mathbb{R}^n$, the set U is called a *convex set* if any arbitrary convex combination of any two points of U is also in U, i.e., U is convex if $\mathbf{x}, \mathbf{y} \in U$ implies $\theta\mathbf{x} + (1-\theta)\mathbf{y} \in U$, $0 \leq \theta \leq 1$. Note that U, the domain of f, is restricted to be a convex set. This is necessary to ensure $\theta\mathbf{x} + (1-\theta)\mathbf{y}$ is also in U, whenever $\mathbf{x}$ and $\mathbf{y}$ are in U, for otherwise $f\big(\theta\mathbf{x} + (1-\theta)\mathbf{y}\big)$ has no meaning. The discussion provided here has been largely taken from Takayama (1994), which should be conferred with for a more rigorous and comprehensive treatment.

Measures of diversity have been employed in probability and information theory [see Shannon (1948)], as well as in mathematical ecology [see Simpson (1949) and Hill (1973)]. Here, Definition 4.5.5 is an adaptation of the concept of a measure of diversity to financial equity markets.

Remark 4.5.6. We shall find that many functions defined on Δ^{n-1} generate portfolios. However, recall that the dimension of Δ^{n-1} is $n-1$, yet there are n stocks. The difficulty is that on Δ^{n-1} there is no natural linear coordinate system which treats all the n stocks in the same manner. Hence, it will be convenient to use the standard coordinate system in $\mathbb{R}^n$, even though it is not a coordinate system on Δ^{n-1}. For this reason we shall consider functions that are defined on an open neighbourhood $U \subset \mathbb{R}^n$ of Δ^{n-1}. Functions defined on Δ^{n-1} can always be extended to an open neighbourhood by making the extension constant along lines parallel to one of the axes of $\mathbb{R}^n$ [Fernholz (2002)].

Having considered all of the above, hereinafter, we provide a comprehensive discussion of the devices that will be employed in demonstrating whether a function is indeed a measure of diversity. This involves showing that a certain function satisfies the properties of an appropriate diversity measure. These devices concentrate heavily on the verification of the concavity property, and is the focus of Appendix E.

4.6 Examples of Measures of Diversity

We shall consider some examples of measures of diversity in this section. Note that in this section we shall not consider asymptotic events, as a consequence we shall restrict the time domain to $[0, T]$.

4.6.1 Entropy

Entropy is a measure of the uniformity of a probability distribution that was first used in thermodynamics and statistical mechanics, and more recently has been used in probability theory and information theory. The notion of entropy was introduced by Shannon (1948) in his mathematical formulation of information theory and as a measure of randomness in probability theory, and later in Shannon & Weaver (1964). The *entropy function*, $\mathbf{S}^{\mathrm{E}} : \Delta^{n-1} \to (0, \infty)$, is defined by

$$\mathbf{S}^{\mathrm{E}}(\mathbf{x}) \quad = \quad \mathbf{S}^{\mathrm{E}}(x_1, \ldots, x_n) \quad \triangleq \quad -\sum_{i=1}^{n} x_i \log x_i, \tag{4.6.1}$$

for all $\mathbf{x}$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\mathbf{x} \in \Delta^{n-1}$,

$$\Delta^{n-1} \quad \triangleq \quad \Big\{ \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \;\Big|\; x_1 + \cdots + x_n = 1,\ 0 < x_i < 1,\ i = 1, 2, \ldots, n \Big\}.$$

Then, the corresponding market diversity measure process (the *market entropy function*) is given below by:

Definition 4.6.1 (Market Entropy)**.** *Let μ be the market portfolio. Then the* ***market entropy process****, $\mathbf{S}^{\mathrm{E}}(\boldsymbol{\mu}) = \big\{ \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big), t \in [0, T] \big\}$, $\mathbf{S}^{\mathrm{E}} : \Delta^{n-1} \to (0, \infty)$, is defined by*

$$\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \quad = \quad \mathbf{S}^{\mathrm{E}}\big(\mu_1(t), \ldots, \mu_n(t)\big) \quad \triangleq \quad -\sum_{i=1}^{n} \mu_i(t) \log \mu_i(t), \qquad t \in [0, T], \tag{4.6.2}$$

for all $\boldsymbol{\mu}(t)$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\boldsymbol{\mu}(t) \in \Delta^{n-1}$,

$$\Delta^{n-1} \quad \triangleq \quad \Big\{ \boldsymbol{\mu}(t) = \big(\mu_1(t), \ldots, \mu_n(t)\big) \in \mathbb{R}^n \;\Big|\; \mu_1(t) + \cdots + \mu_n(t) = 1,\ 0 < \mu_i(t) < 1,\ i = 1, 2, \ldots, n \Big\}.$$

It follows from this definition that $\mathbf{S}^{\mathrm{E}}(\boldsymbol{\mu})$ is a continuous semimartingale. Since, $0 < \mu_i(t) < 1$ for all $i = 1, 2, \ldots, n$, we have $\log \mu_i(t) < 0$ for all $i = 1, 2, \ldots, n$, which in turn implies

$$\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \quad = \quad -\sum_{i=1}^{n} \mu_i(t) \log \mu_i(t) \quad > \quad 0.$$

Recall, that minimum diversity occurs precisely when the entire capital in the market is concentrated into a single stock, i.e., when the weight of the largest stock is 1, $\mu_{(1)}(t) = 1$, and the weights of the remaining smaller ranked stocks are 0, $\mu_{(k)}(t) = 0$ for all $k = 2, \ldots, n$. Thus, we require a measure of diversity to be minimised when all the capital is entirely invested in the single largest stock. For the entropy diversity measure, minimum diversity occurs when the market entropy process is given by

$$\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \quad = \quad -\mu_{(1)}(t)\log\mu_{(1)}(t) \quad = \quad 0.$$

Thus a value of 0 for the market entropy process indicates that the market is not diverse. Hence, in a diverse market, market entropy is bounded away from zero. Recall, that maximum diversity occurs precisely when all the stocks in the market have the same capitalisation, and thus the capital is evenly distributed among all the stocks in the market, i.e., $\mu_{(1)}(t) = \mu_{(2)}(t) = \cdots = \mu_{(n)}(t) = \frac{1}{n}$. We thus require a measure of diversity to be maximised by equal weights. For the entropy diversity measure, maximum diversity occurs when the market entropy process is given by

$$\begin{aligned}\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) &= n\left(-\frac{1}{n}\log\left(\frac{1}{n}\right)\right)\\ &= -\log(n)^{-1}\\ &= \log(n).\end{aligned}$$

Therefore, a value of $\log(n)$ for the market entropy process indicates that the market is completely diverse. Consequently, we have the following bounds on the market entropy process

$$0 \quad < \quad \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \quad \leq \quad \log n, \qquad t \in [0,T], \qquad \text{a.s.} \tag{4.6.3}$$

To show that the market entropy process is indeed a measure of diversity, we need to demonstrate that it satisfies the three primary properties that a sufficient diversity measure should possess, those being twice continuous differentiability, symmetry and concavity. Clearly, the positive market entropy process is both twice continuously differentiable and symmetric. To demonstrate the concavity property we shall verify statement (ii) of Lemma E.2.7 of Appendix E by showing that the market entropy process $\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$ is strictly concave on Δ^{n-1} if and only if its Hessian matrix $\mathbf{H}^{\mathrm{E}} = \mathbf{HS}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) = D^2\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) = \big(D_iD_j\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)\big)_{1\leq i,j\leq n} = \big(D_{ij}\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)\big)_{1\leq i,j\leq n} \triangleq \big(\frac{\partial^2\mathbf{S}^{\mathrm{E}}(\boldsymbol{\mu}(t))}{\partial\mu_i(t)\partial\mu_j(t)}\big)_{1\leq i,j\leq n}$, with $\boldsymbol{\mu}(t) = \big(\mu_1(t), \mu_2(t), \ldots, \mu_n(t)\big)$, is negative definite, i.e., if $\mathbf{x}\mathbf{H}^{\mathrm{E}}\mathbf{x}^{\mathsf{T}} < 0$ for all $\mathbf{x} \in \Delta^{n-1} \subset \mathbb{R}^n$. For the market entropy process (4.6.2) of Definition 4.6.1, we have for all $i = 1, 2, \ldots, n$,

$$D_i\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \quad = \quad \frac{\partial\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}{\partial\mu_i(t)} \quad = \quad -\log\mu_i(t) - 1. \tag{4.6.4}$$

Therefore, for all $i = 1, 2, \ldots, n$, we have

$$D_{ii}\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \quad = \quad \frac{\partial^2\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}{\partial\mu_i(t)^2} \quad = \quad -\frac{1}{\mu_i(t)}, \tag{4.6.5}$$

and for all $i \neq j$, $i, j = 1, 2, \ldots, n$, we have

$$D_{ij}\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \quad = \quad \frac{\partial^2\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}{\partial\mu_i(t)\partial\mu_j(t)} \quad = \quad 0. \tag{4.6.6}$$

Therefore, the $n \times n$ Hessian matrix of $\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$ is given by

$$\mathbf{H}^{\mathrm{E}} \quad = \quad \mathbf{HS}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \quad = \quad \begin{bmatrix} -\dfrac{1}{\mu_1(t)} & 0 & \cdots & 0 \\ 0 & -\dfrac{1}{\mu_2(t)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\dfrac{1}{\mu_n(t)} \end{bmatrix}.$$

Consequently, we have

$$\mathbf{x}\mathbf{H}^{\mathrm{E}}\mathbf{x}^{\mathsf{T}} \;=\; -\Big(\frac{x_1^2}{\mu_1(t)}+\frac{x_2^2}{\mu_2(t)}+\cdots+\frac{x_n^2}{\mu_n(t)}\Big) \;=\; -\sum_{i=1}^{n}\frac{x_i^2}{\mu_i(t)} \;\leq\; 0,$$

since $0<\mu_i(t)<1$, for all $i=1,2,\ldots,n$. Moreover, for $\mathbf{x}=\boldsymbol{\mu}(t)\in\Delta^{n-1}$, we have

$$\boldsymbol{\mu}(t)\mathbf{H}^{\mathrm{E}}\boldsymbol{\mu}^{\mathsf{T}}(t) \;=\; -\Big(\mu_1(t)+\mu_2(t)+\cdots+\mu_n(t)\Big) \;=\; -\sum_{i=1}^{n}\mu_i(t) \;=\; -1 \;<\; 0.$$

Therefore, we have

$$\boldsymbol{\mu}(t)\mathbf{H}^{\mathrm{E}}\boldsymbol{\mu}^{\mathsf{T}}(t) \;<\; 0.$$

Thus, the $n\times n$ Hessian matrix, $\mathbf{H}^{\mathrm{E}}$, is negative definite, and by statement (ii) of Lemma E.2.7 of Appendix E, we can conclude that the market entropy process, $\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$, is strictly concave, and is as a result a measure of diversity. Of course, one can prove the above concavity result by simply noticing that the function $f(x)=-x\log x$ is strictly concave (by Lemma E.2.8 of Appendix E) and by appealing to Lemma E.2.9 of Appendix E which states that the linear combination of concave functions is itself concave, since the market entropy process is a linear combination of the concave functions of the form $f(x)=-x\log x$. In conclusion, the market entropy process satisfies the three requirements of a sufficient diversity measure, and is thus an appropriate measure of diversity for our purposes. Entropy is the archetypical measure of diversity, and is the one that is most frequently adopted in this setting. However, entropy, is by no means the only and last resort when it comes to suitable diversity measures. There are other functions that also measure the extent of diversity in the market (i.e., the extent to which capital is distributed among the securities in the market), and that satisfy the requirements of a sufficient diversity measure. These functions will actually prove more useful than the entropy diversity measure. These are considered next.

4.6.2 Modified Entropy

The *modified entropy function*, $\mathbf{S}^{\mathrm{ME}}=\mathbf{S}_c^{\mathrm{E}}:\Delta^{n-1}\to(0,\infty)$, for any given $c\in(0,\infty)$, is defined by

$$\mathbf{S}^{\mathrm{ME}}(\mathbf{x}) \;=\; \mathbf{S}_c^{\mathrm{E}}(\mathbf{x}) \;=\; \mathbf{S}_c^{\mathrm{E}}(x_1,\ldots,x_n) \;\triangleq\; c+\mathbf{S}^{\mathrm{E}}(\mathbf{x}) \;=\; c-\sum_{i=1}^{n}x_i\log x_i, \tag{4.6.7}$$

for all $\mathbf{x}$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\mathbf{x}\in\Delta^{n-1}$. Then, the corresponding market diversity measure process (the *modified market entropy function*) is given below by:

Definition 4.6.2 (Modified Market Entropy)**.** *Let μ be the market portfolio. Then the* ***modified market entropy process****, $\mathbf{S}^{\mathrm{ME}}(\boldsymbol{\mu})=\mathbf{S}_c^{\mathrm{E}}(\boldsymbol{\mu})=\big\{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big),t\in[0,T]\big\}$, $\mathbf{S}^{\mathrm{ME}}=\mathbf{S}_c^{\mathrm{E}}:\Delta^{n-1}\to(0,\infty)$, is defined for all $t\in[0,T]$ by*

$$\mathbf{S}^{\mathrm{ME}}\big(\boldsymbol{\mu}(t)\big) \;=\; \mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \;=\; \mathbf{S}_c^{\mathrm{E}}\big(\mu_1(t),\ldots,\mu_n(t)\big) \;\triangleq\; c+\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \;=\; c-\sum_{i=1}^{n}\mu_i(t)\log\mu_i(t), \tag{4.6.8}$$

for all $\boldsymbol{\mu}(t)$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\boldsymbol{\mu}(t)\in\Delta^{n-1}$.

Thus, we have the following

$$\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \;=\; \mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)-c, \qquad t\in[0,T]. \tag{4.6.9}$$

Therefore, we also have

$$\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \;=\; c\sum_{i=1}^{n}\mu_i(t)-\sum_{i=1}^{n}\mu_i(t)\log\mu_i(t) \;=\; \sum_{i=1}^{n}\mu_i(t)\big(c-\log\mu_i(t)\big). \tag{4.6.10}$$

It follows from this definition that the modified market entropy, $\mathbf{S}_c^{\mathrm{E}}(\boldsymbol{\mu})$, is a continuous semimartingale and, from (4.6.3), satisfies the following bounds

$$c \;<\; \mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \;\leq\; c + \log n, \qquad t \in [0,T], \qquad \text{a.s.} \tag{4.6.11}$$

Clearly, the positive modified market entropy process is both twice continuously differentiable and symmetric. To demonstrate the concavity property we shall verify statement (ii) of Lemma E.2.7 of Appendix E by showing that the modified market entropy process $\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$ is strictly concave on Δ^{n-1} if and only if its Hessian matrix $\mathbf{H}_c^{\mathrm{E}} = \mathbf{H}\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) = D^2\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) = \big(D_iD_j\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)\big)_{1\leq i,j\leq n} = \big(D_{ij}\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)\big)_{1\leq i,j\leq n} \triangleq \big(\frac{\partial^2\mathbf{S}_c^{\mathrm{E}}(\boldsymbol{\mu}(t))}{\partial\mu_i(t)\partial\mu_j(t)}\big)_{1\leq i,j\leq n}$, with $\boldsymbol{\mu}(t) = \big(\mu_1(t), \mu_2(t), \ldots, \mu_n(t)\big)$, is negative definite, i.e., if $\mathbf{x}\mathbf{H}_c^{\mathrm{E}}\mathbf{x}^{\mathsf{T}} < 0$ for all $\mathbf{x} \in \Delta^{n-1} \subset \mathbb{R}^n$. For the modified market entropy process (4.6.8) of Definition 4.6.2, using equation (4.6.9) together with equation (4.6.4), we have for all $i = 1, 2, \ldots, n$,

$$D_i\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \;=\; D_i\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \;=\; -\log\mu_i(t) - 1. \tag{4.6.12}$$

Therefore, using equation (4.6.9) together with equation (4.6.5), for all $i = 1, 2, \ldots, n$, we have

$$D_{ii}\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \;=\; D_{ii}\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \;=\; -\frac{1}{\mu_i(t)}, \tag{4.6.13}$$

and, using equation (4.6.9) together with equation (4.6.6), for all $i \neq j$, $i, j = 1, 2, \ldots, n$, we have

$$D_{ij}\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \;=\; D_{ij}\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \;=\; 0. \tag{4.6.14}$$

Therefore, the $n \times n$ Hessian matrix of $\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$ is given by

$$\mathbf{H}_c^{\mathrm{E}} \;=\; \mathbf{H}\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \;=\; \begin{bmatrix} -\dfrac{1}{\mu_1(t)} & 0 & \cdots & 0 \\ 0 & -\dfrac{1}{\mu_2(t)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\dfrac{1}{\mu_n(t)} \end{bmatrix}.$$

Therefore, we have

$$\mathbf{H}_c^{\mathrm{E}} \;=\; \mathbf{H}^{\mathrm{E}}.$$

Consequently, we have for all $\boldsymbol{\mu}(t) \in \Delta^{n-1}$,

$$\boldsymbol{\mu}(t)\mathbf{H}_c^{\mathrm{E}}\boldsymbol{\mu}^{\mathsf{T}}(t) \;=\; \boldsymbol{\mu}(t)\mathbf{H}^{\mathrm{E}}\boldsymbol{\mu}^{\mathsf{T}}(t) \;<\; 0.$$

Therefore, we have

$$\boldsymbol{\mu}(t)\mathbf{H}_c^{\mathrm{E}}\boldsymbol{\mu}^{\mathsf{T}}(t) \;<\; 0.$$

Thus, the $n \times n$ Hessian matrix, $\mathbf{H}_c^{\mathrm{E}}$, is negative definite, and by statement (ii) of Lemma E.2.7 of Appendix E, we can conclude that the modified market entropy process, $\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$, is strictly concave, and is as a result a measure of diversity. Of course, one can show the above concavity result by simply recalling that the entropy function $\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$ is concave and by appealing to Lemma E.2.9 of Appendix E which states that the linear combination of concave functions is again concave, since the modified market entropy process is the linear combination of the concave function $\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$ with the positive constant c. In conclusion, the modified market entropy process satisfies the three requirements of a sufficient diversity measure, and is thus an appropriate measure of diversity for our purposes.

4.6.3 The $\mathbf{D}_p$ Index

Besides entropy, the most important measure of diversity for our purposes is the $\mathbf{D}_p$ index. The $\mathbf{D}_p$*-function*, $\mathbf{D}_p : \Delta^{n-1} \to (0, \infty)$, for any arbitrary given $0 < p < 1$, is defined by

$$\mathbf{D}_p(\mathbf{x}) \;=\; \mathbf{D}_p(x_1, \ldots, x_n) \;\triangleq\; \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}}, \tag{4.6.15}$$

for all $\mathbf{x}$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\mathbf{x} \in \Delta^{n-1}$. Then, the corresponding market diversity measure process (the *market* $\mathbf{D}_p$*-function*) is given below by:

Definition 4.6.3 (Market $\mathbf{D}_p$ Index)**.** *Let μ be the market portfolio. Then the* ***market*** $\mathbf{D}_p$ ***index process****,* $\mathbf{D}_p(\boldsymbol{\mu}) = \big\{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big), t \in [0, T]\big\}$*,* $\mathbf{D}_p : \Delta^{n-1} \to (0, \infty)$*, is defined by*

$$\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) \;=\; \mathbf{D}_p\big(\mu_1(t), \ldots, \mu_n(t)\big) \;\triangleq\; \left(\sum_{i=1}^{n} \big(\mu_i(t)\big)^p\right)^{\frac{1}{p}}, \qquad t \in [0, T], \tag{4.6.16}$$

for all $\boldsymbol{\mu}(t)$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\boldsymbol{\mu}(t) \in \Delta^{n-1}$.

This is a measure of diversity, and has in fact been used to construct an institutional equity investment product [see Fernholz, Garvy & Hannon (1998)]. It follows from this definition that the market $\mathbf{D}_p(\boldsymbol{\mu})$ index, is a continuous semimartingale. For the $\mathbf{D}_p$-function, we have the simple bounds

$$1 \;=\; \sum_{i=1}^{n} \mu_i(t) \;=\; \left(\sum_{i=1}^{n} \mu_i(t)\right)^p \;\leq\; \sum_{i=1}^{n} \big(\mu_i(t)\big)^p \;=\; \Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^p \;\leq\; n^{1-p}. \tag{4.6.17}$$

In other words, the minimum of $\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)$ occurs when the entire market is concentrated in one stock, and its maximum when all stocks have the same capitalisation, this justifies considering the $\mathbf{D}_p$-function as a measure of diversity. Recall, that minimum diversity occurs precisely when the entire capital in the market is concentrated into a single stock, i.e., when the weight of the largest stock is 1, $\mu_{(1)}(t) = 1$, and the weights of the remaining smaller ranked stocks are 0, $\mu_{(k)}(t) = 0$ for all $k = 2, \ldots, n$. Thus, we require a measure of diversity to be minimised when all the capital is entirely invested in the single largest stock. For the $\mathbf{D}_p$ diversity measure, minimum diversity occurs when the market $\mathbf{D}_p$ index process is given by

$$\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) \;=\; \Big(\big(\mu_{(1)}(t)\big)^p\Big)^{\frac{1}{p}} \;=\; \mu_{(1)}(t) \;=\; 1.$$

Thus a value of 1 for the market $\mathbf{D}_p$ index process indicates that the market is not diverse. Recall, that maximum diversity occurs precisely when all the stocks in the market have the same capitalisation, and thus the capital is evenly distributed among all the stocks in the market, i.e., $\mu_{(1)}(t) = \mu_{(2)}(t) = \cdots = \mu_{(n)}(t) = \frac{1}{n}$. We thus require a measure of diversity to be maximised by equal weights. For the $\mathbf{D}_p$ diversity measure, maximum diversity occurs when the market $\mathbf{D}_p$ index process is given by

$$\begin{aligned} \mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) \;&=\; \Big(n\big(1/n\big)^p\Big)^{\frac{1}{p}} \\ &=\; \big(n^{1-p}\big)^{\frac{1}{p}} \\ &=\; n^{(1-p)/p}, \end{aligned}$$

where $\frac{1-p}{p} > 1$ for $0 < p < 1$. Therefore, a value of $n^{(1-p)/p}$ for the market $\mathbf{D}_p$ index process indicates that the market is completely diverse. Consequently, we have the following bounds on the market $\mathbf{D}_p$ index process

$$1 \;<\; \mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) \;\leq\; n^{(1-p)/p}, \qquad t \in [0, T], \qquad \text{a.s.} \tag{4.6.18}$$

To show that the market $\mathbf{D}_p$ index process is indeed a measure of diversity, we need to demonstrate that it satisfies the three primary properties that a sufficient diversity measure should possess, as stipulated already. Clearly, the positive market $\mathbf{D}_p$ index process is both twice continuously differentiable and symmetric. To

demonstrate the concavity property we shall verify statement (i) of Lemma E.2.7 of Appendix E by showing that the market $\mathbf{D}_p$ index process $\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)$ is concave on Δ^{n-1} if and only if its Hessian matrix $\mathbf{H}^{\mathbf{D}_p} = \mathbf{H}\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) = D^2\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) = \big(D_iD_j\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)_{1\le i,j\le n} = \big(D_{ij}\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)_{1\le i,j\le n} \triangleq \big(\frac{\partial^2\mathbf{D}_p(\boldsymbol{\mu}(t))}{\partial\mu_i(t)\partial\mu_j(t)}\big)_{1\le i,j\le n}$, with $\boldsymbol{\mu}(t) = \big(\mu_1(t), \mu_2(t), \ldots, \mu_n(t)\big)$, is negative semidefinite, i.e., if $\mathbf{x}\mathbf{H}^{\mathbf{D}_p}\mathbf{x}^\mathsf{T} \le 0$ for all $\mathbf{x} \in \Delta^{n-1} \subset \mathbb{R}^n$. For the market $\mathbf{D}_p$ index process (4.6.16) of Definition 4.6.3, we have for all $i = 1, 2, \ldots, n$,

$$D_i\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) = \frac{\partial\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\partial\mu_i(t)} = \big(\mu_i(t)\big)^{p-1}\Big(\sum_{k=1}^n\big(\mu_k(t)\big)^p\Big)^{\frac{1}{p}-1} \tag{4.6.19}$$

$$= \frac{\big(\mu_i(t)\big)^{p-1}\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\sum_{k=1}^n\big(\mu_k(t)\big)^p}. \tag{4.6.20}$$

Therefore, for all $i = 1, 2, \ldots, n$, we have

$$\begin{aligned}
D_{ii}\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) &= \frac{\partial^2\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\partial\mu_i(t)^2} \\
&= (p-1)\big(\mu_i(t)\big)^{p-2}\Big(\sum_{k=1}^n\big(\mu_k(t)\big)^p\Big)^{\frac{1}{p}-1} + \big(\mu_i(t)\big)^{p-1}\left(\frac{1}{p}-1\right)\Big(\sum_{k=1}^n\big(\mu_k(t)\big)^p\Big)^{\frac{1}{p}-2}p\big(\mu_i(t)\big)^{p-1} \\
&= (p-1)\big(\mu_i(t)\big)^{p-2}\Big(\sum_{k=1}^n\big(\mu_k(t)\big)^p\Big)^{\frac{1}{p}-1} + (1-p)\big(\mu_i(t)\big)^{2p-2}\Big(\sum_{k=1}^n\big(\mu_k(t)\big)^p\Big)^{\frac{1}{p}-2} \\
&= (p-1)\big(\mu_i(t)\big)^{p-2}\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big(\sum_{k=1}^n\big(\mu_k(t)\big)^p\Big)^{-1} + (1-p)\big(\mu_i(t)\big)^{2p-2}\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big(\sum_{k=1}^n\big(\mu_k(t)\big)^p\Big)^{-2} && (4.6.21) \\
&= (p-1)\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big(\mu_i(t)\big)^{p-2}}{\sum_{k=1}^n\big(\mu_k(t)\big)^p} - (p-1)\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big(\mu_i(t)\big)^{2p-2}}{\Big(\sum_{k=1}^n\big(\mu_k(t)\big)^p\Big)^2} && (4.6.22) \\
&= (p-1)\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\left[\frac{\big(\mu_i(t)\big)^{p-2}}{\sum_{k=1}^n\big(\mu_k(t)\big)^p} - \frac{\big(\mu_i(t)\big)^{2p-2}}{\Big(\sum_{k=1}^n\big(\mu_k(t)\big)^p\Big)^2}\right], && (4.6.23)
\end{aligned}$$

and for all $i \ne j$, $i, j = 1, 2, \ldots, n$, we have

$$\begin{aligned}
D_{ij}\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) = \frac{\partial^2\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\partial\mu_i(t)\partial\mu_j(t)} &= \big(\mu_i(t)\big)^{p-1}\left(\frac{1}{p}-1\right)\Big(\sum_{k=1}^n\big(\mu_k(t)\big)^p\Big)^{\frac{1}{p}-2}p\big(\mu_j(t)\big)^{p-1} \\
&= (1-p)\big(\mu_i(t)\big)^{p-1}\big(\mu_j(t)\big)^{p-1}\Big(\sum_{k=1}^n\big(\mu_k(t)\big)^p\Big)^{\frac{1}{p}-2} \\
&= (1-p)\big(\mu_i(t)\big)^{p-1}\big(\mu_j(t)\big)^{p-1}\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big(\sum_{k=1}^n\big(\mu_k(t)\big)^p\Big)^{-2} && (4.6.24) \\
&= -(p-1)\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big(\mu_i(t)\big)^{p-1}\big(\mu_j(t)\big)^{p-1}}{\Big(\sum_{k=1}^n\big(\mu_k(t)\big)^p\Big)^2}. && (4.6.25)
\end{aligned}$$

Define $y\big(\boldsymbol{\mu}(t)\big) := \Big(\sum_{k=1}^n\big(\mu_k(t)\big)^p\Big)^{-1}$ or $y(\boldsymbol{\mu}) := \Big(\sum_{k=1}^n\mu_k^p\Big)^{-1}$, and $\mathfrak{a} := p-1$, then the $n\times n$ Hessian matrix of $\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)$ is given by

$$\mathbf{H}^{\mathbf{D}_p} =$$

$$\begin{bmatrix}
\mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})\Big[y(\boldsymbol{\mu})\mu_1^{p-2} - y^2(\boldsymbol{\mu})\mu_1^{2p-2}\Big] & -\mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})y^2(\boldsymbol{\mu})\mu_1^{p-1}\mu_2^{p-1} & \cdots & -\mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})y^2(\boldsymbol{\mu})\mu_1^{p-1}\mu_n^{p-1} \\
-\mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})y^2(\boldsymbol{\mu})\mu_2^{p-1}\mu_1^{p-1} & \mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})\Big[y(\boldsymbol{\mu})\mu_2^{p-2} - y^2(\boldsymbol{\mu})\mu_2^{2p-2}\Big] & \cdots & -\mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})y^2(\boldsymbol{\mu})\mu_2^{p-1}\mu_n^{p-1} \\
\vdots & \vdots & \ddots & \vdots \\
-\mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})y^2(\boldsymbol{\mu})\mu_n^{p-1}\mu_1^{p-1} & -\mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})y^2(\boldsymbol{\mu})\mu_n^{p-1}\mu_2^{p-1} & \cdots & \mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})\Big[y(\boldsymbol{\mu})\mu_n^{p-2} - y^2(\boldsymbol{\mu})\mu_n^{2p-2}\Big]
\end{bmatrix}.$$

Consequently, we have for all $\boldsymbol{\mu}(t) \in \Delta^{n-1}$,

$$\begin{aligned}
\boldsymbol{\mu}(t)\mathbf{H}^{\mathbf{D}_p}\boldsymbol{\mu}^{\mathsf{T}}(t) &= \mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})\Big[y(\boldsymbol{\mu})\mu_1^p - y^2(\boldsymbol{\mu})\mu_1^{2p}\Big] - \mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})y^2(\boldsymbol{\mu})\mu_1^p\mu_2^p - \cdots - \mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})y^2(\boldsymbol{\mu})\mu_1^p\mu_n^p \\
&\quad - \mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})y^2(\boldsymbol{\mu})\mu_2^p\mu_1^p + \mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})\Big[y(\boldsymbol{\mu})\mu_2^p - y^2(\boldsymbol{\mu})\mu_2^{2p}\Big] - \cdots - \mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})y^2(\boldsymbol{\mu})\mu_2^p\mu_n^p \\
&\quad - \cdots - \mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})y^2(\boldsymbol{\mu})\mu_n^p\mu_1^p - \mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})y^2(\boldsymbol{\mu})\mu_n^p\mu_2^p - \cdots + \mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})\Big[y(\boldsymbol{\mu})\mu_n^p - y^2(\boldsymbol{\mu})\mu_n^{2p}\Big] \\
&= \mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})y(\boldsymbol{\mu})\mu_1^p - \mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})y^2(\boldsymbol{\mu})\mu_1^{2p} - \mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})y^2(\boldsymbol{\mu})\mu_1^p\mu_2^p - \cdots - \mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})y^2(\boldsymbol{\mu})\mu_1^p\mu_n^p \\
&\quad - \mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})y^2(\boldsymbol{\mu})\mu_2^p\mu_1^p + \mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})y(\boldsymbol{\mu})\mu_2^p - \mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})y^2(\boldsymbol{\mu})\mu_2^{2p} - \cdots - \mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})y^2(\boldsymbol{\mu})\mu_2^p\mu_n^p \\
&\quad - \cdots - \mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})y^2(\boldsymbol{\mu})\mu_n^p\mu_1^p - \mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})y^2(\boldsymbol{\mu})\mu_n^p\mu_2^p - \cdots + \mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})y(\boldsymbol{\mu})\mu_n^p - \mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})y^2(\boldsymbol{\mu})\mu_n^{2p} \\
&= \mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})y(\boldsymbol{\mu})\left[\sum_{k=1}^n \mu_k^p\right] - \mathfrak{a}\mathbf{D}_p(\boldsymbol{\mu})y^2(\boldsymbol{\mu})\left[\sum_{k,m=1}^n \mu_k^p\mu_m^p\right] \\
&= \mathfrak{a}\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)y\big(\boldsymbol{\mu}(t)\big)\left[\sum_{k=1}^n \big(\mu_k(t)\big)^p\right] - \mathfrak{a}\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)y^2\big(\boldsymbol{\mu}(t)\big)\left[\sum_{k,m=1}^n \big(\mu_k(t)\big)^p\big(\mu_m(t)\big)^p\right].
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\boldsymbol{\mu}(t)\mathbf{H}^{\mathbf{D}_p}\boldsymbol{\mu}^{\mathsf{T}}(t) \\
&= (p-1)\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\sum_{k=1}^n \big(\mu_k(t)\big)^p}\left[\sum_{k=1}^n \big(\mu_k(t)\big)^p\right] - (p-1)\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\Big(\sum_{k=1}^n \big(\mu_k(t)\big)^p\Big)^2}\left[\sum_{k,m=1}^n \big(\mu_k(t)\big)^p\big(\mu_m(t)\big)^p\right] \\
&= (p-1)\left[\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\frac{\sum_{k=1}^n \big(\mu_k(t)\big)^p}{\sum_{k=1}^n \big(\mu_k(t)\big)^p} - \mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\frac{\sum_{k,m=1}^n \big(\mu_k(t)\big)^p\big(\mu_m(t)\big)^p}{\Big(\sum_{k=1}^n \big(\mu_k(t)\big)^p\Big)^2}\right] \\
&= (p-1)\left[\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\frac{\sum_{k=1}^n \big(\mu_k(t)\big)^p}{\sum_{k=1}^n \big(\mu_k(t)\big)^p} - \mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\frac{\Big(\sum_{k=1}^n \big(\mu_k(t)\big)^p\Big)^2}{\Big(\sum_{k=1}^n \big(\mu_k(t)\big)^p\Big)^2}\right] \\
&= (p-1)\Big[\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) - \mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big] \\
&= 0.
\end{aligned}$$

Alternatively, since $\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p = \sum_{k=1}^n \big(\mu_k(t)\big)^p$, for the market $\mathbf{D}_p$ index process, we have for all $i = 1, 2, \ldots, n$,

$$\begin{aligned}
D_i\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) &= \frac{\partial \mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\partial \mu_i(t)} = \frac{\big(\mu_i(t)\big)^{p-1}\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p} \\
&= \big(\mu_i(t)\big)^{p-1}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p}. \qquad (4.6.26)
\end{aligned}$$

Therefore, for all $i = 1, 2, \ldots, n$, we have

$$\begin{aligned}
D_{ii}\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) &= \frac{\partial^2 \mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\partial \mu_i(t)^2} \\
&= (p-1)\big(\mu_i(t)\big)^{p-2}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p} - (p-1)\big(\mu_i(t)\big)^{p-1}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{-p}D_i\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) \\
&= (p-1)\big(\mu_i(t)\big)^{p-2}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p} - (p-1)\big(\mu_i(t)\big)^{p-1}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{-p}\big(\mu_i(t)\big)^{p-1}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p} \\
&= (p-1)\big(\mu_i(t)\big)^{p-2}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p} - (p-1)\big(\mu_i(t)\big)^{2p-2}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p} \qquad (4.6.27) \\
&= (p-1)\left[\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p}\big(\mu_i(t)\big)^{p-2} - \Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p}\big(\mu_i(t)\big)^{2p-2}\right], \qquad (4.6.28)
\end{aligned}$$

and for all $i \neq j$, $i, j = 1, 2, \ldots, n$, we have

$$\begin{aligned}
D_{ij}\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) &= \frac{\partial^2 \mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\partial \mu_i(t)\partial \mu_j(t)} = -(p-1)\big(\mu_i(t)\big)^{p-1}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{-p} D_j\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) \\
&= -(p-1)\big(\mu_i(t)\big)^{p-1}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{-p}\big(\mu_j(t)\big)^{p-1}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p} \\
&= -(p-1)\big(\mu_i(t)\big)^{p-1}\big(\mu_j(t)\big)^{p-1}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p} \qquad (4.6.29) \\
&= -(p-1)\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p}\big(\mu_i(t)\big)^{p-1}\big(\mu_j(t)\big)^{p-1}. \qquad (4.6.30)
\end{aligned}$$

Then the $n \times n$ Hessian matrix of $\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)$ is given by

$$\mathbf{H}^{\mathbf{D}_p} =$$

$$\begin{bmatrix}
\mathfrak{a}\Big[\mathbf{D}_p^{1-p}(\boldsymbol{\mu})\mu_1^{p-2} - \mathbf{D}_p^{1-2p}(\boldsymbol{\mu})\mu_1^{2p-2}\Big] & -\mathfrak{a}\mathbf{D}_p^{1-2p}(\boldsymbol{\mu})\mu_1^{p-1}\mu_2^{p-1} & \cdots & -\mathfrak{a}\mathbf{D}_p^{1-2p}(\boldsymbol{\mu})\mu_1^{p-1}\mu_n^{p-1} \\
-\mathfrak{a}\mathbf{D}_p^{1-2p}(\boldsymbol{\mu})\mu_2^{p-1}\mu_1^{p-1} & \mathfrak{a}\Big[\mathbf{D}_p^{1-p}(\boldsymbol{\mu})\mu_2^{p-2} - \mathbf{D}_p^{1-2p}(\boldsymbol{\mu})\mu_2^{2p-2}\Big] & \cdots & -\mathfrak{a}\mathbf{D}_p^{1-2p}(\boldsymbol{\mu})\mu_2^{p-1}\mu_n^{p-1} \\
\vdots & \vdots & \ddots & \vdots \\
-\mathfrak{a}\mathbf{D}_p^{1-2p}(\boldsymbol{\mu})\mu_n^{p-1}\mu_1^{p-1} & -\mathfrak{a}\mathbf{D}_p^{1-2p}(\boldsymbol{\mu})\mu_n^{p-1}\mu_2^{p-1} & \cdots & \mathfrak{a}\Big[\mathbf{D}_p^{1-p}(\boldsymbol{\mu})\mu_n^{p-2} - \mathbf{D}_p^{1-2p}(\boldsymbol{\mu})\mu_n^{2p-2}\Big]
\end{bmatrix}.$$

Consequently, we have for all $\boldsymbol{\mu}(t) \in \Delta^{n-1}$,

$$\begin{aligned}
\boldsymbol{\mu}(t)\mathbf{H}^{\mathbf{D}_p}\boldsymbol{\mu}^{\mathsf{T}}(t) &= \mathfrak{a}\Big[\mathbf{D}_p^{1-p}(\boldsymbol{\mu})\mu_1^p - \mathbf{D}_p^{1-2p}(\boldsymbol{\mu})\mu_1^{2p}\Big] - \mathfrak{a}\mathbf{D}_p^{1-2p}(\boldsymbol{\mu})\mu_1^p\mu_2^p - \cdots - \mathfrak{a}\mathbf{D}_p^{1-2p}(\boldsymbol{\mu})\mu_1^p\mu_n^p \\
&\quad - \mathfrak{a}\mathbf{D}_p^{1-2p}(\boldsymbol{\mu})\mu_2^p\mu_1^p + \mathfrak{a}\Big[\mathbf{D}_p^{1-p}(\boldsymbol{\mu})\mu_2^p - \mathbf{D}_p^{1-2p}(\boldsymbol{\mu})\mu_2^{2p}\Big] - \cdots - \mathfrak{a}\mathbf{D}_p^{1-2p}(\boldsymbol{\mu})\mu_2^p\mu_n^p \\
&\quad - \cdots - \mathfrak{a}\mathbf{D}_p^{1-2p}(\boldsymbol{\mu})\mu_n^p\mu_1^p - \mathfrak{a}\mathbf{D}_p^{1-2p}(\boldsymbol{\mu})\mu_n^p\mu_2^p - \cdots + \mathfrak{a}\Big[\mathbf{D}_p^{1-p}(\boldsymbol{\mu})\mu_n^p - \mathbf{D}_p^{1-2p}(\boldsymbol{\mu})\mu_n^{2p}\Big] \\
&= \mathfrak{a}\mathbf{D}_p^{1-p}(\boldsymbol{\mu})\mu_1^p - \mathfrak{a}\mathbf{D}_p^{1-2p}(\boldsymbol{\mu})\mu_1^{2p} - \mathfrak{a}\mathbf{D}_p^{1-2p}(\boldsymbol{\mu})\mu_1^p\mu_2^p - \cdots - \mathfrak{a}\mathbf{D}_p^{1-2p}(\boldsymbol{\mu})\mu_1^p\mu_n^p \\
&\quad - \mathfrak{a}\mathbf{D}_p^{1-2p}(\boldsymbol{\mu})\mu_2^p\mu_1^p + \mathfrak{a}\mathbf{D}_p^{1-p}(\boldsymbol{\mu})\mu_2^p - \mathfrak{a}\mathbf{D}_p^{1-2p}(\boldsymbol{\mu})\mu_2^{2p} - \cdots - \mathfrak{a}\mathbf{D}_p^{1-2p}(\boldsymbol{\mu})\mu_2^p\mu_n^p \\
&\quad - \cdots - \mathfrak{a}\mathbf{D}_p^{1-2p}(\boldsymbol{\mu})\mu_n^p\mu_1^p - \mathfrak{a}\mathbf{D}_p^{1-2p}(\boldsymbol{\mu})\mu_n^p\mu_2^p - \cdots + \mathfrak{a}\mathbf{D}_p^{1-p}(\boldsymbol{\mu})\mu_n^p - \mathfrak{a}\mathbf{D}_p^{1-2p}(\boldsymbol{\mu})\mu_n^{2p} \\
&= \mathfrak{a}\mathbf{D}_p^{1-p}(\boldsymbol{\mu})\left[\sum_{k=1}^n \mu_k^p\right] - \mathfrak{a}\mathbf{D}_p^{1-2p}(\boldsymbol{\mu})\left[\sum_{k,m=1}^n \mu_k^p\mu_m^p\right] \\
&= \mathfrak{a}\mathbf{D}_p^{1-p}\big(\boldsymbol{\mu}(t)\big)\left[\sum_{k=1}^n \big(\mu_k(t)\big)^p\right] - \mathfrak{a}\mathbf{D}_p^{1-2p}\big(\boldsymbol{\mu}(t)\big)\left[\sum_{k,m=1}^n \big(\mu_k(t)\big)^p\big(\mu_m(t)\big)^p\right].
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\boldsymbol{\mu}(t)\mathbf{H}^{\mathbf{D}_p}\boldsymbol{\mu}^{\mathsf{T}}(t) &= (p-1)\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p}\left[\sum_{k=1}^n \big(\mu_k(t)\big)^p\right] - (p-1)\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p}\left[\sum_{k,m=1}^n \big(\mu_k(t)\big)^p\big(\mu_m(t)\big)^p\right] \\
&= (p-1)\left[\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p}\left(\sum_{k=1}^n \big(\mu_k(t)\big)^p\right) - \Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p}\left(\sum_{k,m=1}^n \big(\mu_k(t)\big)^p\big(\mu_m(t)\big)^p\right)\right] \\
&= (p-1)\left[\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p}\left(\sum_{k=1}^n \big(\mu_k(t)\big)^p\right) - \Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p}\left(\sum_{k=1}^n \big(\mu_k(t)\big)^p\right)^2\right] \\
&= (p-1)\left[\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{p} - \Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{2p}\right] \\
&= (p-1)\Big[\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) - \mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big] \\
&= 0.
\end{aligned}$$

Therefore, for all $p > 0$, we have

$$\boldsymbol{\mu}(t)\mathbf{H}^{\mathbf{D}_p}\boldsymbol{\mu}^{\mathsf{T}}(t) = 0.$$

Thus, the $n \times n$ Hessian matrix, $\mathbf{H}^{\mathbf{D}_p}$, is simultaneously negative semidefinite and positive semidefinite, and by statements (i) and (iii) of Lemma E.2.7 of Appendix E, we can draw no conclusions regarding the concavity or convexity of the market $\mathbf{D}_p$ index process. However, by appealing to the bounds $1 < \mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) \leq n^{(1-p)/p}$ as well as the nature and location (i.e., coordinates) of the bounds and the fact that $\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)$ is a symmetric function, we can conclude that the concavity of the market $\mathbf{D}_p$ index process is only given for a limited parameter space, and therefore the market $\mathbf{D}_p$ index process, $\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)$, is concave for the limited parameter space, i.e. for $0 < p < 1$, and is as a result a measure of diversity. The $\mathbf{D}_p$-function first appeared as a measure of diversity in Fernholz, Garvy & Hannon (1998), and Fernholz (1999a). In terms of the capabilities as a sufficient measure of diversity, the market $\mathbf{D}_p$-function has benefits over the market entropy function, since the parameter p can be adjusted to accommodate particular circumstances that may arise in practice. Furthermore, $\mathbf{D}_p$ is scale invariant in the sense that if $x_1, \ldots, x_n$ are positive numbers that do not necessarily add up to 1, then

$$\frac{\mathbf{D}_p(x_1, \ldots, x_n)}{x_1 + \cdots + x_n} = \mathbf{D}_p\left(\frac{x_1}{x_1 + \cdots + x_n}, \ldots, \frac{x_n}{x_1 + \cdots + x_n}\right). \tag{4.6.31}$$

In conclusion, the market $\mathbf{D}_p$ index process satisfies the three requirements of a sufficient diversity measure for $0 < p < 1$, and is thus an appropriate measure of diversity for our purposes.

4.6.4 The Normalised Version of the $\mathbf{D}_p$ Index (The $\widetilde{\mathbf{D}}_p$ Index)

The *normalised version of the* $\mathbf{D}_p$*-function*, i.e. the $\widetilde{\mathbf{D}}_p$*-function*, $\widetilde{\mathbf{D}}_p : \Delta^{n-1} \to (0, \infty)$, for any given $0 < p < 1$, is defined by

$$\widetilde{\mathbf{D}}_p(\mathbf{x}) = \widetilde{\mathbf{D}}_p(x_1, \ldots, x_n) \triangleq \left(n^{p-1} \sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} \equiv n^{(p-1)/p} \mathbf{D}_p(\mathbf{x}), \tag{4.6.32}$$

for all $\mathbf{x}$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\mathbf{x} \in \Delta^{n-1}$. Then, the corresponding market diversity measure process (the *normalised version of the market* $\mathbf{D}_p$*-function* or the *market* $\widetilde{\mathbf{D}}_p$*-function*) is given below by:

Definition 4.6.4 (Normalised Market $\mathbf{D}_p$ Index, Market $\widetilde{\mathbf{D}}_p$ Index)**.** *Let μ be the market portfolio. Then the* ***normalised market*** $\mathbf{D}_p$ ***index process*** *or the* ***market*** $\widetilde{\mathbf{D}}_p$ ***index process****, $\widetilde{\mathbf{D}}_p(\boldsymbol{\mu}) = \big\{\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big), t \in [0, T]\big\}$, $\widetilde{\mathbf{D}}_p : \Delta^{n-1} \to (0, \infty)$, is defined for all $t \in [0, T]$ by*

$$\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big) = \widetilde{\mathbf{D}}_p\big(\mu_1(t), \ldots, \mu_n(t)\big) \triangleq \left(n^{p-1} \sum_{i=1}^{n} \big(\mu_i(t)\big)^p\right)^{\frac{1}{p}} \equiv n^{(p-1)/p} \mathbf{D}_p\big(\boldsymbol{\mu}(t)\big), \tag{4.6.33}$$

for all $\boldsymbol{\mu}(t)$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\boldsymbol{\mu}(t) \in \Delta^{n-1}$.

Therefore, we also have

$$\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) = n^{(1-p)/p} \widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big), \qquad t \in [0, T]. \tag{4.6.34}$$

The normalised market $\mathbf{D}_p$ index or the market $\widetilde{\mathbf{D}}_p$ index attains its maximum value of 1 when all the market weights are equal, and it reaches its minimum value when all the capital is concentrated into a single stock. Consequently, from (4.6.18), we have the following bounds on the normalised market $\mathbf{D}_p$ index process or the market $\widetilde{\mathbf{D}}_p$ index process

$$n^{(p-1)/p} < \widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big) \leq 1, \qquad t \in [0, T], \qquad \text{a.s.} \tag{4.6.35}$$

Clearly, the positive normalised market $\mathbf{D}_p$ index process or the market $\widetilde{\mathbf{D}}_p$ index process is both twice continuously differentiable and symmetric. To demonstrate the concavity property we shall verify statement (i) of Lemma E.2.7 of Appendix E by showing that the market $\widetilde{\mathbf{D}}_p$ index process $\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)$ is concave on Δ^{n-1} if and only if its Hessian matrix $\mathbf{H}^{\widetilde{\mathbf{D}}_p} = \mathbf{H}\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big) = D^2\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big) = \big(D_i D_j \widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)\big)_{1 \leq i,j \leq n} = \big(D_{ij}\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)\big)_{1 \leq i,j \leq n} \triangleq \big(\frac{\partial^2 \widetilde{\mathbf{D}}_p(\boldsymbol{\mu}(t))}{\partial \mu_i(t) \partial \mu_j(t)}\big)_{1 \leq i,j \leq n}$, with $\boldsymbol{\mu}(t) = \big(\mu_1(t), \mu_2(t), \ldots, \mu_n(t)\big)$, is negative semidefinite,

i.e., if $\mathbf{x}\mathbf{H}^{\widetilde{\mathbf{D}}_p}\mathbf{x}^\mathsf{T} \leq 0$ for all $\mathbf{x} \in \Delta^{n-1} \subset \mathbb{R}^n$. For the normalised market $\mathbf{D}_p$ index process or the market $\widetilde{\mathbf{D}}_p$ index process (4.6.33) of Definition 4.6.4, using (4.6.34) together with equation (4.6.26), we have for all $i = 1, 2, \ldots, n$,

$$\begin{aligned} D_i\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big) &= n^{(p-1)/p} D_i\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) &&= n^{(p-1)/p}\big(\mu_i(t)\big)^{p-1}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p} && (4.6.36)\\ &&&= n^{(p-1)/p}n^{(1-p)^2/p}\big(\mu_i(t)\big)^{p-1}\Big(\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p} \\ &&&= n^{(p^2-p)/p}\big(\mu_i(t)\big)^{p-1}\Big(\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p} \\ &&&= n^{p-1}\big(\mu_i(t)\big)^{p-1}\Big(\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p}. && (4.6.37) \end{aligned}$$

Therefore, using (4.6.34) together with equations (4.6.27) and (4.6.28), for all $i = 1, 2, \ldots, n$, we have

$$\begin{aligned} D_{ii}\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big) &= n^{(p-1)/p} D_{ii}\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) && (4.6.38)\\ &= (p-1)n^{(p-1)/p}\big(\mu_i(t)\big)^{p-2}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p} - (p-1)n^{(p-1)/p}\big(\mu_i(t)\big)^{2p-2}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p} && (4.6.39)\\ &= (p-1)n^{(p-1)/p}\left[\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p}\big(\mu_i(t)\big)^{p-2} - \Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p}\big(\mu_i(t)\big)^{2p-2}\right] && (4.6.40)\\ &= (p-1)\left[n^{(p-1)/p}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p}\big(\mu_i(t)\big)^{p-2} - n^{(p-1)/p}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p}\big(\mu_i(t)\big)^{2p-2}\right] && (4.6.41)\\ &= (p-1)\left[n^{(p-1)/p}n^{(1-p)^2/p}\Big(\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p}\big(\mu_i(t)\big)^{p-2} - n^{(p-1)/p}n^{(1-p)(1-2p)/p}\Big(\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p}\big(\mu_i(t)\big)^{2p-2}\right] \\ &= (p-1)\left[n^{(p^2-p)/p}\Big(\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p}\big(\mu_i(t)\big)^{p-2} - n^{(2p^2-2p)/p}\Big(\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p}\big(\mu_i(t)\big)^{2p-2}\right] \\ &= (p-1)\left[n^{p-1}\Big(\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p}\big(\mu_i(t)\big)^{p-2} - n^{2p-2}\Big(\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p}\big(\mu_i(t)\big)^{2p-2}\right] && (4.6.42)\\ &= (p-1)n^{p-1}\big(\mu_i(t)\big)^{p-2}\Big(\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p} - (p-1)n^{2p-2}\big(\mu_i(t)\big)^{2p-2}\Big(\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p}, && (4.6.43) \end{aligned}$$

and, using (4.6.34) together with equations (4.6.29) and (4.6.30), for all $i \neq j$, $i, j = 1, 2, \ldots, n$, we have

$$\begin{aligned} D_{ij}\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big) &= n^{(p-1)/p} D_{ij}\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) && (4.6.44)\\ &= -(p-1)n^{(p-1)/p}\big(\mu_i(t)\big)^{p-1}\big(\mu_j(t)\big)^{p-1}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p} && (4.6.45)\\ &= -(p-1)n^{(p-1)/p}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p}\big(\mu_i(t)\big)^{p-1}\big(\mu_j(t)\big)^{p-1} && (4.6.46)\\ &= -(p-1)n^{(p-1)/p}n^{(1-p)(1-2p)/p}\Big(\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p}\big(\mu_i(t)\big)^{p-1}\big(\mu_j(t)\big)^{p-1} \\ &= -(p-1)n^{(2p^2-2p)/p}\Big(\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p}\big(\mu_i(t)\big)^{p-1}\big(\mu_j(t)\big)^{p-1} \\ &= -(p-1)n^{2p-2}\Big(\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p}\big(\mu_i(t)\big)^{p-1}\big(\mu_j(t)\big)^{p-1} && (4.6.47)\\ &= -(p-1)n^{2p-2}\big(\mu_i(t)\big)^{p-1}\big(\mu_j(t)\big)^{p-1}\Big(\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p}. && (4.6.48) \end{aligned}$$

Then the $n \times n$ Hessian matrix of $\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)$ is given by

$$\mathbf{H}^{\widetilde{\mathbf{D}}_p} = n^{(p-1)/p}\mathbf{H}^{\mathbf{D}_p}.$$

Consequently, we have for all $\boldsymbol{\mu}(t) \in \Delta^{n-1}$,

$$\boldsymbol{\mu}(t)\mathbf{H}^{\widetilde{\mathbf{D}}_p}\boldsymbol{\mu}^\mathsf{T}(t) = n^{(p-1)/p}\boldsymbol{\mu}(t)\mathbf{H}^{\mathbf{D}_p}\boldsymbol{\mu}^\mathsf{T}(t) = 0.$$

Therefore, for all $p > 0$, we have

$$\boldsymbol{\mu}(t)\mathbf{H}^{\widetilde{\mathbf{D}}_p}\boldsymbol{\mu}^\mathsf{T}(t) = 0.$$

Thus, the $n \times n$ Hessian matrix, $\mathbf{H}^{\widetilde{\mathbf{D}}_p}$, is simultaneously negative semidefinite and positive semidefinite, and by statements (i) and (iii) of Lemma E.2.7 of Appendix E, we can draw no conclusions regarding the concavity or convexity of the market $\widetilde{\mathbf{D}}_p$ index process. However, by appealing to the bounds $n^{(p-1)/p} < \widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big) \leq 1$ as well as the nature and location (i.e., coordinates) of the bounds and the fact that $\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)$ is a symmetric function, we can conclude that the concavity of the market $\widetilde{\mathbf{D}}_p$ index process is only given for a limited parameter space, and therefore the market $\widetilde{\mathbf{D}}_p$ index process, $\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)$, is concave for the limited parameter space, i.e. for $0 < p < 1$, and is as a result a measure of diversity. In conclusion, the market $\widetilde{\mathbf{D}}_p$ index process satisfies the three requirements of a sufficient diversity measure for $0 < p < 1$, and is thus an appropriate measure of diversity for our purposes.

4.6.5 Rényi Entropy

The *Rényi entropy* is a measure of diversity that was proposed by Rényi (1961), and is a generalisation of the entropy function we considered above [see Rényi (1961)]. For the parameter $\alpha \neq 1$, the *Rényi entropy function*, $\mathbf{S}^{\mathrm{R}}_{\alpha} : \Delta^{n-1} \to (0, \infty)$, is defined by

$$\mathbf{S}^{\mathrm{R}}_{\alpha}(\mathbf{x}) \;=\; \mathbf{S}^{\mathrm{R}}_{\alpha}(x_1, \ldots, x_n) \;\triangleq\; \frac{1}{1-\alpha} \log\left(\sum_{i=1}^{n} x_i^{\alpha}\right), \tag{4.6.49}$$

for all $\mathbf{x}$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\mathbf{x} \in \Delta^{n-1}$. Then, the corresponding market diversity measure process (the *market Rényi entropy function*) is given below by:

Definition 4.6.5 (Market Rényi Entropy)**.** *Let μ be the market portfolio. Then the* ***market Rényi entropy process****, $\mathbf{S}^{\mathrm{R}}_{\alpha}(\boldsymbol{\mu}) = \big\{\mathbf{S}^{\mathrm{R}}_{\alpha}\big(\boldsymbol{\mu}(t)\big), t \in [0,T]\big\}$, $\mathbf{S}^{\mathrm{R}}_{\alpha} : \Delta^{n-1} \to (0, \infty)$, is defined by*

$$\mathbf{S}^{\mathrm{R}}_{\alpha}\big(\boldsymbol{\mu}(t)\big) \;=\; \mathbf{S}^{\mathrm{R}}_{\alpha}\big(\mu_1(t), \ldots, \mu_n(t)\big) \;\triangleq\; \frac{1}{1-\alpha} \log\left(\sum_{i=1}^{n} \big(\mu_i(t)\big)^{\alpha}\right), \qquad t \in [0,T], \tag{4.6.50}$$

for all $\boldsymbol{\mu}(t)$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\boldsymbol{\mu}(t) \in \Delta^{n-1}$.

The market Rényi entropy process, for $\alpha = p$, is related to the market $\mathbf{D}_p$ index process by the following

$$\mathbf{S}^{\mathrm{R}}_{p}\big(\boldsymbol{\mu}(t)\big) \;=\; \frac{1}{1-p} \log\left(\sum_{i=1}^{n} \big(\mu_i(t)\big)^{p}\right) \;=\; \frac{p}{1-p} \log\left(\sum_{i=1}^{n} \big(\mu_i(t)\big)^{p}\right)^{\frac{1}{p}} \;=\; \frac{p}{1-p} \log \mathbf{D}_p\big(\boldsymbol{\mu}(t)\big).$$

For the Rényi entropy, minimum diversity occurs for the following value of the Rényi entropy process

$$\begin{aligned} \mathbf{S}^{\mathrm{R}}_{\alpha}\big(\boldsymbol{\mu}(t)\big) &= \frac{1}{1-\alpha} \log\Big(\big(\mu_{(1)}(t)\big)^{\alpha}\Big) \\ &= \frac{1}{1-\alpha} \log(1) \\ &= 0. \end{aligned}$$

Thus a value of 0 for the Rényi entropy indicates that the market is not diverse. In addition, maximum diversity occurs for the following value of the Rényi entropy process

$$\begin{aligned} \mathbf{S}^{\mathrm{R}}_{\alpha}\big(\boldsymbol{\mu}(t)\big) &= \frac{1}{1-\alpha} \log\left(n \left(\frac{1}{n}\right)^{\alpha}\right) \\ &= \frac{1}{1-\alpha} \log\left(n^{1-\alpha}\right) \\ &= \frac{1-\alpha}{1-\alpha} \log(n) \\ &= \log(n). \end{aligned}$$

Therefore, a value of $\log(n)$ for the Rényi entropy indicates that the market is completely diverse. Consequently, we have the following bounds on the market Rényi entropy process

$$0 \;<\; \mathbf{S}^{\mathrm{R}}_{\alpha}\big(\boldsymbol{\mu}(t)\big) \;\leq\; \log n, \qquad t \in [0,T], \qquad \text{a.s.} \tag{4.6.51}$$

To show that the market Rényi entropy process is indeed a measure of diversity, we need to demonstrate that it satisfies the three primary properties that a sufficient diversity measure should possess, those being twice continuous differentiability, symmetry and concavity. Clearly, the positive market Rényi entropy process is both twice continuously differentiable and symmetric. However, the concavity of the market Rényi entropy process is only given for a limited parameter space. To demonstrate the concavity property (for the limited parameter space) we shall verify statement (i) of Lemma E.2.7 of Appendix E by showing that the market Rényi entropy process $\mathbf{S}_\alpha^{\mathrm{R}}\big(\boldsymbol{\mu}(t)\big)$ is concave on Δ^{n-1} if and only if its Hessian matrix $\mathbf{H}_\alpha^{\mathrm{R}} = \mathbf{H}\mathbf{S}_\alpha^{\mathrm{R}}\big(\boldsymbol{\mu}(t)\big) = D^2\mathbf{S}_\alpha^{\mathrm{R}}\big(\boldsymbol{\mu}(t)\big) = \big(D_iD_j\mathbf{S}_\alpha^{\mathrm{R}}\big(\boldsymbol{\mu}(t)\big)\big)_{1\le i,j\le n} = \big(D_{ij}\mathbf{S}_\alpha^{\mathrm{R}}\big(\boldsymbol{\mu}(t)\big)\big)_{1\le i,j\le n} \triangleq \big(\frac{\partial^2\mathbf{S}_\alpha^{\mathrm{R}}(\boldsymbol{\mu}(t))}{\partial\mu_i(t)\partial\mu_j(t)}\big)_{1\le i,j\le n}$, with $\boldsymbol{\mu}(t) = \big(\mu_1(t), \mu_2(t), \ldots, \mu_n(t)\big)$, is negative semidefinite, i.e., if $\mathbf{x}\mathbf{H}_\alpha^{\mathrm{R}}\mathbf{x}^\mathsf{T} \le 0$ for all $\mathbf{x} \in \Delta^{n-1} \subset \mathbb{R}^n$. For the market Rényi entropy process (4.6.50) of Definition 4.6.5, we have for all $i = 1, 2, \ldots, n$,

$$D_i\mathbf{S}_\alpha^{\mathrm{R}}\big(\boldsymbol{\mu}(t)\big) = \frac{\partial\mathbf{S}_\alpha^{\mathrm{R}}\big(\boldsymbol{\mu}(t)\big)}{\partial\mu_i(t)} = \frac{\alpha}{1-\alpha}\left[\big(\mu_i(t)\big)^{\alpha-1}\Big(\sum_{k=1}^n\big(\mu_k(t)\big)^\alpha\Big)^{-1}\right] \tag{4.6.52}$$

$$= \frac{\alpha}{1-\alpha}\left[\frac{\big(\mu_i(t)\big)^{\alpha-1}}{\sum_{k=1}^n\big(\mu_k(t)\big)^\alpha}\right]. \tag{4.6.53}$$

Therefore, for all $i = 1, 2, \ldots, n$, we have

$$\begin{aligned} D_{ii}\mathbf{S}_\alpha^{\mathrm{R}}\big(\boldsymbol{\mu}(t)\big) &= \frac{\partial^2\mathbf{S}_\alpha^{\mathrm{R}}\big(\boldsymbol{\mu}(t)\big)}{\partial\mu_i(t)^2} \\ &= \frac{\alpha}{1-\alpha}\left[(\alpha-1)\big(\mu_i(t)\big)^{\alpha-2}\Big(\sum_{k=1}^n\big(\mu_k(t)\big)^\alpha\Big)^{-1} - \alpha\big(\mu_i(t)\big)^{2\alpha-2}\Big(\sum_{k=1}^n\big(\mu_k(t)\big)^\alpha\Big)^{-2}\right] \\ &= -\alpha\,\big(\mu_i(t)\big)^{\alpha-2}\Big(\sum_{k=1}^n\big(\mu_k(t)\big)^\alpha\Big)^{-1} - \frac{\alpha^2}{1-\alpha}\,\big(\mu_i(t)\big)^{2\alpha-2}\Big(\sum_{k=1}^n\big(\mu_k(t)\big)^\alpha\Big)^{-2} \qquad (4.6.54) \\ &= -\alpha\,\frac{\big(\mu_i(t)\big)^{\alpha-2}}{\sum_{k=1}^n\big(\mu_k(t)\big)^\alpha} - \frac{\alpha^2}{1-\alpha}\,\frac{\big(\mu_i(t)\big)^{2\alpha-2}}{\Big(\sum_{k=1}^n\big(\mu_k(t)\big)^\alpha\Big)^2}, \qquad (4.6.55) \end{aligned}$$

and for all $i \ne j$, $i, j = 1, 2, \ldots, n$, we have

$$\begin{aligned} D_{ij}\mathbf{S}_\alpha^{\mathrm{R}}\big(\boldsymbol{\mu}(t)\big) &= \frac{\partial^2\mathbf{S}_\alpha^{\mathrm{R}}\big(\boldsymbol{\mu}(t)\big)}{\partial\mu_i(t)\partial\mu_j(t)} = \frac{\alpha}{1-\alpha}\left[-\alpha\,\big(\mu_i(t)\big)^{\alpha-1}\big(\mu_j(t)\big)^{\alpha-1}\Big(\sum_{k=1}^n\big(\mu_k(t)\big)^\alpha\Big)^{-2}\right] \\ &= \frac{-\alpha^2}{1-\alpha}\,\big(\mu_i(t)\big)^{\alpha-1}\big(\mu_j(t)\big)^{\alpha-1}\Big(\sum_{k=1}^n\big(\mu_k(t)\big)^\alpha\Big)^{-2} \qquad (4.6.56) \\ &= \frac{-\alpha^2}{1-\alpha}\,\frac{\big(\mu_i(t)\big)^{\alpha-1}\big(\mu_j(t)\big)^{\alpha-1}}{\Big(\sum_{k=1}^n\big(\mu_k(t)\big)^\alpha\Big)^2}. \qquad (4.6.57) \end{aligned}$$

Define $z\big(\boldsymbol{\mu}(t)\big) := \Big(\sum_{k=1}^n\big(\mu_k(t)\big)^\alpha\Big)^{-1}$ or $z(\boldsymbol{\mu}) := \Big(\sum_{k=1}^n\mu_k^\alpha\Big)^{-1}$, and $\mathfrak{b} := \alpha^2/(1-\alpha)$, then the $n \times n$ Hessian matrix of $\mathbf{S}_\alpha^{\mathrm{R}}\big(\boldsymbol{\mu}(t)\big)$ is given by

$$\begin{aligned} \mathbf{H}_\alpha^{\mathrm{R}} &= \mathbf{H}\mathbf{S}_\alpha^{\mathrm{R}}(\boldsymbol{\mu}) \\ &= \begin{bmatrix} -\alpha z(\boldsymbol{\mu})\mu_1^{\alpha-2} - \mathfrak{b}z^2(\boldsymbol{\mu})\mu_1^{2\alpha-2} & -\mathfrak{b}z^2(\boldsymbol{\mu})\mu_1^{\alpha-1}\mu_2^{\alpha-1} & \cdots & -\mathfrak{b}z^2(\boldsymbol{\mu})\mu_1^{\alpha-1}\mu_n^{\alpha-1} \\ -\mathfrak{b}z^2(\boldsymbol{\mu})\mu_2^{\alpha-1}\mu_1^{\alpha-1} & -\alpha z(\boldsymbol{\mu})\mu_2^{\alpha-2} - \mathfrak{b}z^2(\boldsymbol{\mu})\mu_2^{2\alpha-2} & \cdots & -\mathfrak{b}z^2(\boldsymbol{\mu})\mu_2^{\alpha-1}\mu_n^{\alpha-1} \\ \vdots & \vdots & \ddots & \vdots \\ -\mathfrak{b}z^2(\boldsymbol{\mu})\mu_n^{\alpha-1}\mu_1^{\alpha-1} & -\mathfrak{b}z^2(\boldsymbol{\mu})\mu_n^{\alpha-1}\mu_2^{\alpha-1} & \cdots & -\alpha z(\boldsymbol{\mu})\mu_n^{\alpha-2} - \mathfrak{b}z^2(\boldsymbol{\mu})\mu_n^{2\alpha-2} \end{bmatrix}. \end{aligned}$$

Consequently, we have for all $\alpha \neq 1$ and for all $\boldsymbol{\mu}(t) \in \Delta^{n-1}$,

$$\begin{aligned}
\boldsymbol{\mu}(t)\mathbf{H}_\alpha^{\mathrm{R}}\boldsymbol{\mu}^{\mathsf{T}}(t) &= -\alpha z(\boldsymbol{\mu})\mu_1^\alpha - \mathfrak{b}z^2(\boldsymbol{\mu})\mu_1^{2\alpha} - \mathfrak{b}z^2(\boldsymbol{\mu})\mu_1^\alpha\mu_2^\alpha - \cdots - \mathfrak{b}z^2(\boldsymbol{\mu})\mu_1^\alpha\mu_n^\alpha \\
&\quad - \mathfrak{b}z^2(\boldsymbol{\mu})\mu_2^\alpha\mu_1^\alpha - \alpha z(\boldsymbol{\mu})\mu_2^\alpha - \mathfrak{b}z^2(\boldsymbol{\mu})\mu_2^{2\alpha} - \cdots - \mathfrak{b}z^2(\boldsymbol{\mu})\mu_2^\alpha\mu_n^\alpha \\
&\quad - \cdots - \mathfrak{b}z^2(\boldsymbol{\mu})\mu_n^\alpha\mu_1^\alpha - \mathfrak{b}z^2(\boldsymbol{\mu})\mu_n^\alpha\mu_2^\alpha - \cdots - \alpha z(\boldsymbol{\mu})\mu_n^\alpha - \mathfrak{b}z^2(\boldsymbol{\mu})\mu_n^{2\alpha} \\
&= -\alpha z(\boldsymbol{\mu})\left[\sum_{k=1}^n \mu_k^\alpha\right] - \mathfrak{b}z^2(\boldsymbol{\mu})\left[\sum_{k,m=1}^n \mu_k^\alpha\mu_m^\alpha\right] \\
&= -\alpha z\big(\boldsymbol{\mu}(t)\big)\left[\sum_{k=1}^n \big(\mu_k(t)\big)^\alpha\right] - \mathfrak{b}z^2\big(\boldsymbol{\mu}(t)\big)\left[\sum_{k,m=1}^n \big(\mu_k(t)\big)^\alpha\big(\mu_m(t)\big)^\alpha\right].
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\boldsymbol{\mu}(t)\mathbf{H}_\alpha^{\mathrm{R}}\boldsymbol{\mu}^{\mathsf{T}}(t) &= -\alpha\,\frac{1}{\sum_{k=1}^n \big(\mu_k(t)\big)^\alpha}\left[\sum_{k=1}^n \big(\mu_k(t)\big)^\alpha\right] - \frac{\alpha^2}{1-\alpha}\,\frac{1}{\Big(\sum_{k=1}^n \big(\mu_k(t)\big)^\alpha\Big)^2}\left[\sum_{k,m=1}^n \big(\mu_k(t)\big)^\alpha\big(\mu_m(t)\big)^\alpha\right] \\
&= \frac{\alpha}{1-\alpha}\left[(\alpha-1)\,\frac{\sum_{k=1}^n \big(\mu_k(t)\big)^\alpha}{\sum_{k=1}^n \big(\mu_k(t)\big)^\alpha} - \alpha\,\frac{\sum_{k,m=1}^n \big(\mu_k(t)\big)^\alpha\big(\mu_m(t)\big)^\alpha}{\Big(\sum_{k=1}^n \big(\mu_k(t)\big)^\alpha\Big)^2}\right] \\
&= \frac{\alpha}{1-\alpha}\left[(\alpha-1)\,\frac{\sum_{k=1}^n \big(\mu_k(t)\big)^\alpha}{\sum_{k=1}^n \big(\mu_k(t)\big)^\alpha} - \alpha\,\frac{\Big(\sum_{k=1}^n \big(\mu_k(t)\big)^\alpha\Big)^2}{\Big(\sum_{k=1}^n \big(\mu_k(t)\big)^\alpha\Big)^2}\right] \\
&= \frac{\alpha}{1-\alpha}\Big[(\alpha-1)-\alpha\Big] \\
&= \frac{-\alpha}{1-\alpha}.
\end{aligned}$$

Therefore, we have

$$\boldsymbol{\mu}(t)\mathbf{H}_\alpha^{\mathrm{R}}\boldsymbol{\mu}^{\mathsf{T}}(t) = \frac{-\alpha}{1-\alpha} \leq 0,$$

if and only if $\alpha \in [0,1)$. Therefore, we have

$$\boldsymbol{\mu}(t)\mathbf{H}_\alpha^{\mathrm{R}}\boldsymbol{\mu}^{\mathsf{T}}(t) \leq 0,$$

if and only if $\alpha \in [0,1)$. Thus, the $n \times n$ Hessian matrix, $\mathbf{H}_\alpha^{\mathrm{R}}$, is negative semidefinite if and only if $\alpha \in [0,1)$, and by statement (i) of Lemma E.2.7 of Appendix E, we can conclude that the market Rényi entropy process, $\mathbf{S}_\alpha^{\mathrm{R}}\big(\boldsymbol{\mu}(t)\big)$, is concave if and only if $\alpha \in [0,1)$, and is as a result a measure of diversity for $\alpha \in [0,1)$. Hence, for $\alpha > 1$ and $\alpha < 0$, the market Rényi entropy process is not concave. In conclusion, note the following:

(i) The market Rényi entropy process, $\mathbf{S}_\alpha^{\mathrm{R}}$, tends to the archetypal entropy function as $\alpha \to 1$;

(ii) For $0 \leq \alpha < 1$, the market Rényi entropy process, $\mathbf{S}_\alpha^{\mathrm{R}}$, is indeed a measure of diversity;

(iii) However, for $\alpha > 1$ and $\alpha < 0$, the market Rényi entropy process, $\mathbf{S}_\alpha^{\mathrm{R}}$, is not concave, and thus fails to satisfy all the properties of a sufficient diversity measure.

4.6.6 The Gini Coefficient

The *Gini coefficient* is frequently adopted by economists to measure the diversity of the distribution of wealth [see Rao (1982a,b,c, 1984) and Simpson (1949)]. The *Gini coefficient*, $\mathbf{S}^{\mathsf{G}} : \Delta^{n-1} \to (0,\infty)$, is defined by

$$\mathbf{S}^{\mathsf{G}}(\mathbf{x}) = \mathbf{S}^{\mathsf{G}}(x_1,\ldots,x_n) \triangleq \frac{1}{2}\sum_{i=1}^n \big|x_i - n^{-1}\big|, \tag{4.6.58}$$

for all $\mathbf{x}$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\mathbf{x} \in \Delta^{n-1}$. Then, the corresponding market measure process (the *market Gini coefficient*) is given below by:

Definition 4.6.6 (Market Gini Coefficient). *Let μ be the market portfolio. Then the* ***market Gini coefficient process****, $\mathbf{S}^{\mathsf{G}}(\boldsymbol{\mu}) = \{\mathbf{S}^{\mathsf{G}}(\boldsymbol{\mu}(t)), t \in [0,T]\}$, $\mathbf{S}^{\mathsf{G}} : \Delta^{n-1} \to (0,\infty)$, is defined by*

$$\mathbf{S}^{\mathsf{G}}(\boldsymbol{\mu}(t)) \;=\; \mathbf{S}^{\mathsf{G}}(\mu_1(t),\ldots,\mu_n(t)) \;\triangleq\; \frac{1}{2}\sum_{i=1}^{n}\left|\mu_i(t) - n^{-1}\right|, \qquad t \in [0,T], \tag{4.6.59}$$

for all $\boldsymbol{\mu}(t)$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\boldsymbol{\mu}(t) \in \Delta^{n-1}$.

The Gini coefficient is frequently modified to $\mathbf{S}^{\mathrm{G}} : \Delta^{n-1} \to (0,\infty)$,

$$\mathbf{S}^{\mathrm{G}}(\mathbf{x}) \;=\; \mathbf{S}^{\mathrm{G}}(x_1,\ldots,x_n) \;\triangleq\; 1 - \frac{1}{2}\sum_{i=1}^{n}\left|x_i - n^{-1}\right|, \tag{4.6.60}$$

this comes closer to Definition 4.5.5 of a measure of diversity, however, it fails to be twice continuously differentiable. Then, the corresponding market measure process (the *market Gini coefficient*) is given below by:

Definition 4.6.7 (Market Gini Coefficient). *Let μ be the market portfolio. Then the* ***market Gini coefficient process****, $\mathbf{S}^{\mathrm{G}}(\boldsymbol{\mu}) = \{\mathbf{S}^{\mathrm{G}}(\boldsymbol{\mu}(t)), t \in [0,T]\}$, $\mathbf{S}^{\mathrm{G}} : \Delta^{n-1} \to (0,\infty)$, is defined by*

$$\mathbf{S}^{\mathrm{G}}(\boldsymbol{\mu}(t)) \;=\; \mathbf{S}^{\mathrm{G}}(\mu_1(t),\ldots,\mu_n(t)) \;\triangleq\; 1 - \frac{1}{2}\sum_{i=1}^{n}\left|\mu_i(t) - n^{-1}\right|, \qquad t \in [0,T], \tag{4.6.61}$$

for all $\boldsymbol{\mu}(t)$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\boldsymbol{\mu}(t) \in \Delta^{n-1}$.

4.6.6.1 The Quadratic Gini Coefficient

Rather than attempt to analyse the Gini coefficient here, we shall settle for a quadratic version of it, the *quadratic Gini coefficient*, $\mathbf{S}^{\mathrm{G}} : \Delta^{n-1} \to (0,\infty)$, given by

$$\mathbf{S}^{\mathrm{G}}(\mathbf{x}) \;=\; \mathbf{S}^{\mathrm{G}}(x_1,\ldots,x_n) \;\triangleq\; 1 - \frac{1}{2}\sum_{i=1}^{n}\left(x_i - n^{-1}\right)^2, \tag{4.6.62}$$

for all $\mathbf{x}$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\mathbf{x} \in \Delta^{n-1}$, which is a measure of diversity. Then, the corresponding market diversity measure process (the *market quadratic Gini coefficient*) is given below by:

Definition 4.6.8 (Market Quadratic Gini Coefficient). *Let μ be the market portfolio. Then the* ***market quadratic Gini coefficient process****, $\mathbf{S}^{\mathrm{G}}(\boldsymbol{\mu}) = \{\mathbf{S}^{\mathrm{G}}(\boldsymbol{\mu}(t)), t \in [0,T]\}$, $\mathbf{S}^{\mathrm{G}} : \Delta^{n-1} \to (0,\infty)$, is defined by*

$$\mathbf{S}^{\mathrm{G}}(\boldsymbol{\mu}(t)) \;=\; \mathbf{S}^{\mathrm{G}}(\mu_1(t),\ldots,\mu_n(t)) \;\triangleq\; 1 - \frac{1}{2}\sum_{i=1}^{n}\left(\mu_i(t) - n^{-1}\right)^2, \qquad t \in [0,T], \tag{4.6.63}$$

for all $\boldsymbol{\mu}(t)$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\boldsymbol{\mu}(t) \in \Delta^{n-1}$.

For the quadratic Gini coefficient, minimum diversity occurs when the quadratic Gini coefficient process is given by

$$\begin{aligned}
\mathbf{S}^{\mathrm{G}}(\boldsymbol{\mu}(t)) &= 1 - \frac{1}{2}\left[\left(\mu_{(1)}(t) - \frac{1}{n}\right)^2 + (n-1)\left(-\frac{1}{n}\right)^2\right] \\
&= 1 - \frac{1}{2}\left[\left(1 - \frac{1}{n}\right)^2 + (n-1)\left(-\frac{1}{n}\right)^2\right] \\
&= 1 - \frac{1}{2}\left[\left(1 - \frac{2}{n} + \frac{1}{n^2}\right) + (n-1)\left(\frac{1}{n^2}\right)\right] \\
&= 1 - \frac{1}{2}\left[1 - \frac{1}{n}\right] \\
&= 1 - \frac{1}{2} + \frac{1}{2n} \;=\; \frac{1}{2} + \frac{1}{2n} \;>\; \frac{1}{2}.
\end{aligned}$$

Thus a value of $\frac{1}{2}$ for the quadratic Gini coefficient indicates that the market is not diverse. In addition, for the quadratic Gini coefficient, maximum diversity occurs when the quadratic Gini coefficient is given by

$$\begin{aligned} \mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) &= 1 - \frac{1}{2}\left[n\left(\frac{1}{n} - \frac{1}{n}\right)^2\right] \\ &= 1. \end{aligned}$$

Therefore, a value of 1 for the quadratic Gini coefficient indicates that the market is completely diverse. Consequently, we have the following bounds on the market quadratic Gini coefficient process

$$\frac{1}{2} \;<\; \mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) \;\leq\; 1, \qquad t \in [0,T], \qquad \text{a.s.} \tag{4.6.64}$$

To show that the quadratic Gini coefficient process is indeed a measure of diversity, we need to demonstrate that it satisfies the three primary properties that a sufficient diversity measure should possess. Clearly, the positive quadratic Gini coefficient process is both twice continuously differentiable and symmetric. To demonstrate the concavity property we shall once again verify statement (ii) of Lemma E.2.7 of Appendix E by showing that this market quadratic Gini coefficient process $\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)$ is strictly concave on Δ^{n-1} if and only if its Hessian matrix $\mathbf{H}^{\mathrm{G}} = \mathbf{H}\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) = D^2\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) = \big(D_iD_j\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)\big)_{1\leq i,j\leq n} = \big(D_{ij}\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)\big)_{1\leq i,j\leq n} \triangleq \big(\frac{\partial^2\mathbf{S}^{\mathrm{G}}(\boldsymbol{\mu}(t))}{\partial\mu_i(t)\partial\mu_j(t)}\big)_{1\leq i,j\leq n}$, with $\boldsymbol{\mu}(t)$, is negative definite, i.e., if $\mathbf{x}\mathbf{H}^{\mathrm{G}}\mathbf{x}^{\mathsf{T}} < 0$ for all $\mathbf{x} \in \Delta^{n-1} \subset \mathbb{R}^n$. For the market quadratic Gini coefficient process (4.6.63) of Definition 4.6.8, we have for all $i = 1, 2, \ldots, n$,

$$D_i\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) \;=\; \frac{\partial\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)}{\partial\mu_i(t)} \;=\; \frac{1}{n} - \mu_i(t). \tag{4.6.65}$$

Therefore, for all $i = 1, 2, \ldots, n$, we have

$$D_{ii}\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) \;=\; \frac{\partial^2\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)}{\partial\mu_i(t)^2} \;=\; -1, \tag{4.6.66}$$

and for all $i \neq j$, $i, j = 1, 2, \ldots, n$, we have

$$D_{ij}\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) \;=\; \frac{\partial^2\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)}{\partial\mu_i(t)\partial\mu_j(t)} \;=\; 0. \tag{4.6.67}$$

Therefore, the $n \times n$ Hessian matrix of $\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)$ is given by

$$\mathbf{H}^{\mathrm{G}} \;=\; \mathbf{H}\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) \;=\; \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{bmatrix} \;=\; -\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \;=\; -\mathbf{I}_n,$$

where $\mathbf{I}_n$ is the $n \times n$ identity matrix. Consequently, we have for all $\boldsymbol{\mu}(t) \in \Delta^{n-1}$,

$$\boldsymbol{\mu}(t)\mathbf{H}^{\mathrm{G}}\boldsymbol{\mu}^{\mathsf{T}}(t) \;=\; -\boldsymbol{\mu}(t)\mathbf{I}_n\boldsymbol{\mu}^{\mathsf{T}}(t) \;=\; -\boldsymbol{\mu}(t)\boldsymbol{\mu}^{\mathsf{T}}(t) \;=\; -\Big(\mu_1^2(t) + \mu_2^2(t) + \cdots + \mu_n^2(t)\Big).$$

Therefore, we have

$$\boldsymbol{\mu}(t)\mathbf{H}^{\mathrm{G}}\boldsymbol{\mu}^{\mathsf{T}}(t) \;=\; -\sum_{i=1}^{n}\mu_i^2(t) \;=\; -\big\|\boldsymbol{\mu}(t)\big\|^2 \;<\; 0.$$

Thus, we have

$$\boldsymbol{\mu}(t)\mathbf{H}^{\mathrm{G}}\boldsymbol{\mu}^{\mathsf{T}}(t) \;<\; 0.$$

Thus, the $n \times n$ Hessian matrix, $\mathbf{H}^{\mathrm{G}}$, is negative definite, and by statement (ii) of Lemma E.2.7 of Appendix E, we can conclude that the quadratic Gini coefficient process, $\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)$, is strictly concave, and is as a result a measure of diversity. In conclusion, the market quadratic Gini coefficient process satisfies the three requirements of a sufficient diversity measure, and is thus an appropriate measure of diversity for our purposes.

4.6.6.2 The Quartic Gini Coefficient

The function $\mathbf{S}^{\mathrm{QG}} : \Delta^{n-1} \to (0, \infty)$, defined by

$$\mathbf{S}^{\mathrm{QG}}(\mathbf{x}) \;=\; \mathbf{S}^{\mathrm{QG}}(x_1, \ldots, x_n) \;\triangleq\; 1 - \frac{1}{4}\sum_{i=1}^{n}\left(x_i - n^{-1}\right)^4, \tag{4.6.68}$$

for all $\mathbf{x}$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\mathbf{x} \in \Delta^{n-1}$, is the *quartic version of the Gini coefficient* (the *quartic Gini coefficient*) and is quite similar to that of the quadratic version of the Gini coefficient (the quadratic Gini coefficient), and is indeed a measure of diversity (this will be verified in a moment). Then, the corresponding market diversity measure process (the *market quartic Gini coefficient*) is given below by:

Definition 4.6.9 (Market Quartic Gini Coefficient)**.** *Let μ be the market portfolio. Then the* ***market quartic Gini coefficient process****, $\mathbf{S}^{\mathrm{QG}}(\boldsymbol{\mu}) = \left\{\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big), t \in [0,T]\right\}$, $\mathbf{S}^{\mathrm{QG}} : \Delta^{n-1} \to (0,\infty)$, is defined by*

$$\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big) \;=\; \mathbf{S}^{\mathrm{QG}}\big(\mu_1(t), \ldots, \mu_n(t)\big) \;\triangleq\; 1 - \frac{1}{4}\sum_{i=1}^{n}\Big(\mu_i(t) - n^{-1}\Big)^4, \qquad t \in [0,T], \tag{4.6.69}$$

for all $\boldsymbol{\mu}(t)$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\boldsymbol{\mu}(t) \in \Delta^{n-1}$.

For this quartic Gini coefficient process, minimum diversity occurs when

$$\begin{aligned}
\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big) &= 1 - \frac{1}{4}\left[\left(\mu_{(1)}(t) - \frac{1}{n}\right)^4 + (n-1)\left(-\frac{1}{n}\right)^4\right] \\
&= 1 - \frac{1}{4}\left[\left(1 - \frac{1}{n}\right)^4 + (n-1)\left(-\frac{1}{n}\right)^4\right] \\
&= 1 - \frac{1}{4}\left[\left(1 - \frac{4}{n} + \frac{6}{n^2} - \frac{4}{n^3} + \frac{1}{n^4}\right) + (n-1)\left(\frac{1}{n^4}\right)\right] \\
&= 1 - \frac{1}{4}\left[1 - \frac{4}{n} + \frac{6}{n^2} - \frac{3}{n^3}\right] \\
&= 1 - \frac{1}{4} + \frac{1}{4}\left[\frac{4}{n} - \frac{6}{n^2} + \frac{3}{n^3}\right] \\
&= \frac{3}{4} + \left[\frac{1}{n} - \frac{3}{2n^2} + \frac{3}{4n^3}\right] \\
&= \frac{3}{4} + \frac{3}{4n}\left[\frac{1}{n^2} - \frac{2}{n} + \frac{4}{3}\right] \;=\; \frac{3}{4} + \frac{3}{4n}\left[\left(\frac{1}{n} - 1\right)^2 + \frac{1}{3}\right] \;>\; \frac{3}{4}.
\end{aligned}$$

Thus a value of $\frac{3}{4}$ indicates that the market is not diverse. In addition, maximum diversity occurs when

$$\begin{aligned}
\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big) &= 1 - \frac{1}{4}\left[n\left(\frac{1}{n} - \frac{1}{n}\right)^4\right] \\
&= 1.
\end{aligned}$$

Therefore, a value of 1 indicates that the market is completely diverse. Consequently, we have the following bounds on the market quartic Gini coefficient process

$$\frac{3}{4} \;<\; \mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big) \;\leq\; 1, \qquad t \in [0,T], \qquad \text{a.s.} \tag{4.6.70}$$

To show that this quartic Gini coefficient process is indeed a measure of diversity, we need to demonstrate that it satisfies the three primary properties that a sufficient diversity measure should possess. Clearly, this positive quartic Gini coefficient process is both twice continuously differentiable and symmetric. To demonstrate the concavity property we shall once again verify statement (i) of Lemma E.2.7 of Appendix E by showing that this market quartic Gini coefficient process $\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)$ is concave on Δ^{n-1} if and only if its Hessian matrix $\mathbf{H}^{\mathrm{QG}} =$

$\mathbf{HS}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big) = D^2\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big) = \big(D_iD_j\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)\big)_{1\le i,j\le n} = \big(D_{ij}\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)\big)_{1\le i,j\le n} \triangleq \big(\frac{\partial^2\mathbf{S}^{\mathrm{QG}}(\boldsymbol{\mu}(t))}{\partial\mu_i(t)\partial\mu_j(t)}\big)_{1\le i,j\le n}$, with $\boldsymbol{\mu}(t) = \big(\mu_1(t), \mu_2(t), \ldots, \mu_n(t)\big)$, is negative semidefinite, i.e., if $\mathbf{x}\mathbf{H}^{\mathrm{QG}}\mathbf{x}^{\mathsf{T}} \le 0$ for all $\mathbf{x} \in \Delta^{n-1} \subset \mathbb{R}^n$. For this market quartic Gini coefficient process (4.6.69) of Definition 4.6.9, we have for all $i = 1, 2, \ldots, n$,

$$D_i\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big) = \frac{\partial\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)}{\partial\mu_i(t)} = -\left(\mu_i(t) - \frac{1}{n}\right)^3. \tag{4.6.71}$$

Therefore, for all $i = 1, 2, \ldots, n$, we have

$$D_{ii}\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big) = \frac{\partial^2\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)}{\partial\mu_i(t)^2} = -3\left(\mu_i(t) - \frac{1}{n}\right)^2, \tag{4.6.72}$$

and for all $i \ne j$, $i, j = 1, 2, \ldots, n$, we have

$$D_{ij}\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big) = \frac{\partial^2\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)}{\partial\mu_i(t)\partial\mu_j(t)} = 0. \tag{4.6.73}$$

Therefore, the $n \times n$ Hessian matrix of $\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)$ is given by

$$\mathbf{H}^{\mathrm{QG}} = \mathbf{HS}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big) = \begin{bmatrix} -3\left(\mu_1(t) - \frac{1}{n}\right)^2 & 0 & \cdots & 0 \\ 0 & -3\left(\mu_2(t) - \frac{1}{n}\right)^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -3\left(\mu_n(t) - \frac{1}{n}\right)^2 \end{bmatrix}.$$

Consequently, we have for all $\boldsymbol{\mu}(t) \in \Delta^{n-1}$

$$\begin{aligned} \boldsymbol{\mu}(t)\mathbf{H}^{\mathrm{QG}}\boldsymbol{\mu}^{\mathsf{T}}(t) &= -3\left(\mu_1^2(t)\left(\mu_1(t) - \frac{1}{n}\right)^2 + \mu_2^2(t)\left(\mu_2(t) - \frac{1}{n}\right)^2 + \cdots + \mu_n^2(t)\left(\mu_n(t) - \frac{1}{n}\right)^2\right) \\ &= -3\sum_{i=1}^n \mu_i^2(t)\left(\mu_i(t) - \frac{1}{n}\right)^2 \\ &= -3\sum_{i=1}^n \left(\mu_i(t)\left(\mu_i(t) - \frac{1}{n}\right)\right)^2 \le 0. \end{aligned}$$

Therefore, we have

$$\boldsymbol{\mu}(t)\mathbf{H}^{\mathrm{QG}}\boldsymbol{\mu}^{\mathsf{T}}(t) \le 0.$$

Thus, the $n \times n$ Hessian matrix, $\mathbf{H}^{\mathrm{QG}}$, is negative semidefinite, and by statement (i) of Lemma E.2.7 of Appendix E, we can conclude that this market quartic Gini coefficient process, $\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)$, is concave, and is thus a sufficient measure of diversity.

4.6.7 The Gini-Simpson Index

Let $\mathcal{M}$ be a market without dividends, then the function $\mathbf{S}^{\mathrm{GS}} : \Delta^{n-1} \to (0, \infty)$, defined by

$$\mathbf{S}^{\mathrm{GS}}(\mathbf{x}) = \mathbf{S}^{\mathrm{GS}}(x_1, \ldots, x_n) \triangleq 1 - \sum_{i=1}^n x_i^2, \tag{4.6.74}$$

for all $\mathbf{x}$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\mathbf{x} \in \Delta^{n-1}$, is the *Gini-Simpson index* and is a measure of diversity [see Rao (1982a,b,c, 1984) and Simpson (1949)]. Then, the corresponding market diversity measure process (the *market Gini-Simpson index*) is given below by:

Definition 4.6.10 (Market Gini-Simpson Index). *Let μ be the market portfolio. Then the* ***market Gini-Simpson index process****, $\mathbf{S}^{\mathrm{GS}}(\boldsymbol{\mu}) = \{\mathbf{S}^{\mathrm{GS}}(\boldsymbol{\mu}(t)), t \in [0,T]\}$, $\mathbf{S}^{\mathrm{GS}} : \Delta^{n-1} \to (0,\infty)$, is defined by*

$$\mathbf{S}^{\mathrm{GS}}(\boldsymbol{\mu}(t)) \;=\; \mathbf{S}^{\mathrm{GS}}(\mu_1(t),\ldots,\mu_n(t)) \;\triangleq\; 1-\sum_{i=1}^{n}\mu_i^2(t), \qquad t\in[0,T], \tag{4.6.75}$$

for all $\boldsymbol{\mu}(t)$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\boldsymbol{\mu}(t)\in\Delta^{n-1}$.

It follows from this definition that $\mathbf{S}^{\mathrm{GS}}(\boldsymbol{\mu})$ is a continuous semimartingale. For the Gini-Simpson index, minimum diversity occurs when the market Gini-Simpson index process is given by

$$\mathbf{S}^{\mathrm{GS}}(\boldsymbol{\mu}(t)) \;=\; 1-\mu_{(1)}^2(t) \;=\; 0.$$

Thus a value of 0 for the market Gini-Simpson index indicates that the market is not diverse. In addition, for the Gini-Simpson index, maximum diversity occurs when the market Gini-Simpson index is given by

$$\begin{aligned}\mathbf{S}^{\mathrm{GS}}(\boldsymbol{\mu}(t)) &= 1-n\left(\frac{1}{n}\right)^2\\ &= 1-\frac{1}{n}\\ &< 1.\end{aligned}$$

Therefore, a value of $\frac{n-1}{n}$ for the market Gini-Simpson index indicates that the market is completely diverse. Consequently, we have the following bounds on the market Gini-Simpson index process

$$0 \;<\; \mathbf{S}^{\mathrm{GS}}(\boldsymbol{\mu}(t)) \;<\; 1, \qquad t\in[0,T], \qquad \text{a.s.} \tag{4.6.76}$$

To show that the market Gini-Simpson index process is indeed a measure of diversity, we need to demonstrate that it satisfies the three primary properties that a sufficient diversity measure should possess. Clearly, the positive market Gini-Simpson index process is both twice continuously differentiable and symmetric. To demonstrate the concavity property we shall verify statement (ii) of Lemma E.2.7 of Appendix E by showing that this market Gini-Simpson index process $\mathbf{S}^{\mathrm{GS}}(\boldsymbol{\mu}(t))$ is strictly concave on Δ^{n-1} if and only if its Hessian matrix $\mathbf{H}^{\mathrm{GS}} = \mathbf{HS}^{\mathrm{GS}}(\boldsymbol{\mu}(t)) = D^2\mathbf{S}^{\mathrm{GS}}(\boldsymbol{\mu}(t)) = \big(D_iD_j\mathbf{S}^{\mathrm{GS}}(\boldsymbol{\mu}(t))\big)_{1\le i,j\le n} = \big(D_{ij}\mathbf{S}^{\mathrm{GS}}(\boldsymbol{\mu}(t))\big)_{1\le i,j\le n} \triangleq \big(\frac{\partial^2\mathbf{S}^{\mathrm{GS}}(\boldsymbol{\mu}(t))}{\partial\mu_i(t)\partial\mu_j(t)}\big)_{1\le i,j\le n}$, with $\boldsymbol{\mu}(t)$, is negative definite, i.e., if $\mathbf{x}\mathbf{H}^{\mathrm{GS}}\mathbf{x}^{\mathsf{T}} < 0$ for all $\mathbf{x}\in\Delta^{n-1}\subset\mathbb{R}^n$. For the market Gini-Simpson index process (4.6.75) of Definition 4.6.10, we have for all $i=1,2,\ldots,n$,

$$D_i\mathbf{S}^{\mathrm{GS}}(\boldsymbol{\mu}(t)) \;=\; \frac{\partial\mathbf{S}^{\mathrm{GS}}(\boldsymbol{\mu}(t))}{\partial\mu_i(t)} \;=\; -2\,\mu_i(t). \tag{4.6.77}$$

Therefore, for all $i=1,2,\ldots,n$, we have

$$D_{ii}\mathbf{S}^{\mathrm{GS}}(\boldsymbol{\mu}(t)) \;=\; \frac{\partial^2\mathbf{S}^{\mathrm{GS}}(\boldsymbol{\mu}(t))}{\partial\mu_i(t)^2} \;=\; -2, \tag{4.6.78}$$

and for all $i\neq j$, $i,j=1,2,\ldots,n$, we have

$$D_{ij}\mathbf{S}^{\mathrm{GS}}(\boldsymbol{\mu}(t)) \;=\; \frac{\partial^2\mathbf{S}^{\mathrm{GS}}(\boldsymbol{\mu}(t))}{\partial\mu_i(t)\partial\mu_j(t)} \;=\; 0. \tag{4.6.79}$$

Therefore, the $n\times n$ Hessian matrix of $\mathbf{S}^{\mathrm{GS}}(\boldsymbol{\mu}(t))$ is given by

$$\mathbf{H}^{\mathrm{GS}} \;=\; \mathbf{HS}^{\mathrm{GS}}(\boldsymbol{\mu}(t)) \;=\; \begin{bmatrix}-2 & 0 & \cdots & 0\\ 0 & -2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & -2\end{bmatrix} \;=\; -2\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{bmatrix} \;=\; -2\,\mathbf{I}_n,$$

where $\mathbf{I}_n$ is the $n\times n$ identity matrix. Consequently, we have for all $\boldsymbol{\mu}(t)\in\Delta^{n-1}$,

$$\boldsymbol{\mu}(t)\mathbf{H}^{\mathrm{GS}}\boldsymbol{\mu}^{\mathsf{T}}(t) \;=\; -2\,\boldsymbol{\mu}(t)\mathbf{I}_n\boldsymbol{\mu}^{\mathsf{T}}(t) \;=\; -2\,\boldsymbol{\mu}(t)\boldsymbol{\mu}^{\mathsf{T}}(t) \;=\; -2\left(\mu_1^2(t)+\mu_2^2(t)+\cdots+\mu_n^2(t)\right).$$

Therefore, we have

$$\boldsymbol{\mu}(t)\mathbf{H}^{\mathrm{GS}}\boldsymbol{\mu}^{\mathsf{T}}(t) \quad = \quad -2\sum_{i=1}^{n}\mu_i^2(t) \quad = \quad -2\left\|\boldsymbol{\mu}(t)\right\|^2 \quad < \quad 0.$$

Thus, we have

$$\boldsymbol{\mu}(t)\mathbf{H}^{\mathrm{GS}}\boldsymbol{\mu}^{\mathsf{T}}(t) \quad < \quad 0.$$

Thus, the $n \times n$ Hessian matrix, $\mathbf{H}^{\mathrm{GS}}$, is negative definite, and by statement (ii) of Lemma E.2.7 of Appendix E, we can conclude that the market Gini-Simpson index process, $\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)$, is strictly concave, and is as a result a sufficient measure of diversity. Of course, one can prove the above concavity result by simply noticing that the function $f(x) = -x^2$ is strictly concave (by Lemma E.2.8 of Appendix E) and by appealing to Lemma E.2.9 of Appendix E which states that the linear combination of concave functions is itself concave, since the market Gini-Simpson index process is a linear combination of the concave functions of the form $f(x) = -x^2$.

4.6.8 An Admissible Market-Dominating Diversity Measure

Let $\mathcal{M}$ be a market without dividends, then the function $\mathbf{S}^{\mathrm{A}} : \Delta^{n-1} \to (0, \infty)$, defined by

$$\mathbf{S}^{\mathrm{A}}(\mathbf{x}) \quad = \quad \mathbf{S}^{\mathrm{A}}(x_1, \ldots, x_n) \quad \triangleq \quad 1 - \frac{1}{2}\sum_{i=1}^{n} x_i^2, \tag{4.6.80}$$

for all $\mathbf{x}$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\mathbf{x} \in \Delta^{n-1}$, is the *admissible, market-dominating diversity measure* and is a measure of diversity [see Fernholz (2002)]. Then, the corresponding market diversity measure process (the *admissible, market-dominating measure*) is given below by:

Definition 4.6.11 (Admissible Market-Dominating Measure)**.** *Let μ be the market portfolio. Then the* ***admissible, market-dominating measure process,*** $\mathbf{S}^{\mathrm{A}}(\boldsymbol{\mu}) = \big\{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big), t \in [0, T]\big\}$, $\mathbf{S}^{\mathrm{A}} : \Delta^{n-1} \to (0, \infty)$, *is defined by*

$$\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big) \quad = \quad \mathbf{S}^{\mathrm{A}}\big(\mu_1(t), \ldots, \mu_n(t)\big) \quad \triangleq \quad 1 - \frac{1}{2}\sum_{i=1}^{n}\mu_i^2(t), \qquad t \in [0, T], \tag{4.6.81}$$

for all $\boldsymbol{\mu}(t)$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\boldsymbol{\mu}(t) \in \Delta^{n-1}$.

It follows from this definition that $\mathbf{S}^{\mathrm{A}}(\boldsymbol{\mu})$ is a continuous semimartingale. Recall, that minimum diversity occurs precisely when the entire capital in the market is concentrated into a single stock, i.e., when the weight of the largest stock is 1, $\mu_{(1)}(t) = 1$, and the weights of the remaining smaller ranked stocks are 0, $\mu_{(k)}(t) = 0$ for all $k = 2, \ldots, n$. Thus, we require a measure of diversity to be minimised when all the capital is entirely invested in the single largest stock. For this admissible, market-dominating diversity measure, minimum diversity occurs when the admissible, market-dominating diversity measure process is given by

$$\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big) \quad = \quad 1 - \frac{1}{2}\mu_{(1)}^2(t) \quad = \quad \frac{1}{2}.$$

Thus a value of $\frac{1}{2}$ for the admissible, market-dominating diversity measure process indicates that the market is not diverse. Recall, that maximum diversity occurs precisely when all the stocks in the market have the same capitalisation, and thus the capital is evenly distributed among all the stocks in the market, i.e., $\mu_{(1)}(t) = \mu_{(2)}(t) = \cdots = \mu_{(n)}(t) = \frac{1}{n}$. We thus require a measure of diversity to be maximised by equal weights. For this admissible, market-dominating diversity measure, maximum diversity occurs when the admissible, market-dominating diversity measure process is given by

$$\begin{aligned}\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big) \quad &= \quad 1 - \frac{1}{2}\left[n\left(\frac{1}{n}\right)^2\right] \\ &= \quad 1 - \frac{1}{2n} \\ &< \quad 1.\end{aligned}$$

Therefore, a value of $\frac{2n-1}{2n}$ for the admissible, market-dominating diversity measure process indicates that the market is completely diverse. Consequently, we have the following bounds on this admissible, market-dominating diversity measure process

$$\frac{1}{2} \quad < \quad \mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big) \quad < \quad 1, \qquad t \in [0,T], \qquad \text{a.s.} \tag{4.6.82}$$

To show that this admissible, market-dominating diversity measure process is indeed a measure of diversity, we need to demonstrate that it satisfies the three primary properties that a sufficient diversity measure should possess. Clearly, this positive admissible, market-dominating diversity measure process is both twice continuously differentiable and symmetric. To demonstrate the concavity property we shall verify statement (ii) of Lemma E.2.7 of Appendix E by showing that this admissible, market-dominating diversity measure process $\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)$ is strictly concave on Δ^{n-1} if and only if its Hessian matrix $\mathbf{H}^{\mathrm{A}} = \mathbf{H}\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big) = D^2\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big) = \big(D_iD_j\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)\big)_{1\le i,j\le n} = \big(D_{ij}\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)\big)_{1\le i,j\le n} \triangleq \big(\frac{\partial^2\mathbf{S}^{\mathrm{A}}(\boldsymbol{\mu}(t))}{\partial\mu_i(t)\partial\mu_j(t)}\big)_{1\le i,j\le n}$, with $\boldsymbol{\mu}(t) = \big(\mu_1(t), \mu_2(t), \ldots, \mu_n(t)\big)$, is negative definite, i.e., if $\mathbf{x}\mathbf{H}^{\mathrm{A}}\mathbf{x}^{\mathsf{T}} < 0$ for all $\mathbf{x} \in \Delta^{n-1} \subset \mathbb{R}^n$. For the admissible, market-dominating diversity measure process (4.6.81) of Definition 4.6.11, we have for all $i = 1, 2, \ldots, n$,

$$D_i\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big) \quad = \quad \frac{\partial\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)}{\partial\mu_i(t)} \quad = \quad -\mu_i(t). \tag{4.6.83}$$

Therefore, for all $i = 1, 2, \ldots, n$, we have

$$D_{ii}\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big) \quad = \quad \frac{\partial^2\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)}{\partial\mu_i(t)^2} \quad = \quad -1, \tag{4.6.84}$$

and for all $i \neq j$, $i, j = 1, 2, \ldots, n$, we have

$$D_{ij}\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big) \quad = \quad \frac{\partial^2\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)}{\partial\mu_i(t)\partial\mu_j(t)} \quad = \quad 0. \tag{4.6.85}$$

Therefore, the $n \times n$ Hessian matrix of $\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)$ is given by

$$\mathbf{H}^{\mathrm{A}} \quad = \quad \mathbf{H}\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big) \quad = \quad \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{bmatrix} \quad = \quad -\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \quad = \quad -\mathbf{I}_n,$$

where $\mathbf{I}_n$ is the $n \times n$ identity matrix. Consequently, we have for all $\boldsymbol{\mu}(t) \in \Delta^{n-1}$,

$$\boldsymbol{\mu}(t)\mathbf{H}^{\mathrm{A}}\boldsymbol{\mu}^{\mathsf{T}}(t) \quad = \quad -\boldsymbol{\mu}(t)\mathbf{I}_n\boldsymbol{\mu}^{\mathsf{T}}(t) \quad = \quad -\boldsymbol{\mu}(t)\boldsymbol{\mu}^{\mathsf{T}}(t) \quad = \quad -\Big(\mu_1^2(t) + \mu_2^2(t) + \cdots + \mu_n^2(t)\Big).$$

Therefore, we have

$$\boldsymbol{\mu}(t)\mathbf{H}^{\mathrm{A}}\boldsymbol{\mu}^{\mathsf{T}}(t) \quad = \quad -\sum_{i=1}^{n}\mu_i^2(t) \quad = \quad -\big\|\boldsymbol{\mu}(t)\big\|^2 \quad < \quad 0.$$

Thus, we have

$$\boldsymbol{\mu}(t)\mathbf{H}^{\mathrm{A}}\boldsymbol{\mu}^{\mathsf{T}}(t) \quad < \quad 0.$$

Thus, the $n \times n$ Hessian matrix, $\mathbf{H}^{\mathrm{A}}$, is negative definite, and by statement (ii) of Lemma E.2.7 of Appendix E, we can conclude that this admissible, market-dominating diversity measure process, $\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)$, is strictly concave, and is as a result a measure of diversity. Once again, one can prove the above concavity result by simply noticing that the function $f(x) = -x^2$ is strictly concave (by Lemma E.2.8 of Appendix E) and by appealing to Lemma E.2.9 of Appendix E. In conclusion, the admissible, market-dominating diversity measure process satisfies the three requirements of a sufficient diversity measure, and is thus an appropriate measure of diversity for our purposes.

4.6.9 Miscellaneous

Here are a few other examples of simple measures of diversity:

(i) $\mathbf{S}^{\mathrm{M}}(\mathbf{x}) \triangleq w$, a positive constant, and the corresponding market diversity measure process $\mathbf{S}^{\mathrm{M}}\big(\boldsymbol{\mu}(t)\big) \triangleq w$;

(ii) $\mathbf{S}^{\mathrm{BH}}(\mathbf{x}) \triangleq w_1x_1 + \cdots + w_nx_n$, where $w_1, \ldots, w_n$ are nonnegative constants at least one of which is positive, is the *buy-and-hold function*, $\mathbf{S}^{\mathrm{BH}} : \Delta^{n-1} \to (0, \infty)$, which is defined by

$$\mathbf{S}^{\mathrm{BH}}(\mathbf{x}) \;=\; \mathbf{S}^{\mathrm{BH}}(x_1, \ldots, x_n) \;\triangleq\; w_1x_1 + \cdots + w_nx_n \;=\; \sum_{i=1}^{n} w_ix_i, \tag{4.6.86}$$

for all $\mathbf{x}$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\mathbf{x} \in \Delta^{n-1}$. Then, the corresponding market measure process (the *market buy-and-hold function*) is given below by:

Definition 4.6.12 (Market Buy-and-Hold Measure)**.** *Let μ be the market portfolio. Then the* ***market buy-and-hold measure process,*** $\mathbf{S}^{\mathrm{BH}}(\boldsymbol{\mu}) = \big\{\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big), t \in [0, T]\big\}$, $\mathbf{S}^{\mathrm{BH}} : \Delta^{n-1} \to (0, \infty)$, *is defined for all $t \in [0, T]$ by*

$$\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big) \;=\; \mathbf{S}^{\mathrm{BH}}\big(\mu_1(t), \ldots, \mu_n(t)\big) \;\triangleq\; w_1\mu_1(t) + \cdots + w_n\mu_n(t) \;=\; \sum_{i=1}^{n} w_i\mu_i(t), \tag{4.6.87}$$

for all $\boldsymbol{\mu}(t)$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\boldsymbol{\mu}(t) \in \Delta^{n-1}$.

For this market buy-and-hold function, we have the simple bounds

$$0 \;\le\; w_{(n)} \;<\; \mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big) \;\le\; w_{(1)}, \qquad t \in [0, T], \qquad \text{a.s.} \tag{4.6.88}$$

Clearly, this positive market buy-and-hold measure process is twice continuously differentiable. To demonstrate the concavity property we shall verify statement (i) of Lemma E.2.7 of Appendix E by showing that this market buy-and-hold measure process $\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)$ is concave on Δ^{n-1} if and only if its Hessian matrix $\mathbf{H}^{\mathrm{BH}} = \mathbf{H}\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big) = D^2\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big) = \big(D_iD_j\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)\big)_{1\le i,j\le n} = \big(D_{ij}\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)\big)_{1\le i,j\le n} \triangleq \big(\frac{\partial^2\mathbf{S}^{\mathrm{BH}}(\boldsymbol{\mu}(t))}{\partial\mu_i(t)\partial\mu_j(t)}\big)_{1\le i,j\le n}$, with $\boldsymbol{\mu}(t)$, is negative semidefinite, i.e., if $\mathbf{x}\mathbf{H}^{\mathrm{BH}}\mathbf{x}^{\mathsf{T}} \le 0$ for all $\mathbf{x} \in \Delta^{n-1} \subset \mathbb{R}^n$. For the market buy-and-hold measure process (4.6.87) of Definition 4.6.12, we have for all $i = 1, 2, \ldots, n$,

$$D_i\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big) \;=\; \frac{\partial\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)}{\partial\mu_i(t)} \;=\; w_i. \tag{4.6.89}$$

Therefore, for all $i, j = 1, 2, \ldots, n$, we have

$$D_{ij}\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big) \;=\; \frac{\partial^2\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)}{\partial\mu_i(t)\partial\mu_j(t)} \;=\; 0. \tag{4.6.90}$$

Therefore, the $n \times n$ Hessian matrix of $\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)$ is given by

$$\mathbf{H}^{\mathrm{BH}} \;=\; \mathbf{H}\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big) \;=\; \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \;=\; \mathbf{0}_n,$$

where $\mathbf{0}_n$ is the $n \times n$ zero matrix. Hence, the $n \times n$ Hessian matrix of $\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)$ is the zero matrix. Consequently, we have for all $\boldsymbol{\mu}(t) \in \Delta^{n-1}$,

$$\boldsymbol{\mu}(t)\mathbf{H}^{\mathrm{BH}}\boldsymbol{\mu}^{\mathsf{T}}(t) \;=\; \boldsymbol{\mu}(t)\mathbf{0}_n\boldsymbol{\mu}^{\mathsf{T}}(t) \;=\; 0.$$

Therefore, we have

$$\boldsymbol{\mu}(t)\mathbf{H}^{\mathrm{BH}}\boldsymbol{\mu}^{\mathsf{T}}(t) \;=\; 0.$$

Thus, the $n \times n$ Hessian matrix, $\mathbf{H}^{\mathrm{BH}}$, is both negative semidefinite and positive semidefinite, and by statements (i) and (iii) of Lemma E.2.7 of Appendix E, we can conclude that the market buy-and-hold measure process, $\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)$, is both concave and convex (since $\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)$ is an affine function, i.e., a linear function). In addition, the market buy-and-hold function $\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)$ fails to be symmetric, and is as a result not a measure of diversity. In conclusion, the market buy-and-hold measure process fails to satisfy all three requirements of a sufficient diversity measure, by failing the symmetry condition, and is thus not an appropriate measure of diversity for our purposes.

(iii) $\mathbf{S}^{\mathrm{GM}}(\mathbf{x}) \triangleq (x_1 \cdots x_n)^{\frac{1}{n}}$, is the *geometric mean function*, $\mathbf{S}^{\mathrm{GM}} : \Delta^{n-1} \to (0, \infty)$, which is defined by

$$\mathbf{S}^{\mathrm{GM}}(\mathbf{x}) \;=\; \mathbf{S}^{\mathrm{GM}}(x_1, \ldots, x_n) \;\triangleq\; (x_1 \cdots x_n)^{\frac{1}{n}} \;=\; \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}, \tag{4.6.91}$$

for all $\mathbf{x}$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\mathbf{x} \in \Delta^{n-1}$. Then, the corresponding market diversity measure process (the *market geometric mean function*) is given below by:

Definition 4.6.13 (Market Geometric Mean)**.** *Let μ be the market portfolio. Then the* ***market geometric mean process****, $\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}) = \big\{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big), t \in [0,T]\big\}$, $\mathbf{S}^{\mathrm{GM}} : \Delta^{n-1} \to (0, \infty)$, is defined for all $t \in [0,T]$ by*

$$\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) \;=\; \mathbf{S}^{\mathrm{GM}}\big(\mu_1(t), \ldots, \mu_n(t)\big) \;\triangleq\; \big(\mu_1(t) \cdots \mu_n(t)\big)^{\frac{1}{n}} \;=\; \left(\prod_{i=1}^{n} \mu_i(t)\right)^{\frac{1}{n}}, \tag{4.6.92}$$

for all $\boldsymbol{\mu}(t)$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\boldsymbol{\mu}(t) \in \Delta^{n-1}$.

For this market geometric mean function, we have the simple bounds

$$0 \;<\; \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) \;\leq\; \frac{1}{n}, \qquad t \in [0,T], \qquad \text{a.s.} \tag{4.6.93}$$

Clearly, the positive market geometric mean process, $\mathbf{S}^{\mathrm{GM}}$, is both twice continuously differentiable and symmetric. To demonstrate the concavity property we shall verify statement (i) of Lemma E.2.7 of Appendix E by showing that the market geometric mean process $\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)$ is concave on Δ^{n-1} if and only if its Hessian matrix $\mathbf{H}^{\mathrm{GM}} = \mathbf{H}\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) = D^2\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) = \big(D_i D_j \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)\big)_{1 \leq i,j \leq n} = \big(D_{ij}\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)\big)_{1 \leq i,j \leq n} \triangleq \big(\frac{\partial^2 \mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}(t))}{\partial \mu_i(t) \partial \mu_j(t)}\big)_{1 \leq i,j \leq n}$, with $\boldsymbol{\mu}(t) = \big(\mu_1(t), \mu_2(t), \ldots, \mu_n(t)\big)$, is negative semidefinite, i.e., if $\mathbf{x}\mathbf{H}^{\mathrm{GM}}\mathbf{x}^{\mathsf{T}} \leq 0$ for all $\mathbf{x} \in \Delta^{n-1} \subset \mathbb{R}^n$. For the market geometric mean process (4.6.92) of Definition 4.6.13, we have for all $i = 1, 2, \ldots, n$,

$$\begin{aligned}
D_i\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) \;=\; \frac{\partial \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\partial \mu_i(t)} \;&=\; \frac{1}{n}\left(\prod_{k=1}^{n} \mu_k(t)\right)^{\frac{1}{n}-1}\left(\prod_{k \neq i}^{n} \mu_k(t)\right) \\
&=\; \frac{1}{n}\left(\prod_{k=1}^{n} \mu_k(t)\right)^{\frac{1}{n}}\left(\prod_{k \neq i}^{n} \mu_k(t)\right)\Bigg/\left(\prod_{k=1}^{n} \mu_k(t)\right) \\
&=\; \frac{1}{n}\,\frac{1}{\mu_i(t)}\left(\prod_{k=1}^{n} \mu_k(t)\right)^{\frac{1}{n}} \\
&=\; \frac{1}{n}\,\frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)}.
\end{aligned} \tag{4.6.94}$$

Therefore, for all $i = 1, 2, \ldots, n$, we have

$$\begin{aligned}
D_{ii}\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) &= \frac{\partial^2 \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\partial \mu_i(t)^2} = \frac{1}{n}\left(\frac{1}{n}-1\right)\left(\prod_{k=1}^{n}\mu_k(t)\right)^{\frac{1}{n}-2}\left(\prod_{k\neq i}^{n}\mu_k(t)\right)^2 \\
&= \frac{1}{n}\left(\frac{1}{n}-1\right)\left(\prod_{k=1}^{n}\mu_k(t)\right)^{\frac{1}{n}}\left(\prod_{k\neq i}^{n}\mu_k(t)\right)^2 \Bigg/ \left(\prod_{k=1}^{n}\mu_k(t)\right)^2 \\
&= \frac{1}{n}\left(\frac{1}{n}-1\right)\frac{1}{\mu_i^2(t)}\left(\prod_{k=1}^{n}\mu_k(t)\right)^{\frac{1}{n}} \\
&= \frac{1}{n}\left(\frac{1}{n}-1\right)\frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} \qquad (4.6.95) \\
&= \frac{1}{n^2}\frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} - \frac{1}{n}\frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)}, \qquad (4.6.96)
\end{aligned}$$

and for all $i \neq j$, $i, j = 1, 2, \ldots, n$, we have

$$\begin{aligned}
D_{ij}\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) &= \frac{\partial^2 \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\partial \mu_i(t)\partial \mu_j(t)} = \frac{1}{n^2}\frac{1}{\mu_i(t)}\left(\prod_{k=1}^{n}\mu_k(t)\right)^{\frac{1}{n}-1}\left(\prod_{k\neq j}^{n}\mu_k(t)\right) \\
&= \frac{1}{n^2}\frac{1}{\mu_i(t)}\left(\prod_{k=1}^{n}\mu_k(t)\right)^{\frac{1}{n}}\left(\prod_{k\neq j}^{n}\mu_k(t)\right) \Bigg/ \left(\prod_{k=1}^{n}\mu_k(t)\right) \\
&= \frac{1}{n^2}\frac{1}{\mu_i(t)\mu_j(t)}\left(\prod_{k=1}^{n}\mu_k(t)\right)^{\frac{1}{n}} \\
&= \frac{1}{n^2}\frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)\mu_j(t)}. \qquad (4.6.97)
\end{aligned}$$

Then the $n \times n$ Hessian matrix of $\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)$ is given by

$$\begin{aligned}
\mathbf{H}^{\mathrm{GM}} &= \mathbf{H}\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}) \\
&= \begin{bmatrix}
\frac{1}{n}\left(\frac{1}{n}-1\right)\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_1^2} & \frac{1}{n^2}\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_1\mu_2} & \cdots & \frac{1}{n^2}\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_1\mu_n} \\
\frac{1}{n^2}\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_2\mu_1} & \frac{1}{n}\left(\frac{1}{n}-1\right)\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_2^2} & \cdots & \frac{1}{n^2}\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_2\mu_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{n^2}\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_n\mu_1} & \frac{1}{n^2}\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_n\mu_2} & \cdots & \frac{1}{n}\left(\frac{1}{n}-1\right)\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_n^2}
\end{bmatrix} \\
&= \begin{bmatrix}
\frac{1}{n^2}\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_1^2} - \frac{1}{n}\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_1^2} & \frac{1}{n^2}\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_1\mu_2} & \cdots & \frac{1}{n^2}\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_1\mu_n} \\
\frac{1}{n^2}\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_2\mu_1} & \frac{1}{n^2}\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_2^2} - \frac{1}{n}\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_2^2} & \cdots & \frac{1}{n^2}\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_2\mu_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{n^2}\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_n\mu_1} & \frac{1}{n^2}\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_n\mu_2} & \cdots & \frac{1}{n^2}\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_n^2} - \frac{1}{n}\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_n^2}
\end{bmatrix}.
\end{aligned}$$

Consequently, we have for all $\boldsymbol{\mu}(t) \in \Delta^{n-1}$,

$$\boldsymbol{\mu}(t)\mathbf{H}^{\mathrm{GM}}\boldsymbol{\mu}^{\mathsf{T}}(t) = n^2\left(\frac{1}{n^2}\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})\right) - n\left(\frac{1}{n}\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})\right) = n\left(\frac{1}{n}\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}) - \frac{1}{n}\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})\right) = 0.$$

Therefore, we have

$$\boldsymbol{\mu}(t)\mathbf{H}^{\mathrm{GM}}\boldsymbol{\mu}^{\mathsf{T}}(t) = 0.$$

Thus, the $n \times n$ Hessian matrix, $\mathbf{H}^{\mathrm{GM}}$, is simultaneously negative semidefinite and positive semidefinite, and by statements (i) and (iii) of Lemma E.2.7 of Appendix E, we can draw no conclusions regarding the concavity or convexity of the market geometric mean process, $\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)$. However, by appealing to the bounds $0 < \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) \leq \frac{1}{n}$ as well as the nature and location (i.e., coordinates) of the bounds and the fact that $\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)$ is a symmetric function, we can conclude that the market geometric mean process, $\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)$, is concave, and is as a result a measure of diversity. In conclusion, the market geometric mean process satisfies all three requirements of a sufficient diversity measure, and is thus an appropriate measure of diversity for our purposes.

(iv) $\mathbf{S}^{\mathrm{MGM}}(\mathbf{x}) = \mathbf{S}_c^{\mathrm{GM}}(\mathbf{x}) \triangleq c + \mathbf{S}^{\mathrm{GM}}(\mathbf{x}) = c + (x_1 \cdots x_n)^{\frac{1}{n}}$ for any sufficiently large real constant $c \in (0, \infty)$, is the *modified geometric mean function*, $\mathbf{S}^{\mathrm{MGM}} = \mathbf{S}_c^{\mathrm{GM}} : \Delta^{n-1} \to (0, \infty)$, which is defined by

$$\begin{aligned} \mathbf{S}^{\mathrm{MGM}}(\mathbf{x}) &= \mathbf{S}_c^{\mathrm{GM}}(\mathbf{x}) = \mathbf{S}_c^{\mathrm{GM}}(x_1, \ldots, x_n) \\ &\triangleq c + \mathbf{S}^{\mathrm{GM}}(\mathbf{x}) = c + (x_1 \cdots x_n)^{\frac{1}{n}} = c + \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}, \end{aligned} \tag{4.6.98}$$

for all $\mathbf{x}$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\mathbf{x} \in \Delta^{n-1}$. Then, the corresponding market diversity measure process (the *modified market geometric mean function*) is given below by:

Definition 4.6.14 (Modified Market Geometric Mean). *Let μ be the market portfolio. Then the **modified market geometric mean process**, $\mathbf{S}^{\mathrm{MGM}}(\boldsymbol{\mu}) = \mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}) = \big\{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big), t \in [0, T]\big\}$, $\mathbf{S}^{\mathrm{MGM}} = \mathbf{S}_c^{\mathrm{GM}} : \Delta^{n-1} \to (0, \infty)$, is defined for all $t \in [0, T]$ by*

$$\begin{aligned} \mathbf{S}^{\mathrm{MGM}}\big(\boldsymbol{\mu}(t)\big) &= \mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) = \mathbf{S}_c^{\mathrm{GM}}\big(\mu_1(t), \ldots, \mu_n(t)\big) \\ &\triangleq c + \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) = c + \big(\mu_1(t) \cdots \mu_n(t)\big)^{\frac{1}{n}} = c + \left(\prod_{i=1}^{n} \mu_i(t)\right)^{\frac{1}{n}}, \end{aligned} \tag{4.6.99}$$

for all $\boldsymbol{\mu}(t)$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\boldsymbol{\mu}(t) \in \Delta^{n-1}$.

Thus, we also have the following

$$\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) = \mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - c, \qquad t \in [0, T]. \tag{4.6.100}$$

For this modified market geometric mean function, from (4.6.93), we have the simple bounds

$$c < \mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) \leq c + \frac{1}{n}, \qquad t \in [0, T], \qquad \text{a.s.} \tag{4.6.101}$$

Clearly, the positive modified market geometric mean process, $\mathbf{S}_c^{\mathrm{GM}}$, is both twice continuously differentiable and symmetric. To demonstrate the concavity property we shall verify statement (i) of Lemma E.2.7 of Appendix E by showing that the modified market geometric mean process $\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)$ is concave on Δ^{n-1} if and only if its Hessian matrix $\mathbf{H}_c^{\mathrm{GM}} = \mathbf{H}\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) = D^2\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) = \big(D_i D_j \mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)\big)_{1 \leq i,j \leq n} = \big(D_{ij}\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)\big)_{1 \leq i,j \leq n} \triangleq \big(\frac{\partial^2 \mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))}{\partial \mu_i(t) \partial \mu_j(t)}\big)_{1 \leq i,j \leq n}$, with $\boldsymbol{\mu}(t) = \big(\mu_1(t), \mu_2(t), \ldots, \mu_n(t)\big)$, is negative semidefinite, i.e., if $\mathbf{x}\mathbf{H}_c^{\mathrm{GM}}\mathbf{x}^{\top} \leq 0$ for all $\mathbf{x} \in \Delta^{n-1} \subset \mathbb{R}^n$. For the modified market geometric mean process (4.6.99) of Definition 4.6.14, using (4.6.100) together with equation (4.6.94), we have for all $i = 1, 2, \ldots, n$,

$$D_i\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) = D_i\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) = \frac{1}{n}\frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)} = \frac{1}{n}\frac{\big(\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - c\big)}{\mu_i(t)}. \tag{4.6.102}$$

Therefore, using (4.6.100) together with equation (4.6.95) and (4.6.96), for all $i = 1, 2, \ldots, n$, we have

$$D_{ii}\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) = D_{ii}\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) = \frac{1}{n}\left(\frac{1}{n} - 1\right)\frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} = \frac{1}{n}\left(\frac{1}{n} - 1\right)\frac{\big(\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - c\big)}{\mu_i^2(t)} \tag{4.6.103}$$

$$= \frac{1}{n^2}\frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} - \frac{1}{n}\frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} \tag{4.6.104}$$

$$= \frac{1}{n^2}\frac{\big(\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - c\big)}{\mu_i^2(t)} - \frac{1}{n}\frac{\big(\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - c\big)}{\mu_i^2(t)}, \tag{4.6.105}$$

and, using (4.6.100) together with equation (4.6.97), for all $i \neq j$, $i, j = 1, 2, \ldots, n$, we have

$$D_{ij}\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) \;=\; D_{ij}\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) \;=\; \frac{1}{n^2}\,\frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)\mu_j(t)} \;=\; \frac{1}{n^2}\,\frac{\big(\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - c\big)}{\mu_i(t)\mu_j(t)}. \tag{4.6.106}$$

Then the $n \times n$ Hessian matrix of $\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)$ is given by

$$\begin{aligned}
\mathbf{H}_c^{\mathrm{GM}} &= \mathbf{H}\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}) \\
&= \begin{bmatrix}
\frac{1}{n}\left(\frac{1}{n}-1\right)\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_1^2} & \frac{1}{n^2}\,\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_1\mu_2} & \cdots & \frac{1}{n^2}\,\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_1\mu_n} \\
\frac{1}{n^2}\,\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_2\mu_1} & \frac{1}{n}\left(\frac{1}{n}-1\right)\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_2^2} & \cdots & \frac{1}{n^2}\,\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_2\mu_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{n^2}\,\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_n\mu_1} & \frac{1}{n^2}\,\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_n\mu_2} & \cdots & \frac{1}{n}\left(\frac{1}{n}-1\right)\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu})}{\mu_n^2}
\end{bmatrix} \\
&= \begin{bmatrix}
\frac{1}{n}\left(\frac{1}{n}-1\right)\frac{\big(\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu})-c\big)}{\mu_1^2} & \frac{1}{n^2}\,\frac{\big(\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu})-c\big)}{\mu_1\mu_2} & \cdots & \frac{1}{n^2}\,\frac{\big(\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu})-c\big)}{\mu_1\mu_n} \\
\frac{1}{n^2}\,\frac{\big(\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu})-c\big)}{\mu_2\mu_1} & \frac{1}{n}\left(\frac{1}{n}-1\right)\frac{\big(\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu})-c\big)}{\mu_2^2} & \cdots & \frac{1}{n^2}\,\frac{\big(\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu})-c\big)}{\mu_2\mu_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{n^2}\,\frac{\big(\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu})-c\big)}{\mu_n\mu_1} & \frac{1}{n^2}\,\frac{\big(\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu})-c\big)}{\mu_n\mu_2} & \cdots & \frac{1}{n}\left(\frac{1}{n}-1\right)\frac{\big(\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu})-c\big)}{\mu_n^2}
\end{bmatrix}.
\end{aligned}$$

Therefore, we have

$$\mathbf{H}_c^{\mathrm{GM}} \;=\; \mathbf{H}^{\mathrm{GM}}.$$

Consequently, we have for all $\boldsymbol{\mu}(t) \in \Delta^{n-1}$,

$$\boldsymbol{\mu}(t)\mathbf{H}_c^{\mathrm{GM}}\boldsymbol{\mu}^{\mathsf{T}}(t) \;=\; \boldsymbol{\mu}(t)\mathbf{H}^{\mathrm{GM}}\boldsymbol{\mu}^{\mathsf{T}}(t) \;=\; 0.$$

Therefore, we have

$$\boldsymbol{\mu}(t)\mathbf{H}_c^{\mathrm{GM}}\boldsymbol{\mu}^{\mathsf{T}}(t) \;=\; 0.$$

Thus, the $n \times n$ Hessian matrix, $\mathbf{H}_c^{\mathrm{GM}}$, is simultaneously negative semidefinite and positive semidefinite, and by statements (i) and (iii) of Lemma E.2.7 of Appendix E, we can draw no conclusions regarding the concavity or convexity of the modified market geometric mean process, $\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)$. However, by appealing to the bounds $c < \mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) \leq c + \frac{1}{n}$ as well as the nature and location (i.e., coordinates) of the bounds and the fact that $\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)$ is a symmetric function, we can conclude that the modified market geometric mean process, $\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)$, is concave, and is as a result a measure of diversity. Of course, one can show the above concavity result by simply recalling that the geometric mean function $\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)$ is concave and by appealing to Lemma E.2.9 of Appendix E which states that the linear combination of concave functions is again concave, since the modified market geometric mean process is the linear combination of the concave geometric mean function $\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)$ with the positive constant c. In conclusion, the modified market geometric mean process satisfies all three requirements of a sufficient diversity measure, and is thus an appropriate measure of diversity for our purposes.

(v) $\mathbf{S}^{\mathrm{C}}(\mathbf{x}) \triangleq x_1^{p_1} \cdots x_n^{p_n}$, where $p_1, \ldots, p_n$ are constants and $p_1 + \cdots + p_n = 1$, is given by the following function, $\mathbf{S}^{\mathrm{C}} : \Delta^{n-1} \to (0, \infty)$, which is defined by

$$\mathbf{S}^{\mathrm{C}}(\mathbf{x}) \;=\; \mathbf{S}^{\mathrm{C}}(x_1, \ldots, x_n) \;\triangleq\; x_1^{p_1} \cdots x_n^{p_n} \;=\; \prod_{i=1}^{n} x_i^{p_i}, \tag{4.6.107}$$

for all $\mathbf{x}$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\mathbf{x} \in \Delta^{n-1}$. Then, the corresponding market measure process is given below by:

Definition 4.6.15. *Let μ be the market portfolio. Then the market process, $\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu}) = \big\{\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big), t \in [0,T]\big\}$, $\mathbf{S}^{\mathrm{C}} : \Delta^{n-1} \to (0,\infty)$, is defined for all $t \in [0,T]$ by*

$$\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big) \;=\; \mathbf{S}^{\mathrm{C}}\big(\mu_1(t),\ldots,\mu_n(t)\big) \;\triangleq\; \big(\mu_1(t)\big)^{p_1}\cdots\big(\mu_n(t)\big)^{p_n} \;=\; \prod_{i=1}^{n}\big(\mu_i(t)\big)^{p_i}, \tag{4.6.108}$$

for all $\boldsymbol{\mu}(t)$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\boldsymbol{\mu}(t) \in \Delta^{n-1}$.

For this function, we have the simple bounds

$$0 \;<\; \mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big) \;<\; \infty, \qquad t \in [0,T], \qquad \text{a.s.} \tag{4.6.109}$$

Clearly, the positive market process $\mathbf{S}^{\mathrm{C}}$, is twice continuously differentiable. To demonstrate the concavity property we shall verify statement (i) of Lemma E.2.7 of Appendix E by showing that the market process $\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)$ is concave on Δ^{n-1} if and only if its Hessian matrix $\mathbf{H}^{\mathrm{C}} = \mathbf{H}\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big) = D^2\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big) = \big(D_iD_j\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)\big)_{1\le i,j\le n} = \big(D_{ij}\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)\big)_{1\le i,j\le n} \triangleq \big(\frac{\partial^2\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu}(t))}{\partial\mu_i(t)\partial\mu_j(t)}\big)_{1\le i,j\le n}$, with $\boldsymbol{\mu}(t)$, is negative semidefinite, i.e., if $\mathbf{x}\mathbf{H}^{\mathrm{C}}\mathbf{x}^{\mathsf{T}} \le 0$ for all $\mathbf{x} \in \Delta^{n-1} \subset \mathbb{R}^n$. For the market process $\mathbf{S}^{\mathrm{C}}$ (4.6.108) of Definition 4.6.15, we have for all $i = 1,2,\ldots,n$,

$$\begin{aligned} D_i\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big) \;=\; \frac{\partial\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)}{\partial\mu_i(t)} \;&=\; p_i\big(\mu_i(t)\big)^{p_i-1}\left(\prod_{k\ne i}^{n}\big(\mu_k(t)\big)^{p_k}\right) \\ &=\; p_i\,\frac{1}{\mu_i(t)}\left(\prod_{k=1}^{n}\big(\mu_k(t)\big)^{p_k}\right) \\ &=\; p_i\,\frac{\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)}. \end{aligned} \tag{4.6.110}$$

Therefore, for all $i = 1,2,\ldots,n$, we have

$$\begin{aligned} D_{ii}\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big) \;=\; \frac{\partial^2\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)}{\partial\mu_i(t)^2} \;&=\; p_i(p_i-1)\big(\mu_i(t)\big)^{p_i-2}\left(\prod_{k\ne i}^{n}\big(\mu_k(t)\big)^{p_k}\right) \\ &=\; p_i(p_i-1)\,\frac{1}{\mu_i^2(t)}\left(\prod_{k=1}^{n}\big(\mu_k(t)\big)^{p_k}\right) \\ &=\; p_i(p_i-1)\,\frac{\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} \qquad (4.6.111) \\ &=\; p_i^2\,\frac{\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} - p_i\,\frac{\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)}, \qquad (4.6.112) \end{aligned}$$

and for all $i \ne j$, $i,j = 1,2,\ldots,n$, we have

$$\begin{aligned} D_{ij}\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big) \;=\; \frac{\partial^2\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)}{\partial\mu_i(t)\partial\mu_j(t)} \;&=\; p_i\big(\mu_i(t)\big)^{p_i-1}p_j\big(\mu_j(t)\big)^{p_j-1}\left(\prod_{k\ne i,j}^{n}\big(\mu_k(t)\big)^{p_k}\right) \\ &=\; p_ip_j\,\frac{1}{\mu_i(t)\mu_j(t)}\left(\prod_{k=1}^{n}\big(\mu_k(t)\big)^{p_k}\right) \\ &=\; p_ip_j\,\frac{\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)\mu_j(t)}. \end{aligned} \tag{4.6.113}$$

Then the $n \times n$ Hessian matrix of $\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)$ is given by

$$
\begin{aligned}
\mathbf{H}^{\mathrm{C}} &= \mathbf{H}\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu}) \\
&= \begin{bmatrix}
p_1(p_1-1)\,\dfrac{\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu})}{\mu_1^2} & p_1p_2\,\dfrac{\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu})}{\mu_1\mu_2} & \cdots & p_1p_n\,\dfrac{\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu})}{\mu_1\mu_n} \\
p_2p_1\,\dfrac{\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu})}{\mu_2\mu_1} & p_2(p_2-1)\,\dfrac{\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu})}{\mu_2^2} & \cdots & p_2p_n\,\dfrac{\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu})}{\mu_2\mu_n} \\
\vdots & \vdots & \ddots & \vdots \\
p_np_1\,\dfrac{\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu})}{\mu_n\mu_1} & p_np_2\,\dfrac{\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu})}{\mu_n\mu_2} & \cdots & p_n(p_n-1)\,\dfrac{\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu})}{\mu_n^2}
\end{bmatrix} \\
&= \begin{bmatrix}
p_1^2\,\dfrac{\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu})}{\mu_1^2} - p_1\,\dfrac{\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu})}{\mu_1^2} & p_1p_2\,\dfrac{\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu})}{\mu_1\mu_2} & \cdots & p_1p_n\,\dfrac{\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu})}{\mu_1\mu_n} \\
p_2p_1\,\dfrac{\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu})}{\mu_2\mu_1} & p_2^2\,\dfrac{\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu})}{\mu_2^2} - p_2\,\dfrac{\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu})}{\mu_2^2} & \cdots & p_2p_n\,\dfrac{\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu})}{\mu_2\mu_n} \\
\vdots & \vdots & \ddots & \vdots \\
p_np_1\,\dfrac{\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu})}{\mu_n\mu_1} & p_np_2\,\dfrac{\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu})}{\mu_n\mu_2} & \cdots & p_n^2\,\dfrac{\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu})}{\mu_n^2} - p_n\,\dfrac{\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu})}{\mu_n^2}
\end{bmatrix}.
\end{aligned}
$$

Consequently, we have for all $\boldsymbol{\mu}(t) \in \Delta^{n-1}$,

$$
\begin{aligned}
\boldsymbol{\mu}(t)\mathbf{H}^{\mathrm{C}}\boldsymbol{\mu}^{\mathsf{T}}(t) &= p_1(p_1-1)\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu}) + p_1p_2\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu}) + \cdots + p_1p_n\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu}) \\
&\quad + p_2p_1\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu}) + p_2(p_2-1)\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu}) + \cdots + p_2p_n\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu}) \\
&\quad + \cdots + p_np_1\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu}) + p_np_2\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu}) + \cdots + p_n(p_n-1)\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu}) \\
&= p_1^2\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu}) - p_1\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu}) + p_1p_2\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu}) + \cdots + p_1p_n\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu}) \\
&\quad + p_2p_1\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu}) + p_2^2\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu}) - p_2\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu}) + \cdots + p_2p_n\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu}) \\
&\quad + \cdots + p_np_1\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu}) + p_np_2\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu}) + \cdots + p_n^2\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu}) - p_n\mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu}) \\
&= \mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu})\left[\sum_{k,m=1}^{n} p_kp_m\right] - \mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu})\left[\sum_{k=1}^{n} p_k\right] \\
&= \mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu})\left[\sum_{k,m=1}^{n} p_kp_m - \sum_{k=1}^{n} p_k\right] \\
&= \mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)\left[\sum_{k,m=1}^{n} p_kp_m - \sum_{k=1}^{n} p_k\right] \\
&= \mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)\left[\left(\sum_{k=1}^{n} p_k\right)^2 - \sum_{k=1}^{n} p_k\right].
\end{aligned}
$$

Since $p_1 + \cdots + p_n = 1$, we have

$$
\boldsymbol{\mu}(t)\mathbf{H}^{\mathrm{C}}\boldsymbol{\mu}^{\mathsf{T}}(t) = 0.
$$

Thus, the $n \times n$ Hessian matrix, $\mathbf{H}^{\mathrm{C}}$, is simultaneously negative semidefinite and positive semidefinite, and by statements (i) and (iii) of Lemma E.2.7 of Appendix E, we can draw no conclusions regarding the concavity or convexity of the market process, $\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)$. The concavity or convexity of the function depends on the combination of the values that the constants $p_1, \ldots, p_n$ assume. In addition, the function $\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)$ fails to be symmetric, and is as a result not a measure of diversity. In conclusion, the process $\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)$ fails to satisfy all three requirements of a sufficient diversity measure, by failing the symmetry condition and by potentially failing the concavity condition, and is thus not an appropriate measure of diversity for our purposes.

4.7 Entropy and Stock Market Diversity

Recall that the notion of diversity provided a criterion for establishing whether or not a market is diverse, whereas the concept of entropy (or any diversity measure for that matter) provides a measure of the extent to which the market is diverse. These two concepts are united in the following proposition, where diversity is characterised in terms of the market entropy process.

Proposition 4.7.1 ([**Fernholz (1998b, 2002)**]). *The market $\mathcal{M}$ is diverse if and only if there is a $\zeta > 0$ such that*

$$\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \quad \geq \quad \zeta, \qquad t \in [0,T], \qquad a.s. \tag{4.7.1}$$

Proof. The proof is elementary and depends on the continuity of the market entropy function (4.6.2) when extended to the closed nonnegative unit $(n-1)-$simplex $\overline{\Delta}^{n-1} \subset \mathbb{R}^n$,

$$\overline{\Delta}^{n-1} \quad \triangleq \quad \Big\{\boldsymbol{\mu}(t) = \big(\mu_1(t), \ldots, \mu_n(t)\big) \in \mathbb{R}^n \;\Big|\; \mu_1(t) + \cdots + \mu_n(t) = 1,\ 0 \leq \mu_i(t) \leq 1,\ i = 1, 2, \ldots, n\Big\}.$$

The function $\mathbf{S}^{\mathrm{E}}(\boldsymbol{\mu})$ is nonnegative on the compact set $\overline{\Delta}^{n-1}$, and $\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) = 0$ only on the vertices, i.e., $\mu_i(t) = 0$ and $\mu_i(t) = 1$ for all $i = 1, 2, \ldots, n$. If a neighbourhood of the vertices in $\overline{\Delta}^{n-1}$ is withdrawn, then the market entropy process, $\mathbf{S}^{\mathrm{E}}(\boldsymbol{\mu})$ is bounded away from 0 on the rest of $\overline{\Delta}^{n-1}$. ■

This also follows from the bounds obtained for the market entropy process, given by (4.6.3). These bounds revealed that a value of 0 for the market entropy process is an indication that the market is not diverse. However, a value other than 0 for the market entropy process would suggest that the market exhibits diverse behaviour. Clearly, if the market entropy process is bounded away from 0 the market is diverse.

4.8 Summary and Conclusion

In this chapter, we explored the notion of diversity in equity markets. Diversity of a stock market is a measure of how uniformly the capital is apportioned among the stocks in the market. This is tantamount to saying that diversity is a measure of the level of the concentration of capital into each stock in the market. With this in mind, a market is diverse if it avoids the extreme concentration of capital into single stocks and if at no time the largest stock in the market accounts for almost the entire market capitalisation. We offered a formal definition of the notion of diversity in financial equity markets, together with the formal definitions of the allied, successively weaker notions of weak diversity, uniform weak diversity and asymptotic weak diversity. The market $\mathcal{M}$ is diverse if there exists a number $\delta > 0$, $\delta \in (0,1)$ such that

$$\mu_{(1)}(t) \quad \leq \quad 1 - \delta, \qquad t \in [0, \infty), \qquad \text{a.s.} \tag{4.8.1}$$

The market $\mathcal{M}$ is diverse on the time horizon $[0,T]$, with $T > 0$ a given real number, if there exists a number $\delta > 0$, $\delta \in (0,1)$ such that

$$\mu_{(1)}(t) \quad \leq \quad 1 - \delta, \qquad t \in [0,T], \qquad \text{a.s.}, \tag{4.8.2}$$

in the reverse-order-statistics notation (2.4.53) of Definition 2.4.10, where $\mu_{\max}(t) \triangleq \max_{1 \leq i \leq n} \mu_i(t) \triangleq \mu_{(1)}(t)$, i.e., $\mu_{(1)}$ represents the largest market weight. The market $\mathcal{M}$ is weakly diverse on the time horizon $[0,T]$, with $T > 0$ a given real number, if there exists a number $\delta > 0$, $\delta \in (0,1)$ such that we have

$$\frac{1}{T} \int_0^T \mu_{(1)}(t)\, dt \quad \leq \quad 1 - \delta, \qquad \text{a.s.} \tag{4.8.3}$$

We say that $\mathcal{M}$ is uniformly weakly diverse on $[T_0, \infty)$, for some real number $T_0 > 0$, if there exists a number $\delta \in (0,1)$ such that the above holds a.s. for every $T \in [T_0, \infty)$. Moreover, the market $\mathcal{M}$ is called asymptotically weakly diverse if, for some $\delta \in (0,1)$, we have

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \mu_{(1)}(t)\, dt \quad \leq \quad 1 - \delta, \qquad \text{a.s.}, \tag{4.8.4}$$

alternatively, if

$$\overline{\lim_{T\to\infty}} \frac{1}{T}\int_0^T \mu_{(1)}(t)\,dt \quad = \quad \limsup_{T\to\infty} \frac{1}{T}\int_0^T \mu_{(1)}(t)\,dt \quad \leq \quad 1-\delta, \qquad \text{a.s.} \tag{4.8.5}$$

We also presented some consequences of stock market diversity. Specifically, we showed that diversity can be characterised in terms of the excess growth rate of the market portfolio (the market excess growth rate). Thus, there exists a relationship between the excess growth rate of the market and diversity. This relationship amounts to imposing a condition on the excess growth rate of the market in order for the equity market to be diverse. If the market $\mathcal{M}$ is nondegenerate and diverse (resp., weakly diverse on the time horizon $[0,T]$), then there is a number $\zeta > 0$ such that the following is satisfied

$$\gamma_\mu^*(t) \quad \geq \quad \zeta, \qquad t \in [0,\infty), \qquad \text{a.s.}, \tag{4.8.6}$$

$$\left(\textit{resp.}, \quad \frac{1}{T}\int_0^T \gamma_\mu^*(t)\,dt \quad \geq \quad \zeta, \qquad \text{a.s.}\right). \tag{4.8.7}$$

The above result can then be applied to look at diversity (or rather the lack thereof) in both the equal-growth-rate equity market and the constant-growth-rate equity market. Suppose that the market $\mathcal{M}$ is nondegenerate. If all the stocks in the market $\mathcal{M}$ have the same growth rate ($\gamma_i(t) \equiv \gamma(t)$, for all $i = 1, 2, \ldots, n$), then the equity market $\mathcal{M}$ is not diverse. That is, such an equal-growth-rate market cannot be diverse, even weakly diverse, over long time horizons, provided that the strong nondegeneracy condition is also satisfied. Furthermore, if all the stocks in the market $\mathcal{M}$ have constant growth rates (though not necessarily equal), then the equity market $\mathcal{M}$ is not diverse. That is, such a constant-growth-rate market cannot be diverse, even weakly diverse, over long time horizons, provided that the strong nondegeneracy condition is also satisfied. We then discussed the notion that dividends can be employed as a means to maintain stock market diversity in equity markets. We then diverted from this to discuss the concept of the distribution of equity capital of the market as well as the related capital distribution curve. The distribution of equity capital in the market, also the spread of equity capital in the market, is just a representation of the allocation of the available capital in the equity market to each of the stocks in the equity market. We provided a formal definition the equity capital distribution and the capital distribution curve in terms of the market weights. This essentially entails positioning the ranking of these market weights in decreasing order from the largest market weight through to the smallest market weight. The capital distribution curve of the market is then the graph representation of the family of the ranked market weights. We then defined those functions that are measures of diversity, which required us to establish and formulate the requisite fundamental properties that a sufficient measure of diversity should possess, namely the symmetry property and the concavity property. Let U be an open neighbourhood in $\mathbb{R}^n$ of the open positive unit simplex Δ^{n-1}, where

$$\Delta^{n-1} \quad = \quad \Big\{\boldsymbol{\mu}(t) = \big(\mu_1(t), \ldots, \mu_n(t)\big) \in \mathbb{R}^n \;\Big|\; \mu_1(t) + \cdots + \mu_n(t) = 1,\ 0 < \mu_i(t) < 1,\ i = 1, \ldots, n\Big\}.$$

Then, a positive twice continuously differentiable function, i.e., $f \in C^2$ or $\mathbf{S} \in C^2$, of the market weights, $f(\boldsymbol{\mu}) \in C^2$ or $\mathbf{S}(\boldsymbol{\mu}) \in C^2$, defined on $U \subset \mathbb{R}^n$ of Δ^{n-1}, $f : \Delta^{n-1} \to \mathbb{R}^+$ or $f : U \to \mathbb{R}^+$, alternatively $\mathbf{S} : \Delta^{n-1} \to \mathbb{R}^+$ or $\mathbf{S} : U \to \mathbb{R}^+$, is a measure of diversity if it is both symmetric and concave. A real-valued function of multiple variables is said to be symmetric if its value is independent of the order of the variables, i.e. is invariant under any permutations of the variables. The symmetry property ensures that all the stocks in the market receive equal treatment so that the degree of diversity in the market is not affected by the ordering of the stocks. The concavity property ensures that diversity increases by mixing the equity capital distribution, as it is required that transferring capital from a larger stock to a smaller stock increases the value of the diversity measure, and analogously transferring capital from a smaller stock to a larger stock decreases the value of that measure. Furthermore, since maximum diversity is associated with equal weights, we require a measure of diversity to be maximised by equal weights. Likewise, minimum diversity must be attained when the weight of the largest stock is 1 and the weights of the remaining smaller stocks are 0. Thus, a measure of diversity must be minimised when all the capital is concentrated into a single (largest) stock. Armed with this definition of measures of diversity, we then went on to present and study several examples of potential measures of diversity. These include, but are not limited to, the following examples of diversity measures: entropy, modified

entropy, the $\mathbf{D}_p$ index, the normalised version of the $\mathbf{D}_p$ index, Rényi entropy, the quadratic Gini coefficient, the quartic Gini coefficient, the Gini-Simpson index, an admissible market-dominating diversity measure, the geometric mean and the modified geometric mean. We also put forth the buy-and-hold function in this section, however, it is not a valid measure of diversity as it fails to satisfy the required symmetry condition, which is one of the sufficient diversity measure requirements. In each case, we presented and defined the (diversity measure) function of the market weights under consideration, established maximum and minimum diversity bounds imposed on the (diversity measure) function through the diversity constraints on the market weights, and then demonstrated that these functions are indeed appropriate measures of diversity for our purposes, by appealing to the definition of diversity measures that requires a positive twice continuously differentiable real-valued function of the market weights to satisfy both the symmetry and concavity conditions. Showing that these functions satisfy the concavity condition invoked a calculation of the corresponding Hessian matrices and a determination that these Hessian matrices are negative semidefinite, or negative definite for strictly concave functions. We started this presentation of diversity measures with the entropy function $\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$, defined by

$$\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \;=\; \mathbf{S}^{\mathrm{E}}\big(\mu_1(t),\ldots,\mu_n(t)\big) \;\triangleq\; -\sum_{i=1}^{n}\mu_i(t)\log\mu_i(t), \qquad t\in[0,T]. \tag{4.8.8}$$

We thus established the following bounds on the market entropy process

$$0 \;<\; \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \;\leq\; \log n, \qquad t\in[0,T], \qquad \text{a.s.} \tag{4.8.9}$$

The modified market entropy function, $\mathbf{S}^{\mathrm{E}}_c\big(\boldsymbol{\mu}(t)\big)$, is defined by

$$\mathbf{S}^{\mathrm{ME}}\big(\boldsymbol{\mu}(t)\big) \;=\; \mathbf{S}^{\mathrm{E}}_c\big(\boldsymbol{\mu}(t)\big) \;=\; \mathbf{S}^{\mathrm{E}}_c\big(\mu_1(t),\ldots,\mu_n(t)\big) \;\triangleq\; c+\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \;=\; c-\sum_{i=1}^{n}\mu_i(t)\log\mu_i(t). \tag{4.8.10}$$

The modified market entropy process satisfies the following bounds

$$c \;<\; \mathbf{S}^{\mathrm{E}}_c\big(\boldsymbol{\mu}(t)\big) \;\leq\; c+\log n, \qquad t\in[0,T], \qquad \text{a.s.} \tag{4.8.11}$$

The $\mathbf{D}_p$-function, $\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)$, is defined, for $0<p<1$, by

$$\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) \;=\; \mathbf{D}_p\big(\mu_1(t),\ldots,\mu_n(t)\big) \;\triangleq\; \left(\sum_{i=1}^{n}\big(\mu_i(t)\big)^p\right)^{\frac{1}{p}}, \qquad t\in[0,T]. \tag{4.8.12}$$

We have the following bounds on the market $\mathbf{D}_p$ index process

$$1 \;<\; \mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) \;\leq\; n^{(1-p)/p}, \qquad t\in[0,T], \qquad \text{a.s.} \tag{4.8.13}$$

The normalised version of the $\mathbf{D}_p$-function, $\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)$, is defined, for $0<p<1$, by

$$\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big) \;=\; \widetilde{\mathbf{D}}_p\big(\mu_1(t),\ldots,\mu_n(t)\big) \;\triangleq\; \left(n^{p-1}\sum_{i=1}^{n}\big(\mu_i(t)\big)^p\right)^{\frac{1}{p}} \;\equiv\; n^{(p-1)/p}\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big). \tag{4.8.14}$$

We have the following bounds on the normalised market $\mathbf{D}_p$ index process or the market $\widetilde{\mathbf{D}}_p$ index process

$$n^{(p-1)/p} \;<\; \widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big) \;\leq\; 1, \qquad t\in[0,T], \qquad \text{a.s.} \tag{4.8.15}$$

The Rényi entropy function, $\mathbf{S}^{\mathrm{R}}_\alpha\big(\boldsymbol{\mu}(t)\big)$, is defined by

$$\mathbf{S}^{\mathrm{R}}_\alpha\big(\boldsymbol{\mu}(t)\big) \;=\; \mathbf{S}^{\mathrm{R}}_\alpha\big(\mu_1(t),\ldots,\mu_n(t)\big) \;\triangleq\; \frac{1}{1-\alpha}\log\left(\sum_{i=1}^{n}\big(\mu_i(t)\big)^\alpha\right), \qquad t\in[0,T]. \tag{4.8.16}$$

We have the following bounds on the market Rényi entropy process

$$0 \;<\; \mathbf{S}^{\mathrm{R}}_\alpha\big(\boldsymbol{\mu}(t)\big) \;\leq\; \log n, \qquad t\in[0,T], \qquad \text{a.s.} \tag{4.8.17}$$

The quadratic Gini coefficient, $\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)$, is defined by

$$\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) \;=\; \mathbf{S}^{\mathrm{G}}\big(\mu_1(t),\ldots,\mu_n(t)\big) \;\triangleq\; 1-\frac{1}{2}\sum_{i=1}^{n}\Big(\mu_i(t)-n^{-1}\Big)^2, \qquad t\in[0,T]. \tag{4.8.18}$$

We have the following bounds on the market quadratic Gini coefficient process

$$\frac{1}{2} \;<\; \mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) \;\leq\; 1, \qquad t\in[0,T], \qquad \text{a.s.} \tag{4.8.19}$$

The quartic Gini coefficient, $\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)$, is defined by

$$\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big) \;=\; \mathbf{S}^{\mathrm{QG}}\big(\mu_1(t),\ldots,\mu_n(t)\big) \;\triangleq\; 1-\frac{1}{4}\sum_{i=1}^{n}\Big(\mu_i(t)-n^{-1}\Big)^4, \qquad t\in[0,T]. \tag{4.8.20}$$

We have the following bounds on the market quartic Gini coefficient process

$$\frac{3}{4} \;<\; \mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big) \;\leq\; 1, \qquad t\in[0,T], \qquad \text{a.s.} \tag{4.8.21}$$

The Gini-Simpson index, $\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)$, is defined by

$$\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big) \;=\; \mathbf{S}^{\mathrm{GS}}\big(\mu_1(t),\ldots,\mu_n(t)\big) \;\triangleq\; 1-\sum_{i=1}^{n}\mu_i^2(t), \qquad t\in[0,T]. \tag{4.8.22}$$

We have the following bounds on the market Gini-Simpson index process

$$0 \;<\; \mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big) \;<\; 1, \qquad t\in[0,T], \qquad \text{a.s.} \tag{4.8.23}$$

The admissible, market-dominating measure, $\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)$, is defined by

$$\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big) \;=\; \mathbf{S}^{\mathrm{A}}\big(\mu_1(t),\ldots,\mu_n(t)\big) \;\triangleq\; 1-\frac{1}{2}\sum_{i=1}^{n}\mu_i^2(t), \qquad t\in[0,T]. \tag{4.8.24}$$

We have the following bounds on this admissible, market-dominating diversity measure process

$$\frac{1}{2} \;<\; \mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big) \;<\; 1, \qquad t\in[0,T], \qquad \text{a.s.} \tag{4.8.25}$$

The geometric mean function, $\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)$, is defined by

$$\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) \;=\; \mathbf{S}^{\mathrm{GM}}\big(\mu_1(t),\ldots,\mu_n(t)\big) \;\triangleq\; \big(\mu_1(t)\cdots\mu_n(t)\big)^{\frac{1}{n}} \;=\; \left(\prod_{i=1}^{n}\mu_i(t)\right)^{\frac{1}{n}}. \tag{4.8.26}$$

For this market geometric mean process, we have the simple bounds

$$0 \;<\; \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) \;\leq\; \frac{1}{n}, \qquad t\in[0,T], \qquad \text{a.s.} \tag{4.8.27}$$

Finally, we characterised stock market diversity in terms of the market entropy process. The market $\mathcal{M}$ is diverse if and only if there is a $\zeta>0$ such that

$$\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \;\geq\; \zeta, \qquad t\in[0,T], \qquad \text{a.s.} \tag{4.8.28}$$

Thus, the equity market is diverse if and only if the market entropy process is sufficiently bounded below by a positive constant, i.e. it is sufficiently bounded away from zero. This stems from the fact (i.e., from the bounds of the market entropy function) that a value of zero for the market entropy process signifies an equity market that is not diverse, whereas a positive value other than zero for the market entropy process is indicative of an equity market displaying some form of diversity at varying levels.

Chapter 5

Portfolio Generating Functions and Functionally Generated Portfolios

5.1 Introduction

In this chapter, we shall demonstrate that a broad range of functions can be used to construct a wide variety of portfolios. In particular, we shall show that, in a continuously-traded equity market, certain real-valued functions (i.e., positive twice continuously differentiable functions) of the market weights $\mu_1(t), \ldots, \mu_n(t)$ can be used to construct and generate dynamic equity portfolios that behave in a controlled manner. Thus, a general method will be presented for constructing dynamic equity portfolios through the use of mathematical portfolio generating functions. The portfolio generating functions that are of particular concern to us fall into two categories: smooth functions of the market weights and smooth functions of the ranked market weights. Here, we are interested in and will focus on the portfolio generating functions that are smooth functions of the market weights. Functionally generated portfolios constitute one of the basic tools of stochastic portfolio theory, and portfolio generating functions are a powerful tool for the creation of portfolios with well-defined return characteristics. These notions and ideas of portfolio generating functions and functionally generated portfolios first appeared in Fernholz (1999c).

These functionally generated portfolios are of interest to us because under certain appropriate market conditions the return of the functionally generated portfolios relative to the market portfolio is governed by a stochastic differential equation. This equation decomposes this relative return into two components: the change in the value of the generating function, and a drift process that is of finite (bounded) variation. Consequently, the logarithmic return of this functionally generated portfolio relative to the market portfolio can be expressed as the sum of the change in the value of the logarithm of the generating function plus a monotonically increasing drift process. Portfolios generated by measures of diversity have nondecreasing (increasing) drift processes and have positive weights for which the weight ratios decrease with increasing market weight. There is typically a positive lower bound on the value of the generating function we use, and the nondegeneracy condition ensures then that the rate of increase of the monotonic drift process is bounded away from zero. This combination of lower bounds implies that the functionally generated portfolio will dominate the market portfolio. Hence, this stochastic differential equation can be used to establish the existence of a dominance relationship between the two portfolios, i.e., between such a functionally generated portfolio and the market portfolio. So that for this new class of portfolios, the derived decomposition of their relative return proves useful in the construction and study of arbitrage opportunities relative to the market portfolio. By appropriate selection of the portfolio generating function, the components can be structured in such a way so that the functionally generated portfolio will guarantee to have desirable return characteristics.

Functionally generated equity portfolios were also used in Fernholz (1999a) to study stock market diversity and was adopted in Fernholz (1998b) to establish the conditions under which relative arbitrage opportunities will

exist in equity markets. The results of the previous chapter imply that in a diverse equity market, certain measures of stock market diversity can be used to generate portfolios that dominate the market portfolio. A general discussion of portfolio generating functions, including several key examples, can be found in Fernholz (1999c). This chapter will be devoted to the study of the basic properties of these portfolio generating functions and the functionally generated portfolios they generate.

This chapter begins with Section 5.2, in which we shall introduce and define the concept of portfolio generating functions in conjunction with the allied notion of the functionally generated portfolios that these functions generate. Basically, the essential premise is that there exist certain (generating) functions of the market weights that can be employed to generate certain portfolios, some specifically designed to have desirable and worthwhile investment potential. It must be pointed out at this juncture, that generating functions are not only encountered in stochastic portfolio theory, but they can also be found elsewhere in other areas of mathematical finance, mathematical statistics and applied mathematics where they exhibit a wide and varied use, in particular, its applications to portfolios again can be located in classical portfolio theory. There are many references that attest to this fact. We shall determine that such functions of the market weights are required to be positive, twice continuously differentiable real-valued functions, in order to be able to generate portfolios. These positive twice continuously differentiable real-valued functions of the market weights that are utilised to generate portfolios are called portfolio generating functions, i.e. the generating functions of the portfolios in question. Moreover, the portfolios in question that are generated by these functions of the market weights are said to be functionally generated, and are called the functionally generated portfolios corresponding to these said portfolio generating functions. Portfolio generating functions and their associated functionally generated portfolios are considerably important within the realm of stochastic portfolio theory. In this section, we shall explore the behaviour of these functionally generated portfolios by gaining insight into the behaviour of the functions that generated them, the portfolio generating functions. We shall present the so-called master formula, which is given by the logarithmic relative return process of this functionally generated portfolio with respect to the market portfolio. The logarithmic return of this functionally generated portfolio relative to the market portfolio is connected to the behaviour of the function that generates it, i.e. to the portfolio generating function. We shall demonstrate that the logarithmic return on such a functionally generated portfolio relative to the market portfolio follows a stochastic differential equation associated with this functionally generated portfolio, which provides a decomposition of the logarithmic relative return of the functionally generated portfolio with respect to the reference benchmark market portfolio, specifically into two separate constituents. We shall find that the first component is given by the logarithmic change in the value of the portfolio generating function, to wit, the portfolio generating function component. This component depends only on the market weights which are observable; hence its value is always known. The second component is determined to be the (bounded) drift process corresponding to the functionally generated portfolio generated by the portfolio generating function. The main result of this chapter presented in the primary theorem of portfolio generating functions provides explicit formulas for the weights and the drift of functionally generated portfolios. The weights of these functionally generated portfolios are shown to only depend on the market weights themselves, and do not rely on any prior knowledge of the covariance structure of the equity market at all. This allows for the relatively easy implementation of these functionally generated portfolios. However, the computation of the drift process corresponding to the functionally generated portfolio does rely on the explicit knowledge of the covariance structure of the equity market, as it contains the relative covariance processes. It turns out, however, that this problem can be easily averted by simply considering the cumulative effect of the drift process over a certain period of time, the computation of which does not require any estimation or knowledge whatsoever of the covariance structure of the equity market and it is expressed in terms of market observable quantities. A very interesting point is that not all portfolios can be generated by means of a function of the market weights, i.e. by means of a portfolio generating function. This suggests that not all portfolios are functionally generated, and they cannot be referred to as functionally generated portfolios. For this reason, the ability to characterise such portfolios that are functionally generated, i.e. that are functionally generated portfolios, is crucial for the study that occurs in this chapter. In what follows in the first part of Section 5.3, we shall concern ourselves with the characterisation of functionally generated portfolios, in which we determine the conditions necessary for a portfolio to be classified as functionally generated. Here, we shall also investigate the circumstances under which two different portfolio generating functions will both generate the same functionally generated portfolio. In particular, we shall find that the

necessary and sufficient condition required for two different portfolio generating functions to both generate the same functionally generated portfolio, is that the ratio of these portfolio generating functions be constant. The second part of this section is dedicated to a study of the portfolio generating functions that generate functionally generated portfolios that possess increasing drift processes. We shall provide a characterisation of portfolio generating functions, in which we establish the conditions necessary for a portfolio generating function to generate a functionally generated portfolio with an increasing drift process. The reason why a study of these functionally generated portfolios exhibiting drift processes that are increasing, is of so much importance and of utmost relevance to us, is that, with this drift property, these functionally generated portfolios tend to outperform the market portfolio. Section 5.4 presents a few examples of portfolio generating functions along with their associated functionally generated portfolios that they generate. The drift process accompanying these functionally generated portfolios shall also be calculated. In addition, we shall explore and examine the performance of these portfolios relative to the market portfolio by taking a look at the logarithmic relative return process of these portfolios with respect to the market portfolio. In particular, we shall offer the following examples of functionally generated portfolios that are each generated by their respective portfolio generating functions: the constant-weighted portfolio which comprises constant weights, the buy-and-hold portfolio, the weighted-average capitalisation generated portfolio, the price-to-book ratio generated portfolio, and a single stock with leverage. Measures of diversity, that were discussed in the previous chapter, are positive twice continuously differentiable real-valued functions of the market weights that can be employed to generate portfolios with specific inherent characteristics, and are thus also referred to as being portfolio generating functions, in particular, diversity portfolio generating functions. This topic is put under scrutiny in Section 5.5, and is also focused upon in the rest of this chapter. We shall introduce and define the concept of diversity portfolio generating functions, which are just the measures of diversity, in conjunction with the allied notion of the diversity generated portfolios that these measures of diversity generate. The portfolios in question that are generated by these measures of diversity are said to be diversity generated, and are called the diversity generated portfolios corresponding to these measures of diversity, or diversity portfolio generating functions. Diversity portfolio generating functions and their associated diversity generated portfolios are incredibly poignant within the stochastic portfolio theory sphere, as the notion of diversity and of diverse equity markets is such an essential and invaluable tool in examining relative stock and portfolio performance. In this regard, we shall show that measures of diversity generate diversity generated portfolios that are endowed with increasing drift processes, in accordance with the results laid out in an aforementioned section of this chapter. Consequently, these diversity generated portfolios tend to outperform the market portfolio. More so, the corresponding weights of these diversity generated portfolios are positive and the affiliated weight ratios are determined to decrease with increasing market weight. Section 5.6 presents a plenitude of examples of generating functions that are measures of diversity, i.e. diversity portfolio generating functions, encountered already in the preceding chapter, along with their associated diversity generated portfolios that they generate. Upon examining the weight ratios associated with these diversity generated portfolios relative to the market portfolio, we notice, according to the previous section, that they decrease with increasing market weight. Hence, compared to the market portfolio, these diversity generated portfolios are less concentrated than the market portfolio in those stocks with the highest weights and is more concentrated than the market portfolio in those stocks with the lowest weights. The drift process accompanying these diversity generated portfolios shall also be calculated, and it shall be demonstrated that, in a nondegenerate and diverse equity market, these drift processes of these examples of diversity generated portfolios are all increasing in nature. Consequently, ownership of this increasing drift process means that these examples of diversity generated portfolios, that shall be divulged here, tend to outperform the market portfolio. In addition, we shall explore and examine the performance of these portfolios relative to the market portfolio by taking a look at the logarithmic relative return process of these portfolios with respect to the market portfolio. In particular, we shall offer the following examples of diversity generated portfolios that are each generated by their respective measures of diversity: the entropy-weighted portfolio, the modified entropy-weighted portfolio, the diversity-weighted index portfolio, the normalised version of the diversity-weighted index portfolio, the market portfolio which is the quintessential buy-and-hold strategy, the equally-weighted portfolio, the modified equally-weighted portfolio, the quadratic Gini-coefficient-weighted portfolio, the quartic Gini-coefficient-weighted portfolio, the Gini-Simpson-weighted index portfolio, and lastly, an admissible market-dominating portfolio. The next section, Section 5.7, generalises and extends upon the theory of portfolio generating functions relayed at the outset

of this chapter, to include time-dependent portfolio generating functions and their corresponding time-based functionally generated portfolios that they generate, by incorporating a time domain into the mix. Thus, the portfolio generating functions of focus is now a function of both the market weights and of time, and must be continuously differentiable in the time variable. These functions of both the market weights and of time that are utilised to generate portfolios are called time-dependent portfolio generating functions, i.e. the time-dependent generating functions of the portfolios in question. Moreover, the portfolios in question that are generated by these functions of the market weights and time are said to be time-based functionally generated, and are called the time-based functionally generated portfolios corresponding to these time-dependent portfolio generating functions. Also, in this section, we shall briefly mention a very neat Black-Scholes analogy that establishes a connection between the time-dependent portfolio generating functions within stochastic portfolio theory and the Black-Scholes option pricing theory. We shall include dividends into the equation in Section 5.8 by considering the total return processes of functionally generated portfolios, and study portfolio generating functions in combination with dividends. We do so specifically for both the entropy-weighted portfolio and the diversity-weighted index portfolio. A summary and conclusion of this chapter in its entirety is given in Section 5.9.

5.2 Portfolio Generating Functions

In this section, we shall introduce portfolio generating functions as well as the notion of functionally generated portfolios. The basic idea is that certain (i.e., those functions that are positive and twice continuously differentiable) real-valued functions, defined on the set Δ^{n-1}, can be adopted to generate portfolios. In fact, it will be shown that there exist many such functions that not only generate portfolios, but generate portfolios with desirable investment properties. Furthermore, insight into the behaviour of functionally generated portfolios can be derived by examining the behaviour of the functions that generated them, in this regard, these functions are extremely useful. We shall begin with a formal definition of such generating functions and the portfolios they generate.

Definition 5.2.1 (Generating Functions). *Let U be an open neighbourhood in $\mathbb{R}^n$ ($U \subset \mathbb{R}^n$) of the open positive unit $(n-1)-$simplex Δ^{n-1}*

$$\Delta^{n-1} = \Big\{\boldsymbol{\mu}(t) = \big(\mu_1(t), \ldots, \mu_n(t)\big) \in \mathbb{R}^n \;\Big|\; \mu_1(t) + \cdots + \mu_n(t) = 1,\ 0 < \mu_i(t) < 1,\ i = 1, \ldots, n\Big\},$$

*and let $\mathbf{G} : U \to (0, \infty)$ be a positive twice continuously differentiable function defined on some open neighbourhood U of Δ^{n-1}. Then $\mathbf{G}$ **generates** a portfolio φ if there exist continuous, measurable and adapted processes of bounded variation $\Theta = \{\Theta(t), t \in [0, \infty)\}$ and $\mathfrak{g} = \{\mathfrak{g}(t), t \in [0, \infty)\}$, such that*

$$\log\Big(Z_\varphi(t)/Z_\mu(t)\Big) = \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) + \Theta(t), \qquad t \in [0, T], \qquad a.s., \tag{5.2.1}$$

or such that we have the equivalent differential form

$$d\log\Big(Z_\varphi(t)/Z_\mu(t)\Big) = d\log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) + d\Theta(t), \qquad t \in [0, T], \qquad a.s., \tag{5.2.2}$$

$$d\log\Big(Z_\varphi(t)/Z_\mu(t)\Big) = d\log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) + \mathfrak{g}(t)\,dt, \qquad t \in [0, T], \qquad a.s., \tag{5.2.3}$$

*where $d\Theta(t) = \mathfrak{g}(t)\,dt$, $\Theta(\cdot) = \int_0^\cdot d\Theta(t) = \int_0^\cdot \mathfrak{g}(t)\,dt$ or $\mathfrak{g}(t) = \frac{d\Theta(t)}{dt} = \Theta'(t)$. The process Θ is called the **drift process** corresponding to the **generating function** $\mathbf{G}$. If $\mathbf{G}$ generates the portfolio φ, then $\mathbf{G}$ is called the **generating function** of the portfolio φ, and the portfolio φ is said to be **functionally generated**, or is said to be the **functionally generated portfolio** corresponding to the **portfolio generating function** $\mathbf{G}$.*

Since the drift process Θ is of bounded variation, $\log \mathbf{G}(\boldsymbol{\mu})$ is a continuous semimartingale. This is what ultimately allows us to express (5.2.1) in the differential form (5.2.2). It is in this differential form that we shall consider generating functions. The functionally generated portfolio φ has the essential property that its return relative to the market is connected to the behaviour of the function that generates it, the portfolio generating

function. Note that the existence of the Itô differential $d\log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)$ does not necessarily imply that $\mathbf{G}$ is differentiable. The integral form of (5.2.2) for this functionally generated portfolio φ, is given by

$$\log\left(\frac{Z_\varphi(T)}{Z_\mu(T)}\right) - \log\left(\frac{Z_\varphi(0)}{Z_\mu(0)}\right) = \log\left(\frac{\mathbf{G}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{G}\big(\boldsymbol{\mu}(0)\big)}\right) + \int_0^T d\Theta(t), \qquad T \in [0,\infty), \tag{5.2.4}$$

or,

$$\log\left(\frac{Z_\varphi(T)}{Z_\mu(T)}\right) = \log\left(\frac{Z_\varphi(0)}{Z_\mu(0)}\right) + \log\left(\frac{\mathbf{G}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{G}\big(\boldsymbol{\mu}(0)\big)}\right) + \int_0^T d\Theta(t), \qquad T \in [0,\infty), \tag{5.2.5}$$

which can be equivalently expressed as

$$\log\left(\frac{Z_\varphi(T)}{Z_\varphi(0)}\right) - \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) = \log\left(\frac{\mathbf{G}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{G}\big(\boldsymbol{\mu}(0)\big)}\right) + \int_0^T d\Theta(t), \qquad T \in [0,\infty), \tag{5.2.6}$$

or,

$$\log\left(\frac{Z_\varphi(T)}{Z_\varphi(0)}\right) = \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log\left(\frac{\mathbf{G}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{G}\big(\boldsymbol{\mu}(0)\big)}\right) + \int_0^T d\Theta(t), \qquad T \in [0,\infty). \tag{5.2.7}$$

Where $Z_\varphi(0) = Z_\mu(0)$, the logarithmic relative return process of this functionally generated portfolio φ, with respect to the market, is given by the so-called *"master formula"* or *"master equation"* [Fernholz & Karatzas (2009)], for all $T \in [0,\infty)$,

$$\log\left(\frac{Z_\varphi(T)}{Z_\mu(T)}\right) = \log\left(\frac{\mathbf{G}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{G}\big(\boldsymbol{\mu}(0)\big)}\right) + \int_0^T d\Theta(t), \tag{5.2.8}$$

$$\log\left(\frac{Z_\varphi(T)}{Z_\mu(T)}\right) = \log\left(\frac{\mathbf{G}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{G}\big(\boldsymbol{\mu}(0)\big)}\right) + \Theta(T), \tag{5.2.9}$$

$$\log\left(\frac{Z_\varphi(T)}{Z_\mu(T)}\right) = \log\left(\frac{\mathbf{G}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{G}\big(\boldsymbol{\mu}(0)\big)}\right) + \int_0^T \mathfrak{g}(t)\,dt, \tag{5.2.10}$$

alternatively expressed, for all $T \in [0,\infty)$, as

$$\Theta(T) = \int_0^T d\Theta(t) = \int_0^T \mathfrak{g}(t)\,dt = \log\left(\frac{Z_\varphi(T)}{Z_\mu(T)}\right) - \log\left(\frac{\mathbf{G}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{G}\big(\boldsymbol{\mu}(0)\big)}\right) \tag{5.2.11}$$

$$= \log\left(\frac{Z_\varphi(T)\mathbf{G}\big(\boldsymbol{\mu}(0)\big)}{Z_\mu(T)\mathbf{G}\big(\boldsymbol{\mu}(T)\big)}\right), \tag{5.2.12}$$

which gives the cumulative effect of the drift process. The stochastic differential equation, (5.2.2), associated with a functionally generated portfolio, decomposes the logarithmic relative return of the functionally generated portfolio φ, with respect to the benchmark market portfolio, into two components. The first component is the generating function component, or more precisely the logarithmic change or variation in the value of the generating function $\mathbf{G}$. It is also important to note that the first component depends only on the market weights which are observable; hence its value is always known. The second component is the drift process Θ. Since the drift process is of bounded variation, the generating function component includes the local martingale part of the relative return process. This decomposition is useful because the variation of the generating function component can be controlled by bounds on the logarithm of the portfolio generating function itself $\log \mathbf{G}$. So, under certain conditions the drift process will dominate the behaviour of the relative return process. In fact, in specific examples the generating function component of the relative return often dominates the *short-term behaviour* of the relative return, and the drift process often dominates the *long-term behaviour* of the relative return, especially if the logarithm of the portfolio generating function itself $\log \mathbf{G}$ is *bounded*. The following main theorem of portfolio generating functions gives explicit formulas for the weights and the drift of a functionally generated portfolio.

Theorem 5.2.2 ([**Fernholz (2002)**]). *Let U be an open neighbourhood in $\mathbb{R}^n$ ($U \subset \mathbb{R}^n$) of the open positive unit $(n-1)-$simplex Δ^{n-1} and let $\mathbf{G} : U \to (0,\infty)$ be a positive twice continuously differentiable function defined on some open neighbourhood U of Δ^{n-1}, such that for all $i = 1,2,\ldots,n$, $x_i D_i \log \mathbf{G}(\mathbf{x})$ is bounded on Δ^{n-1}. Then for $t \in [0,T]$ and for $i = 1,2,\ldots,n$, the generating function $\mathbf{G}$ generates the (functionally generated) portfolio φ with* ***weights***

$$\varphi_i(t) \quad = \quad \left(D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{j=1}^{n} \mu_j(t)\, D_j \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) \right) \mu_i(t), \tag{5.2.13}$$

and with ***drift process*** Θ, *for $t \in [0,T]$, a.s.*

$$d\Theta(t) \quad = \quad \frac{-1}{2\mathbf{G}\big(\boldsymbol{\mu}(t)\big)} \sum_{i,j=1}^{n} D_{ij}\mathbf{G}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\, dt, \tag{5.2.14}$$

alternatively, with ***drift process*** $\mathfrak{g}$, *for $t \in [0,T]$, a.s.*

$$\mathfrak{g}(t) \quad = \quad \frac{d\Theta(t)}{dt} \quad = \quad \frac{-1}{2\mathbf{G}\big(\boldsymbol{\mu}(t)\big)} \sum_{i,j=1}^{n} D_{ij}\mathbf{G}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t). \tag{5.2.15}$$

Let μ be the market portfolio and φ be the functionally generated portfolio, and let Z_μ and Z_φ be their portfolio value processes, respectively. Then, a.s., for $t \in [0,T]$,

$$d \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) \quad = \quad d \log \Big(Z_\varphi(t)/Z_\mu(t) \Big) + \frac{1}{2\mathbf{G}\big(\boldsymbol{\mu}(t)\big)} \sum_{i,j=1}^{n} D_{ij}\mathbf{G}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\, dt. \tag{5.2.16}$$

Proof. Firstly, since $\boldsymbol{\mu} \in \Delta^{n-1}$, then by assumption $\mu_i(t)\, D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)$ is bounded on Δ^{n-1} for all $i = 1,2,\ldots,n$. In addition, the market weights $\mu_i(t)$ are bounded, $0 \le \mu_i(t) \le 1$ for all $i = 1,2,\ldots,n$ and the market weights sum to 1. Thus, the weights $\varphi_i(t)$ are also bounded. These conditions on the generating function $\mathbf{G}$ ensure that $\boldsymbol{\varphi}$ is a portfolio. Let us define

$$\phi\big(\boldsymbol{\mu}(t)\big) \quad \triangleq \quad 1 - \sum_{j=1}^{n} \mu_j(t)\, D_j \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big), \qquad t \in [0,T], \tag{5.2.17}$$

then the weights in (5.2.13) are given by

$$\varphi_i(t) \quad = \quad \Big(D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) + \phi\big(\boldsymbol{\mu}(t)\big) \Big) \mu_i(t), \qquad t \in [0,T], \tag{5.2.18}$$

for all $i = 1,2,\ldots,n$. Therefore,

$$\begin{aligned} \sum_{i=1}^{n} \varphi_i(t) \quad &= \quad \sum_{i=1}^{n} \mu_i(t)\, D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) + \sum_{i=1}^{n} \mu_i(t)\phi\big(\boldsymbol{\mu}(t)\big) \\ &= \quad \sum_{i=1}^{n} \mu_i(t)\, D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) + \phi\big(\boldsymbol{\mu}(t)\big) \\ &= \quad 1. \end{aligned}$$

Thus the weights $\varphi_i(t)$ sum to 1, consequently it follows that

$$\varphi_i(t) \quad = \quad \left(D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{j=1}^{n} \mu_j(t)\, D_j \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) \right) \mu_i(t), \tag{5.2.19}$$

is satisfied. Now, for a positive twice continuously differentiable function, $\mathbf{G}$, and for all $\mathbf{x} \in \Delta^{n-1}$, we have

$$D_i \log \mathbf{G}(\mathbf{x}) \quad = \quad \frac{\partial \log \mathbf{G}(\mathbf{x})}{\partial x_i} \quad = \quad \frac{1}{\mathbf{G}(\mathbf{x})} \frac{\partial \mathbf{G}(\mathbf{x})}{\partial x_i} \quad = \quad \frac{D_i \mathbf{G}(\mathbf{x})}{\mathbf{G}(\mathbf{x})}, \tag{5.2.20}$$

and employing (5.2.20), we have

$$\begin{aligned} D_{ij}\log \mathbf{G}(\mathbf{x}) &= \frac{\partial^2 \log \mathbf{G}(\mathbf{x})}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_j}\left(\frac{1}{\mathbf{G}(\mathbf{x})}\frac{\partial \mathbf{G}(\mathbf{x})}{\partial x_i}\right) = D_j\left(\frac{D_i\mathbf{G}(\mathbf{x})}{\mathbf{G}(\mathbf{x})}\right) \\ &= \frac{1}{\mathbf{G}(\mathbf{x})}\frac{\partial^2 \mathbf{G}(\mathbf{x})}{\partial x_i \partial x_j} - \frac{\partial \mathbf{G}(\mathbf{x})}{\partial x_i}\frac{1}{\mathbf{G}^2(\mathbf{x})}\frac{\partial \mathbf{G}(\mathbf{x})}{\partial x_j} \\ &= \frac{D_{ij}\mathbf{G}(\mathbf{x})}{\mathbf{G}(\mathbf{x})} - \frac{D_i\mathbf{G}(\mathbf{x})\, D_j\mathbf{G}(\mathbf{x})}{\mathbf{G}^2(\mathbf{x})} \\ &= \frac{D_{ij}\mathbf{G}(\mathbf{x})}{\mathbf{G}(\mathbf{x})} - D_i\log \mathbf{G}(\mathbf{x})\, D_j \log \mathbf{G}(\mathbf{x}). \end{aligned} \tag{5.2.21}$$

Therefore for the market portfolio, $\boldsymbol{\mu} \in \Delta^{n-1}$, we have

$$D_{ij}\log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) = \frac{D_{ij}\mathbf{G}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{G}\big(\boldsymbol{\mu}(t)\big)} - D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\, D_j \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big). \tag{5.2.22}$$

Recall that by equation (2.12.48), we have

$$d\left\langle \mu_i, \mu_j\right\rangle_t = \mu_i(t)\mu_j(t)\,\tau_{ij}(t)\,dt, \qquad t \in [0,T], \qquad \text{a.s.} \tag{5.2.23}$$

Now the above equation (5.2.23) in conjunction with an application of Itô's formula to $\log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)$, yields a.s., for $t \in [0,T]$,

$$\begin{aligned} d\log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) &= \sum_{i=1}^n D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\, d\mu_i(t) + \frac{1}{2}\sum_{i,j=1}^n D_{ij}\log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\, d\left\langle \mu_i, \mu_j\right\rangle_t \\ &= \sum_{i=1}^n D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\, d\mu_i(t) + \frac{1}{2}\sum_{i,j=1}^n D_{ij}\log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt, \end{aligned} \tag{5.2.24}$$

which when combined with (5.2.22), yields a.s. for $t \in [0,T]$,

$$\begin{aligned} d\log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) &= \sum_{i=1}^n D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\, d\mu_i(t) \\ &\quad + \frac{1}{2}\sum_{i,j=1}^n \left[\frac{D_{ij}\mathbf{G}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{G}\big(\boldsymbol{\mu}(t)\big)} - D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\, D_j \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\right]\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt \\ &= \sum_{i=1}^n D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\, d\mu_i(t) + \frac{1}{2\mathbf{G}\big(\boldsymbol{\mu}(t)\big)}\sum_{i,j=1}^n D_{ij}\mathbf{G}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt \\ &\quad - \frac{1}{2}\sum_{i,j=1}^n D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\, D_j \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt. \end{aligned} \tag{5.2.25}$$

The first term on the right-hand side of the last equation is the local martingale component of $\log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)$. In order for (5.2.2) to hold, the local martingale components of $\log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)$ and $\log\big(Z_\varphi(t)/Z_\mu(t)\big)$ must be equal. By setting $\boldsymbol{\pi} := \boldsymbol{\varphi}$ in Proposition 2.12.8 we have a.s., for $t \in [0,T]$,

$$d\log\Big(Z_\varphi(t)/Z_\mu(t)\Big) = \sum_{i=1}^n \varphi_i(t)\, d\log \mu_i(t) + \gamma_\varphi^*(t)\,dt. \tag{5.2.26}$$

From (2.12.36), we have

$$d\log \mu_i(t) = \frac{d\mu_i(t)}{\mu_i(t)} - \frac{1}{2}\,\tau_{ii}(t)\,dt,$$

and from Lemma 2.4.5 we have the numéraire invariance property (2.4.26),

$$\begin{aligned} \gamma_\varphi^*(t) &= \frac{1}{2}\left(\sum_{i=1}^n \varphi_i(t)\tau_{ii}^\mu(t) - \sum_{i,j=1}^n \varphi_i(t)\tau_{ij}^\mu(t)\varphi_j(t)\right) \\ &= \frac{1}{2}\left(\sum_{i=1}^n \varphi_i(t)\tau_{ii}(t) - \sum_{i,j=1}^n \varphi_i(t)\tau_{ij}(t)\varphi_j(t)\right). \end{aligned}$$

Substituting the above expressions into (5.2.26) yields

$$\begin{aligned} d\log\Big(Z_\varphi(t)/Z_\mu(t)\Big) &= \sum_{i=1}^n \varphi_i(t)\Big[\frac{d\mu_i(t)}{\mu_i(t)} - \frac{1}{2}\,\tau_{ii}(t)\,dt\Big] \\ &\quad + \frac{1}{2}\sum_{i=1}^n \varphi_i(t)\tau_{ii}(t)\,dt - \frac{1}{2}\sum_{i,j=1}^n \varphi_i(t)\varphi_j(t)\tau_{ij}(t)\,dt \\ &= \sum_{i=1}^n \frac{\varphi_i(t)}{\mu_i(t)}\,d\mu_i(t) - \frac{1}{2}\sum_{i,j=1}^n \varphi_i(t)\varphi_j(t)\tau_{ij}(t)\,dt. \end{aligned} \tag{5.2.27}$$

With ϕ defined as in (5.2.17), we have by (5.2.18),

$$\frac{\varphi_i(t)}{\mu_i(t)} = D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) + \phi\big(\boldsymbol{\mu}(t)\big), \tag{5.2.28}$$

then the first term on the right-hand side of (5.2.27) becomes

$$\begin{aligned} \sum_{i=1}^n \frac{\varphi_i(t)}{\mu_i(t)}\,d\mu_i(t) &= \sum_{i=1}^n \Big[D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) + \phi\big(\boldsymbol{\mu}(t)\big)\Big]\,d\mu_i(t) \\ &= \sum_{i=1}^n D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\,d\mu_i(t) + \phi\big(\boldsymbol{\mu}(t)\big)\sum_{i=1}^n d\mu_i(t) \\ &= \sum_{i=1}^n D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\,d\mu_i(t) + \phi\big(\boldsymbol{\mu}(t)\big)\,d\Big(\sum_{i=1}^n \mu_i(t)\Big) \\ &= \sum_{i=1}^n D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\,d\mu_i(t), \end{aligned} \tag{5.2.29}$$

since $\sum_{i=1}^n d\mu_i(t) = d\big(\sum_{i=1}^n \mu_i(t)\big) = 0$. Therefore, we have a.s., for $t \in [0,T]$,

$$d\log\Big(Z_\varphi(t)/Z_\mu(t)\Big) = \sum_{i=1}^n D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\,d\mu_i(t) - \frac{1}{2}\sum_{i,j=1}^n \varphi_i(t)\varphi_j(t)\tau_{ij}(t)\,dt. \tag{5.2.30}$$

Hence, the local martingale components of $\log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)$ and $\log\big(Z_\varphi(t)/Z_\mu(t)\big)$ are equal, so that (5.2.2) is satisfied. With $\varphi_i(t)$ defined as in (5.2.18), the second term on the right-hand side of the last equation is given a.s. by, for $t \in [0,T]$,

$$\begin{aligned} \sum_{i,j=1}^n \varphi_i(t)\varphi_j(t)\tau_{ij}(t) &= \sum_{i,j=1}^n \Big[D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) + \phi\big(\boldsymbol{\mu}(t)\big)\Big]\Big[D_j \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) + \phi\big(\boldsymbol{\mu}(t)\big)\Big]\mu_i(t)\mu_j(t)\tau_{ij}(t) \\ &= \sum_{i,j=1}^n D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\,D_j \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t) \\ &\quad + 2\,\phi\big(\boldsymbol{\mu}(t)\big)\sum_{i,j=1}^n D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t) \\ &\quad + \phi^2\big(\boldsymbol{\mu}(t)\big)\sum_{i,j=1}^n \mu_i(t)\mu_j(t)\tau_{ij}(t) \\ &= \sum_{i,j=1}^n D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\,D_j \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t) \\ &\quad + 2\,\phi\big(\boldsymbol{\mu}(t)\big)\sum_{i=1}^n D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\Big[\sum_{j=1}^n \mu_j(t)\tau_{ij}(t)\Big] \\ &\quad + \phi^2\big(\boldsymbol{\mu}(t)\big)\sum_{i,j=1}^n \mu_i(t)\mu_j(t)\tau_{ij}(t). \end{aligned}$$

Thus, by (2.12.54) of Lemma 2.12.4 and (2.12.56) (since $\boldsymbol{\mu}(t)$ is in the null space of $\boldsymbol{\tau}(t)$), we have a.s., for $t \in [0,T]$,

$$\sum_{i,j=1}^{n} \varphi_i(t)\varphi_j(t)\tau_{ij}(t) \quad = \quad \sum_{i,j=1}^{n} D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\, D_j \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t). \tag{5.2.31}$$

Hence, a.s., for $t \in [0,T]$, we have

$$d\log\Big(Z_\varphi(t)/Z_\mu(t)\Big) = \sum_{i=1}^{n} D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\, d\mu_i(t) - \frac{1}{2}\sum_{i,j=1}^{n} D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\, D_j \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\, dt. \tag{5.2.32}$$

Comparing this last expression with (5.2.25),

$$\begin{aligned} d\log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) \quad &= \quad \sum_{i=1}^{n} D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\, d\mu_i(t) - \frac{1}{2}\sum_{i,j=1}^{n} D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\, D_j \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\, dt \\ &\quad + \frac{1}{2\mathbf{G}\big(\boldsymbol{\mu}(t)\big)} \sum_{i,j=1}^{n} D_{ij}\mathbf{G}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\, dt, \end{aligned}$$

we obtain for $t \in [0,T]$, a.s.,

$$d\log\Big(Z_\varphi(t)/Z_\mu(t)\Big) \quad = \quad d\log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) - \frac{1}{2\mathbf{G}\big(\boldsymbol{\mu}(t)\big)} \sum_{i,j=1}^{n} D_{ij}\mathbf{G}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\, dt. \tag{5.2.33}$$

This expression yields (5.2.16). Now, by recalling (5.2.2) of Definition 5.2.1, it is clear from (5.2.33) that the logarithmic relative return of the portfolio φ with respect to the market has the form provided in equation (5.2.2), where the differential of the drift process, $d\Theta(t)$, is given by the second term on the right-hand side of (5.2.33). Since by definition $d\log\big(Z_\varphi(t)/Z_\mu(t)\big) = d\log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) + d\Theta(t)$, we deduce that the drift process Θ is given by

$$d\Theta(t) \quad = \quad \mathfrak{g}(t)\, dt \quad = \quad \frac{-1}{2\mathbf{G}\big(\boldsymbol{\mu}(t)\big)} \sum_{i,j=1}^{n} D_{ij}\mathbf{G}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\, dt,$$

which completes the proof. ■

Note that the quantities $\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)$ depend only on the market weights $\mu_1(t), \mu_2(t), \ldots, \mu_n(t)$, and not on the covariance structure of the market. Thus, the portfolio can be implemented, and its associated value process $Z_\varphi(t)$ observed through time purely in terms of these market weights over the time period $[0,T]$. The covariance structure of the market only comes into play when computing the expression for the drift term, where the relative covariance term $\tau_{ij}(t)$ makes an appearance. Thus, the computation of the drift process requires the covariance structure of the market, through the relative covariances, to be known. However, if we rather consider the *cumulative effect* of the drift process over a period of time $[0,T]$, $\int_0^T d\Theta(t) = \int_0^T \mathfrak{g}(t)\, dt$, given in equations (5.2.11) and (5.2.12), we need only compute

$$\begin{aligned} \int_0^T d\Theta(t) \quad = \quad \int_0^T \mathfrak{g}(t)\, dt \quad &= \quad \log\left(\frac{Z_\varphi(T)}{Z_\mu(T)}\right) - \log\left(\frac{\mathbf{G}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{G}\big(\boldsymbol{\mu}(0)\big)}\right) \\ &= \quad \log\left(\frac{Z_\varphi(T)\mathbf{G}\big(\boldsymbol{\mu}(0)\big)}{Z_\mu(T)\mathbf{G}\big(\boldsymbol{\mu}(T)\big)}\right). \end{aligned}$$

Hence, the cumulative effect of the drift process reveals the remarkable observation that in order to compute this effect over a period of time $[0,T]$ using past data, there is no need to estimate or even know the covariance structure at all. Moreover, the above expression for the cumulative effect of the drift process is in terms of quantities that are all observable. This technique has its advantages in reducing management fees, as selection

and weighting decisions are often the product of a highly sophisticated, continuous and complex analysis. In addition, the decision-making process is rather difficult and quite costly to implement, this is without even considering that the analysis needs to be executed on a routine basis and that, in that event, we also need to take into account the need for effective and close human supervision and intervention.

By appealing to the long-term time average, it can be shown that the growth rates of the functionally generated portfolio φ and that of the market portfolio μ are related to the drift process Θ. This is demonstrated in the following proposition.

Proposition 5.2.3 ([**Fernholz (2002)**]). *Let* $\mathbf{G}$ *generate a portfolio* φ *with drift process* Θ (*or,* $\mathfrak{g}$), *and suppose that*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) = 0, \qquad t \in [0,T], \qquad a.s. \tag{5.2.34}$$

Equivalently, supposing that, for $T \in [0,\infty)$,

$$\lim_{T\to\infty} \frac{1}{T} \log \mathbf{G}\big(\boldsymbol{\mu}(T)\big) = 0, \qquad T \in [0,\infty), \qquad a.s. \tag{5.2.35}$$

Then, we have

$$\lim_{T\to\infty} \frac{1}{T} \left(\int_0^T \gamma_\varphi(t)\,dt - \int_0^T \gamma_\mu(t)\,dt - \Theta(T) \right) = 0, \qquad a.s., \tag{5.2.36}$$

or,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \Big(\gamma_\varphi(t) - \gamma_\mu(t) - \mathfrak{g}(t) \Big)\,dt = 0, \qquad a.s. \tag{5.2.37}$$

Proof. Equation (5.2.2) of Definition 5.2.1 implies that the semimartingale decomposition of the logarithm of the portfolio generating function satisfies

$$d\log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) = d\log\Big(Z_\varphi(t)/Z_\mu(t)\Big) - d\Theta(t), \qquad t \in [0,T], \qquad \text{a.s.} \tag{5.2.38}$$

By appealing to (2.6.11) of Lemma 2.6.4, we obtain, a.s., for $t \in [0,T]$,

$$\begin{aligned} d\log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) &= \big(\gamma_\varphi(t) - \gamma_\mu(t)\big)\,dt + \sum_{\nu=1}^n \big(\xi_{\varphi\nu}(t) - \xi_{\mu\nu}(t)\big)\,dW_\nu(t) - d\Theta(t) \\ &= \big(\gamma_\varphi(t) - \gamma_\mu(t)\big)\,dt + \sum_{\nu=1}^n \big(\xi_{\varphi\nu}(t) - \xi_{\mu\nu}(t)\big)\,dW_\nu(t) - \mathfrak{g}(t)\,dt. \end{aligned} \tag{5.2.39}$$

Therefore by (5.2.1),

$$\log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) = \log\Big(Z_\varphi(t)/Z_\mu(t)\Big) - \Theta(t), \qquad t \in [0,T], \qquad \text{a.s.}, \tag{5.2.40}$$

we have, a.s., for $T \in [0,\infty)$,

$$\begin{aligned} \log \mathbf{G}\big(\boldsymbol{\mu}(T)\big) &= \log Z_\varphi(T) - \log Z_\mu(T) - \Theta(T) \\ &= \int_0^T \big(\gamma_\varphi(t) - \gamma_\mu(t)\big)\,dt + \int_0^T \sum_{\nu=1}^n \big(\xi_{\varphi\nu}(t) - \xi_{\mu\nu}(t)\big)\,dW_\nu(t) - \Theta(T) \qquad (5.2.41) \\ &= \int_0^T \big(\gamma_\varphi(t) - \gamma_\mu(t)\big)\,dt + \int_0^T \sum_{\nu=1}^n \big(\xi_{\varphi\nu}(t) - \xi_{\mu\nu}(t)\big)\,dW_\nu(t) - \int_0^T \mathfrak{g}(t)\,dt. \qquad (5.2.42) \end{aligned}$$

Taking the long-term time average, i.e., the limit as $T \to \infty$ of the time average on both sides of (5.2.41), gives

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{T} \log \mathbf{G}\big(\boldsymbol{\mu}(T)\big) &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \big(\gamma_\varphi(t) - \gamma_\mu(t)\big)\,dt - \lim_{T\to\infty} \frac{1}{T}\,\Theta(T) \\ &\quad + \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{\nu=1}^n \big(\xi_{\varphi\nu}(t) - \xi_{\mu\nu}(t)\big)\,dW_\nu(t). \end{aligned}$$

Then, $\log \mathbf{G}\big(\boldsymbol{\mu}(T)\big)$ vanishes by (5.2.34), and the last term in the above expression vanishes by equation (3.2.2) of Lemma 3.2.1, since the second term in (5.2.39) is a continuous local martingale. Thus, we have the desired result

$$\lim_{T\to\infty} \frac{1}{T}\left(\int_0^T \gamma_\varphi(t)\,dt - \int_0^T \gamma_\mu(t)\,dt - \Theta(T)\right) \quad = \quad 0, \qquad \text{a.s.} \tag{5.2.43}$$

Alternatively, taking the long-term time average on both sides of (5.2.42), gives

$$\begin{aligned}\lim_{T\to\infty} \frac{1}{T} \log \mathbf{G}\big(\boldsymbol{\mu}(T)\big) \quad &= \quad \lim_{T\to\infty} \frac{1}{T}\int_0^T \big(\gamma_\varphi(t) - \gamma_\mu(t)\big)\,dt - \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathfrak{g}(t)\,dt \\ &\quad + \lim_{T\to\infty} \frac{1}{T}\int_0^T \sum_{\nu=1}^n \big(\xi_{\varphi\nu}(t) - \xi_{\mu\nu}(t)\big)\,dW_\nu(t).\end{aligned}$$

Then, using the same results, we obtain the similar result in terms of the drift process $\mathfrak{g}$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \Big(\gamma_\varphi(t) - \gamma_\mu(t) - \mathfrak{g}(t)\Big)\,dt \quad = \quad 0, \qquad \text{a.s.} \tag{5.2.44}$$

■

Note that the long-term time average condition (5.2.34) for the portfolio generating function, in the aforementioned proposition, will be satisfied if, for example, $\log \mathbf{G}$ is bounded on Δ^{n-1}. Indeed, this bound condition for the portfolio generating function is the condition that pertains in most applications.

Proposition 5.2.4 ([**Fernholz (1999c)**]). *Suppose that* $\mathbf{G}$ *generates the portfolio* φ *and* G *generates the portfolio* η*, with drift processes* Θ_φ *and* Θ_η*, respectively. Then*

$$\log\Big(Z_\varphi(t)/Z_\eta(t)\Big) \quad = \quad \log\Big(\mathbf{G}\big(\boldsymbol{\mu}(t)\big)/\mathsf{G}\big(\boldsymbol{\mu}(t)\big)\Big) + \Theta_\varphi(t) - \Theta_\eta(t), \qquad t\in[0,T], \qquad \textit{a.s.} \tag{5.2.45}$$

Alternatively, with drift processes $\mathfrak{g}_\varphi$ *and* $\mathfrak{g}_\eta$*, in differential form we have,*

$$d\log\Big(Z_\varphi(t)/Z_\eta(t)\Big) \quad = \quad d\log\Big(\mathbf{G}\big(\boldsymbol{\mu}(t)\big)/\mathsf{G}\big(\boldsymbol{\mu}(t)\big)\Big) + d\Theta_\varphi(t) - d\Theta_\eta(t), \qquad t\in[0,T], \qquad \textit{a.s.} \tag{5.2.46}$$

$$= \quad d\log\Big(\mathbf{G}\big(\boldsymbol{\mu}(t)\big)/\mathsf{G}\big(\boldsymbol{\mu}(t)\big)\Big) + \big(\mathfrak{g}_\varphi(t) - \mathfrak{g}_\eta(t)\big)\,dt, \qquad t\in[0,T], \qquad \textit{a.s.} \tag{5.2.47}$$

Proof. The proof follows directly from Definition 5.2.1, with

$$\log\Big(Z_\varphi(t)/Z_\mu(t)\Big) \quad = \quad \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) + \Theta_\varphi(t), \qquad t\in[0,T], \qquad \text{a.s.},$$

and

$$\log\Big(Z_\eta(t)/Z_\mu(t)\Big) \quad = \quad \log \mathsf{G}\big(\boldsymbol{\mu}(t)\big) + \Theta_\eta(t), \qquad t\in[0,T], \qquad \text{a.s.},$$

by subtracting the second expression from the first. ■

The corollary that follows demonstrates that several generating functions can be combined in such a way that enables hybrid characteristics to be induced in the portfolios that they generate.

Corollary 5.2.5 ([**Fernholz (1999c, 2002)**]). *Let* $\mathsf{G}_1, \mathsf{G}_2, \ldots, \mathsf{G}_m$ *denote portfolio generating functions that generate portfolios* $\varphi_1, \varphi_2, \ldots, \varphi_m$*, respectively. Then for constants* $p_1, p_2, \ldots, p_m$ *such that* $p_1+p_2+\cdots+p_m = 1$*, the function*

$$\mathbf{G} \quad = \quad \mathsf{G}_1^{p_1}\mathsf{G}_2^{p_2}\cdots\mathsf{G}_m^{p_m}, \tag{5.2.48}$$

generates a portfolio ζ *with weights*

$$\zeta_i(t) \quad = \quad p_1\varphi_{1i}(t) + p_2\varphi_{2i}(t) + \cdots + p_m\varphi_{mi}(t), \tag{5.2.49}$$

for $i = 1, 2, \ldots, n$*, and* $t\in[0,T]$*, where* φ_{ki} *is the ith weight of* φ_k*, for* $k = 1, 2, \ldots, m$*.*

Proof. From equation (5.2.13) of Theorem 5.2.2, for the generating function provided in equation (5.2.48), $\mathbf{G}\big(\boldsymbol{\mu}(t)\big) = \mathsf{G}_1^{p_1}\big(\boldsymbol{\mu}(t)\big)\mathsf{G}_2^{p_2}\big(\boldsymbol{\mu}(t)\big)\cdots\mathsf{G}_m^{p_m}\big(\boldsymbol{\mu}(t)\big)$, we have the following

$$\begin{aligned}
\zeta_i(t) &= \left(D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{j=1}^{n} \mu_j(t)\, D_j \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\right)\mu_i(t) \\
&= \left(D_i \log\Big(\mathsf{G}_1^{p_1}\big(\boldsymbol{\mu}(t)\big)\cdots\mathsf{G}_m^{p_m}\big(\boldsymbol{\mu}(t)\big)\Big) + 1 - \sum_{j=1}^{n} \mu_j(t)\, D_j \log\Big(\mathsf{G}_1^{p_1}\big(\boldsymbol{\mu}(t)\big)\cdots\mathsf{G}_m^{p_m}\big(\boldsymbol{\mu}(t)\big)\Big)\right)\mu_i(t) \\
&= \Bigg(D_i\Big(\log \mathsf{G}_1^{p_1}\big(\boldsymbol{\mu}(t)\big) + \cdots + \log \mathsf{G}_m^{p_m}\big(\boldsymbol{\mu}(t)\big)\Big) + 1 \\
&\qquad - \sum_{j=1}^{n} \mu_j(t)\, D_j\Big(\log \mathsf{G}_1^{p_1}\big(\boldsymbol{\mu}(t)\big) + \cdots + \log \mathsf{G}_m^{p_m}\big(\boldsymbol{\mu}(t)\big)\Big)\Bigg)\mu_i(t) \\
&= \Bigg(D_i\Big(p_1 \log \mathsf{G}_1\big(\boldsymbol{\mu}(t)\big) + \cdots + p_m \log \mathsf{G}_m\big(\boldsymbol{\mu}(t)\big)\Big) + 1 \\
&\qquad - \sum_{j=1}^{n} \mu_j(t)\, D_j\Big(p_1 \log \mathsf{G}_1\big(\boldsymbol{\mu}(t)\big) + \cdots + p_m \log \mathsf{G}_m\big(\boldsymbol{\mu}(t)\big)\Big)\Bigg)\mu_i(t) \\
&= \Bigg(\Big(p_1 D_i \log \mathsf{G}_1\big(\boldsymbol{\mu}(t)\big) + \cdots + p_m D_i \log \mathsf{G}_m\big(\boldsymbol{\mu}(t)\big)\Big) + 1 \\
&\qquad - \sum_{j=1}^{n} \mu_j(t)\Big(p_1 D_j \log \mathsf{G}_1\big(\boldsymbol{\mu}(t)\big) + \cdots + p_m D_j \log \mathsf{G}_m\big(\boldsymbol{\mu}(t)\big)\Big)\Bigg)\mu_i(t) \\
&= \left(\sum_{k=1}^{m} p_k D_i \log \mathsf{G}_k\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{k=1}^{m} p_k\left[\sum_{j=1}^{n} \mu_j(t)\, D_j \log \mathsf{G}_k\big(\boldsymbol{\mu}(t)\big)\right]\right)\mu_i(t) \\
&= \left(\sum_{k=1}^{m} p_k D_i \log \mathsf{G}_k\big(\boldsymbol{\mu}(t)\big) + \sum_{k=1}^{m} p_k - \sum_{k=1}^{m} p_k\left[\sum_{j=1}^{n} \mu_j(t)\, D_j \log \mathsf{G}_k\big(\boldsymbol{\mu}(t)\big)\right]\right)\mu_i(t),
\end{aligned}$$

since $\sum_{k=1}^{m} p_k = 1$. Therefore,

$$\begin{aligned}
\zeta_i(t) &= \left(\sum_{k=1}^{m} p_k\left[D_i \log \mathsf{G}_k\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{j=1}^{n} \mu_j(t)\, D_j \log \mathsf{G}_k\big(\boldsymbol{\mu}(t)\big)\right]\right)\mu_i(t) \\
&= \sum_{k=1}^{m} p_k\left[\left(D_i \log \mathsf{G}_k\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{j=1}^{n} \mu_j(t)\, D_j \log \mathsf{G}_k\big(\boldsymbol{\mu}(t)\big)\right)\mu_i(t)\right] \\
&= \sum_{k=1}^{m} p_k \varphi_{ki}(t),
\end{aligned}$$

which follows easily from Definition 5.2.1. Thus, we have the required result, $\zeta_i(t) = p_1\varphi_{1i}(t) + \cdots + p_m\varphi_{mi}(t)$. ■

5.3 Functionally Generated Portfolios

5.3.1 Characterisation of Functionally Generated Portfolios

Not all portfolios can be generated by means of a portfolio generating function, i.e., not all portfolios are functionally generated. Portfolio generating functions and the portfolios they generate are of critical importance in stochastic portfolio theory, and as a result functionally generated portfolios are of particular interest. Thus, it is essential that we are able to characterise those portfolios that are functionally generated, this is carried out in the proposition below.

Proposition 5.3.1 ([**Fernholz (1999c, 2002)**]). *Suppose that $f_1, \ldots, f_n$ are continuously differentiable real-valued functions defined on a neighbourhood U of Δ^{n-1} such that $\sum_{i=1}^n f_i(\mathbf{x}) = 1$ for all $\mathbf{x} \in \Delta^{n-1}$. Then the portfolio φ defined by $\varphi_i(t) = f_i\big(\boldsymbol{\mu}(t)\big)$ for $t \in [0,T]$ and $i = 1, 2, \ldots, n$ is* ***functionally generated*** *if and only if there exists a continuously differentiable real-valued function F defined on a neighbourhood of Δ^{n-1} such that*

$$\sum_{i=1}^n \left(\frac{f_i(\mathbf{x})}{x_i} + F(\mathbf{x}) \right) dx_i, \tag{5.3.1}$$

is an ***exact differential****.*[1] *Alternatively, in terms of the market portfolio, such that*

$$\sum_{i=1}^n \left(\frac{f_i\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)} + F\big(\boldsymbol{\mu}(t)\big) \right) d\mu_i(t), \tag{5.3.2}$$

is an ***exact differential****.*

Proof. Firstly, assume that φ, defined by $\varphi_i(t) = f_i\big(\boldsymbol{\mu}(t)\big)$ for $t \in [0,T]$ and $i = 1, 2, \ldots, n$, is a functionally generated portfolio and has a generating function $\mathbf{G}$ defined on a neighbourhood U of Δ^{n-1}. Then, for $\mathbf{x} \in U$, let

$$f_i(\mathbf{x}) \quad = \quad \left(D_i \log \mathbf{G}(\mathbf{x}) + 1 - \sum_{j=1}^n x_j \, D_j \log \mathbf{G}(\mathbf{x}) \right) x_i,$$

for $i = 1, 2, \ldots, n$ and let

$$F(\mathbf{x}) \quad = \quad -1 + \sum_{j=1}^n x_j \, D_j \log \mathbf{G}(\mathbf{x}).$$

Then clearly, by Theorem 5.2.2, the weights φ_i satisfy $\varphi_i(t) = f_i\big(\boldsymbol{\mu}(t)\big)$, for $i = 1, 2, \ldots, n$, for $t \in [0,T]$, a.s. From the above expressions, we have

$$f_i(\mathbf{x}) \quad = \quad \Big(D_i \log \mathbf{G}(\mathbf{x}) - F(\mathbf{x}) \Big) \, x_i.$$

Thus,

$$\frac{f_i(\mathbf{x})}{x_i} \quad = \quad D_i \log \mathbf{G}(\mathbf{x}) - F(\mathbf{x}).$$

In terms of the market portfolio, we obtain

$$f_i\big(\boldsymbol{\mu}(t)\big) \quad = \quad \Big(D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) - F\big(\boldsymbol{\mu}(t)\big) \Big) \, \mu_i(t), \tag{5.3.3}$$

and

$$\frac{f_i\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)} \quad = \quad D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) - F\big(\boldsymbol{\mu}(t)\big), \tag{5.3.4}$$

which are directly comparable to (5.2.18) and (5.2.28), respectively, where

$$F\big(\boldsymbol{\mu}(t)\big) \quad = \quad -\phi\big(\boldsymbol{\mu}(t)\big) \quad = \quad -1 + \sum_{j=1}^n \mu_j(t) \, D_j \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big).$$

Thus, employing (5.3.4), for $\boldsymbol{\mu} \in \Delta^{n-1}$, the differential (5.3.2) is given by

$$\begin{aligned} \sum_{i=1}^n \left(\frac{f_i\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)} + F\big(\boldsymbol{\mu}(t)\big) \right) d\mu_i(t) \quad &= \quad \sum_{i=1}^n \Big(D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) - F\big(\boldsymbol{\mu}(t)\big) + F\big(\boldsymbol{\mu}(t)\big) \Big) \, d\mu_i(t) \\ &= \quad \sum_{i=1}^n D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) \, d\mu_i(t) \\ &= \quad \sum_{i=1}^n D_i \, H\big(\boldsymbol{\mu}(t)\big) \, d\mu_i(t), \end{aligned}$$

[1] A differential is *exact* if it is of the form $\sum_{i=1}^n D_i \, H(\mathbf{x}) \, dx_i$ for some differentiable function H [see Spivak (1965)].

where $H = \log \mathbf{G}$. Since the differential (5.3.2) is of the form $\sum_{i=1}^n D_i\, H(\mathbf{x})\, dx_i$, it is exact. Hence, the first implication statement is proved. Now, assume that the differential (5.3.1) is an exact differential and suppose that such functions $F, f_1, \ldots, f_n$ exist and that $\varphi_i(t) = f_i\big(\boldsymbol{\mu}(t)\big)$ for $i = 1, 2, \ldots, n$. Then, there exists a function H such that for $\mathbf{x} \in \Delta^{n-1}$,

$$\sum_{i=1}^n D_i\, H(\mathbf{x})\, dx_i \quad = \quad \sum_{i=1}^n \left(\frac{f_i(\mathbf{x})}{x_i} + F(\mathbf{x}) \right) dx_i.$$

Therefore, for $i = 1, 2, \ldots, n$, we have

$$D_i\, H(\mathbf{x}) \quad = \quad \frac{f_i(\mathbf{x})}{x_i} + F(\mathbf{x}),$$

and in terms of the market portfolio $\boldsymbol{\mu} \in \Delta^{n-1}$, we have for $i = 1, 2, \ldots, n$,

$$D_i\, H\big(\boldsymbol{\mu}(t)\big) \quad = \quad \frac{f_i\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)} + F\big(\boldsymbol{\mu}(t)\big). \tag{5.3.5}$$

Thus, from (5.2.28), we have

$$D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) \quad = \quad \frac{\varphi_i(t)}{\mu_i(t)} - \phi\big(\boldsymbol{\mu}(t)\big),$$

which when compared with (5.3.5) implies that the function $\mathbf{G} = e^H$ generates the portfolio φ, with $H\big(\boldsymbol{\mu}(t)\big) = \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)$ and $F\big(\boldsymbol{\mu}(t)\big) = -\phi\big(\boldsymbol{\mu}(t)\big)$. Applying Itô's formula to the function H gives for all $t \in [0, T]$, a.s.,

$$dH\big(\boldsymbol{\mu}(t)\big) \quad = \quad \sum_{i=1}^n D_i\, H\big(\boldsymbol{\mu}(t)\big)\, d\mu_i(t) + \frac{1}{2} \sum_{i,j=1}^n D_{ij}\, H\big(\boldsymbol{\mu}(t)\big)\, d\big\langle \mu_i, \mu_j \big\rangle_t.$$

Substituting (5.3.5) into the above expression and using the fact that $\varphi_i(t) = f_i\big(\boldsymbol{\mu}(t)\big)$ for all $i = 1, 2, \ldots, n$, gives

$$\begin{aligned}
dH\big(\boldsymbol{\mu}(t)\big) \quad &= \quad \sum_{i=1}^n \left(\frac{f_i\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)} + F\big(\boldsymbol{\mu}(t)\big) \right) d\mu_i(t) + \frac{1}{2} \sum_{i,j=1}^n D_{ij}\, H\big(\boldsymbol{\mu}(t)\big) \mu_i(t) \mu_j(t) \tau_{ij}(t)\, dt \\
&= \quad \sum_{i=1}^n \frac{\varphi_i(t)}{\mu_i(t)}\, d\mu_i(t) + F\big(\boldsymbol{\mu}(t)\big) \sum_{i=1}^n d\mu_i(t) + \frac{1}{2} \sum_{i,j=1}^n D_{ij}\, H\big(\boldsymbol{\mu}(t)\big) \mu_i(t) \mu_j(t) \tau_{ij}(t)\, dt \\
&= \quad \sum_{i=1}^n \frac{\varphi_i(t)}{\mu_i(t)}\, d\mu_i(t) + \frac{1}{2} \sum_{i,j=1}^n D_{ij}\, H\big(\boldsymbol{\mu}(t)\big) \mu_i(t) \mu_j(t) \tau_{ij}(t)\, dt,
\end{aligned}$$

since $\sum_{i=1}^n d\mu_i(t) = 0$. By (5.2.27) it follows that a.s., for $t \in [0, T]$,

$$dH\big(\boldsymbol{\mu}(t)\big) \quad = \quad d \log \Big(Z_\varphi(t) / Z_\mu(t) \Big) + \frac{1}{2} \sum_{i,j=1}^n \Big(\varphi_i(t) \varphi_j(t) \tau_{ij}(t) + D_{ij}\, H\big(\boldsymbol{\mu}(t)\big) \mu_i(t) \mu_j(t) \tau_{ij}(t) \Big)\, dt,$$

equivalently,

$$d \log \Big(Z_\varphi(t) / Z_\mu(t) \Big) \quad = \quad dH\big(\boldsymbol{\mu}(t)\big) - \frac{1}{2} \sum_{i,j=1}^n \Big(\varphi_i(t) \varphi_j(t) \tau_{ij}(t) + D_{ij}\, H\big(\boldsymbol{\mu}(t)\big) \mu_i(t) \mu_j(t) \tau_{ij}(t) \Big)\, dt.$$

Then with the function $\mathbf{G} = e^H$, the above expression can be written as

$$d \log \Big(Z_\varphi(t) / Z_\mu(t) \Big) \quad = \quad d \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) - \frac{1}{2} \sum_{i,j=1}^n \Big(\varphi_i(t) \varphi_j(t) \tau_{ij}(t) + D_{ij} \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) \mu_i(t) \mu_j(t) \tau_{ij}(t) \Big)\, dt.$$

Hence, the function $\mathbf{G} = e^H$ generates the portfolio φ with $\varphi_i(t) = f_i\big(\boldsymbol{\mu}(t)\big)$, for all $i = 1, 2, \ldots, n$, since the differential equation above has the form given in (5.2.2) of Definition 5.2.1, with the drift process of bounded variation Θ corresponding to $\mathbf{G}$, given by

$$d\Theta(t) \quad = \quad \mathfrak{g}(t)\, dt \quad = \quad -\frac{1}{2} \sum_{i,j=1}^n \Big(\varphi_i(t) \varphi_j(t) \tau_{ij}(t) + D_{ij} \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) \mu_i(t) \mu_j(t) \tau_{ij}(t) \Big)\, dt.$$

Thus, by definition, $\mathbf{G}$ generates φ and the portfolio φ defined by $\varphi_i(t) = f_i\big(\boldsymbol{\mu}(t)\big)$, for all $i = 1, 2, \ldots, n$ is functionally generated. Hence, the second implication statement is proved. ■

Refer to Fernholz (2002) for an example of a portfolio whose weights depend on the market portfolio weights, but is not functionally generated. By appealing to the preceding result, Fernholz (2002) reveals that this portfolio has no generating function. It would seem reasonable that only the values of a generating function on Δ^{n-1} should affect the portfolio it generates. To prove this, we first need to prove a few lemmas.

Lemma 5.3.2 ([**Fernholz (1999c)**]). *Let $h_1, h_2, \ldots, h_n$ be continuous real-valued functions on Δ^{n-1} and ϱ be a continuous real-valued function on $\mathbb{R}^n \times \Delta^{n-1}$. If*

$$\sum_{i=1}^{n} h_i(\mathbf{x})\, dx_i \quad = \quad \varrho(t, \mathbf{x}), \qquad t \in [0, T], \qquad a.s., \tag{5.3.6}$$

equivalently,

$$\sum_{i=1}^{n} h_i\big(\boldsymbol{\mu}(t)\big)\, d\mu_i(t) \quad = \quad \varrho\big(t, \boldsymbol{\mu}(t)\big), \qquad t \in [0, T], \qquad a.s., \tag{5.3.7}$$

then for $\mathbf{x} \in \Delta^{n-1}$, $h_i(\mathbf{x}) = h_j(\mathbf{x})$ for all $i, j = 1, 2, \ldots, n$, and $\varrho(t, \mathbf{x}) = 0$.

Proof. Let

$$Y(T) \quad = \quad \int_0^T \sum_{i=1}^{n} h_i\big(\boldsymbol{\mu}(t)\big)\, d\mu_i(t), \qquad T \in [0, \infty).$$

Then Y is a semimartingale with

$$\begin{aligned}
\langle Y \rangle_t \quad &= \quad \left\langle \int_0^{\cdot} \sum_{i=1}^{n} h_i\big(\boldsymbol{\mu}_s\big)\, d\mu_{i,s}, \int_0^{\cdot} \sum_{j=1}^{n} h_j\big(\boldsymbol{\mu}_s\big)\, d\mu_{j,s} \right\rangle_t \\
&= \quad \sum_{i,j=1}^{n} \left\langle \int_0^{\cdot} h_i\big(\boldsymbol{\mu}_s\big)\, d\mu_{i,s}, \int_0^{\cdot} h_j\big(\boldsymbol{\mu}_s\big)\, d\mu_{j,s} \right\rangle_t \\
&= \quad \sum_{i,j=1}^{n} \int_0^t h_i\big(\boldsymbol{\mu}(s)\big) h_j\big(\boldsymbol{\mu}(s)\big)\, d\left\langle \mu_i, \mu_j \right\rangle_s \\
&= \quad \int_0^t \sum_{i,j=1}^{n} h_i\big(\boldsymbol{\mu}(s)\big) h_j\big(\boldsymbol{\mu}(s)\big) \mu_i(s) \mu_j(s) \tau_{ij}(s)\, ds.
\end{aligned}$$

Consequently, we obtain

$$d\langle Y \rangle_t \quad = \quad \sum_{i,j=1}^{n} h_i\big(\boldsymbol{\mu}(t)\big) h_j\big(\boldsymbol{\mu}(t)\big) \mu_i(t) \mu_j(t) \tau_{ij}(t)\, dt, \qquad t \in [0, T], \qquad \text{a.s.}$$

Moreover, by (5.3.7), we have for $T \in [0, \infty)$,

$$Y(T) \quad = \quad \int_0^T \varrho\big(t, \boldsymbol{\mu}(t)\big)\, dt.$$

As a consequence, the martingale component of Y is null and this implies that $\langle Y \rangle_t = 0$. Therefore,

$$\sum_{i,j=1}^{n} h_i\big(\boldsymbol{\mu}(t)\big) h_j\big(\boldsymbol{\mu}(t)\big) \mu_i(t) \mu_j(t) \tau_{ij}(t) \quad = \quad 0, \qquad t \in [0, T], \qquad \text{a.s.}$$

Let $\mathbf{y}(t) = \big(h_1\big(\boldsymbol{\mu}(t)\big)\mu_1(t), h_2\big(\boldsymbol{\mu}(t)\big)\mu_2(t), \ldots, h_n\big(\boldsymbol{\mu}(t)\big)\mu_n(t)\big)$, then the last expression implies

$$\mathbf{y}(t) \boldsymbol{\tau}(t) \mathbf{y}^{\mathsf{T}}(t) \quad = \quad 0, \qquad t \in [0, T], \qquad \text{a.s.}$$

Thus $\mathbf{y}(t)$ is in the null space of $\boldsymbol{\tau}(t)$. In addition, we know by Lemma 2.12.7 that the null space of $\boldsymbol{\tau}(t)$ is spanned by $\boldsymbol{\mu}(t) = \big(\mu_1(t), \mu_2(t), \ldots, \mu_n(t)\big)$. It follows that

$$h_1\big(\boldsymbol{\mu}(t)\big) \;=\; h_2\big(\boldsymbol{\mu}(t)\big) \;=\; \cdots \;=\; h_n\big(\boldsymbol{\mu}(t)\big) \;=\; h\big(\boldsymbol{\mu}(t)\big),$$

so that $h_i\big(\boldsymbol{\mu}(t)\big) = h_j\big(\boldsymbol{\mu}(t)\big) \equiv h\big(\boldsymbol{\mu}(t)\big)$ for all $i, j = 1, 2, \ldots, n$, which proves the first part of the lemma. Thus for the second part of the lemma, we have a.s., for $t \in [0, T]$,

$$\sum_{i=1}^{n} h_i\big(\boldsymbol{\mu}(t)\big)\, d\mu_i(t) \;=\; h\big(\boldsymbol{\mu}(t)\big) \sum_{i=1}^{n} d\mu_i(t) \;=\; 0,$$

since $\sum_{i=1}^{n} d\mu_i(t) = 0$. Therefore, from (5.3.7), we deduce $\varrho\big(t, \boldsymbol{\mu}(t)\big) = 0$. ■

Theorem 5.2.2 shows that a positive twice continuously differentiable function $\mathbf{G}$ defined on an open neighbourhood of U of the open positive unit $(n-1)-$simplex Δ^{n-1}, generates a portfolio. Thus, it would seem reasonable that the values of $\mathbf{G}$ on Δ^{n-1} should uniquely determine the portfolio, and that the values of $\mathbf{G}$ in the complement of Δ^{n-1} should be irrelevant [Fernholz (2002)]. We shall show that this is indeed the case, but a preliminary result is initially required.

Lemma 5.3.3 ([**Fernholz (1999c, 2002)**]). *Let f be a continuously differentiable real-valued function defined on a neighbourhood U of Δ^{n-1}. Then f is constant on Δ^{n-1} if and only if for all $\mathbf{x} \in \Delta^{n-1}$, $D_i f(\mathbf{x}) = D_j f(\mathbf{x})$ for all $i, j = 1, 2, \ldots, n$.*

Proof. Parameterise Δ^{n-1} by positive real numbers $y_1, \ldots, y_{n-1}$ with $y_1 + \cdots + y_{n-1} < 1$, such that $x_i = y_i$ (i.e., $x_i(y_i) = y_i$) for $i = 1, 2, \ldots, n-1$ and $x_n = 1 - y_1 - \cdots - y_{n-1}$ (i.e., $x_n(y_1, \ldots, y_{n-1}) = 1 - y_1 - \cdots - y_{n-1}$). Then, for $i = 1, \ldots, n-1$, we have

$$\frac{\partial x_i}{\partial y_i} \;=\; 1,$$

and

$$\frac{\partial x_n}{\partial y_i} \;=\; -1.$$

Therefore, for the following function

$$f(\mathbf{x}) \;=\; f\Big(x_1(y_1), x_2(y_2), \ldots, x_i(y_i), \ldots, x_{n-1}(y_{n-1}), x_n(y_1, \ldots, y_i, \ldots, y_{n-1})\Big),$$

we have for $i = 1, \ldots, n-1$ and for all $\mathbf{x} \in \Delta^{n-1}$,

$$\begin{aligned}
\frac{\partial f(\mathbf{x})}{\partial y_i} &= \frac{\partial f(\mathbf{x})}{\partial x_i}\frac{\partial x_i}{\partial y_i} + \frac{\partial f(\mathbf{x})}{\partial x_n}\frac{\partial x_n}{\partial y_i} \\
&= \frac{\partial f(\mathbf{x})}{\partial x_i} - \frac{\partial f(\mathbf{x})}{\partial x_n} \\
&= D_i f(\mathbf{x}) - D_n f(\mathbf{x}).
\end{aligned}$$

Consequently, if f is constant on Δ^{n-1}, then all its first partial derivatives with respect to the parameters y_i are zero, which implies that $D_i f(\mathbf{x}) = D_n f(\mathbf{x})$ for all $i = 1, \ldots, n$. Likewise, if $D_i f(\mathbf{x}) = D_j f(\mathbf{x})$ for all $i, j = 1, \ldots, n$, then the first partial derivatives with respect to all the y_i are zero. Hence, f is constant on Δ^{n-1}. ■

We shall now consider generating functions that generate the same portfolio for all values of the market weights $\mu_1(t), \mu_2(t), \ldots, \mu_n(t)$, for $t \in [0, T]$. The initial value of the market portfolio can assume any value in the set Δ^{n-1}. Consequently, Theorem 5.2.2 insinuates that the positive twice continuously differentiable functions $\mathbf{G}_1$

and $\mathbf{G}_2$ will both generate the same portfolio for all possible realisations of the market portfolio provided that for all $i = 1, 2, \ldots, n$, and $\boldsymbol{\mu}(t) \in \Delta^{n-1}$, the following equation is satisfied

$$D_i \log \mathbf{G}_1\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{j=1}^{n} \mu_j(t)\, D_j \log \mathbf{G}_1\big(\boldsymbol{\mu}(t)\big) \quad = \quad D_i \log \mathbf{G}_2\big(\boldsymbol{\mu}(t)\big) - 1 + \sum_{j=1}^{n} \mu_j(t)\, D_j \log \mathbf{G}_2\big(\boldsymbol{\mu}(t)\big).$$

The subsequent proposition deals with this concept, and reveals a necessary and sufficient condition for the generating functions $\mathbf{G}_1$ and $\mathbf{G}_2$ to generate the same portfolio.

Proposition 5.3.4 ([**Fernholz (1999c, 2002)**]). *Let $\mathbf{G}_1$ and $\mathbf{G}_2$ be positive twice continuously differentiable functions defined on an open neighbourhood of Δ^{n-1}. Then $\mathbf{G}_1$ and $\mathbf{G}_2$ generate the same portfolio if and only if $\mathbf{G}_1/\mathbf{G}_2$ is constant on Δ^{n-1}.*

Proof. Suppose $\mathbf{G}_1$ and $\mathbf{G}_2$ are defined on an open neighbourhood U of Δ^{n-1}, and that $\mathbf{G}_1/\mathbf{G}_2$ is constant on Δ^{n-1}. Define the function f for $\mathbf{x} \in U$, by

$$f(\mathbf{x}) \quad = \quad \log\Big(\mathbf{G}_1(\mathbf{x})/\mathbf{G}_2(\mathbf{x})\Big) \quad = \quad \log \mathbf{G}_1(\mathbf{x}) - \log \mathbf{G}_2(\mathbf{x}).$$

Then f is constant on Δ^{n-1} since $\mathbf{G}_1(\mathbf{x})/\mathbf{G}_2(\mathbf{x})$ is constant. Thus, by Lemma 5.3.3 if f is constant then for all $\mathbf{x} \in \Delta^{n-1}$, $D_i f(\mathbf{x}) = D_j f(\mathbf{x})$, for all $i, j = 1, 2, \ldots, n$. Therefore, for $\boldsymbol{\mu} \in \Delta^{n-1}$, we have $D_i f\big(\boldsymbol{\mu}(t)\big) = D_j f\big(\boldsymbol{\mu}(t)\big)$ for all $i, j = 1, 2, \ldots, n$ and $t \in [0, T]$, which implies

$$\begin{aligned} D_i \Big(\log \mathbf{G}_1\big(\boldsymbol{\mu}(t)\big) - \log \mathbf{G}_2\big(\boldsymbol{\mu}(t)\big)\Big) &= D_j \Big(\log \mathbf{G}_1\big(\boldsymbol{\mu}(t)\big) - \log \mathbf{G}_2\big(\boldsymbol{\mu}(t)\big)\Big) \\ D_i \log \mathbf{G}_1\big(\boldsymbol{\mu}(t)\big) - D_i \log \mathbf{G}_2\big(\boldsymbol{\mu}(t)\big) &= D_j \log \mathbf{G}_1\big(\boldsymbol{\mu}(t)\big) - D_j \log \mathbf{G}_2\big(\boldsymbol{\mu}(t)\big). \end{aligned} \tag{5.3.8}$$

Hence, the difference $D_i \log \mathbf{G}_1\big(\boldsymbol{\mu}(t)\big) - D_i \log \mathbf{G}_2\big(\boldsymbol{\mu}(t)\big)$ is the same for all $i = 1, 2, \ldots, n$. Let φ_1 denote the portfolio generated by $\mathbf{G}_1$ and let φ_2 denote the portfolio generated by $\mathbf{G}_2$, then $\varphi_{1i}(t)$ is the ith weight of φ_1, for $i = 1, 2, \ldots, n$ and $\varphi_{2i}(t)$ is the ith weight of φ_2, for $i = 1, 2, \ldots, n$. Then from (5.2.13) of Theorem 5.2.2, we deduce the following for all $i = 1, 2, \ldots, n$ and $t \in [0, T]$,

$$\begin{aligned} &\varphi_{1i}(t) - \varphi_{2i}(t) \\ &= \left(D_i \log \mathbf{G}_1\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{j=1}^{n} \mu_j(t)\, D_j \log \mathbf{G}_1\big(\boldsymbol{\mu}(t)\big) - D_i \log \mathbf{G}_2\big(\boldsymbol{\mu}(t)\big) - 1 + \sum_{j=1}^{n} \mu_j(t)\, D_j \log \mathbf{G}_2\big(\boldsymbol{\mu}(t)\big) \right) \mu_i(t) \\ &= \left(D_i \log \mathbf{G}_1\big(\boldsymbol{\mu}(t)\big) - D_i \log \mathbf{G}_2\big(\boldsymbol{\mu}(t)\big) - \sum_{j=1}^{n} \mu_j(t) \Big[D_j \log \mathbf{G}_1\big(\boldsymbol{\mu}(t)\big) - D_j \log \mathbf{G}_2\big(\boldsymbol{\mu}(t)\big) \Big] \right) \mu_i(t) \\ &= \left(\sum_{j=1}^{n} \mu_j(t) \Big[\Big(D_i \log \mathbf{G}_1\big(\boldsymbol{\mu}(t)\big) - D_i \log \mathbf{G}_2\big(\boldsymbol{\mu}(t)\big) \Big) - \Big(D_j \log \mathbf{G}_1\big(\boldsymbol{\mu}(t)\big) - D_j \log \mathbf{G}_2\big(\boldsymbol{\mu}(t)\big) \Big) \Big] \right) \mu_i(t). \end{aligned}$$

Thus, from (5.3.8), we have for all $i = 1, 2, \ldots, n$ and $t \in [0, T]$,

$$\varphi_{1i}(t) - \varphi_{2i}(t) \quad = \quad 0.$$

This implies that the weights generated by $\mathbf{G}_1$ and $\mathbf{G}_2$ are the same, therefore $\varphi_1 = \varphi_2$. It follows that the generating functions $\mathbf{G}_1$ and $\mathbf{G}_2$ generate the same functionally generated portfolio. Now, suppose that $\mathbf{G}_1$ and $\mathbf{G}_2$ generate the same portfolio. Applying Itô's formula to $\log(\mathbf{G}_1/\mathbf{G}_2)$, we obtain a.s., for all $t \in [0, T]$,

$$\begin{aligned} d \log \Big(\mathbf{G}_1\big(\boldsymbol{\mu}(t)\big)/\mathbf{G}_2\big(\boldsymbol{\mu}(t)\big)\Big) \quad &= \quad \sum_{i=1}^{n} D_i \log \Big(\mathbf{G}_1\big(\boldsymbol{\mu}(t)\big)/\mathbf{G}_2\big(\boldsymbol{\mu}(t)\big)\Big)\, d\mu_i(t) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^{n} D_{ij} \log \Big(\mathbf{G}_1\big(\boldsymbol{\mu}(t)\big)/\mathbf{G}_2\big(\boldsymbol{\mu}(t)\big)\Big) \mu_i(t) \mu_j(t) \tau_{ij}(t)\, dt. \end{aligned}$$

Since the values of the portfolios (i.e., Z_{φ_1} and Z_{φ_2}) generated by $\mathbf{G}_1$ and $\mathbf{G}_2$ are equal, i.e., $Z_{\varphi_1}(t) = Z_{\varphi_2}(t) = Z_\varphi(t)$ for all $t \in [0,T]$, (5.2.47) of Proposition 5.2.4 implies that a.s., for $t \in [0,T]$,

$$\begin{aligned} d\log\Big(\mathbf{G}_1\big(\boldsymbol{\mu}(t)\big)/\mathbf{G}_2\big(\boldsymbol{\mu}(t)\big)\Big) &= d\log\Big(Z_{\varphi_1}(t)/Z_{\varphi_2}(t)\Big) - \big(\mathfrak{g}_{\varphi_1}(t) - \mathfrak{g}_{\varphi_2}(t)\big)\,dt \\ &= d\log\Big(Z_\varphi(t)/Z_\varphi(t)\Big) - \big(\mathfrak{g}_1(t) - \mathfrak{g}_2(t)\big)\,dt \\ &= \big(\mathfrak{g}_2(t) - \mathfrak{g}_1(t)\big)\,dt. \end{aligned}$$

Hence, combining these two expressions gives a.s., for all $t \in [0,T]$,

$$\begin{aligned} &\sum_{i=1}^n D_i \log\Big(\mathbf{G}_1\big(\boldsymbol{\mu}(t)\big)/\mathbf{G}_2\big(\boldsymbol{\mu}(t)\big)\Big)\,d\mu_i(t) \\ &\qquad = \Big(\mathfrak{g}_2(t) - \mathfrak{g}_1(t) - \frac{1}{2}\sum_{i,j=1}^n D_{ij}\log\Big(\mathbf{G}_1\big(\boldsymbol{\mu}(t)\big)/\mathbf{G}_2\big(\boldsymbol{\mu}(t)\big)\Big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\Big)\,dt. \end{aligned}$$

Then, with

$$h_i\big(\boldsymbol{\mu}(t)\big) \triangleq D_i \log\Big(\mathbf{G}_1\big(\boldsymbol{\mu}(t)\big)/\mathbf{G}_2\big(\boldsymbol{\mu}(t)\big)\Big),$$

and

$$\varrho\big(t,\boldsymbol{\mu}(t)\big) \triangleq \Big(\mathfrak{g}_2(t) - \mathfrak{g}_1(t) - \frac{1}{2}\sum_{i,j=1}^n D_{ij}\log\Big(\mathbf{G}_1\big(\boldsymbol{\mu}(t)\big)/\mathbf{G}_2\big(\boldsymbol{\mu}(t)\big)\Big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\Big)\,dt,$$

we have a.s., for all $t \in [0,T]$,

$$\sum_{i=1}^n h_i\big(\boldsymbol{\mu}(t)\big)\,d\mu_i(t) = \varrho\big(t,\boldsymbol{\mu}(t)\big).$$

Therefore, Lemma 5.3.2 implies that for all $\mathbf{x} \in \Delta^{n-1}$, $h_i\big(\boldsymbol{\mu}(t)\big) = h_j\big(\boldsymbol{\mu}(t)\big)$, so

$$D_i \log\Big(\mathbf{G}_1\big(\boldsymbol{\mu}(t)\big)/\mathbf{G}_2\big(\boldsymbol{\mu}(t)\big)\Big) = D_j \log\Big(\mathbf{G}_1\big(\boldsymbol{\mu}(t)\big)/\mathbf{G}_2\big(\boldsymbol{\mu}(t)\big)\Big),$$

for all $i,j = 1,2,\ldots,n$. Let $f\big(\boldsymbol{\mu}(t)\big) = \log\big(\mathbf{G}_1\big(\boldsymbol{\mu}(t)\big)/\mathbf{G}_2\big(\boldsymbol{\mu}(t)\big)\big)$, then $D_i f\big(\boldsymbol{\mu}(t)\big) = D_j f\big(\boldsymbol{\mu}(t)\big)$ for all $i,j = 1,2,\ldots,n$. It follows by Lemma 5.3.3 that $\log(\mathbf{G}_1/\mathbf{G}_2)$ is constant on Δ^{n-1}. ■

Hence, this proposition implies that the portfolio generating function could rather have been defined on Δ^{n-1}, as opposed to being defined on an open neighbourhood of Δ^{n-1}.

5.3.2 Functionally Generated Portfolios with Increasing Drift Processes

Let us now consider generating functions that generate portfolios with increasing drift processes, and, in the next proposition, we shall provide a characterisation for those generating functions and establish conditions under which a generating function will have an increasing drift process. Under appropriate market conditions, it can be shown that a dominance relationship holds between functionally generated portfolios and the market portfolio. Thus, functionally generated portfolios are of utmost interest as they will tend to outperform the market portfolio under these market conditions. In particular, functionally generated portfolios that exhibit an increasing drift are of further specific concern, as they will tend to outperform the market portfolio over periods in which the portfolio generating function either remains constant or increases.

Proposition 5.3.5 ([**Fernholz (1999c, 2002)**]). *Let $\mathbf{G}$ be a generating function such that for all $\mathbf{x} \in \Delta^{n-1}$, the Hessian matrix $\mathbf{HG}(\mathbf{x}) = \nabla^2\mathbf{G}(\mathbf{x}) = D^2\mathbf{G}(\mathbf{x}) = \big(D_iD_j\mathbf{G}(\mathbf{x})\big)_{1\le i,j\le n} = \big(D_{ij}\mathbf{G}(\mathbf{x})\big)_{1\le i,j\le n} \triangleq \big(\frac{\partial^2\mathbf{G}(\mathbf{x})}{\partial x_i \partial x_j}\big)_{1\le i,j\le n}$ has at most one positive eigenvalue for each $\mathbf{x} \in \Delta^{n-1}$, and if there is such a positive eigenvalue, it corresponds to an eigenvector orthogonal to Δ^{n-1}. Let φ be the portfolio generated by $\mathbf{G}$. Then $\varphi_i(t) \ge 0$, for $i = 1,2,\ldots,n$*

(i.e., the portfolio φ has only nonnegative weights and is thus long-only), and $\mathfrak{g}(t) = \Theta'(t) \geq 0$ (i.e., the drift process is nonnegative) and Θ is nondecreasing, for all $t \in [0,T]$, a.s. If for all $\mathbf{x} \in \Delta^{n-1}$, $\operatorname{rank}\big(\mathbf{HG}(\mathbf{x})\big) > 1$, then $\mathfrak{g}(t) = \Theta'(t) > 0$ (i.e., the drift process is strictly positive) and Θ is strictly increasing, for all $t \in [0,T]$, a.s.

Proof. Suppose that $\mathbf{G}$ is a generating function such that for all $\mathbf{x} = (x_1, \ldots, x_n) \in \Delta^{n-1}$, the Hessian matrix $\mathbf{HG}(\mathbf{x}) = \big(D_{ij}\mathbf{G}(\mathbf{x})\big)_{1\leq i,j\leq n}$ has at most one positive eigenvalue, and if there is a positive eigenvalue it corresponds to an eigenvector that is orthogonal to Δ^{n-1}. Let $\mathbf{e}_k$ be the kth unit vector in $\mathbb{R}^n$, i.e., $\mathbf{e}_k = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the kth coordinate is 1 and the other coordinates are all 0, then for any $\mathbf{x} \in \Delta^{n-1}$ and for $0 \leq u < 1$, define $\mathbf{y}(u) \in \Delta^{n-1}$ by

$$\begin{aligned} \mathbf{y}(u) &\triangleq u\mathbf{e}_k + (1-u)\mathbf{x} \\ &= \mathbf{x} + \big(\mathbf{e}_k - \mathbf{x}\big)u. \end{aligned} \tag{5.3.9}$$

Therefore, we have

$$\begin{aligned} \mathbf{y}(u) &= \Big(y_1(u), \ldots, y_{k-1}(u), y_k(u), y_{k+1}(u), \ldots, y_n(u)\Big) \\ &= \Big((1-u)x_1, \ldots, (1-u)x_{k-1}, x_k + (1-x_k)u, (1-u)x_{k+1}, \ldots, (1-u)x_n\Big), \end{aligned}$$

consequently, for $i = 1, 2, \ldots, n$, we have

$$y_i(u) = \begin{cases} (1-u)x_i & \text{if} \quad i \neq k, \\ x_k + (1-x_k)u & \text{if} \quad i = k. \end{cases}$$

So, for $i = 1, 2, \ldots, n$, we obtain

$$\frac{dy_i(u)}{du} = \begin{cases} -x_i & \text{if} \quad i \neq k, \\ 1 - x_k & \text{if} \quad i = k, \end{cases} \tag{5.3.10}$$

equivalently from (5.3.9), in vector form, we have

$$\frac{d\mathbf{y}(u)}{du} = \mathbf{e}_k - \mathbf{x}. \tag{5.3.11}$$

Let

$$f(u) = \mathbf{G}\big(\mathbf{y}(u)\big) = \mathbf{G}\Big(y_1(u), \ldots, y_{k-1}(u), y_k(u), y_{k+1}(u), \ldots, y_n(u)\Big),$$

then, by the chain rule for functions of several variables, for one independent variable u, we obtain

$$\begin{aligned} f'(u) = \frac{df(u)}{du} &= \frac{\partial f(u)}{\partial y_1(u)}\frac{dy_1(u)}{du} + \frac{\partial f(u)}{\partial y_2(u)}\frac{dy_2(u)}{du} + \cdots + \frac{\partial f(u)}{\partial y_n(u)}\frac{dy_n(u)}{du} \\ &= \frac{\partial \mathbf{G}\big(\mathbf{y}(u)\big)}{\partial y_1(u)}\frac{dy_1(u)}{du} + \frac{\partial \mathbf{G}\big(\mathbf{y}(u)\big)}{\partial y_2(u)}\frac{dy_2(u)}{du} + \cdots + \frac{\partial \mathbf{G}\big(\mathbf{y}(u)\big)}{\partial y_n(u)}\frac{dy_n(u)}{du} \\ &= D_1\mathbf{G}\big(\mathbf{y}(u)\big)\frac{dy_1(u)}{du} + D_2\mathbf{G}\big(\mathbf{y}(u)\big)\frac{dy_2(u)}{du} + \cdots + D_n\mathbf{G}\big(\mathbf{y}(u)\big)\frac{dy_n(u)}{du} \\ &= \sum_{i=1}^{n}\frac{dy_i(u)}{du}D_i\mathbf{G}\big(\mathbf{y}(u)\big). \end{aligned}$$

Hence, employing (5.3.10), we deduce that

$$\begin{aligned} f'(u) &= (1-x_k)D_k\mathbf{G}\big(\mathbf{y}(u)\big) - \sum_{i\neq k}^{n} x_i D_i\mathbf{G}\big(\mathbf{y}(u)\big) \\ &= D_k\mathbf{G}\big(\mathbf{y}(u)\big) - x_k D_k\mathbf{G}\big(\mathbf{y}(u)\big) - \sum_{i\neq k}^{n} x_i D_i\mathbf{G}\big(\mathbf{y}(u)\big) \\ &= D_k\mathbf{G}\big(\mathbf{y}(u)\big) - \sum_{i=1}^{n} x_i D_i\mathbf{G}\big(\mathbf{y}(u)\big). \end{aligned} \tag{5.3.12}$$

Furthermore, by the product rule for functions of several variables, for one independent variable u, we obtain

$$\begin{aligned} f''(u) &= \frac{d^2 f(u)}{du^2} = \frac{d}{du}\left(\frac{df(u)}{du}\right) \\ &= \frac{d}{du}\left(\frac{\partial f(u)}{\partial y_1(u)}\frac{dy_1(u)}{du} + \frac{\partial f(u)}{\partial y_2(u)}\frac{dy_2(u)}{du} + \cdots + \frac{\partial f(u)}{\partial y_n(u)}\frac{dy_n(u)}{du}\right) \\ &= \frac{d}{du}\left(\frac{\partial f(u)}{\partial y_1(u)}\right)\frac{dy_1(u)}{du} + \frac{\partial f(u)}{\partial y_1(u)}\frac{d^2 y_1(u)}{du^2} + \cdots + \frac{d}{du}\left(\frac{\partial f(u)}{\partial y_n(u)}\right)\frac{dy_n(u)}{du} + \frac{\partial f(u)}{\partial y_n(u)}\frac{d^2 y_n(u)}{du^2}. \end{aligned}$$

From (5.3.10) it is clear that $\frac{d^2 y_i(u)}{du^2} = 0$, for all $i = 1, 2, \ldots, n$, consequently we obtain

$$f''(u) = \frac{d}{du}\left(\frac{\partial f(u)}{\partial y_1(u)}\right)\frac{dy_1(u)}{du} + \cdots + \frac{d}{du}\left(\frac{\partial f(u)}{\partial y_n(u)}\right)\frac{dy_n(u)}{du}.$$

Now, let $h_i(u) = h_i\big(y_1(u), \ldots, y_n(u)\big) = \frac{\partial f(u)}{\partial y_i(u)}$ for $i = 1, 2, \ldots, n$, then

$$f''(u) = \left(\frac{dh_1(u)}{du}\right)\frac{dy_1(u)}{du} + \cdots + \left(\frac{dh_n(u)}{du}\right)\frac{dy_n(u)}{du}.$$

Once again by the chain rule for functions of several variables, for one independent variable u, we have for all $i = 1, 2, \ldots, n$,

$$\begin{aligned} h_i'(u) &= \frac{dh_i(u)}{du} = \frac{\partial h_i(u)}{\partial y_1(u)}\frac{dy_1(u)}{du} + \frac{\partial h_i(u)}{\partial y_2(u)}\frac{dy_2(u)}{du} + \cdots + \frac{\partial h_i(u)}{\partial y_n(u)}\frac{dy_n(u)}{du} \\ &= \sum_{j=1}^{n}\frac{\partial h_i(u)}{\partial y_j(u)}\frac{dy_j(u)}{du}. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} f''(u) &= \left(\sum_{j=1}^{n}\frac{\partial h_1(u)}{\partial y_j(u)}\frac{dy_j(u)}{du}\right)\frac{dy_1(u)}{du} + \cdots + \left(\sum_{j=1}^{n}\frac{\partial h_n(u)}{\partial y_j(u)}\frac{dy_j(u)}{du}\right)\frac{dy_n(u)}{du} \\ &= \sum_{i,j=1}^{n}\frac{\partial h_i(u)}{\partial y_j(u)}\frac{dy_i(u)}{du}\frac{dy_j(u)}{du} \\ &= \sum_{i,j=1}^{n}\frac{\partial}{\partial y_j(u)}\left(\frac{\partial f(u)}{\partial y_i(u)}\right)\frac{dy_i(u)}{du}\frac{dy_j(u)}{du} \\ &= \sum_{i,j=1}^{n}\frac{\partial^2 f(u)}{\partial y_i(u)\partial y_j(u)}\frac{dy_i(u)}{du}\frac{dy_j(u)}{du} \\ &= \sum_{i,j=1}^{n}\frac{\partial^2 \mathbf{G}\big(\mathbf{y}(u)\big)}{\partial y_i(u)\partial y_j(u)}\frac{dy_i(u)}{du}\frac{dy_j(u)}{du} \\ &= \sum_{i,j=1}^{n}\frac{dy_i(u)}{du}D_{ij}\mathbf{G}\big(\mathbf{y}(u)\big)\frac{dy_j(u)}{du} \\ &= \left(\frac{d\mathbf{y}(u)}{du}\right)\Big(D_{ij}\mathbf{G}\big(\mathbf{y}(u)\big)\Big)\left(\frac{d\mathbf{y}(u)}{du}\right)^{\mathsf{T}}. \end{aligned}$$

Thus, from (5.3.11), we have

$$f''(u) = \big(\mathbf{e}_k - \mathbf{x}\big)\Big(D_{ij}\mathbf{G}\big(\mathbf{y}(u)\big)\Big)\big(\mathbf{e}_k - \mathbf{x}\big)^{\mathsf{T}} \tag{5.3.13}$$

$$= \big(\mathbf{e}_k - \mathbf{x}\big)\mathbf{HG}\big(\mathbf{y}(u)\big)\big(\mathbf{e}_k - \mathbf{x}\big)^{\mathsf{T}}. \tag{5.3.14}$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the Hessian matrix above $\mathbf{HG}\big(\mathbf{y}(u)\big)$ and let $\mathbf{q}_k = (q_{k1}, q_{k2}, \ldots, q_{kn})^{\mathsf{T}}$ be a normalised eigenvector of $\mathbf{HG}\big(\mathbf{y}(u)\big)$ corresponding to the eigenvalue λ_k for $k = 1, 2, \ldots, n$. Then, the square

symmetric $n \times n$ Hessian matrix above $\mathbf{HG}\big(\mathbf{y}(u)\big)$ has the following eigenvalue-eigenvector decomposition[2]

$$\mathbf{HG}\big(\mathbf{y}(u)\big) \quad = \quad \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{\mathsf{T}}, \tag{5.3.15}$$

where $\mathbf{Q} = \big(\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\big)$, equivalently

$$\mathbf{Q} \quad = \quad \begin{bmatrix} q_{11} & q_{21} & \cdots & q_{n1} \\ q_{12} & q_{22} & \cdots & q_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ q_{1n} & q_{2n} & \cdots & q_{nn} \end{bmatrix},$$

is the square orthogonal $n \times n$ matrix whose ith column is the ith eigenvector $\mathbf{q}_i$ of $\mathbf{HG}\big(\mathbf{y}(u)\big)$, and

$$\boldsymbol{\Lambda} \quad = \quad \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix},$$

is the diagonal matrix whose diagonal elements are the corresponding eigenvalues λ_i, for $i = 1, 2, \ldots, n$. Therefore, (5.3.14) can be cast in the following form

$$\begin{aligned} f''(u) &= \big(\mathbf{e}_k - \mathbf{x}\big)\mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{\mathsf{T}}\big(\mathbf{e}_k - \mathbf{x}\big)^{\mathsf{T}} \\ &= \Big(\big(\mathbf{e}_k - \mathbf{x}\big)\mathbf{Q}\Big)\boldsymbol{\Lambda}\Big(\big(\mathbf{e}_k - \mathbf{x}\big)\mathbf{Q}\Big)^{\mathsf{T}} \\ &= \Big(\big(\mathbf{e}_k - \mathbf{x}\big)\big(\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\big)\Big)\boldsymbol{\Lambda}\Big(\big(\mathbf{e}_k - \mathbf{x}\big)\big(\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n\big)\Big)^{\mathsf{T}}. \end{aligned} \tag{5.3.16}$$

Now, recall the assumption that the Hessian matrix has at most one positive eigenvalue, and if one of the λ_i is indeed positive, then its corresponding eigenvector, $\mathbf{q}_i$, is orthogonal to Δ^{n-1}. Since, for all $\mathbf{x} \in \Delta^{n-1}$, $\mathbf{e}_k - \mathbf{x}$ is a vector parallel to Δ^{n-1}, it is clear that the eigenvector $\mathbf{q}_i$ is also orthogonal to $\mathbf{e}_k - \mathbf{x}$. Moreover, we can assume without loss of generality that the positive eigenvalue is λ_1 with corresponding eigenvector $\mathbf{q}_1 = \pm\big(n^{-\frac{1}{2}}, \ldots, n^{-\frac{1}{2}}\big)^{\mathsf{T}}$, then since $\mathbf{q}_1$ is orthogonal[3] to $\mathbf{e}_k - \mathbf{x}$, we have $\big(\mathbf{e}_k - \mathbf{x}\big)\mathbf{q}_1 = 0$. Define $\big(\mathbf{e}_k - \mathbf{x}\big)\mathbf{q}_i \triangleq p_i$ for all $i = 2, \ldots, n$, then (5.3.16) becomes

$$\begin{aligned} f''(u) &= \big(0, p_2, p_3, \ldots, p_n\big)\boldsymbol{\Lambda}\big(0, p_2, p_3, \ldots, p_n\big)^{\mathsf{T}} \\ &= p_2^2\lambda_2 + p_3^2\lambda_3 + \cdots + p_n^2\lambda_n \qquad (5.3.17) \\ &= \sum_{i=2}^{n} p_i^2\lambda_i \quad \leq \quad 0, \qquad (5.3.18) \end{aligned}$$

since the remaining eigenvalues are all nonpositive, i.e., $\lambda_i \leq 0$ for all $i = 2, \ldots, n$, and hence is composed of eigenvectors of $\mathbf{HG}\big(\mathbf{y}(u)\big)$ that have nonpositive eigenvalues. Furthermore, since $f''(u) \leq 0$, statement (i) of Lemma E.2.8 implies that the function f is concave on $[0, 1)$. Consequently, by statement (i), equation (E.2.18) of Lemma E.2.6, the concavity of f implies that for $x_0 = 0$ and $0 \leq u < 1$, we have

$$f(u) \quad \leq \quad f(0) + uf'(0),$$

and since $f(u) = \mathbf{G}\big(\mathbf{y}(u)\big)$, where $\mathbf{G}$ is a positive function by definition of a generating function, we have for $0 \leq u < 1$,

$$0 \quad < \quad f(0) + uf'(0).$$

[2]A square symmetric $n \times n$ matrix $\mathbf{A}$ with n orthogonal eigenvectors $\mathbf{q}_i$, for $i = 1, 2, \ldots, n$ can be factorised as $\mathbf{A} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{-1} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{\mathsf{T}}$, where $\mathbf{Q}$ is the square orthogonal (i.e., $\mathbf{Q}^{-1} = \mathbf{Q}^{\mathsf{T}}$) $n \times n$ matrix whose ith column is the eigenvector $\mathbf{q}_i$ of $\mathbf{A}$ and $\boldsymbol{\Lambda}$ is the diagonal matrix whose diagonal elements are the corresponding eigenvalues λ_i, for $i = 1, 2, \ldots, n$.

[3]Two vectors, $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ are called *orthogonal* if the dot product $\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n = \mathbf{x}\mathbf{y}^{\mathsf{T}} = 0$.

Now, from (5.3.9), $u = 0$ implies that $\mathbf{y}(0) = \mathbf{x}$, so that $f(0) = \mathbf{G}\big(\mathbf{y}(0)\big) = \mathbf{G}(\mathbf{x})$. It thus follows that this, in conjunction with both (5.3.12) and the fact that $0 \le u < 1$, yields

$$0 \quad \le \quad \mathbf{G}(\mathbf{x}) + u\left(D_k\mathbf{G}(\mathbf{x}) - \sum_{i=1}^n x_i D_i\mathbf{G}(\mathbf{x})\right) \quad < \quad \mathbf{G}(\mathbf{x}) + D_k\mathbf{G}(\mathbf{x}) - \sum_{i=1}^n x_i D_i\mathbf{G}(\mathbf{x}).$$

Hence, by setting $\mathbf{x} := \boldsymbol{\mu}(t) \in \Delta^{n-1}$ for all $t \in [0,T]$, we have

$$\mathbf{G}\big(\boldsymbol{\mu}(t)\big) + D_k\mathbf{G}\big(\boldsymbol{\mu}(t)\big) - \sum_{i=1}^n \mu_i(t)\, D_i\mathbf{G}\big(\boldsymbol{\mu}(t)\big) \quad \ge \quad 0. \tag{5.3.19}$$

Thus, by (5.2.13) of Theorem 5.2.2, for $k = 1, 2, \ldots, n$ and $t \in [0,T]$, we deduce the following

$$\begin{aligned}
\varphi_k(t) &= \left(D_k \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{i=1}^n \mu_i(t)\, D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\right)\mu_k(t) \\
&= \left(\frac{D_k\mathbf{G}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{G}\big(\boldsymbol{\mu}(t)\big)} + 1 - \sum_{i=1}^n \mu_i(t)\, \frac{D_i\mathbf{G}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{G}\big(\boldsymbol{\mu}(t)\big)}\right)\mu_k(t) \\
&= \frac{\mu_k(t)}{\mathbf{G}\big(\boldsymbol{\mu}(t)\big)}\left(D_k\mathbf{G}\big(\boldsymbol{\mu}(t)\big) + \mathbf{G}\big(\boldsymbol{\mu}(t)\big) - \sum_{i=1}^n \mu_i(t)\, D_i\mathbf{G}\big(\boldsymbol{\mu}(t)\big)\right).
\end{aligned}$$

From (5.3.19) and the fact that $0 < \mu_k(t) < 1$ for all $t \in [0,T]$ and $\mathbf{G}$ is positive, we have $\varphi_k(t) \ge 0$, for all $k = 1, 2, \ldots, n$, $t \in [0,T]$, a.s. This proves the first part of the proposition. To prove the second part, recall (5.3.15), i.e., $D^2\mathbf{G}\big(\mathbf{y}(u)\big) = \big(D_{ij}\mathbf{G}\big(\mathbf{y}(u)\big)\big)_{1\le i,j\le n} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^\mathsf{T}$, so that for $i, j = 1, 2, \ldots, n$, we have

$$D_{ij}\mathbf{G}\big(\mathbf{y}(u)\big) \quad = \quad \sum_{k=1}^n \lambda_k q_{ki} q_{kj},$$

and for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$\begin{aligned}
\mathbf{u}\Big(D_{ij}\mathbf{G}\big(\mathbf{y}(u)\big)\Big)\mathbf{v}^\mathsf{T} &= \mathbf{u}\mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^\mathsf{T}\mathbf{v}^\mathsf{T} \\
&= \sum_{k=1}^n \lambda_k \mathbf{u}\mathbf{q}_k\mathbf{q}_k^\mathsf{T}\mathbf{v}^\mathsf{T} \\
&= \sum_{k=1}^n \lambda_k \sum_{i,j=1}^n u_i q_{ki} q_{kj} v_j \\
&= \sum_{k=1}^n \lambda_k \sum_{i,j=1}^n q_{ki} q_{kj} u_i v_j.
\end{aligned}$$

Therefore,

$$\sum_{i,j=1}^n u_i D_{ij}\mathbf{G}\big(\mathbf{y}(u)\big) v_j \quad = \quad \sum_{i,j=1}^n D_{ij}\mathbf{G}\big(\mathbf{y}(u)\big) u_i v_j \quad = \quad \sum_{k=1}^n \lambda_k \sum_{i,j=1}^n q_{ki} q_{kj} u_i v_j.$$

It follows that for $\mathbf{y}(u) = \boldsymbol{\mu}(t) \in \Delta^{n-1}$ and $\mathbf{u} = \mathbf{v} = \boldsymbol{\mu}(t) \in \mathbb{R}^n$, for all $t \in [0,T]$,

$$\sum_{i,j=1}^n D_{ij}\mathbf{G}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t) \quad = \quad \sum_{k=1}^n \lambda_k \sum_{i,j=1}^n q_{ki} q_{kj} \mu_i(t)\mu_j(t)\tau_{ij}(t). \tag{5.3.20}$$

Now, since we assumed that the only positive eigenvalue is λ_1 which corresponds to the eigenvector $\mathbf{q}_1 = \pm\big(n^{-\frac{1}{2}}, \ldots, n^{-\frac{1}{2}}\big)^\mathsf{T}$ (i.e., $q_{1i} = \pm n^{-\frac{1}{2}}$ for all $i = 1, 2, \ldots, n$), orthogonal to Δ^{n-1}, we have that $\mathbf{q}_1$ is also orthogonal to $\boldsymbol{\mu}(t)$. Moreover, Lemma 2.4.2 in conjunction with Lemma 2.12.7 imply that a.s., for all $t \in [0,T]$, $\boldsymbol{\tau}(t)$ is positive semidefinite, with its null space spanned by $\boldsymbol{\mu}(t)$, i.e., we have $\boldsymbol{\mu}(t)\boldsymbol{\tau}(t)\boldsymbol{\mu}^\mathsf{T}(t) = 0$, a.s. for all $t \in [0,T]$. Therefore,

$$\sum_{i,j=1}^n q_{1i} q_{1j} \mu_i(t)\mu_j(t)\tau_{ij}(t) \quad = \quad \frac{1}{n}\sum_{i,j=1}^n \mu_i(t)\tau_{ij}(t)\mu_j(t) \quad = \quad \frac{1}{n}\boldsymbol{\mu}(t)\boldsymbol{\tau}(t)\boldsymbol{\mu}^\mathsf{T}(t) \quad = \quad 0. \tag{5.3.21}$$

For $k = 2, \ldots, n$, define

$$\mathfrak{Q}_k \quad = \quad \mathrm{diag}(\mathbf{q}_k) \quad = \quad \begin{bmatrix} q_{k1} & 0 & \cdots & 0 \\ 0 & q_{k2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & q_{kn} \end{bmatrix},$$

then

$$\begin{aligned} \sum_{i,j=1}^{n} q_{ki} q_{kj} \mu_i(t) \mu_j(t) \tau_{ij}(t) \quad &= \quad \sum_{i,j=1}^{n} \mu_i(t) q_{ki} \tau_{ij}(t) q_{kj} \mu_j(t) \quad = \quad \Big(\boldsymbol{\mu}(t)\mathfrak{Q}_k\Big)\boldsymbol{\tau}(t)\Big(\mathfrak{Q}_k \boldsymbol{\mu}^{\mathsf{T}}(t)\Big) \\ &= \quad \Big(\boldsymbol{\mu}(t)\mathfrak{Q}_k\Big)\boldsymbol{\tau}(t)\Big(\boldsymbol{\mu}(t)\mathfrak{Q}_k\Big)^{\mathsf{T}}. \end{aligned}$$

Now, since $\boldsymbol{\tau}(t)$ is positive semidefinite, we know that $\mathbf{x}\boldsymbol{\tau}(t)\mathbf{x}^{\mathsf{T}} \geq 0$, for all $\mathbf{x} \in \mathbb{R}^n$. Consequently, since $\boldsymbol{\mu}(t)\mathfrak{Q}_k \in \mathbb{R}^n$, we have $\big(\boldsymbol{\mu}(t)\mathfrak{Q}_k\big)\boldsymbol{\tau}(t)\big(\boldsymbol{\mu}(t)\mathfrak{Q}_k\big)^{\mathsf{T}} \geq 0$. Hence, for $k = 2, \ldots, n$, we have

$$\sum_{i,j=1}^{n} q_{ki} q_{kj} \mu_i(t) \mu_j(t) \tau_{ij}(t) \quad = \quad \Big(\boldsymbol{\mu}(t)\mathfrak{Q}_k\Big)\boldsymbol{\tau}(t)\Big(\boldsymbol{\mu}(t)\mathfrak{Q}_k\Big)^{\mathsf{T}} \quad \geq \quad 0. \tag{5.3.22}$$

By hypothesis, none of the eigenvalues $\lambda_2, \ldots, \lambda_n$ are positive, i.e., $\lambda_i \leq 0$ for $i = 2, \ldots, n$. Thus, (5.3.21) together with (5.3.22) implies that (5.3.20) becomes

$$\sum_{i,j=1}^{n} D_{ij} \mathbf{G}\big(\boldsymbol{\mu}(t)\big) \mu_i(t) \mu_j(t) \tau_{ij}(t) \quad \leq \quad 0. \tag{5.3.23}$$

This inequality in conjunction with (5.2.15) of Theorem 5.2.2,

$$\frac{d\Theta(t)}{dt} \quad = \quad \frac{-1}{2\mathbf{G}\big(\boldsymbol{\mu}(t)\big)} \sum_{i,j=1}^{n} D_{ij} \mathbf{G}\big(\boldsymbol{\mu}(t)\big) \mu_i(t) \mu_j(t) \tau_{ij}(t), \tag{5.3.24}$$

and the fact that $\mathbf{G}\big(\boldsymbol{\mu}(t)\big)$ is a positive function, implies that

$$\mathfrak{g}(t) \quad = \quad \Theta'(t) \quad = \quad \frac{d\Theta(t)}{dt} \quad \geq \quad 0.$$

It follows that the drift process Θ is nondecreasing for all $t \in [0, T]$, a.s. If for all $\mathbf{x} \in \Delta^{n-1}$, $\mathrm{rank}\big(\mathbf{HG}(\mathbf{x})\big) > 1$, then at least one of the eigenvalues $\lambda_2, \ldots, \lambda_n$ is strictly negative, i.e., $\lambda_k < 0$ for at least one $k \geq 2$. Since the eigenvectors $\mathbf{q}_k$ are pairwise orthogonal, only $\mathbf{q}_1$ can be orthogonal to Δ^{n-1}, so equation (5.3.22) will be strictly positive for at least one $k \geq 2$, i.e., we have for at least one $k \geq 2$,

$$\sum_{i,j=1}^{n} q_{ki} q_{kj} \mu_i(t) \mu_j(t) \tau_{ij}(t) \quad > \quad 0.$$

So, for at least one $k \geq 2$, λ_k is strictly negative ($\lambda_k < 0$) and we have

$$\lambda_k \sum_{i,j=1}^{n} q_{ki} q_{kj} \mu_i(t) \mu_j(t) \tau_{ij}(t) \quad < \quad 0,$$

and for the remaining k, the λ_k are nonpositive ($\lambda_k \leq 0$), and we have

$$\lambda_k \sum_{i,j=1}^{n} q_{ki} q_{kj} \mu_i(t) \mu_j(t) \tau_{ij}(t) \quad \leq \quad 0.$$

The strict inequality above implies that the expression on the right-hand side of (5.3.20) can never be equal to 0, and is thus always strictly less than 0,

$$\sum_{i,j=1}^{n} D_{ij} \mathbf{G}\big(\boldsymbol{\mu}(t)\big) \mu_i(t) \mu_j(t) \tau_{ij}(t) \quad = \quad \sum_{k=1}^{n} \lambda_k \sum_{i,j=1}^{n} q_{ki} q_{kj} \mu_i(t) \mu_j(t) \tau_{ij}(t) \quad < \quad 0. \tag{5.3.25}$$

This inequality in conjunction with (5.2.15) of Theorem 5.2.2, and the fact that $\mathbf{G}\big(\boldsymbol{\mu}(t)\big)$ is a strictly positive function, implies that

$$\mathfrak{g}(t) \;=\; \Theta'(t) \;=\; \frac{d\Theta(t)}{dt} \;>\; 0.$$

It follows that the drift process Θ is strictly increasing for all $t \in [0,T]$, a.s. ■

5.4 Examples of Portfolio Generating Functions and their Functionally Generated Portfolios

5.4.1 The Constant-Weighted Portfolio

Consider the following function, $\mathbf{G}(\boldsymbol{\mu}) \equiv \mathbf{S}^{\mathrm{C}}(\boldsymbol{\mu}) = \big\{\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big), t \in [0,T]\big\}$, $\mathbf{G} \equiv \mathbf{S}^{\mathrm{C}} : \Delta^{n-1} \to (0,\infty)$, introduced in (4.6.108) of Definition 4.6.15, where $p_1,\ldots,p_n$ are constants and $p_1 + \cdots + p_n = 1$,

$$\mathbf{G}\big(\boldsymbol{\mu}(t)\big) \;\equiv\; \mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big) \;=\; \big(\mu_1(t)\big)^{p_1} \cdots \big(\mu_n(t)\big)^{p_n} \;=\; \prod_{k=1}^{n} \big(\mu_k(t)\big)^{p_k}, \qquad t \in [0,T]. \tag{5.4.1}$$

From equation (4.6.110), we obtain for all $i = 1,2,\ldots,n$, and $t \in [0,T]$,

$$D_i \log \mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big) \;=\; \frac{D_i \mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)} \;=\; \frac{p_i}{\mu_i(t)}.$$

Therefore, from (5.2.13) of Theorem 5.2.2, with $\mathbf{G}(\mathbf{x}) = \mathbf{S}^{\mathrm{C}}(\mathbf{x})$, we obtain the following weights for the portfolio generated by $\mathbf{S}^{\mathrm{C}}$,

$$\begin{aligned}
\varphi_i^{\mathsf{constant}}(t) &= \left(D_i \log \mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{j=1}^{n} \mu_j(t)\, D_j \log \mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big) \right) \mu_i(t) \\
&= \left(\frac{p_i}{\mu_i(t)} + 1 - \sum_{j=1}^{n} \mu_j(t) \left[\frac{p_j}{\mu_j(t)} \right] \right) \mu_i(t) \\
&= \left(\frac{p_i}{\mu_i(t)} + 1 - \sum_{j=1}^{n} p_j \right) \mu_i(t) \\
&= \mu_i(t) \left(\frac{p_i}{\mu_i(t)} + 1 - 1 \right) \\
&= \mu_i(t) \left(\frac{p_i}{\mu_i(t)} \right) \\
&= p_i,
\end{aligned}$$

since $\sum_{i=1}^{n} p_i = 1$. Hence, for all $i = 1,2,\ldots,n$ and $t \in [0,T]$, the above expression simplifies to

$$\varphi_i^{\mathsf{constant}}(t) \;=\; p_i, \tag{5.4.2}$$

which gives the weights corresponding to the generating function $\mathbf{S}^{\mathrm{C}}$. Thus, the weights corresponding to the generating function $\mathbf{S}^{\mathrm{C}}$ can be formalised in the following definition.

Definition 5.4.1 (Constant-Weighted Portfolio)**.** *Let μ be the market portfolio. The portfolio process $\varphi^{\mathsf{constant}} = \big\{\varphi^{\mathsf{constant}}(t) = \big(\varphi_1^{\mathsf{constant}}(t), \varphi_2^{\mathsf{constant}}(t), \ldots, \varphi_n^{\mathsf{constant}}(t)\big), t \in [0,T]\big\}$, with weights defined by* (5.4.2)*, i.e.,*

$$\varphi_i^{\mathsf{constant}}(t) \;\triangleq\; p_i, \tag{5.4.3}$$

and with $p_i \in \mathbb{R}$, for all $i = 1,2,\ldots,n$, and $t \in [0,T]$, such that $\sum_{i=1}^{n} p_i = 1$, is called the ***constant-weighted portfolio (process)****.*

Thus, the generating function $\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big) = \big(\mu_1(t)\big)^{p_1} \cdots \big(\mu_n(t)\big)^{p_n}$, $\mathbf{S}^{\mathrm{C}} : \Delta^{n-1} \to (0,\infty)$, generates a portfolio with constant weights for all $t \in [0,T]$, i.e., the *constant-weighted portfolio*. It can be easily verified that $\varphi^{\mathsf{constant}}$ satisfies the requirements of Definition 2.2.16, since the weights are, of course, bounded and satisfy the following

$$\sum_{i=1}^{n} \varphi_i^{\mathsf{constant}}(t) \quad = \quad \sum_{i=1}^{n} p_i \quad = \quad 1.$$

Now, from (4.6.111), (4.6.112) and (4.6.113), we have for all $i,j = 1,2,\ldots,n$, and $t \in [0,T]$,

$$D_{ij}\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big) \quad = \quad \begin{cases} p_i(p_i-1)\,\dfrac{\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} & \text{if} \quad i=j \\ p_i p_j\,\dfrac{\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)\mu_j(t)} & \text{if} \quad i \neq j \end{cases} \quad = \quad \begin{cases} p_i^2\,\dfrac{\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} - p_i\,\dfrac{\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} & \text{if} \quad i=j, \\ p_i p_j\,\dfrac{\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)\mu_j(t)} & \text{if} \quad i \neq j. \end{cases} \tag{5.4.4}$$

The above expression in conjunction with (5.2.14) of Theorem 5.2.2 with $\mathbf{G}(\mathbf{x}) = \mathbf{S}^{\mathrm{C}}(\mathbf{x})$, yields the following associated drift process which corresponds to the constant-weighted portfolio process

$$\begin{aligned}
d\Theta^{\mathsf{constant}}(t) \quad &= \quad d\Theta_{\varphi^{\mathsf{constant}}}(t) \\
&= \quad \frac{-1}{2\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i,j=1}^{n} D_{ij}\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt \\
&= \quad \frac{-1}{2\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)} \left(\sum_{i=1}^{n} \left[p_i(p_i-1)\,\frac{\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} \right] \mu_i^2(t)\tau_{ii}(t) + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \left[p_i p_j\,\frac{\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)\mu_j(t)} \right] \mu_i(t)\mu_j(t)\tau_{ij}(t) \right) dt \\
&= \quad \frac{-1}{2\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)} \left(\sum_{i=1}^{n} \left[p_i^2\,\frac{\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} - p_i\,\frac{\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} \right] \mu_i^2(t)\tau_{ii}(t) + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \left[p_i p_j\,\frac{\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)\mu_j(t)} \right] \mu_i(t)\mu_j(t)\tau_{ij}(t) \right) dt \\
&= \quad -\frac{1}{2} \left(\sum_{i=1}^{n} (p_i^2 - p_i)\tau_{ii}(t) + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} p_i p_j \tau_{ij}(t) \right) dt \\
&= \quad -\frac{1}{2} \left(\sum_{i=1}^{n} p_i^2\tau_{ii}(t) - \sum_{i=1}^{n} p_i\tau_{ii}(t) + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} p_i p_j \tau_{ij}(t) \right) dt \\
&= \quad \frac{1}{2} \left(\sum_{i=1}^{n} p_i\tau_{ii}(t) - \sum_{i=1}^{n} p_i^2\tau_{ii}(t) - \sum_{\substack{i,j=1 \\ i\neq j}}^{n} p_i p_j \tau_{ij}(t) \right) dt \\
&= \quad \frac{1}{2} \left(\sum_{i=1}^{n} p_i\tau_{ii}(t) - \left[\sum_{i=1}^{n} p_i^2\tau_{ii}(t) + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} p_i p_j \tau_{ij}(t) \right] \right) dt \\
&= \quad \frac{1}{2} \left(\sum_{i=1}^{n} p_i\tau_{ii}(t) - \sum_{i,j=1}^{n} p_i p_j \tau_{ij}(t) \right) dt,
\end{aligned}$$

which, by (5.4.2), becomes

$$d\Theta^{\mathsf{constant}}(t) \quad = \quad d\Theta_{\varphi^{\mathsf{constant}}}(t) \quad = \quad \frac{1}{2} \left(\sum_{i=1}^{n} \varphi_i^{\mathsf{constant}}(t)\tau_{ii}(t) - \sum_{i,j=1}^{n} \varphi_i^{\mathsf{constant}}(t)\tau_{ij}(t)\varphi_j^{\mathsf{constant}}(t) \right) dt.$$

Therefore, by the numéraire invariance property of the excess growth rate of the constant-weighted portfolio (2.4.26) of Lemma 2.4.5, we arrive at the following

$$d\Theta^{\mathsf{constant}}(t) \quad = \quad d\Theta_{\varphi^{\mathsf{constant}}}(t) \quad = \quad \gamma^*_{\varphi^{\mathsf{constant}}}(t)\,dt, \tag{5.4.5}$$

equivalently,

$$\mathfrak{g}^{\mathsf{constant}}(t) \;=\; \mathfrak{g}_{\varphi^{\mathsf{constant}}}(t) \;=\; \gamma^*_{\varphi^{\mathsf{constant}}}(t). \tag{5.4.6}$$

Furthermore, by (5.4.5) in conjunction with Theorem 5.2.2 and equation (5.2.2), the logarithmic return process of the constant-weighted portfolio $\varphi^{\mathsf{constant}}$, relative to the market portfolio satisfies, for all $t \in [0,T]$, a.s.,

$$\begin{aligned}
d\log\Big(Z_{\varphi^{\mathsf{constant}}}(t)/Z_\mu(t)\Big) &= d\log \mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(t)\big) + \gamma^*_{\varphi^{\mathsf{constant}}}(t)\,dt && (5.4.7)\\
&= d\log\Big(\big(\mu_1(t)\big)^{p_1}\cdots\big(\mu_n(t)\big)^{p_n}\Big) + \gamma^*_{\varphi^{\mathsf{constant}}}(t)\,dt && (5.4.8)\\
&= d\left(\sum_{i=1}^n \log\big(\mu_i(t)\big)^{p_i}\right) + \gamma^*_{\varphi^{\mathsf{constant}}}(t)\,dt\\
&= d\left(\sum_{i=1}^n p_i\log\mu_i(t)\right) + \gamma^*_{\varphi^{\mathsf{constant}}}(t)\,dt\\
&= \sum_{i=1}^n d\Big(p_i\log\mu_i(t)\Big) + \gamma^*_{\varphi^{\mathsf{constant}}}(t)\,dt\\
&= \sum_{i=1}^n p_i\,d\log\mu_i(t) + \gamma^*_{\varphi^{\mathsf{constant}}}(t)\,dt. && (5.4.9)
\end{aligned}$$

Thus, by appealing to (2.12.63) of Proposition 2.12.8, we have

$$d\log\Big(Z_{\varphi^{\mathsf{constant}}}(t)/Z_\mu(t)\Big) \;=\; \sum_{i=1}^n \varphi_i^{\mathsf{constant}}(t)\,d\log\mu_i(t) + \gamma^*_{\varphi^{\mathsf{constant}}}(t)\,dt,$$

where $\varphi_i^{\mathsf{constant}}(t) = p_i$ for all $i = 1,2,\ldots,n$, which demonstrates that $\mathbf{S}^{\mathrm{C}}$ does indeed generate the constant-weighted portfolio. Alternatively, by (5.2.8), for all $T \in [0,\infty)$, the relative performance of the constant-weighted portfolio $\varphi^{\mathsf{constant}}$, can be represented in integral form as

$$\begin{aligned}
\log\left(\frac{Z_{\varphi^{\mathsf{constant}}}(T)}{Z_\mu(T)}\right) &= \log\left(\frac{\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(0)\big)}\right) + \int_0^T \gamma^*_{\varphi^{\mathsf{constant}}}(t)\,dt && (5.4.10)\\
&= \log\left(\frac{\big(\mu_1(T)\big)^{p_1}\cdots\big(\mu_n(T)\big)^{p_n}}{\big(\mu_1(0)\big)^{p_1}\cdots\big(\mu_n(0)\big)^{p_n}}\right) + \int_0^T \gamma^*_{\varphi^{\mathsf{constant}}}(t)\,dt && (5.4.11)\\
&= \left(\sum_{i=1}^n \log\big(\mu_i(T)\big)^{p_i} - \sum_{i=1}^n \log\big(\mu_i(0)\big)^{p_i}\right) + \int_0^T \gamma^*_{\varphi^{\mathsf{constant}}}(t)\,dt\\
&= \left(\sum_{i=1}^n p_i\log\mu_i(T) - \sum_{i=1}^n p_i\log\mu_i(0)\right) + \int_0^T \gamma^*_{\varphi^{\mathsf{constant}}}(t)\,dt\\
&= \sum_{i=1}^n \Big(p_i\log\mu_i(T) - p_i\log\mu_i(0)\Big) + \int_0^T \gamma^*_{\varphi^{\mathsf{constant}}}(t)\,dt\\
&= \sum_{i=1}^n p_i\Big(\log\mu_i(T) - \log\mu_i(0)\Big) + \int_0^T \gamma^*_{\varphi^{\mathsf{constant}}}(t)\,dt\\
&= \sum_{i=1}^n p_i\,\log\left(\frac{\mu_i(T)}{\mu_i(0)}\right) + \int_0^T \gamma^*_{\varphi^{\mathsf{constant}}}(t)\,dt, && (5.4.12)
\end{aligned}$$

if $Z_{\varphi^{\mathsf{constant}}}(0) = Z_\mu(0)$, otherwise the integral representation is given by

$$\begin{aligned}
\log\left(\frac{Z_{\varphi^{\mathsf{constant}}}(T)}{Z_\mu(T)}\right) &= \log\left(\frac{Z_{\varphi^{\mathsf{constant}}}(0)}{Z_\mu(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(0)\big)}\right) + \int_0^T \gamma^*_{\varphi^{\mathsf{constant}}}(t)\,dt && (5.4.13)\\
&= \log\left(\frac{Z_{\varphi^{\mathsf{constant}}}(0)}{Z_\mu(0)}\right) + \sum_{i=1}^n p_i\,\log\left(\frac{\mu_i(T)}{\mu_i(0)}\right) + \int_0^T \gamma^*_{\varphi^{\mathsf{constant}}}(t)\,dt, && (5.4.14)
\end{aligned}$$

or, as per equation (5.2.7), we have

$$\log\left(\frac{Z_{\varphi^{\mathsf{constant}}}(T)}{Z_{\varphi^{\mathsf{constant}}}(0)}\right) = \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{C}}\big(\boldsymbol{\mu}(0)\big)}\right) + \int_0^T \gamma^*_{\varphi^{\mathsf{constant}}}(t)\,dt \tag{5.4.15}$$

$$= \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \sum_{i=1}^n p_i \log\left(\frac{\mu_i(T)}{\mu_i(0)}\right) + \int_0^T \gamma^*_{\varphi^{\mathsf{constant}}}(t)\,dt. \tag{5.4.16}$$

5.4.2 The Buy-and-Hold Portfolio

Consider the following function, $\mathbf{G}(\boldsymbol{\mu}) \equiv \mathbf{S}^{\mathrm{BH}}(\boldsymbol{\mu}) = \big\{\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big), t \in [0,T]\big\}$, $\mathbf{G} \equiv \mathbf{S}^{\mathrm{BH}} : \Delta^{n-1} \to (0,\infty)$, introduced in (4.6.87) of Definition 4.6.12,

$$\mathbf{G}\big(\boldsymbol{\mu}(t)\big) \equiv \mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big) = w_1\mu_1(t) + \cdots + w_n\mu_n(t) = \sum_{k=1}^n w_k\mu_k(t), \qquad t \in [0,T], \tag{5.4.17}$$

where $w_1, \ldots, w_n$ are nonnegative constants, i.e., $w_k \geq 0$ for all $k = 1, 2, \ldots, n$, at least one of which is strictly positive. From equation (4.6.89), we obtain for all $i = 1, 2, \ldots, n$, and $t \in [0,T]$,

$$D_i \log \mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big) = \frac{D_i\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)} = \frac{w_i}{\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)}.$$

Therefore, from (5.2.13) of Theorem 5.2.2, with $\mathbf{G}(\mathbf{x}) = \mathbf{S}^{\mathrm{BH}}(\mathbf{x})$, we obtain the following weights for the portfolio generated by the buy-and-hold function $\mathbf{S}^{\mathrm{BH}}$,

$$\begin{aligned}
\varphi_i^{\mathsf{bh}}(t) &= \left(D_i \log \mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{j=1}^n \mu_j(t)\, D_j \log \mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)\right)\mu_i(t) \\
&= \left(\frac{w_i}{\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)} + 1 - \sum_{j=1}^n \mu_j(t)\left[\frac{w_j}{\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)}\right]\right)\mu_i(t) \\
&= \left(\frac{w_i}{\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)} + 1 - \sum_{j=1}^n \frac{w_j\mu_j(t)}{\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)}\right)\mu_i(t) \\
&= \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)}\left(w_i + \mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big) - \sum_{j=1}^n w_j\mu_j(t)\right) \\
&= \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)}\left(w_i + \mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big) - \mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)\right) \\
&= \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)}\,(w_i) \\
&= \frac{w_i\mu_i(t)}{\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)}.
\end{aligned}$$

Hence, for all $i = 1, 2, \ldots, n$ and $t \in [0,T]$, we have the weights corresponding to the generating function $\mathbf{S}^{\mathrm{BH}}$,

$$\varphi_i^{\mathsf{bh}}(t) = \frac{w_i\mu_i(t)}{\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)} = \frac{w_i\mu_i(t)}{\sum_{j=1}^n w_j\mu_j(t)} = \frac{w_i\mu_i(t)}{w_1\mu_1(t) + \cdots + w_n\mu_n(t)}. \tag{5.4.18}$$

Thus, the weights corresponding to the generating function $\mathbf{S}^{\mathrm{BH}}$ can be formalised in the following definition.

Definition 5.4.2 (Buy-and-Hold Portfolio). *Let μ be the market portfolio. The portfolio process $\varphi^{\mathsf{bh}} = \big\{\varphi^{\mathsf{bh}}(t) = \big(\varphi_1^{\mathsf{bh}}(t), \varphi_2^{\mathsf{bh}}(t), \ldots, \varphi_n^{\mathsf{bh}}(t)\big), t \in [0,T]\big\}$, with weights defined by* (5.4.18)*, i.e.,*

$$\varphi_i^{\mathsf{bh}}(t) \triangleq \frac{w_i\mu_i(t)}{\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)}, \tag{5.4.19}$$

for all $i = 1, 2, \ldots, n$, and $t \in [0,T]$, is called the ***buy-and-hold portfolio (process)****.*

Thus, the generating function $\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big) = w_1\mu_1(t) + \cdots + w_n\mu_n(t)$, $\mathbf{S}^{\mathrm{BH}} : \Delta^{n-1} \to (0,\infty)$, generates the *buy-and-hold portfolio*, that holds w_i shares of the ith stock, i.e., it generates the passive buy-and-hold strategy that buys at time $t = 0$ and holds until time $t = T$, a fixed number of shares w_i in each stock. The market portfolio corresponds to the special case where $w_1 = \cdots = w_n = w$ of equal numbers of shares across stocks. It can be easily verified that φ^{bh} satisfies the requirements of Definition 2.2.16, since we know that the market weights are bounded on $[0,\infty)$ and the $\mathbf{S}^{\mathrm{BH}}$ process, $\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)$, is also bounded on $[0,\infty)$ according to the bounds derived in (4.6.88), i.e., $0 \le w_{(n)} < \mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big) \le w_{(1)}$. Moreover, the weights, (5.4.19), satisfy the following

$$\begin{aligned}
\sum_{i=1}^{n} \varphi_i^{\mathsf{bh}}(t) &= \sum_{i=1}^{n} \frac{w_i\mu_i(t)}{\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)} \\
&= \frac{1}{\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^{n} w_i\mu_i(t) \\
&= \frac{\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)} \\
&= 1.
\end{aligned}$$

Now, from (4.6.90), we have for all $i, j = 1, 2, \ldots, n$, and $t \in [0, T]$,

$$D_{ij}\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big) = \frac{\partial^2 \mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)}{\partial\mu_i(t)\partial\mu_j(t)} = 0. \tag{5.4.20}$$

The above expression in conjunction with (5.2.14) of Theorem 5.2.2 with $\mathbf{G}(\mathbf{x}) = \mathbf{S}^{\mathrm{BH}}(\mathbf{x})$, yields the following drift process corresponding to the buy-and-hold function, and thus to the buy-and-hold portfolio process

$$d\Theta^{\mathsf{bh}}(t) = d\Theta_{\varphi^{\mathsf{bh}}}(t) = \frac{-1}{2\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i,j=1}^{n} D_{ij}\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt = 0, \tag{5.4.21}$$

equivalently, for all $t \in [0, T]$, we have

$$\mathfrak{g}^{\mathsf{bh}}(t) = \mathfrak{g}_{\varphi^{\mathsf{bh}}}(t) = 0. \tag{5.4.22}$$

Furthermore, by (5.4.21) in conjunction with Theorem 5.2.2 and equation (5.2.2), the logarithmic return process of the buy-and-hold portfolio φ^{bh}, relative to the market portfolio satisfies, for all $t \in [0, T]$, a.s.,

$$d\log\Big(Z_{\varphi^{\mathsf{bh}}}(t)/Z_\mu(t)\Big) = d\log\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(t)\big), \tag{5.4.23}$$

or, by (5.2.8), for all $T \in [0,\infty)$, the relative performance of the buy-and-hold portfolio φ^{bh}, can alternatively be represented in integral form as

$$\log\left(\frac{Z_{\varphi^{\mathsf{bh}}}(T)}{Z_\mu(T)}\right) = \log\left(\frac{\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(0)\big)}\right), \tag{5.4.24}$$

if $Z_{\varphi^{\mathsf{bh}}}(0) = Z_\mu(0)$, otherwise the integral representation is given by

$$\log\left(\frac{Z_{\varphi^{\mathsf{bh}}}(T)}{Z_\mu(T)}\right) = \log\left(\frac{Z_{\varphi^{\mathsf{bh}}}(0)}{Z_\mu(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(0)\big)}\right), \tag{5.4.25}$$

or, as per equation (5.2.7), we have

$$\log\left(\frac{Z_{\varphi^{\mathsf{bh}}}(T)}{Z_{\varphi^{\mathsf{bh}}}(0)}\right) = \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}(0)\big)}\right). \tag{5.4.26}$$

5.4.3 The Weighted-Average Capitalisation Generated Portfolio

The *weighted-average capitalisation* of the market is used sometimes as a measure of the concentration of capital in the market. The value of this weighted-average capitalisation would be

$$\sum_{i=1}^{n} \mu_i(t) X_i(t), \qquad t \in [0,T].$$

Thus, the weighted average of the capitalisation weights is given by

$$\sum_{i=1}^{n} \mu_i^2(t), \qquad t \in [0,T],$$

which is proportional to the weighted-average capitalisation. Then, the *weighted-average capitalisation*, $\mathbf{S}^{\mathrm{WC}} : \Delta^{n-1} \to (0,\infty)$, is given by the square root of the weighted-average capitalisation weights, and is defined by

$$\mathbf{S}^{\mathrm{WC}}(\mathbf{x}) \;=\; \mathbf{S}^{\mathrm{WC}}(x_1,\ldots,x_n) \;\triangleq\; \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}, \tag{5.4.27}$$

for all $\mathbf{x}$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\mathbf{x} \in \Delta^{n-1}$. The corresponding market measure (the *weighted-average market capitalisation*) is given below by:

Definition 5.4.3 (Weighted-Average Market Capitalisation)**.** *Let μ be the market portfolio. Then the **weighted-average market capitalisation process**, $\mathbf{S}^{\mathrm{WC}}(\boldsymbol{\mu}) = \big\{\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big), t \in [0,T]\big\}$, $\mathbf{S}^{\mathrm{WC}} : \Delta^{n-1} \to (0,\infty)$, is defined by*

$$\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big) \;=\; \mathbf{S}^{\mathrm{WC}}\big(\mu_1(t),\ldots,\mu_n(t)\big) \;\triangleq\; \left(\sum_{i=1}^{n} \mu_i^2(t)\right)^{\frac{1}{2}}, \qquad t \in [0,T], \tag{5.4.28}$$

for all $\boldsymbol{\mu}(t)$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\boldsymbol{\mu}(t) \in \Delta^{n-1}$.

Recall from (5.4.28) of Definition 5.4.3 that the weighted-average market capitalisation process, $\mathbf{G}(\boldsymbol{\mu}) \equiv \mathbf{S}^{\mathrm{WC}}(\boldsymbol{\mu}) = \big\{\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big), t \in [0,T]\big\}$, $\mathbf{G} \equiv \mathbf{S}^{\mathrm{WC}} : \Delta^{n-1} \to (0,\infty)$, is given by

$$\mathbf{G}\big(\boldsymbol{\mu}(t)\big) \;\equiv\; \mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big) \;=\; \left(\sum_{k=1}^{n} \mu_k^2(t)\right)^{\frac{1}{2}}, \qquad t \in [0,T]. \tag{5.4.29}$$

Subsequently, we deduce the following for all $i = 1,2,\ldots,n$,

$$\begin{aligned} D_i \mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big) \;=\; \frac{\partial \mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)}{\partial \mu_i(t)} \;&=\; \mu_i(t)\left(\sum_{k=1}^{n} \mu_k^2(t)\right)^{-\frac{1}{2}} \\ &=\; \frac{\mu_i(t)}{\big(\sum_{k=1}^{n} \mu_k^2(t)\big)^{\frac{1}{2}}} \\ &=\; \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)}. \end{aligned} \tag{5.4.30}$$

It follows that for all $i = 1,2,\ldots,n$, and $t \in [0,T]$,

$$D_i \log \mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big) \;=\; \frac{D_i \mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)} \;=\; \frac{\mu_i(t)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^2}.$$

Therefore, from (5.2.13) of Theorem 5.2.2, with $\mathbf{G}(\mathbf{x}) = \mathbf{S}^{\mathrm{WC}}(\mathbf{x})$, we obtain the following weights for the

portfolio generated by the weighted-average market capitalisation process $\mathbf{S}^{\mathrm{WC}}$,

$$\begin{aligned}
\varphi_i^{\varpi}(t) &= \left(D_i \log \mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{j=1}^{n} \mu_j(t)\, D_j \log \mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\right) \mu_i(t) \\
&= \left(\frac{\mu_i(t)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^2} + 1 - \sum_{j=1}^{n} \mu_j(t) \left[\frac{\mu_j(t)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^2}\right]\right) \mu_i(t) \\
&= \left(\frac{\mu_i(t)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^2} + 1 - \sum_{j=1}^{n} \frac{\mu_j^2(t)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^2}\right) \mu_i(t) \\
&= \frac{\mu_i(t)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^2} \left(\mu_i(t) + \Big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\Big)^2 - \sum_{j=1}^{n} \mu_j^2(t)\right) \\
&= \frac{\mu_i(t)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^2} \left(\mu_i(t) + \Big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\Big)^2 - \Big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\Big)^2\right) \\
&= \frac{\mu_i(t)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^2} \Big(\mu_i(t)\Big) \\
&= \frac{\mu_i^2(t)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^2},
\end{aligned}$$

since $\sum_{k=1}^{n} \mu_k^2(t) = \big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^2$. Hence, for all $i = 1, 2, \ldots, n$ and $t \in [0, T]$, the above expression simplifies to

$$\varphi_i^{\varpi}(t) = \frac{\mu_i^2(t)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^2} = \frac{\mu_i^2(t)}{\sum_{j=1}^{n} \mu_j^2(t)} = \frac{\mu_i^2(t)}{\mu_1^2(t) + \cdots + \mu_n^2(t)}, \tag{5.4.31}$$

which gives the weights corresponding to the generating function $\mathbf{S}^{\mathrm{WC}}$. Thus, the weights corresponding to the generating function $\mathbf{S}^{\mathrm{WC}}$ can be formalised in the following definition.

Definition 5.4.4 (Weighted-Average Capitalisation Portfolio)**.** *Let μ be the market portfolio. The portfolio process $\varphi^{\varpi} = \big\{\varphi^{\varpi}(t) = \big(\varphi_1^{\varpi}(t), \varphi_2^{\varpi}(t), \ldots, \varphi_n^{\varpi}(t)\big), t \in [0, T]\big\}$, with weights defined by* (5.4.31)*, i.e.,*

$$\varphi_i^{\varpi}(t) \triangleq \frac{\mu_i^2(t)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^2} \tag{5.4.32}$$

$$= \left(\frac{\mu_i(t)}{\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)}\right)^2, \tag{5.4.33}$$

for all $i = 1, 2, \ldots, n$, and $t \in [0, T]$, is called the ***weighted-average capitalisation generated portfolio (process)****.*

Thus, the generating function $\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)$, $\mathbf{S}^{\mathrm{WC}} : \Delta^{n-1} \to (0, \infty)$, generates the *weighted-average capitalisation portfolio*. It can be easily verified that φ^{ϖ} satisfies the requirements of Definition 2.2.16, since we know that the market weights are bounded on $[0, \infty)$ and the $\mathbf{S}^{\mathrm{WC}}$ process, $\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)$, is also bounded on $[0, \infty)$. Moreover, the weights satisfy the following

$$\begin{aligned}
\sum_{i=1}^{n} \varphi_i^{\varpi}(t) &= \sum_{i=1}^{n} \frac{\mu_i^2(t)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^2} \\
&= \frac{1}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^2} \sum_{i=1}^{n} \mu_i^2(t) \\
&= \frac{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^2}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^2} \\
&= 1.
\end{aligned}$$

The weight ratios, for $i = 1, 2, \ldots, n$, satisfy

$$\frac{\varphi_i^{\varpi}(t)}{\mu_i(t)} = \frac{\mu_i(t)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^2}, \qquad t \in [0, T], \tag{5.4.34}$$

which increase with increasing market weight $\mu_i(t)$, i.e., if $\mu_i(t) \geq \mu_j(t)$ for some $i > j$, then we have for all $t \in [0, T]$,

$$\frac{\mu_i(t)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^2} \geq \frac{\mu_j(t)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^2}.$$

Thus, if $\mu_i(t) \geq \mu_j(t)$ for some $i > j$, (5.4.34) implies

$$\frac{\varphi_i^{\varpi}(t)}{\mu_i(t)} \geq \frac{\varphi_j^{\varpi}(t)}{\mu_j(t)}.$$

This means that relative to the market, φ^{ϖ} is overweighted in the larger stocks and underweighted in the smaller stocks [Fernholz (2002)]. From (5.4.30), we have for all $i = 1, 2, \ldots, n$,

$$\begin{aligned} D_{ii}\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big) &= \frac{\partial^2 \mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)}{\partial \mu_i(t)^2} = \frac{1}{\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)} - \frac{\mu_i(t)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^2} D_i \mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big) \\ &= \frac{1}{\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)} - \frac{\mu_i(t)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^2} \left[\frac{\mu_i(t)}{\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)}\right] \\ &= \frac{1}{\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)} - \frac{\mu_i^2(t)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^3} \\ &= \frac{1}{\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)} \left[1 - \frac{\mu_i^2(t)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^2}\right], \end{aligned} \tag{5.4.35}$$

which by (5.4.32) of Definition 5.4.4 reduces to

$$\begin{aligned} D_{ii}\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big) &= \frac{\partial^2 \mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)}{\partial \mu_i(t)^2} = \frac{1}{\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)} \left[1 - \varphi_i^{\varpi}(t)\right] \\ &= \frac{1 - \varphi_i^{\varpi}(t)}{\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)}. \end{aligned} \tag{5.4.36}$$

Moreover, for all $i \neq j$, $i, j = 1, 2, \ldots, n$, from (5.4.30) we obtain

$$\begin{aligned} D_{ij}\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big) &= \frac{\partial^2 \mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)}{\partial \mu_i(t) \partial \mu_j(t)} = \frac{-\mu_i(t)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^2} D_j \mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big) \\ &= \frac{-\mu_i(t)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^2} \left[\frac{\mu_j(t)}{\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)}\right] \\ &= \frac{-\mu_i(t)\mu_j(t)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^3}. \end{aligned} \tag{5.4.37}$$

Therefore, from (5.4.36) and (5.4.37), we have for all $i, j = 1, 2, \ldots, n$, and $t \in [0, T]$,

$$D_{ij}\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big) = \begin{cases} \dfrac{1 - \varphi_i^{\varpi}(t)}{\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)} & \text{if} \quad i = j, \\ \dfrac{-\mu_i(t)\mu_j(t)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^3} & \text{if} \quad i \neq j. \end{cases} \tag{5.4.38}$$

The above expression in conjunction with (5.2.14) of Theorem 5.2.2 with $\mathbf{G}(\mathbf{x}) = \mathbf{S}^{\mathrm{WC}}(\mathbf{x})$, yields the following

associated drift process that corresponds to the weighted-average capitalisation portfolio process

$$
\begin{aligned}
d\Theta^{\varpi}(t) &= d\Theta_{\varphi^{\varpi}}(t) \\
&= \frac{-1}{2\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i,j=1}^{n} D_{ij}\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt \\
&= \frac{-1}{2\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)} \left(\sum_{i=1}^{n} \left[\frac{1-\varphi_i^{\varpi}(t)}{\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)} \right] \mu_i^2(t)\tau_{ii}(t) + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \left[\frac{-\mu_i(t)\mu_j(t)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^3} \right] \mu_i(t)\mu_j(t)\tau_{ij}(t) \right) dt \\
&= -\frac{1}{2} \left(\sum_{i=1}^{n} \left[\frac{\mu_i^2(t)\big(1-\varphi_i^{\varpi}(t)\big)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^2} \right] \tau_{ii}(t) - \sum_{\substack{i,j=1\\ i\neq j}}^{n} \left[\frac{\mu_i^2(t)\mu_j^2(t)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^4} \right] \tau_{ij}(t) \right) dt \\
&= -\frac{1}{2} \left(\sum_{i=1}^{n} \left[\frac{\mu_i^2(t)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^2} \right] \big(1-\varphi_i^{\varpi}(t)\big)\tau_{ii}(t) - \sum_{\substack{i,j=1\\ i\neq j}}^{n} \left[\frac{\mu_i^2(t)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^2} \right] \left[\frac{\mu_j^2(t)}{\big(\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big)\big)^2} \right] \tau_{ij}(t) \right) dt,
\end{aligned}
$$

which, by (5.4.32) of Definition 5.4.4, becomes

$$
\begin{aligned}
d\Theta^{\varpi}(t) = d\Theta_{\varphi^{\varpi}}(t) &= -\frac{1}{2} \left(\sum_{i=1}^{n} \varphi_i^{\varpi}(t)\big(1-\varphi_i^{\varpi}(t)\big)\tau_{ii}(t) - \sum_{\substack{i,j=1\\ i\neq j}}^{n} \varphi_i^{\varpi}(t)\varphi_j^{\varpi}(t)\tau_{ij}(t) \right) dt \\
&= -\frac{1}{2} \left(\sum_{i=1}^{n} \Big(\varphi_i^{\varpi}(t) - \big(\varphi_i^{\varpi}(t)\big)^2\Big)\tau_{ii}(t) - \sum_{\substack{i,j=1\\ i\neq j}}^{n} \varphi_i^{\varpi}(t)\varphi_j^{\varpi}(t)\tau_{ij}(t) \right) dt \\
&= -\frac{1}{2} \left(\sum_{i=1}^{n} \varphi_i^{\varpi}(t)\tau_{ii}(t) - \sum_{i=1}^{n} \big(\varphi_i^{\varpi}(t)\big)^2\tau_{ii}(t) - \sum_{\substack{i,j=1\\ i\neq j}}^{n} \varphi_i^{\varpi}(t)\varphi_j^{\varpi}(t)\tau_{ij}(t) \right) dt \\
&= -\frac{1}{2} \left(\sum_{i=1}^{n} \varphi_i^{\varpi}(t)\tau_{ii}(t) - \left[\sum_{i=1}^{n} \big(\varphi_i^{\varpi}(t)\big)^2\tau_{ii}(t) + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \varphi_i^{\varpi}(t)\varphi_j^{\varpi}(t)\tau_{ij}(t) \right] \right) dt \\
&= -\frac{1}{2} \left(\sum_{i=1}^{n} \varphi_i^{\varpi}(t)\tau_{ii}(t) - \sum_{i,j=1}^{n} \varphi_i^{\varpi}(t)\tau_{ij}(t)\varphi_j^{\varpi}(t) \right) dt.
\end{aligned}
$$

Thus, by the numéraire invariance property of the excess growth rate of the weighted-average capitalisation portfolio, (2.4.26) of Lemma 2.4.5, we arrive at the following

$$
d\Theta^{\varpi}(t) = d\Theta_{\varphi^{\varpi}}(t) = -\gamma^*_{\varphi^{\varpi}}(t)\,dt, \tag{5.4.39}
$$

equivalently,

$$
\mathfrak{g}^{\varpi}(t) = \mathfrak{g}_{\varphi^{\varpi}}(t) = -\gamma^*_{\varphi^{\varpi}}(t). \tag{5.4.40}
$$

Since the weights of the weighted-average capitalisation portfolio are all nonnegative (positive) and equal to (5.4.33), we have by (2.4.37) of Proposition 2.4.8 that for a strictly long-only portfolio the excess growth rate of the weighted-average capitalisation portfolio is nonnegative (positive), i.e., $\gamma^*_{\varphi^{\varpi}}(t) \geq 0$ for all $t \in [0,T]$. In addition, Lemma 2.4.12, (2.4.55), and Lemma 2.4.14, (2.4.56) demonstrate that in a *nondegenerate market*, the excess growth rate of the weighted-average capitalisation portfolio, $\gamma^*_{\varphi^{\varpi}}(t)$, has a positive lower bound. Consequently, the drift process of the weighted-average capitalisation portfolio, in (5.4.40), must have a negative upper bound. Therefore, the drift process is nonpositive (negative) $\mathfrak{g}^{\varpi}(t) = \frac{d\Theta^{\varpi}(t)}{dt} \leq 0$, i.e.,

$$
\mathfrak{g}^{\varpi}(t) = \mathfrak{g}_{\varphi^{\varpi}}(t) = -\gamma^*_{\varphi^{\varpi}}(t) \leq 0,
$$

and the drift process Θ^{ϖ} is nonincreasing (decreasing) since $\mathfrak{g}^{\varpi}(t) = \frac{d\Theta^{\varpi}(t)}{dt} \leq 0$, i.e., the gradient of the drift process is nonpositive (negative). Since,

$$
\log \mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big) = \frac{1}{2}\log\left(\sum_{i=1}^{n}\mu_i^2(t)\right), \qquad t \in [0,T],
$$

if the weighted average of the capitalisation weights is the same at the beginning and end of a given period of time, then φ^{ϖ} will have a lower return than the market over that period. Furthermore, by (5.4.39) in conjunction with Theorem 5.2.2 and equation (5.2.2), the logarithmic return process of the weighted-average capitalisation portfolio φ^{ϖ}, relative to the market portfolio satisfies, for all $t \in [0, T]$, a.s.,

$$d \log \Big(Z_{\varphi^{\varpi}}(t) / Z_{\mu}(t) \Big) \quad = \quad d \log \mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(t)\big) - \gamma^*_{\varphi^{\varpi}}(t)\, dt, \tag{5.4.41}$$

or, by (5.2.8), for all $T \in [0, \infty)$, the relative performance of the weighted-average capitalisation portfolio φ^{ϖ}, can alternatively be represented in integral form as

$$\log \left(\frac{Z_{\varphi^{\varpi}}(T)}{Z_{\mu}(T)} \right) \quad = \quad \log \left(\frac{\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(0)\big)} \right) - \int_0^T \gamma^*_{\varphi^{\varpi}}(t)\, dt, \tag{5.4.42}$$

if $Z_{\varphi^{\varpi}}(0) = Z_{\mu}(0)$, otherwise the integral representation is given by

$$\log \left(\frac{Z_{\varphi^{\varpi}}(T)}{Z_{\mu}(T)} \right) \quad = \quad \log \left(\frac{Z_{\varphi^{\varpi}}(0)}{Z_{\mu}(0)} \right) + \log \left(\frac{\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(0)\big)} \right) - \int_0^T \gamma^*_{\varphi^{\varpi}}(t)\, dt, \tag{5.4.43}$$

or, as per equation (5.2.7), we have

$$\log \left(\frac{Z_{\varphi^{\varpi}}(T)}{Z_{\varphi^{\varpi}}(0)} \right) \quad = \quad \log \left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)} \right) + \log \left(\frac{\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{WC}}\big(\boldsymbol{\mu}(0)\big)} \right) - \int_0^T \gamma^*_{\varphi^{\varpi}}(t)\, dt. \tag{5.4.44}$$

5.4.4 The Price-to-Book Ratio Generated Portfolio

Suppose that $b_i > 0$ represents the book value of the ith company, and suppose that b_i is constant. Then the *price-to-book ratio* at time $t \in [0, \infty)$ for this company is $\frac{X_i(t)}{b_i}$. This ratio is frequently used to distinguish *growth stocks*, those with higher price-to-book ratios, from *value stocks*, those with lower price-to-book ratios. For our purposes, it is convenient to consider the ratio $\frac{\mu_i(t)}{b_i}$, but let us continue to call it the price-to-book ratio, since the two ratios have similar properties. We shall assume that $b_i > 0$ is constant, for $i = 1, \ldots, n$. The weighted-average price-to-book ratio of the market is given by

$$\sum_{i=1}^{n} \frac{\mu_i^2(t)}{b_i}, \qquad t \in [0, T].$$

Then, the *price-to-book ratio*, $\mathbf{S}^{\mathrm{PBR}} : \Delta^{n-1} \to (0, \infty)$, is given by the square root of the weighted-average price-to-book ratio, and is defined by

$$\mathbf{S}^{\mathrm{PBR}}(\mathbf{x}) \quad = \quad \mathbf{S}^{\mathrm{PBR}}(x_1, \ldots, x_n) \quad \triangleq \quad \left(\sum_{i=1}^{n} \frac{x_i^2}{b_i} \right)^{\frac{1}{2}}, \tag{5.4.45}$$

for all $\mathbf{x}$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\mathbf{x} \in \Delta^{n-1}$. The corresponding market measure (the *market price-to-book ratio*) is given below by:

Definition 5.4.5 (Market Price-to-Book Ratio)**.** *Let μ be the market portfolio. Then the* ***market price-to-book ratio process****, $\mathbf{S}^{\mathrm{PBR}}(\boldsymbol{\mu}) = \big\{ \mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big), t \in [0, T] \big\}$, $\mathbf{S}^{\mathrm{PBR}} : \Delta^{n-1} \to (0, \infty)$, is defined by*

$$\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big) \quad = \quad \mathbf{S}^{\mathrm{PBR}}\big(\mu_1(t), \ldots, \mu_n(t)\big) \quad \triangleq \quad \left(\sum_{i=1}^{n} \frac{\mu_i^2(t)}{b_i} \right)^{\frac{1}{2}}, \qquad t \in [0, T], \tag{5.4.46}$$

for all $\boldsymbol{\mu}(t)$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\boldsymbol{\mu}(t) \in \Delta^{n-1}$.

Recall from (5.4.46) of Definition 5.4.5 that the market price-to-book ratio process, $\mathbf{G}(\boldsymbol{\mu}) \equiv \mathbf{S}^{\mathrm{PBR}}(\boldsymbol{\mu}) = \big\{ \mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big), t \in [0, T] \big\}$, $\mathbf{G} \equiv \mathbf{S}^{\mathrm{PBR}} : \Delta^{n-1} \to (0, \infty)$, is given by

$$\mathbf{G}\big(\boldsymbol{\mu}(t)\big) \quad \equiv \quad \mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big) \quad = \quad \left(\sum_{k=1}^{n} \frac{\mu_k^2(t)}{b_k} \right)^{\frac{1}{2}}, \qquad t \in [0, T]. \tag{5.4.47}$$

Subsequently, we deduce the following for all $i = 1, 2, \ldots, n$,

$$\begin{aligned} D_i \mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big) &= \frac{\partial \mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)}{\partial \mu_i(t)} = \frac{\mu_i(t)}{b_i}\left(\sum_{k=1}^{n} \frac{\mu_k^2(t)}{b_k}\right)^{-\frac{1}{2}} \\ &= \frac{\mu_i(t)}{b_i\Big(\sum_{k=1}^{n} \frac{\mu_k^2(t)}{b_k}\Big)^{\frac{1}{2}}} \\ &= \frac{\mu_i(t)}{b_i \mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)}. \end{aligned} \tag{5.4.48}$$

It follows that for all $i = 1, 2, \ldots, n$, and $t \in [0, T]$,

$$D_i \log \mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big) = \frac{D_i \mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)} = \frac{\mu_i(t)}{b_i\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^2}.$$

Therefore, from (5.2.13) of Theorem 5.2.2, with $\mathbf{G}(\mathbf{x}) = \mathbf{S}^{\mathrm{PBR}}(\mathbf{x})$, we obtain the following weights for the portfolio generated by the market price-to-book ratio process $\mathbf{S}^{\mathrm{PBR}}$,

$$\begin{aligned} \varphi_i^{\varsigma}(t) &= \left(D_i \log \mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{j=1}^{n} \mu_j(t)\, D_j \log \mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\right) \mu_i(t) \\ &= \left(\frac{\mu_i(t)}{b_i\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^2} + 1 - \sum_{j=1}^{n} \mu_j(t) \left[\frac{\mu_j(t)}{b_j\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^2}\right]\right) \mu_i(t) \\ &= \left(\frac{\mu_i(t)}{b_i\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^2} + 1 - \sum_{j=1}^{n} \frac{\mu_j^2(t)}{b_j\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^2}\right) \mu_i(t) \\ &= \frac{\mu_i(t)}{\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^2}\left(\frac{\mu_i(t)}{b_i} + \Big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\Big)^2 - \sum_{j=1}^{n} \frac{\mu_j^2(t)}{b_j}\right) \\ &= \frac{\mu_i(t)}{\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^2}\left(\frac{\mu_i(t)}{b_i} + \Big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\Big)^2 - \Big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\Big)^2\right) \\ &= \frac{\mu_i(t)}{\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^2}\left(\frac{\mu_i(t)}{b_i}\right) \\ &= \frac{\mu_i^2(t)}{b_i\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^2}, \end{aligned}$$

since $\sum_{k=1}^{n} \frac{\mu_k^2(t)}{b_k} = \big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^2$. Hence, for all $i = 1, 2, \ldots, n$ and $t \in [0, T]$, the above expression simplifies to

$$\varphi_i^{\varsigma}(t) = \frac{\mu_i^2(t)}{b_i\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^2} = \frac{\mu_i^2(t)/b_i}{\sum_{j=1}^{n} \mu_j^2(t)/b_j}, \tag{5.4.49}$$

which gives the weights corresponding to the generating function $\mathbf{S}^{\mathrm{PBR}}$. Thus, the weights corresponding to the generating function $\mathbf{S}^{\mathrm{PBR}}$ can be formalised in the following definition.

Definition 5.4.6 (Price-to-Book Ratio Portfolio)**.** *Let μ be the market portfolio. The portfolio process $\varphi^{\varsigma} = \big\{\varphi^{\varsigma}(t) = \big(\varphi_1^{\varsigma}(t), \varphi_2^{\varsigma}(t), \ldots, \varphi_n^{\varsigma}(t)\big), t \in [0, T]\big\}$, with weights defined by* (5.4.49)*, i.e.,*

$$\varphi_i^{\varsigma}(t) \triangleq \frac{\mu_i^2(t)}{b_i\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^2} \tag{5.4.50}$$

$$= \frac{1}{b_i}\left(\frac{\mu_i(t)}{\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)}\right)^2, \tag{5.4.51}$$

for all $i = 1, 2, \ldots, n$, and $t \in [0, T]$, is called the ***price-to-book ratio generated portfolio (process)****.*

Thus, the generating function $\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)$, $\mathbf{S}^{\mathrm{PBR}} : \Delta^{n-1} \to (0,\infty)$, generates the *price-to-book ratio portfolio*. It can be easily verified that φ^{ς} satisfies the requirements of Definition 2.2.16, since we know that the market weights are bounded on $[0,\infty)$ and the $\mathbf{S}^{\mathrm{PBR}}$ process, $\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)$, is also bounded on $[0,\infty)$. Moreover, the weights satisfy the following

$$\begin{aligned}
\sum_{i=1}^{n} \varphi_i^{\varsigma}(t) &= \sum_{i=1}^{n} \frac{\mu_i^2(t)}{b_i\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^2} \\
&= \frac{1}{\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^2} \sum_{i=1}^{n} \frac{\mu_i^2(t)}{b_i} \\
&= \frac{\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^2}{\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^2} \\
&= 1.
\end{aligned}$$

The weight ratios, for $i = 1, 2, \ldots, n$, satisfy

$$\frac{\varphi_i^{\varsigma}(t)}{\mu_i(t)} = \frac{\mu_i(t)}{b_i\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^2}, \qquad t \in [0,T], \tag{5.4.52}$$

which increase with increasing price-to-book ratio $\frac{\mu_i(t)}{b_i}$, i.e., if $\frac{\mu_i(t)}{b_i} \geq \frac{\mu_j(t)}{b_j}$ for some $i > j$, then we have for all $t \in [0,T]$,

$$\frac{\mu_i(t)}{b_i\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^2} \geq \frac{\mu_j(t)}{b_j\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^2}.$$

Thus, if $\frac{\mu_i(t)}{b_i} \geq \frac{\mu_j(t)}{b_j}$ for some $i > j$, (5.4.52) implies

$$\frac{\varphi_i^{\varsigma}(t)}{\mu_i(t)} \geq \frac{\varphi_j^{\varsigma}(t)}{\mu_j(t)}.$$

This means that relative to the market, φ^{ς} is overweighted in the growth stocks and underweighted in the value stocks [Fernholz (2002)]. From (5.4.48), we have for all $i = 1, 2, \ldots, n$,

$$\begin{aligned}
D_{ii}\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big) = \frac{\partial^2 \mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)}{\partial \mu_i(t)^2} &= \frac{1}{b_i\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)} - \frac{\mu_i(t)}{b_i\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^2} D_i\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big) \\
&= \frac{1}{b_i\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)} - \frac{\mu_i(t)}{b_i\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^2} \left[\frac{\mu_i(t)}{b_i\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)}\right] \\
&= \frac{1}{b_i\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)} - \frac{\mu_i^2(t)}{b_i^2\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^3} \\
&= \frac{1}{b_i\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)} \left[1 - \frac{\mu_i^2(t)}{b_i\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^2}\right],
\end{aligned} \tag{5.4.53}$$

which by (5.4.50) of Definition 5.4.6 reduces to

$$\begin{aligned}
D_{ii}\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big) = \frac{\partial^2 \mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)}{\partial \mu_i(t)^2} &= \frac{1}{b_i\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)} \Big[1 - \varphi_i^{\varsigma}(t)\Big] \\
&= \frac{1 - \varphi_i^{\varsigma}(t)}{b_i\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)}.
\end{aligned} \tag{5.4.54}$$

Moreover, for all $i \neq j$, $i, j = 1, 2, \ldots, n$, from (5.4.48) we obtain

$$\begin{aligned} D_{ij}\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big) &= \frac{\partial^2 \mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)}{\partial \mu_i(t)\partial \mu_j(t)} = \frac{-\mu_i(t)}{b_i\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^2} D_j\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big) \\ &= \frac{-\mu_i(t)}{b_i\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^2}\left[\frac{\mu_j(t)}{b_j\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)}\right] \\ &= \frac{-\mu_i(t)\mu_j(t)}{b_ib_j\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^3}. \end{aligned} \tag{5.4.55}$$

Therefore, from (5.4.54) and (5.4.55), we have for all $i, j = 1, 2, \ldots, n$, and $t \in [0, T]$,

$$D_{ij}\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big) = \begin{cases} \dfrac{1-\varphi_i^{\varsigma}(t)}{b_i\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)} & \text{if} \quad i = j, \\ \dfrac{-\mu_i(t)\mu_j(t)}{b_ib_j\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^3} & \text{if} \quad i \neq j. \end{cases} \tag{5.4.56}$$

The above expression in conjunction with (5.2.14) of Theorem 5.2.2 with $\mathbf{G}(\mathbf{x}) = \mathbf{S}^{\mathrm{PBR}}(\mathbf{x})$, yields the following associated drift process that corresponds to the price-to-book ratio generated portfolio process

$$\begin{aligned} d\Theta^{\varsigma}(t) &= d\Theta_{\varphi^{\varsigma}}(t) \\ &= \frac{-1}{2\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)}\sum_{i,j=1}^{n} D_{ij}\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt \\ &= \frac{-1}{2\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)}\left(\sum_{i=1}^{n}\left[\frac{1-\varphi_i^{\varsigma}(t)}{b_i\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)}\right]\mu_i^2(t)\tau_{ii}(t) + \sum_{\substack{i,j=1\\ i\neq j}}^{n}\left[\frac{-\mu_i(t)\mu_j(t)}{b_ib_j\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^3}\right]\mu_i(t)\mu_j(t)\tau_{ij}(t)\right)dt \\ &= -\frac{1}{2}\left(\sum_{i=1}^{n}\left[\frac{\mu_i^2(t)\big(1-\varphi_i^{\varsigma}(t)\big)}{b_i\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^2}\right]\tau_{ii}(t) - \sum_{\substack{i,j=1\\ i\neq j}}^{n}\left[\frac{\mu_i^2(t)\mu_j^2(t)}{b_ib_j\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^4}\right]\tau_{ij}(t)\right)dt \\ &= -\frac{1}{2}\left(\sum_{i=1}^{n}\left[\frac{\mu_i^2(t)}{b_i\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^2}\right]\big(1-\varphi_i^{\varsigma}(t)\big)\tau_{ii}(t) - \sum_{\substack{i,j=1\\ i\neq j}}^{n}\left[\frac{\mu_i^2(t)}{b_i\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^2}\right]\left[\frac{\mu_j^2(t)}{b_j\big(\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big)\big)^2}\right]\tau_{ij}(t)\right)dt, \end{aligned}$$

which, by (5.4.50) of Definition 5.4.6, becomes

$$\begin{aligned} d\Theta^{\varsigma}(t) = d\Theta_{\varphi^{\varsigma}}(t) &= -\frac{1}{2}\left(\sum_{i=1}^{n}\varphi_i^{\varsigma}(t)\big(1-\varphi_i^{\varsigma}(t)\big)\tau_{ii}(t) - \sum_{\substack{i,j=1\\ i\neq j}}^{n}\varphi_i^{\varsigma}(t)\varphi_j^{\varsigma}(t)\tau_{ij}(t)\right)dt \\ &= -\frac{1}{2}\left(\sum_{i=1}^{n}\Big(\varphi_i^{\varsigma}(t) - \big(\varphi_i^{\varsigma}(t)\big)^2\Big)\tau_{ii}(t) - \sum_{\substack{i,j=1\\ i\neq j}}^{n}\varphi_i^{\varsigma}(t)\varphi_j^{\varsigma}(t)\tau_{ij}(t)\right)dt \\ &= -\frac{1}{2}\left(\sum_{i=1}^{n}\varphi_i^{\varsigma}(t)\tau_{ii}(t) - \sum_{i=1}^{n}\big(\varphi_i^{\varsigma}(t)\big)^2\tau_{ii}(t) - \sum_{\substack{i,j=1\\ i\neq j}}^{n}\varphi_i^{\varsigma}(t)\varphi_j^{\varsigma}(t)\tau_{ij}(t)\right)dt \\ &= -\frac{1}{2}\left(\sum_{i=1}^{n}\varphi_i^{\varsigma}(t)\tau_{ii}(t) - \left[\sum_{i=1}^{n}\big(\varphi_i^{\varsigma}(t)\big)^2\tau_{ii}(t) + \sum_{\substack{i,j=1\\ i\neq j}}^{n}\varphi_i^{\varsigma}(t)\varphi_j^{\varsigma}(t)\tau_{ij}(t)\right]\right)dt \\ &= -\frac{1}{2}\left(\sum_{i=1}^{n}\varphi_i^{\varsigma}(t)\tau_{ii}(t) - \sum_{i,j=1}^{n}\varphi_i^{\varsigma}(t)\tau_{ij}(t)\varphi_j^{\varsigma}(t)\right)dt. \end{aligned}$$

Thus, by the numéraire invariance property of the excess growth rate of the price-to-book ratio generated portfolio, (2.4.26) of Lemma 2.4.5, we arrive at the following

$$d\Theta^{\varsigma}(t) = d\Theta_{\varphi^{\varsigma}}(t) = -\gamma^*_{\varphi^{\varsigma}}(t)\,dt, \tag{5.4.57}$$

equivalently,

$$\mathfrak{g}^{\varsigma}(t) \quad = \quad \mathfrak{g}_{\varphi^{\varsigma}}(t) \quad = \quad -\gamma^{*}_{\varphi^{\varsigma}}(t). \tag{5.4.58}$$

Since the weights of the price-to-book ratio generated portfolio are all nonnegative (positive) (as $b_i > 0$ for all $i = 1, 2, \ldots, n$) and equal to (5.4.51), we have by (2.4.37) of Proposition 2.4.8 that for a strictly long-only portfolio the excess growth rate of the price-to-book ratio generated portfolio is nonnegative (positive), i.e., $\gamma^{*}_{\varphi^{\varsigma}}(t) \geq 0$ for all $t \in [0, T]$. In addition, Lemma 2.4.12, (2.4.55), and Lemma 2.4.14, (2.4.56) demonstrate that in a *nondegenerate market*, the excess growth rate of the price-to-book ratio generated portfolio, $\gamma^{*}_{\varphi^{\varsigma}}(t)$, has a positive lower bound. Consequently, the drift process of the price-to-book ratio generated portfolio, in (5.4.58), must have a negative upper bound. Therefore, the drift process is nonpositive (negative) $\mathfrak{g}^{\varsigma}(t) = \frac{d\Theta^{\varsigma}(t)}{dt} \leq 0$, i.e.,

$$\mathfrak{g}^{\varsigma}(t) \quad = \quad \mathfrak{g}_{\varphi^{\varsigma}}(t) \quad = \quad -\gamma^{*}_{\varphi^{\varsigma}}(t) \quad \leq \quad 0,$$

and the drift process Θ^{ς} is nonincreasing (decreasing) since $\mathfrak{g}^{\varsigma}(t) = \frac{d\Theta^{\varsigma}(t)}{dt} \leq 0$, i.e., the gradient of the drift process is nonpositive (negative). A similar reasoning to that employed in the previous example, shows that over a period of time in which the weighted-average price-to-book ratio of the market remains fixed, the portfolio φ^{ς} will have a lower return than the market. Furthermore, by (5.4.57) in conjunction with Theorem 5.2.2 and equation (5.2.2), the logarithmic return process of the price-to-book ratio portfolio φ^{ς}, relative to the market portfolio satisfies, for all $t \in [0, T]$, a.s.,

$$d \log \Big(Z_{\varphi^{\varsigma}}(t) / Z_{\mu}(t) \Big) \quad = \quad d \log \mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big) - \gamma^{*}_{\varphi^{\varsigma}}(t)\, dt, \tag{5.4.59}$$

or, by (5.2.8), for all $T \in [0, \infty)$, the relative performance of the price-to-book ratio portfolio φ^{ς}, can alternatively be represented in integral form as

$$\log \left(\frac{Z_{\varphi^{\varsigma}}(T)}{Z_{\mu}(T)} \right) \quad = \quad \log \left(\frac{\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(0)\big)} \right) - \int_0^T \gamma^{*}_{\varphi^{\varsigma}}(t)\, dt, \tag{5.4.60}$$

if $Z_{\varphi^{\varsigma}}(0) = Z_{\mu}(0)$, otherwise the integral representation is given by

$$\log \left(\frac{Z_{\varphi^{\varsigma}}(T)}{Z_{\mu}(T)} \right) \quad = \quad \log \left(\frac{Z_{\varphi^{\varsigma}}(0)}{Z_{\mu}(0)} \right) + \log \left(\frac{\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(0)\big)} \right) - \int_0^T \gamma^{*}_{\varphi^{\varsigma}}(t)\, dt, \tag{5.4.61}$$

or, as per equation (5.2.7), we have

$$\log \left(\frac{Z_{\varphi^{\varsigma}}(T)}{Z_{\varphi^{\varsigma}}(0)} \right) \quad = \quad \log \left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)} \right) + \log \left(\frac{\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(0)\big)} \right) - \int_0^T \gamma^{*}_{\varphi^{\varsigma}}(t)\, dt. \tag{5.4.62}$$

5.4.5 A Single Stock with Leverage

Consider the following function $\mathbf{S}^{\mathrm{SSL}} : \Delta^{n-1} \to (0, \infty)$, which is defined by

$$\mathbf{S}^{\mathrm{SSL}}(\mathbf{x}) \quad = \quad \mathbf{S}^{\mathrm{SSL}}(x_1, \ldots, x_n) \quad \triangleq \quad x_1^2, \tag{5.4.63}$$

for all $\mathbf{x}$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\mathbf{x} \in \Delta^{n-1}$, which can be referred to as the *square function*. Then, the associated market measure process (the *market square function*) is given below by:

Definition 5.4.7 (Market Square Measure). *Let μ be the market portfolio. Then the* ***market square measure process****, $\mathbf{S}^{\mathrm{SSL}}(\boldsymbol{\mu}) = \big\{ \mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big), t \in [0, T] \big\}$, $\mathbf{S}^{\mathrm{SSL}} : \Delta^{n-1} \to (0, \infty)$, is defined by*

$$\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big) \quad = \quad \mathbf{S}^{\mathrm{SSL}}\big(\mu_1(t), \ldots, \mu_n(t)\big) \quad \triangleq \quad \mu_1^2(t), \qquad t \in [0, T], \tag{5.4.64}$$

for all $\boldsymbol{\mu}(t)$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\boldsymbol{\mu}(t) \in \Delta^{n-1}$.

Recall from equation (5.4.64) of Definition 5.4.7 that the market square measure process, $\mathbf{G}(\boldsymbol{\mu}) \equiv \mathbf{S}^{\mathrm{SSL}}(\boldsymbol{\mu}) = \big\{\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big), t \in [0,T]\big\}$, $\mathbf{G} \equiv \mathbf{S}^{\mathrm{SSL}} : \Delta^{n-1} \to (0,\infty)$, is given by

$$\mathbf{G}\big(\boldsymbol{\mu}(t)\big) \equiv \mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big) = \mu_1^2(t), \qquad t \in [0,T]. \tag{5.4.65}$$

Therefore, we have

$$D_1\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big) = \frac{\partial \mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)}{\partial \mu_1(t)} = 2\mu_1(t), \tag{5.4.66}$$

and for $i = 2, \ldots, n$, we have

$$D_i\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big) = \frac{\partial \mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)}{\partial \mu_i(t)} = 0. \tag{5.4.67}$$

Hence, for $i = 1, 2, \ldots, n$, and $t \in [0,T]$, we obtain

$$D_i\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big) = \begin{cases} 2\mu_1(t) & \text{if} \quad i = 1, \\ 0 & \text{if} \quad i \neq 1. \end{cases} \tag{5.4.68}$$

So that for $i = 1, 2, \ldots, n$, and $t \in [0,T]$, we obtain

$$D_i \log \mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big) = \frac{D_i\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)} = \begin{cases} \dfrac{2\mu_1(t)}{\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)} & \text{if} \quad i = 1 \\ 0 & \text{if} \quad i \neq 1 \end{cases} = \begin{cases} \dfrac{2}{\mu_1(t)} & \text{if} \quad i = 1, \\ 0 & \text{if} \quad i \neq 1. \end{cases}$$

Therefore, from (5.2.13) of Theorem 5.2.2, with $\mathbf{G}(\mathbf{x}) = \mathbf{S}^{\mathrm{SSL}}(\mathbf{x})$, we obtain the following weights for the portfolio generated by the square function $\mathbf{S}^{\mathrm{SSL}}$,

$$\begin{aligned}
\varphi_1^{\mathsf{X}_1}(t) &= \left(D_1 \log \mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{j=1}^{n} \mu_j(t)\, D_j \log \mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)\right) \mu_1(t) \\
&= \left(D_1 \log \mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big) + 1 - \mu_1(t)\, D_1 \log \mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)\right) \mu_1(t) \\
&= \left(\frac{2\mu_1(t)}{\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)} + 1 - \mu_1(t)\left[\frac{2\mu_1(t)}{\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)}\right]\right) \mu_1(t) \\
&= \left(\frac{2\mu_1(t)}{\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)} + 1 - \frac{2\mu_1^2(t)}{\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)}\right) \mu_1(t) \\
&= \frac{\mu_1(t)}{\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)} \left(2\mu_1(t) + \mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big) - 2\mu_1^2(t)\right) \\
&= \frac{\mu_1(t)}{\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)} \left(2\mu_1(t) + \mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big) - 2\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)\right) \\
&= \frac{\mu_1(t)}{\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)} \left(2\mu_1(t) - \mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)\right) \\
&= \frac{2\mu_1^2(t)}{\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)} - \mu_1(t) \\
&= \frac{2\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)} - \mu_1(t) \\
&= 2 - \mu_1(t),
\end{aligned} \tag{5.4.69}$$

and for $i = 2, \ldots, n$, the weights of the portfolio generated by (5.4.64) are given by

$$\begin{aligned}
\varphi_i^{\chi_1}(t) &= \left(D_i \log \mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{j=1}^{n} \mu_j(t)\, D_j \log \mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big) \right) \mu_i(t) \\
&= \left(1 - \mu_1(t)\, D_1 \log \mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big) \right) \mu_i(t) \\
&= \left(1 - \mu_1(t) \left[\frac{2\mu_1(t)}{\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)} \right] \right) \mu_i(t) \\
&= \left(1 - \frac{2\mu_1^2(t)}{\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)} \right) \mu_i(t) \\
&= \left(1 - \frac{2\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)} \right) \mu_i(t) \\
&= (1-2)\, \mu_i(t) \\
&= -\mu_i(t). \qquad (5.4.70)
\end{aligned}$$

Thus, for $i = 1, 2, \ldots, n$, and $t \in [0, T]$, we have

$$\varphi_i^{\chi_1}(t) = \begin{cases} 2 - \mu_1(t) & \text{if} \quad i = 1, \\ -\mu_i(t) & \text{if} \quad i \neq 1. \end{cases} \qquad (5.4.71)$$

Thus, the weights corresponding to the generating function $\mathbf{S}^{\mathrm{SSL}}$ can be formalised in the following definition.

Definition 5.4.8 (Single Stock with Leverage)**.** *Let μ be the market portfolio. The portfolio process $\varphi^{\chi_1} = \big\{\varphi^{\chi_1}(t) = \big(\varphi_1^{\chi_1}(t), \varphi_2^{\chi_1}(t), \ldots, \varphi_n^{\chi_1}(t)\big), t \in [0, T]\big\}$, with weights defined by* (5.4.71)*, i.e.,*

$$\varphi_i^{\chi_1}(t) \triangleq \begin{cases} 2 - \mu_1(t) & \text{if} \quad i = 1, \\ -\mu_i(t) & \text{if} \quad i \neq 1, \end{cases} \qquad (5.4.72)$$

for all $i = 1, 2, \ldots, n$, and $t \in [0, T]$, is called the ***single stock with leverage (process)****.*

Thus, the generating function $\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big) = \mu_1^2(t)$, $\mathbf{S}^{\mathrm{SSL}} : \Delta^{n-1} \to (0, \infty)$, generates a *single stock with leverage*. These portfolio weights correspond to a continuously rebalanced portfolio in which for each \$2 invested in the single stock X_1 (i.e., in the portfolio χ_1), $-$\$1 is invested in the market portfolio μ (as a form of leverage) [Fernholz (2002)]. It can be easily verified that φ^{χ_1} satisfies the requirements of Definition 2.2.16, since we know that the market weights are bounded on $[0, \infty)$. Moreover, the weights satisfy the following

$$\begin{aligned}
\sum_{i=1}^{n} \varphi_i^{\chi_1}(t) &= 2 - \mu_1(t) - \sum_{i=2}^{n} \mu_i(t) \\
&= 2 - \mu_1(t) - \big(1 - \mu_1(t)\big) \\
&= 2 - \mu_1(t) - 1 + \mu_1(t) \\
&= 1.
\end{aligned}$$

To establish the drift process of the portfolio comprising the single stock X_1 with leverage, from (5.4.66), (5.4.67) and (5.4.68), first notice the following

$$D_{11}\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big) = \frac{\partial^2 \mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)}{\partial \mu_1(t)^2} = 2, \qquad (5.4.73)$$

otherwise, we have

$$D_{ij}\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big) = \frac{\partial^2 \mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)}{\partial \mu_i(t) \mu_j(t)} = 0. \qquad (5.4.74)$$

Hence, for all $i, j = 1, 2, \ldots, n$, and $t \in [0, T]$, we obtain

$$D_{ij}\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big) = \begin{cases} 2 & \text{if} \quad i = j = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.4.75}$$

Then, the above expression in conjunction with (5.2.14) of Theorem 5.2.2 with $\mathbf{G}(\mathbf{x}) = \mathbf{S}^{\mathrm{SSL}}(\mathbf{x})$, yields the following associated drift process which corresponds to the single stock with leverage process

$$\begin{aligned} d\Theta^{\chi_1}(t) = d\Theta_{\varphi^{\chi_1}}(t) &= \frac{-1}{2\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i,j=1}^{n} D_{ij}\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt \\ &= \frac{-1}{2\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)} D_{11}\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)\mu_1^2(t)\tau_{11}(t)\,dt \\ &= \frac{-1}{2\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)} (2)\,\mu_1^2(t)\tau_{11}(t)\,dt \\ &= \frac{-1}{\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)} \mu_1^2(t)\tau_{11}(t)\,dt \\ &= \frac{-1}{\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)} \Big(\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big)\Big)\tau_{11}(t)\,dt \\ &= -\tau_{11}(t)\,dt, \end{aligned} \tag{5.4.76}$$

equivalently,

$$\mathfrak{g}^{\chi_1}(t) = \mathfrak{g}_{\varphi^{\chi_1}}(t) = -\tau_{11}(t). \tag{5.4.77}$$

According to (2.4.40) of Lemma 2.4.9 and (2.4.54) of Lemma 2.4.11 (which both demonstrate that in a *nondegenerate market*, the variance of the ith stock relative to the market, $\tau_{ii}(t)$, has a positive lower bound), the drift process of the single stock with leverage, in (5.4.77), must have a negative upper bound. Thus, the drift process is nonpositive (negative) $\mathfrak{g}^{\chi_1}(t) = \frac{d\Theta^{\chi_1}(t)}{dt} \leq 0$, i.e.,

$$\mathfrak{g}^{\chi_1}(t) = \mathfrak{g}_{\varphi^{\chi_1}}(t) = -\tau_{11}(t) \leq 0,$$

and the drift process Θ^{χ_1} is nonincreasing (decreasing) since $\mathfrak{g}^{\chi_1}(t) = \frac{d\Theta^{\chi_1}(t)}{dt} \leq 0$, i.e., the gradient of the drift process is nonpositive (negative). This suggests that although investment in a single stock may be reasonable, it may not be wise to leverage this investment by shorting the market, at least over the long term [Fernholz (2002)]. Furthermore, by (5.4.76) in conjunction with Theorem 5.2.2 and equation (5.2.2), the logarithmic return process of the single stock portfolio comprising the stock X_1 with leverage, φ^{χ_1}, relative to the market portfolio satisfies, for all $t \in [0, T]$, a.s.,

$$d\log\Big(Z_{\varphi^{\chi_1}}(t)/Z_{\mu}(t)\Big) = d\log\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big) - \tau_{11}(t)\,dt \tag{5.4.78}$$

$$= d\log\mu_1^2(t) - \tau_{11}(t)\,dt, \tag{5.4.79}$$

or, by (5.2.8), for all $T \in [0, \infty)$, the relative performance of the single stock portfolio comprising the stock X_1 with leverage, φ^{χ_1}, can alternatively be represented in integral form as

$$\log\left(\frac{Z_{\varphi^{\chi_1}}(T)}{Z_{\mu}(T)}\right) = \log\left(\frac{\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(0)\big)}\right) - \int_0^T \tau_{11}(t)\,dt \tag{5.4.80}$$

$$= \log\left(\frac{\mu_1^2(T)}{\mu_1^2(0)}\right) - \int_0^T \tau_{11}(t)\,dt \tag{5.4.81}$$

$$= \log\left(\frac{\mu_1(T)}{\mu_1(0)}\right)^2 - \int_0^T \tau_{11}(t)\,dt$$

$$= 2\,\log\left(\frac{\mu_1(T)}{\mu_1(0)}\right) - \int_0^T \tau_{11}(t)\,dt, \tag{5.4.82}$$

if $Z_{\varphi^{\mathrm{x}_1}}(0) = Z_\mu(0)$, otherwise the integral representation is given by

$$\log\left(\frac{Z_{\varphi^{\mathrm{x}_1}}(T)}{Z_\mu(T)}\right) = \log\left(\frac{Z_{\varphi^{\mathrm{x}_1}}(0)}{Z_\mu(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{SSL}}(\boldsymbol{\mu}(T))}{\mathbf{S}^{\mathrm{SSL}}(\boldsymbol{\mu}(0))}\right) - \int_0^T \tau_{11}(t)\,dt \tag{5.4.83}$$

$$= \log\left(\frac{Z_{\varphi^{\mathrm{x}_1}}(0)}{Z_\mu(0)}\right) + 2\log\left(\frac{\mu_1(T)}{\mu_1(0)}\right) - \int_0^T \tau_{11}(t)\,dt, \tag{5.4.84}$$

or, as per equation (5.2.7), we have

$$\log\left(\frac{Z_{\varphi^{\mathrm{x}_1}}(T)}{Z_{\varphi^{\mathrm{x}_1}}(0)}\right) = \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{SSL}}(\boldsymbol{\mu}(T))}{\mathbf{S}^{\mathrm{SSL}}(\boldsymbol{\mu}(0))}\right) - \int_0^T \tau_{11}(t)\,dt \tag{5.4.85}$$

$$= \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + 2\log\left(\frac{\mu_1(T)}{\mu_1(0)}\right) - \int_0^T \tau_{11}(t)\,dt. \tag{5.4.86}$$

5.5 Diversity Portfolio Generating Functions and Diversity Generated Portfolios

The results of Section 5.2 imply that measures of diversity can be employed to generate portfolios. This concept is the focus of the remainder of this chapter.

Definition 5.5.1 (Diversity Portfolio Generating Functions). *A portfolio φ generated by a measure of diversity, henceforth also referred to as a **diversity portfolio generating function**, is called a **diversity-weighted portfolio**, and its proportions are called **diversity weights**. If a measure of diversity $\mathbf{G}$ generates the portfolio φ, then $\mathbf{G}$ is called the **diversity generating function** of the portfolio φ, and the portfolio φ is said to be **diversity generated**, or is said to be the diversity generated portfolio corresponding to the diversity portfolio generating function (or, the measure of diversity) $\mathbf{G}$.*

Proposition 5.5.2 ([**Fernholz (1999c, 2002)**]). *Suppose that $\mathbf{G}$ is a measure of diversity that generates a portfolio φ (the diversity generated portfolio) with drift process Θ. Then $\mathfrak{g}(t) = \frac{d\Theta(t)}{dt} = \Theta'(t) \geq 0$ is nonnegative and Θ is **nondecreasing** for all $t \in [0,T]$, a.s. Moreover, $\mu_i(t) \leq \mu_j(t)$ for all $t \in [0,T]$, implies that*

$$\frac{\varphi_i(t)}{\mu_i(t)} \geq \frac{\varphi_j(t)}{\mu_j(t)}, \tag{5.5.1}$$

for all $t \in [0,T]$, a.s. Equivalently, $\mu_i(t) \geq \mu_j(t)$ for all $t \in [0,T]$, implies that

$$\frac{\varphi_i(t)}{\mu_i(t)} \leq \frac{\varphi_j(t)}{\mu_j(t)}, \tag{5.5.2}$$

for all $t \in [0,T]$, a.s.

Proof. If the generating function $\mathbf{G}$ is a measure of diversity then by Definition 4.5.5, $\mathbf{G}$ is a positive twice continuously differentiable function that is both symmetric and concave. By statement (i) of Lemma E.2.7, if $\mathbf{G}$ is a twice continuously differentiable, concave function then its Hessian matrix, $\mathbf{HG}(\mathbf{x}) = \big(D_{ij}\mathbf{G}(\mathbf{x})\big)_{1\leq i,j\leq n}$ is negative semidefinite. It is well known that a symmetric matrix is negative semidefinite if and only if all its eigenvalues are nonpositive. Thus, the negative semidefinite Hessian matrix of the generating function $\mathbf{G}$ has no positive eigenvalues. Since all the eigenvalues are nonpositive, Proposition 5.3.5 implies that $\mathfrak{g}(t) = \frac{d\Theta(t)}{dt} = \Theta'(t) \geq 0$ is nonnegative (positive) and Θ is nondecreasing (increasing) for all $t \in [0,T]$, a.s., since $\mathfrak{g}(t) = \frac{d\Theta(t)}{dt} = \Theta'(t) \geq 0$, i.e., the gradient of the drift process is nonnegative (positive). Now, suppose that $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \Delta^{n-1}$ with $x_i \leq x_j$ for some $i < j$. For $0 \leq u < 1$, define $\mathbf{y}(u) \in \Delta^{n-1}$, by

$$\begin{aligned}\mathbf{y}(u) &= \Big(y_1(u), \ldots, y_{i-1}(u), y_i(u), y_{i+1}(u), \ldots, y_{j-1}(u), y_j(u), y_{j+1}(u), \ldots, y_n(u)\Big) \\ &\triangleq \Big(x_1, \ldots, x_{i-1}, (1-u)x_i + ux_j, x_{i+1}, \ldots, x_{j-1}, ux_i + (1-u)x_j, x_{j+1}, \ldots, x_n\Big).\end{aligned} \tag{5.5.3}$$

Hence, for $k = 1, 2, \ldots, n$, we have

$$y_k(u) \quad = \quad \begin{cases} x_k & \text{if} \quad k \neq i, j, \\ (1-u)x_i + ux_j & \text{if} \quad k = i, \\ ux_i + (1-u)x_j & \text{if} \quad k = j. \end{cases}$$

So, for $k = 1, 2, \ldots, n$, we obtain

$$\frac{dy_k(u)}{du} \quad = \quad \begin{cases} 0 & \text{if} \quad k \neq i, j, \\ x_j - x_i & \text{if} \quad k = i, \\ x_i - x_j & \text{if} \quad k = j. \end{cases} \tag{5.5.4}$$

Define

$$f(u) \quad = \quad \mathbf{G}\big(\mathbf{y}(u)\big) \quad = \quad \mathbf{G}\Big(y_1(u), \ldots, y_{i-1}(u), y_i(u), y_{i+1}(u), \ldots, y_{j-1}(u), y_j(u), y_{j+1}(u), \ldots, y_n(u)\Big),$$

so f is a positive twice continuously differentiable, concave and symmetric function. Then, by the chain rule for functions of several variables, for one independent variable u, we obtain

$$\begin{aligned} f'(u) \quad = \quad \frac{df(u)}{du} \quad &= \quad D_1\mathbf{G}\big(\mathbf{y}(u)\big)\frac{dy_1(u)}{du} + D_2\mathbf{G}\big(\mathbf{y}(u)\big)\frac{dy_2(u)}{du} + \cdots + D_n\mathbf{G}\big(\mathbf{y}(u)\big)\frac{dy_n(u)}{du} \\ &= \quad \sum_{k=1}^{n} \frac{dy_k(u)}{du} D_k\mathbf{G}\big(\mathbf{y}(u)\big). \end{aligned}$$

Hence, employing (5.5.4), we deduce that

$$\begin{aligned} f'(u) \quad &= \quad \big(x_j - x_i\big)D_i\mathbf{G}\big(\mathbf{y}(u)\big) + \big(x_i - x_j\big)D_j\mathbf{G}\big(\mathbf{y}(u)\big) \\ &= \quad \big(x_j - x_i\big)\Big(D_i\mathbf{G}\big(\mathbf{y}(u)\big) - D_j\mathbf{G}\big(\mathbf{y}(u)\big)\Big). \end{aligned} \tag{5.5.5}$$

From (5.5.3), $u = 0$ implies that

$$\mathbf{y}(0) \quad = \quad \Big(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_n\Big) \quad \equiv \quad \mathbf{x}, \tag{5.5.6}$$

so that for $u = 0$, $f(0) = \mathbf{G}\big(\mathbf{y}(0)\big) = \mathbf{G}(\mathbf{x})$. Moreover, from (5.5.3), $u = 1$ implies that

$$\mathbf{y}(1) \quad = \quad \Big(x_1, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_n\Big) \quad \equiv \quad \widetilde{\mathbf{x}},$$

which is just $\mathbf{x}$ with the ith and jth coordinates reversed, i.e., $\widetilde{\mathbf{x}}$ is a trivial permutation of $\mathbf{x}$. So, for $u = 1$, $f(1) = \mathbf{G}\big(\mathbf{y}(1)\big) = \mathbf{G}(\widetilde{\mathbf{x}})$. Now, since the generating function $\mathbf{G}$ is a measure of diversity which is symmetric (i.e., invariant under any permutation of the variables), we have $f(0) = \mathbf{G}\big(\mathbf{y}(0)\big) = \mathbf{G}(\mathbf{x}) \equiv \mathbf{G}(\widetilde{\mathbf{x}}) = \mathbf{G}\big(\mathbf{y}(1)\big) = f(1)$. Then, by *Rolle's Theorem*,[4] for any continuously differentiable function f, if $f(a) = f(b)$ then there exists a $c \in (a, b)$ such that $f'(c) = 0$. Hence, for any continuously differentiable, concave function f where $f(a) = f(b)$, and there exists a $c \in (a, b)$ such that $f'(c) = 0$ then $f'(z) \geq 0$ for all $z \in [a, c]$ (in particular, $f'(a) \geq 0$) and $f'(z) \leq 0$ for all $z \in [c, b]$ (in particular, $f'(b) \leq 0$). Consequently, the concavity of f and the fact that $f(0) = f(1)$, implies that $f'(0) \geq 0$. This concavity property of f in conjunction with (5.5.5) and (5.5.6) yields

$$f'(0) \quad = \quad \big(x_j - x_i\big)\Big(D_i\mathbf{G}(\mathbf{x}) - D_j\mathbf{G}(\mathbf{x})\Big) \quad \geq \quad 0. \tag{5.5.7}$$

Since $x_i \leq x_j$, i.e., $x_j - x_i \geq 0$, it follows from the above inequality that

$$D_i\mathbf{G}(\mathbf{x}) - D_j\mathbf{G}(\mathbf{x}) \quad \geq \quad 0,$$

which implies that

$$D_i\mathbf{G}(\mathbf{x}) \quad \geq \quad D_j\mathbf{G}(\mathbf{x}).$$

[4] Let f be a continuous function on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a) = f(b)$, then there is at least one number $c \in (a, b)$ such that $f'(c) = 0$.

It follows that for $\mathbf{x} = \boldsymbol{\mu}(t) \in \Delta^{n-1}$ with $\mu_i(t) \leq \mu_j(t)$, for all $t \in [0, T]$,

$$D_i \mathbf{G}\big(\boldsymbol{\mu}(t)\big) \quad \geq \quad D_j \mathbf{G}\big(\boldsymbol{\mu}(t)\big),$$

which implies that

$$\mathbf{G}\big(\boldsymbol{\mu}(t)\big)\, D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) \quad \geq \quad \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\, D_j \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big).$$

Since $\mathbf{G}\big(\boldsymbol{\mu}(t)\big) \geq 0$ for all $t \in [0, T]$, we have for some $i < j$,

$$D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) \quad \geq \quad D_j \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big).$$

Thus, the above inequality together with (5.2.13) of Theorem 5.2.2, implies that for $\mu_i(t) \leq \mu_j(t)$ for some $i < j$, we have

$$\begin{aligned}
\frac{\varphi_i(t)}{\mu_i(t)} &= D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{k=1}^{n} \mu_k(t)\, D_k \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) \\
&\geq D_j \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{k=1}^{n} \mu_k(t)\, D_k \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) \\
&= \frac{\varphi_j(t)}{\mu_j(t)}.
\end{aligned}$$

Therefore, $\mu_i(t) \leq \mu_j(t)$ implies that

$$\frac{\varphi_i(t)}{\mu_i(t)} \quad \geq \quad \frac{\varphi_j(t)}{\mu_j(t)}.$$

Now, suppose that $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \Delta^{n-1}$ with $x_i \geq x_j$ for some $i > j$. For $0 \leq u < 1$, define $\mathbf{y}(u) \in \Delta^{n-1}$, by

$$\begin{aligned}
\mathbf{y}(u) &= \Big(y_1(u), \ldots, y_{j-1}(u), y_j(u), y_{j+1}(u), \ldots, y_{i-1}(u), y_i(u), y_{i+1}(u), \ldots, y_n(u)\Big) \\
&\triangleq \Big(x_1, \ldots, x_{j-1}, (1-u)x_j + u x_i, x_{j+1}, \ldots, x_{i-1}, u x_j + (1-u) x_i, x_{i+1}, \ldots, x_n\Big). \qquad (5.5.8)
\end{aligned}$$

From (5.5.8), $u = 0$ implies that

$$\mathbf{y}(0) \quad = \quad \Big(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n\Big) \quad \equiv \quad \mathbf{x}, \qquad (5.5.9)$$

so that for $u = 0$, $f(0) = \mathbf{G}\big(\mathbf{y}(0)\big) = \mathbf{G}(\mathbf{x})$. Since $x_i \geq x_j$, i.e., $x_j - x_i \leq 0$, it follows from (5.5.7) that

$$D_i \mathbf{G}(\mathbf{x}) - D_j \mathbf{G}(\mathbf{x}) \quad \leq \quad 0,$$

which implies that

$$D_i \mathbf{G}(\mathbf{x}) \quad \leq \quad D_j \mathbf{G}(\mathbf{x}).$$

It follows that for $\mathbf{x} = \boldsymbol{\mu}(t) \in \Delta^{n-1}$ with $\mu_i(t) \geq \mu_j(t)$, for all $t \in [0, T]$,

$$D_i \mathbf{G}\big(\boldsymbol{\mu}(t)\big) \quad \leq \quad D_j \mathbf{G}\big(\boldsymbol{\mu}(t)\big),$$

which implies that

$$\mathbf{G}\big(\boldsymbol{\mu}(t)\big)\, D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) \quad \leq \quad \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\, D_j \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big),$$

so that for all $t \in [0, T]$ and for some $i > j$, we have

$$D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) \quad \leq \quad D_j \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big).$$

Thus, the above inequality together with (5.2.13) of Theorem 5.2.2, implies that for $\mu_i(t) \geq \mu_j(t)$ for some $i > j$, we have

$$\begin{aligned}
\frac{\varphi_i(t)}{\mu_i(t)} &= D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{k=1}^{n} \mu_k(t)\, D_k \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) \\
&\leq D_j \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{k=1}^{n} \mu_k(t)\, D_k \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) \\
&= \frac{\varphi_j(t)}{\mu_j(t)}.
\end{aligned}$$

Therefore, $\mu_i(t) \geq \mu_j(t)$ implies that

$$\frac{\varphi_i(t)}{\mu_i(t)} \leq \frac{\varphi_j(t)}{\mu_j(t)}.$$

■

This proposition demonstrates that portfolios generated by measures of diversity have nondecreasing drift processes, and positive weights for which the weight ratios $\frac{\varphi_i(t)}{\mu_i(t)}$ decrease with increasing market weight. Consequently, if the market weight of the stock increases, in other words, the value of the stock goes up relative to the market, then the portfolio φ sells some (fractional) shares of stock.

5.6 Examples of Diversity Portfolio Generating Functions and their Diversity Generated Portfolios

Here we present some examples of generating functions, that are diversity measures, and the portfolios they generate.

5.6.1 The Entropy-Weighted Portfolio

Recall from (4.6.2) of Definition 4.6.1 that the market entropy process, $\mathbf{G}(\boldsymbol{\mu}) \equiv \mathbf{S}^{\mathrm{E}}(\boldsymbol{\mu}) = \big\{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big), t \in [0,T]\big\}$, $\mathbf{G} \equiv \mathbf{S}^{\mathrm{E}} : \Delta^{n-1} \to (0,\infty)$, is given by

$$\mathbf{G}\big(\boldsymbol{\mu}(t)\big) \equiv \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) = -\sum_{k=1}^{n} \mu_k(t) \log \mu_k(t), \qquad t \in [0,T], \tag{5.6.1}$$

and is a measure of stock market diversity that can be used to generate a portfolio. Thus the market entropy process is a portfolio generating function since it is a positive twice continuously differentiable function defined on Δ^{n-1} and satisfies the requirements of Definition 5.2.1. From equations (5.2.20) and (4.6.4), we have for the market entropy process for all $i = 1, 2, \ldots, n$ and $t \in [0,T]$,

$$D_i \log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) = \frac{D_i \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} = \frac{-\log \mu_i(t) - 1}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}.$$

Therefore, from (5.2.13) of Theorem 5.2.2, with $\mathbf{G}(\mathbf{x}) = \mathbf{S}^{\mathrm{E}}(\mathbf{x})$, we obtain the following weights for the portfolio generated by the entropy function $\mathbf{S}^{\mathrm{E}}$,

$$\begin{aligned}
\varphi_i^{\mathsf{e}}(t) &= \left(D_i \log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{j=1}^{n} \mu_j(t)\, D_j \log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \right) \mu_i(t) \\
&= \left(\frac{-\log \mu_i(t) - 1}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} + 1 - \sum_{j=1}^{n} \mu_j(t) \left[\frac{-\log \mu_j(t) - 1}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \right] \right) \mu_i(t) \\
&= \left(\frac{-\log \mu_i(t) - 1}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} + 1 - \sum_{j=1}^{n} \frac{\mu_j(t)\big(-\log \mu_j(t) - 1\big)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \right) \mu_i(t) \\
&= \left(\frac{-\log \mu_i(t) - 1}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} + 1 - \sum_{j=1}^{n} \frac{-\mu_j(t)\big(\log \mu_j(t) + 1\big)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \right) \mu_i(t) \\
&= \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \left(-\log \mu_i(t) - 1 + \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) + \sum_{j=1}^{n} \mu_j(t)\big(\log \mu_j(t) + 1\big) \right) \\
&= \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \left(-\log \mu_i(t) - 1 + \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) + \sum_{j=1}^{n} \big(\mu_j(t) \log \mu_j(t) + \mu_j(t)\big) \right) \\
&= \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \left(-\log \mu_i(t) - 1 + \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) + \sum_{j=1}^{n} \mu_j(t) \log \mu_j(t) + \sum_{j=1}^{n} \mu_j(t) \right) \\
&= \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \left(-\log \mu_i(t) - 1 + \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) + \sum_{j=1}^{n} \mu_j(t) \log \mu_j(t) + 1 \right) \\
&= \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \left(-\log \mu_i(t) + \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) + \sum_{j=1}^{n} \mu_j(t) \log \mu_j(t) \right),
\end{aligned}$$

since $\sum_{i=1}^{n} \mu_i(t) = 1$. Now, since $\sum_{k=1}^{n} \mu_k(t) \log \mu_k(t) = -\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$, the above expression simplifies to

$$\begin{aligned}
\varphi_i^{\mathsf{e}}(t) &= \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \left(-\log \mu_i(t) + \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) - \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \right) \\
&= \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \left(-\log \mu_i(t) \right) \\
&= \frac{-\mu_i(t) \log \mu_i(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)},
\end{aligned}$$

so that for all $i = 1, 2, \ldots, n$ and $t \in [0, T]$, we have the *entropy weights*, i.e. the weights corresponding to the generating function $\mathbf{S}^{\mathrm{E}}$,

$$\varphi_i^{\mathsf{e}}(t) = \frac{-\mu_i(t) \log \mu_i(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} = \frac{-\mu_i(t) \log \mu_i(t)}{-\sum_{j=1}^{n} \mu_j(t) \log \mu_j(t)} = \frac{\mu_i(t) \log \mu_i(t)}{\sum_{j=1}^{n} \mu_j(t) \log \mu_j(t)}. \tag{5.6.2}$$

Thus, the weights corresponding to the generating function $\mathbf{S}^{\mathrm{E}}$ can be formalised in the following definition.

Definition 5.6.1 (Entropy-Weighted Portfolio)**.** *Let μ be the market portfolio. The portfolio process $\varphi^{\mathsf{e}} = \big\{\varphi^{\mathsf{e}}(t) = \big(\varphi_1^{\mathsf{e}}(t), \varphi_2^{\mathsf{e}}(t), \ldots, \varphi_n^{\mathsf{e}}(t)\big), t \in [0, T]\big\}$, with weights defined by* (5.6.2)*, i.e.,*

$$\varphi_i^{\mathsf{e}}(t) \triangleq \frac{-\mu_i(t) \log \mu_i(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}, \tag{5.6.3}$$

for all $i = 1, 2, \ldots, n$, and $t \in [0, T]$, is called the ***entropy-weighted portfolio (process)****.*

Thus, the generating function $\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$, $\mathbf{S}^{\mathrm{E}} : \Delta^{n-1} \to (0, \infty)$, generates the *entropy-weighted portfolio*. It can be easily verified that φ^{e} satisfies the requirements of Definition 2.2.16, since we know that the market weights

are bounded on $[0,\infty)$ and the market entropy process, $\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$, is also bounded on $[0,\infty)$ according to the bounds derived in (4.6.3), i.e., $0 < \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \leq \log(n)$. Moreover, the *entropy weights* satisfy the following

$$\begin{aligned}
\sum_{i=1}^{n} \varphi_i^{\mathsf{e}}(t) &= -\sum_{i=1}^{n} \frac{\mu_i(t)\log\mu_i(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \\
&= \frac{-1}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^{n} \mu_i(t)\log\mu_i(t) \\
&= \frac{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \\
&= 1.
\end{aligned}$$

The weight ratios, for $i = 1, 2, \ldots, n$,

$$\frac{\varphi_i^{\mathsf{e}}(t)}{\mu_i(t)} = \frac{-\log\mu_i(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}, \qquad t \in [0,T], \tag{5.6.4}$$

decrease with increasing market weight $\mu_i(t)$, this follows from Proposition 5.5.2, since $\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$ is a measure of diversity that generates the diversity generated portfolio, the entropy-weighted portfolio, i.e., if $\mu_i(t) \geq \mu_j(t)$ for some $i > j$, then since the logarithm function is monotone increasing, we have for all $t \in [0,T]$,

$$\log\mu_i(t) \geq \log\mu_j(t),$$

therefore since $\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$ is strictly positive for all $t \in [0,T]$, we obtain for some $i > j$, and for all $t \in [0,T]$,

$$\frac{-\log\mu_i(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \leq \frac{-\log\mu_j(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)},$$

which by (5.6.4), suggests that if $\mu_i(t) \geq \mu_j(t)$ for some $i > j$, we have the result specified in Proposition 5.5.2, for some $i > j$,

$$\frac{\varphi_i^{\mathsf{e}}(t)}{\mu_i(t)} \leq \frac{\varphi_j^{\mathsf{e}}(t)}{\mu_j(t)}.$$

Hence, compared to the market portfolio μ, the entropy-weighted portfolio φ^{e} is less concentrated than the market portfolio μ in those stocks with the highest market weights (i.e., φ^{e}, relative to the market, is underweighted in the larger stocks), and is more concentrated than the market portfolio μ in those stocks with the lowest market weights (i.e., φ^{e}, relative to the market, is overweighted in the smaller stocks) [Fernholz (2002)]. In particular, as $\mu_i(t) \to 1$, we have

$$\lim_{\mu_i(t)\to 1} \frac{\varphi_i^{\mathsf{e}}(t)}{\mu_i(t)} = -\lim_{\mu_i(t)\to 1} \frac{\log\mu_i(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} = 0.$$

Now, from (4.6.5) and (4.6.6), we have for all $i, j = 1, 2, \ldots, n$, and $t \in [0,T]$,

$$D_{ij}\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) = \begin{cases} -\dfrac{1}{\mu_i(t)} & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{5.6.5}$$

The above expression in conjunction with (5.2.14) of Theorem 5.2.2 with $\mathbf{G}(\mathbf{x}) = \mathbf{S}^{\mathrm{E}}(\mathbf{x})$, yields the following drift process corresponding to the entropy function, and thus to the entropy-weighted portfolio process

$$\begin{aligned}
d\Theta^{\mathsf{e}}(t) = d\Theta_{\varphi^{\mathsf{e}}}(t) &= \frac{-1}{2\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i,j=1}^{n} D_{ij}\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt \\
&= \frac{-1}{2\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^{n} \left[-\frac{1}{\mu_i(t)}\right] \mu_i^2(t)\tau_{ii}(t)\,dt \\
&= \frac{1}{2\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^{n} \mu_i(t)\tau_{ii}(t)\,dt,
\end{aligned}$$

which by (2.4.32) of Lemma 2.4.7, or by equations (2.4.36) and (2.12.95), reduces to

$$d\Theta^{\mathsf{e}}(t) \quad = \quad d\Theta_{\varphi^{\mathsf{e}}}(t) \quad = \quad \frac{\gamma_\mu^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\, dt, \tag{5.6.6}$$

equivalently,

$$\mathfrak{g}^{\mathsf{e}}(t) \quad = \quad \mathfrak{g}_{\varphi^{\mathsf{e}}}(t) \quad = \quad \frac{\gamma_\mu^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}. \tag{5.6.7}$$

Since the weights of the market portfolio are all nonnegative (positive), we have by (2.4.37) of Proposition 2.4.8 that for a strictly long-only portfolio the excess growth rate of the market is nonnegative (positive), i.e., $\gamma_\mu^*(t) \geq 0$ for all $t \in [0, T]$. In addition, Lemma 2.4.12, (2.4.55), and Lemma 2.4.14, (2.4.56) demonstrate that in a *nondegenerate market*, the excess growth rate of the market $\gamma_\mu^*(t)$ has a positive lower bound. Furthermore, (4.2.6) of Proposition 4.2.3 demonstrates that in a market that is both *nondegenerate* and *diverse*, the market excess growth rate has a positive lower bound. If the market is diverse then Proposition 4.7.1 in conjunction with (4.6.3) imply that the market entropy process is bounded and thus also has a positive lower bound, given by $\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \geq \zeta$. Hence, these two concepts together indicate that in a nondegenerate and diverse market, the drift process of the entropy-weighted portfolio (5.6.7), must also have a positive lower bound. Therefore, the drift process is nonnegative (positive) $\mathfrak{g}^{\mathsf{e}}(t) = \frac{d\Theta^{\mathsf{e}}(t)}{dt} \geq 0$ and Θ^{e} is nondecreasing (increasing) since $\mathfrak{g}^{\mathsf{e}}(t) = \frac{d\Theta^{\mathsf{e}}(t)}{dt} \geq 0$, i.e., the gradient of the drift process is nonnegative (positive), which follows from Proposition 5.5.2 since the entropy function $\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$ is a measure of diversity that generates the diversity generated portfolio φ^{e} with drift Θ^{e}. Thus, the entropy-weighted portfolio has an increasing drift process.

Theorem 5.6.2 ([**Fernholz (2002)**]). *Let μ be the market portfolio and φ^{e} be the entropy-weighted portfolio, and let Z_μ and $Z_{\varphi^{\mathsf{e}}}$ be their portfolio value processes, respectively. Then, a.s., for $t \in [0, T]$,*

$$d \log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \quad = \quad d \log \Big(Z_{\varphi^{\mathsf{e}}}(t)/Z_\mu(t)\Big) - \frac{\gamma_\mu^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\, dt. \tag{5.6.8}$$

Proof. For a positive twice continuously differentiable function, $\mathbf{S}^{\mathrm{E}}$, and for all $\mathbf{x} \in \Delta^{n-1}$, by (5.2.20) we have

$$D_i \log \mathbf{S}^{\mathrm{E}}(\mathbf{x}) \quad = \quad \frac{D_i \mathbf{S}^{\mathrm{E}}(\mathbf{x})}{\mathbf{S}^{\mathrm{E}}(\mathbf{x})}, \tag{5.6.9}$$

where $D_i \mathbf{S}^{\mathrm{E}}(\mathbf{x}) = -\log x_i - 1$, and by (5.2.21) we have

$$D_{ij} \log \mathbf{S}^{\mathrm{E}}(\mathbf{x}) \quad = \quad \frac{D_{ij} \mathbf{S}^{\mathrm{E}}(\mathbf{x})}{\mathbf{S}^{\mathrm{E}}(\mathbf{x})} - D_i \log \mathbf{S}^{\mathrm{E}}(\mathbf{x})\, D_j \log \mathbf{S}^{\mathrm{E}}(\mathbf{x}). \tag{5.6.10}$$

Therefore for the market portfolio, $\boldsymbol{\mu} \in \Delta^{n-1}$, we have

$$D_{ij} \log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \quad = \quad \frac{D_{ij} \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} - D_i \log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)\, D_j \log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big). \tag{5.6.11}$$

Equation (5.2.23) in conjunction with an application of Itô's formula to $\log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$, yields a.s., for $t \in [0, T]$,

$$\begin{aligned} d \log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \quad &= \quad \sum_{i=1}^n D_i \log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)\, d\mu_i(t) + \frac{1}{2} \sum_{i,j=1}^n D_{ij} \log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)\, d\big\langle \mu_i, \mu_j \big\rangle_t \\ &= \quad \sum_{i=1}^n D_i \log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)\, d\mu_i(t) + \frac{1}{2} \sum_{i,j=1}^n D_{ij} \log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \mu_i(t) \mu_j(t) \tau_{ij}(t)\, dt, \end{aligned} \tag{5.6.12}$$

which when combined with (5.6.9) and (5.6.11), yields a.s. for $t \in [0, T]$,

$$\begin{aligned} & d \log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \\ &= \quad \sum_{i=1}^n D_i \log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)\, d\mu_i(t) + \frac{1}{2} \sum_{i,j=1}^n \left[\frac{D_{ij} \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} - D_i \log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)\, D_j \log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \right] \mu_i(t) \mu_j(t) \tau_{ij}(t)\, dt \\ &= \quad \sum_{i=1}^n \frac{D_i \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\, d\mu_i(t) + \frac{1}{2} \sum_{i,j=1}^n \left[\frac{D_{ij} \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} - \frac{D_i \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)\, D_j \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}{\big(\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)\big)^2} \right] \mu_i(t) \mu_j(t) \tau_{ij}(t)\, dt. \end{aligned} \tag{5.6.13}$$

Thus, (4.6.4) together with (5.6.5) imply that, a.s. for $t \in [0,T]$,

$$\begin{aligned} d\log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) &= -\sum_{i=1}^n \frac{\log \mu_i(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\, d\mu_i(t) - \frac{1}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^n d\mu_i(t) + \frac{1}{2\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^n \left[-\frac{1}{\mu_i(t)}\right] \mu_i^2(t)\tau_{ii}(t)\, dt \\ &\quad - \frac{1}{2\big(\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)\big)^2} \sum_{i,j=1}^n \big(\log \mu_i(t)+1\big)\big(\log \mu_j(t)+1\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\, dt \\ &= -\sum_{i=1}^n \frac{\log \mu_i(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\, d\mu_i(t) - \frac{1}{2\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^n \mu_i(t)\tau_{ii}(t)\, dt \\ &\quad - \frac{1}{2\big(\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)\big)^2} \sum_{i,j=1}^n \big(\log \mu_i(t)+1\big)\big(\log \mu_j(t)+1\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\, dt, \end{aligned} \tag{5.6.14}$$

since $\sum_{i=1}^n d\mu_i(t) = d\big(\sum_{i=1}^n \mu_i(t)\big) = 0$. The first term on the right-hand side of the last equation is the local martingale component of $\log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$. In order for (5.2.2) to hold, the local martingale components of $\log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$ and $\log\big(Z_{\varphi^{\mathrm{e}}}(t)/Z_\mu(t)\big)$ must be equal. By setting $\boldsymbol{\pi} := \boldsymbol{\varphi}^{\mathrm{e}}$ in Proposition 2.12.8 we have a.s., for $t \in [0,T]$,

$$d\log\Big(Z_{\varphi^{\mathrm{e}}}(t)/Z_\mu(t)\Big) = \sum_{i=1}^n \varphi_i^{\mathrm{e}}(t)\, d\log \mu_i(t) + \gamma^*_{\varphi^{\mathrm{e}}}(t)\, dt.$$

Thus, from (5.2.27), we similarly have

$$d\log\Big(Z_{\varphi^{\mathrm{e}}}(t)/Z_\mu(t)\Big) = \sum_{i=1}^n \frac{\varphi_i^{\mathrm{e}}(t)}{\mu_i(t)}\, d\mu_i(t) - \frac{1}{2}\sum_{i,j=1}^n \varphi_i^{\mathrm{e}}(t)\varphi_j^{\mathrm{e}}(t)\tau_{ij}(t)\, dt.$$

Substituting the entropy weights (5.6.2) into the above equation yields

$$d\log\Big(Z_{\varphi^{\mathrm{e}}}(t)/Z_\mu(t)\Big) = -\sum_{i=1}^n \frac{\log \mu_i(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\, d\mu_i(t) - \frac{1}{2\big(\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)\big)^2} \sum_{i,j=1}^n \log \mu_i(t)\log \mu_j(t)\mu_i(t)\mu_j(t)\tau_{ij}(t)\, dt. \tag{5.6.15}$$

Hence, the local martingale components of $\log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$ and $\log\big(Z_{\varphi^{\mathrm{e}}}(t)/Z_\mu(t)\big)$ are equal, so that (5.2.2) is satisfied. The summation part of the third term on the right-hand side of equation (5.6.14) is given a.s. by, for $t \in [0,T]$,

$$\begin{aligned} \sum_{i,j=1}^n \big(\log \mu_i(t)+1\big)\big(\log \mu_j(t)+1\big)\mu_i(t)\mu_j(t)\tau_{ij}(t) &= \sum_{i,j=1}^n \log \mu_i(t)\log \mu_j(t)\mu_i(t)\mu_j(t)\tau_{ij}(t) \\ &\quad + 2\sum_{i,j=1}^n \log \mu_i(t)\mu_i(t)\mu_j(t)\tau_{ij}(t) \\ &\quad + \sum_{i,j=1}^n \mu_i(t)\mu_j(t)\tau_{ij}(t) \\ &= \sum_{i,j=1}^n \log \mu_i(t)\log \mu_j(t)\mu_i(t)\mu_j(t)\tau_{ij}(t) \\ &\quad + 2\sum_{i=1}^n \log \mu_i(t)\mu_i(t)\left[\sum_{j=1}^n \mu_j(t)\tau_{ij}(t)\right] \\ &\quad + \sum_{i,j=1}^n \mu_i(t)\mu_j(t)\tau_{ij}(t). \end{aligned}$$

Thus, by (2.12.54) of Lemma 2.12.4 and (2.12.56) (since $\boldsymbol{\mu}(t)$ is in the null space of $\boldsymbol{\tau}(t)$), we have a.s., for $t \in [0,T]$,

$$\sum_{i,j=1}^n \big(\log \mu_i(t)+1\big)\big(\log \mu_j(t)+1\big)\mu_i(t)\mu_j(t)\tau_{ij}(t) = \sum_{i,j=1}^n \log \mu_i(t)\log \mu_j(t)\mu_i(t)\mu_j(t)\tau_{ij}(t). \tag{5.6.16}$$

Hence, (5.6.14) becomes

$$\begin{aligned} d\log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) &= -\sum_{i=1}^{n} \frac{\log \mu_i(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\, d\mu_i(t) - \frac{1}{2\big(\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)\big)^2} \sum_{i,j=1}^{n} \log \mu_i(t) \log \mu_j(t) \mu_i(t) \mu_j(t) \tau_{ij}(t)\, dt \\ &\quad - \frac{1}{2\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^{n} \mu_i(t) \tau_{ii}(t)\, dt. \end{aligned} \tag{5.6.17}$$

a.s., for $t \in [0, T]$. Comparing this last expression with (5.6.15), we obtain for $t \in [0, T]$, a.s.,

$$d\log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) = d\log\Big(Z_{\varphi^{\mathrm{e}}}(t)/Z_{\mu}(t)\Big) - \frac{1}{2\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^{n} \mu_i(t)\tau_{ii}(t)\, dt. \tag{5.6.18}$$

The desired result is then obtained by appealing to (2.4.32) of Lemma 2.4.7, or equations (2.4.36) and (2.12.95), so that (5.6.18) simplifies to

$$d\log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) = d\log\Big(Z_{\varphi^{\mathrm{e}}}(t)/Z_{\mu}(t)\Big) - \frac{\gamma_{\mu}^{*}(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\, dt.$$

■

Therefore, for all $t \in [0, T]$, a.s., the logarithmic return process of the entropy-weighted portfolio φ^{e}, relative to the market portfolio satisfies

$$d\log\Big(Z_{\varphi^{\mathrm{e}}}(t)/Z_{\mu}(t)\Big) = d\log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) + \frac{\gamma_{\mu}^{*}(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\, dt. \tag{5.6.19}$$

Thus, the logarithmic return process of the entropy-weighted portfolio φ^{e}, relative to the market portfolio, has the following form

$$d\log\Big(Z_{\varphi^{\mathrm{e}}}(t)/Z_{\mu}(t)\Big) = d\log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) + d\Theta_{\varphi^{\mathrm{e}}}(t) = d\log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) + \mathfrak{g}_{\varphi^{\mathrm{e}}}(t)\, dt,$$

which indicates that the entropy function is a portfolio generating function that generates a functionally generated portfolio, that being the entropy-weighted portfolio, as its relative return has the required decomposition as stipulated in Definition 5.2.1, by equations (5.2.2) and (5.2.3). In particular, the entropy function is a measure of diversity that generates a diversity generated portfolio. In addition, by (5.2.8), for all $T \in [0, \infty)$, the relative performance of the entropy-weighted portfolio φ^{e}, can alternatively be represented in integral form as

$$\log\left(\frac{Z_{\varphi^{\mathrm{e}}}(T)}{Z_{\mu}(T)}\right) = \log\left(\frac{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(0)\big)}\right) + \int_0^T \frac{\gamma_{\mu}^{*}(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\, dt, \tag{5.6.20}$$

if $Z_{\varphi^{\mathrm{e}}}(0) = Z_{\mu}(0)$, otherwise the integral representation is given by

$$\log\left(\frac{Z_{\varphi^{\mathrm{e}}}(T)}{Z_{\mu}(T)}\right) = \log\left(\frac{Z_{\varphi^{\mathrm{e}}}(0)}{Z_{\mu}(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(0)\big)}\right) + \int_0^T \frac{\gamma_{\mu}^{*}(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\, dt, \tag{5.6.21}$$

or, as per equation (5.2.7), we have

$$\log\left(\frac{Z_{\varphi^{\mathrm{e}}}(T)}{Z_{\varphi^{\mathrm{e}}}(0)}\right) = \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(0)\big)}\right) + \int_0^T \frac{\gamma_{\mu}^{*}(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\, dt. \tag{5.6.22}$$

Furthermore, recall from (5.2.11) and (5.2.12), that we can calculate the drift process of the entropy-weighted portfolio as follows, for all $T \in [0, \infty)$,

$$\Theta^{\mathrm{e}}(T) = \int_0^T d\Theta^{\mathrm{e}}(t) = \int_0^T \mathfrak{g}^{\mathrm{e}}(t)\, dt = \log\left(\frac{Z_{\varphi^{\mathrm{e}}}(T)}{Z_{\mu}(T)}\right) - \log\left(\frac{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(0)\big)}\right) \tag{5.6.23}$$

$$= \log\left(\frac{Z_{\varphi^{\mathrm{e}}}(T)\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(0)\big)}{Z_{\mu}(T)\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(T)\big)}\right). \tag{5.6.24}$$

Hence, the cumulative effect of the drift process reveals the remarkable observation that in order to compute this effect over a period of time $[0, T]$ using past data, there is no need to estimate or even know the covariance structure at all. Moreover, the above expression for the cumulative effect of the drift process is in terms of observable quantities.

Proposition 5.6.3. *Let $\mathbf{S}^{\mathrm{E}}$ generate the portfolio φ^{e} (the entropy-weighted portfolio) with drift process $\Theta_{\varphi^{\mathrm{e}}}$, and suppose that*

$$\lim_{T\to\infty} \frac{1}{T} \log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(T)\big) \quad = \quad 0, \qquad a.s. \tag{5.6.25}$$

Then,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\gamma_{\varphi^{\mathrm{e}}}(t) - \gamma_\mu(t) - \frac{\gamma_\mu^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \right) dt \quad = \quad 0, \qquad a.s. \tag{5.6.26}$$

Proof. Equation (5.6.8) of Theorem 5.6.2 implies that the semimartingale decomposition of the logarithm of the market entropy process satisfies

$$d \log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \quad = \quad d \log \Big(Z_{\varphi^{\mathrm{e}}}(t) / Z_\mu(t) \Big) - \frac{\gamma_\mu^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\, dt, \qquad t \in [0, T], \qquad \text{a.s.} \tag{5.6.27}$$

By appealing to (2.6.11) of Lemma 2.6.4, we obtain, a.s., for $t \in [0, T]$,

$$d \log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \quad = \quad \big(\gamma_{\varphi^{\mathrm{e}}}(t) - \gamma_\mu(t)\big)\, dt + \sum_{\nu=1}^n \big(\xi_{\varphi^{\mathrm{e}}\nu}(t) - \xi_{\mu\nu}(t)\big)\, dW_\nu(t) - \frac{\gamma_\mu^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\, dt \tag{5.6.28}$$

$$= \quad \left(\gamma_{\varphi^{\mathrm{e}}}(t) - \gamma_\mu(t) - \frac{\gamma_\mu^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \right) dt + \sum_{\nu=1}^n \big(\xi_{\varphi^{\mathrm{e}}\nu}(t) - \xi_{\mu\nu}(t)\big)\, dW_\nu(t). \tag{5.6.29}$$

Therefore, we have, a.s., for $T \in [0, \infty)$,

$$\log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(T)\big) \quad = \quad \int_0^T \left(\gamma_{\varphi^{\mathrm{e}}}(t) - \gamma_\mu(t) - \frac{\gamma_\mu^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \right) dt + \int_0^T \sum_{\nu=1}^n \big(\xi_{\varphi^{\mathrm{e}}\nu}(t) - \xi_{\mu\nu}(t)\big)\, dW_\nu(t). \tag{5.6.30}$$

Taking the long-term time average, i.e., the limit as $T \to \infty$ of the time average on both sides of (5.6.30), gives

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{T} \log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(T)\big) \quad &= \quad \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\gamma_{\varphi^{\mathrm{e}}}(t) - \gamma_\mu(t) - \frac{\gamma_\mu^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \right) dt \\ &\quad + \lim_{T\to\infty} \frac{1}{T} \int_0^T \sum_{\nu=1}^n \big(\xi_{\varphi^{\mathrm{e}}\nu}(t) - \xi_{\mu\nu}(t)\big)\, dW_\nu(t). \end{aligned}$$

Now, the second term in the above expression vanishes by equation (3.2.2) of Lemma 3.2.1, since the second term in (5.6.28) is a continuous local martingale, so that a.s.,

$$\lim_{T\to\infty} \frac{1}{T} \log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(T)\big) \quad = \quad \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\gamma_{\varphi^{\mathrm{e}}}(t) - \gamma_\mu(t) - \frac{\gamma_\mu^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \right) dt. \tag{5.6.31}$$

Hence, (5.6.25) implies that $\log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(T)\big)$ vanishes, and we have the desired result

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\gamma_{\varphi^{\mathrm{e}}}(t) - \gamma_\mu(t) - \frac{\gamma_\mu^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \right) dt \quad = \quad 0, \qquad \text{a.s.} \tag{5.6.32}$$

■

Therefore, in order to have long-term stability, it is necessary that, on average, $\gamma_{\varphi^{\mathrm{e}}}(t)$ be greater than $\gamma_\mu(t)$. Consequently, over the long term, we can expect that the entropy-weighted portfolio will outperform the market portfolio. The following series of graphs demonstrates this.

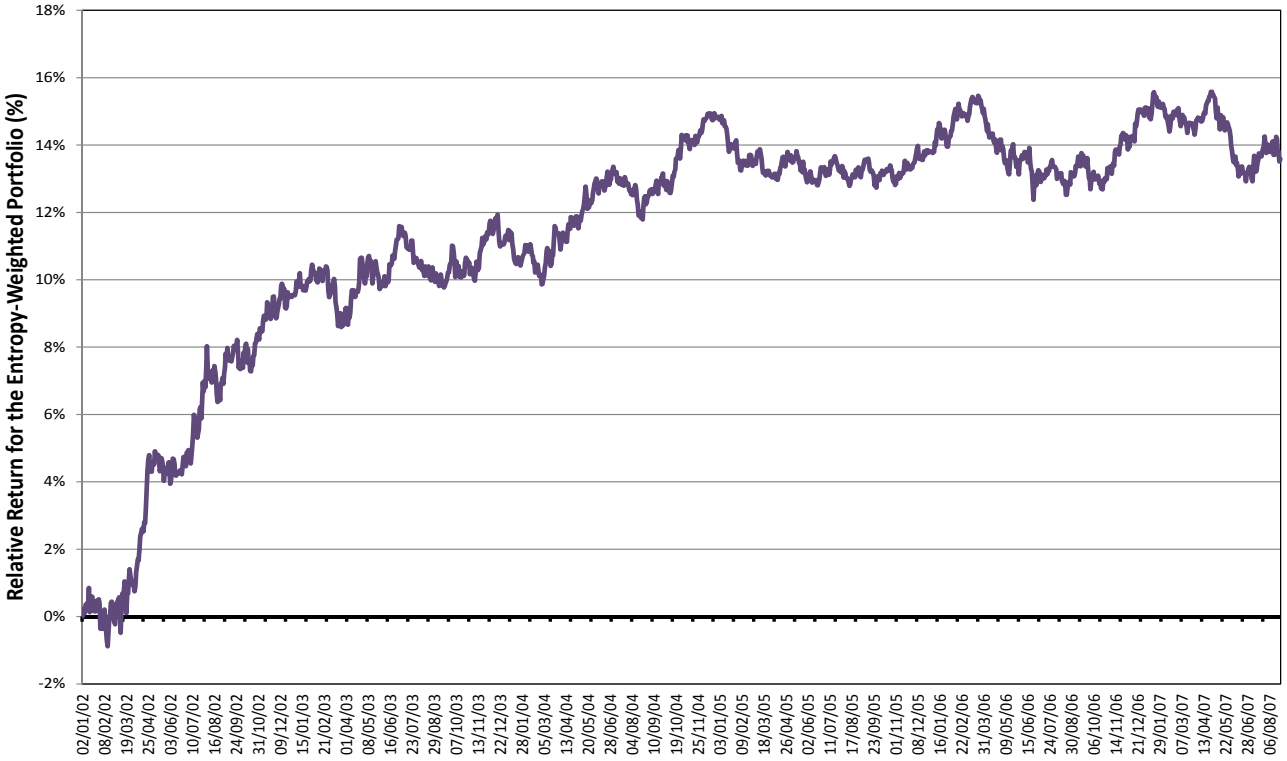

Figure 5.1: $\log\left(Z_{\varphi^e}(t)/Z_\mu(t)\right)$ for the Entropy-Weighted Portfolio.

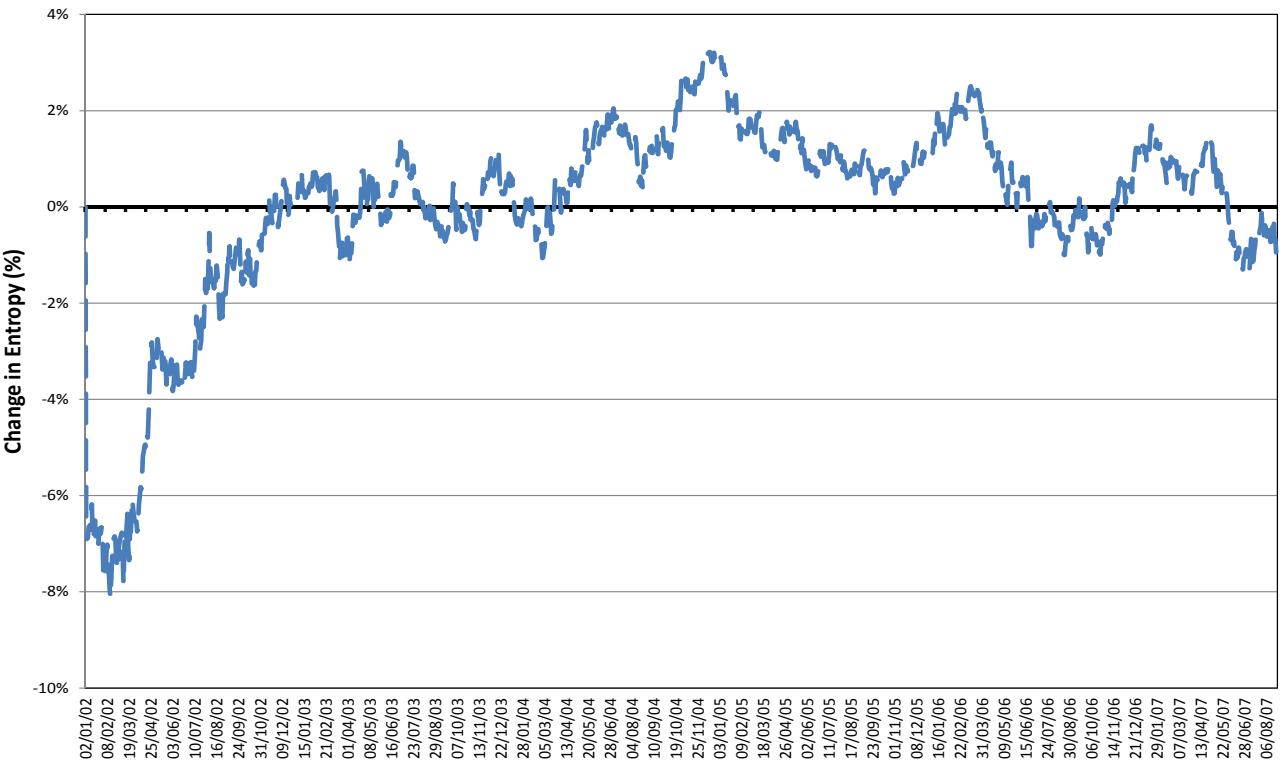

Figure 5.2: Change in Market Entropy (Centered to have zero sample mean).

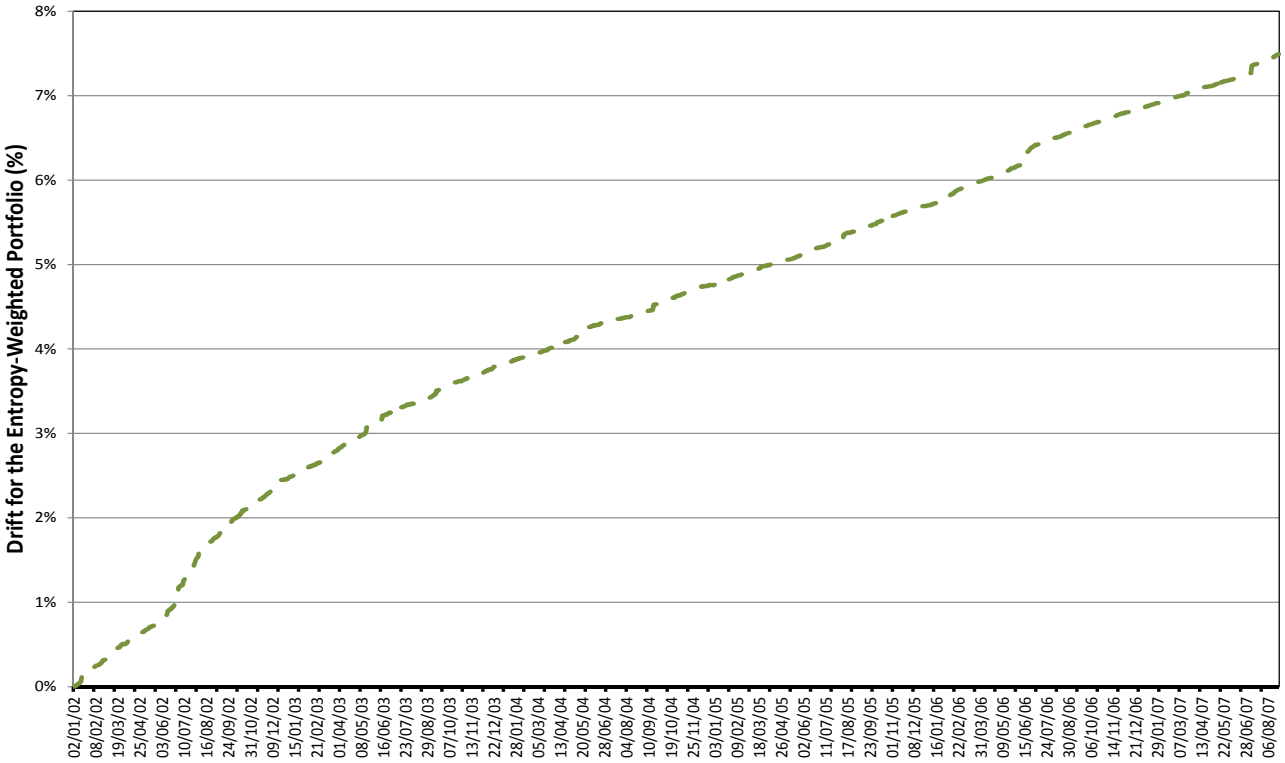

Figure 5.3: Drift Process for the Entropy-Weighted Portfolio.

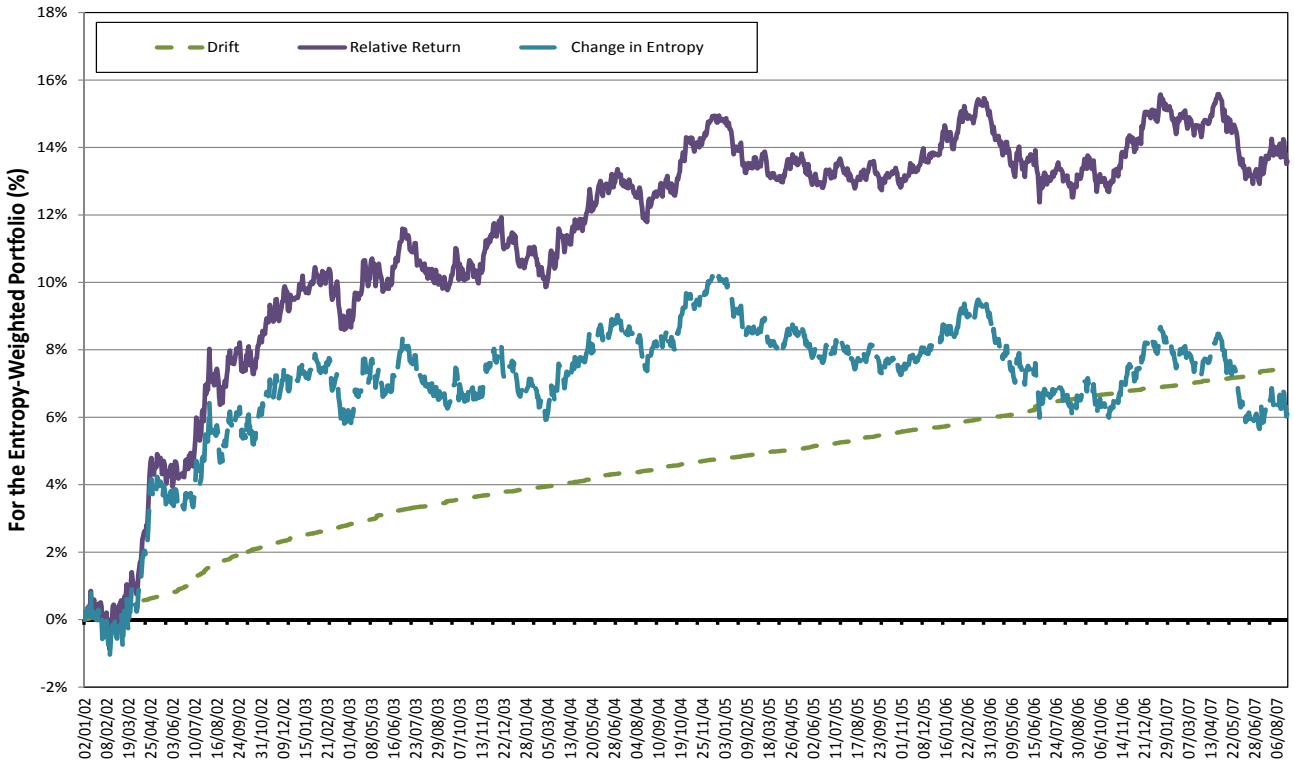

Figure 5.4: The Performance of the Entropy-Weighted Portfolio.

Note that the data utilised in the production of these graphs (and the remaining ones that follow in this chapter) are taken from I-Net Bridge for the South African JSE ALSI Top40 Index, over the time period from the 2nd January 2002 to the 23rd August 2007.[5] Any new stocks entering or exiting the equity market over the time period under consideration have been ignored, for the sake of simplicity. The assumption that there are only 36 stocks within the South African JSE ALSI Index has also been made. Within the context of the South African equity market, after rebalancing every day at the end of each day, then the closing share prices, the corresponding free float factors[6] and the number of shares in issue on that particular day, can be observed for the JSE ALSI Top40 Index. Then, the changes in value of all the capitalisation-related time series for the subsequent day by using these values can be calculated. For each of the time series, the cumulative value of these changes is obtained in the graphs. Let us briefly recall the assumptions we made at the outset of this dissertation: the number of companies in the market is finite and fixed; neither are there new companies founded nor do existing companies go bankrupt; companies neither enter nor leave the market; the total number of shares of each company remains constant and companies do not merge or break up.

Figure 5.1 reveals that the cumulative relative return process of the entropy-weighted portfolio is generally increasing over the entire duration of the time period under investigation and it also reveals that the entropy-weighted portfolio φ^{e} generated a greater return than the market portfolio μ over the time period under consideration. Now, Figure 5.2 shows the cumulative change in the entropy function over the considered time period, which has been adjusted (centered) to have zero sample mean. From the figure, we can see that the cumulative change in the market entropy is in the region of -0.886% or -0.161% a year, over the time horizon under consideration. Thus, the change in the entropy is negative, which is comparably small to the total variation over the considered time period. We can also see that the time series evolves in a mean-reverting fashion. Since the market entropy function is a measure of diversity, it is completely reasonable to expect that the diversity component of the equity market would maintain some form of stability over the entire duration of the time period under investigation and over the long term. Thus, it is reasonable to assume that the equity market remains fairly stable in its nature over the long term. Thus, the cumulative changes in the entropy-weighted portfolio demonstrates a marked stability. Let us now turn to the drift process of the entropy-weighted portfolio, given in Figure 5.3. The drift process is rather close to being a pure trend process. In total, Figure 5.4 combines all three results of the performance of the entropy-weighted portfolio into a single figure. Here, each process shows the cumulative value of the daily changes induced in the corresponding process by capital gains or losses in the stocks. From the equations above, it is clear that the relative return curve is given by adding the drift process curve to the changes in the entropy curve. Now, notice that the relative return process, is the dominant term over the time horizon under consideration. Due to the fact that we are making use of the *free float market capitalisation*[7] in the calculation concerned, any changes in the *free float factor* or any changes of the *shares in issue* has an effect on entropy, which will subsequently affect the figures.

5.6.2 The Modified Entropy-Weighted Portfolio

Recall from (4.6.8) of Definition 4.6.2 and equation (4.6.10) that the modified market entropy process, $\mathbf{G}(\boldsymbol{\mu}) \equiv \mathbf{S}_c^{\mathrm{E}}(\boldsymbol{\mu}) = \{\mathbf{S}_c^{\mathrm{E}}(\boldsymbol{\mu}(t)), t \in [0,T]\}$, $\mathbf{G} \equiv \mathbf{S}_c^{\mathrm{E}} : \Delta^{n-1} \to (0,\infty)$, is given by

$$\mathbf{G}\big(\boldsymbol{\mu}(t)\big) \;\equiv\; \mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \;=\; c + \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \;=\; \sum_{k=1}^{n} \mu_k(t)\big(c - \log \mu_k(t)\big), \qquad t \in [0,T]. \tag{5.6.33}$$

For the modified market entropy process for all $i = 1, 2, \ldots, n$ and $t \in [0,T]$, from (4.6.12) we have

$$D_i \log \mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \;=\; \frac{D_i \mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \;=\; \frac{D_i \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \;=\; \frac{-\log \mu_i(t) - 1}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}.$$

[5] Note that the graphs presented here were also put forth by Wan (2007) on the same set of South African equity market data.

[6] The *free float factor* is the percentage of shares remaining after the block ownership and restricted shares are subtracted from the total number of shares

[7] The shares of a particular company can be held by various investors. For instance, the shares may be held by a government faction or, some or most of the shares may be held by the shareholders, executives or directors of the company.

Therefore, from (5.2.13) of Theorem 5.2.2, with $\mathbf{G}(\mathbf{x}) = \mathbf{S}_c^{\mathrm{E}}(\mathbf{x})$, we obtain the following weights for the portfolio generated by the modified entropy function $\mathbf{S}_c^{\mathrm{E}}$,

$$\begin{aligned}
\varphi_i^{(\mathsf{e},c)}(t) &= \left(D_i \log \mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{j=1}^n \mu_j(t)\, D_j \log \mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \right) \mu_i(t) \\
&= \left(\frac{-\log \mu_i(t) - 1}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} + 1 - \sum_{j=1}^n \mu_j(t) \left[\frac{-\log \mu_j(t) - 1}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \right] \right) \mu_i(t) \\
&= \left(\frac{-\log \mu_i(t) - 1}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} + 1 - \sum_{j=1}^n \frac{\mu_j(t)\big(-\log \mu_j(t) - 1\big)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \right) \mu_i(t) \\
&= \left(\frac{-\log \mu_i(t) - 1}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} + 1 - \sum_{j=1}^n \frac{-\mu_j(t)\big(\log \mu_j(t) + 1\big)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \right) \mu_i(t) \\
&= \frac{\mu_i(t)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \left(-\log \mu_i(t) - 1 + \mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) + \sum_{j=1}^n \mu_j(t)\big(\log \mu_j(t) + 1\big) \right) \\
&= \frac{\mu_i(t)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \left(-\log \mu_i(t) - 1 + \mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) + \sum_{j=1}^n \big(\mu_j(t) \log \mu_j(t) + \mu_j(t)\big) \right) \\
&= \frac{\mu_i(t)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \left(-\log \mu_i(t) - 1 + \mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) + \sum_{j=1}^n \mu_j(t) \log \mu_j(t) + \sum_{j=1}^n \mu_j(t) \right) \\
&= \frac{\mu_i(t)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \left(-\log \mu_i(t) - 1 + \mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) + \sum_{j=1}^n \mu_j(t) \log \mu_j(t) + 1 \right) \\
&= \frac{\mu_i(t)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \left(-\log \mu_i(t) + \mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) + \sum_{j=1}^n \mu_j(t) \log \mu_j(t) \right),
\end{aligned}$$

since $\sum_{i=1}^n \mu_i(t) = 1$. Now, since $\sum_{k=1}^n \mu_k(t) \log \mu_k(t) = c - \mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$, the above expression simplifies to

$$\begin{aligned}
\varphi_i^{(\mathsf{e},c)}(t) &= \frac{\mu_i(t)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \left(-\log \mu_i(t) + \mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) + c - \mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \right) \\
&= \frac{\mu_i(t)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \Big(-\log \mu_i(t) + c \Big) \\
&= \frac{\mu_i(t)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \Big(c - \log \mu_i(t) \Big) \\
&= \frac{\mu_i(t)\big(c - \log \mu_i(t)\big)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)},
\end{aligned}$$

so that for all $i = 1, 2, \ldots, n$ and $t \in [0, T]$, we have the *modified entropy weights*, i.e. the weights corresponding to the generating function $\mathbf{S}_c^{\mathrm{E}}$,

$$\varphi_i^{(\mathsf{e},c)}(t) = \frac{\mu_i(t)\big(c - \log \mu_i(t)\big)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} = \frac{\mu_i(t)\big(c - \log \mu_i(t)\big)}{\sum_{j=1}^n \mu_j(t)\big(c - \log \mu_j(t)\big)}. \tag{5.6.34}$$

Thus, the weights corresponding to the generating function $\mathbf{S}_c^{\mathrm{E}}$ can be formalised in the following definition.

Definition 5.6.4 (Modified Entropy-Weighted Portfolio)**.** *Let μ be the market portfolio. The portfolio process $\varphi^{(\mathsf{e},c)} = \big\{\varphi^{(\mathsf{e},c)}(t) = \big(\varphi_1^{(\mathsf{e},c)}(t), \varphi_2^{(\mathsf{e},c)}(t), \ldots, \varphi_n^{(\mathsf{e},c)}(t)\big), t \in [0, T]\big\}$, with weights defined by* (5.6.34)*, i.e.,*

$$\varphi_i^{(\mathsf{e},c)}(t) \triangleq \frac{\mu_i(t)\big(c - \log \mu_i(t)\big)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} = \frac{\mu_i(t)\big(c - \log \mu_i(t)\big)}{\sum_{j=1}^n \mu_j(t)\big(c - \log \mu_j(t)\big)}, \tag{5.6.35}$$

for all $i = 1, 2, \ldots, n$, and $t \in [0, T]$, is called the ***modified entropy-weighted portfolio (process)****.*

Thus, the generating function $\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$, $\mathbf{S}_c^{\mathrm{E}} : \Delta^{n-1} \to (0,\infty)$, generates the *modified entropy-weighted portfolio*. It can be easily verified that $\varphi^{(\mathsf{e},c)}$ satisfies the requirements of Definition 2.2.16, since we know that the market weights are bounded on $[0,\infty)$ and the modified market entropy process, $\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$, is also bounded on $[0,\infty)$ according to the bounds derived in (4.6.11), i.e., $c < \mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \le c + \log(n)$. Moreover, the *modified entropy weights* satisfy the following

$$\begin{aligned}
\sum_{i=1}^{n} \varphi_i^{(\mathsf{e},c)}(t) &= \sum_{i=1}^{n} \frac{\mu_i(t)\big(c - \log \mu_i(t)\big)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \\
&= \frac{1}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^{n} \mu_i(t)\big(c - \log \mu_i(t)\big) \\
&= \frac{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \\
&= 1.
\end{aligned}$$

The weight ratios, for $i = 1, 2, \ldots, n$,

$$\frac{\varphi_i^{(\mathsf{e},c)}(t)}{\mu_i(t)} = \frac{c - \log \mu_i(t)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}, \qquad t \in [0,T], \tag{5.6.36}$$

decrease with increasing market weight $\mu_i(t)$, this follows from Proposition 5.5.2, since $\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$ is a measure of diversity that generates the diversity generated portfolio, the modified entropy-weighted portfolio, i.e., if $\mu_i(t) \ge \mu_j(t)$ for some $i > j$, then since the logarithm function is monotone increasing, we have for all $t \in [0,T]$,

$$\log \mu_i(t) \ge \log \mu_j(t),$$

therefore since $\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$ is strictly positive for all $t \in [0,T]$ (as $c > 0$), we obtain for some $i > j$, and for all $t \in [0,T]$,

$$\frac{c - \log \mu_i(t)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \le \frac{c - \log \mu_j(t)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)},$$

which by (5.6.36), suggests that if $\mu_i(t) \ge \mu_j(t)$ for some $i > j$, we have the result specified in Proposition 5.5.2, for some $i > j$,

$$\frac{\varphi_i^{(\mathsf{e},c)}(t)}{\mu_i(t)} \le \frac{\varphi_j^{(\mathsf{e},c)}(t)}{\mu_j(t)}.$$

Hence, compared to the market portfolio μ, the modified entropy-weighted portfolio $\varphi^{(\mathsf{e},c)}$ is less concentrated than the market portfolio μ in those stocks with the highest market weights (i.e., $\varphi^{(\mathsf{e},c)}$, relative to the market, is underweighted in the larger stocks), and is more concentrated than the market portfolio μ in those stocks with the lowest market weights (i.e., $\varphi^{(\mathsf{e},c)}$, relative to the market, is overweighted in the smaller stocks). In particular, as $\mu_i(t) \to 1$, we have

$$\lim_{\mu_i(t)\to 1} \frac{\varphi_i^{(\mathsf{e},c)}(t)}{\mu_i(t)} = \lim_{\mu_i(t)\to 1} \frac{c - \log \mu_i(t)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} = 1.$$

Now, from (4.6.13), (4.6.14) and (5.6.5), we have for all $i, j = 1, 2, \ldots, n$, and $t \in [0,T]$,

$$D_{ij}\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) = D_{ij}\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) = \begin{cases} -\dfrac{1}{\mu_i(t)} & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \ne j. \end{cases} \tag{5.6.37}$$

The above expression in conjunction with (5.2.14) of Theorem 5.2.2 with $\mathbf{G}(\mathbf{x}) = \mathbf{S}_c^{\mathrm{E}}(\mathbf{x})$, yields the following drift process corresponding to the modified entropy function, and thus to the modified entropy-weighted portfolio

process

$$\begin{aligned} d\Theta^{(\mathsf{e},c)}(t) \;=\; d\Theta_{\varphi^{(\mathsf{e},c)}}(t) &= \frac{-1}{2\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i,j=1}^{n} D_{ij}\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt \\ &= \frac{-1}{2\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i,j=1}^{n} D_{ij}\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt \\ &= \frac{-1}{2\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^{n} \left[-\frac{1}{\mu_i(t)}\right] \mu_i^2(t)\tau_{ii}(t)\,dt \\ &= \frac{1}{2\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^{n} \mu_i(t)\tau_{ii}(t)\,dt, \end{aligned}$$

which by (2.4.32) of Lemma 2.4.7, or by equations (2.4.36) and (2.12.95), reduces to

$$d\Theta^{(\mathsf{e},c)}(t) \;=\; d\Theta_{\varphi^{(\mathsf{e},c)}}(t) \;=\; \frac{\gamma_\mu^*(t)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\,dt, \tag{5.6.38}$$

equivalently,

$$\mathfrak{g}^{(\mathsf{e},c)}(t) \;=\; \mathfrak{g}_{\varphi^{(\mathsf{e},c)}}(t) \;=\; \frac{\gamma_\mu^*(t)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}. \tag{5.6.39}$$

Since the weights of the market portfolio are all nonnegative (positive), we have by (2.4.37) of Proposition 2.4.8 that for a strictly long-only portfolio the excess growth rate of the market is nonnegative (positive), i.e., $\gamma_\mu^*(t) \geq 0$ for all $t \in [0,T]$. In addition, Lemma 2.4.12, (2.4.55), and Lemma 2.4.14, (2.4.56) demonstrate that in a *nondegenerate market*, the excess growth rate of the market $\gamma_\mu^*(t)$ has a positive lower bound. Furthermore, (4.2.6) of Proposition 4.2.3 demonstrates that in a market that is both *nondegenerate* and *diverse*, the market excess growth rate has a positive lower bound. If the market is diverse then Proposition 4.7.1 in conjunction with (4.6.11) imply that the modified market entropy process is bounded and thus also has a positive lower bound, given by $\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) > c$ (since $c > 0$). Hence, these two concepts together indicate that in a nondegenerate and diverse market, the drift process of the modified entropy-weighted portfolio (5.6.39), must also have a positive lower bound. Therefore, the drift process is nonnegative (positive) $\mathfrak{g}^{(\mathsf{e},c)}(t) = \frac{d\Theta^{(\mathsf{e},c)}(t)}{dt} \geq 0$ and $\Theta^{(\mathsf{e},c)}$ is nondecreasing (increasing) since $\mathfrak{g}^{(\mathsf{e},c)}(t) = \frac{d\Theta^{(\mathsf{e},c)}(t)}{dt} \geq 0$, i.e., the gradient of the drift process is nonnegative (positive), which follows from Proposition 5.5.2 since the modified entropy function $\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$ is a measure of diversity that generates the diversity generated portfolio $\varphi^{(\mathsf{e},c)}$ with drift $\Theta^{(\mathsf{e},c)}$. Thus, the modified entropy-weighted portfolio has an increasing drift process. Moreover, by (5.6.38) in conjunction with Theorem 5.2.2 and equation (5.2.2), the logarithmic return process of the modified entropy-weighted portfolio $\varphi^{(\mathsf{e},c)}$, relative to the market portfolio satisfies, for all $t \in [0,T]$, a.s.,

$$d\log\Big(Z_{\varphi^{(\mathsf{e},c)}}(t)/Z_\mu(t)\Big) \;=\; d\log\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) + \frac{\gamma_\mu^*(t)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\,dt, \tag{5.6.40}$$

or, by (5.2.8), for all $T \in [0,\infty)$, the relative performance of the modified entropy-weighted portfolio $\varphi^{(\mathsf{e},c)}$, can alternatively be represented in integral form as

$$\log\left(\frac{Z_{\varphi^{(\mathsf{e},c)}}(T)}{Z_\mu(T)}\right) \;=\; \log\left(\frac{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(0)\big)}\right) + \int_0^T \frac{\gamma_\mu^*(t)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\,dt, \tag{5.6.41}$$

if $Z_{\varphi^{(\mathsf{e},c)}}(0) = Z_\mu(0)$, otherwise the integral representation is given by

$$\log\left(\frac{Z_{\varphi^{(\mathsf{e},c)}}(T)}{Z_\mu(T)}\right) \;=\; \log\left(\frac{Z_{\varphi^{(\mathsf{e},c)}}(0)}{Z_\mu(0)}\right) + \log\left(\frac{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(0)\big)}\right) + \int_0^T \frac{\gamma_\mu^*(t)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\,dt, \tag{5.6.42}$$

or, as per equation (5.2.7), we have

$$\log\left(\frac{Z_{\varphi^{(\mathsf{e},c)}}(T)}{Z_{\varphi^{(\mathsf{e},c)}}(0)}\right) \;=\; \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log\left(\frac{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(0)\big)}\right) + \int_0^T \frac{\gamma_\mu^*(t)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\,dt. \tag{5.6.43}$$

5.6.3 The $\mathbf{D}_p$-Weighted (Diversity-Weighted) Index Portfolio

Recall from (4.6.16) of Definition 4.6.3 that the market $\mathbf{D}_p$ index process, $\mathbf{G}(\boldsymbol{\mu}) \equiv \mathbf{D}_p(\boldsymbol{\mu}) = \big\{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big), t \in [0,T]\big\}$, $\mathbf{G} \equiv \mathbf{D}_p : \Delta^{n-1} \to (0,\infty)$, is given, for $0 < p < 1$, by

$$\mathbf{G}\big(\boldsymbol{\mu}(t)\big) \equiv \mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) = \left(\sum_{k=1}^{n} \big(\mu_k(t)\big)^p\right)^{\frac{1}{p}}, \qquad t \in [0,T]. \tag{5.6.44}$$

Thus $\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p = \sum_{k=1}^{n} \big(\mu_k(t)\big)^p$, so that from (4.6.20) and (4.6.26), we obtain for all $i = 1, 2, \ldots, n$, and $t \in [0,T]$,

$$D_i\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) = \frac{\big(\mu_i(t)\big)^{p-1}\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p} = \big(\mu_i(t)\big)^{p-1}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p}. \tag{5.6.45}$$

Subsequently, from equations (4.6.20), (4.6.26) and (5.6.45), we obtain for all $i = 1, 2, \ldots, n$, and $t \in [0,T]$,

$$D_i \log \mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) = \frac{D_i\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)} = \frac{\big(\mu_i(t)\big)^{p-1}}{\sum_{k=1}^{n}\big(\mu_k(t)\big)^p} = \frac{\big(\mu_i(t)\big)^{p-1}}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p}.$$

Therefore, from (5.2.13) of Theorem 5.2.2, with $\mathbf{G}(\mathbf{x}) = \mathbf{D}_p(\mathbf{x})$, we obtain the following weights for the portfolio generated by the $\mathbf{D}_p$-function, $\mathbf{D}_p$,

$$\begin{aligned}
\varphi_i^{(p)}(t) &= \left(D_i \log \mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{j=1}^{n} \mu_j(t)\, D_j \log \mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\right)\mu_i(t) \qquad (5.6.46)\\
&= \left(\frac{\big(\mu_i(t)\big)^{p-1}}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p} + 1 - \sum_{j=1}^{n} \mu_j(t)\left[\frac{\big(\mu_j(t)\big)^{p-1}}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p}\right]\right)\mu_i(t)\\
&= \left(\frac{\big(\mu_i(t)\big)^{p-1}}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p} + 1 - \sum_{j=1}^{n} \frac{\big(\mu_j(t)\big)^{p}}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p}\right)\mu_i(t)\\
&= \frac{\mu_i(t)}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p}\left(\big(\mu_i(t)\big)^{p-1} + \Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^p - \sum_{j=1}^{n}\big(\mu_j(t)\big)^p\right).
\end{aligned}$$

Now, since $\sum_{k=1}^{n}\big(\mu_k(t)\big)^p = \big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p$, the above expression simplifies to

$$\begin{aligned}
\varphi_i^{(p)}(t) &= \frac{\mu_i(t)}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p}\left(\big(\mu_i(t)\big)^{p-1} + \Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^p - \Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^p\right)\\
&= \frac{\mu_i(t)}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p}\left(\big(\mu_i(t)\big)^{p-1}\right)\\
&= \frac{\big(\mu_i(t)\big)^p}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p}, \qquad (5.6.47)
\end{aligned}$$

so that for all $i = 1, 2, \ldots, n$ and $t \in [0,T]$, we have the $\mathbf{D}_p$-*weights*, i.e. the weights corresponding to the generating function $\mathbf{D}_p$,

$$\varphi_i^{(p)}(t) = \frac{\big(\mu_i(t)\big)^p}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p} = \frac{\big(\mu_i(t)\big)^p}{\sum_{j=1}^{n}\big(\mu_j(t)\big)^p} = \frac{\mu_i^p(t)}{\sum_{j=1}^{n}\mu_j^p(t)} = \frac{\mu_i^p(t)}{\mu_1^p(t) + \cdots + \mu_n^p(t)}. \tag{5.6.48}$$

Thus, the weights corresponding to the generating function $\mathbf{D}_p$ can be formalised in the following definition.

Definition 5.6.5 ($\mathbf{D}_p$-Weighted Index Portfolio)**.** *Let μ be the market portfolio. The portfolio process $\varphi^{(p)} = \big\{\varphi^{(p)}(t) = \big(\varphi_1^{(p)}(t), \varphi_2^{(p)}(t), \ldots, \varphi_n^{(p)}(t)\big), t \in [0,T]\big\}$, with weights defined by* (5.6.48)*, i.e.,*

$$\varphi_i^{(p)}(t) \triangleq \frac{\big(\mu_i(t)\big)^p}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p} = \frac{\mu_i^p(t)}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p} \tag{5.6.49}$$

$$= \left(\frac{\mu_i(t)}{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}\right)^p, \tag{5.6.50}$$

for all $i = 1, 2, \ldots, n$, and $t \in [0, T]$, and defined for some arbitrary but fixed $p \in (0, 1)$ in terms of the market portfolio μ, is called the ***$\mathbf{D}_p$-weighted index portfolio (process)****, or generically, the* ***diversity-weighted index portfolio (process)****.*

Thus, the generating function $\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)$, $\mathbf{D}_p : \Delta^{n-1} \to (0, \infty)$, generates the *diversity-weighted portfolio*. It can be easily verified that $\varphi^{(p)}$ satisfies the requirements of Definition 2.2.16, since we know that the market weights are bounded on $[0, \infty)$ and the $\mathbf{D}_p$ index process, $\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)$, is also bounded on $[0, \infty)$ according to the bounds derived in (4.6.17), i.e., $1 < \big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p \leq n^{1-p}$, and (4.6.18), i.e., $1 < \mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) \leq n^{(1-p)/p}$. Moreover, the $\mathbf{D}_p$-*weights* satisfy the following

$$\begin{aligned}
\sum_{i=1}^{n} \varphi_i^{(p)}(t) &= \sum_{i=1}^{n} \frac{\big(\mu_i(t)\big)^p}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p} \\
&= \frac{1}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p} \sum_{i=1}^{n} \big(\mu_i(t)\big)^p \\
&= \frac{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p} \\
&= 1.
\end{aligned}$$

The weight ratios, for $i = 1, 2, \ldots, n$, satisfy

$$\frac{\varphi_i^{(p)}(t)}{\mu_i(t)} = \frac{\big(\mu_i(t)\big)^{p-1}}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p}, \qquad t \in [0, T], \tag{5.6.51}$$

which decrease with increasing market weight $\mu_i(t)$, this follows from Proposition 5.5.2, since $\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)$ is a measure of diversity that generates the diversity generated portfolio, the $\mathbf{D}_p$-weighted index portfolio (or, the diversity-weighted index portfolio), i.e., if $\mu_i(t) \geq \mu_j(t)$ for some $i > j$, then since $0 < \mu_i(t) < 1$ for all $i = 1, 2, \ldots, n$, and $0 < p < 1$ (i.e., $-1 < p - 1 < 0$), we deduce that for all $t \in [0, T]$,

$$\big(\mu_i(t)\big)^{p-1} \leq \big(\mu_j(t)\big)^{p-1},$$

therefore since $\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)$ is strictly positive for all $t \in [0, T]$ and $p \in (0, 1)$, $\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p$ is strictly positive for all $t \in [0, T]$ and $p \in (0, 1)$, and we obtain for some $i > j$, and for all $t \in [0, T]$,

$$\frac{\big(\mu_i(t)\big)^{p-1}}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p} \leq \frac{\big(\mu_j(t)\big)^{p-1}}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p},$$

which by (5.6.51), suggests that if $\mu_i(t) \geq \mu_j(t)$ for some $i > j$, we have the result specified in Proposition 5.5.2, for some $i > j$, namely

$$\frac{\varphi_i^{(p)}(t)}{\mu_i(t)} \leq \frac{\varphi_j^{(p)}(t)}{\mu_j(t)}.$$

Hence, compared to the market portfolio μ, the $\mathbf{D}_p$-weighted index portfolio $\varphi^{(p)}$ is less concentrated than the market portfolio μ in those stocks with the highest market weights, i.e., $\varphi^{(p)}$, relative to the market, is underweighted in the larger stocks and thus the $\mathbf{D}_p$-weighted index portfolio decreases the proportion(s) held in the largest stock(s), and is more concentrated than the market portfolio μ in those stocks with the lowest market weights, i.e., $\varphi^{(p)}$, relative to the market, is overweighted in the smaller stocks and thus the $\mathbf{D}_p$-weighted index portfolio increases those proportion(s) placed in the smallest stock(s), whilst preserving the relative rankings of all the stocks. In particular, the largest weight of the $\mathbf{D}_p$-weighted index portfolio does not exceed the largest market weight, this is demonstrated as follows, suppose again that $\mu_i(t) \geq \mu_j(t)$ for some $i > j$, then since $0 < \mu_i(t) < 1$ for all $i = 1, 2, \ldots, n$, and $0 < p < 1$ (i.e., $0 < 1 - p < 1$), we deduce that for all $t \in [0, T]$,

$$\big(\mu_i(t)\big)^{1-p} \geq \big(\mu_j(t)\big)^{1-p}.$$

This implies that for the largest market weight, $\mu_{(1)}(t) = \mu_{\max}(t) := \max_{1\le i\le n} \mu_i(t)$, where $\mu_{(1)}(t) \ge \mu_j(t)$ for all $j = 1, 2, \ldots, n$, we have for all $j = 1, 2, \ldots, n$,

$$\big(\mu_{(1)}(t)\big)^{1-p} \quad \ge \quad \big(\mu_j(t)\big)^{1-p}.$$

Consequently, we have for all $t \in [0, T]$,

$$\begin{aligned}
\big(\mu_{(1)}(t)\big)^p \quad = \quad \sum_{j=1}^n \mu_j(t)\big(\mu_{(1)}(t)\big)^p \quad &= \quad \sum_{j=1}^n \big(\mu_j(t)\big)^{1-p}\big(\mu_j(t)\big)^p\big(\mu_{(1)}(t)\big)^p \\
&\le \quad \sum_{j=1}^n \big(\mu_{(1)}(t)\big)^{1-p}\big(\mu_j(t)\big)^p\big(\mu_{(1)}(t)\big)^p \\
&= \quad \sum_{j=1}^n \mu_{(1)}(t)\big(\mu_j(t)\big)^p \\
&= \quad \mu_{(1)}(t)\sum_{j=1}^n \big(\mu_j(t)\big)^p.
\end{aligned}$$

Thus, we have the following inequality

$$\big(\mu_{(1)}(t)\big)^p \quad \le \quad \mu_{(1)}(t)\sum_{j=1}^n \big(\mu_j(t)\big)^p. \tag{5.6.52}$$

Therefore by (5.6.52), we have for the largest weight of the $\mathbf{D}_p$-weighted index portfolio,

$$\begin{aligned}
\varphi_{(1)}^{(p)}(t) \quad = \quad \varphi_{\max}^{(p)}(t) \quad := \quad \max_{1\le i\le n} \varphi_i^{(p)}(t) \quad = \quad \frac{\big(\mu_{(1)}(t)\big)^p}{\big(\mathbf{D}_p(\boldsymbol{\mu}(t))\big)^p} \quad &= \quad \frac{\big(\mu_{(1)}(t)\big)^p}{\sum_{\ell=1}^n \big(\mu_{(\ell)}(t)\big)^p} \\
&= \quad \frac{\big(\mu_{(1)}(t)\big)^p}{\sum_{j=1}^n \big(\mu_j(t)\big)^p} \\
&\le \quad \frac{\mu_{(1)}(t)\sum_{j=1}^n \big(\mu_j(t)\big)^p}{\sum_{j=1}^n \big(\mu_j(t)\big)^p} \\
&= \quad \mu_{(1)}(t),
\end{aligned}$$

so that for all $t \in [0, T]$, we have

$$\varphi_{\max}^{(p)}(t) \quad \equiv \quad \varphi_{(1)}^{(p)}(t) \quad \le \quad \mu_{(1)}(t) \quad \equiv \quad \mu_{\max}(t). \tag{5.6.53}$$

Analogously, the smallest weight of the $\mathbf{D}_p$-weighted index portfolio does exceed the smallest market weight. Thus, for the smallest market weight, $\mu_{(n)}(t) = \mu_{\min}(t) := \min_{1\le i\le n} \mu_i(t)$, where $\mu_j(t) \ge \mu_{(n)}(t)$ for all $j = 1, 2, \ldots, n$, we have for all $j = 1, 2, \ldots, n$,

$$\big(\mu_j(t)\big)^{1-p} \quad \ge \quad \big(\mu_{(n)}(t)\big)^{1-p}.$$

Consequently, we have for all $t \in [0, T]$,

$$\begin{aligned}
\big(\mu_{(n)}(t)\big)^p \quad = \quad \sum_{j=1}^n \mu_j(t)\big(\mu_{(n)}(t)\big)^p \quad &= \quad \sum_{j=1}^n \big(\mu_j(t)\big)^{1-p}\big(\mu_j(t)\big)^p\big(\mu_{(n)}(t)\big)^p \\
&\ge \quad \sum_{j=1}^n \big(\mu_{(n)}(t)\big)^{1-p}\big(\mu_j(t)\big)^p\big(\mu_{(n)}(t)\big)^p \\
&= \quad \sum_{j=1}^n \mu_{(n)}(t)\big(\mu_j(t)\big)^p \\
&= \quad \mu_{(n)}(t)\sum_{j=1}^n \big(\mu_j(t)\big)^p.
\end{aligned}$$

Thus, we have the following inequality

$$\big(\mu_{(n)}(t)\big)^p \quad \geq \quad \mu_{(n)}(t)\sum_{j=1}^{n}\big(\mu_j(t)\big)^p. \tag{5.6.54}$$

Therefore by (5.6.54), we have for the smallest weight of the $\mathbf{D}_p$-weighted index portfolio,

$$\begin{aligned}
\varphi_{(n)}^{(p)}(t) \quad = \quad \varphi_{\min}^{(p)}(t) \quad := \quad \min_{1\leq i\leq n}\varphi_i^{(p)}(t) \quad = \quad \frac{\big(\mu_{(n)}(t)\big)^p}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p} \quad &= \quad \frac{\big(\mu_{(n)}(t)\big)^p}{\sum_{\ell=1}^{n}\big(\mu_{(\ell)}(t)\big)^p} \\
&= \quad \frac{\big(\mu_{(n)}(t)\big)^p}{\sum_{j=1}^{n}\big(\mu_j(t)\big)^p} \\
&\geq \quad \frac{\mu_{(n)}(t)\sum_{j=1}^{n}\big(\mu_j(t)\big)^p}{\sum_{j=1}^{n}\big(\mu_j(t)\big)^p} \\
&= \quad \mu_{(n)}(t),
\end{aligned}$$

so that for all $t \in [0,T]$, we have

$$\varphi_{\min}^{(p)}(t) \quad \equiv \quad \varphi_{(n)}^{(p)}(t) \quad \geq \quad \mu_{(n)}(t) \quad \equiv \quad \mu_{\min}(t). \tag{5.6.55}$$

Now, from (4.6.27) and (4.6.29), we have for all $i,j = 1,2,\ldots,n$, and $t \in [0,T]$,

$$\begin{aligned}
&D_{ij}\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) \\
&\quad = \quad \begin{cases} (p-1)\big(\mu_i(t)\big)^{p-2}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p} - (p-1)\big(\mu_i(t)\big)^{2p-2}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p} & \text{if} \quad i=j, \\ -(p-1)\big(\mu_i(t)\big)^{p-1}\big(\mu_j(t)\big)^{p-1}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p} & \text{if} \quad i\neq j. \end{cases}
\end{aligned} \tag{5.6.56}$$

The above expression in conjunction with (5.2.14) of Theorem 5.2.2 with $\mathbf{G}(\mathbf{x}) = \mathbf{D}_p(\mathbf{x})$, yields the following

drift process corresponding to the $\mathbf{D}_p$-function, and thus to the $\mathbf{D}_p$-weighted index portfolio process

$$
\begin{aligned}
d\Theta^{(p)}(t) &= d\Theta_{\varphi^{(p)}}(t) && (5.6.57)\\
&= \frac{-1}{2\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)} \sum_{i,j=1}^{n} D_{ij}\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt && (5.6.58)\\
&= \frac{-1}{2\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)} \Bigg(\sum_{i=1}^{n} \Big[(p-1)\big(\mu_i(t)\big)^{p-2}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p} - (p-1)\big(\mu_i(t)\big)^{2p-2}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p}\Big]\mu_i^2(t)\tau_{ii}(t)\\
&\qquad + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \Big[-(p-1)\big(\mu_i(t)\big)^{p-1}\big(\mu_j(t)\big)^{p-1}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p}\Big]\mu_i(t)\mu_j(t)\tau_{ij}(t)\Bigg)\,dt\\
&= \frac{-(p-1)}{2} \Bigg(\sum_{i=1}^{n} \Big[\big(\mu_i(t)\big)^{p}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{-p} - \big(\mu_i(t)\big)^{2p}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{-2p}\Big]\tau_{ii}(t)\\
&\qquad - \sum_{\substack{i,j=1\\ i\neq j}}^{n} \big(\mu_i(t)\big)^{p}\big(\mu_j(t)\big)^{p}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{-2p}\tau_{ij}(t)\Bigg)\,dt\\
&= \frac{(1-p)}{2} \Bigg(\sum_{i=1}^{n} \big(\mu_i(t)\big)^{p}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{-p}\tau_{ii}(t) - \sum_{i=1}^{n} \big(\mu_i(t)\big)^{2p}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{-2p}\tau_{ii}(t)\\
&\qquad - \sum_{\substack{i,j=1\\ i\neq j}}^{n} \big(\mu_i(t)\big)^{p}\big(\mu_j(t)\big)^{p}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{-2p}\tau_{ij}(t)\Bigg)\,dt\\
&= \frac{(1-p)}{2} \Bigg(\sum_{i=1}^{n} \big(\mu_i(t)\big)^{p}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{-p}\tau_{ii}(t) - \Bigg[\sum_{i=1}^{n} \big(\mu_i(t)\big)^{2p}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{-2p}\tau_{ii}(t)\\
&\qquad + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \big(\mu_i(t)\big)^{p}\big(\mu_j(t)\big)^{p}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{-2p}\tau_{ij}(t)\Bigg]\Bigg)\,dt\\
&= \frac{(1-p)}{2} \Bigg(\sum_{i=1}^{n} \big(\mu_i(t)\big)^{p}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{-p}\tau_{ii}(t) - \sum_{i,j=1}^{n} \big(\mu_i(t)\big)^{p}\big(\mu_j(t)\big)^{p}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{-2p}\tau_{ij}(t)\Bigg)\,dt\\
&= \frac{(1-p)}{2} \Bigg(\sum_{i=1}^{n} \left[\frac{\big(\mu_i(t)\big)^{p}}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^{p}}\right]\tau_{ii}(t) - \sum_{i,j=1}^{n} \left[\frac{\big(\mu_i(t)\big)^{p}\big(\mu_j(t)\big)^{p}}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^{2p}}\right]\tau_{ij}(t)\Bigg)\,dt\\
&= \frac{(1-p)}{2} \Bigg(\sum_{i=1}^{n} \left[\frac{\big(\mu_i(t)\big)^{p}}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^{p}}\right]\tau_{ii}(t) - \sum_{i,j=1}^{n} \left[\frac{\big(\mu_i(t)\big)^{p}}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^{p}}\right]\left[\frac{\big(\mu_j(t)\big)^{p}}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^{p}}\right]\tau_{ij}(t)\Bigg)\,dt,
\end{aligned}
$$

which, in accordance with (5.6.49) of Definition 5.6.5, simplifies to

$$
d\Theta^{(p)}(t) = d\Theta_{\varphi^{(p)}}(t) = \frac{(1-p)}{2}\left(\sum_{i=1}^{n}\varphi_i^{(p)}(t)\tau_{ii}(t) - \sum_{i,j=1}^{n}\varphi_i^{(p)}(t)\tau_{ij}(t)\varphi_j^{(p)}(t)\right)dt,
$$

which, in turn, by the numéraire invariance property of the excess growth rate of the $\mathbf{D}_p$-weighted index portfolio (2.4.26) of Lemma 2.4.5, reduces to

$$
d\Theta^{(p)}(t) = d\Theta_{\varphi^{(p)}}(t) = (1-p)\gamma^*_{\varphi^{(p)}}(t)\,dt, \tag{5.6.59}
$$

equivalently,

$$
\mathfrak{g}^{(p)}(t) = \mathfrak{g}_{\varphi^{(p)}}(t) = (1-p)\gamma^*_{\varphi^{(p)}}(t). \tag{5.6.60}
$$

Alternatively, the drift process corresponding to the $\mathbf{D}_p$-function, and thus to the $\mathbf{D}_p$-weighted index portfolio

process can also be determined as follows

$$
\begin{aligned}
d\Theta^{(p)}(t) &= d\Theta_{\varphi^{(p)}}(t) \\
&= \frac{-1}{2\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)} \sum_{i,j=1}^{n} D_{ij}\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt \\
&= \frac{-1}{2\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)} \Bigg(\sum_{i=1}^{n} \Big[(p-1)\big(\mu_i(t)\big)^{p-2}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p} - (p-1)\big(\mu_i(t)\big)^{2p-2}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p}\Big]\mu_i^2(t)\tau_{ii}(t) \\
&\qquad + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \Big[-(p-1)\big(\mu_i(t)\big)^{p-1}\big(\mu_j(t)\big)^{p-1}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p}\Big]\mu_i(t)\mu_j(t)\tau_{ij}(t)\Bigg)\,dt \\
&= \frac{-(p-1)}{2} \Bigg(\sum_{i=1}^{n} \Big[\big(\mu_i(t)\big)^{p}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{-p} - \big(\mu_i(t)\big)^{2p}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{-2p}\Big]\tau_{ii}(t) \\
&\qquad - \sum_{\substack{i,j=1\\ i\neq j}}^{n} \big(\mu_i(t)\big)^{p}\big(\mu_j(t)\big)^{p}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{-2p}\tau_{ij}(t)\Bigg)\,dt \\
&= \frac{(1-p)}{2} \left(\sum_{i=1}^{n} \left[\frac{\big(\mu_i(t)\big)^{p}}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^{p}} - \frac{\big(\mu_i(t)\big)^{2p}}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^{2p}} \right]\tau_{ii}(t) - \sum_{\substack{i,j=1\\ i\neq j}}^{n} \left[\frac{\big(\mu_i(t)\big)^{p}\big(\mu_j(t)\big)^{p}}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^{2p}} \right]\tau_{ij}(t) \right)\,dt \\
&= \frac{(1-p)}{2} \left(\sum_{i=1}^{n} \left[\frac{\big(\mu_i(t)\big)^{p}}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^{p}} - \frac{\big(\mu_i(t)\big)^{2p}}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^{2p}} \right]\tau_{ii}(t) - \sum_{\substack{i,j=1\\ i\neq j}}^{n} \left[\frac{\big(\mu_i(t)\big)^{p}}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^{p}} \right]\left[\frac{\big(\mu_j(t)\big)^{p}}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^{p}} \right]\tau_{ij}(t) \right)\,dt \\
&= \frac{(1-p)}{2} \left(\sum_{i=1}^{n} \left[\frac{\big(\mu_i(t)\big)^{p}}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^{p}} \left(1 - \frac{\big(\mu_i(t)\big)^{p}}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^{p}} \right) \right]\tau_{ii}(t) - \sum_{\substack{i,j=1\\ i\neq j}}^{n} \left[\frac{\big(\mu_i(t)\big)^{p}}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^{p}} \right]\left[\frac{\big(\mu_j(t)\big)^{p}}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^{p}} \right]\tau_{ij}(t) \right)\,dt,
\end{aligned}
$$

which, by (5.6.49) of Definition 5.6.5, becomes

$$
\begin{aligned}
d\Theta^{(p)}(t) = d\Theta_{\varphi^{(p)}}(t) &= \frac{(1-p)}{2} \left(\sum_{i=1}^{n} \varphi_i^{(p)}(t)\big(1-\varphi_i^{(p)}(t)\big)\tau_{ii}(t) - \sum_{\substack{i,j=1\\ i\neq j}}^{n} \varphi_i^{(p)}(t)\varphi_j^{(p)}(t)\tau_{ij}(t) \right)\,dt \\
&= \frac{(1-p)}{2} \left(\sum_{i=1}^{n} \Big(\varphi_i^{(p)}(t) - \big(\varphi_i^{(p)}(t)\big)^2\Big)\tau_{ii}(t) - \sum_{\substack{i,j=1\\ i\neq j}}^{n} \varphi_i^{(p)}(t)\varphi_j^{(p)}(t)\tau_{ij}(t) \right)\,dt \\
&= \frac{(1-p)}{2} \left(\sum_{i=1}^{n} \varphi_i^{(p)}(t)\tau_{ii}(t) - \sum_{i=1}^{n} \big(\varphi_i^{(p)}(t)\big)^2\tau_{ii}(t) - \sum_{\substack{i,j=1\\ i\neq j}}^{n} \varphi_i^{(p)}(t)\varphi_j^{(p)}(t)\tau_{ij}(t) \right)\,dt \\
&= \frac{(1-p)}{2} \left(\sum_{i=1}^{n} \varphi_i^{(p)}(t)\tau_{ii}(t) - \left[\sum_{i=1}^{n} \big(\varphi_i^{(p)}(t)\big)^2\tau_{ii}(t) + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \varphi_i^{(p)}(t)\varphi_j^{(p)}(t)\tau_{ij}(t) \right] \right)\,dt \\
&= \frac{(1-p)}{2} \left(\sum_{i=1}^{n} \varphi_i^{(p)}(t)\tau_{ii}(t) - \sum_{i,j=1}^{n} \varphi_i^{(p)}(t)\tau_{ij}(t)\varphi_j^{(p)}(t) \right)\,dt.
\end{aligned}
$$

Alternatively, from (4.6.27) and (5.6.49) of Definition 5.6.5, we have for all $i = 1, 2, \ldots, n$, and $t \in [0, T]$,

$$\begin{aligned}
D_{ii}\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) &= (p-1)\big(\mu_i(t)\big)^{p-2}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p} - (p-1)\big(\mu_i(t)\big)^{2p-2}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p} \\
&= (p-1)\big(\mu_i(t)\big)^{p-2}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p}\left[1 - \big(\mu_i(t)\big)^p\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{-p}\right] \\
&= (p-1)\left[\frac{\big(\mu_i(t)\big)^p}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p}\right]\left[1 - \frac{\big(\mu_i(t)\big)^p}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p}\right]\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} \\
&= (p-1)\varphi_i^{(p)}(t)\Big[1 - \varphi_i^{(p)}(t)\Big]\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} \\
&= (p-1)\varphi_i^{(p)}(t)\big(1 - \varphi_i^{(p)}(t)\big)\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} \qquad (5.6.61) \\
&= (p-1)\varphi_i^{(p)}(t)\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} - (p-1)\big(\varphi_i^{(p)}(t)\big)^2\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)}, \qquad (5.6.62)
\end{aligned}$$

and, from (4.6.29) and (5.6.49) of Definition 5.6.5, we have for all $i \neq j$, $i, j = 1, 2, \ldots, n$, and $t \in [0, T]$,

$$\begin{aligned}
D_{ij}\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) &= -(p-1)\big(\mu_i(t)\big)^{p-1}\big(\mu_j(t)\big)^{p-1}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-2p} \\
&= -(p-1)\left[\frac{\big(\mu_i(t)\big)^p\big(\mu_j(t)\big)^p}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^{2p}}\right]\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)\mu_j(t)} \\
&= -(p-1)\left[\frac{\big(\mu_i(t)\big)^p}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p}\right]\left[\frac{\big(\mu_j(t)\big)^p}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p}\right]\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)\mu_j(t)} \\
&= -(p-1)\varphi_i^{(p)}(t)\varphi_j^{(p)}(t)\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)\mu_j(t)}. \qquad (5.6.63)
\end{aligned}$$

Now, from (5.6.61), (5.6.62) and (5.6.63), we have for all $i, j = 1, 2, \ldots, n$, and $t \in [0, T]$,

$$D_{ij}\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) = \begin{cases} (p-1)\varphi_i^{(p)}(t)\big(1 - \varphi_i^{(p)}(t)\big)\dfrac{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} & \text{if} \quad i = j, \\ -(p-1)\varphi_i^{(p)}(t)\varphi_j^{(p)}(t)\dfrac{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)\mu_j(t)} & \text{if} \quad i \neq j, \end{cases} \qquad (5.6.64)$$

$$= \begin{cases} (p-1)\varphi_i^{(p)}(t)\dfrac{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} - (p-1)\big(\varphi_i^{(p)}(t)\big)^2\dfrac{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} & \text{if} \quad i = j, \\ -(p-1)\varphi_i^{(p)}(t)\varphi_j^{(p)}(t)\dfrac{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)\mu_j(t)} & \text{if} \quad i \neq j. \end{cases} \qquad (5.6.65)$$

The above expression in conjunction with (5.2.14) of Theorem 5.2.2 with $\mathbf{G}(\mathbf{x}) = \mathbf{D}_p(\mathbf{x})$, yields the following

drift process corresponding to the $\mathbf{D}_p$-function, and thus to the $\mathbf{D}_p$-weighted index portfolio process

$$\begin{aligned}
d\Theta^{(p)}(t) &= d\Theta_{\varphi^{(p)}}(t) \\
&= \frac{-1}{2\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)} \sum_{i,j=1}^{n} D_{ij}\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt \\
&= \frac{-1}{2\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)} \left(\sum_{i=1}^{n} \left[(p-1)\varphi_i^{(p)}(t)\big(1-\varphi_i^{(p)}(t)\big)\,\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} \right] \mu_i^2(t)\tau_{ii}(t) \right. \\
&\qquad \left. + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \left[-(p-1)\varphi_i^{(p)}(t)\varphi_j^{(p)}(t)\,\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)\mu_j(t)} \right] \mu_i(t)\mu_j(t)\tau_{ij}(t) \right) dt \\
&= \frac{-1}{2\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)} \left(\sum_{i=1}^{n} \left[(p-1)\varphi_i^{(p)}(t)\,\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} - (p-1)\big(\varphi_i^{(p)}(t)\big)^2\,\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} \right] \mu_i^2(t)\tau_{ii}(t) \right. \\
&\qquad \left. + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \left[-(p-1)\varphi_i^{(p)}(t)\varphi_j^{(p)}(t)\,\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)\mu_j(t)} \right] \mu_i(t)\mu_j(t)\tau_{ij}(t) \right) dt \\
&= \frac{-(p-1)}{2\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)} \left(\sum_{i=1}^{n} \left[\varphi_i^{(p)}(t)\,\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} - \big(\varphi_i^{(p)}(t)\big)^2\,\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} \right] \mu_i^2(t)\tau_{ii}(t) \right. \\
&\qquad \left. - \sum_{\substack{i,j=1\\ i\neq j}}^{n} \left[\varphi_i^{(p)}(t)\varphi_j^{(p)}(t)\,\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)\mu_j(t)} \right] \mu_i(t)\mu_j(t)\tau_{ij}(t) \right) dt,
\end{aligned}$$

which, by (5.6.49) of Definition 5.6.5, becomes

$$\begin{aligned}
d\Theta^{(p)}(t) = d\Theta_{\varphi^{(p)}}(t) &= \frac{(1-p)}{2} \left(\sum_{i=1}^{n} \Big(\varphi_i^{(p)}(t) - \big(\varphi_i^{(p)}(t)\big)^2\Big)\tau_{ii}(t) - \sum_{\substack{i,j=1\\ i\neq j}}^{n} \varphi_i^{(p)}(t)\varphi_j^{(p)}(t)\tau_{ij}(t) \right) dt \\
&= \frac{(1-p)}{2} \left(\sum_{i=1}^{n} \varphi_i^{(p)}(t)\tau_{ii}(t) - \sum_{i=1}^{n} \big(\varphi_i^{(p)}(t)\big)^2\tau_{ii}(t) - \sum_{\substack{i,j=1\\ i\neq j}}^{n} \varphi_i^{(p)}(t)\varphi_j^{(p)}(t)\tau_{ij}(t) \right) dt \\
&= \frac{(1-p)}{2} \left(\sum_{i=1}^{n} \varphi_i^{(p)}(t)\tau_{ii}(t) - \left[\sum_{i=1}^{n} \big(\varphi_i^{(p)}(t)\big)^2\tau_{ii}(t) + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \varphi_i^{(p)}(t)\varphi_j^{(p)}(t)\tau_{ij}(t) \right] \right) dt \\
&= \frac{(1-p)}{2} \left(\sum_{i=1}^{n} \varphi_i^{(p)}(t)\tau_{ii}(t) - \sum_{i,j=1}^{n} \varphi_i^{(p)}(t)\tau_{ij}(t)\varphi_j^{(p)}(t) \right) dt.
\end{aligned}$$

Now, since the $\mathbf{D}_p$ index is a measure of diversity, it is concave and thus its Hessian matrix has no positive eigenvalues. Thus, by Proposition 5.3.5, if $\varphi^{(p)}$ is the portfolio generated by $\mathbf{D}_p$, i.e., the $\mathbf{D}_p$-weighted index portfolio, then $\varphi_i^{(p)}(t) \geq 0$ for all $i = 1, 2, \ldots, n$ and $t \in [0, T]$ and the corresponding drift process is nonnegative (positive), i.e., $\mathfrak{g}^{(p)}(t) \geq 0$, and thus $\Theta^{(p)}$ is nondecreasing (increasing). Moreover, due to fact that the weights of the $\mathbf{D}_p$-weighted index portfolio are all nonnegative (positive) and equal to (5.6.50), we have by (2.4.37) of Proposition 2.4.8 that for a strictly long-only portfolio the excess growth rate of the $\mathbf{D}_p$-weighted index portfolio is nonnegative (positive), i.e., $\gamma^*_{\varphi^{(p)}}(t) \geq 0$ for all $t \in [0, T]$. In addition, Lemma 2.4.12, (2.4.55), and Lemma 2.4.14, (2.4.56) demonstrate that in a *nondegenerate market*, the excess growth rate of the $\mathbf{D}_p$-weighted index portfolio $\gamma^*_{\varphi^{(p)}}(t)$ has a positive lower bound. Hence, since $0 < p < 1$ (i.e., $0 < 1 - p < 1$), the drift process of the $\mathbf{D}_p$-weighted index portfolio, in (5.6.60), must also have a positive lower bound. Therefore, the drift process is nonnegative (positive) $\mathfrak{g}^{(p)}(t) = \frac{d\Theta^{(p)}(t)}{dt} \geq 0$, i.e.,

$$\mathfrak{g}^{(p)}(t) = \mathfrak{g}_{\varphi^{(p)}}(t) = (1-p)\gamma^*_{\varphi^{(p)}}(t) \geq 0,$$

and the drift process $\Theta^{(p)}$ is nondecreasing (increasing) since $\mathfrak{g}^{(p)}(t) = \frac{d\Theta^{(p)}(t)}{dt} \geq 0$, i.e., the gradient of the drift process is nonnegative (positive), which follows from Proposition 5.5.2 since the $\mathbf{D}_p$-function $\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)$ is a measure of diversity that generates the diversity generated portfolio $\varphi^{(p)}$ with drift $\Theta^{(p)}$. Note that as $p \to 1$, the $\mathbf{D}_p$-weighted index portfolio approaches the market portfolio, this is because for all $i = 1, 2, \ldots, n$ and $t \in [0, T]$, we have by (5.6.49) of Definition 5.6.5,

$$\varphi_i^{(1)}(t) \quad = \quad \frac{\mu_i(t)}{\mathbf{D_1}\big(\boldsymbol{\mu}(t)\big)} \quad = \quad \frac{\mu_i(t)}{\sum_{k=1}^n \mu_k(t)} \quad = \quad \mu_i(t).$$

Thus, with $p = 1$, the $\mathbf{D}_p$-weighted index portfolio corresponds to the market portfolio. Furthermore, note that as $p \to 0$, the $\mathbf{D}_p$-weighted index portfolio approaches the equal-weighted portfolio, this is because for all $i = 1, 2, \ldots, n$ and $t \in [0, T]$, we have by (5.6.49) of Definition 5.6.5,

$$\varphi_i^{(0)}(t) \quad = \quad \frac{1}{\big(\mathbf{D_0}\big(\boldsymbol{\mu}(t)\big)\big)^0} \quad = \quad \frac{1}{\sum_{k=1}^n 1} \quad = \quad \frac{1}{n}.$$

Thus, with $p = 0$, the $\mathbf{D}_p$-weighted index portfolio gives the equally-weighted portfolio. The $\mathbf{D}_p$-weighted index portfolios $\varphi^{(p)}$ with $0 < p < 1$ stand between these two extremes, those of capitalisation weighting (i.e., the market portfolio) and of equal weighting (i.e., the equal-weighted portfolio as in the Value-Line Index). With this adjustable parameter p, we may ask ourselves and want to know whether or not there is an ideal or optimal value for p for the $\mathbf{D}_p$-weighted index portfolio. The adjustment of the parameter p purely and simply makes it possible to control the excess return and tracking error of the associated $\mathbf{D}_p$-weighted index portfolio. For example, p can be used to set the tracking error between the $\mathbf{D}_p$-weighted index portfolio and the base benchmark market portfolio. As p decreases from a value of 1 to a value of 0 the "leverage" increases, with the corresponding $\mathbf{D}_p$-weighted index portfolio resembling the capitalisation-weighted market portfolio for $p = 1$ and resembling the equally-weighted portfolio for $p = 0$. Thus no true single optimal value is achieved, it depends on the tracking error. Moreover, by (5.6.59) in conjunction with Theorem 5.2.2 and equation (5.2.2), the logarithmic return process of the $\mathbf{D}_p$-weighted index portfolio $\varphi^{(p)}$, relative to the market portfolio satisfies, for all $t \in [0, T]$, a.s.,

$$d \log \Big(Z_{\varphi^{(p)}}(t) / Z_\mu(t) \Big) \quad = \quad d \log \mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) + (1-p)\gamma^*_{\varphi^{(p)}}(t)\, dt, \tag{5.6.66}$$

or, by (5.2.8), for all $T \in [0, \infty)$, the relative performance of the $\mathbf{D}_p$-weighted index portfolio $\varphi^{(p)}$, can alternatively be represented in integral form as

$$\log \left(\frac{Z_{\varphi^{(p)}}(T)}{Z_\mu(T)} \right) \quad = \quad \log \left(\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(T)\big)}{\mathbf{D}_p\big(\boldsymbol{\mu}(0)\big)} \right) + (1-p) \int_0^T \gamma^*_{\varphi^{(p)}}(t)\, dt, \tag{5.6.67}$$

if $Z_{\varphi^{(p)}}(0) = Z_\mu(0)$, otherwise the integral representation is given by

$$\log \left(\frac{Z_{\varphi^{(p)}}(T)}{Z_\mu(T)} \right) \quad = \quad \log \left(\frac{Z_{\varphi^{(p)}}(0)}{Z_\mu(0)} \right) + \log \left(\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(T)\big)}{\mathbf{D}_p\big(\boldsymbol{\mu}(0)\big)} \right) + (1-p) \int_0^T \gamma^*_{\varphi^{(p)}}(t)\, dt, \tag{5.6.68}$$

or, as per equation (5.2.7), we have

$$\log \left(\frac{Z_{\varphi^{(p)}}(T)}{Z_{\varphi^{(p)}}(0)} \right) \quad = \quad \log \left(\frac{Z_\mu(T)}{Z_\mu(0)} \right) + \log \left(\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(T)\big)}{\mathbf{D}_p\big(\boldsymbol{\mu}(0)\big)} \right) + (1-p) \int_0^T \gamma^*_{\varphi^{(p)}}(t)\, dt. \tag{5.6.69}$$

One big advantage of the above expression is that it is free of stochastic integrals, and thus lends itself to pathwise (almost sure) comparisons. Furthermore, recall from (5.2.11) and (5.2.12), that we can calculate the drift process of the $\mathbf{D}_p$-weighted (diversity-weighted) index portfolio as follows, for all $T \in [0, \infty)$,

$$\Theta^{(p)}(T) \quad = \quad \int_0^T d\Theta^{(p)}(t) \quad = \quad \int_0^T \mathfrak{g}^{(p)}(t)\, dt \quad = \quad \log \left(\frac{Z_{\varphi^{(p)}}(T)}{Z_\mu(T)} \right) - \log \left(\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(T)\big)}{\mathbf{D}_p\big(\boldsymbol{\mu}(0)\big)} \right) \tag{5.6.70}$$

$$= \quad \log \left(\frac{Z_{\varphi^{(p)}}(T) \mathbf{D}_p\big(\boldsymbol{\mu}(0)\big)}{Z_\mu(T) \mathbf{D}_p\big(\boldsymbol{\mu}(T)\big)} \right). \tag{5.6.71}$$

Consider the following series of graphs.

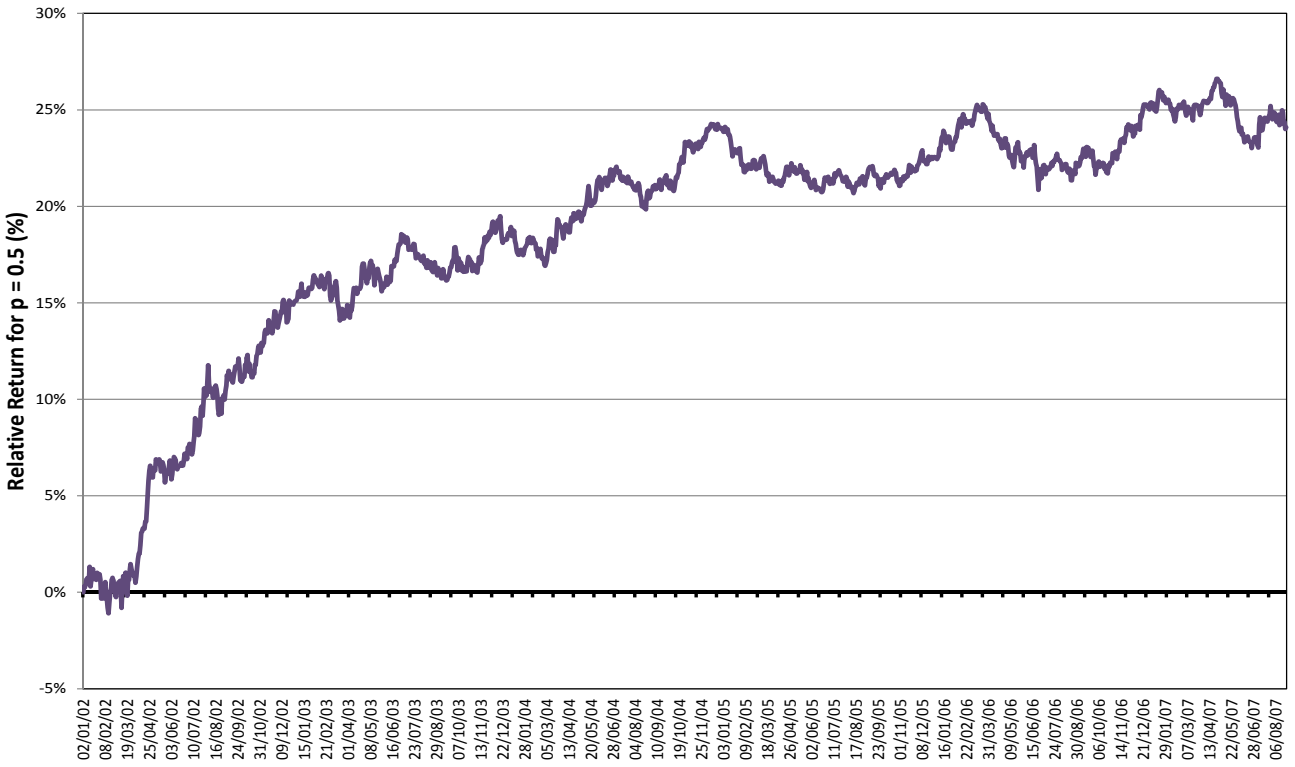

Figure 5.5: $\log\left(Z_{\varphi^{(p)}}(t)/Z_{\mu}(t)\right)$ for the $\mathbf{D}_p$-Weighted (Diversity-Weighted) Index Portfolio with $p = 0.5$.

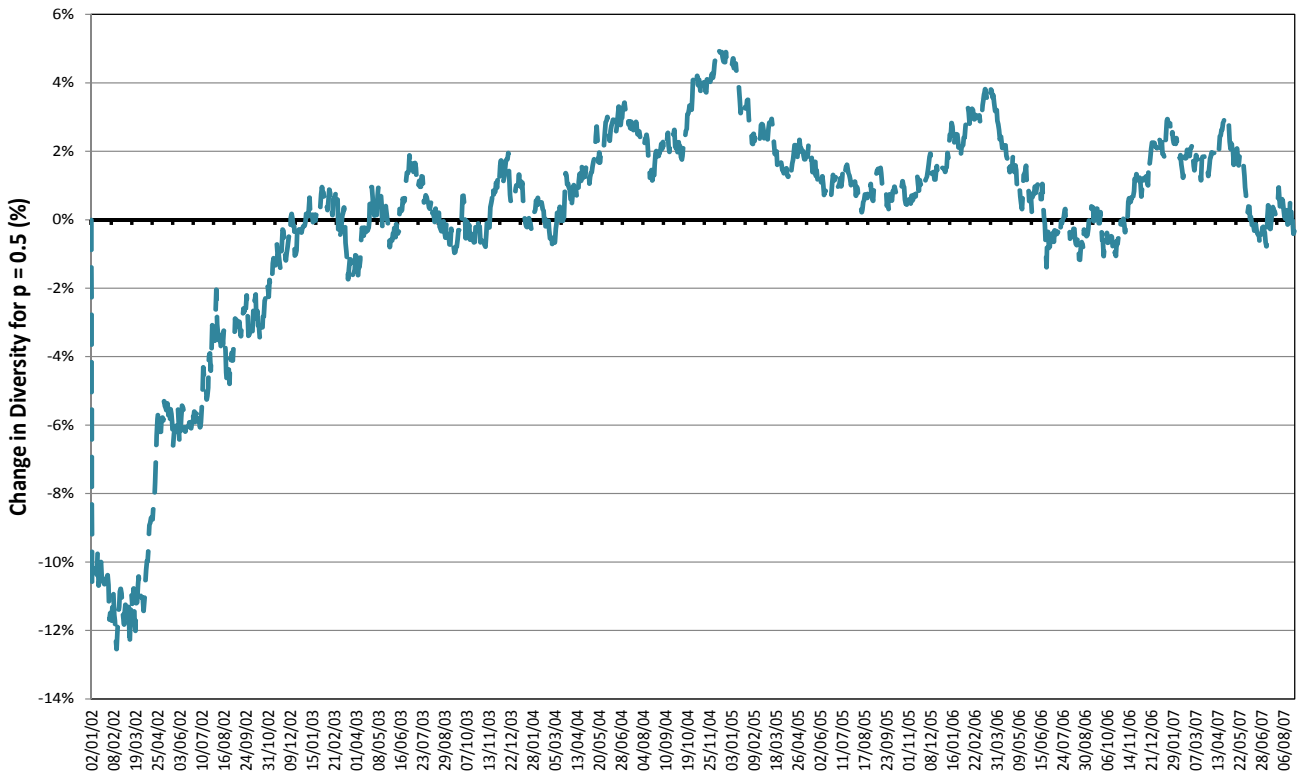

Figure 5.6: Change in Market $\mathbf{D}_p$ for $p = 0.5$ (Centered to have zero sample mean).

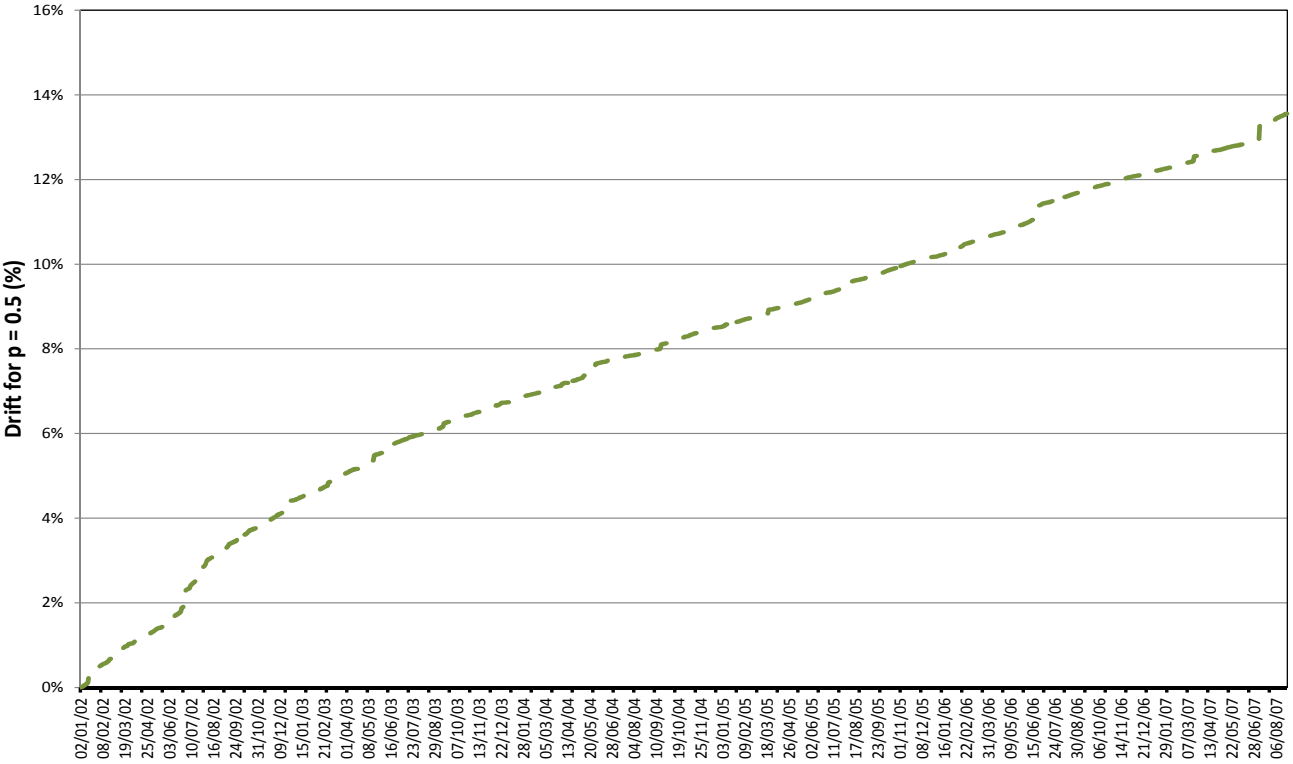

Figure 5.7: Drift Process for the $\mathbf{D}_p$-Weighted (Diversity-Weighted) Index Portfolio with $p = 0.5$.

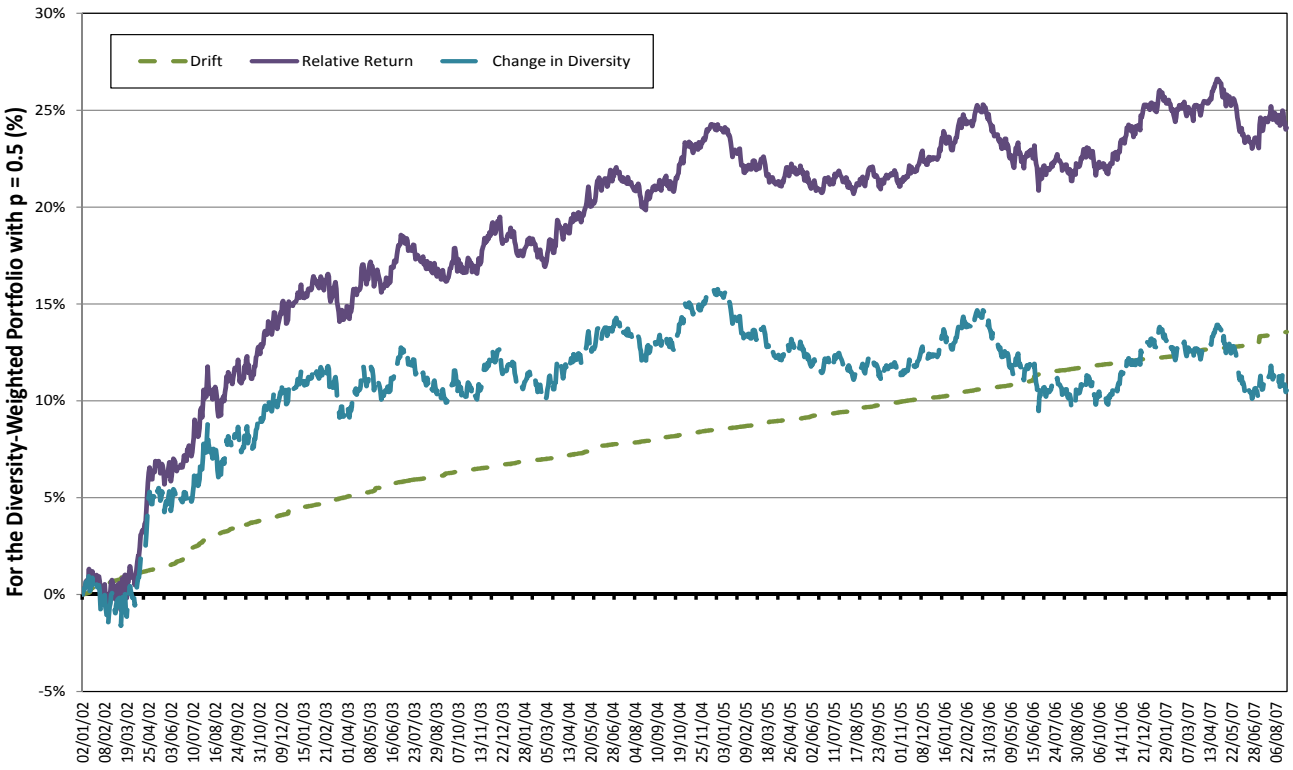

Figure 5.8: The Performance of the $\mathbf{D}_p$-Weighted (Diversity-Weighted) Index Portfolio with $p = 0.5$.

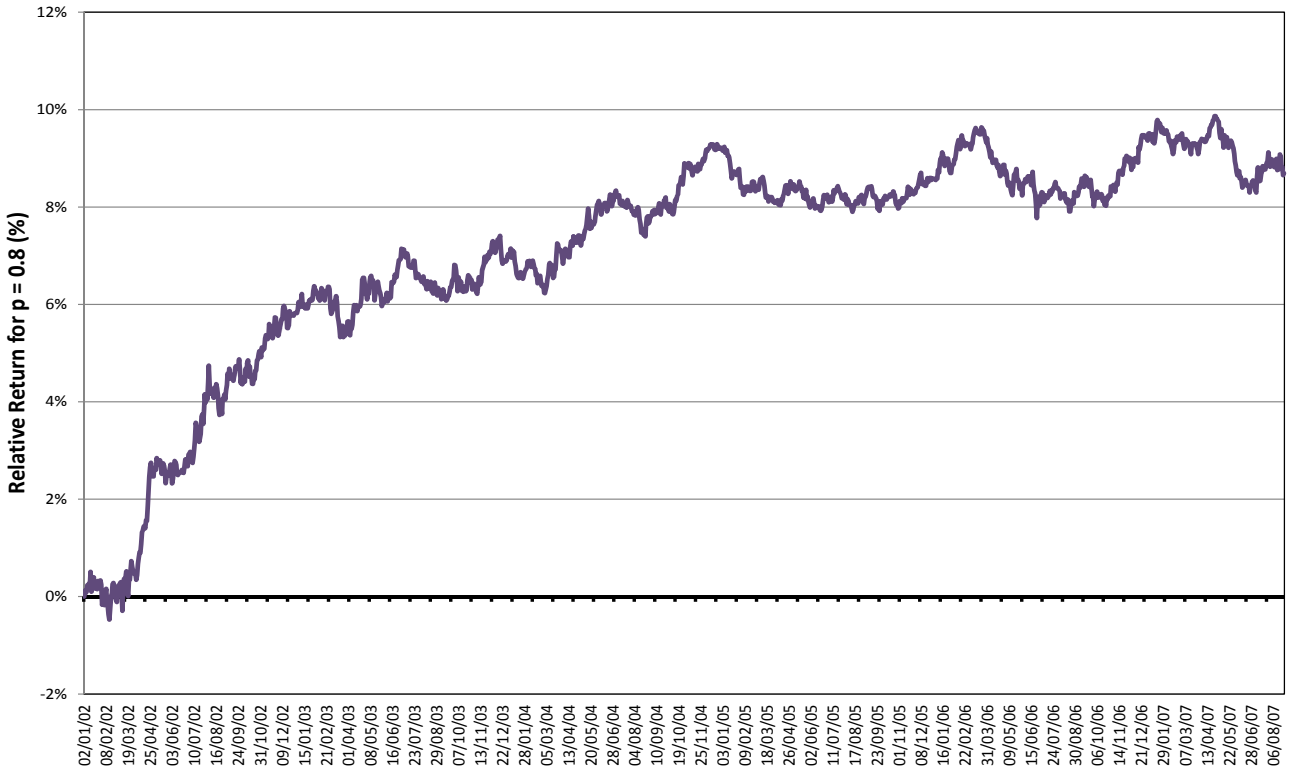

Figure 5.9: $\log\left(Z_{\varphi^{(p)}}(t)/Z_{\mu}(t)\right)$ for the $\mathbf{D}_p$-Weighted (Diversity-Weighted) Index Portfolio with $p = 0.8$.

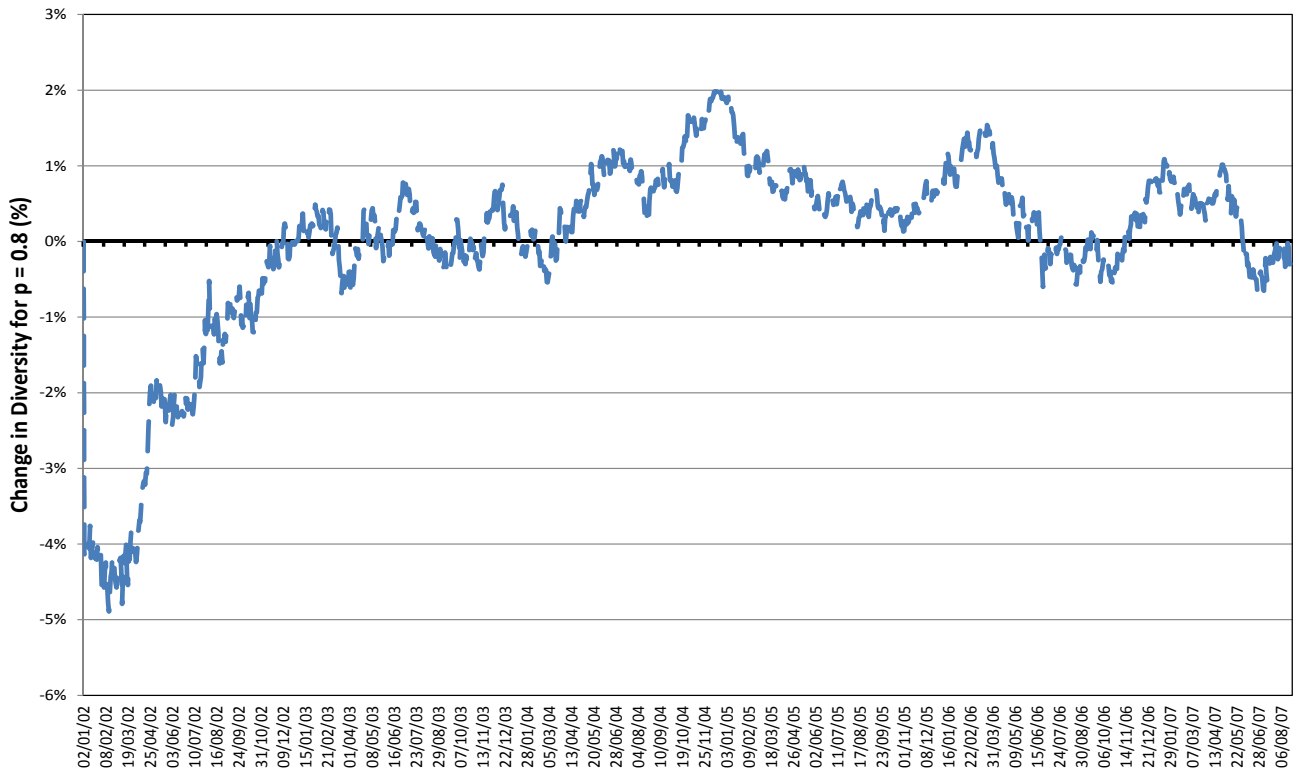

Figure 5.10: Change in Market $\mathbf{D}_p$ for $p = 0.8$ (Centered to have zero sample mean).

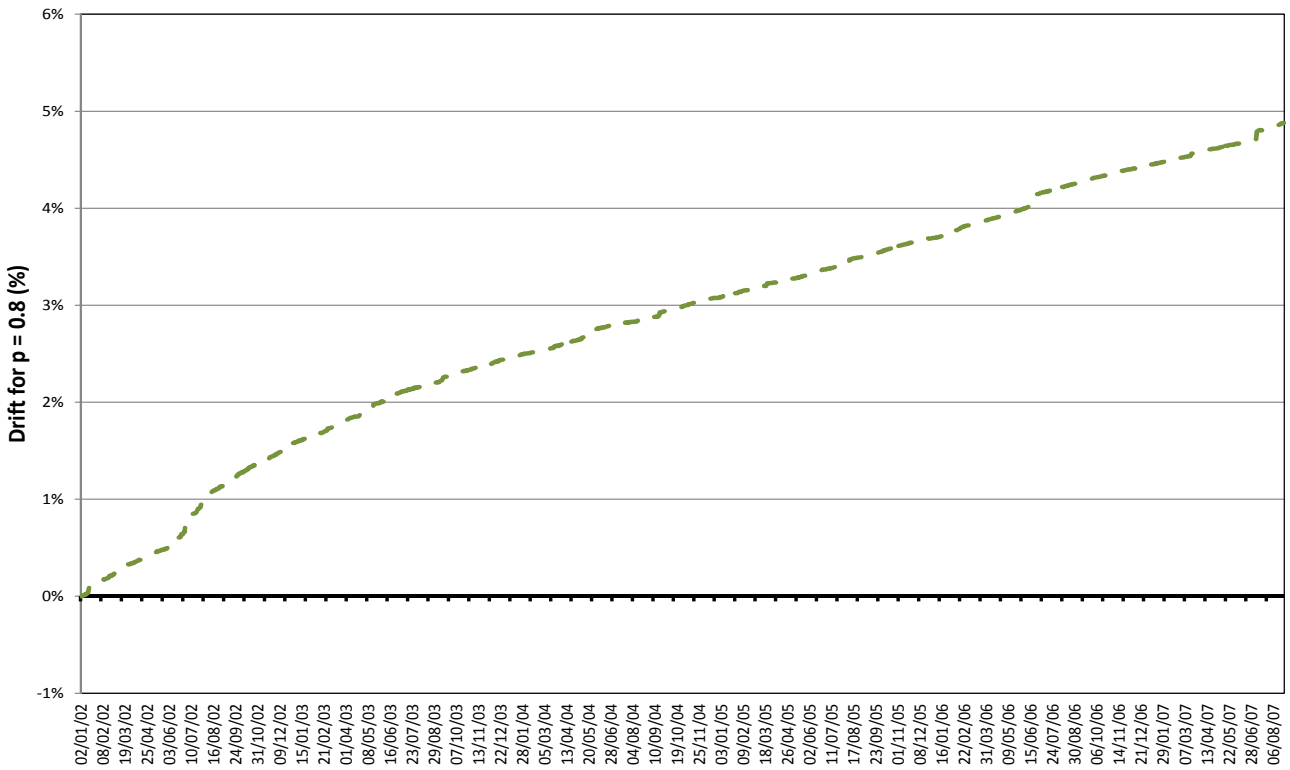

Figure 5.11: Drift Process for the $\mathbf{D}_p$-Weighted (Diversity-Weighted) Index Portfolio with $p = 0.8$.

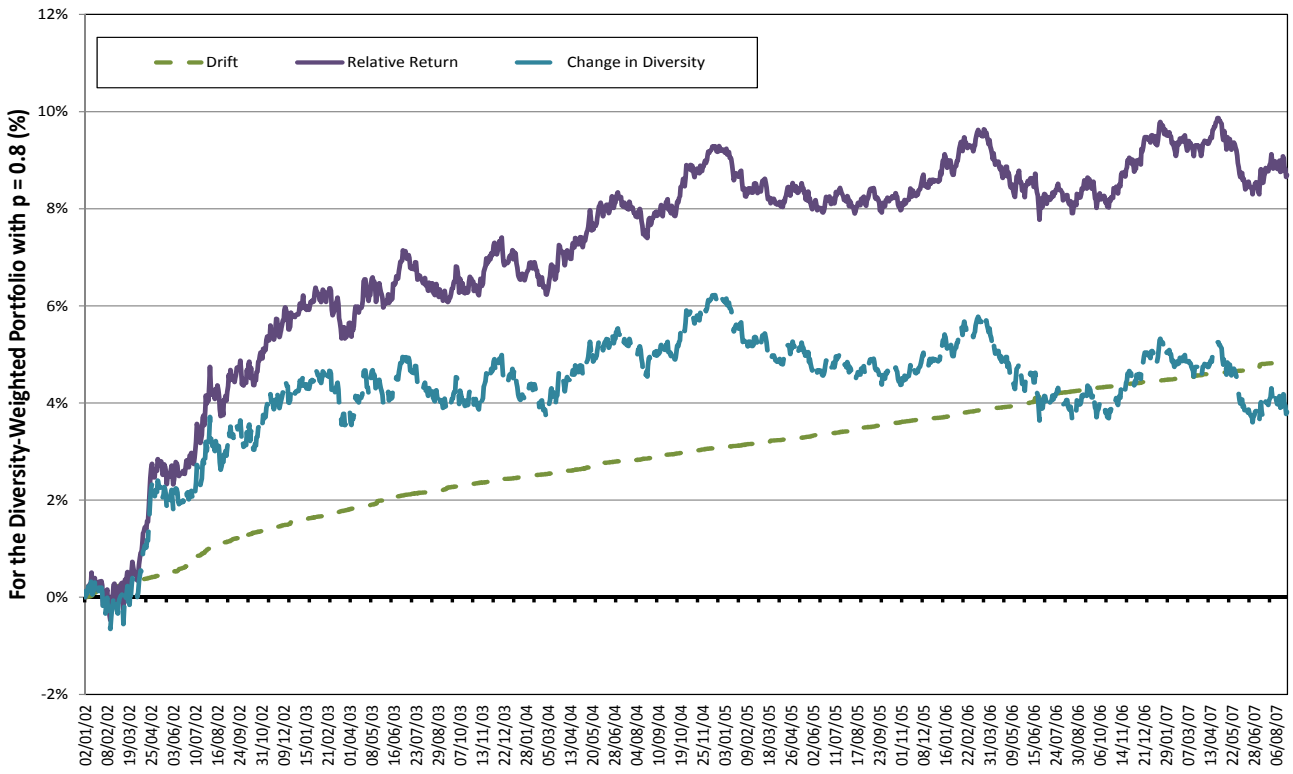

Figure 5.12: The Performance of the $\mathbf{D}_p$-Weighted (Diversity-Weighted) Index Portfolio with $p = 0.8$.

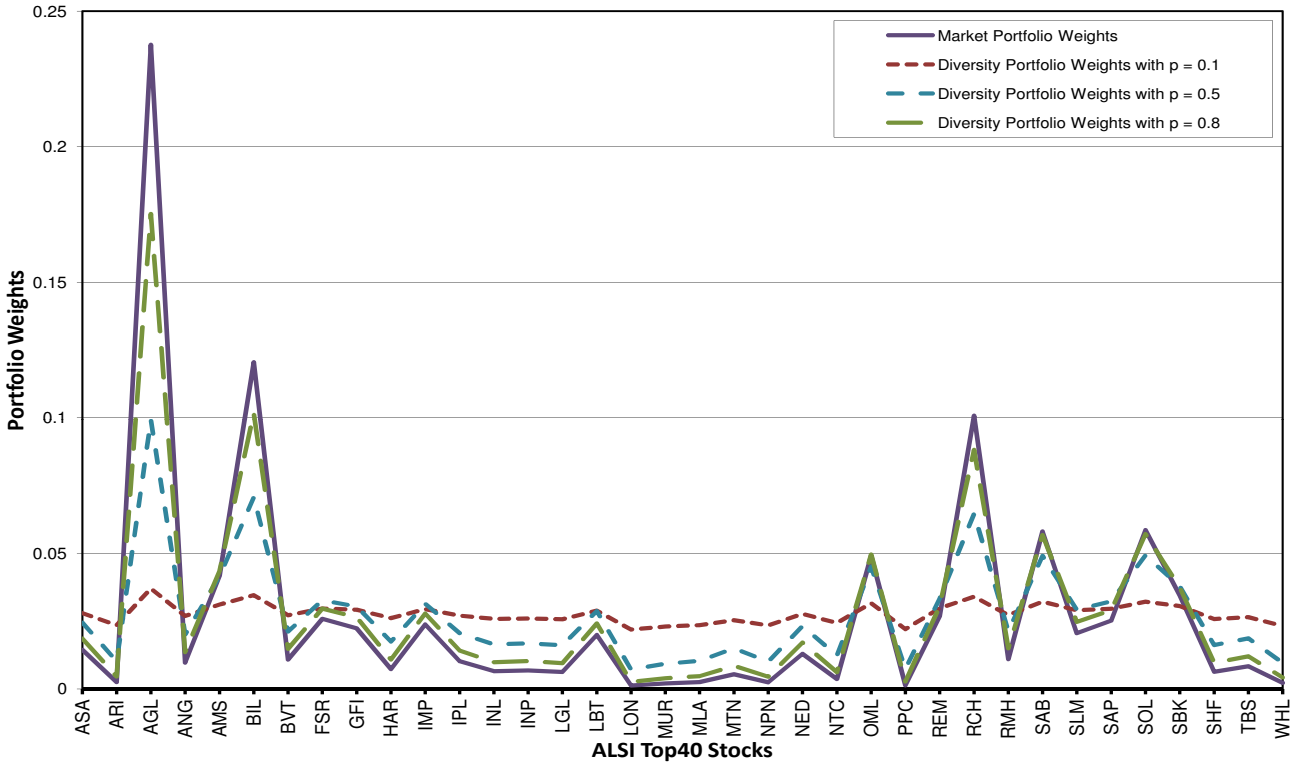

Figure 5.13: Market Weights and $\mathbf{D}_p$ Portfolio Weights on 2nd January 2002.

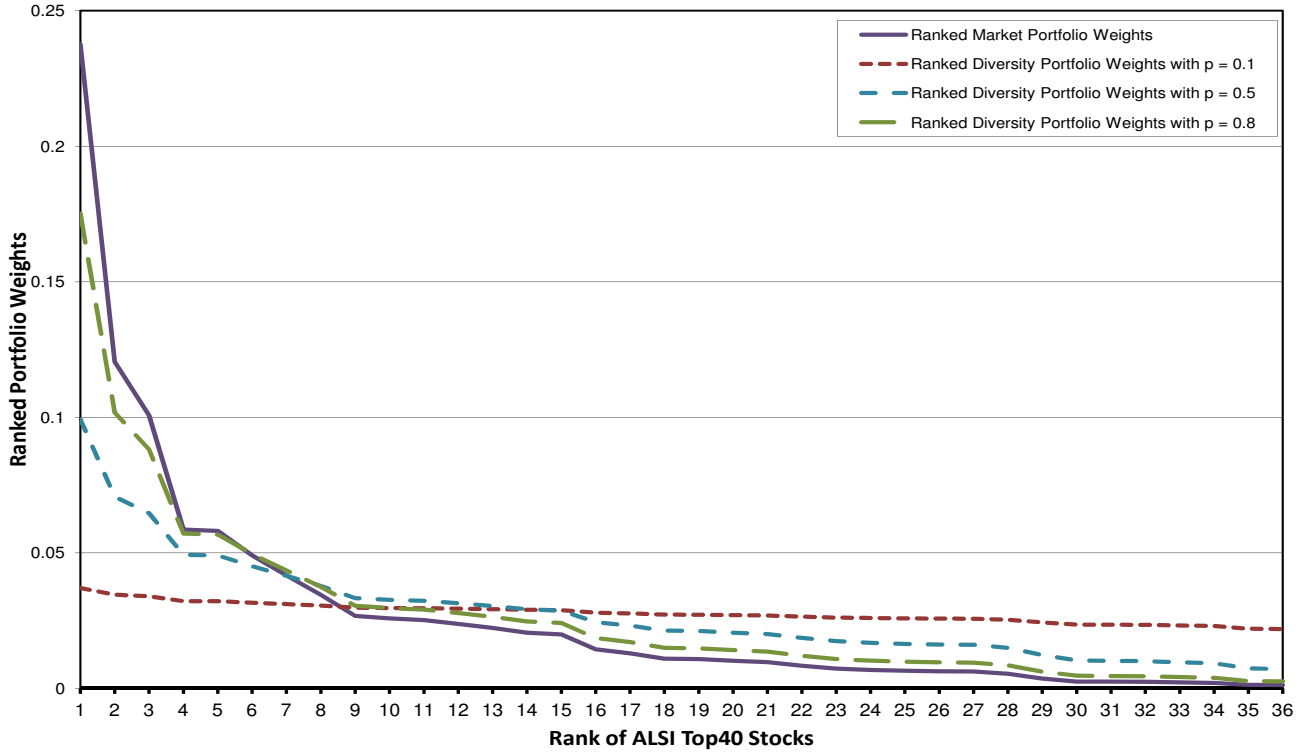

Figure 5.14: Ranked Market Weights and Ranked $\mathbf{D}_p$ Portfolio Weights on 2nd January 2002.

Hence, the cumulative effect of the drift process reveals the remarkable observation that in order to compute this effect over a period of time $[0, T]$ using past data, there is no need to estimate or even know the covariance structure at all. Moreover, the above expression for the cumulative effect of the drift process is in terms of quantities that are all observable.

For the parameter $p = 0.5$, Figure 5.5 reveals that the $\mathbf{D}_p$-weighted (diversity-weighted) index portfolio $\varphi^{(p)}$ generated a greater return than the market portfolio μ over the time period under consideration. Now, Figure 5.6 shows the cumulative change in the $\mathbf{D}_p$-function over the considered time period, which has been adjusted (centered) to have zero sample mean. From the figure, we can see that the cumulative change in the market $\mathbf{D}_p$-function is in the region of -0.3265% or -0.0594% a year, over the time horizon under consideration. Hence, the change in the market $\mathbf{D}_p$-function is wholly responsible for almost all of the volatility in the relative return of the $\mathbf{D}_p$-weighted (diversity-weighted) index portfolio. Thus, the change in the $\mathbf{D}_p$ is negative, however it is small in relation to the total variation over the time period. We can also see that the time series evolves in a mean-reverting fashion, similar to that of market entropy in Figure 5.2. Furthermore, the cumulative changes in the $\mathbf{D}_p$-weighted (diversity-weighted) index portfolio demonstrates a higher volatility than when compared to the entropy-weighted portfolio. This suggests that the $\mathbf{D}_p$-weighted (diversity-weighted) index portfolio is a riskier alternative to the entropy-weighted portfolio, and depending on the value of the parameter p, this inherent riskiness can be majorly offset by a much higher relative return. Thus, the $\mathbf{D}_p$-weighted (diversity-weighted) index portfolio certainly appears to be a rather appropriate choice for investors who are risk-seeking in the sense that they will take on this increased risk profile if it comes hand-in-hand with a higher relative return. If this is indeed the case, then the $\mathbf{D}_p$-weighted (diversity-weighted) index portfolio is the preferred investment choice in this regard. Let us now concern ourselves with the drift process of the $\mathbf{D}_p$-weighted (diversity-weighted) index portfolio, given in Figure 5.7. The drift process is rather close to being a pure trend process. In total, Figure 5.8 combines all three results of the performance of the $\mathbf{D}_p$-weighted (diversity-weighted) index portfolio into a single figure. Here, each process shows the cumulative value of the daily changes induced in the corresponding process by capital gains or losses in the stocks. From the equations above, it is clear that the relative return curve is given by adding the drift process curve to the changes in the $\mathbf{D}_p$ curve. Now, notice that the relative return, is the dominant term over the time horizon under consideration. Now, if we were to adjust the value of the parameter p in the $\mathbf{D}_p$-function, then these results will, of course, change slightly as a different value for the parameter p results in a different $\mathbf{D}_p$-weighted (diversity-weighted) index portfolio corresponding to that value of p. Thus, the results for the case where $p = 0.8$ are also given due consideration and are considered in Figure 5.9 through to Figure 5.12. Please refer to these figures for the appropriate analysis, which is similar in nature to that provided for the case where $p = 0.5$. Also, recall that as the parameter $p \to 1$, the $\mathbf{D}_p$-weighted (diversity-weighted) index portfolio tends to the archetypal market portfolio. As a mini test for this result, refer to Figure 5.13 which directly confirms this result that the larger the value of p, the closer the $\mathbf{D}_p$-weighted (diversity-weighted) index portfolio weights are to the market portfolio weights, and the smaller the value of p, the further away the $\mathbf{D}_p$-weighted (diversity-weighted) index portfolio weights are from the market portfolio weights. Figure 5.13 also confirms that when compared to the market portfolio μ, the $\mathbf{D}_p$-weighted (diversity-weighted) index portfolio $\varphi^{(p)}$ is less concentrated than the market portfolio μ in those stocks with the highest market weights, i.e., $\varphi^{(p)}$, relative to the market, is underweighted in the larger stocks and thus the $\mathbf{D}_p$-weighted (diversity-weighted) index portfolio decreases the proportion(s) held in the largest stock(s), and is more concentrated than the market portfolio μ in those stocks with the lowest market weights, i.e., $\varphi^{(p)}$, relative to the market, is overweighted in the smaller stocks and thus the $\mathbf{D}_p$-weighted (diversity-weighted) index portfolio increases those proportion(s) placed in the smallest stock(s), whilst preserving the relative rankings of all the stocks. In particular, the largest weight of the $\mathbf{D}_p$-weighted (diversity-weighted) index portfolio does not exceed the largest market weight, whilst the smallest weight of the $\mathbf{D}_p$-weighted (diversity-weighted) index portfolio does exceed the smallest market weight.

5.6.4 The Normalised $\mathbf{D}_p$-Weighted (Diversity-Weighted) Index Portfolio (The $\widetilde{\mathbf{D}}_p$-Weighted Index Portfolio)

Consider the following measure of diversity introduced in (4.6.33) of Definition 4.6.4, i.e., the normalised version of the $\mathbf{D}_p$-function or the normalised market $\mathbf{D}_p$ index process, i.e. the market $\widetilde{\mathbf{D}}_p$ index process, $\mathbf{G}(\boldsymbol{\mu}) \equiv \widetilde{\mathbf{D}}_p(\boldsymbol{\mu}) = \big\{\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big), t \in [0,T]\big\}$, $\mathbf{G} \equiv \widetilde{\mathbf{D}}_p : \Delta^{n-1} \to (0,\infty)$, for $p \in (0,1)$,

$$\mathbf{G}\big(\boldsymbol{\mu}(t)\big) \equiv \widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big) = \left(n^{p-1}\sum_{k=1}^{n}\big(\mu_k(t)\big)^p\right)^{\frac{1}{p}} = n^{(p-1)/p}\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big), \qquad t \in [0,T]. \tag{5.6.72}$$

Therefore, from equations (4.6.36) and (4.6.37), we have for all $i = 1, 2, \ldots, n$,

$$D_i\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big) = n^{(p-1)/p}D_i\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) = n^{(p-1)/p}\big(\mu_i(t)\big)^{p-1}\Big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p} \tag{5.6.73}$$

$$= n^{p-1}\big(\mu_i(t)\big)^{p-1}\Big(\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)\Big)^{1-p}. \tag{5.6.74}$$

Thus, from equations (5.6.72) and (5.6.73), we arrive at the following for the normalised version of the $\mathbf{D}_p$-function, for all $i = 1, 2, \ldots, n$ and $t \in [0,T]$,

$$D_i \log\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big) = \frac{D_i\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)}{\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)} = \frac{n^{(p-1)/p}D_i\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{n^{(p-1)/p}\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)} = \frac{D_i\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)} \equiv D_i \log\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big).$$

Therefore, from (5.2.13) of Theorem 5.2.2 and (5.6.46), with $\mathbf{G}(\mathbf{x}) = \widetilde{\mathbf{D}}_p(\mathbf{x})$, we obtain the following weights for the portfolio generated by the normalised version of the $\mathbf{D}_p$-function, $\widetilde{\mathbf{D}}_p$,

$$\begin{aligned}\widetilde{\varphi}_i^{(p)}(t) &= \left(D_i \log\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{j=1}^{n}\mu_j(t)\, D_j \log\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)\right)\mu_i(t)\\ &\equiv \left(D_i \log\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{j=1}^{n}\mu_j(t)\, D_j \log\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\right)\mu_i(t)\\ &\equiv \varphi_i^{(p)}(t).\end{aligned} \tag{5.6.75}$$

Therefore from equation (4.6.34), so that, $\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p = n^{1-p}\big(\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)\big)^p$, and from (5.6.47), we have

$$\widetilde{\varphi}_i^{(p)}(t) \equiv \varphi_i^{(p)}(t) = \frac{\big(\mu_i(t)\big)^p}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p} = \frac{\big(\mu_i(t)\big)^p}{n^{1-p}\big(\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)\big)^p} = \frac{n^{p-1}\big(\mu_i(t)\big)^p}{\big(\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)\big)^p}, \tag{5.6.76}$$

so that for all $i = 1, 2, \ldots, n$ and $t \in [0,T]$, we have, from (5.6.48), the $\widetilde{\mathbf{D}}_p$-*weights*, i.e. the weights corresponding to the generating function $\widetilde{\mathbf{D}}_p$,

$$\widetilde{\varphi}_i^{(p)}(t) \equiv \varphi_i^{(p)}(t) = \frac{\big(\mu_i(t)\big)^p}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p} = \frac{\big(\mu_i(t)\big)^p}{\sum_{j=1}^{n}\big(\mu_j(t)\big)^p} = \frac{\mu_i^p(t)}{\sum_{j=1}^{n}\mu_j^p(t)} = \frac{\mu_i^p(t)}{\mu_1^p(t) + \cdots + \mu_n^p(t)}. \tag{5.6.77}$$

Thus, the weights corresponding to the generating function $\widetilde{\mathbf{D}}_p$ can be formalised in the following definition.

Definition 5.6.6 (Normalised $\mathbf{D}_p$-Weighted Index Portfolio, $\widetilde{\mathbf{D}}_p$-Weighted Index Portfolio)**.** *Let μ be the market portfolio. The portfolio process $\widetilde{\varphi}^{(p)} = \big\{\widetilde{\varphi}^{(p)}(t) = \big(\widetilde{\varphi}_1^{(p)}(t), \widetilde{\varphi}_2^{(p)}(t), \ldots, \widetilde{\varphi}_n^{(p)}(t)\big), t \in [0,T]\big\}$, with weights defined by* (5.6.75) *and* (5.6.76)*, and using* (5.6.49) *and* (5.6.50) *of Definition* 5.6.5*, i.e.,*

$$\widetilde{\varphi}_i^{(p)}(t) \triangleq \varphi_i^{(p)}(t) = \frac{\big(\mu_i(t)\big)^p}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p} = \frac{\mu_i^p(t)}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p} = \left(\frac{\mu_i(t)}{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}\right)^p \tag{5.6.78}$$

$$= \frac{n^{p-1}\big(\mu_i(t)\big)^p}{\big(\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)\big)^p} = \frac{n^{p-1}\mu_i^p(t)}{\big(\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)\big)^p} = n^{p-1}\left(\frac{\mu_i(t)}{\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big)}\right)^p, \tag{5.6.79}$$

for all $i = 1, 2, \ldots, n$, and $t \in [0, T]$, is called the ***normalised $\mathbf{D}_p$-weighted index portfolio (process)*** *or the* ***$\widetilde{\mathbf{D}}_p$-weighted index portfolio (process)****, or also generically, the* ***normalised diversity-weighted index portfolio (process)****.*

Thus, the generating function $\widetilde{\mathbf{D}}_p(\boldsymbol{\mu}(t))$, $\widetilde{\mathbf{D}}_p : \Delta^{n-1} \to (0, \infty)$, generates the *normalised diversity-weighted portfolio*. In addition, $\mathbf{D}_p$ and $\widetilde{\mathbf{D}}_p$ both generate the same portfolio. Proposition 5.3.4 reaffirms this, since the ratio of these two generating functions is constant on Δ^{n-1}. Now, from (5.6.73), (4.6.36), (4.6.38) and (4.6.44), we have for all $i, j = 1, 2, \ldots, n$, and $t \in [0, T]$,

$$D_{ij}\widetilde{\mathbf{D}}_p(\boldsymbol{\mu}(t)) \quad \equiv \quad n^{(p-1)/p} D_{ij}\mathbf{D}_p(\boldsymbol{\mu}(t)). \tag{5.6.80}$$

The above expression in conjunction with (5.2.14) of Theorem 5.2.2, (5.6.72), (5.6.57) and (5.6.58), with $\mathbf{G}(\mathbf{x}) = \widetilde{\mathbf{D}}_p(\mathbf{x})$, yields the following drift process corresponding to the normalised version of the $\mathbf{D}_p$-function, and thus to the normalised $\mathbf{D}_p$-weighted index portfolio process

$$\begin{aligned} d\widetilde{\Theta}^{(p)}(t) \quad = \quad d\Theta_{\widetilde{\varphi}^{(p)}}(t) \quad &= \quad \frac{-1}{2\widetilde{\mathbf{D}}_p(\boldsymbol{\mu}(t))} \sum_{i,j=1}^{n} D_{ij}\widetilde{\mathbf{D}}_p(\boldsymbol{\mu}(t))\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt \\ &= \quad \frac{-1}{2\,n^{(p-1)/p}\mathbf{D}_p(\boldsymbol{\mu}(t))} \sum_{i,j=1}^{n} n^{(p-1)/p} D_{ij}\mathbf{D}_p(\boldsymbol{\mu}(t))\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt \\ &\equiv \quad \frac{-1}{2\mathbf{D}_p(\boldsymbol{\mu}(t))} \sum_{i,j=1}^{n} D_{ij}\mathbf{D}_p(\boldsymbol{\mu}(t))\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt \\ &\equiv \quad d\Theta_{\varphi^{(p)}}(t) \quad = \quad d\Theta^{(p)}(t), \end{aligned} \tag{5.6.81}$$

equivalently,

$$\widetilde{\mathfrak{g}}^{(p)}(t) \quad = \quad \mathfrak{g}_{\widetilde{\varphi}^{(p)}}(t) \quad \equiv \quad \mathfrak{g}_{\varphi^{(p)}}(t) \quad = \quad \mathfrak{g}^{(p)}(t). \tag{5.6.82}$$

Therefore, from equations (5.6.59) and (5.6.60), we have

$$d\widetilde{\Theta}^{(p)}(t) \quad = \quad d\Theta^{(p)}(t) \quad \equiv \quad (1-p)\gamma^*_{\varphi^{(p)}}(t)\,dt, \tag{5.6.83}$$

equivalently,

$$\widetilde{\mathfrak{g}}^{(p)}(t) \quad = \quad \mathfrak{g}^{(p)}(t) \quad \equiv \quad (1-p)\gamma^*_{\varphi^{(p)}}(t). \tag{5.6.84}$$

Moreover, by (5.6.83) in conjunction with Theorem 5.2.2 and equation (5.2.2), the logarithmic return process of the $\widetilde{\mathbf{D}}_p$-weighted index portfolio $\widetilde{\varphi}^{(p)}$, relative to the market portfolio satisfies, for all $t \in [0, T]$, a.s.,

$$d\log\Big(Z_{\widetilde{\varphi}^{(p)}}(t)/Z_\mu(t)\Big) \quad = \quad d\log\widetilde{\mathbf{D}}_p(\boldsymbol{\mu}(t)) + (1-p)\gamma^*_{\varphi^{(p)}}(t)\,dt, \tag{5.6.85}$$

or, by (5.2.8), for all $T \in [0, \infty)$, the relative performance of the $\widetilde{\mathbf{D}}_p$-weighted index portfolio $\widetilde{\varphi}^{(p)}$, can alternatively be represented in integral form as

$$\log\left(\frac{Z_{\widetilde{\varphi}^{(p)}}(T)}{Z_\mu(T)}\right) \quad = \quad \log\left(\frac{\widetilde{\mathbf{D}}_p(\boldsymbol{\mu}(T))}{\widetilde{\mathbf{D}}_p(\boldsymbol{\mu}(0))}\right) + (1-p)\int_0^T \gamma^*_{\varphi^{(p)}}(t)\,dt, \tag{5.6.86}$$

if $Z_{\widetilde{\varphi}^{(p)}}(0) = Z_\mu(0)$, otherwise the integral representation is given by

$$\log\left(\frac{Z_{\widetilde{\varphi}^{(p)}}(T)}{Z_\mu(T)}\right) \quad = \quad \log\left(\frac{Z_{\widetilde{\varphi}^{(p)}}(0)}{Z_\mu(0)}\right) + \log\left(\frac{\widetilde{\mathbf{D}}_p(\boldsymbol{\mu}(T))}{\widetilde{\mathbf{D}}_p(\boldsymbol{\mu}(0))}\right) + (1-p)\int_0^T \gamma^*_{\varphi^{(p)}}(t)\,dt, \tag{5.6.87}$$

or, as per equation (5.2.7), we have

$$\log\left(\frac{Z_{\widetilde{\varphi}^{(p)}}(T)}{Z_{\widetilde{\varphi}^{(p)}}(0)}\right) \quad = \quad \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log\left(\frac{\widetilde{\mathbf{D}}_p(\boldsymbol{\mu}(T))}{\widetilde{\mathbf{D}}_p(\boldsymbol{\mu}(0))}\right) + (1-p)\int_0^T \gamma^*_{\varphi^{(p)}}(t)\,dt. \tag{5.6.88}$$

5.6.5 The Market Portfolio

Consider the following measure of diversity, $\mathbf{G}(\boldsymbol{\mu}) \equiv \mathbf{S}^{\mathrm{M}}(\boldsymbol{\mu}) = \big\{\mathbf{S}^{\mathrm{M}}\big(\boldsymbol{\mu}(t)\big), t \in [0,T]\big\}$, $\mathbf{G} \equiv \mathbf{S}^{\mathrm{M}} : \Delta^{n-1} \to (0,\infty)$, for a positive constant $w > 0$,

$$\mathbf{G}\big(\boldsymbol{\mu}(t)\big) \equiv \mathbf{S}^{\mathrm{M}}\big(\boldsymbol{\mu}(t)\big) = w, \qquad t \in [0,T]. \tag{5.6.89}$$

Therefore, we have for all $i = 1, 2, \ldots, n$,

$$D_i\mathbf{S}^{\mathrm{M}}\big(\boldsymbol{\mu}(t)\big) = \frac{\partial \mathbf{S}^{\mathrm{M}}\big(\boldsymbol{\mu}(t)\big)}{\partial \mu_i(t)} = 0, \tag{5.6.90}$$

so that, we obtain for all $i = 1, 2, \ldots, n$, and $t \in [0,T]$,

$$D_i \log \mathbf{S}^{\mathrm{M}}\big(\boldsymbol{\mu}(t)\big) = \frac{D_i\mathbf{S}^{\mathrm{M}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}^{\mathrm{M}}\big(\boldsymbol{\mu}(t)\big)} = 0.$$

Therefore, from (5.2.13) of Theorem 5.2.2, with $\mathbf{G}(\mathbf{x}) = \mathbf{S}^{\mathrm{M}}(\mathbf{x})$, we obtain the following weights for the portfolio generated by $\mathbf{S}^{\mathrm{M}}$,

$$\varphi_i^{\mathsf{m}}(t) = \left(D_i \log \mathbf{S}^{\mathrm{M}}\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{j=1}^{n} \mu_j(t)\, D_j \log \mathbf{S}^{\mathrm{M}}\big(\boldsymbol{\mu}(t)\big)\right) \mu_i(t) = \mu_i(t).$$

Hence, for all $i = 1, 2, \ldots, n$ and $t \in [0,T]$, we have the weights corresponding to the generating function $\mathbf{S}^{\mathrm{M}}$,

$$\varphi_i^{\mathsf{m}}(t) = \mu_i(t). \tag{5.6.91}$$

Thus, the weights corresponding to the generating function $\mathbf{S}^{\mathrm{M}}$ can be formalised in the following definition.

Definition 5.6.7 (The Market Portfolio). *Let μ be the market portfolio. The portfolio process $\varphi^{\mathsf{m}} = \big\{\varphi^{\mathsf{m}}(t) = \big(\varphi_1^{\mathsf{m}}(t), \varphi_2^{\mathsf{m}}(t), \ldots, \varphi_n^{\mathsf{m}}(t)\big), t \in [0,T]\big\}$, with weights defined by* (5.6.91)*, i.e.,*

$$\varphi_i^{\mathsf{m}}(t) \triangleq \mu_i(t), \tag{5.6.92}$$

for all $i = 1, 2, \ldots, n$, and $t \in [0,T]$, is called the ***market portfolio (process)****.*

Thus, the generating function $\mathbf{S}^{\mathrm{M}}\big(\boldsymbol{\mu}(t)\big) = w$, $\mathbf{S}^{\mathrm{M}} : \Delta^{n-1} \to (0,\infty)$, generates the *market portfolio*, μ. The market portfolio buys at time $t = 0$ the same number of shares in all companies of the market and holds them until the end of the investing horizon. It represents the quintessential buy-and-hold strategy. Now, from (5.6.89) and (5.6.90), we have for all $i, j = 1, 2, \ldots, n$, and $t \in [0,T]$,

$$D_{ij}\mathbf{S}^{\mathrm{M}}\big(\boldsymbol{\mu}(t)\big) = \frac{\partial^2 \mathbf{S}^{\mathrm{M}}\big(\boldsymbol{\mu}(t)\big)}{\partial \mu_i(t) \partial \mu_j(t)} = 0. \tag{5.6.93}$$

The above expression in conjunction with (5.2.14) of Theorem 5.2.2 with $\mathbf{G}(\mathbf{x}) = \mathbf{S}^{\mathrm{M}}(\mathbf{x})$, yields the following drift process corresponding to the measure of diversity $\mathbf{S}^{\mathrm{M}}$, and thus to the market portfolio process

$$d\Theta^{\mathsf{m}}(t) = d\Theta_{\varphi^{\mathsf{m}}}(t) = \frac{-1}{2\mathbf{S}^{\mathrm{M}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i,j=1}^{n} D_{ij}\mathbf{S}^{\mathrm{M}}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\, dt = 0, \tag{5.6.94}$$

equivalently, for all $t \in [0,T]$, we have

$$\mathfrak{g}^{\mathsf{m}}(t) = \mathfrak{g}_{\varphi^{\mathsf{m}}}(t) = 0. \tag{5.6.95}$$

5.6.6 The Equal-Weighted Portfolio

Recall from equation (4.6.92) of Definition 4.6.13 that the market geometric mean process, $\mathbf{G}(\boldsymbol{\mu}) \equiv \mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}) = \big\{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big), t \in [0,T]\big\}$, $\mathbf{G} \equiv \mathbf{S}^{\mathrm{GM}} : \Delta^{n-1} \to (0,\infty)$, is given by

$$\mathbf{G}\big(\boldsymbol{\mu}(t)\big) \equiv \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) = \big(\mu_1(t) \cdots \mu_n(t)\big)^{\frac{1}{n}} = \left(\prod_{k=1}^{n} \mu_k(t)\right)^{\frac{1}{n}}, \qquad t \in [0,T]. \tag{5.6.96}$$

From equation (4.6.94), we have for the market geometric mean process for all $i = 1, 2, \ldots, n$, and $t \in [0,T]$,

$$D_i \log \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) = \frac{D_i \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} = \frac{1}{n\,\mu_i(t)}.$$

Therefore, from (5.2.13) of Theorem 5.2.2, with $\mathbf{G}(\mathbf{x}) = \mathbf{S}^{\mathrm{GM}}(\mathbf{x})$, we obtain the following weights for the portfolio generated by the geometric mean function $\mathbf{S}^{\mathrm{GM}}$,

$$\begin{aligned}
\varphi_i^{\mathsf{equal}}(t) &= \left(D_i \log \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{j=1}^{n} \mu_j(t)\, D_j \log \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)\right) \mu_i(t) \\
&= \left(\frac{1}{n\,\mu_i(t)} + 1 - \sum_{j=1}^{n} \mu_j(t) \left[\frac{1}{n\,\mu_j(t)}\right]\right) \mu_i(t) \\
&= \left(\frac{1}{n\,\mu_i(t)} + 1 - \sum_{j=1}^{n} \frac{1}{n}\right) \mu_i(t) \\
&= \mu_i(t) \left(\frac{1}{n\,\mu_i(t)} + 1 - 1\right) \\
&= \mu_i(t) \left(\frac{1}{n\,\mu_i(t)}\right) \\
&= \frac{1}{n}.
\end{aligned}$$

Hence, for all $i = 1, 2, \ldots, n$, and $t \in [0,T]$, we have the weights corresponding to the generating function $\mathbf{S}^{\mathrm{GM}}$,

$$\varphi_i^{\mathsf{equal}}(t) = \frac{1}{n}. \tag{5.6.97}$$

Thus, the weights corresponding to the generating function $\mathbf{S}^{\mathrm{GM}}$ can be formalised in the following definition.

Definition 5.6.8 (Equally-Weighted Portfolio)**.** *Let μ be the market portfolio. The portfolio process $\varphi^{\mathsf{equal}} = \big\{\varphi^{\mathsf{equal}}(t) = \big(\varphi_1^{\mathsf{equal}}(t), \varphi_2^{\mathsf{equal}}(t), \ldots, \varphi_n^{\mathsf{equal}}(t)\big), t \in [0,T]\big\}$, with weights defined by* (5.6.97)*, i.e.,*

$$\varphi_i^{\mathsf{equal}}(t) \triangleq \frac{1}{n}, \tag{5.6.98}$$

for all $i = 1, 2, \ldots, n$, and $t \in [0,T]$, is called the ***equally-weighted portfolio (process)****.*

Thus, the generating function $\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) = \big(\mu_1(t) \cdots \mu_n(t)\big)^{\frac{1}{n}}$, $\mathbf{S}^{\mathrm{GM}} : \Delta^{n-1} \to (0,\infty)$, generates a portfolio with all the weights equal to $\frac{1}{n}$, i.e., the *equal-weighted portfolio*, $\varphi_i(t) = \frac{1}{n}$, for all $i = 1, 2, \ldots, n$. That is, the investor holds each stock equally in the market. The equally-weighted portfolio maintains equal weights in all stocks at all times; it accomplishes this by selling those stocks whose price rises relative to the rest, and by buying those stocks whose price falls relative to the others. Because of this built-in aspect of "buying-low-and-selling-high", the equally-weighted portfolio can be used as a simple prototype for studying systematically the performance of statistical arbitrage strategies in equity markets [see Fernholz & Maguire (2007) for details]. It can be easily verified that φ^{equal} satisfies the requirements of Definition 2.2.16, since the weights are, of course, bounded and satisfy the following

$$\sum_{i=1}^{n} \varphi_i^{\mathsf{equal}}(t) = \sum_{i=1}^{n} \frac{1}{n} = 1.$$

The weight ratios, for $i = 1, 2, \ldots, n$, satisfy

$$\frac{\varphi_i^{\mathsf{equal}}(t)}{\mu_i(t)} = \frac{1}{n\,\mu_i(t)}, \qquad t \in [0, T]. \tag{5.6.99}$$

It is obvious that the weight ratios decrease with increasing market weight $\mu_i(t)$, and that compared to the market portfolio μ, the equal-weighted portfolio φ^{equal} is less concentrated than the market portfolio μ in those stocks with the highest market weights (i.e., φ^{equal}, relative to the market, is underweighted in the larger stocks), and is more concentrated than the market portfolio μ in those stocks with the lowest market weights (i.e., φ^{equal}, relative to the market, is overweighted in the smaller stocks). This result also follows from Proposition 5.5.2, since $\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)$ is a measure of diversity that generates a diversity generated portfolio, the equally-weighted portfolio. Alternatively, if $\mu_i(t) \geq \mu_j(t)$ for some $i > j$, then $\frac{1}{\mu_i(t)} \leq \frac{1}{\mu_j(t)}$ also follows and we have for all $t \in [0, T]$,

$$\frac{1}{n\,\mu_i(t)} \leq \frac{1}{n\,\mu_j(t)}.$$

Thus, if $\mu_i(t) \geq \mu_j(t)$ for some $i > j$, (5.6.99) implies

$$\frac{\varphi_i^{\mathsf{equal}}(t)}{\mu_i(t)} \leq \frac{\varphi_j^{\mathsf{equal}}(t)}{\mu_j(t)}. \tag{5.6.100}$$

Now, from (4.6.95), (4.6.96) and (4.6.97), we have for all $i, j = 1, 2, \ldots, n$, and $t \in [0, T]$,

$$D_{ij}\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) = \begin{cases} \dfrac{1}{n}\left(\dfrac{1}{n} - 1\right)\dfrac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} & \text{if} \quad i = j \\ \dfrac{1}{n^2}\,\dfrac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)\mu_j(t)} & \text{if} \quad i \neq j \end{cases} = \begin{cases} \dfrac{1}{n^2}\,\dfrac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} - \dfrac{1}{n}\,\dfrac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} & \text{if} \quad i = j, \\ \dfrac{1}{n^2}\,\dfrac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)\mu_j(t)} & \text{if} \quad i \neq j. \end{cases} \tag{5.6.101}$$

The above expression in conjunction with (5.2.14) of Theorem 5.2.2 with $\mathbf{G}(\mathbf{x}) = \mathbf{S}^{\mathrm{GM}}(\mathbf{x})$, yields the following

drift process corresponding to the geometric mean function, and thus to the equally-weighted portfolio process

$$
\begin{aligned}
d\Theta^{\mathsf{equal}}(t) &= d\Theta_{\varphi^{\mathsf{equal}}}(t) \\
&= \frac{-1}{2\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i,j=1}^{n} D_{ij}\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt \\
&= \frac{-1}{2\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \left(\sum_{i=1}^{n} \left[\frac{1}{n}\left(\frac{1}{n}-1\right) \frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} \right] \mu_i^2(t)\tau_{ii}(t) + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \left[\frac{1}{n^2} \frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)\mu_j(t)} \right] \mu_i(t)\mu_j(t)\tau_{ij}(t) \right) dt \\
&= \frac{-1}{2\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \left(\sum_{i=1}^{n} \left[\frac{1}{n^2} \frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} - \frac{1}{n} \frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} \right] \mu_i^2(t)\tau_{ii}(t) + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \left[\frac{1}{n^2} \frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)\mu_j(t)} \right] \mu_i(t)\mu_j(t)\tau_{ij}(t) \right) dt \\
&= -\frac{1}{2} \left(\sum_{i=1}^{n} \left(\frac{1}{n^2} - \frac{1}{n} \right) \tau_{ii}(t) + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \frac{1}{n^2}\tau_{ij}(t) \right) dt \\
&= -\frac{1}{2} \left(\sum_{i=1}^{n} \frac{1}{n^2}\tau_{ii}(t) - \sum_{i=1}^{n} \frac{1}{n}\tau_{ii}(t) + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \frac{1}{n^2}\tau_{ij}(t) \right) dt \\
&= \frac{1}{2} \left(\sum_{i=1}^{n} \frac{1}{n}\tau_{ii}(t) - \sum_{i=1}^{n} \frac{1}{n^2}\tau_{ii}(t) - \sum_{\substack{i,j=1\\ i\neq j}}^{n} \frac{1}{n^2}\tau_{ij}(t) \right) dt \\
&= \frac{1}{2} \left(\sum_{i=1}^{n} \frac{1}{n}\tau_{ii}(t) - \left[\sum_{i=1}^{n} \frac{1}{n^2}\tau_{ii}(t) + \sum_{\substack{i,j=1\\ i\neq j}}^{n} \frac{1}{n^2}\tau_{ij}(t) \right] \right) dt \\
&= \frac{1}{2} \left(\sum_{i=1}^{n} \frac{1}{n}\tau_{ii}(t) - \sum_{i,j=1}^{n} \frac{1}{n^2}\tau_{ij}(t) \right) dt \\
&= \frac{1}{2} \left(\sum_{i=1}^{n} \frac{1}{n}\tau_{ii}(t) - \sum_{i,j=1}^{n} \left(\frac{1}{n}\right)^2 \tau_{ij}(t) \right) dt,
\end{aligned}
$$

which, by (5.6.97), becomes

$$
d\Theta^{\mathsf{equal}}(t) = d\Theta_{\varphi^{\mathsf{equal}}}(t) = \frac{1}{2} \left(\sum_{i=1}^{n} \varphi_i^{\mathsf{equal}}(t)\tau_{ii}(t) - \sum_{i,j=1}^{n} \varphi_i^{\mathsf{equal}}(t)\tau_{ij}(t)\varphi_j^{\mathsf{equal}}(t) \right) dt.
$$

Therefore, by the numéraire invariance property of the excess growth rate of the equal-weighted portfolio (2.4.26) of Lemma 2.4.5, we arrive at the following

$$
d\Theta^{\mathsf{equal}}(t) = d\Theta_{\varphi^{\mathsf{equal}}}(t) = \gamma^*_{\varphi^{\mathsf{equal}}}(t)\,dt, \tag{5.6.102}
$$

equivalently, for all $t \in [0,T]$, we have

$$
\mathfrak{g}^{\mathsf{equal}}(t) = \mathfrak{g}_{\varphi^{\mathsf{equal}}}(t) = \gamma^*_{\varphi^{\mathsf{equal}}}(t). \tag{5.6.103}
$$

Since the weights of the equal-weighted portfolio are all nonnegative (positive) and equal to $\frac{1}{n}$, we have by (2.4.37) of Proposition 2.4.8 that for a strictly long-only portfolio the excess growth rate of the equal-weighted portfolio is nonnegative (positive), i.e., $\gamma^*_{\varphi^{\mathsf{equal}}}(t) \geq 0$ for all $t \in [0,T]$. In addition, Lemma 2.4.12, (2.4.55), and Lemma 2.4.14, (2.4.56) demonstrate that in a *nondegenerate market*, the excess growth rate of the equal-weighted portfolio, $\gamma^*_{\varphi^{\mathsf{equal}}}(t)$, has a positive lower bound. Consequently, the drift process of the equal-weighted portfolio, in (5.6.103), must also have a positive lower bound. Therefore, the drift process is nonnegative (positive) $\mathfrak{g}^{\mathsf{equal}}(t) = \frac{d\Theta^{\mathsf{equal}}(t)}{dt} \geq 0$, i.e.,

$$
\mathfrak{g}^{\mathsf{equal}}(t) = \mathfrak{g}_{\varphi^{\mathsf{equal}}}(t) = \gamma^*_{\varphi^{\mathsf{equal}}}(t) \geq 0,
$$

and the drift process Θ^{equal} is nondecreasing (increasing) since $\mathfrak{g}^{\mathsf{equal}}(t) = \frac{d\Theta^{\mathsf{equal}}(t)}{dt} \geq 0$, i.e., the gradient of the drift process is nonnegative (positive), which follows from Proposition 5.5.2 since the geometric mean $\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)$ is a measure of diversity that generates the diversity generated portfolio φ^{equal} with drift Θ^{equal}. Moreover, by (5.6.102) in conjunction with Theorem 5.2.2 and equation (5.2.2), the logarithmic return process of the equal-weighted portfolio φ^{equal}, relative to the market portfolio satisfies, for all $t \in [0, T]$, a.s.,

$$d \log \Big(Z_{\varphi^{\mathsf{equal}}}(t) / Z_{\mu}(t) \Big) \quad = \quad d \log \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) + \gamma^*_{\varphi^{\mathsf{equal}}}(t)\, dt, \tag{5.6.104}$$

or, by (5.2.8), for all $T \in [0, \infty)$, the relative performance of the equally-weighted portfolio φ^{equal}, can alternatively be represented in integral form as

$$\begin{aligned}
\log \left(\frac{Z_{\varphi^{\mathsf{equal}}}(T)}{Z_{\mu}(T)} \right) &= \log \left(\frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(0)\big)} \right) + \int_0^T \gamma^*_{\varphi^{\mathsf{equal}}}(t)\, dt && (5.6.105)\\
&= \log \left(\frac{\big(\mu_1(T) \cdots \mu_n(T)\big)^{\frac{1}{n}}}{\big(\mu_1(0) \cdots \mu_n(0)\big)^{\frac{1}{n}}} \right) + \int_0^T \gamma^*_{\varphi^{\mathsf{equal}}}(t)\, dt && (5.6.106)\\
&= \log \left(\frac{\mu_1(T) \cdots \mu_n(T)}{\mu_1(0) \cdots \mu_n(0)} \right)^{\frac{1}{n}} + \int_0^T \gamma^*_{\varphi^{\mathsf{equal}}}(t)\, dt \\
&= \frac{1}{n} \log \left(\frac{\mu_1(T) \cdots \mu_n(T)}{\mu_1(0) \cdots \mu_n(0)} \right) + \int_0^T \gamma^*_{\varphi^{\mathsf{equal}}}(t)\, dt && (5.6.107)\\
&= \frac{1}{n} \left(\sum_{i=1}^n \log \mu_i(T) - \sum_{i=1}^n \log \mu_i(0) \right) + \int_0^T \gamma^*_{\varphi^{\mathsf{equal}}}(t)\, dt \\
&= \frac{1}{n} \sum_{i=1}^n \Big(\log \mu_i(T) - \log \mu_i(0) \Big) + \int_0^T \gamma^*_{\varphi^{\mathsf{equal}}}(t)\, dt \\
&= \frac{1}{n} \sum_{i=1}^n \log \left(\frac{\mu_i(T)}{\mu_i(0)} \right) + \int_0^T \gamma^*_{\varphi^{\mathsf{equal}}}(t)\, dt, && (5.6.108)
\end{aligned}$$

if $Z_{\varphi^{\mathsf{equal}}}(0) = Z_{\mu}(0)$, otherwise the integral representation is given by

$$\begin{aligned}
\log \left(\frac{Z_{\varphi^{\mathsf{equal}}}(T)}{Z_{\mu}(T)} \right) &= \log \left(\frac{Z_{\varphi^{\mathsf{equal}}}(0)}{Z_{\mu}(0)} \right) + \log \left(\frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(0)\big)} \right) + \int_0^T \gamma^*_{\varphi^{\mathsf{equal}}}(t)\, dt && (5.6.109)\\
&= \log \left(\frac{Z_{\varphi^{\mathsf{equal}}}(0)}{Z_{\mu}(0)} \right) + \frac{1}{n} \sum_{i=1}^n \log \left(\frac{\mu_i(T)}{\mu_i(0)} \right) + \int_0^T \gamma^*_{\varphi^{\mathsf{equal}}}(t)\, dt, && (5.6.110)
\end{aligned}$$

or, as per equation (5.2.7), we have

$$\begin{aligned}
\log \left(\frac{Z_{\varphi^{\mathsf{equal}}}(T)}{Z_{\varphi^{\mathsf{equal}}}(0)} \right) &= \log \left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)} \right) + \log \left(\frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(0)\big)} \right) + \int_0^T \gamma^*_{\varphi^{\mathsf{equal}}}(t)\, dt && (5.6.111)\\
&= \log \left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)} \right) + \frac{1}{n} \sum_{i=1}^n \log \left(\frac{\mu_i(T)}{\mu_i(0)} \right) + \int_0^T \gamma^*_{\varphi^{\mathsf{equal}}}(t)\, dt. && (5.6.112)
\end{aligned}$$

We can confirm that the geometric mean does indeed generate the equal-weighted portfolio, by deducing the

following from the above expression

$$\begin{aligned}
d\log\Big(Z_{\varphi^{\mathsf{equal}}}(t)/Z_{\mu}(t)\Big) &= d\log\Big(\mu_1(t)\cdots\mu_n(t)\Big)^{\frac{1}{n}} + \gamma^*_{\varphi^{\mathsf{equal}}}(t)\,dt \qquad (5.6.113)\\
&= d\log\Big(\sqrt[n]{\mu_1(t)\cdots\mu_n(t)}\Big) + \gamma^*_{\varphi^{\mathsf{equal}}}(t)\,dt\\
&= d\left(\frac{1}{n}\log\big(\mu_1(t)\cdots\mu_n(t)\big)\right) + \gamma^*_{\varphi^{\mathsf{equal}}}(t)\,dt\\
&= d\left(\frac{1}{n}\sum_{i=1}^{n}\log\mu_i(t)\right) + \gamma^*_{\varphi^{\mathsf{equal}}}(t)\,dt\\
&= \frac{1}{n}\,d\left(\sum_{i=1}^{n}\log\mu_i(t)\right) + \gamma^*_{\varphi^{\mathsf{equal}}}(t)\,dt\\
&= \frac{1}{n}\sum_{i=1}^{n}d\log\mu_i(t) + \gamma^*_{\varphi^{\mathsf{equal}}}(t)\,dt\\
&= \sum_{i=1}^{n}\frac{1}{n}\,d\log\mu_i(t) + \gamma^*_{\varphi^{\mathsf{equal}}}(t)\,dt. \qquad (5.6.114)
\end{aligned}$$

Thus, by appealing to (2.12.63) of Proposition 2.12.8, we have

$$d\log\Big(Z_{\varphi^{\mathsf{equal}}}(t)/Z_{\mu}(t)\Big) = \sum_{i=1}^{n}\varphi_i^{\mathsf{equal}}(t)\,d\log\mu_i(t) + \gamma^*_{\varphi^{\mathsf{equal}}}(t)\,dt,$$

where $\varphi_i^{\mathsf{equal}}(t) = \frac{1}{n}$ for all $i = 1, 2, \ldots, n$, which demonstrates that the geometric mean does indeed generate the equal-weighted portfolio.

5.6.7 The Modified Equal-Weighted Portfolio

Consider the following measure of diversity introduced in equation (4.6.99) of Definition 4.6.14, i.e., the modified form of the geometric mean function or the modified market geometric mean process, $\mathbf{G}(\boldsymbol{\mu}) \equiv \mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}) = \big\{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big), t \in [0,T]\big\}$, $\mathbf{G} \equiv \mathbf{S}_c^{\mathrm{GM}} : \Delta^{n-1} \to (0,\infty)$, for any sufficiently large real constant $c \in (0,\infty)$, and for all $t \in [0,T]$,

$$\mathbf{G}\big(\boldsymbol{\mu}(t)\big) \equiv \mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) = c + \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) = c + \big(\mu_1(t)\cdots\mu_n(t)\big)^{\frac{1}{n}} = c + \left(\prod_{k=1}^{n}\mu_k(t)\right)^{\frac{1}{n}}. \qquad (5.6.115)$$

Thus, by equations (4.6.94), (4.6.102) and (5.6.115), we arrive at the following for the modified geometric mean, for all $i = 1, 2, \ldots, n$ and $t \in [0,T]$,

$$D_i\log\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) = \frac{D_i\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} = \frac{D_i\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{c + \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} = \frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{n\,\mu_i(t)\big(c + \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)\big)}.$$

Alternatively, by appealing to (4.6.100) and (5.6.115), or by directly employing (4.6.102) , we equivalently obtain

$$D_i\log\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) = \frac{D_i\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} = \frac{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - c}{n\,\mu_i(t)\,\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}.$$

Therefore, from (5.2.13) of Theorem 5.2.2, with $\mathbf{G}(\mathbf{x}) = \mathbf{S}_c^{\mathrm{GM}}(\mathbf{x})$, we obtain the following weights for the portfolio generated by the modified geometric mean function $\mathbf{S}_c^{\mathrm{GM}}$,

$$\begin{aligned}
\varphi_i^{(\mathsf{equal},c)}(t) &= \left(D_i \log \mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{j=1}^n \mu_j(t)\, D_j \log \mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) \right) \mu_i(t) \\
&= \left(\frac{D_i \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{c + \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} + 1 - \sum_{j=1}^n \mu_j(t) \left[\frac{D_j \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{c + \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \right] \right) \mu_i(t) \\
&= \left(\frac{D_i \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{c + \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} + 1 - \sum_{j=1}^n \frac{\mu_j(t)\, D_j \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{c + \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \right) \mu_i(t) \\
&= \frac{\mu_i(t)}{c + \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \left(D_i \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) + c + \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - \sum_{j=1}^n \mu_j(t)\, D_j \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) \right).
\end{aligned}$$

Employing (4.6.94) yields the following

$$\begin{aligned}
\varphi_i^{(\mathsf{equal},c)}(t) &= \frac{\mu_i(t)}{c + \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \left(\frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{n\, \mu_i(t)} + c + \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - \sum_{j=1}^n \mu_j(t) \left[\frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{n\, \mu_j(t)} \right] \right) \\
&= \frac{\mu_i(t)}{c + \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \left(\frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{n\, \mu_i(t)} + c + \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - \sum_{j=1}^n \frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{n} \right) \\
&= \frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{c + \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \cdot \mu_i(t) \left(\frac{1}{n\, \mu_i(t)} + \frac{c}{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} + 1 - \sum_{j=1}^n \frac{1}{n} \right) \\
&= \frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{c + \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \cdot \mu_i(t) \left(\frac{1}{n\, \mu_i(t)} + \frac{c}{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} + 1 - 1 \right) \\
&= \frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{c + \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \cdot \mu_i(t) \left(\frac{1}{n\, \mu_i(t)} + \frac{c}{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \right).
\end{aligned}$$

Hence, by appealing to (4.6.100) and (5.6.115), for all $i = 1, 2, \ldots, n$ and $t \in [0, T]$, we have the following weights generated by the modified geometric mean function $\mathbf{S}_c^{\mathrm{GM}}$

$$\varphi_i^{(\mathsf{equal},c)}(t) = \frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{c + \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \cdot \frac{1}{n} + \frac{c}{c + \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \cdot \mu_i(t) \tag{5.6.116}$$

$$= \frac{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - c}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \cdot \frac{1}{n} + \frac{c}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \cdot \mu_i(t). \tag{5.6.117}$$

Therefore, from the previous section, we have

$$\varphi_i^{(\mathsf{equal},c)}(t) = \frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{c + \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \cdot \varphi_i^{\mathsf{equal}}(t) + \frac{c}{c + \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \cdot \mu_i(t) \tag{5.6.118}$$

$$= \frac{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - c}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \cdot \varphi_i^{\mathsf{equal}}(t) + \frac{c}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \cdot \mu_i(t). \tag{5.6.119}$$

Clearly, the modified geometric mean function generates the convex combination, (5.6.116), of the equally-weighted portfolio and the market portfolio. Alternatively, also from (5.2.13) of Theorem 5.2.2, with $\mathbf{G}(\mathbf{x}) = \mathbf{S}_c^{\mathrm{GM}}(\mathbf{x})$, we obtain the following weights for the portfolio generated by the modified geometric mean function

$\mathbf{S}_c^{\mathrm{GM}}$,

$$\begin{aligned}
\varphi_i^{(\mathsf{equal},c)}(t) &= \left(D_i \log \mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{j=1}^{n} \mu_j(t)\, D_j \log \mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) \right) \mu_i(t) \\
&= \left(\frac{D_i \mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} + 1 - \sum_{j=1}^{n} \mu_j(t) \left[\frac{D_j \mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \right] \right) \mu_i(t) \\
&= \left(\frac{D_i \mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} + 1 - \sum_{j=1}^{n} \frac{\mu_j(t)\, D_j \mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \right) \mu_i(t) \\
&= \frac{\mu_i(t)}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \left(D_i \mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) + \mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - \sum_{j=1}^{n} \mu_j(t)\, D_j \mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) \right).
\end{aligned}$$

Employing (4.6.102) yields the following

$$\begin{aligned}
\varphi_i^{(\mathsf{equal},c)}(t) &= \frac{\mu_i(t)}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \left(\frac{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - c}{n\,\mu_i(t)} + \mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - \sum_{j=1}^{n} \mu_j(t) \left[\frac{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - c}{n\,\mu_j(t)} \right] \right) \\
&= \frac{\mu_i(t)}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \left(\frac{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - c}{n\,\mu_i(t)} + \mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - \sum_{j=1}^{n} \frac{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - c}{n} \right) \\
&= \frac{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - c}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \cdot \mu_i(t) \left(\frac{1}{n\,\mu_i(t)} + \frac{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - c} - \sum_{j=1}^{n} \frac{1}{n} \right) \\
&= \frac{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - c}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \cdot \mu_i(t) \left(\frac{1}{n\,\mu_i(t)} + \frac{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - c} - 1 \right) \\
&= \frac{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - c}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \cdot \mu_i(t) \left(\frac{1}{n\,\mu_i(t)} + \frac{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - \mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) + c}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - c} \right) \\
&= \frac{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - c}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \cdot \mu_i(t) \left(\frac{1}{n\,\mu_i(t)} + \frac{c}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - c} \right).
\end{aligned}$$

Hence, by appealing to (4.6.100) and (5.6.115), for all $i = 1, 2, \ldots, n$ and $t \in [0, T]$, we have the following weights generated by the modified geometric mean function $\mathbf{S}_c^{\mathrm{GM}}$

$$\varphi_i^{(\mathsf{equal},c)}(t) = \frac{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - c}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \cdot \frac{1}{n} + \frac{c}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \cdot \mu_i(t) \tag{5.6.120}$$

$$= \frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{c + \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \cdot \frac{1}{n} + \frac{c}{c + \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \cdot \mu_i(t). \tag{5.6.121}$$

Therefore, from the previous section, we have

$$\varphi_i^{(\mathsf{equal},c)}(t) = \frac{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - c}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \cdot \varphi_i^{\mathsf{equal}}(t) + \frac{c}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \cdot \mu_i(t) \tag{5.6.122}$$

$$= \frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{c + \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \cdot \varphi_i^{\mathsf{equal}}(t) + \frac{c}{c + \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \cdot \mu_i(t). \tag{5.6.123}$$

Thus, the weights corresponding to the generating function $\mathbf{S}_c^{\mathrm{GM}}$ can be formalised in the following definition.

Definition 5.6.9 (Modified Equally-Weighted Portfolio)**.** *Let μ be the market portfolio. The portfolio process $\varphi^{(\mathsf{equal},c)} = \big\{\varphi^{(\mathsf{equal},c)}(t) = \big(\varphi_1^{(\mathsf{equal},c)}(t), \varphi_2^{(\mathsf{equal},c)}(t), \ldots, \varphi_n^{(\mathsf{equal},c)}(t)\big), t \in [0, T]\big\}$, with weights defined by* (5.6.118) *or* (5.6.119)*, i.e.,*

$$\varphi_i^{(\mathsf{equal},c)}(t) \triangleq \frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{c + \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \cdot \varphi_i^{\mathsf{equal}}(t) + \frac{c}{c + \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \cdot \mu_i(t) \tag{5.6.124}$$

$$= \frac{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) - c}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \cdot \varphi_i^{\mathsf{equal}}(t) + \frac{c}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \cdot \mu_i(t), \tag{5.6.125}$$

for all $i = 1, 2, \ldots, n$, and $t \in [0, T]$, is called the ***modified equally-weighted portfolio (process)****.*

Thus, the generating function $\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t)) = c + \big(\mu_1(t) \cdots \mu_n(t)\big)^{\frac{1}{n}}$, $\mathbf{S}_c^{\mathrm{GM}} : \Delta^{n-1} \to (0, \infty)$, generates the *modified equal-weighted portfolio*. It can be easily verified that $\varphi^{(\mathsf{equal},c)}$ satisfies the requirements of Definition 2.2.16, since we know that the market weights and the equal weights from the previous section $\varphi_i^{\mathsf{equal}}(t)$ are both bounded on $[0, \infty)$ and the modified market geometric mean process, $\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))$, is also bounded on $[0, \infty)$ according to the bounds derived in (4.6.101), i.e., $c < \mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t)) \leq c + \frac{1}{n}$. Alternatively, we also know that the market geometric mean process, $\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}(t))$, is bounded on $[0, \infty)$ according to the bounds derived in (4.6.93), i.e., $0 < \mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}(t)) \leq \frac{1}{n}$. Moreover, using (5.6.124) and the result from the previous section that $\sum_{i=1}^n \varphi_i^{\mathsf{equal}}(t) = 1$, the weights satisfy the following

$$\begin{aligned}
\sum_{i=1}^n \varphi_i^{(\mathsf{equal},c)}(t) &= \frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}(t))}{c + \mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}(t))} \sum_{i=1}^n \varphi_i^{\mathsf{equal}}(t) + \frac{c}{c + \mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}(t))} \sum_{i=1}^n \mu_i(t) \\
&= \frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}(t))}{c + \mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}(t))} + \frac{c}{c + \mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}(t))} \\
&= \frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}(t)) + c}{c + \mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}(t))} \\
&= 1.
\end{aligned}$$

Alternatively, using (5.6.125) and the result from the previous section that $\sum_{i=1}^n \varphi_i^{\mathsf{equal}}(t) = 1$, the weights satisfy the following

$$\begin{aligned}
\sum_{i=1}^n \varphi_i^{(\mathsf{equal},c)}(t) &= \frac{\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t)) - c}{\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))} \sum_{i=1}^n \varphi_i^{\mathsf{equal}}(t) + \frac{c}{\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))} \sum_{i=1}^n \mu_i(t) \\
&= \frac{\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t)) - c}{\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))} + \frac{c}{\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))} \\
&= \frac{\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t)) - c + c}{\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))} \\
&= \frac{\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))}{\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))} \\
&= 1.
\end{aligned}$$

The weight ratios, for $i = 1, 2, \ldots, n$ and $t \in [0, T]$, satisfy

$$\frac{\varphi_i^{(\mathsf{equal},c)}(t)}{\mu_i(t)} = \frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}(t))}{c + \mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}(t))} \cdot \frac{\varphi_i^{\mathsf{equal}}(t)}{\mu_i(t)} + \frac{c}{c + \mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}(t))} = \frac{\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t)) - c}{\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))} \cdot \frac{\varphi_i^{\mathsf{equal}}(t)}{\mu_i(t)} + \frac{c}{\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))}, \tag{5.6.126}$$

which decrease with increasing market weight $\mu_i(t)$. This follows from Proposition 5.5.2, since $\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))$ is a measure of diversity that generates a diversity generated portfolio, the modified equally-weighted portfolio. Alternatively, if $\mu_i(t) \geq \mu_j(t)$ for some $i > j$, then since (5.6.100) holds from the previous section and since $\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}(t))$ is strictly positive for all $t \in [0, T]$, we have for all $t \in [0, T]$,

$$\frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}(t))}{c + \mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}(t))} \cdot \frac{\varphi_i^{\mathsf{equal}}(t)}{\mu_i(t)} + \frac{c}{c + \mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}(t))} \leq \frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}(t))}{c + \mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}(t))} \cdot \frac{\varphi_j^{\mathsf{equal}}(t)}{\mu_j(t)} + \frac{c}{c + \mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}(t))}.$$

In the same vein, if $\mu_i(t) \geq \mu_j(t)$ for some $i > j$, then since (5.6.100) holds from the previous section and since $\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))$ is strictly positive for all $t \in [0, T]$, we have for all $t \in [0, T]$,

$$\frac{\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t)) - c}{\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))} \cdot \frac{\varphi_i^{\mathsf{equal}}(t)}{\mu_i(t)} + \frac{c}{\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))} \leq \frac{\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t)) - c}{\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))} \cdot \frac{\varphi_j^{\mathsf{equal}}(t)}{\mu_j(t)} + \frac{c}{\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))}.$$

Thus, if $\mu_i(t) \geq \mu_j(t)$ for some $i > j$, (5.6.126) implies

$$\frac{\varphi_i^{(\mathsf{equal},c)}(t)}{\mu_i(t)} \leq \frac{\varphi_j^{(\mathsf{equal},c)}(t)}{\mu_j(t)}.$$

This means that compared to the market portfolio μ, the modified equal-weighted portfolio $\varphi^{(\mathsf{equal},c)}$ is less concentrated than the market portfolio μ in those stocks with the highest market weights (i.e., $\varphi^{(\mathsf{equal},c)}$, relative to the market, is underweighted in the larger stocks), and is more concentrated than the market portfolio μ in those stocks with the lowest market weights (i.e., $\varphi^{(\mathsf{equal},c)}$, relative to the market, is overweighted in the smaller stocks). Now, from (4.6.95), (4.6.96), (4.6.97), (4.6.103), (4.6.104) and (4.6.106), we have for all $i, j = 1, 2, \ldots, n$, and $t \in [0, T]$,

$$D_{ij}\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) = D_{ij}\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) = \begin{cases} \dfrac{1}{n}\left(\dfrac{1}{n}-1\right)\dfrac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} & \text{if} \quad i=j, \\ \dfrac{1}{n^2}\dfrac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)\mu_j(t)} & \text{if} \quad i \neq j, \end{cases} \tag{5.6.127}$$

$$= \begin{cases} \dfrac{1}{n^2}\dfrac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} - \dfrac{1}{n}\dfrac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} & \text{if} \quad i=j, \\ \dfrac{1}{n^2}\dfrac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)\mu_j(t)} & \text{if} \quad i \neq j. \end{cases} \tag{5.6.128}$$

The above expression in conjunction with (5.2.14) of Theorem 5.2.2 with $\mathbf{G}(\mathbf{x}) = \mathbf{S}_c^{\mathrm{GM}}(\mathbf{x})$, yields the following drift process corresponding to the modified geometric mean function, and thus to the modified equally-weighted

portfolio process

$$
\begin{aligned}
d\Theta^{(\mathsf{equal},c)}(t) &= d\Theta_{\varphi^{(\mathsf{equal},c)}}(t) \\
&= \frac{-1}{2\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i,j=1}^n D_{ij}\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt \\
&= \frac{-1}{2\big(c+\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)\big)} \sum_{i,j=1}^n D_{ij}\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt \\
&= \frac{-1}{2\big(c+\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)\big)} \left(\sum_{i=1}^n \left[\frac{1}{n}\left(\frac{1}{n}-1\right)\frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)}\right]\mu_i^2(t)\tau_{ii}(t) + \sum_{\substack{i,j=1\\ i\neq j}}^n \left[\frac{1}{n^2}\,\frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)\mu_j(t)}\right]\mu_i(t)\mu_j(t)\tau_{ij}(t)\right) dt \\
&= \frac{-1}{2\big(c+\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)\big)} \left(\sum_{i=1}^n \left[\frac{1}{n^2}\,\frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)} - \frac{1}{n}\,\frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i^2(t)}\right]\mu_i^2(t)\tau_{ii}(t) + \sum_{\substack{i,j=1\\ i\neq j}}^n \left[\frac{1}{n^2}\,\frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{\mu_i(t)\mu_j(t)}\right]\mu_i(t)\mu_j(t)\tau_{ij}(t)\right) dt \\
&= \frac{-\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{2\big(c+\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)\big)} \left(\sum_{i=1}^n \left(\frac{1}{n^2}-\frac{1}{n}\right)\tau_{ii}(t) + \sum_{\substack{i,j=1\\ i\neq j}}^n \frac{1}{n^2}\tau_{ij}(t)\right) dt \\
&= \frac{-\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{2\big(c+\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)\big)} \left(\sum_{i=1}^n \frac{1}{n^2}\tau_{ii}(t) - \sum_{i=1}^n \frac{1}{n}\tau_{ii}(t) + \sum_{\substack{i,j=1\\ i\neq j}}^n \frac{1}{n^2}\tau_{ij}(t)\right) dt \\
&= \frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{2\big(c+\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)\big)} \left(\sum_{i=1}^n \frac{1}{n}\tau_{ii}(t) - \sum_{i=1}^n \frac{1}{n^2}\tau_{ii}(t) - \sum_{\substack{i,j=1\\ i\neq j}}^n \frac{1}{n^2}\tau_{ij}(t)\right) dt \\
&= \frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{2\big(c+\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)\big)} \left(\sum_{i=1}^n \frac{1}{n}\tau_{ii}(t) - \left[\sum_{i=1}^n \frac{1}{n^2}\tau_{ii}(t) + \sum_{\substack{i,j=1\\ i\neq j}}^n \frac{1}{n^2}\tau_{ij}(t)\right]\right) dt \\
&= \frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{2\big(c+\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)\big)} \left(\sum_{i=1}^n \frac{1}{n}\tau_{ii}(t) - \sum_{i,j=1}^n \frac{1}{n^2}\tau_{ij}(t)\right) dt \\
&= \frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{2\big(c+\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)\big)} \left(\sum_{i=1}^n \frac{1}{n}\tau_{ii}(t) - \sum_{i,j=1}^n \left(\frac{1}{n}\right)^2\tau_{ij}(t)\right) dt \\
&= \frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{2\big(c+\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)\big)} \left(\sum_{i=1}^n \varphi_i^{\mathsf{equal}}(t)\tau_{ii}(t) - \sum_{i,j=1}^n \varphi_i^{\mathsf{equal}}(t)\tau_{ij}(t)\varphi_j^{\mathsf{equal}}(t)\right) dt,
\end{aligned}
$$

which, by the numéraire invariance property of the excess growth rate of the equal-weighted portfolio (2.4.26) of Lemma 2.4.5, and (5.6.102), (4.6.100) and (5.6.115), reduces to

$$
d\Theta^{(\mathsf{equal},c)}(t) = d\Theta_{\varphi^{(\mathsf{equal},c)}}(t) = \frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{c+\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}\,\gamma^*_{\varphi^{\mathsf{equal}}}(t)\,dt = \frac{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)-c}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}\,\gamma^*_{\varphi^{\mathsf{equal}}}(t)\,dt, \tag{5.6.129}
$$

equivalently,

$$
\mathfrak{g}^{(\mathsf{equal},c)}(t) = \mathfrak{g}_{\varphi^{(\mathsf{equal},c)}}(t) = \frac{\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}{c+\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}\,\gamma^*_{\varphi^{\mathsf{equal}}}(t) = \frac{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)-c}{\mathbf{S}_c^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big)}\,\gamma^*_{\varphi^{\mathsf{equal}}}(t). \tag{5.6.130}
$$

Alternatively, from (4.6.103), (4.6.105) and (4.6.106), or by employing (4.6.100), we have for all $i,j=1,2,\ldots,n$,

and $t \in [0,T]$,

$$D_{ij}\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t)) = D_{ij}\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}(t)) = \begin{cases} \frac{1}{n}\left(\frac{1}{n}-1\right)\frac{(\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))-c)}{\mu_i^2(t)} & \text{if} \quad i=j, \\ \frac{1}{n^2}\frac{(\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))-c)}{\mu_i(t)\mu_j(t)} & \text{if} \quad i \neq j, \end{cases} \tag{5.6.131}$$

$$= \begin{cases} \frac{1}{n^2}\frac{(\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))-c)}{\mu_i^2(t)} - \frac{1}{n}\frac{(\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))-c)}{\mu_i^2(t)} & \text{if} \quad i=j, \\ \frac{1}{n^2}\frac{(\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))-c)}{\mu_i(t)\mu_j(t)} & \text{if} \quad i \neq j. \end{cases} \tag{5.6.132}$$

The above expression in conjunction with (5.2.14) of Theorem 5.2.2 with $\mathbf{G}(\mathbf{x}) = \mathbf{S}_c^{\mathrm{GM}}(\mathbf{x})$, yields the following drift process corresponding to the modified geometric mean function

$$\begin{aligned}
d\Theta^{(\mathsf{equal},c)}(t) &= d\Theta_{\varphi^{(\mathsf{equal},c)}}(t) \\
&= \frac{-1}{2\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))}\sum_{i,j=1}^n D_{ij}\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt \\
&= \frac{-1}{2\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))}\left(\sum_{i=1}^n\left[\frac{1}{n}\left(\frac{1}{n}-1\right)\frac{(\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))-c)}{\mu_i^2(t)}\right]\mu_i^2(t)\tau_{ii}(t) + \sum_{\substack{i,j=1\\ i\neq j}}^n\left[\frac{1}{n^2}\frac{(\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))-c)}{\mu_i(t)\mu_j(t)}\right]\mu_i(t)\mu_j(t)\tau_{ij}(t)\right)dt \\
&= \frac{-(\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))-c)}{2\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))}\left(\sum_{i=1}^n\left(\frac{1}{n^2}-\frac{1}{n}\right)\tau_{ii}(t) + \sum_{\substack{i,j=1\\ i\neq j}}^n\frac{1}{n^2}\tau_{ij}(t)\right)dt \\
&= \frac{-(\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))-c)}{2\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))}\left(\sum_{i=1}^n\frac{1}{n^2}\tau_{ii}(t) - \sum_{i=1}^n\frac{1}{n}\tau_{ii}(t) + \sum_{\substack{i,j=1\\ i\neq j}}^n\frac{1}{n^2}\tau_{ij}(t)\right)dt \\
&= \frac{\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))-c}{2\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))}\left(\sum_{i=1}^n\frac{1}{n}\tau_{ii}(t) - \sum_{i=1}^n\frac{1}{n^2}\tau_{ii}(t) - \sum_{\substack{i,j=1\\ i\neq j}}^n\frac{1}{n^2}\tau_{ij}(t)\right)dt \\
&= \frac{\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))-c}{2\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))}\left(\sum_{i=1}^n\frac{1}{n}\tau_{ii}(t) - \left[\sum_{i=1}^n\frac{1}{n^2}\tau_{ii}(t) + \sum_{\substack{i,j=1\\ i\neq j}}^n\frac{1}{n^2}\tau_{ij}(t)\right]\right)dt \\
&= \frac{\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))-c}{2\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))}\left(\sum_{i=1}^n\frac{1}{n}\tau_{ii}(t) - \sum_{i,j=1}^n\frac{1}{n^2}\tau_{ij}(t)\right)dt \\
&= \frac{\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))-c}{2\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))}\left(\sum_{i=1}^n\frac{1}{n}\tau_{ii}(t) - \sum_{i,j=1}^n\left(\frac{1}{n}\right)^2\tau_{ij}(t)\right)dt \\
&= \frac{\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))-c}{2\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))}\left(\sum_{i=1}^n\varphi_i^{\mathsf{equal}}(t)\tau_{ii}(t) - \sum_{i,j=1}^n\varphi_i^{\mathsf{equal}}(t)\tau_{ij}(t)\varphi_j^{\mathsf{equal}}(t)\right)dt,
\end{aligned}$$

which, by the numéraire invariance property of the excess growth rate of the equal-weighted portfolio (2.4.26) of Lemma 2.4.5, and (5.6.102), (4.6.100) and (5.6.115), reduces to

$$d\Theta^{(\mathsf{equal},c)}(t) = d\Theta_{\varphi^{(\mathsf{equal},c)}}(t) = \frac{\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))-c}{\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))}\gamma^*_{\varphi^{\mathsf{equal}}}(t)\,dt = \frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}(t))}{c+\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}(t))}\gamma^*_{\varphi^{\mathsf{equal}}}(t)\,dt, \tag{5.6.133}$$

equivalently,

$$\mathfrak{g}^{(\mathsf{equal},c)}(t) = \mathfrak{g}_{\varphi^{(\mathsf{equal},c)}}(t) = \frac{\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))-c}{\mathbf{S}_c^{\mathrm{GM}}(\boldsymbol{\mu}(t))}\gamma^*_{\varphi^{\mathsf{equal}}}(t) = \frac{\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}(t))}{c+\mathbf{S}^{\mathrm{GM}}(\boldsymbol{\mu}(t))}\gamma^*_{\varphi^{\mathsf{equal}}}(t). \tag{5.6.134}$$

Since the weights of the equal-weighted portfolio are all nonnegative (positive) and equal to $\frac{1}{n}$, we have by (2.4.37) of Proposition 2.4.8 that for a strictly long-only portfolio the excess growth rate of the equal-weighted portfolio is nonnegative (positive), i.e., $\gamma^*_{\varphi^{\mathsf{equal}}}(t) \geq 0$ for all $t \in [0,T]$. In addition, Lemma 2.4.12, (2.4.55), and Lemma 2.4.14, (2.4.56) demonstrate that in a *nondegenerate market*, the excess growth rate of the equal-weighted portfolio, $\gamma^*_{\varphi^{\mathsf{equal}}}(t)$, has a positive lower bound. In addition, by (4.6.101), the modified market geometric mean process also has a positive lower bound, given by $\mathbf{S}^{\mathrm{GM}}_c\big(\boldsymbol{\mu}(t)\big) > c$. Alternatively, by (4.6.93), the market geometric mean process also has a positive lower bound, given by $\mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) > 0$. Hence, these two concepts together indicate that in a nondegenerate market, the drift process of the modified equally-weighted portfolio (5.6.130) or (5.6.134), must also have a positive lower bound. Therefore, the drift process is nonnegative (positive) $\mathfrak{g}^{(\mathsf{equal},c)}(t) = \frac{d\Theta^{(\mathsf{equal},c)}(t)}{dt} \geq 0$ and $\Theta^{(\mathsf{equal},c)}$ is nondecreasing (increasing) since $\mathfrak{g}^{(\mathsf{equal},c)}(t) = \frac{d\Theta^{(\mathsf{equal},c)}(t)}{dt} \geq 0$, i.e., the gradient of the drift process is nonnegative (positive), which follows from Proposition 5.5.2 since the modified geometric mean $\mathbf{S}^{\mathrm{GM}}_c\big(\boldsymbol{\mu}(t)\big)$ is a measure of diversity that generates the diversity generated portfolio $\varphi^{(\mathsf{equal},c)}$ with drift $\Theta^{(\mathsf{equal},c)}$. Moreover, by (5.6.129) or (5.6.133) in conjunction with Theorem 5.2.2 and equation (5.2.2), the logarithmic return process of the modified equal-weighted portfolio $\varphi^{(\mathsf{equal},c)}$, relative to the market portfolio satisfies, for all $t \in [0,T]$, a.s.,

$$d\log\Big(Z_{\varphi^{(\mathsf{equal},c)}}(t)/Z_\mu(t)\Big) \;=\; d\log \mathbf{S}^{\mathrm{GM}}_c\big(\boldsymbol{\mu}(t)\big) + \frac{\mathbf{S}^{\mathrm{GM}}_c\big(\boldsymbol{\mu}(t)\big) - c}{\mathbf{S}^{\mathrm{GM}}_c\big(\boldsymbol{\mu}(t)\big)}\,\gamma^*_{\varphi^{\mathsf{equal}}}(t)\,dt, \tag{5.6.135}$$

or, by (5.2.8), for all $T \in [0,\infty)$, the relative performance of the modified equally-weighted portfolio $\varphi^{(\mathsf{equal},c)}$, can alternatively be represented in integral form as

$$\log\left(\frac{Z_{\varphi^{(\mathsf{equal},c)}}(T)}{Z_\mu(T)}\right) \;=\; \log\left(\frac{\mathbf{S}^{\mathrm{GM}}_c\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{GM}}_c\big(\boldsymbol{\mu}(0)\big)}\right) + \int_0^T \frac{\mathbf{S}^{\mathrm{GM}}_c\big(\boldsymbol{\mu}(t)\big) - c}{\mathbf{S}^{\mathrm{GM}}_c\big(\boldsymbol{\mu}(t)\big)}\,\gamma^*_{\varphi^{\mathsf{equal}}}(t)\,dt, \tag{5.6.136}$$

if $Z_{\varphi^{(\mathsf{equal},c)}}(0) = Z_\mu(0)$, otherwise the integral representation is given by

$$\log\left(\frac{Z_{\varphi^{(\mathsf{equal},c)}}(T)}{Z_\mu(T)}\right) \;=\; \log\left(\frac{Z_{\varphi^{(\mathsf{equal},c)}}(0)}{Z_\mu(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{GM}}_c\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{GM}}_c\big(\boldsymbol{\mu}(0)\big)}\right) + \int_0^T \frac{\mathbf{S}^{\mathrm{GM}}_c\big(\boldsymbol{\mu}(t)\big) - c}{\mathbf{S}^{\mathrm{GM}}_c\big(\boldsymbol{\mu}(t)\big)}\,\gamma^*_{\varphi^{\mathsf{equal}}}(t)\,dt, \tag{5.6.137}$$

or, as per equation (5.2.7), we have

$$\log\left(\frac{Z_{\varphi^{(\mathsf{equal},c)}}(T)}{Z_{\varphi^{(\mathsf{equal},c)}}(0)}\right) \;=\; \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{GM}}_c\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{GM}}_c\big(\boldsymbol{\mu}(0)\big)}\right) + \int_0^T \frac{\mathbf{S}^{\mathrm{GM}}_c\big(\boldsymbol{\mu}(t)\big) - c}{\mathbf{S}^{\mathrm{GM}}_c\big(\boldsymbol{\mu}(t)\big)}\,\gamma^*_{\varphi^{\mathsf{equal}}}(t)\,dt. \tag{5.6.138}$$

5.6.8 The Gini-Coefficient-Weighted Portfolio

5.6.8.1 The Quadratic Gini-Coefficient-Weighted Portfolio

Recall from (4.6.63) of Definition 4.6.8 that the quadratic version of the market Gini coefficient or the market quadratic Gini coefficient process, $\mathbf{G}(\boldsymbol{\mu}) \equiv \mathbf{S}^{\mathrm{G}}(\boldsymbol{\mu}) = \big\{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big), t \in [0,T]\big\}$, $\mathbf{G} \equiv \mathbf{S}^{\mathrm{G}} : \Delta^{n-1} \to (0,\infty)$, is given by

$$\mathbf{G}\big(\boldsymbol{\mu}(t)\big) \;\equiv\; \mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) \;=\; 1 - \frac{1}{2}\sum_{k=1}^n \Big(\mu_k(t) - n^{-1}\Big)^2, \qquad t \in [0,T], \tag{5.6.139}$$

and is a measure of stock market diversity that can be used to generate a portfolio. It follows that

$$\begin{aligned}
\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) &= 1 - \frac{1}{2}\sum_{k=1}^n \left(\mu_k^2(t) - 2\,\frac{\mu_k(t)}{n} + \frac{1}{n^2}\right) \\
&= 1 - \frac{1}{2}\sum_{k=1}^n \mu_k^2(t) + \frac{1}{n}\sum_{k=1}^n \mu_k(t) - \frac{1}{2}\sum_{k=1}^n \frac{1}{n^2} \\
&= 1 - \frac{1}{2}\sum_{k=1}^n \mu_k^2(t) + \frac{1}{n} - \frac{1}{2n} \\
&= 1 - \frac{1}{2}\sum_{k=1}^n \mu_k^2(t) + \frac{1}{2n},
\end{aligned}$$

and consequently we have

$$\sum_{k=1}^{n} \mu_k^2(t) \quad = \quad 2 + \frac{1}{n} - 2\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big). \tag{5.6.140}$$

From equation (4.6.65), we have for all $i = 1, 2, \ldots, n$ and $t \in [0, T]$,

$$D_i \log \mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) \quad = \quad \frac{D_i \mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} \quad = \quad \frac{n^{-1} - \mu_i(t)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)}.$$

Therefore, from (5.2.13) of Theorem 5.2.2, with $\mathbf{G}(\mathbf{x}) = \mathbf{S}^{\mathrm{G}}(\mathbf{x})$, we obtain the following weights for the portfolio generated by the quadratic Gini coefficient $\mathbf{S}^{\mathrm{G}}$,

$$\begin{aligned}
\varphi_i^{\mathrm{g}}(t) &= \left(D_i \log \mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{j=1}^{n} \mu_j(t)\, D_j \log \mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) \right) \mu_i(t) \\
&= \left(\frac{n^{-1} - \mu_i(t)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} + 1 - \sum_{j=1}^{n} \mu_j(t) \left[\frac{\big(n^{-1} - \mu_j(t)\big)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} \right] \right) \mu_i(t) \qquad (5.6.141) \\
&= \left(\frac{n^{-1} - \mu_i(t)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} + 1 - \sum_{j=1}^{n} \frac{\mu_j(t)\big(n^{-1} - \mu_j(t)\big)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} \right) \mu_i(t) \qquad (5.6.142) \\
&= \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} \left(\frac{1}{n} - \mu_i(t) + \mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) - \sum_{j=1}^{n} \mu_j(t)\big(n^{-1} - \mu_j(t)\big) \right) \\
&= \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} \left(\frac{1}{n} - \mu_i(t) + \mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) - \sum_{j=1}^{n} \big(\mu_j(t) n^{-1} - \mu_j^2(t)\big) \right) \\
&= \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} \left(\frac{1}{n} - \mu_i(t) + \mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) - \frac{1}{n} \sum_{j=1}^{n} \mu_j(t) + \sum_{j=1}^{n} \mu_j^2(t) \right) \\
&= \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} \left(\frac{1}{n} - \mu_i(t) + \mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) - \frac{1}{n} + \sum_{j=1}^{n} \mu_j^2(t) \right) \\
&= \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} \left(- \mu_i(t) + \mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) + \sum_{j=1}^{n} \mu_j^2(t) \right).
\end{aligned}$$

Now, since $\sum_{k=1}^{n} \mu_k^2(t) = 2 + \frac{1}{n} - 2\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)$ from (5.6.140), the above expression becomes

$$\begin{aligned}
\varphi_i^{\mathrm{g}}(t) &= \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} \left(- \mu_i(t) + \mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) + 2 + \frac{1}{n} - 2\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) \right) \\
&= \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} \left(- \mu_i(t) + 2 + \frac{1}{n} - \mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) \right) \\
&= \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} \left(2 + \frac{1}{n} - \mu_i(t) - \mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) \right) \\
&= \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} \left(2 + \big(n^{-1} - \mu_i(t)\big) - \mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) \right) \\
&= \left(\frac{2 + \big(n^{-1} - \mu_i(t)\big)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} - 1 \right) \mu_i(t) \qquad (5.6.143) \\
&= \left(\frac{2 - \big(\mu_i(t) - n^{-1}\big)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} - 1 \right) \mu_i(t), \qquad (5.6.144)
\end{aligned}$$

for $i = 1, 2, \ldots, n$, and $t \in [0, T]$. Thus, the weights corresponding to the generating function $\mathbf{S}^{\mathrm{G}}$ can be formalised in the following definition.

Definition 5.6.10 (Quadratic Gini-Coefficient-Weighted Portfolio). *Let μ be the market portfolio. The portfolio process $\varphi^{\mathrm{g}} = \big\{\varphi^{\mathrm{g}}(t) = \big(\varphi_1^{\mathrm{g}}(t), \varphi_2^{\mathrm{g}}(t), \ldots, \varphi_n^{\mathrm{g}}(t)\big), t \in [0,T]\big\}$, with weights defined by* (5.6.142), (5.6.143) *or* (5.6.144), *i.e.,*

$$\begin{aligned}
\varphi_i^{\mathrm{g}}(t) &\triangleq \left(\frac{n^{-1} - \mu_i(t)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} + 1 - \sum_{j=1}^{n} \frac{\mu_j(t)\big(n^{-1} - \mu_j(t)\big)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)}\right)\mu_i(t) && (5.6.145)\\
&= \left(\frac{2 + \big(n^{-1} - \mu_i(t)\big)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} - 1\right)\mu_i(t) && (5.6.146)\\
&= \left(\frac{2 - \big(\mu_i(t) - n^{-1}\big)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} - 1\right)\mu_i(t), && (5.6.147)
\end{aligned}$$

for all $i = 1, 2, \ldots, n$, and $t \in [0,T]$, is called the ***quadratic Gini-coefficient-weighted portfolio (process)****.*

Thus, the generating function $\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)$, $\mathbf{S}^{\mathrm{G}} : \Delta^{n-1} \to (0,\infty)$, generates the *quadratic Gini-coefficient-weighted portfolio*. It can be easily verified that φ^{g} satisfies the requirements of Definition 2.2.16, since we know that the market weights are bounded on $[0,\infty)$ and the market quadratic Gini coefficient process, $\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)$, is also bounded on $[0,\infty)$ according to the bounds derived in (4.6.64), i.e., $\frac{1}{2} < \mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) \le 1$. Moreover, the *quadratic Gini coefficient weights* satisfy the following

$$\sum_{i=1}^{n} \varphi_i^{\mathrm{g}}(t) = \sum_{i=1}^{n} \frac{\big(n^{-1} - \mu_i(t)\big)\mu_i(t)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} + \sum_{i=1}^{n} \mu_i(t) - \sum_{j=1}^{n} \frac{\mu_j(t)\big(n^{-1} - \mu_j(t)\big)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)}\left[\sum_{i=1}^{n} \mu_i(t)\right] = 1.$$

The weight ratios, for $i = 1, 2, \ldots, n$, satisfy

$$\frac{\varphi_i^{\mathrm{g}}(t)}{\mu_i(t)} = \frac{2 + \big(n^{-1} - \mu_i(t)\big)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} - 1 = \frac{2 + n^{-1} - \mu_i(t)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} - 1, \qquad t \in [0,T], \tag{5.6.148}$$

which decrease with increasing market weight $\mu_i(t)$. This follows from Proposition 5.5.2, since $\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)$ is a measure of diversity that generates a diversity generated portfolio, the quadratic Gini-coefficient-weighted portfolio. Alternatively, if $\mu_i(t) \ge \mu_j(t)$ for some $i > j$, then since $\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)$ is strictly positive for all $t \in [0,T]$, we have for all $t \in [0,T]$,

$$\frac{2 + n^{-1} - \mu_i(t)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} - 1 \le \frac{2 + n^{-1} - \mu_j(t)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} - 1.$$

Thus, if $\mu_i(t) \ge \mu_j(t)$ for some $i > j$, (5.6.148) implies

$$\frac{\varphi_i^{\mathrm{g}}(t)}{\mu_i(t)} \le \frac{\varphi_j^{\mathrm{g}}(t)}{\mu_j(t)}.$$

This means that relative to the market, φ^{g} is underweighted in the larger stocks and overweighted in the smaller stocks. Now, from (4.6.66) and (4.6.67), we have for all $i, j = 1, 2, \ldots, n$, and $t \in [0,T]$,

$$D_{ij}\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) = \begin{cases} -1 & \text{if} \quad i = j,\\ 0 & \text{if} \quad i \ne j. \end{cases} \tag{5.6.149}$$

The above expression in conjunction with (5.2.14) of Theorem 5.2.2 with $\mathbf{G}(\mathbf{x}) = \mathbf{S}^{\mathrm{G}}(\mathbf{x})$, yields the following drift process corresponding to the quadratic Gini-coefficient-weighted portfolio process

$$\begin{aligned}
d\Theta^{\mathrm{g}}(t) = d\Theta_{\varphi^{\mathrm{g}}}(t) &= \frac{-1}{2\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i,j=1}^{n} D_{ij}\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt\\
&= \frac{-1}{2\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^{n} (-1)\,\mu_i^2(t)\tau_{ii}(t)\,dt\\
&= \frac{1}{2\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^{n} \mu_i^2(t)\tau_{ii}(t)\,dt, && (5.6.150)
\end{aligned}$$

equivalently,

$$\mathfrak{g}^{\mathrm{g}}(t) \;=\; \mathfrak{g}_{\varphi^{\mathrm{g}}}(t) \;=\; \frac{1}{2\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)}\sum_{i=1}^{n}\mu_i^2(t)\tau_{ii}(t). \tag{5.6.151}$$

Moreover, by (5.6.150) in conjunction with Theorem 5.2.2 and equation (5.2.2), the logarithmic return process of the quadratic Gini-coefficient-weighted portfolio φ^{g}, relative to the market portfolio satisfies, for all $t \in [0,T]$, a.s.,

$$d\log\Big(Z_{\varphi^{\mathrm{g}}}(t)/Z_{\mu}(t)\Big) \;=\; d\log\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) + \frac{1}{2\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)}\sum_{i=1}^{n}\mu_i^2(t)\tau_{ii}(t)\,dt, \tag{5.6.152}$$

or, by (5.2.8), for all $T \in [0,\infty)$, the relative performance of the quadratic Gini-coefficient-weighted portfolio φ^{g}, can alternatively be represented in integral form as

$$\log\left(\frac{Z_{\varphi^{\mathrm{g}}}(T)}{Z_{\mu}(T)}\right) \;=\; \log\left(\frac{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(0)\big)}\right) + \frac{1}{2}\int_0^T \frac{1}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)}\sum_{i=1}^{n}\mu_i^2(t)\tau_{ii}(t)\,dt, \tag{5.6.153}$$

if $Z_{\varphi^{\mathrm{g}}}(0) = Z_{\mu}(0)$, otherwise the integral representation is given by

$$\log\left(\frac{Z_{\varphi^{\mathrm{g}}}(T)}{Z_{\mu}(T)}\right) \;=\; \log\left(\frac{Z_{\varphi^{\mathrm{g}}}(0)}{Z_{\mu}(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(0)\big)}\right) + \frac{1}{2}\int_0^T \frac{1}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)}\sum_{i=1}^{n}\mu_i^2(t)\tau_{ii}(t)\,dt, \tag{5.6.154}$$

or, as per equation (5.2.7), we have

$$\log\left(\frac{Z_{\varphi^{\mathrm{g}}}(T)}{Z_{\varphi^{\mathrm{g}}}(0)}\right) \;=\; \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(0)\big)}\right) + \frac{1}{2}\int_0^T \frac{1}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)}\sum_{i=1}^{n}\mu_i^2(t)\tau_{ii}(t)\,dt. \tag{5.6.155}$$

Recall that Lemma 2.4.9, (2.4.40), and Lemma 2.4.11, (2.4.54) demonstrate that in a *nondegenerate market*, the variance of the ith stock relative to the market, $\tau_{ii}(t)$, has a positive lower bound. Furthermore, (2.4.32) of Lemma 2.4.7 in conjunction with (4.2.6) of Proposition 4.2.3 demonstrate that in a market that is both *nondegenerate* and *diverse*, the variance of the ith stock relative to the market has a positive lower bound. In addition, by (4.6.64), the market quadratic Gini coefficient process also has a positive lower bound, given by $\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) > \frac{1}{2}$. Hence, these two concepts together indicate that in a diverse market, the drift process of the quadratic Gini-coefficient-weighted portfolio (5.6.151), must also have a positive lower bound. Therefore, the drift process is nonnegative (positive) $\mathfrak{g}^{\mathrm{g}}(t) = \frac{d\Theta^{\mathrm{g}}(t)}{dt} \geq 0$ and Θ^{g} is nondecreasing (increasing) since $\mathfrak{g}^{\mathrm{g}}(t) = \frac{d\Theta^{\mathrm{g}}(t)}{dt} \geq 0$, i.e., the gradient of the drift process is nonnegative (positive), which follows from Proposition 5.5.2 since $\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)$ is a measure of diversity that generates the diversity generated portfolio φ^{g} with drift Θ^{g}. Furthermore, recall from (5.2.11) and (5.2.12), that we can calculate the drift process of the quadratic Gini-coefficient-weighted portfolio as follows, for all $T \in [0,\infty)$,

$$\Theta^{\mathrm{g}}(T) \;=\; \int_0^T d\Theta^{\mathrm{g}}(t) \;=\; \int_0^T \mathfrak{g}^{\mathrm{g}}(t)\,dt \;=\; \log\left(\frac{Z_{\varphi^{\mathrm{g}}}(T)}{Z_{\mu}(T)}\right) - \log\left(\frac{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(0)\big)}\right) \tag{5.6.156}$$

$$\;=\; \log\left(\frac{Z_{\varphi^{\mathrm{g}}}(T)\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(0)\big)}{Z_{\mu}(T)\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(T)\big)}\right). \tag{5.6.157}$$

Hence, the cumulative effect of the drift process reveals the remarkable observation that in order to compute this effect over a period of time $[0,T]$ using past data, there is no need to estimate or even know the covariance structure at all. Moreover, the above expression for the cumulative effect of the drift process is in terms of quantities that are all observable. Consider the following series of graphs.

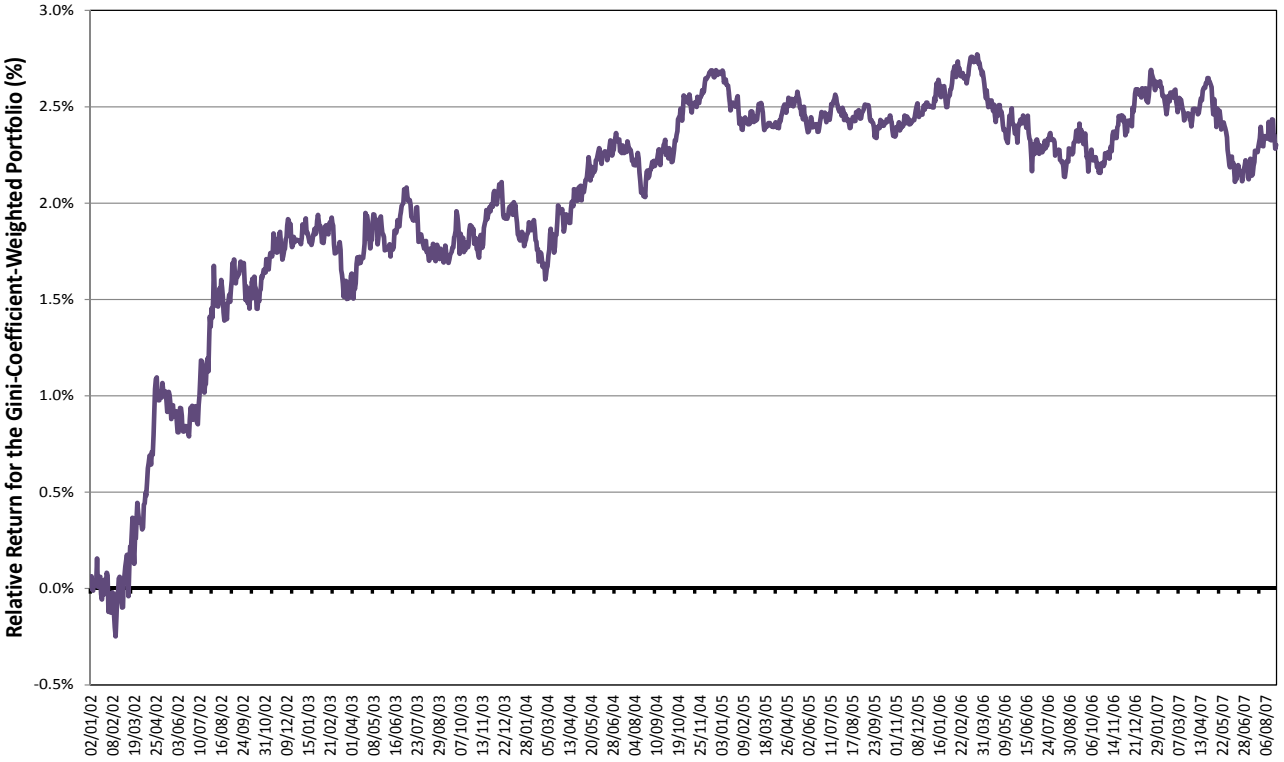

Figure 5.15: $\log\left(Z_{\varphi^g}(t)/Z_\mu(t)\right)$ for the Quadratic Gini-Coefficient-Weighted Portfolio.

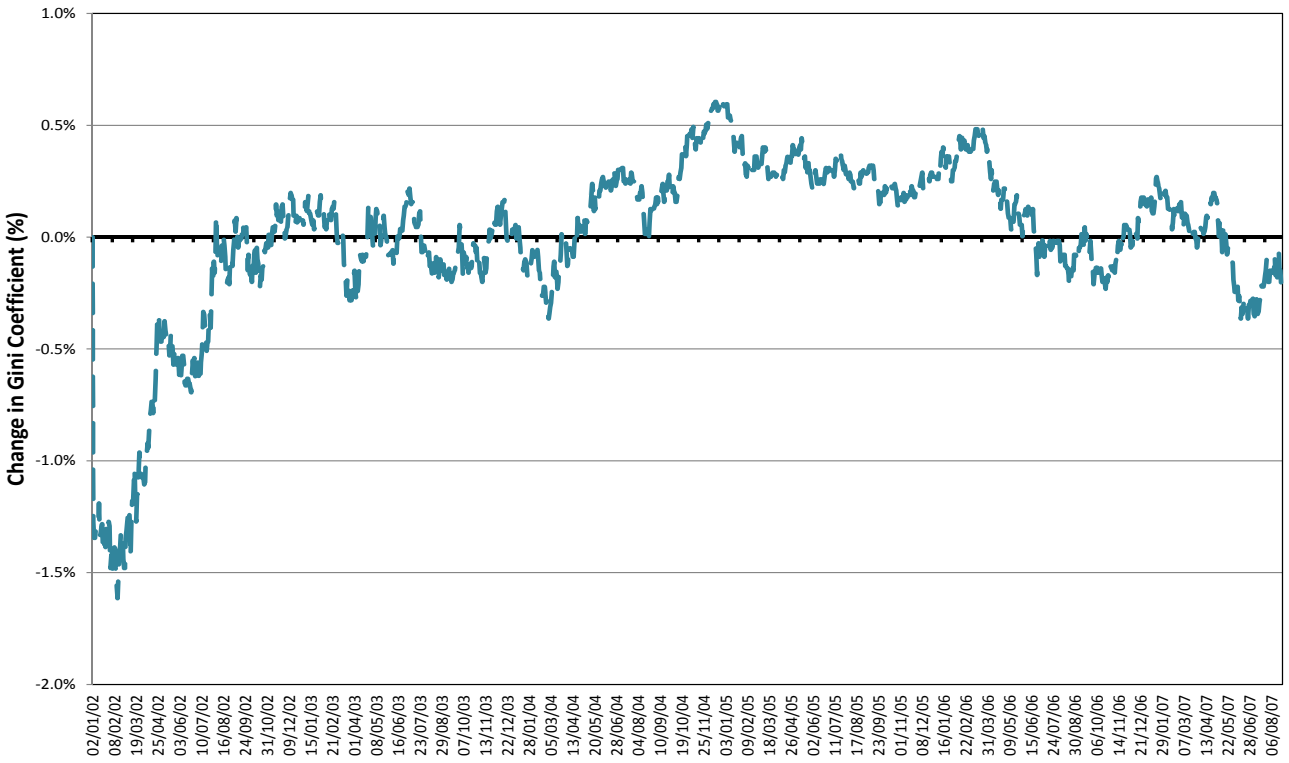

Figure 5.16: Change in the Market Quadratic Gini Coefficient (Centered to have zero sample mean).

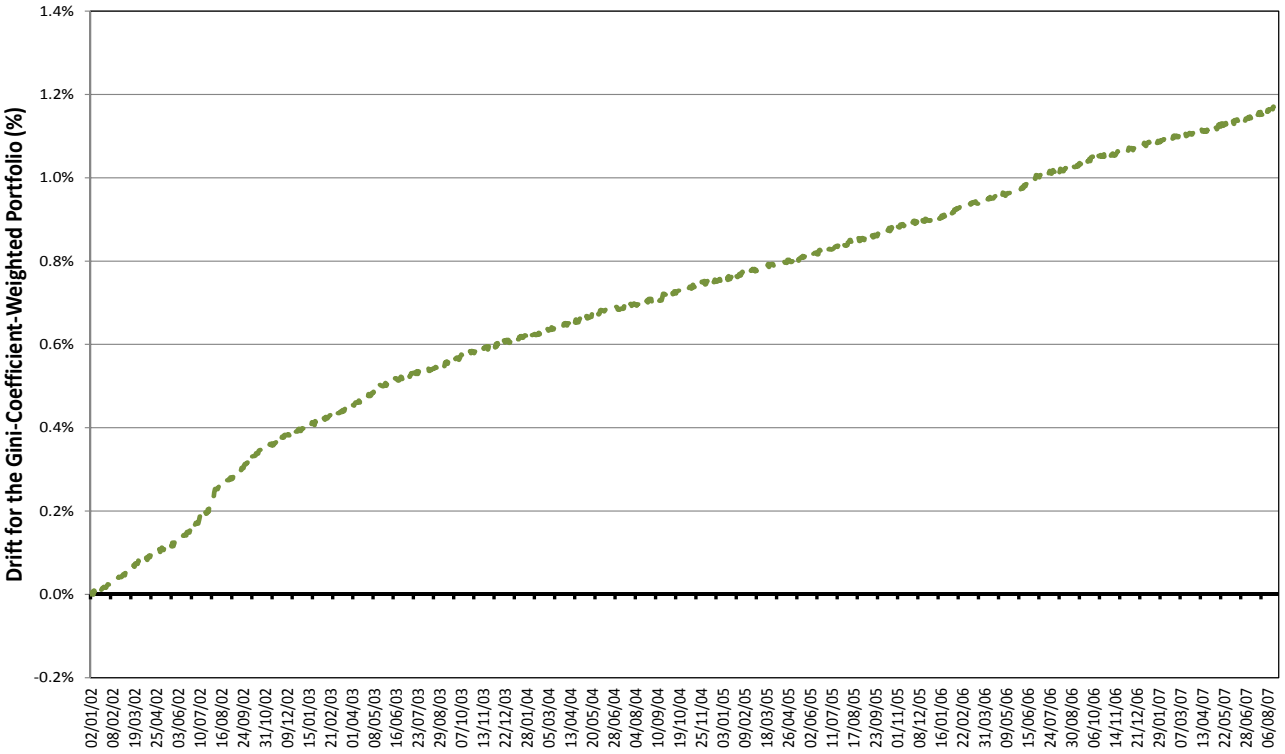

Figure 5.17: Drift Process for the Quadratic Gini-Coefficient-Weighted Portfolio.

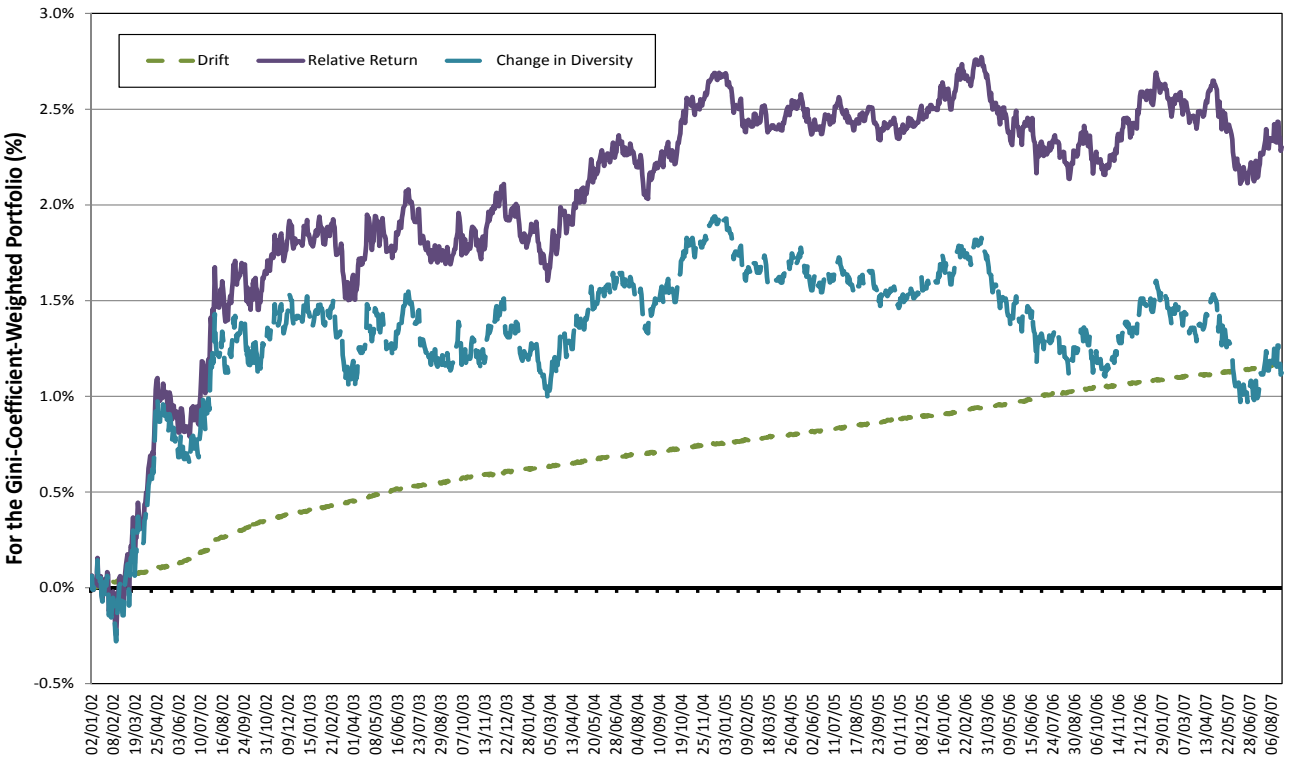

Figure 5.18: The Performance of the Quadratic Gini-Coefficient-Weighted Portfolio.

Figure 5.15 reveals that the quadratic Gini-coefficient-weighted portfolio φ^{g} generated a greater return than the market portfolio μ over the time period under consideration. Thus, the quadratic Gini-coefficient-weighted portfolio φ^{g} will outperform the market portfolio μ over the time horizon under consideration. Now, Figure 5.16 shows the cumulative change in the quadratic Gini coefficient function over the considered time period, which has been adjusted (centered) to have zero sample mean. From the figure, we can see that the cumulative change in the market quadratic Gini coefficient is in the region of -0.21116% or -0.0384% a year, over the time horizon under consideration. Thus, the change in the quadratic Gini coefficient is negative, however it is small in relation to the total variation over the time period. We can also see that the time series evolves in a mean-reverting fashion, similar to that of market entropy and the $\mathbf{D}_p$ function in Figure 5.2 and Figure 5.5. Furthermore, the cumulative changes in the quadratic Gini-coefficient-weighted portfolio demonstrates a stability which is more apparent than for the previous portfolios considered. The quadratic Gini-coefficient-weighted portfolio appears to be a rather appropriate choice for investors who are not keen on taking on a lot of risk, as this portfolio has low risk but backs this up by generating a relative return that is higher (however, small) than the market portfolio. Let us now turn to the drift process of the quadratic Gini-coefficient-weighted portfolio, given in Figure 5.17. The drift process is rather close to being a pure trend process. In total, Figure 5.18 combines all three results of the performance of the quadratic Gini-Coefficient-weighted portfolio into a single figure. Here, each process shows the cumulative value of the daily changes induced in the corresponding process by capital gains or losses in the stocks. From the equations above, it is clear that the relative return curve is given by adding the drift process curve to the changes in the Gini coefficient curve. Now, notice that the relative return process, is the dominant term over the time horizon under consideration.

5.6.8.2 The Quartic Gini-Coefficient-Weighted Portfolio

Consider the following measure of diversity introduced in (4.6.69) of Definition 4.6.9, i.e., the quartic version of the Gini coefficient or the market quartic Gini coefficient process, $\mathbf{G}(\boldsymbol{\mu}) \equiv \mathbf{S}^{\mathrm{QG}}(\boldsymbol{\mu}) = \big\{\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big), t \in [0,T]\big\}$, $\mathbf{G} \equiv \mathbf{S}^{\mathrm{QG}} : \Delta^{n-1} \to (0,\infty)$,

$$\mathbf{G}\big(\boldsymbol{\mu}(t)\big) \equiv \mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big) = 1 - \frac{1}{4}\sum_{k=1}^{n}\Big(\mu_k(t) - n^{-1}\Big)^4, \qquad t \in [0,T]. \tag{5.6.158}$$

From equation (4.6.71), we have for all $i = 1,2,\ldots,n$ and $t \in [0,T]$,

$$D_i \log \mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big) = \frac{D_i\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)} = \frac{-\big(\mu_i(t) - n^{-1}\big)^3}{\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)} = \frac{\big(n^{-1} - \mu_i(t)\big)^3}{\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)}.$$

Therefore, from (5.2.13) of Theorem 5.2.2, with $\mathbf{G}(\mathbf{x}) = \mathbf{S}^{\mathrm{QG}}(\mathbf{x})$, we obtain the following weights for the portfolio generated by the quartic Gini coefficient $\mathbf{S}^{\mathrm{QG}}$,

$$\begin{aligned}
\varphi_i^{\mathsf{qg}}(t) &= \left(D_i \log \mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{j=1}^{n}\mu_j(t)\, D_j \log \mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)\right)\mu_i(t) \\
&= \left(\frac{-\big(\mu_i(t) - n^{-1}\big)^3}{\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)} + 1 - \sum_{j=1}^{n}\mu_j(t)\left[\frac{-\big(\mu_j(t) - n^{-1}\big)^3}{\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)}\right]\right)\mu_i(t) \\
&= \left(\frac{-\big(\mu_i(t) - n^{-1}\big)^3}{\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)} + 1 + \sum_{j=1}^{n}\frac{\mu_j(t)\big(\mu_j(t) - n^{-1}\big)^3}{\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)}\right)\mu_i(t) \qquad (5.6.159) \\
&= \left(\frac{\big(n^{-1} - \mu_i(t)\big)^3}{\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)} + 1 - \sum_{j=1}^{n}\frac{\mu_j(t)\big(n^{-1} - \mu_j(t)\big)^3}{\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)}\right)\mu_i(t), \qquad (5.6.160)
\end{aligned}$$

for $i = 1,2,\ldots,n$, and $t \in [0,T]$. Thus, the weights corresponding to the generating function $\mathbf{S}^{\mathrm{QG}}$ can be formalised in the following definition.

Definition 5.6.11 (Quartic Gini-Coefficient-Weighted Portfolio). *Let μ be the market portfolio. The portfolio process $\varphi^{\mathsf{qg}} = \big\{\varphi^{\mathsf{qg}}(t) = \big(\varphi_1^{\mathsf{qg}}(t), \varphi_2^{\mathsf{qg}}(t), \ldots, \varphi_n^{\mathsf{qg}}(t)\big), t \in [0,T]\big\}$, with weights defined by* (5.6.159) *or*

(5.6.160), *i.e.,*

$$\varphi_i^{\mathsf{qg}}(t) \triangleq \left(\frac{-\big(\mu_i(t)-n^{-1}\big)^3}{\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)} + 1 + \sum_{j=1}^n \frac{\mu_j(t)\big(\mu_j(t)-n^{-1}\big)^3}{\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)} \right) \mu_i(t) \tag{5.6.161}$$

$$= \left(\frac{\big(n^{-1}-\mu_i(t)\big)^3}{\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)} + 1 - \sum_{j=1}^n \frac{\mu_j(t)\big(n^{-1}-\mu_j(t)\big)^3}{\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)} \right) \mu_i(t), \tag{5.6.162}$$

for all $i = 1, 2, \ldots, n$, and $t \in [0, T]$, is called the ***quartic Gini-coefficient-weighted portfolio (process)****.*

Thus, the generating function $\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)$, $\mathbf{S}^{\mathrm{QG}} : \Delta^{n-1} \to (0, \infty)$, generates the *quartic Gini-coefficient-weighted portfolio*. It can be easily verified that φ^{qg} satisfies the requirements of Definition 2.2.16, since we know that the market weights are bounded on $[0, \infty)$ and the market quartic Gini coefficient process, $\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)$, is also bounded on $[0, \infty)$ according to the bounds derived in (4.6.70), i.e., $\frac{3}{4} < \mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big) \leq 1$. Moreover, the *quartic Gini coefficient weights* satisfy the following

$$\sum_{i=1}^n \varphi_i^{\mathsf{qg}}(t) = -\sum_{i=1}^n \frac{\big(\mu_i(t)-n^{-1}\big)^3 \mu_i(t)}{\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)} + \sum_{i=1}^n \mu_i(t) + \sum_{j=1}^n \frac{\mu_j(t)\big(\mu_j(t)-n^{-1}\big)^3}{\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)} \left[\sum_{i=1}^n \mu_i(t) \right] = 1.$$

Now, from (4.6.72) and (4.6.73), we have for all $i, j = 1, 2, \ldots, n$, and $t \in [0, T]$,

$$D_{ij}\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big) = \begin{cases} -3\left(\mu_i(t)-n^{-1}\right)^2 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} = \begin{cases} -3\left(n^{-1}-\mu_i(t)\right)^2 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{5.6.163}$$

The above expression in conjunction with (5.2.14) of Theorem 5.2.2 with $\mathbf{G}(\mathbf{x}) = \mathbf{S}^{\mathrm{QG}}(\mathbf{x})$, yields the following drift process corresponding to the quartic Gini-coefficient-weighted portfolio process

$$\begin{aligned} d\Theta^{\mathsf{qg}}(t) = d\Theta_{\varphi^{\mathsf{qg}}}(t) &= \frac{-1}{2\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i,j=1}^n D_{ij}\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt \\ &= \frac{-1}{2\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^n \left[-3\Big(\mu_i(t)-n^{-1}\Big)^2 \right] \mu_i^2(t)\tau_{ii}(t)\,dt \\ &= \frac{3}{2\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^n \Big(\mu_i(t)-n^{-1}\Big)^2 \mu_i^2(t)\tau_{ii}(t)\,dt \qquad (5.6.164) \\ &= \frac{3}{2\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^n \Big(n^{-1}-\mu_i(t)\Big)^2 \mu_i^2(t)\tau_{ii}(t)\,dt, \qquad (5.6.165) \end{aligned}$$

equivalently,

$$\mathfrak{g}^{\mathsf{qg}}(t) = \mathfrak{g}_{\varphi^{\mathsf{qg}}}(t) = \frac{3}{2\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^n \Big(\mu_i(t)-n^{-1}\Big)^2 \mu_i^2(t)\tau_{ii}(t) \tag{5.6.166}$$

$$= \frac{3}{2\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^n \Big(n^{-1}-\mu_i(t)\Big)^2 \mu_i^2(t)\tau_{ii}(t). \tag{5.6.167}$$

Moreover, by (5.6.164) in conjunction with Theorem 5.2.2 and equation (5.2.2), the logarithmic return process of the quartic Gini-coefficient-weighted portfolio φ^{qg}, relative to the market portfolio satisfies, for all $t \in [0, T]$, a.s.,

$$d\log\Big(Z_{\varphi^{\mathsf{qg}}}(t)/Z_\mu(t)\Big) = d\log\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big) + \frac{3}{2\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^n \Big(\mu_i(t)-n^{-1}\Big)^2 \mu_i^2(t)\tau_{ii}(t)\,dt, \tag{5.6.168}$$

or, by (5.2.8), for all $T \in [0, \infty)$, the relative performance of the quartic Gini-coefficient-weighted portfolio φ^{qg}, can alternatively be represented in integral form as

$$\log\left(\frac{Z_{\varphi^{\mathsf{qg}}}(T)}{Z_\mu(T)}\right) = \log\left(\frac{\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(0)\big)}\right) + \frac{3}{2}\int_0^T \frac{1}{\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^n \Big(\mu_i(t)-n^{-1}\Big)^2 \mu_i^2(t)\tau_{ii}(t)\,dt, \tag{5.6.169}$$

if $Z_{\varphi^{\mathtt{qg}}}(0) = Z_\mu(0)$, otherwise the integral representation is given by

$$\log\left(\frac{Z_{\varphi^{\mathtt{qg}}}(T)}{Z_\mu(T)}\right) = \log\left(\frac{Z_{\varphi^{\mathtt{qg}}}(0)}{Z_\mu(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(0)\big)}\right) + \frac{3}{2}\int_0^T \frac{1}{\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^n \Big(\mu_i(t) - n^{-1}\Big)^2 \mu_i^2(t)\tau_{ii}(t)\,dt, \tag{5.6.170}$$

or, as per equation (5.2.7), we have

$$\log\left(\frac{Z_{\varphi^{\mathtt{qg}}}(T)}{Z_{\varphi^{\mathtt{qg}}}(0)}\right) = \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(0)\big)}\right) + \frac{3}{2}\int_0^T \frac{1}{\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^n \Big(\mu_i(t) - n^{-1}\Big)^2 \mu_i^2(t)\tau_{ii}(t)\,dt. \tag{5.6.171}$$

Recall that Lemma 2.4.9, (2.4.40), and Lemma 2.4.11, (2.4.54) demonstrate that in a *nondegenerate market*, the variance of the ith stock relative to the market, $\tau_{ii}(t)$, has a positive lower bound. Furthermore, (2.4.32) of Lemma 2.4.7 in conjunction with (4.2.6) of Proposition 4.2.3 demonstrate that in a market that is both *nondegenerate* and *diverse*, the variance of the ith stock relative to the market has a positive lower bound. In addition, by (4.6.70), the market quartic Gini coefficient process also has a positive lower bound, given by $\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big) > \frac{3}{4}$. Hence, these two concepts together indicate that in a diverse market, the drift process of the quartic Gini-coefficient-weighted portfolio (5.6.166), must also have a positive lower bound. Therefore, the drift process is nonnegative (positive) $\mathfrak{g}^{\mathtt{qg}}(t) = \frac{d\Theta^{\mathtt{qg}}(t)}{dt} \geq 0$ and $\Theta^{\mathtt{qg}}$ is nondecreasing (increasing) since $\mathfrak{g}^{\mathtt{qg}}(t) = \frac{d\Theta^{\mathtt{qg}}(t)}{dt} \geq 0$, i.e., the gradient of the drift process is nonnegative (positive), which follows from Proposition 5.5.2 since $\mathbf{S}^{\mathrm{QG}}\big(\boldsymbol{\mu}(t)\big)$ is a measure of diversity that generates the diversity generated portfolio $\varphi^{\mathtt{qg}}$ with drift $\Theta^{\mathtt{qg}}$.

5.6.9 The Gini-Simpson-Weighted Index Portfolio

Recall from (4.6.75) of Definition 4.6.10 that the market Gini-Simpson index process, $\mathbf{G}(\boldsymbol{\mu}) \equiv \mathbf{S}^{\mathrm{GS}}(\boldsymbol{\mu}) = \big\{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big), t \in [0,T]\big\}$, $\mathbf{G} \equiv \mathbf{S}^{\mathrm{GS}} : \Delta^{n-1} \to (0,\infty)$, is given by

$$\mathbf{G}\big(\boldsymbol{\mu}(t)\big) \equiv \mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big) = 1 - \sum_{k=1}^n \mu_k^2(t), \qquad t \in [0,T]. \tag{5.6.172}$$

From equation (4.6.77), we have for all $i = 1, 2, \ldots, n$ and $t \in [0,T]$,

$$D_i \log \mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big) = \frac{D_i \mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)} = \frac{-2\mu_i(t)}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)}.$$

Therefore, from (5.2.13) of Theorem 5.2.2, with $\mathbf{G}(\mathbf{x}) = \mathbf{S}^{\mathrm{GS}}(\mathbf{x})$, we obtain the following weights for the portfolio generated by the Gini-Simpson index $\mathbf{S}^{\mathrm{GS}}$,

$$\begin{aligned}
\varphi_i^{\mathtt{gs}}(t) &= \left(D_i \log \mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{j=1}^n \mu_j(t)\, D_j \log \mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)\right)\mu_i(t) \\
&= \left(\frac{-2\mu_i(t)}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)} + 1 - \sum_{j=1}^n \mu_j(t)\left[\frac{-2\mu_j(t)}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)}\right]\right)\mu_i(t) \\
&= \left(\frac{-2\mu_i(t)}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)} + 1 - \sum_{j=1}^n \frac{-2\mu_j^2(t)}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)}\right)\mu_i(t) \\
&= \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)}\left(-2\mu_i(t) + \mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big) + 2\sum_{j=1}^n \mu_j^2(t)\right).
\end{aligned}$$

Now, since $\sum_{k=1}^{n} \mu_k^2(t) = 1 - \mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)$, the above expression becomes

$$\begin{aligned}
\varphi_i^{\mathrm{gs}}(t) &= \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)}\left(-2\mu_i(t) + \mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big) + 2\left[1 - \mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)\right]\right) \\
&= \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)}\left(-2\mu_i(t) + \mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big) + 2 - 2\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)\right) \\
&= \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)}\left(-2\mu_i(t) + 2 - \mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)\right) \\
&= \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)}\left(2 - 2\mu_i(t) - \mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)\right) \\
&= \left(\frac{2 - 2\mu_i(t)}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)} - 1\right)\mu_i(t), \qquad (5.6.173)
\end{aligned}$$

for $i = 1, 2, \ldots, n$, and $t \in [0, T]$. Thus, the weights corresponding to the generating function $\mathbf{S}^{\mathrm{GS}}$ can be formalised in the following definition.

Definition 5.6.12 (Gini-Simpson-Weighted Index Portfolio)**.** *Let μ be the market portfolio. The portfolio process $\varphi^{\mathrm{gs}} = \big\{\varphi^{\mathrm{gs}}(t) = \big(\varphi_1^{\mathrm{gs}}(t), \varphi_2^{\mathrm{gs}}(t), \ldots, \varphi_n^{\mathrm{gs}}(t)\big), t \in [0, T]\big\}$, with weights defined by* (5.6.173)*, i.e.,*

$$\varphi_i^{\mathrm{gs}}(t) \triangleq \left(\frac{2 - 2\mu_i(t)}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)} - 1\right)\mu_i(t), \qquad (5.6.174)$$

for all $i = 1, 2, \ldots, n$, and $t \in [0, T]$, is called the ***Gini-Simpson-weighted index portfolio (process)****.*

Thus, the generating function $\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)$, $\mathbf{S}^{\mathrm{GS}} : \Delta^{n-1} \to (0, \infty)$, generates the *Gini-Simpson-weighted index portfolio*. It can be easily verified that φ^{gs} satisfies the requirements of Definition 2.2.16, since we know that the market weights are bounded on $[0, \infty)$ and the $\mathbf{S}^{\mathrm{GS}}$ process, $\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)$, is also bounded on $[0, \infty)$ according to the bounds derived in (4.6.76), i.e., $0 < \mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big) < 1$. Moreover, the weights satisfy the following

$$\begin{aligned}
\sum_{i=1}^{n} \varphi_i^{\mathrm{gs}}(t) &= \sum_{i=1}^{n} \frac{\big(2 - 2\mu_i(t)\big)\mu_i(t)}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)} - \sum_{i=1}^{n} \mu_i(t) \\
&= \sum_{i=1}^{n} \frac{2\big(1 - \mu_i(t)\big)\mu_i(t)}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)} - \sum_{i=1}^{n} \mu_i(t) \\
&= \frac{2}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^{n} \big(\mu_i(t) - \mu_i^2(t)\big) - 1 \\
&= \frac{2}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)}\left(\sum_{i=1}^{n} \mu_i(t) - \sum_{i=1}^{n} \mu_i^2(t)\right) - 1 \\
&= \frac{2}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)}\left(1 - \sum_{i=1}^{n} \mu_i^2(t)\right) - 1 \\
&= \frac{2}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)}\Big(\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)\Big) - 1 \\
&= 1.
\end{aligned}$$

The weight ratios, for $i = 1, 2, \ldots, n$, satisfy

$$\frac{\varphi_i^{\mathrm{gs}}(t)}{\mu_i(t)} = \frac{2 - 2\mu_i(t)}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)} - 1, \qquad t \in [0, T], \qquad (5.6.175)$$

which decrease with increasing market weight $\mu_i(t)$. This follows from Proposition 5.5.2, since $\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)$ is a measure of diversity that generates a diversity generated portfolio, the Gini-Simpson-weighted index portfolio.

Alternatively, if $\mu_i(t) \geq \mu_j(t)$ for some $i > j$, then since $\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)$ is strictly positive for all $t \in [0,T]$, we have for all $t \in [0,T]$,

$$\frac{2-2\mu_i(t)}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)} - 1 \quad \leq \quad \frac{2-2\mu_j(t)}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)} - 1.$$

Thus, if $\mu_i(t) \geq \mu_j(t)$ for some $i > j$, (5.6.175) implies

$$\frac{\varphi_i^{\mathrm{gs}}(t)}{\mu_i(t)} \quad \leq \quad \frac{\varphi_j^{\mathrm{gs}}(t)}{\mu_j(t)}.$$

This means that relative to the market, φ^{gs} is underweighted in the larger stocks and overweighted in the smaller stocks. Now, from (4.6.78) and (4.6.79), we have for all $i,j = 1,2,\ldots,n$, and $t \in [0,T]$,

$$D_{ij}\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big) \quad = \quad \begin{cases} -2 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{5.6.176}$$

The above expression in conjunction with (5.2.14) of Theorem 5.2.2 with $\mathbf{G}(\mathbf{x}) = \mathbf{S}^{\mathrm{GS}}(\mathbf{x})$, yields the following drift process corresponding to the Gini-Simpson-weighted index portfolio process

$$\begin{aligned} d\Theta^{\mathrm{gs}}(t) \quad = \quad d\Theta_{\varphi^{\mathrm{gs}}}(t) \quad &= \quad \frac{-1}{2\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i,j=1}^{n} D_{ij}\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt \\ &= \quad \frac{-1}{2\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^{n} (-2)\,\mu_i^2(t)\tau_{ii}(t)\,dt \\ &= \quad \frac{1}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^{n} \mu_i^2(t)\tau_{ii}(t)\,dt, \end{aligned} \tag{5.6.177}$$

equivalently,

$$\mathfrak{g}^{\mathrm{gs}}(t) \quad = \quad \mathfrak{g}_{\varphi^{\mathrm{gs}}}(t) \quad = \quad \frac{1}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^{n} \mu_i^2(t)\tau_{ii}(t). \tag{5.6.178}$$

Furthermore, by (5.6.177) in conjunction with Theorem 5.2.2 and equation (5.2.2), the logarithmic return process of the Gini-Simpson-weighted index portfolio φ^{gs}, relative to the market portfolio satisfies, for all $t \in [0,T]$, a.s.,

$$d\log\Big(Z_{\varphi^{\mathrm{gs}}}(t)/Z_{\mu}(t)\Big) \quad = \quad d\log\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big) + \frac{1}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^{n} \mu_i^2(t)\tau_{ii}(t)\,dt, \tag{5.6.179}$$

or, by (5.2.8), for all $T \in [0,\infty)$, the relative performance of the Gini-Simpson-weighted index portfolio φ^{gs}, can alternatively be represented in integral form as

$$\log\left(\frac{Z_{\varphi^{\mathrm{gs}}}(T)}{Z_{\mu}(T)}\right) \quad = \quad \log\left(\frac{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(0)\big)}\right) + \int_0^T \frac{1}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^{n} \mu_i^2(t)\tau_{ii}(t)\,dt, \tag{5.6.180}$$

if $Z_{\varphi^{\mathrm{gs}}}(0) = Z_{\mu}(0)$, otherwise the integral representation is given by

$$\log\left(\frac{Z_{\varphi^{\mathrm{gs}}}(T)}{Z_{\mu}(T)}\right) \quad = \quad \log\left(\frac{Z_{\varphi^{\mathrm{gs}}}(0)}{Z_{\mu}(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(0)\big)}\right) + \int_0^T \frac{1}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^{n} \mu_i^2(t)\tau_{ii}(t)\,dt, \tag{5.6.181}$$

or, as per equation (5.2.7), we have

$$\log\left(\frac{Z_{\varphi^{\mathrm{gs}}}(T)}{Z_{\varphi^{\mathrm{gs}}}(0)}\right) \quad = \quad \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(0)\big)}\right) + \int_0^T \frac{1}{\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^{n} \mu_i^2(t)\tau_{ii}(t)\,dt. \tag{5.6.182}$$

Recall that Lemma 2.4.9, (2.4.40), and Lemma 2.4.11, (2.4.54) demonstrate that in a *nondegenerate market*, the variance of the ith stock relative to the market, $\tau_{ii}(t)$, has a positive lower bound. Furthermore, (2.4.32)

of Lemma 2.4.7 in conjunction with (4.2.6) of Proposition 4.2.3 demonstrate that in a market that is both *nondegenerate* and *diverse*, the variance of the ith stock relative to the market has a positive lower bound. In addition, by (4.6.76), the market Gini-Simpson index process also has a positive lower bound, given by $\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big) > 0$. Hence, these two concepts together indicate that in a diverse market, the drift process of the Gini-Simpson-weighted index portfolio (5.6.178), must also have a positive lower bound. Therefore, the drift process is nonnegative (positive) $\mathfrak{g}^{\mathrm{gs}}(t) = \frac{d\Theta^{\mathrm{gs}}(t)}{dt} \geq 0$ and Θ^{gs} is nondecreasing (increasing) since $\mathfrak{g}^{\mathrm{gs}}(t) = \frac{d\Theta^{\mathrm{gs}}(t)}{dt} \geq 0$, i.e., the gradient of the drift process is nonnegative (positive), which follows from Proposition 5.5.2 since $\mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big)$ is a measure of diversity that generates the diversity generated portfolio φ^{gs} with drift Θ^{gs}.

5.6.10 An Admissible Market-Dominating Portfolio

Recall from (4.6.81) of Definition 4.6.11 that the admissible, market-dominating diversity measure process, $\mathbf{G}(\boldsymbol{\mu}) \equiv \mathbf{S}^{\mathrm{A}}(\boldsymbol{\mu}) = \big\{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big), t \in [0,T]\big\}$, $\mathbf{G} \equiv \mathbf{S}^{\mathrm{A}} : \Delta^{n-1} \to (0,\infty)$, is given by

$$\mathbf{G}\big(\boldsymbol{\mu}(t)\big) \equiv \mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big) = 1 - \frac{1}{2}\sum_{k=1}^{n} \mu_k^2(t), \qquad t \in [0,T]. \tag{5.6.183}$$

From equation (4.6.83), we have for all $i = 1,2,\ldots,n$ and $t \in [0,T]$,

$$D_i \log \mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big) = \frac{D_i \mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)} = \frac{-\mu_i(t)}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)}.$$

Therefore, from (5.2.13) of Theorem 5.2.2, with $\mathbf{G}(\mathbf{x}) = \mathbf{S}^{\mathrm{A}}(\mathbf{x})$, we obtain the following weights for the portfolio generated by the admissible, market-dominating diversity measure process $\mathbf{S}^{\mathrm{A}}$,

$$\begin{aligned}
\varphi_i^{\mathsf{a}}(t) &= \left(D_i \log \mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{j=1}^{n} \mu_j(t)\, D_j \log \mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)\right) \mu_i(t) \\
&= \left(\frac{-\mu_i(t)}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)} + 1 - \sum_{j=1}^{n} \mu_j(t) \left[\frac{-\mu_j(t)}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)}\right]\right) \mu_i(t) \\
&= \left(\frac{-\mu_i(t)}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)} + 1 - \sum_{j=1}^{n} \frac{-\mu_j^2(t)}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)}\right) \mu_i(t) \\
&= \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)} \left(-\mu_i(t) + \mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big) + \sum_{j=1}^{n} \mu_j^2(t)\right).
\end{aligned}$$

Now, since $\sum_{k=1}^{n} \mu_k^2(t) = 2 - 2\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)$, the above expression becomes

$$\begin{aligned}
\varphi_i^{\mathsf{a}}(t) &= \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)} \left(-\mu_i(t) + \mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big) + 2 - 2\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)\right) \\
&= \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)} \left(-\mu_i(t) + 2 - \mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)\right) \\
&= \frac{\mu_i(t)}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)} \left(2 - \mu_i(t) - \mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)\right) \\
&= \left(\frac{2 - \mu_i(t)}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)} - 1\right) \mu_i(t),
\end{aligned} \tag{5.6.184}$$

for $i = 1,2,\ldots,n$, and $t \in [0,T]$. Thus, the weights corresponding to the generating function $\mathbf{S}^{\mathrm{A}}$ can be formalised in the following definition.

Definition 5.6.13 (Admissible Market-Dominating Portfolio). *Let μ be the market portfolio. The portfolio process $\varphi^{\mathsf{a}} = \big\{\varphi^{\mathsf{a}}(t) = \big(\varphi_1^{\mathsf{a}}(t), \varphi_2^{\mathsf{a}}(t), \ldots, \varphi_n^{\mathsf{a}}(t)\big), t \in [0,T]\big\}$, with weights defined by* (5.6.184)*, i.e.,*

$$\varphi_i^{\mathsf{a}}(t) \triangleq \left(\frac{2 - \mu_i(t)}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)} - 1\right) \mu_i(t), \tag{5.6.185}$$

for all $i = 1, 2, \ldots, n$, and $t \in [0, T]$, is called the ***admissible market-dominating portfolio (process)****.*

Thus, the generating function $\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)$, $\mathbf{S}^{\mathrm{A}} : \Delta^{n-1} \to (0, \infty)$, generates the *admissible market-dominating portfolio*. It can be easily verified that φ^{a} satisfies the requirements of Definition 2.2.16, since we know that the market weights are bounded on $[0, \infty)$ and the $\mathbf{S}^{\mathrm{A}}$ process, $\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)$, is also bounded on $[0, \infty)$ according to the bounds derived in (4.6.82), i.e., $\frac{1}{2} < \mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big) < 1$. Moreover, the weights satisfy the following

$$\begin{aligned}
\sum_{i=1}^{n} \varphi_i^{\mathsf{a}}(t) &= \sum_{i=1}^{n} \frac{\big(2 - \mu_i(t)\big)\mu_i(t)}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)} - \sum_{i=1}^{n} \mu_i(t) \\
&= \frac{1}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^{n} \big(2\mu_i(t) - \mu_i^2(t)\big) - 1 \\
&= \frac{1}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)} \left(2 \sum_{i=1}^{n} \mu_i(t) - \sum_{i=1}^{n} \mu_i^2(t)\right) - 1 \\
&= \frac{1}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)} \left(2 - \sum_{i=1}^{n} \mu_i^2(t)\right) - 1 \\
&= \frac{1}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)} \Big(2\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)\Big) - 1 \\
&= 1.
\end{aligned}$$

The weight ratios, for $i = 1, 2, \ldots, n$, satisfy

$$\frac{\varphi_i^{\mathsf{a}}(t)}{\mu_i(t)} = \frac{2 - \mu_i(t)}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)} - 1, \qquad t \in [0, T], \tag{5.6.186}$$

which decrease with increasing market weight $\mu_i(t)$. This follows from Proposition 5.5.2, since $\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)$ is a measure of diversity that generates a diversity generated portfolio, the admissible market-dominating portfolio. Alternatively, if $\mu_i(t) \geq \mu_j(t)$ for some $i > j$, then since $\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)$ is strictly positive for all $t \in [0, T]$, we have for all $t \in [0, T]$,

$$\frac{2 - \mu_i(t)}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)} - 1 \leq \frac{2 - \mu_j(t)}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)} - 1.$$

Thus, if $\mu_i(t) \geq \mu_j(t)$ for some $i > j$, (5.6.186) implies

$$\frac{\varphi_i^{\mathsf{a}}(t)}{\mu_i(t)} \leq \frac{\varphi_j^{\mathsf{a}}(t)}{\mu_j(t)}.$$

This means that relative to the market, φ^{a} is underweighted in the larger stocks and overweighted in the smaller stocks. Now, from (4.6.84) and (4.6.85), we have for all $i, j = 1, 2, \ldots, n$, and $t \in [0, T]$,

$$D_{ij}\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big) = \begin{cases} -1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{5.6.187}$$

The above expression in conjunction with (5.2.14) of Theorem 5.2.2 with $\mathbf{G}(\mathbf{x}) = \mathbf{S}^{\mathrm{A}}(\mathbf{x})$, yields the following drift process corresponding to the admissible, market-dominating portfolio process

$$\begin{aligned}
d\Theta^{\mathsf{a}}(t) = d\Theta_{\varphi^{\mathsf{a}}}(t) &= \frac{-1}{2\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i,j=1}^{n} D_{ij}\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt \\
&= \frac{-1}{2\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^{n} (-1)\,\mu_i^2(t)\tau_{ii}(t)\,dt \\
&= \frac{1}{2\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^{n} \mu_i^2(t)\tau_{ii}(t)\,dt,
\end{aligned} \tag{5.6.188}$$

equivalently,

$$\mathfrak{g}^{\mathsf{a}}(t) \quad = \quad \mathfrak{g}_{\varphi^{\mathsf{a}}}(t) \quad = \quad \frac{1}{2\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^{n} \mu_i^2(t)\tau_{ii}(t). \tag{5.6.189}$$

Furthermore, by (5.6.188) in conjunction with Theorem 5.2.2 and equation (5.2.2), the logarithmic return process of the admissible, market-dominating portfolio φ^{a}, relative to the market portfolio satisfies, for all $t \in [0, T]$, a.s.,

$$d\log\Big(Z_{\varphi^{\mathsf{a}}}(t)/Z_{\mu}(t)\Big) \quad = \quad d\log \mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big) + \frac{1}{2\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^{n} \mu_i^2(t)\tau_{ii}(t)\, dt, \tag{5.6.190}$$

or, by (5.2.8), for all $T \in [0, \infty)$, the relative performance of the admissible, market-dominating portfolio φ^{a}, can alternatively be represented in integral form as

$$\log\left(\frac{Z_{\varphi^{\mathsf{a}}}(T)}{Z_{\mu}(T)}\right) \quad = \quad \log\left(\frac{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(0)\big)}\right) + \frac{1}{2}\int_0^T \frac{1}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^{n} \mu_i^2(t)\tau_{ii}(t)\, dt, \tag{5.6.191}$$

if $Z_{\varphi^{\mathsf{a}}}(0) = Z_{\mu}(0)$, otherwise the integral representation is given by

$$\log\left(\frac{Z_{\varphi^{\mathsf{a}}}(T)}{Z_{\mu}(T)}\right) \quad = \quad \log\left(\frac{Z_{\varphi^{\mathsf{a}}}(0)}{Z_{\mu}(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(0)\big)}\right) + \frac{1}{2}\int_0^T \frac{1}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^{n} \mu_i^2(t)\tau_{ii}(t)\, dt, \tag{5.6.192}$$

or, as per equation (5.2.7), we have

$$\log\left(\frac{Z_{\varphi^{\mathsf{a}}}(T)}{Z_{\varphi^{\mathsf{a}}}(0)}\right) \quad = \quad \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(0)\big)}\right) + \frac{1}{2}\int_0^T \frac{1}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^{n} \mu_i^2(t)\tau_{ii}(t)\, dt. \tag{5.6.193}$$

Recall that Lemma 2.4.9, (2.4.40), and Lemma 2.4.11, (2.4.54) demonstrate that in a *nondegenerate market*, the variance of the ith stock relative to the market, $\tau_{ii}(t)$, has a positive lower bound. Furthermore, (2.4.32) of Lemma 2.4.7 in conjunction with (4.2.6) of Proposition 4.2.3 demonstrate that in a market that is both *nondegenerate* and *diverse*, the variance of the ith stock relative to the market has a positive lower bound. In addition, by (4.6.82), the admissible, market-dominating diversity measure process also has a positive lower bound, given by $\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big) > \frac{1}{2}$. Hence, these two concepts together indicate that in a diverse market, the drift process of the admissible, market-dominating portfolio (5.6.189), must also have a positive lower bound. Therefore, the drift process is nonnegative (positive) $\mathfrak{g}^{\mathsf{a}}(t) = \frac{d\Theta^{\mathsf{a}}(t)}{dt} \geq 0$ and Θ^{a} is nondecreasing (increasing) since $\mathfrak{g}^{\mathsf{a}}(t) = \frac{d\Theta^{\mathsf{a}}(t)}{dt} \geq 0$, i.e., the gradient of the drift process is nonnegative (positive), which follows from Proposition 5.5.2 since $\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)$ is a measure of diversity that generates the diversity generated portfolio φ^{a} with drift Θ^{a}.

5.7 Time-Dependent Portfolio Generating Functions and Time-Based Functionally Generated Portfolios

In this section, we generalise and extend the definition of portfolio generating functions to include time-dependent portfolio generating functions defined on $U \times [0, T]$, with $\log \mathbf{G}\big(\boldsymbol{\mu}(t), t\big)$ replacing $\log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)$. We shall introduce the concept of time-dependent portfolio generating functions as well as the notion of time-based functionally generated portfolios. We shall begin with a formal definition of such time-dependent generating functions and the time-based portfolios they generate.

Definition 5.7.1 (Time-Dependent Generating Functions). *Let U be an open neighbourhood in $\mathbb{R}^n$ ($U \subset \mathbb{R}^n$) of the open positive unit $(n-1)-$simplex Δ^{n-1}*

$$\Delta^{n-1} \quad = \quad \Big\{\boldsymbol{\mu}(t) = \big(\mu_1(t), \ldots, \mu_n(t)\big) \in \mathbb{R}^n \;\Big|\; \mu_1(t) + \cdots + \mu_n(t) = 1,\ 0 < \mu_i(t) < 1,\ i = 1, \ldots, n\Big\},$$

and let $\mathbf{G} : U \times [0,T] \to (0,\infty)$ *be a positive continuous* $C^{2,1}$ *real-valued function defined on some open neighbourhood* U *of* Δ^{n-1}*, i.e. defined on* $U \times [0,T]$ *or* $\Delta^{n-1} \times [0,T]$*, where a positive continuous* $C^{2,1}$ *real-valued function defined on some open neighbourhood* U *of* Δ^{n-1}*, i.e. defined on* $U \times [0,T]$ *or* $\Delta^{n-1} \times [0,T]$*, where* $U \subset \mathbb{R}^n$ *is an open set, is of class* $C^{2,1}$ *if it is twice continuously differentiable in the first* n *variables, and continuously differentiable in the last time variable. Then* $\mathbf{G}$ ***generates*** *a portfolio* φ *if there exist continuous, measurable and adapted processes of bounded variation* $\Theta = \{\Theta(t), t \in [0,\infty)\}$ *and* $\mathfrak{g} = \{\mathfrak{g}(t), t \in [0,\infty)\}$*, such that*

$$\log\Big(Z_\varphi(t)/Z_\mu(t)\Big) = \log \mathbf{G}\big(\boldsymbol{\mu}(t),t\big) + \Theta(t), \qquad t \in [0,T], \qquad a.s., \tag{5.7.1}$$

or such that we have the equivalent differential form

$$d\log\Big(Z_\varphi(t)/Z_\mu(t)\Big) = d\log \mathbf{G}\big(\boldsymbol{\mu}(t),t\big) + d\Theta(t), \qquad t \in [0,T], \qquad a.s., \tag{5.7.2}$$

$$d\log\Big(Z_\varphi(t)/Z_\mu(t)\Big) = d\log \mathbf{G}\big(\boldsymbol{\mu}(t),t\big) + \mathfrak{g}(t)\,dt, \qquad t \in [0,T], \qquad a.s., \tag{5.7.3}$$

where $d\Theta(t) = \mathfrak{g}(t)\,dt$*,* $\Theta(\cdot) = \int_0^\cdot d\Theta(t) = \int_0^\cdot \mathfrak{g}(t)\,dt$ *or* $\mathfrak{g}(t) = \frac{d\Theta(t)}{dt} = \Theta'(t)$*. The process* Θ *is called the* ***drift process*** *corresponding to the* ***time-dependent generating function*** $\mathbf{G}$*. If* $\mathbf{G}$ *generates the portfolio* φ*, then* $\mathbf{G}$ *is called the* ***time-dependent generating function*** *of the portfolio* φ*, and the portfolio* φ *is said to be* ***time-based functionally generated****, or is said to be the* ***time-based functionally generated portfolio*** *corresponding to the* ***time-dependent portfolio generating function*** $\mathbf{G}$*.*

Theorem 5.2.2 can be extended to time-dependent portfolio generating functions.

Theorem 5.7.2 ([**Fernholz (2002)**])**.** *Let* U *be an open neighbourhood in* $\mathbb{R}^n$ $(U \subset \mathbb{R}^n)$ *of the open positive unit* $(n-1)-$*simplex* Δ^{n-1} *and let* $\mathbf{G} : U \times [0,T] \to (0,\infty)$ *be a positive continuous* $C^{2,1}$ *real-valued function defined on some open neighbourhood* U *of* Δ^{n-1}*, i.e. defined on* $U \times [0,T]$ *or* $\Delta^{n-1} \times [0,T]$*, and suppose that for all* $i = 1,2,\ldots,n$*,* $x_i D_i \log \mathbf{G}(\mathbf{x},t)$ *is bounded on* $\Delta^{n-1} \times [0,T]$*. Then for* $t \in [0,T]$ *and for* $i = 1,2,\ldots,n$*, the generating function* $\mathbf{G}$ *generates the (time-based functionally generated) portfolio* φ *with* ***weights***

$$\varphi_i(t) = \left(D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t),t\big) + 1 - \sum_{j=1}^n \mu_j(t)\, D_j \log \mathbf{G}\big(\boldsymbol{\mu}(t),t\big)\right)\mu_i(t), \tag{5.7.4}$$

and with ***drift process*** Θ*, for* $t \in [0,T]$*, a.s., that satisfies*

$$d\Theta(t) = \frac{-1}{2\mathbf{G}\big(\boldsymbol{\mu}(t),t\big)} \sum_{i,j=1}^n D_{ij}\mathbf{G}\big(\boldsymbol{\mu}(t),t\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt - D_t \log \mathbf{G}\big(\boldsymbol{\mu}(t),t\big)\,dt, \tag{5.7.5}$$

alternatively, with ***drift process*** $\mathfrak{g}$*, for* $t \in [0,T]$*, a.s.*

$$\mathfrak{g}(t) = \frac{d\Theta(t)}{dt} = \frac{-1}{2\mathbf{G}\big(\boldsymbol{\mu}(t),t\big)} \sum_{i,j=1}^n D_{ij}\mathbf{G}\big(\boldsymbol{\mu}(t),t\big)\mu_i(t)\mu_j(t)\tau_{ij}(t) - D_t \log \mathbf{G}\big(\boldsymbol{\mu}(t),t\big), \tag{5.7.6}$$

where D_t *represents the first partial derivative with respect to the last variable, the time variable. Let* μ *be the market portfolio and* φ *be the time-based functionally generated portfolio, and let* Z_μ *and* Z_φ *be their portfolio value processes, respectively. Then, a.s., for* $t \in [0,T]$*,*

$$d\log \mathbf{G}\big(\boldsymbol{\mu}(t),t\big) = d\log\Big(Z_\varphi(t)/Z_\mu(t)\Big) + \frac{1}{2\mathbf{G}\big(\boldsymbol{\mu}(t),t\big)} \sum_{i,j=1}^n D_{ij}\mathbf{G}\big(\boldsymbol{\mu}(t),t\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt + D_t \log \mathbf{G}\big(\boldsymbol{\mu}(t),t\big)\,dt. \tag{5.7.7}$$

Proof. Refer to Fernholz (2002) for a short proof, partially based on the proof provided for Theorem 5.2.2. ■

For a few examples of time-dependent portfolio generating functions refer to Fernholz (2002).

5.7.1 Time-Dependent Portfolio Generating Functions: The Connection Between Stochastic Portfolio Theory and the Black-Scholes Option Pricing Theory

Here, we present the connection between time-dependent portfolio generating functions and the Black-Scholes option pricing model, that allows us to view portfolio generating functions in the general setting of option pricing theory [Fernholz (2002)]. If a $C^{2,1}$ time-dependent portfolio generating function $\mathbf{G}$ satisfies for $t \in [0,T]$,

$$D_t \log \mathbf{G}\big(\boldsymbol{\mu}(t),t\big) = \frac{-1}{2\mathbf{G}\big(\boldsymbol{\mu}(t),t\big)} \sum_{i,j=1}^{n} D_{ij}\mathbf{G}\big(\boldsymbol{\mu}(t),t\big)\mu_i(t)\mu_j(t)\tau_{ij}(t), \tag{5.7.8}$$

or, equivalently

$$D_t\mathbf{G}\big(\boldsymbol{\mu}(t),t\big) = -\frac{1}{2}\sum_{i,j=1}^{n} D_{ij}\mathbf{G}\big(\boldsymbol{\mu}(t),t\big)\mu_i(t)\mu_j(t)\tau_{ij}(t), \tag{5.7.9}$$

i.e., $d\Theta(t) = 0$, then (5.7.5) implies that for $t \in [0,T]$,

$$d\log\Big(Z_\varphi(t)/Z_\mu(t)\Big) = d\log \mathbf{G}\big(\boldsymbol{\mu}(t),t\big), \tag{5.7.10}$$

and (5.7.2) becomes a variation of the Black-Scholes option pricing model. Here, $\log \mathbf{G}\big(\boldsymbol{\mu}(t),t\big)$ corresponds to the Black-Scholes option price, and $\log\big(Z_\varphi(t)/Z_\mu(t)\big)$ corresponds to the value of the hedging portfolio. Hence, a time-dependent portfolio generating function with $d\Theta(t) = 0$ or $\mathfrak{g}(t) = 0$ can be considered to be an option pricing function with hedging portfolio φ [see Karatzas and Shreve (1998)].

5.8 Dividends and Portfolio Generating Functions

The next proposition considers the total return process of a functionally generated portfolio and brings the dividend rate into the equation.

Proposition 5.8.1 ([**Fernholz (2002)**]). *Suppose that* $\mathbf{G}$ *generates the portfolio* φ *with drift process* Θ. *Then, a.s., for* $t \in [0,T]$,

$$\log\Big(\widehat{Z}_\varphi(t)/\widehat{Z}_\mu(t)\Big) = \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) + \int_0^t \big(\delta_\varphi(s) - \delta_\mu(s)\big)\,ds + \Theta(t), \tag{5.8.1}$$

or in differential form,

$$d\log\Big(\widehat{Z}_\varphi(t)/\widehat{Z}_\mu(t)\Big) = d\log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) + \big(\delta_\varphi(t) - \delta_\mu(t)\big)\,dt + d\Theta(t) \tag{5.8.2}$$

$$= d\log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) + \Big(\delta_\varphi(t) - \delta_\mu(t) + \mathfrak{g}(t)\Big)\,dt. \tag{5.8.3}$$

Proof. This follows immediately from (5.2.1) or (5.2.2) and (5.2.3) (for the differential form), and from (2.2.157) and Definition 2.2.37 of the total return processes $\widehat{Z}_\varphi$ and $\widehat{Z}_\mu$. ■

The above proposition generalises equations (5.2.1), (5.2.2) and (5.2.3) of Definition 5.2.1 to the total return processes.

5.8.1 Dividends and the Entropy-Weighted Portfolio

The following corollary is akin to Proposition 5.8.1, but it is specific to the entropy-weighted portfolio.

Corollary 5.8.2 ([**Fernholz (2002)**]). *Let* μ *be the market portfolio and* φ^{e} *be the entropy-weighted portfolio, and let* $\widehat{Z}_\mu$ *and* $\widehat{Z}_{\varphi^{\mathrm{e}}}$ *be their total return processes, respectively. Then, a.s., for all* $t \in [0,T]$,

$$d\log\Big(\widehat{Z}_{\varphi^{\mathrm{e}}}(t)/\widehat{Z}_\mu(t)\Big) = d\log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) + \left(\delta_{\varphi^{\mathrm{e}}}(t) - \delta_\mu(t) + \frac{\gamma_\mu^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\right)dt. \tag{5.8.4}$$

Proof. In a similar fashion to the proof of Proposition 5.8.1, this proof follows immediately from Theorem 5.6.2, and from (2.2.157) and Definition 2.2.37 of the total return processes $\widehat{Z}_{\varphi^{\mathrm{e}}}$ and $\widehat{Z}_{\mu}$. ■

If the market is nondegenerate and diverse, then $\log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$ is bounded and hence contributes only negligibly to the long-term variation of $\log\big(\widehat{Z}_{\varphi^{\mathrm{e}}}(t)/\widehat{Z}_{\mu}(t)\big)$. In this case, the long-term risk level of φ^{e} is the same as that of μ, and therefore it could be argued that the return of φ^{e} should not be greater than that of μ. In this case, if T is large enough, we should expect that

$$\int_0^T \big(\delta_{\mu}(t) - \delta_{\varphi^{\mathrm{e}}}(t)\big)\, dt \quad \geq \quad \int_0^T \frac{\gamma_{\mu}^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\, dt.$$

The right-hand side of this inequality is positive, and for the left-hand side to be positive, the dividends of μ must be greater than the dividends of φ^{e}, at least on average over $[0,T]$. Since μ is more concentrated than φ^{e} in the larger stocks in the market, this means that, on average, the larger stocks must have the higher dividend rates than the smaller stocks. We could then conclude that for diversity to be maintained, the larger stocks must redistribute capital by paying dividends [Fernholz (2002)].

5.8.2 Dividends and the $\mathbf{D}_p$-Weighted (Diversity-Weighted) Index Portfolio

The following corollary is akin to Proposition 5.8.1, but it is specific to the $\mathbf{D}_p$-weighted index portfolio.

Corollary 5.8.3. *Let μ be the market portfolio and $\varphi^{(p)}$ be the $\mathbf{D}_p$-weighted index portfolio, and let $\widehat{Z}_{\mu}$ and $\widehat{Z}_{\varphi^{(p)}}$ be their total return processes, respectively. Then, a.s., for all $t \in [0,T]$,*

$$d\log\Big(\widehat{Z}_{\varphi^{(p)}}(t)/\widehat{Z}_{\mu}(t)\Big) \quad = \quad d\log \mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) + \Big(\delta_{\varphi^{(p)}}(t) - \delta_{\mu}(t) + (1-p)\gamma_{\varphi^{(p)}}^*(t)\Big)\, dt. \tag{5.8.5}$$

If the differential dividend rate $\delta_{\varphi^{(p)}}(t) - \delta_{\mu}(t)$ can offset the drift process $(1-p)\gamma_{\varphi^{(p)}}^*(t)$, then the dominance of the $\mathbf{D}_p$-weighted index portfolio over the market portfolio may be prevented. Similarly, the dividends of μ must be greater than the dividends of $\varphi^{(p)}$, at least on average over $[0,T]$, and the larger stocks are required to pay out dividends in order to maintain diversity in the market.

5.9 Summary and Conclusion

This chapter demonstrated that there exist certain (generating) functions of the market weights that can be employed to generate certain portfolios, some specifically designed to have desirable and worthwhile investment properties. In this chapter, we first introduced and defined the concept of portfolio generating functions in conjunction with the allied notion of the functionally generated portfolios that these functions generate. Portfolio generating functions and their associated functionally generated portfolios are considerably important within the realm of stochastic portfolio theory. We provided the stochastic differential equation, associated with functionally generated portfolios. Let U be an open neighbourhood in $\mathbb{R}^n$ ($U \subset \mathbb{R}^n$) of the open positive unit $(n-1)-$simplex Δ^{n-1}

$$\Delta^{n-1} \quad = \quad \Big\{\boldsymbol{\mu}(t) = \big(\mu_1(t), \ldots, \mu_n(t)\big) \in \mathbb{R}^n \;\Big|\; \mu_1(t) + \cdots + \mu_n(t) = 1,\ 0 < \mu_i(t) < 1,\ i = 1, \ldots, n\Big\},$$

and let $\mathbf{G} : U \to (0,\infty)$ be a positive twice continuously differentiable function defined on some open neighbourhood U of Δ^{n-1}. Then $\mathbf{G}$ generates a portfolio φ if there exist continuous, measurable and adapted processes of bounded variation $\Theta = \{\Theta(t), t \in [0,\infty)\}$ and $\mathfrak{g} = \{\mathfrak{g}(t), t \in [0,\infty)\}$, such that

$$\log\Big(Z_{\varphi}(t)/Z_{\mu}(t)\Big) \quad = \quad \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) + \Theta(t), \qquad t \in [0,T], \qquad \text{a.s.}, \tag{5.9.1}$$

or such that we have the equivalent differential form

$$d\log\Big(Z_{\varphi}(t)/Z_{\mu}(t)\Big) \quad = \quad d\log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) + d\Theta(t), \qquad t \in [0,T], \qquad \text{a.s.}, \tag{5.9.2}$$

$$d\log\Big(Z_{\varphi}(t)/Z_{\mu}(t)\Big) \quad = \quad d\log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) + \mathfrak{g}(t)\, dt, \qquad t \in [0,T], \qquad \text{a.s.}, \tag{5.9.3}$$

where $d\Theta(t) = \mathfrak{g}(t)\,dt$, $\Theta(\cdot) = \int_0^{\cdot} d\Theta(t) = \int_0^{\cdot} \mathfrak{g}(t)\,dt$ or $\mathfrak{g}(t) = \frac{d\Theta(t)}{dt} = \Theta'(t)$. The process Θ is called the drift process corresponding to the generating function $\mathbf{G}$. If $\mathbf{G}$ generates the portfolio φ, then $\mathbf{G}$ is called the generating function of the portfolio φ, and the portfolio φ is said to be functionally generated, or is said to be the functionally generated portfolio corresponding to the portfolio generating function $\mathbf{G}$. We determined that such functions of the market weights are required to be positive, twice continuously differentiable real-valued functions, in order to be able to generate portfolios. These positive twice continuously differentiable real-valued functions of the market weights that are utilised to generate portfolios are called portfolio generating functions, i.e. the generating functions of the portfolios in question. Moreover, the portfolios in question that are generated by these functions of the market weights are said to be functionally generated, and are called the functionally generated portfolios corresponding to these said portfolio generating functions. The stochastic differential equation above, associated with the functionally generated portfolio, decomposes the logarithmic relative return process of the functionally generated portfolio with respect to the reference benchmark market portfolio, specifically into two separate constituents. The first component is given by the logarithmic change in the value of the portfolio generating function, to wit, the portfolio generating function component. This component depends only on the market weights which are observable; hence its value is always known. The second component is determined to be the (bounded) drift process corresponding to the functionally generated portfolio generated by the portfolio generating function. The integral form for this functionally generated portfolio φ, is given by

$$\log\left(\frac{Z_\varphi(T)}{Z_\mu(T)}\right) - \log\left(\frac{Z_\varphi(0)}{Z_\mu(0)}\right) = \log\left(\frac{\mathbf{G}(\boldsymbol{\mu}(T))}{\mathbf{G}(\boldsymbol{\mu}(0))}\right) + \int_0^T d\Theta(t), \qquad T \in [0,\infty), \tag{5.9.4}$$

or,

$$\log\left(\frac{Z_\varphi(T)}{Z_\mu(T)}\right) = \log\left(\frac{Z_\varphi(0)}{Z_\mu(0)}\right) + \log\left(\frac{\mathbf{G}(\boldsymbol{\mu}(T))}{\mathbf{G}(\boldsymbol{\mu}(0))}\right) + \int_0^T d\Theta(t), \qquad T \in [0,\infty), \tag{5.9.5}$$

which can be equivalently expressed as

$$\log\left(\frac{Z_\varphi(T)}{Z_\varphi(0)}\right) - \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) = \log\left(\frac{\mathbf{G}(\boldsymbol{\mu}(T))}{\mathbf{G}(\boldsymbol{\mu}(0))}\right) + \int_0^T d\Theta(t), \qquad T \in [0,\infty), \tag{5.9.6}$$

or,

$$\log\left(\frac{Z_\varphi(T)}{Z_\varphi(0)}\right) = \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log\left(\frac{\mathbf{G}(\boldsymbol{\mu}(T))}{\mathbf{G}(\boldsymbol{\mu}(0))}\right) + \int_0^T d\Theta(t), \qquad T \in [0,\infty). \tag{5.9.7}$$

Where $Z_\varphi(0) = Z_\mu(0)$, the logarithmic relative return process of this functionally generated portfolio φ, with respect to the market, is given by the so-called "master formula" or "master equation" [Fernholz & Karatzas (2009)], for all $T \in [0,\infty)$,

$$\log\left(\frac{Z_\varphi(T)}{Z_\mu(T)}\right) = \log\left(\frac{\mathbf{G}(\boldsymbol{\mu}(T))}{\mathbf{G}(\boldsymbol{\mu}(0))}\right) + \int_0^T d\Theta(t), \tag{5.9.8}$$

$$\log\left(\frac{Z_\varphi(T)}{Z_\mu(T)}\right) = \log\left(\frac{\mathbf{G}(\boldsymbol{\mu}(T))}{\mathbf{G}(\boldsymbol{\mu}(0))}\right) + \Theta(T), \tag{5.9.9}$$

$$\log\left(\frac{Z_\varphi(T)}{Z_\mu(T)}\right) = \log\left(\frac{\mathbf{G}(\boldsymbol{\mu}(T))}{\mathbf{G}(\boldsymbol{\mu}(0))}\right) + \int_0^T \mathfrak{g}(t)\,dt, \tag{5.9.10}$$

alternatively expressed, for all $T \in [0,\infty)$, as

$$\Theta(T) = \int_0^T d\Theta(t) = \int_0^T \mathfrak{g}(t)\,dt = \log\left(\frac{Z_\varphi(T)}{Z_\mu(T)}\right) - \log\left(\frac{\mathbf{G}(\boldsymbol{\mu}(T))}{\mathbf{G}(\boldsymbol{\mu}(0))}\right) \tag{5.9.11}$$

$$= \log\left(\frac{Z_\varphi(T)\mathbf{G}(\boldsymbol{\mu}(0))}{Z_\mu(T)\mathbf{G}(\boldsymbol{\mu}(T))}\right), \tag{5.9.12}$$

which gives the cumulative effect of the drift process. Next, the main result of this chapter presented in the primary theorem of portfolio generating functions provides explicit formulas for the weights and the drift of functionally generated portfolios. Let U be an open neighbourhood in $\mathbb{R}^n$ ($U \subset \mathbb{R}^n$) of the open positive unit $(n-1)-$simplex Δ^{n-1} and let $\mathbf{G} : U \to (0, \infty)$ be a positive twice continuously differentiable function defined on some open neighbourhood U of Δ^{n-1}, such that for all $i = 1, 2, \ldots, n$, $x_i D_i \log \mathbf{G}(\mathbf{x})$ is bounded on Δ^{n-1}. Then for $t \in [0, T]$ and for $i = 1, 2, \ldots, n$, the generating function $\mathbf{G}$ generates the (functionally generated) portfolio φ with weights

$$\varphi_i(t) \quad = \quad \left(D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) + 1 - \sum_{j=1}^{n} \mu_j(t)\, D_j \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) \right) \mu_i(t), \tag{5.9.13}$$

and with drift process Θ, for $t \in [0, T]$, a.s.

$$d\Theta(t) \quad = \quad \frac{-1}{2\mathbf{G}\big(\boldsymbol{\mu}(t)\big)} \sum_{i,j=1}^{n} D_{ij}\mathbf{G}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\, dt, \tag{5.9.14}$$

alternatively, with drift process $\mathfrak{g}$, for $t \in [0, T]$, a.s.

$$\mathfrak{g}(t) \quad = \quad \frac{d\Theta(t)}{dt} \quad = \quad \frac{-1}{2\mathbf{G}\big(\boldsymbol{\mu}(t)\big)} \sum_{i,j=1}^{n} D_{ij}\mathbf{G}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t). \tag{5.9.15}$$

Let μ be the market portfolio and φ be the functionally generated portfolio, and let Z_μ and Z_φ be their portfolio value processes, respectively. Then, a.s., for $t \in [0, T]$,

$$d \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) \quad = \quad d \log \Big(Z_\varphi(t)/Z_\mu(t) \Big) + \frac{1}{2\mathbf{G}\big(\boldsymbol{\mu}(t)\big)} \sum_{i,j=1}^{n} D_{ij}\mathbf{G}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\, dt, \tag{5.9.16}$$

or,

$$d \log \Big(Z_\varphi(t)/Z_\mu(t) \Big) \quad = \quad d \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) - \frac{1}{2\mathbf{G}\big(\boldsymbol{\mu}(t)\big)} \sum_{i,j=1}^{n} D_{ij}\mathbf{G}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\, dt. \tag{5.9.17}$$

The weights of these functionally generated portfolios are shown to only depend on the market weights themselves, and do not rely on any prior knowledge of the covariance structure of the equity market at all. This allows for the relatively easy implementation of these functionally generated portfolios. However, the computation of the drift process corresponding to the functionally generated portfolio does rely on the explicit knowledge of the covariance structure of the equity market, as it contains the relative covariance processes. Thus, the computation of the drift process requires the former knowledge of the covariance structure of the market, through the relative covariances. It turns out, however, that this problem can be easily averted by simply considering the cumulative effect of the drift process over a certain period of time, the computation of which does not require any estimation or knowledge whatsoever of the covariance structure of the equity market and it is expressed in terms of market observable quantities. We then provided a characterisation of functionally generated portfolios, in which the conditions necessary for a portfolio to be classified as functionally generated are obtained. We also investigated the circumstances under which two different portfolio generating functions will both generate the same functionally generated portfolio. In fact, in order for two different portfolio generating functions to both generate the same functionally generated portfolio, the necessary and sufficient condition required, is that the ratio of these portfolio generating functions be constant. Then a very important characterisation, that being the characterisation of portfolio generating functions, in which we establish the conditions necessary for a portfolio generating function to generate a functionally generated portfolio with an increasing drift process, is presented. Let $\mathbf{G}$ be a generating function such that for all $\mathbf{x} \in \Delta^{n-1}$, the Hessian matrix $\mathbf{HG}(\mathbf{x}) = \nabla^2\mathbf{G}(\mathbf{x}) = D^2\mathbf{G}(\mathbf{x}) = \big(D_i D_j \mathbf{G}(\mathbf{x})\big)_{1 \le i,j \le n} = \big(D_{ij}\mathbf{G}(\mathbf{x})\big)_{1 \le i,j \le n} \triangleq \big(\frac{\partial^2 \mathbf{G}(\mathbf{x})}{\partial x_i \partial x_j}\big)_{1 \le i,j \le n}$ has at most one positive eigenvalue for each $\mathbf{x} \in \Delta^{n-1}$, and if there is such a positive eigenvalue, it corresponds to an eigenvector orthogonal to Δ^{n-1}. Let φ be the portfolio generated by $\mathbf{G}$. Then $\varphi_i(t) \ge 0$, for $i = 1, 2, \ldots, n$ (i.e., the portfolio φ has only nonnegative weights and is thus long-only), and $\mathfrak{g}(t) = \Theta'(t) \ge 0$ (i.e., the drift process is nonnegative) and Θ is nondecreasing, for all $t \in [0, T]$, a.s. If for all $\mathbf{x} \in \Delta^{n-1}$, $\operatorname{rank}\big(\mathbf{HG}(\mathbf{x})\big) > 1$,

then $\mathfrak{g}(t) = \Theta'(t) > 0$ (i.e., the drift process is strictly positive) and Θ is strictly increasing, for all $t \in [0, T]$, a.s. This characterisation of the possession of an increasing drift process by the functionally generated portfolio, is a crucial one as it forms the foundations for the concept of relative arbitrage that occurs between a functionally generated portfolio, generated by a measure of diversity, and the market portfolio. A few examples of portfolio generating functions along with their associated functionally generated portfolios that they generate, were considered next. We also calculated the drift process accompanying these functionally generated portfolios. The performance of these portfolios relative to the market portfolio, provided in terms of the logarithmic relative return process of these portfolios with respect to the market portfolio, is also determined. The following examples of functionally generated portfolios that are each generated by their respective portfolio generating functions are presented here: the constant-weighted portfolio which comprises constant weights, the buy-and-hold portfolio, the weighted-average capitalisation generated portfolio, the price-to-book ratio generated portfolio, and a single stock with leverage. We introduced and defined the concept of diversity portfolio generating functions, which are just measures of diversity, in conjunction with the allied notion of the diversity generated portfolios that these measures of diversity generate. The portfolios in question that are generated by these measures of diversity are said to be diversity generated, and are called the diversity generated portfolios corresponding to these measures of diversity, or diversity portfolio generating functions. A portfolio φ generated by a measure of diversity, henceforth also referred to as a diversity portfolio generating function, is called a diversity-weighted portfolio, and its proportions are called diversity weights. If a measure of diversity $\mathbf{G}$ generates the portfolio φ, then $\mathbf{G}$ is called the diversity generating function of the portfolio φ, and the portfolio φ is said to be diversity generated, or is said to be the diversity generated portfolio corresponding to the diversity portfolio generating function (or, the measure of diversity) $\mathbf{G}$. We then supplied the following result concerning the drift process of diversity generated portfolios. Suppose that $\mathbf{G}$ is a measure of diversity that generates a portfolio φ (the diversity generated portfolio) with drift process Θ. Then $\mathfrak{g}(t) = \frac{d\Theta(t)}{dt} = \Theta'(t) \geq 0$ is nonnegative and Θ is nondecreasing for all $t \in [0, T]$, a.s. Moreover, $\mu_i(t) \leq \mu_j(t)$ for all $t \in [0, T]$, implies that

$$\frac{\varphi_i(t)}{\mu_i(t)} \geq \frac{\varphi_j(t)}{\mu_j(t)}, \tag{5.9.18}$$

for all $t \in [0, T]$, a.s. Equivalently, $\mu_i(t) \geq \mu_j(t)$ for all $t \in [0, T]$, implies that

$$\frac{\varphi_i(t)}{\mu_i(t)} \leq \frac{\varphi_j(t)}{\mu_j(t)}, \tag{5.9.19}$$

for all $t \in [0, T]$, a.s. This result shows that measures of diversity that generate diversity generated portfolios are endowed with increasing drift processes. Consequently, these diversity generated portfolios tend to outperform the market portfolio. This result also revealed that the corresponding weights of these diversity generated portfolios are positive and the affiliated weight ratios decrease with increasing market weight. A vast presentation of examples of generating functions that are measures of diversity, i.e. diversity portfolio generating functions, along with their associated diversity generated portfolios that they generate, is given next. In particular, we offered the following examples of diversity generated portfolios that are each generated by their respective measures of diversity: the entropy-weighted portfolio, the modified entropy-weighted portfolio, the diversity-weighted index portfolio, the normalised version of the diversity-weighted index portfolio, the market portfolio which is the quintessential buy-and-hold strategy, the equally-weighted portfolio, the modified equally-weighted portfolio, the quadratic Gini-coefficient-weighted portfolio, the quartic Gini-coefficient-weighted portfolio, the Gini-Simpson-weighted index portfolio, and lastly, an admissible market-dominating portfolio. In each case, we determined that the weight ratios associated with these diversity generated portfolios relative to the market portfolio, decrease with increasing market weight. Hence, compared to the market portfolio, these diversity generated portfolios are less concentrated than the market portfolio in those stocks with the highest weights and is more concentrated than the market portfolio in those stocks with the lowest weights. We calculated the drift process accompanying these diversity generated portfolios, and demonstrated that, in a nondegenerate and diverse equity market, these drift processes of these examples of diversity generated portfolios are all increasing in nature. Consequently, ownership of this increasing drift process means that these examples of diversity generated portfolios tend to outperform the market portfolio. An examination of the performance of these portfolios relative to the market portfolio, obtained through the logarithmic relative return process of these

portfolios with respect to the market portfolio, is reflected upon as well. The entropy function $\mathbf{S}^{\mathrm{E}}$ is given by,

$$\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \;=\; \mathbf{S}^{\mathrm{E}}\big(\mu_1(t),\ldots,\mu_n(t)\big) \;\triangleq\; -\sum_{i=1}^{n}\mu_i(t)\log\mu_i(t), \qquad t\in[0,T], \tag{5.9.20}$$

which generates the entropy-weighted portfolio, φ^{e}, with the following weights

$$\varphi_i^{\mathsf{e}}(t) \;\triangleq\; \frac{-\mu_i(t)\log\mu_i(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}, \tag{5.9.21}$$

for all $i=1,2,\ldots,n$, and $t\in[0,T]$, and with the following drift process corresponding to the entropy-weighted portfolio process

$$d\Theta^{\mathsf{e}}(t) \;=\; d\Theta_{\varphi^{\mathsf{e}}}(t) \;=\; \frac{\gamma_\mu^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\,dt, \tag{5.9.22}$$

equivalently,

$$\mathfrak{g}^{\mathsf{e}}(t) \;=\; \mathfrak{g}_{\varphi^{\mathsf{e}}}(t) \;=\; \frac{\gamma_\mu^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}. \tag{5.9.23}$$

The entropy-weighted portfolio satisfies the following, for all $t\in[0,T]$, a.s.,

$$d\log\Big(Z_{\varphi^{\mathsf{e}}}(t)/Z_\mu(t)\Big) \;=\; d\log\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) + \frac{\gamma_\mu^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\,dt, \tag{5.9.24}$$

and we have

$$\log\left(\frac{Z_{\varphi^{\mathsf{e}}}(T)}{Z_\mu(T)}\right) \;=\; \log\left(\frac{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(0)\big)}\right) + \int_0^T \frac{\gamma_\mu^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\,dt, \tag{5.9.25}$$

we also have

$$\log\left(\frac{Z_{\varphi^{\mathsf{e}}}(T)}{Z_\mu(T)}\right) \;=\; \log\left(\frac{Z_{\varphi^{\mathsf{e}}}(0)}{Z_\mu(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(0)\big)}\right) + \int_0^T \frac{\gamma_\mu^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\,dt, \tag{5.9.26}$$

and, lastly, we also have,

$$\log\left(\frac{Z_{\varphi^{\mathsf{e}}}(T)}{Z_{\varphi^{\mathsf{e}}}(0)}\right) \;=\; \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(0)\big)}\right) + \int_0^T \frac{\gamma_\mu^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\,dt. \tag{5.9.27}$$

The $\mathbf{D}_p$-function, $\mathbf{D}_p$, is given by

$$\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) \;=\; \mathbf{D}_p\big(\mu_1(t),\ldots,\mu_n(t)\big) \;\triangleq\; \left(\sum_{i=1}^{n}\big(\mu_i(t)\big)^p\right)^{\frac{1}{p}}, \qquad t\in[0,T], \tag{5.9.28}$$

which generates the $\mathbf{D}_p$-weighted (diversity-weighted) index portfolio, $\varphi^{(p)}$, with the following weights

$$\varphi_i^{(p)}(t) \;\triangleq\; \frac{\big(\mu_i(t)\big)^p}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p} \;=\; \frac{\mu_i^p(t)}{\big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p} \tag{5.9.29}$$

$$\;=\; \left(\frac{\mu_i(t)}{\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)}\right)^p, \tag{5.9.30}$$

for all $i=1,2,\ldots,n$, and $t\in[0,T]$, and with the following drift process corresponding to the $\mathbf{D}_p$-weighted (diversity-weighted) index portfolio process

$$d\Theta^{(p)}(t) \;=\; d\Theta_{\varphi^{(p)}}(t) \;=\; (1-p)\gamma_{\varphi^{(p)}}^*(t)\,dt, \tag{5.9.31}$$

equivalently,

$$\mathfrak{g}^{(p)}(t) \quad = \quad \mathfrak{g}_{\varphi^{(p)}}(t) \quad = \quad (1-p)\gamma^*_{\varphi^{(p)}}(t). \tag{5.9.32}$$

Thus, the $\mathbf{D}_p$-weighted (diversity-weighted) index portfolio satisfies the following, for all $t \in [0,T]$, a.s.,

$$d\log\Big(Z_{\varphi^{(p)}}(t)/Z_\mu(t)\Big) \quad = \quad d\log \mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) + (1-p)\gamma^*_{\varphi^{(p)}}(t)\,dt, \tag{5.9.33}$$

and we have

$$\log\left(\frac{Z_{\varphi^{(p)}}(T)}{Z_\mu(T)}\right) \quad = \quad \log\left(\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(T)\big)}{\mathbf{D}_p\big(\boldsymbol{\mu}(0)\big)}\right) + (1-p)\int_0^T \gamma^*_{\varphi^{(p)}}(t)\,dt, \tag{5.9.34}$$

we also have

$$\log\left(\frac{Z_{\varphi^{(p)}}(T)}{Z_\mu(T)}\right) \quad = \quad \log\left(\frac{Z_{\varphi^{(p)}}(0)}{Z_\mu(0)}\right) + \log\left(\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(T)\big)}{\mathbf{D}_p\big(\boldsymbol{\mu}(0)\big)}\right) + (1-p)\int_0^T \gamma^*_{\varphi^{(p)}}(t)\,dt, \tag{5.9.35}$$

and, lastly, we also have,

$$\log\left(\frac{Z_{\varphi^{(p)}}(T)}{Z_{\varphi^{(p)}}(0)}\right) \quad = \quad \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log\left(\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(T)\big)}{\mathbf{D}_p\big(\boldsymbol{\mu}(0)\big)}\right) + (1-p)\int_0^T \gamma^*_{\varphi^{(p)}}(t)\,dt. \tag{5.9.36}$$

The quadratic Gini coefficient function $\mathbf{S}^{\mathrm{G}}$ is given by

$$\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) \quad = \quad \mathbf{S}^{\mathrm{G}}\big(\mu_1(t),\ldots,\mu_n(t)\big) \quad \triangleq \quad 1 - \frac{1}{2}\sum_{i=1}^n \Big(\mu_i(t) - n^{-1}\Big)^2, \qquad t \in [0,T], \tag{5.9.37}$$

which generates the quadratic Gini-coefficient-weighted portfolio, φ^{g}, with the following weights

$$\varphi^{\mathrm{g}}_i(t) \quad \triangleq \quad \left(\frac{n^{-1} - \mu_i(t)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} + 1 - \sum_{j=1}^n \frac{\mu_j(t)\big(n^{-1} - \mu_j(t)\big)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)}\right)\mu_i(t) \tag{5.9.38}$$

$$= \quad \left(\frac{2 + \big(n^{-1} - \mu_i(t)\big)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} - 1\right)\mu_i(t) \tag{5.9.39}$$

$$= \quad \left(\frac{2 - \big(\mu_i(t) - n^{-1}\big)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} - 1\right)\mu_i(t), \tag{5.9.40}$$

for all $i = 1,2,\ldots,n$, and $t \in [0,T]$, and with the following drift process corresponding to the quadratic Gini-coefficient-weighted portfolio process

$$d\Theta^{\mathrm{g}}(t) \quad = \quad d\Theta_{\varphi^{\mathrm{g}}}(t) \quad = \quad \frac{1}{2\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)}\sum_{i=1}^n \mu_i^2(t)\tau_{ii}(t)\,dt, \tag{5.9.41}$$

equivalently,

$$\mathfrak{g}^{\mathrm{g}}(t) \quad = \quad \mathfrak{g}_{\varphi^{\mathrm{g}}}(t) \quad = \quad \frac{1}{2\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)}\sum_{i=1}^n \mu_i^2(t)\tau_{ii}(t). \tag{5.9.42}$$

Thus, the quadratic Gini-coefficient-weighted portfolio satisfies the following, for all $t \in [0,T]$, a.s.,

$$d\log\Big(Z_{\varphi^{\mathrm{g}}}(t)/Z_\mu(t)\Big) \quad = \quad d\log \mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) + \frac{1}{2\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)}\sum_{i=1}^n \mu_i^2(t)\tau_{ii}(t)\,dt, \tag{5.9.43}$$

and we have

$$\log\left(\frac{Z_{\varphi^{\mathrm{g}}}(T)}{Z_\mu(T)}\right) \quad = \quad \log\left(\frac{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(0)\big)}\right) + \frac{1}{2}\int_0^T \frac{1}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)}\sum_{i=1}^n \mu_i^2(t)\tau_{ii}(t)\,dt, \tag{5.9.44}$$

we also have

$$\log\left(\frac{Z_{\varphi^{\mathsf{g}}}(T)}{Z_{\mu}(T)}\right) = \log\left(\frac{Z_{\varphi^{\mathsf{g}}}(0)}{Z_{\mu}(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(0)\big)}\right) + \frac{1}{2}\int_0^T \frac{1}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)}\sum_{i=1}^n \mu_i^2(t)\tau_{ii}(t)\,dt, \tag{5.9.45}$$

and, lastly, we also have

$$\log\left(\frac{Z_{\varphi^{\mathsf{g}}}(T)}{Z_{\varphi^{\mathsf{g}}}(0)}\right) = \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(0)\big)}\right) + \frac{1}{2}\int_0^T \frac{1}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)}\sum_{i=1}^n \mu_i^2(t)\tau_{ii}(t)\,dt. \tag{5.9.46}$$

The admissible market-dominating diversity measure function $\mathbf{S}^{\mathrm{A}}$ is given by

$$\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big) = \mathbf{S}^{\mathrm{A}}\big(\mu_1(t),\ldots,\mu_n(t)\big) \triangleq 1 - \frac{1}{2}\sum_{i=1}^n \mu_i^2(t), \qquad t\in[0,T], \tag{5.9.47}$$

which generates the admissible market-dominating portfolio, φ^{a}, with the following weights

$$\varphi_i^{\mathsf{a}}(t) \triangleq \left(\frac{2-\mu_i(t)}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)} - 1\right)\mu_i(t), \tag{5.9.48}$$

for all $i = 1,2,\ldots,n$, and $t\in[0,T]$, and with the following drift process corresponding to the admissible market-dominating portfolio process

$$d\Theta^{\mathsf{a}}(t) = d\Theta_{\varphi^{\mathsf{a}}}(t) = \frac{1}{2\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)}\sum_{i=1}^n \mu_i^2(t)\tau_{ii}(t)\,dt, \tag{5.9.49}$$

equivalently,

$$\mathfrak{g}^{\mathsf{a}}(t) = \mathfrak{g}_{\varphi^{\mathsf{a}}}(t) = \frac{1}{2\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)}\sum_{i=1}^n \mu_i^2(t)\tau_{ii}(t). \tag{5.9.50}$$

Thus, the admissible market-dominating portfolio satisfies the following, for all $t\in[0,T]$, a.s.,

$$d\log\Big(Z_{\varphi^{\mathsf{a}}}(t)/Z_{\mu}(t)\Big) = d\log\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big) + \frac{1}{2\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)}\sum_{i=1}^n \mu_i^2(t)\tau_{ii}(t)\,dt, \tag{5.9.51}$$

and we have

$$\log\left(\frac{Z_{\varphi^{\mathsf{a}}}(T)}{Z_{\mu}(T)}\right) = \log\left(\frac{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(0)\big)}\right) + \frac{1}{2}\int_0^T \frac{1}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)}\sum_{i=1}^n \mu_i^2(t)\tau_{ii}(t)\,dt, \tag{5.9.52}$$

we also have

$$\log\left(\frac{Z_{\varphi^{\mathsf{a}}}(T)}{Z_{\mu}(T)}\right) = \log\left(\frac{Z_{\varphi^{\mathsf{a}}}(0)}{Z_{\mu}(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(0)\big)}\right) + \frac{1}{2}\int_0^T \frac{1}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)}\sum_{i=1}^n \mu_i^2(t)\tau_{ii}(t)\,dt, \tag{5.9.53}$$

and, lastly, we also have

$$\log\left(\frac{Z_{\varphi^{\mathsf{a}}}(T)}{Z_{\varphi^{\mathsf{a}}}(0)}\right) = \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(0)\big)}\right) + \frac{1}{2}\int_0^T \frac{1}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)}\sum_{i=1}^n \mu_i^2(t)\tau_{ii}(t)\,dt. \tag{5.9.54}$$

Chapter 6

Rank-Dependent Portfolio Generating Functions and Rank-Based Functionally Generated Portfolios

6.1 Introduction

Size is one of the most important and crucial descriptive characteristics of financial assets. A greater and deeper understanding of financial equity markets can be garnered by observing, and attempting to make sense of, the continual ebb and flow of small-, medium-, and large-capitalisation stocks. By looking at the evolution of the capital distribution curve, which is the logarithms of the market weights arranged in descending order versus the logarithms of their respective ranks, we can study the size feature in equity markets. An important generalisation of the ideas and methods in this chapter concerns portfolio generating functions that record market weights not according to their name (or subscript, or index), but rather according to their rank.

The capital distribution of the market, i.e. the family of ranked market weights, is of paramount importance in stochastic portfolio theory and so too are portfolio generating functions and the portfolios they generate. This chapter will integrate these two concepts, i.e., the distribution of equity capital and the functionally generated portfolios of the previous chapter. Portfolio generating functions were introduced in the previous chapter but as functions of the market weights, not as functions of the ranked market weights. Recall that positive twice continuously differentiable functions of the market weights will generate functionally generated portfolios. The essential feature of functionally generated portfolios is that the relative return of such a portfolio can be decomposed naturally into two components: the change in the value of the generating function and a finite (bounded) variation drift process. However, these positive twice continuously differentiable functions are functions of the market weights, and not functions of the ranked market weights. Although these functionally generated portfolios had useful theoretical properties, the construction was not sufficiently general to allow for the study of portfolios composed of stocks selected by market capitalisation, as occurs in many equity indices.

In this chapter, we shall consider functions of the ranked market weights, and show that under appropriate conditions they also generate portfolios. In particular, the technique introduced in this chapter will allow us to measure the effect that changes in the distribution of capital have on the performance of portfolios that are generated by functions of the ranked market weights. Portfolios generated by functions of the ranked market weights first made an appearance in Fernholz (2001c). We show that a positive twice continuously differentiable function of the ranked market weights of an equity market generates dynamic equity portfolios. In order to characterise the changes in the ranked market weights in terms of the relative returns of the stocks, we need to introduce the concept of a semimartingale local time, a measure of the amount of time a process spends near the origin. This will allow us to consider portfolios composed exclusively of large stocks, which are identified

by ranked market weights. The return on such a portfolio relative to the market portfolio follows a stochastic differential equation similar to that derived in the previous chapter that decomposes the relative return into the two similar components: the logarithmic change in the value of the generating function, and a drift process that is of bounded variation, however now the drift process includes semimartingale local times that account for changes in rank that occur or take place among the market weights in the equity market. Although a change in a market weight represents the relative return of the corresponding stock, changes in the ranked market weights do not have such a simple interpretation. Stocks selected by size depend on the ranks of the stocks in the market, and rank functions are not differentiable. Here we shall extend the results of Fernholz (1999c) to portfolios that are generated by functions of the ranked market weights. The case that involves generating functions as functions of the ranked market weights is mathematically more complicated than Theorem 5.2.2 of the previous chapter. This is because the generating function is not differentiable since the transformation from market weights to ranked market weights is not differentiable, and hence Itô's formula cannot be directly applied. Instead, semimartingale local times must be introduced to account for the changes in rank that occur randomly among the stock price processes. As a consequence, the drift process for a portfolio generated by a function of the ranked market weights is composed of two distinct constituents: a smooth component similar to the drift term in (5.2.14) of Theorem 5.2.2, and a local time component which is the constituent involving local times for changes in rank among the market weights. Some preliminary introductory material regarding semimartingale local time is presented first, and we then prove a few results that are required for rank processes, that are derived from continuous semimartingales. We then proceed to prove the central theorem of this chapter, and finally we present a few examples and applications of the theory of rank-dependent portfolio generating functions.

Portfolio generating functions allow us to construct broad classes of equity portfolios, some of which are interesting theoretically, and some of which are useful for actual investment purposes. Therefore, it has both theoretical and practical applications. For theoretical applications, the use of ranked market weights makes it possible to analyse the effect of company size on portfolio returns. For practical applications, one is frequently interested in a specific family of stocks determined by company size, e.g., the S&P 500 Index or the Russell 1000 Index, so generating functions based on ranked market weights have much wider applicability. As a practical application, we shall show that diversity-weighted large-stock indices may have favourable characteristics as long-term investments.

We shall present two applications: one application offers an explanation for the size effect, the observed tendency of the smaller stocks to have higher long-term returns than the larger stocks, and the other application provides a rigorous analysis of the behaviour of diversity-weighted index portfolios, equity indices with weights that lie between market weights and equal weights. Banz (1981) and Reinganum (1981) observed that in the U.S. equity market, over the long term, smaller stocks tend to have higher returns on average than larger stocks, even when an adjustment is made for the amount of risk. Over the long term, small stocks have a tendency to outperform large stocks. This phenomenon has been called the size effect, and there have been many attempts to justify it in a number of ways and a number of explanations have been proposed to account for it [see, e.g., Roll (1981)]. Conventionally, this phenomenon has been explained by the putatively greater risk of smaller stocks, but an alternative explanation was proposed in Fernholz (2001b). Here we follow this alternative explanation. Since smaller stocks are more difficult and more costly to trade than are larger stocks [see, for instance, Banz (1981) and Reinganum (1981a,b) for explanations in this regard], they must consequently offer higher returns than do larger stocks.

The first application is a theoretical explanation of the size effect. In a stable equity market, a small-stock index portfolio is likely to outperform a large-stock index portfolio, even if the two portfolios have about the same level of riskiness. This phenomenon, the size effect, can be characterised in terms of the semimartingale local times of the ranked market weights. We show that in the context of large, diversified index portfolios, that the smaller stocks do not seem to exhibit higher risk than the larger stocks, at least over the long term. In this case, the higher return on the smaller stocks cannot be attributed to risk, and can rather be attributed to some other factor.

The second application is to diversity-weighted large-stock index portfolios, a new type of passive equity strategy

that was introduced by Fernholz, Garvy, and Hannon (1998). The diversity-weighted index portfolio is currently being used for actual investment purposes [see Fernholz, Garvy, and Hannon (1998)]. The second application provides a rigorous mathematical handling of diversity-weighted large-stock index portfolios. Most equity indices are either capitalisation-weighted, as with the S&P 500 Index in the American equity market listed on the New York Stock Exchange (NYSE) and the South African All Share Index (ALSI) and the Top40 Index in the South African equity market listed on the Johannesburg Stock Exchange (JSE), or they are equally-weighted, as with the Value Line Index in the American equity market listed on the New York Stock Exchange (NYSE) and the South African BettaBeta Equally-Weighted Top40 Index in the South African equity market listed on the Johannesburg Stock Exchange (JSE). Diversity weighting introduces a class of portfolios with weights that lie between these two extremes, those of capitalisation weights and equal weights.

This chapter shall commence with Section 6.2, in which we shall identify stocks by their rankings in the equity market as opposed to identifying them by their name (i.e., by their subscript or index), which we have done up until now. This identification of relative rankings of stocks in the equity market is necessary for selecting stocks for certain rank-dependent portfolios or indices according to their rank in the market. This stock selection by rank is imperative in constructing equity portfolios that depend or rely on the rank of the stocks in the market. Thus, instead of looking at the stock price processes as is, we will now look at the ranked stock price processes, i.e., the stock price processes in their ranked ordering. It is in this section that we shall formally define and express the rank process, in terms of the original named processes, as well as in terms of the mathematically amenable maximum and minimum functions. Thus, at any given moment, the values of the rank processes represent the values of the original named processes arranged in decreasing order. In Section 6.3, we shall introduce the concept of local time. To this end, we shall define and discuss the following notions of local time: local time for Brownian motion, i.e. Brownian local time (due to Lévy (1948)); local time for any arbitrary, general semimartingale; and local time for continuous semimartingales, i.e. semimartingale local time in the continuous-time context, which is a generalisation of Brownian local time. Simply put, we can interpret local time for a particular process as a measure of the quantity of time that a particular process (be it Brownian motion or any arbitrary semimartingale) spends within a certain locale, i.e. within a local neighbourhood of some level (near or at some level). The definition of local time then yields the associated well-known and well-established Tanaka formulae for Brownian motion, and the Tanaka-Meyer formulae for more general semimartingales, which shall also be presented in this section. The Tanaka formulae are variants of Itô's formula for the positive and negative parts of a function and the absolute value function. These functions are also defined here. Furthermore, several results and fundamental properties involving local times are also provided. Section 6.4 considers the fundamental nondegeneracy conditions that are required in order to be able to apply local times in obtaining a decomposition of the ranked stock price processes. Here, we shall assume that the original stock price processes that we shall consider exhibit a certain level of nondegeneracy. Armed with this assumption and premise, we shall define the requisite nondegeneracy conditions in what is known as pathwise mutual nondegeneracy. One such condition ensures that a triple collision event does not exist, i.e. not more than two stock price processes can ever collide with one another at the same exact point in time. In addition to this condition, we must also consider a restricted class of continuous semimartingales so that we can utilise local times effectively. Fortunately, the continuous semimartingales in question, the stock price processes, possess certain properties that facilitate the application of local times. Thus, in Section 6.5, we shall consider one such fundamental property of continuous semimartingales that will effectively perform this task, namely, the absolute continuity property. Thus, the limited class of continuous semimartingales that we shall explore further are referred to as absolutely continuous semimartingales. Moreover, we shall show that certain functions of absolutely continuous semimartingales also exhibit absolute continuity, and are thus also absolutely continuous semimartingales. We end this section off with a brief look at local times for these absolutely continuous semimartingales. We shall provide representations for the dynamics of the rank processes, i.e. the ranked stock price processes, in Section 6.6, by deriving and offering the decomposition of the rank processes derived from either pathwise mutually nondegenerate absolutely continuous semimartingales, absolutely continuous semimartingales, continuous semimartingales, or from more general semimartingales. These representations of the dynamics of the ranked stock price processes demonstrate that the ranked stock price processes, which are derived from these semimartingales, can be expressed in terms of the original named stock price processes, adjusted by semimartingale local times. We start things off, however, with a representation

of the dynamics for the maximum and minimum processes since the architecture of the rank processes depends on the maximum and minimum operations. In Section 6.7, we shall introduce and define the rank market weight process. We shall also apply and extend the results of the previous section to the rank market weight processes, in order to be able to obtain representations for the dynamics of the ranked market weight processes, by offering the decomposition of the ranked market weight processes derived from either pathwise mutually nondegenerate absolutely continuous semimartingales, absolutely continuous semimartingales, continuous semimartingales, or from more general semimartingales. Again, these representations of the dynamics of the ranked market weight processes demonstrate that the ranked market weight processes, which are derived from these semimartingales, can be expressed in terms of the original named market weight processes, adjusted by semimartingale local times. The theorem presented in Section 6.8, which is an extended version of the main theorem in the previous chapter which held for generating functions of the named market weight processes, holds for generating functions of the ranked market weight processes, and contains the main result and crux of this entire chapter. Here, we shall employ the results of the preceding section to consider equity portfolios that are generated by functions of the ranked market weights, referred to as rank-based functionally generated portfolios. Thus, it is revealed that there exists a wide range of portfolio generating functions that are dependent on the ranked market weights. We shall also introduce and define the relative rank covariance process, i.e., the ranked covariance process relative to the equity market, in this section. We shall discuss rank-dependent portfolio generating functions, i.e. rank-dependent generating functions that generate rank-based functionally generated portfolios, in further detail in Section 6.9. It is here, that we provide the stochastic differential equation, associated with rank-based functionally generated portfolios, that decomposes the logarithmic relative return process of the rank-based functionally generated portfolio with respect to the reference benchmark market portfolio, specifically into two separate constituents. We shall find that the first component is given by the logarithmic change in the value of the rank-dependent portfolio generating function, to wit, the rank-dependent portfolio generating function component. The second component is determined to be the drift process corresponding to the rank-based functionally generated portfolio generated by the rank-dependent portfolio generating function. In turn, this drift process can further be divided up into its own two distinguishable parts: the first part being the so-called smooth component of the drift process, and the second part being the local time component of the drift process. The smooth component of the drift process specified here, is very much like the entire drift component encountered in the preceding chapter. The local time component of the drift process is the other component that incorporates local times to account for changes in rank that arise among the stock price processes. Section 6.10 offers a generalisation of the main theorem of this chapter for rank-based functionally generated portfolios. In the main theorem of this chapter, we assumed pathwise mutual nondegeneracy for the stock price processes which guaranteed that triple collision events do not exist, whereas here, we shall generalise this to the case that includes the existence of triple collision events, i.e. three or more stock price processes may collide with one another at the same exact point in time, at any given time. In Section 6.11, we shall offer another generalisation of the main theorem of this chapter. Thus, we shall present a general theorem for portfolio generating functions attributed to Pamen (2011). We shall do so for both the time-independent case and the time-dependent case. In the next section, Section 6.12, we shall provide a few examples of rank-dependent portfolio generating functions together with their affiliated rank-based functionally generated portfolios that they generate. Both the smooth component and the local time component of the drift process accompanying these rank-based functionally generated portfolios shall also be calculated. In addition, we shall explore and examine the performance of these portfolios relative to the market portfolio. In particular, we shall offer the following examples of rank-based functionally generated portfolios that are each generated by their respective rank-dependent portfolio generating functions: the biggest stock which holds only the largest stock in the equity market, the large-stock index portfolio which consists of the largest stocks in the equity market, the small-stock index portfolio which consists of the smallest stocks in the equity market, the diversity-weighted version of the large-stock index portfolio, and a portfolio with fixed weight ratios. Section 6.13 is devoted to the application of the aforementioned rank-based functionally generated portfolios of the previous section. We shall first apply the results obtained for the large-stock index portfolio and the small-stock index portfolio, by contrasting their performances relative to one another, to investigate what is coined the size effect phenomenon in equity markets. The size effect, ascribed to Banz (1981) and Reinganum (1981a,b), is typically described as the observed tendency of smaller stocks to have higher long-term returns on average than those of their larger counterparts. Secondly, we shall apply the

knowledge acquired for the diversity-weighted large-stock index portfolio, by contrasting its performance relative to the performance of the somewhat similar large-stock index portfolio, to introduce the concept of leakage in equity markets. Leakage refers to the scenario in which the smaller stocks contained within a large-stock index portfolio are dropped from this portfolio and thus move out of this portfolio and cross over to the rest of the market portfolio, since their relative standings are no longer large enough to remain within this large-stock index portfolio. In such a situation, these smaller stocks which are dropped from the large-stock index portfolio are said to "leak" out of the large-stock index portfolio, hence, the terminology leakage. Thus, leakage explains the effect that these crossovers have on the equity market. Another extension of portfolio generating functions can also be considered, other than what has already been considered so far in this chapter. Therefore, in Section 6.14, we shall explore an extension that considers the original Gini coefficient that was displayed in Chapter 4, and determine its corresponding generated Gini-coefficient-weighted portfolio in this case, along with the allied resultant drift process. Here, we shall find that the drift process of this Gini-coefficient-weighted portfolio is entirely made up of local times, and thus only exhibits a local time component, with no smooth component. Next, we jump on the local time bandwagon again, by noticing that local times can be estimated in practice from market observable quantities, with relative ease. Thus, the estimation of local time is put forth in Section 6.15. This chapter is closed off with a summary and conclusion presented in Section 6.16.

6.2 Stock Selection by Rank and Rank Processes

It is commonplace to identify stocks by their names, i.e., their subscripts or indices, as in X_1, X_2, X_3, etc. However, when concerned with the capital distribution of the market, it will be a necessary advantage to identify stocks by their ranks as in $X_{(1)}, X_{(2)}, X_{(3)}$, etc., rather than by their names. The stock price processes are then referred to as the ranked stock price processes. The intention is to derive a representation for these rank processes, as well as those corresponding to the market weights. In order to obtain this representation, we must utilise available mathematical tools that yield a precise definition of what we mean when we speak of the rank process. In this section, we shall formally define the rank process using these tools. Thus, we obtain the following definition.

Definition 6.2.1 (Rank Process). *Let $X_1, X_2, \ldots, X_n$ be processes (in particular, semimartingales). Then, for $k = 1, 2, \ldots, n$, the **kth rank process**, $X_{(k)} = \big\{X_{(k)}(t), t \in [0,T]\big\}$ of $\mathcal{M} = \big\{X_1(\cdot), \ldots, X_n(\cdot)\big\}$ is defined by*

$$X_{(k)}(t) \;\triangleq\; \max_{1 \le i_1 < \cdots < i_k \le n} \min\Big(X_{i_1}(t), \ldots, X_{i_k}(t)\Big), \qquad t \in [0,T], \tag{6.2.1}$$

where $i_1 \ge 1$ and $i_k \le n$. In addition, we shall adopt the convention that $X_{(0)}(\cdot)$ and $X_{(n+1)}(\cdot)$ are defined such that for $t \in [0,T]$, $X_{(0)}(t) > X_{(1)}(t)$ and $X_{(n+1)}(t) < X_{(n)}(t)$.

It is not difficult to see that $X_{(k)} \in \{X_1, X_2, \ldots, X_n\}$, and for $t \in [0,T]$, we have the reverse-order-statistics notation

$$\max_{1 \le i \le n} X_i(t) \;\triangleq\; X_{(1)}(t) \;\ge\; X_{(2)}(t) \;\ge\; \cdots \;\ge\; X_{(n-1)}(t) \;\ge\; X_{(n)}(t) \;\triangleq\; \min_{1 \le i \le n} X_i(t), \tag{6.2.2}$$

and for $k = 1, 2, \ldots, n$, we have

$$X_{(k-1)}(t) \;\ge\; X_{(k)}(t) \;\ge\; X_{(k+1)}(t).$$

Thus, at any given time, the values of the rank processes represent the values of the original processes arranged in descending order. The benefit derived from representing the rank process as (6.2.1) stems from the fact that it expresses the ranked processes in terms of mathematically tractable functions, i.e., in terms of the maximum and minimum functions.

Definition 6.2.2 (Rank Process and the Permutation Vector). *Let $X_1, X_2, \ldots, X_n$ be processes, and for $t \in [0,T]$, let $\boldsymbol{p}_t = \big(p_t(1), p_t(2), \ldots, p_t(n)\big)$ denote the **random permutation** of $\{1, 2, \ldots, n\}$, then for $k = 1, 2, \ldots, n$, the following hold*

$$X_{p_t(k)}(t) \;\triangleq\; X_{(k)}(t), \tag{6.2.3}$$

and,

$$\textit{if} \quad X_{(k)}(t) \quad = \quad X_{(k+1)}(t) \quad \textit{then} \quad p_t(k) \quad < \quad p_t(k+1). \tag{6.2.4}$$

The *random permutation vector* $\boldsymbol{p}_t = \big(p_t(1), p_t(2), \ldots, p_t(n)\big)$ represents the random permutation of the set of name elements $\{1, 2, \ldots, n\}$, which describes the relation between the indices of $X(t)$ and the ranks of $X_{(\cdot)}(t)$. Hence, $p_t(1)$ is the 1st element of the permutation vector $\boldsymbol{p}_t$, $p_t(2)$ is the 2nd element of the permutation vector, and so on. In general, $p_t(k)$ represents the kth element of the permutation vector. Moreover, $p_t(k)$ signifies the name (i.e., index or subscript) of the stock with the kth largest capitalisation at time $t \in [0, T]$. In other words, $p_t(k)$ is the name (i.e., index or subscript) of the stock that occupies the kth rank (the kth ranked stock) in terms of total capitalisation at time $t \in [0, T]$. The permutation vector associates each rank process with one of the original processes that has the same value at time t. From (6.2.4), we gather that in the event of a tie, we resort to the lowest index for the kth ranked stock. Thus, ties are resolved by assigning the lower index to the higher-ranked participant of the tie.

6.3 Local Time

One of the main aims of this chapter is to derive a representation for the rank processes, as well as those corresponding to the market weights. In order to achieve this, we must introduce the concept of a semimartingale local time. The notion of local time for Brownian motion was first introduced by Lévy (1948). In fact, the concept and several constructions of local time in the case of Brownian motion are due to Lévy (1948). His idea was to think of local times as a "measure", i.e., a measure of the amount of time a process spends within the neighbourhood of some level (near or at some level). The continuity results for Brownian local times are due to Trotter (1958). In the first part of this chapter, we present the definition of local time and the associated Tanaka formulae [Tanaka (1963)], first for Brownian motion, then the Tanaka-Meyer formulae [Tanaka (1963) and Meyer (1976)] for more general continuous semimartingales. An excellent introduction to and an extensive discussion of local time can be found in Karatzas & Shreve (1991, §3.6–3.7, pp. 201–226) and Protter (2004, Chap. IV, §7, pp. 210–227). The books by Itô & McKean (1996) and Revuz & Yor (1999, Chap. VI, §1–2, pp. 221–243), contain an abundance of further information on the subject.

6.3.1 Local Time for Brownian Motion

Consider the following set $\big\{0 \le s \le t \ : \ \big|W(s) - a\big| \le \varepsilon\big\} = \big\{0 \le s \le t \ : \ W(s) \in (a - \varepsilon, a + \varepsilon)\big\}$, which represents the set of times in $[0, t]$ during which the Brownian motion process, $W = \{W(t), \mathcal{F}_t, t \in [0, \infty)\}$, spends in the vicinity of a point $a \in \mathbb{R}$. Then, the *local time* for the Brownian motion process, $W(t)$, at the point $a \in \mathbb{R}$ (or, in the vicinity of the point $a \in \mathbb{R}$, i.e., in $(a - \varepsilon, a + \varepsilon)$) up to and including time t, is defined as follows for all $t \in [0, \infty)$ and for all $a \in \mathbb{R}$,

$$L_W(a, t) \quad \equiv \quad L(a, t) \quad = \quad L^a(t) \quad \triangleq \quad \lim_{\varepsilon \downarrow 0} \frac{1}{4\varepsilon} \operatorname{Leb}\Big\{0 \le s \le t \ : \ \big|W(s) - a\big| \le \varepsilon\Big\} \tag{6.3.1}$$

$$= \quad \lim_{\varepsilon \downarrow 0} \frac{1}{4\varepsilon} \operatorname{Leb}\Big\{0 \le s \le t \ : \ W(s) \in (a - \varepsilon, a + \varepsilon)\Big\}, \tag{6.3.2}$$

where $\operatorname{Leb}(\cdot)$ denotes the Lebesgue measure. Hence, we can interpret the local time for the Brownian motion process, $L(a, t)$, as a measure of the amount of time the Brownian motion process spends at or near the level a, up to and including time t. Observe that for $t \in [0, \infty)$, we have a.s.,[1]

$$\int_0^t \mathbb{1}_{(a-\varepsilon, a+\varepsilon)}\big(W(s)\big)\, d\langle W \rangle_s \quad = \quad \int_0^t \mathbb{1}_{(a-\varepsilon, a+\varepsilon)}\big(W(s)\big)\, ds \quad = \quad \operatorname{Leb}\Big\{0 \le s \le t \ : \ W(s) \in (a - \varepsilon, a + \varepsilon)\Big\}. \tag{6.3.3}$$

[1]Here $\mathbb{1}_A$ denotes the indicator function of the set A, i.e.

$$\mathbb{1}_A(x) \quad = \quad \mathbb{1}_{\{x \in A\}} \quad \triangleq \quad \begin{cases} 1 & \text{if} \quad x \in A, \\ 0 & \text{if} \quad x \notin A. \end{cases}$$

Therefore, for all $t \in [0, \infty)$, we have a.s.,

$$L(a,t) \;=\; L^a(t) \;=\; \lim_{\varepsilon \downarrow 0} \frac{1}{4\varepsilon} \int_0^t \mathbb{1}_{\{|W(s)-a| \le \varepsilon\}}\, ds \;=\; \lim_{\varepsilon \downarrow 0} \frac{1}{4\varepsilon} \int_0^t \mathbb{1}_{(a-\varepsilon, a+\varepsilon)}\big(W(s)\big)\, ds, \tag{6.3.4}$$

alternatively, a.s. we have the *"right Brownian local time"*,

$$L(a,t) \;=\; L^a(t) \;=\; \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{\{a \le W(s) < a+\varepsilon\}}\, ds \;=\; \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{[a, a+\varepsilon)}\big(W(s)\big)\, ds, \tag{6.3.5}$$

and the *"left Brownian local time"*,

$$L(a-,t) \;=\; L^{a-}(t) \;=\; \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{\{a-\varepsilon < W(s) \le a\}}\, ds \;=\; \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{(a-\varepsilon, a]}\big(W(s)\big)\, ds, \tag{6.3.6}$$

where

$$L(a-,t) \;=\; L^{a-}(t) \;\triangleq\; \lim_{b \uparrow a} L^b(t) \;=\; \lim_{b \to a-} L^b(t). \tag{6.3.7}$$

Note that there is no universal agreement in the literature regarding the normalisation of local time. Some authors choose a different normalisation of local time, i.e., they choose local time to be $2L^a(t)$, as in Protter (2004, Chap. IV, §7, pp. 210–227) and Revuz & Yor (1999, Chap. VI, §1–2). We follow the normalisation adopted by Karatzas & Shreve (1991, §3.6–3.7), who in turn follow the normalisation used by Ikeda & Watanabe (1989). We adopt this normalisation approach as it conforms to the normalisation employed by Fernholz (2002).

Definition 6.3.1 (Signum Function). *For $x \in \mathbb{R}$, the **signum function** is defined to be*

$$\operatorname{sgn}(x) \;\triangleq\; \begin{cases} 1 & \text{if} \quad x > 0, \\ -1 & \text{if} \quad x \le 0. \end{cases} \tag{6.3.8}$$

Notice that we define sgn *so as to make it left-continuous at* 0*, i.e.,* $\operatorname{sgn}(0) \equiv -1$*. Note further that our definition of* sgn *is not symmetric. The* sgn *function can also be expressed in the following ways*

$$\begin{aligned} \operatorname{sgn}(x) \;&\equiv\; -\mathbb{1}_{(-\infty,0]}(x) + \mathbb{1}_{(0,\infty)}(x) && (6.3.9)\\ &=\; -\mathbb{1}_{\{x \le 0\}} + \mathbb{1}_{\{x > 0\}} && (6.3.10)\\ &=\; 2\mathbb{1}_{(0,\infty)}(x) - 1 && (6.3.11)\\ &=\; 2\mathbb{1}_{\{x>0\}} - 1. && (6.3.12) \end{aligned}$$

*In addition, for $x \in \mathbb{R}$ and $a \in \mathbb{R}$, the **signum function** is defined to be*

$$\operatorname{sgn}(x-a) \;\triangleq\; \begin{cases} 1 & \text{if} \quad x > a, \\ -1 & \text{if} \quad x \le a. \end{cases} \tag{6.3.13}$$

Definition 6.3.2 (Positive Part of a Function). *For x, y real-valued variables, let $x \vee y \triangleq \max(x,y)$ and $x \wedge y \triangleq \min(x,y)$. For x a real-valued variable let x^+ be the function $x^+ \triangleq x \vee 0 = \max(x,0)$, which is referred to as the **positive part** of x and is also given by*

$$x^+ \;\triangleq\; \begin{cases} x & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x \le 0. \end{cases} \tag{6.3.14}$$

In addition, for x a real-valued variable and $a \in \mathbb{R}$, let $(x-a)^+$ be the function

$$(x-a)^+ \;\triangleq\; \begin{cases} x-a & \text{if} \quad x > a, \\ 0 & \text{if} \quad x \le a. \end{cases} \tag{6.3.15}$$

Definition 6.3.3 (Negative Part of a Function). *For x, y real-valued variables, let $x \vee y \triangleq \max(x, y)$ and $x \wedge y \triangleq \min(x, y)$. For x a real-valued variable let x^- be the function $x^- \triangleq -(x \wedge 0) = -\min(x, 0) = \max(-x, 0)$, which is referred to as the **negative part** of x and is also given by*

$$x^- \triangleq \begin{cases} 0 & \text{if } x > 0, \\ -x & \text{if } x \le 0. \end{cases} \tag{6.3.16}$$

In addition, for x a real-valued variable and $a \in \mathbb{R}$, let $(x-a)^-$ be the function

$$(x-a)^- \triangleq \begin{cases} 0 & \text{if } x > a, \\ a-x & \text{if } x \le a. \end{cases} \tag{6.3.17}$$

Definition 6.3.4 (Absolute Value Function). *For $x \in \mathbb{R}$, the **absolute value function** is defined to be*

$$|x| \triangleq \begin{cases} x & \text{if } x > 0, \\ -x & \text{if } x \le 0. \end{cases} \tag{6.3.18}$$

*In addition, for $x \in \mathbb{R}$ and $a \in \mathbb{R}$, the **absolute value function** is defined to be*

$$|x-a| \triangleq \begin{cases} x-a & \text{if } x > a, \\ a-x & \text{if } x \le a. \end{cases} \tag{6.3.19}$$

Proposition 6.3.5 (Tanaka Formulae [**Karatzas & Shreve (1991, §3.6, Proposition 3.6.8, p. 205)**]). *Let $W = \{W(t), \mathcal{F}_t, t \in [0,\infty)\}$ be a **one-dimensional Brownian motion** and let $L^a = \{L^a(t), \mathcal{F}_t, t \in [0,\infty)\}$ be its **local time** at the level a between 0 and t, where $a \in \mathbb{R}$. Then for all $t \in [0,\infty)$, we have the **Tanaka formulae** for Brownian motion,*

$$\big(W(t)-a\big)^+ = \big(W(0)-a\big)^+ + \int_0^t \mathbb{1}_{(a,\infty)}\big(W(s)\big)\, dW(s) + L^a(t), \tag{6.3.20}$$

$$\big(W(t)-a\big)^- = \big(W(0)-a\big)^- - \int_0^t \mathbb{1}_{(-\infty,a]}\big(W(s)\big)\, dW(s) + L^a(t), \tag{6.3.21}$$

$$\big|W(t)-a\big| = \big|W(0)-a\big| + \int_0^t \operatorname{sgn}\big(W(s)-a\big)\, dW(s) + 2L^a(t). \tag{6.3.22}$$

Proof. See Karatzas & Shreve (1991, §3.6, Proof of Proposition 3.6.8, p. 206). ■

Any of the formulae above is referred to as the *Tanaka formula* for Brownian motion. The Tanaka formulae are variants of Itô's formula for the absolute value and the positive and negative parts of Brownian motion. The process $\big\{\big(W(t)-a\big)^+, \mathcal{F}_t, t \in [0,\infty)\big\}$ is a continuous, nonnegative submartingale; it admits, therefore, a unique *Doob-Meyer decomposition*,

$$\big(W(t)-a\big)^+ = \big(W(0)-a\big)^+ + M_a^+(t) + V_a^+(t), \qquad t \in [0,\infty), \tag{6.3.23}$$

where $V_a^+ = \big\{V_a^+(t), \mathcal{F}_t, t \in [0,\infty)\big\}$ is a continuous, increasing process and $M_a^+ = \big\{M_a^+(t), \mathcal{F}_t, t \in [0,\infty)\big\}$ is a martingale. The Tanaka formula (6.3.20) identifies both parts of this decomposition as $V_a^+(t) \equiv L^a(t)$ and

$$M_a^+(t) = \int_0^t \mathbb{1}_{(a,\infty)}\big(W(s)\big)\, dW(s), \qquad t \in [0,\infty).$$

Similar remarks apply to representations (6.3.21) and (6.3.22) [Karatzas & Shreve (1991, §3.6)]. Thus, for a one-dimensional Brownian motion W and $a = 0$, the *Tanaka formula* in (6.3.22) becomes

$$\big|W(t)\big| = \big|W(0)\big| + \int_0^t \operatorname{sgn}\big(W(s)\big)\, dW(s) + 2L^0(t) \tag{6.3.24}$$

$$= \big|W(0)\big| + \int_0^t \operatorname{sgn}\big(W(s)\big)\, dW(s) + 2L(t), \tag{6.3.25}$$

where $L(t) = L^0(t)$ denotes the local time for the Brownian motion W at the origin.

Definition 6.3.6 (Local Time for Brownian Motion). *Let $W = \{W(t), \mathcal{F}_t, t \in [0,\infty)\}$ be a* ***one-dimensional Brownian motion****, and let $a \in \mathbb{R}$. The* ***local time*** *for the Brownian motion W at a, denoted $L^a \equiv L(a) = \{L^a(t) \equiv L(a,t), \mathcal{F}_t, t \in [0,\infty)\}$, is defined to be the process given by*

$$L^a(t) \quad = \quad \frac{1}{2}\left(\big|W(t)-a\big| - \big|W(0)-a\big| - \int_0^t \operatorname{sgn}\big(W(s)-a\big)\, dW(s)\right). \tag{6.3.26}$$

Equivalently, the ***local time*** *for the Brownian motion W at the level 0 (i.e., at the origin), denoted $L^0 \equiv L(0) = \{L^0(t) \equiv L(0,t), \mathcal{F}_t, t \in [0,\infty)\}$, is defined to be the process given by*

$$L^0(t) \quad = \quad L(t) \quad = \quad \frac{1}{2}\left(\big|W(t)\big| - \big|W(0)\big| - \int_0^t \operatorname{sgn}\big(W(s)\big)\, dW(s)\right). \tag{6.3.27}$$

The next theorem demonstrates that it is possible to obtain a generalised Itô formula for convex functions which are not necessarily twice differentiable. This possibility was explored by Meyer (1976) and Wang (1977). Recall that Itô's formula requires the existence of a second derivative. For a convex function f, we use instead of its second derivative the *second derivative measure* μ (the second-order derivative measure in the sense of distributions or the second-order derivative in the sense of generalised functions) on $\big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big)$, defined by

$$\mu\big([a,b)\big) \quad \triangleq \quad D^- f(b) - D^- f(a) \quad = \quad f'_-(b) - f'_-(a), \qquad -\infty < a < b < \infty, \tag{6.3.28}$$

where $D^- f \equiv f'_-$ is the left-hand derivative[2] of the function f. This means that [see Chesney, Jeanblanc & Yor (2009) and Rogers & Williams (2000b)],

$$\int_{-\infty}^{\infty} g(a)\, \mu(da) \quad = \quad \int_{-\infty}^{\infty} g''(a) f(a)\, da. \tag{6.3.29}$$

Of course, if f'', the second derivative of the convex function f, exists, then $\mu(da) = f''(a)\, da$. Even without the existence of f'', we may compute the Riemann-Stieltjes integral by parts, to obtain the formula

$$\int_{-\infty}^{\infty} g(a)\, \mu(da) \quad = \quad -\int_{-\infty}^{\infty} g'(a) D^- f(a)\, da, \tag{6.3.30}$$

for every function $g : \mathbb{R} \to \mathbb{R}$ which is piecewise continuous and has compact support.

Theorem 6.3.7 (A Generalised Itô Formula for Convex Functions: Meyer-Itô Formula **[Karatzas & Shreve (1991, §3.6, Theorem 3.6.22, p. 214)]**). *Let $W = \{W(t), \mathcal{F}_t, t \in [0,\infty)\}$ be a* ***one-dimensional Brownian motion****. Then there exists a Brownian local time for W at the level a, $L^a = \{L^a(t), \mathcal{F}_t, t \in [0,\infty)\}$, such that the following hold:*

(*i*) (**[Karatzas & Shreve (1991, §3.6, Statement (ii) of Problem 3.6.13, p. 209)]**). *For every $a \in \mathbb{R}$, the mapping $t \mapsto L^a(t,\omega)$ is continuous and nondecreasing with $L^a(0,\omega) = 0$ where $\omega \in \Omega$, and*

$$\int_0^{\infty} \mathbb{1}_{\mathbb{R}\setminus\{a\}}\big(W(t)\big)\, dL^a(t) \quad = \quad 0, \qquad a.s. \tag{6.3.31}$$

[2] In accordance with the notation adopted in Karatzas & Shreve (1991, §3.6, p. 213) and Revuz & Yor (1999, Chap. VI, §1, p. 221), the *left-hand derivative* for the function $f : \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$, is defined as

$$D^- f(x) \quad \equiv \quad f'_-(x) \quad \triangleq \quad \lim_{h \to 0-} \frac{f(x+h) - f(x)}{h} \quad = \quad \lim_{h \uparrow 0} \frac{f(x+h) - f(x)}{h},$$

where $\lim_{h\to 0-} = \lim_{h\uparrow 0}$ denotes the *left-hand one-sided limit*, where h approaches 0 from the left, or from below. $D^- f$ is left-continuous and nondecreasing on $\mathbb{R}$. The *right-hand derivative* for the function $f : \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$, is defined as

$$D^+ f(x) \quad \equiv \quad f'_+(x) \quad \triangleq \quad \lim_{h \to 0+} \frac{f(x+h) - f(x)}{h} \quad = \quad \lim_{h \downarrow 0} \frac{f(x+h) - f(x)}{h},$$

where $\lim_{h\to 0+} = \lim_{h\downarrow 0}$ denotes the *right-hand one-sided limit*, where h approaches 0 from the right, or from above. $D^+ f$ is right-continuous and nondecreasing on $\mathbb{R}$.

(*ii*) (Occupation Time Density Formula [**Karatzas & Shreve (1991, §3.6, Statement (i) of Problem 3.6.7, p. 204)**]). *For every positive Borel-measurable function $k : \mathbb{R} \to [0, \infty)$, the identity (known as the* ***occupation time density formula for Brownian motion****) holds*

$$\int_0^t k\big(W(s)\big)\, ds \;=\; 2\int_{-\infty}^{\infty} k(a) L^a(t)\, da, \qquad t \in [0,\infty), \qquad a.s. \tag{6.3.32}$$

(*iii*) *For every convex function (or a linear combination of convex functions) $f : \mathbb{R} \to \mathbb{R}$, we get the* ***generalised change of variable formula****, also known as the* ***Meyer-Itô formula****,*

$$f\big(W(t)\big) \;=\; f\big(W(0)\big) + \int_0^t D^- f\big(W(s)\big)\, dW(s) + \int_{-\infty}^{\infty} L^a(t)\, \mu(da), \qquad t \in [0,\infty), \qquad a.s., \tag{6.3.33}$$

where $D^- f$ denotes the left-hand derivative of f and μ denotes the second derivative measure (i.e., the second distributional derivative).

Proof. See Karatzas & Shreve (1991, §3.6, Proof of Theorem 3.6.22, pp. 213–214). ■

Let $a_1 < a_2 < \cdots < a_n$ be real numbers and suppose that $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable on $\mathbb{R} \setminus \{a_1, \ldots, a_n\}$, i.e., f is continuous and f' and f'' exist and are continuous on $\mathbb{R} \setminus \{a_1, \ldots, a_n\}$, and the limits

$$f'(a_k\pm) \;\triangleq\; \lim_{x \to a_k\pm} f'(x), \tag{6.3.34}$$

and

$$f''(a_k\pm) \;\triangleq\; \lim_{x \to a_k\pm} f''(x), \tag{6.3.35}$$

exist and are finite. If f is the difference of two convex functions, then for all $t \in [0,\infty)$, we have

$$f\big(W(t)\big) \;=\; f\big(W(0)\big) + \int_0^t f'\big(W(s)\big)\, dW(s) + \frac{1}{2}\int_0^t f''\big(W(s)\big)\, ds + \sum_{k=1}^{n} L^{a_k}(t)\Big[f'(a_k+) - f'(a_k-)\Big]. \tag{6.3.36}$$

See Karatzas & Shreve (1991, §3.6, Problem 3.6.24, p. 215) and Chesney, Jeanblanc & Yor (2009).

The next result is a fundamental property of local times that embodies a remarkable change of space and time integration (i.e., converts an integral over time into an integral over space), and exhibits doubled local time as the true *density with respect to Lebesgue measure for occupation time*.

Theorem 6.3.8 ([**Karatzas & Shreve (1991, p. 203)**]). *Let $W = \{W(t), \mathcal{F}_t, t \in [0,\infty)\}$ be a* ***one-dimensional Brownian motion*** *with local time $L^a = \{L^a(t), \mathcal{F}_t, t \in [0,\infty)\}$, with $a \in \mathbb{R}$. Then, for every fixed Borel set $B \in \mathcal{B}(\mathbb{R})$, we have for all $t \in [0,\infty)$, a.s.*

$$\int_0^t \mathbb{1}_B\big(W(s)\big)\, ds \;=\; 2\int_B L^a(t)\, da. \tag{6.3.37}$$

Proof. The proof follows easily from (6.3.32) of Theorem 6.3.7, where we let $k(x) = \mathbb{1}_B(x)$,

$$\begin{aligned}\int_0^t \mathbb{1}_B\big(W(s)\big)\, ds \;&=\; 2\int_{-\infty}^{\infty} \mathbb{1}_B(a) L^a(t)\, da \\ &=\; 2\int_B L^a(t)\, da.\end{aligned}$$

■

The time-integral in (6.3.37) of Theorem 6.3.8, is referred to as the *occupation time* of B by the Brownian motion path up to an including time $t \in [0,\infty)$, since

$$\int_0^t \mathbb{1}_B\big(W(s)\big)\, ds \;=\; \mathrm{Leb}\Big\{0 \le s \le t \;:\; W(s) \in B\Big\}. \tag{6.3.38}$$

It is a fact in the theory of generalised functions that for $a \in \mathbb{R}$, we have

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \mathbb{1}_{(a-\varepsilon, a+\varepsilon)}\big(W(t)\big) \quad = \quad \delta_a\big(W(t)\big) \quad = \quad \delta\big(W(t) - a\big), \tag{6.3.39}$$

where $\delta_a(x)$ is the *Dirac delta function* at a and is equivalent to $\delta(x-a)$ which is the Dirac delta function at 0 evaluated at $x - a$. The Dirac delta function at a is a generalised function or distribution defined in a heuristic fashion by

$$\delta_a(x) \quad \triangleq \quad \begin{cases} \infty & \text{if} \quad x = a, \\ 0 & \text{if} \quad x \neq a, \end{cases} \tag{6.3.40}$$

similarly, we have the Dirac delta function at 0, $\delta_0(x) \equiv \delta(x)$,

$$\delta(x) \quad \triangleq \quad \begin{cases} \infty & \text{if} \quad x = 0, \\ 0 & \text{if} \quad x \neq 0. \end{cases} \tag{6.3.41}$$

Hence, by appealing to (6.3.4), the local time for Brownian motion at a can be expressed as

$$L^a(t) \quad = \quad \frac{1}{2} \int_0^t \delta_a\big(W(s)\big)\, ds \quad = \quad \frac{1}{2} \int_0^t \delta\big(W(s) - a\big)\, ds. \tag{6.3.42}$$

Alternatively, we could consider the nondecreasing, convex function $g_a(x) = (x-a)^+$, which is continuously differentiable on $\mathbb{R} \setminus \{a\}$. The left-hand derivative of g_a is $D^- g_a(x) = \mathbb{1}_{\{x > a\}} = \mathbb{1}_{(a,\infty)}(x)$, so that the second derivative of g_a in the distributional sense is $g_a''(x) = \delta_a(x) = \delta(x-a)$. Thus, by letting $k(x) = g_a''(x)$, (6.3.32) of Theorem 6.3.7 gives the intuitive interpretation of local time as (6.3.42). Then, by (6.3.20) of Proposition 6.3.5, we can write

$$\big(W(t) - a\big)^+ \quad = \quad \big(W(0) - a\big)^+ + \int_0^t \mathbb{1}_{(a,\infty)}\big(W(s)\big)\, dW(s) + \frac{1}{2} \int_0^t \delta\big(W(s) - a\big)\, ds. \tag{6.3.43}$$

The above result also follows from (6.3.33) of Theorem 6.3.7 with $f(x) = g_a(x) = (x-a)^+$, where $D^- g_a(x) = \mathbb{1}_{(a,\infty)}(x)$ and $\mu(da) = g_a''(a)\, da$ and by setting $k(x) := g_a''(x) = \delta(x-a)$ in (6.3.32), we have

$$\int_{-\infty}^{\infty} L^a(t)\, \mu(da) \quad = \quad \int_{-\infty}^{\infty} L^a(t)\, g_a''(a)\, da \quad = \quad \frac{1}{2} \int_0^t g_a''\big(W(s)\big)\, ds \quad = \quad \frac{1}{2} \int_0^t \delta(W(s) - a)\, ds,$$

which when substituted into (6.3.33) gives (6.3.43). Similarly, we could consider the nonincreasing, convex function $h_a(x) = (x-a)^-$, which is continuously differentiable on $\mathbb{R} \setminus \{a\}$. The left-hand derivative of h_a is $D^- h_a(x) = -\mathbb{1}_{\{x \leq a\}} = -\mathbb{1}_{(-\infty,a]}(x)$, so that the second derivative of h_a in the distributional sense is $h_a''(x) = \delta_a(x) = \delta(x-a)$. Then, by (6.3.21) of Proposition 6.3.5, we can write

$$\big(W(t) - a\big)^- \quad = \quad \big(W(0) - a\big)^- - \int_0^t \mathbb{1}_{(-\infty,a]}\big(W(s)\big)\, dW(s) + \frac{1}{2} \int_0^t \delta\big(W(s) - a\big)\, ds. \tag{6.3.44}$$

The above result also follows from (6.3.33) of Theorem 6.3.7 with $f(x) = h_a(x) = (x-a)^-$, where $D^- h_a(x) = -\mathbb{1}_{(-\infty,a]}(x)$ and $\mu(da) = h_a''(a)\, da$ and by setting $k(x) := h_a''(x) = \delta(x-a)$ in (6.3.32). Moreover, we could consider the convex function $f_a(x) = |x-a|$, which is continuously differentiable on $\mathbb{R} \setminus \{a\}$. The left-hand derivative of f_a is $D^- f_a(x) = \operatorname{sgn}(x-a)$, so that the second derivative of f_a in the distributional sense is $f_a''(x) = 2\delta(x-a)$. So that, once again, (6.3.32) of Theorem 6.3.7 gives the intuitive interpretation of local time as (6.3.42). Then, by (6.3.22) of Proposition 6.3.5, we can write

$$\big|W(t) - a\big| \quad = \quad \big|W(0) - a\big| + \int_0^t \operatorname{sgn}\big(W(s) - a\big)\, dW(s) + \int_0^t \delta\big(W(s) - a\big)\, ds. \tag{6.3.45}$$

The above result also follows from (6.3.33) of Theorem 6.3.7 with $f(x) = f_a(x) = |x-a|$, where $D^- f_a(x) = \operatorname{sgn}(x-a)$ and $\mu(da) = f_a''(a)\, da$ and by setting $k(x) := f_a''(x) = 2\delta(x-a)$ in (6.3.32).

6.3.2 Local Time for General Semimartingales

Local time can also be defined in greater generality for arbitrary semimartingales. Recall that Itô's formula shows that if a function $f : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable and $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ is a semimartingale then $f(X)$ is again a semimartingale. In other words, semimartingales are preserved under C^2 transformations. This property extends slightly to semimartingales that are preserved under *convex* transformations, as the following theorem, attributed to Protter (2004, Chap. IV, §7, p. 210), shows.

Theorem 6.3.9 ([**Protter (2004, Chap. IV, §7, Theorem 66, p. 210)**]). *Let $f : \mathbb{R} \to \mathbb{R}$ be convex and let $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ be a* ***semimartingale****. Then $f(X)$ is a* ***semimartingale*** *and one has*[3]

$$f\big(X(t)\big) = f\big(X(0)\big) + \int_{0+}^{t} f'_-\big(X(s-)\big)\, dX(s) + A^f(t) \tag{6.3.46}$$

$$= f\big(X(0)\big) + \int_{0+}^{t} D^- f\big(X(s-)\big)\, dX(s) + A^f(t), \qquad t \in [0,\infty), \tag{6.3.47}$$

where $f'_- = D^- f$ is the left-hand derivative of f and A^f is an adapted, right-continuous, increasing process. Moreover, $\Delta A^f(t) = f\big(X(t)\big) - f\big(X(t-)\big) - f'_-\big(X(t-)\big)\,\Delta X(t)$, where $\Delta X(t) \triangleq X(t) - X(t-)$,[4] *(this represents the jump of X in t)*[5] *and the semimartingale X satisfies the following condition*

$$\sum_{0<s\le t} \big|\Delta X(s)\big| < \infty, \qquad \text{for all } t \in [0,\infty), \qquad a.s. \tag{6.3.48}$$

Proof. See Protter (2004, Chap. IV, §7, Proof of Theorem 66, pp. 210–211). ■

Corollary 6.3.10 ([**Protter (2004, Chap. IV, §7, Corollary 1 of Theorem 66, p. 211)**]). *Let $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ be a* ***semimartingale****. Then $|X| = \big\{\big|X(t)\big|, \mathcal{F}_t, t \in [0,\infty)\big\}$, $X^+ = \big\{X^+(t), \mathcal{F}_t, t \in [0,\infty)\big\}$ and $X^- = \big\{X^-(t), \mathcal{F}_t, t \in [0,\infty)\big\}$, are all* ***semimartingales****.*

[3] We usually write

$$\int_0^t H(s)\, dX(s) = \int_{[0,t]} H(s)\, dX(s).$$

To exclude 0 in the integral, we write

$$\int_{0+}^t H(s)\, dX(s) = \int_{(0,t]} H(s)\, dX(s).$$

The notation $\int_{0+}^t = \int_{(0,t]}$ denotes the integral over the half open interval $(0,t]$. Note that

$$\int_0^t H(s)\, dX(s) = H(0)X(0) + \int_{0+}^t H(s)\, dX(s).$$

[4] The *left-hand one-sided limit*, $X_- = \{(X_-)_t = X(t-), \mathcal{F}_t, t \in [0,\infty)\}$, as s decreases in value approaching t (s approaches t from the left or from below), is defined as

$$(X_-)_t = X(t-) \triangleq \lim_{s\to t-} X(s) = \lim_{s\uparrow t} X(s),$$

or for a function $f : \mathbb{R} \to \mathbb{R}$,

$$f\big((X_-)_t\big) = f\big(X(t-)\big) \triangleq \lim_{s\to t-} f\big(X(s)\big) = \lim_{s\uparrow t} f\big(X(s)\big).$$

Similarly, the *right-hand one-sided limit*, $X_+ = \{(X_+)_t = X(t+), \mathcal{F}_t, t \in [0,\infty)\}$, as s increases in value approaching t (s approaches t from the right or from above), is defined as

$$(X_+)_t = X(t+) \triangleq \lim_{s\to t+} X(s) = \lim_{s\downarrow t} X(s),$$

or for a function $f : \mathbb{R} \to \mathbb{R}$,

$$f\big((X_+)_t\big) = f\big(X(t+)\big) \triangleq \lim_{s\to t+} f\big(X(s)\big) = \lim_{s\downarrow t} f\big(X(s)\big).$$

[5] Since $(X_-)_0 = X(0-) = X(0) = 0$, we have $\Delta X(0) = X(0) = 0$, by convention.

Proof. The functions $f(x) = |x|$, $g(x) = x^+$ and $h(x) = x^-$ are all convex, so the result follows by Theorem 6.3.9. ■

Corollary 6.3.11 ([**Protter (2004, Chap. IV, §7, Corollary 2 of Theorem 66, p. 212)**]). *Let* $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ *and* $Y = \{Y(t), \mathcal{F}_t, t \in [0,\infty)\}$ *be* ***semimartingales****. Then* $X \vee Y = \{(X \vee Y)(t), \mathcal{F}_t, t \in [0,\infty)\}$ *and* $X \wedge Y = \{(X \wedge Y)(t), \mathcal{F}_t, t \in [0,\infty)\}$ *are* ***semimartingales****.*

Proof. Since semimartingales form a vector space and noticing that $x \vee y = \frac{1}{2}\big(|x-y| + x + y\big)$ and $x \wedge y = \frac{1}{2}\big(x + y - |x-y|\big)$, the result is an immediate consequence of Corollary 6.3.10. ■

Now let us define the following functions,

$$f_0(x) \quad = \quad |x|, \tag{6.3.49}$$

and

$$f_a(x) \quad = \quad |x-a|. \tag{6.3.50}$$

Then the signum function $\operatorname{sgn}(x)$ is the left derivative of the absolute value function $f_0(x)$, i.e.,

$$f'_{0,-}\big(X(s-)\big) \quad \equiv \quad D^- f_0\big(X(s-)\big) \quad = \quad \operatorname{sgn}\big(X(s-)\big), \tag{6.3.51}$$

and $\operatorname{sgn}(x-a)$ is the left derivative of $f_a(x)$, i.e.,

$$f'_{a,-}\big(X(s-)\big) \quad \equiv \quad D^- f_a\big(X(s-)\big) \quad = \quad \operatorname{sgn}\big(X(s-) - a\big). \tag{6.3.52}$$

Since $f_a(x)$ is convex, by Theorem 6.3.9 we have for a semimartingale X and process $A^{f_a} = \big\{A^{f_a}(t), \mathcal{F}_t, t \in [0,\infty)\big\}$,

$$f_a\big(X(t)\big) \quad \equiv \quad \big|X(t) - a\big| \quad = \quad \big|X(0) - a\big| + \int_{0+}^{t} \operatorname{sgn}\big(X(s-) - a\big)\, dX(s) + A^{f_a}(t) \tag{6.3.53}$$

$$= \quad \big|X(0) - a\big| + \int_{0+}^{t} \operatorname{sgn}\big(X(s-) - a\big)\, dX(s) + A_a(t), \tag{6.3.54}$$

where $A^{f_a} \equiv A_a = \big\{A_a(t), \mathcal{F}_t, t \in [0,\infty)\big\}$ is the increasing process of Theorem 6.3.9. Employing (6.3.50) and (6.3.53) as defined above we can define the concept of a *local time* for an *arbitrary, general semimartingale*, in part due to Protter (2004, Chap. IV, §7, pp. 210–227). We provide a modified version of the definition of local time provided by Protter (2004, Chap. IV, §7, pp. 210–227) to maintain consistency with the notation employed throughout this chapter.

Definition 6.3.12 (Local Time for a General Semimartingale). *Let* $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ *be a* ***semimartingale****, and let* f_a *and* $A^{f_a} = A_a$ *be defined as in* (6.3.50) *and* (6.3.53) *above, where* $a \in \mathbb{R}$*. The* ***local time of the semimartingale*** X *at* a*, denoted* $\Lambda_X^a \equiv \Lambda_X(a) = \big\{\Lambda_X^a(t) \equiv \Lambda_X(a,t), \mathcal{F}_t, t \in [0,\infty)\big\}$*, is defined to be the process given by*

$$\Lambda_X^a(t) \quad \equiv \quad \Lambda_X(a,t) \quad \triangleq \quad \frac{1}{2}\left(A^{f_a}(t) - \sum_{s \le t}\Big[f_a\big(X(s)\big) - f_a\big(X(s-)\big) - D^- f_a\big(X(s-)\big)\,\Delta X(s)\Big]\right) \tag{6.3.55}$$

$$= \quad \frac{1}{2}\left(A_a(t) - \sum_{s \le t}\Big[\big|X(s) - a\big| - \big|X(s-) - a\big| - \operatorname{sgn}\big(X(s-) - a\big)\,\Delta X(s)\Big]\right). \tag{6.3.56}$$

Therefore, we have

$$A_a(t) \quad = \quad 2\Lambda_X^a(t) + \sum_{s \le t}\Big[\big|X(s) - a\big| - \big|X(s-) - a\big| - \operatorname{sgn}\big(X(s-) - a\big)\,\Delta X(s)\Big]. \tag{6.3.57}$$

Thus, the definition of local time for general semimartingales provided in Definition 6.3.12 above, combined with (6.3.53) yields the well-known *Tanaka-Meyer formula*,

$$\begin{aligned} \big|X(t)-a\big| &= \big|X(0)-a\big| + \int_{0+}^{t} \operatorname{sgn}\big(X(s-)-a\big)\, dX(s) + 2\Lambda_X^a(t) \\ &\quad + \sum_{s\le t}\Big[\big|X(s)-a\big| - \big|X(s-)-a\big| - \operatorname{sgn}\big(X(s-)-a\big)\,\Delta X(s)\Big]. \end{aligned} \tag{6.3.58}$$

Protter (2004, Chap. IV, §7, pp. 212–213) points out that the local time for general semimartingales, $\Lambda_X^a = \big\{\Lambda_X^a(t), \mathcal{F}_t, t\in[0,\infty)\big\}$, has a version which is measurable in (a,t,ω) and càdlàg and continuous in t. It is further mentioned that this measurable, càdlàg version of the local time is always chosen. We shall deal exclusively with this version. Further, it is mentioned in Protter (2004, Chap. IV, §7, p. 213) that the local time is the continuous part of the increasing process A^{f_a}.

Theorem 6.3.13 (Tanaka-Meyer Formulae **[Protter (2004, Chap. IV, §7, Theorem 68, p. 213)]**). *Let $X = \{X(t), \mathcal{F}_t, t\in[0,\infty)\}$ be a **semimartingale** and let $\Lambda_X^a = \big\{\Lambda_X^a(t), \mathcal{F}_t, t\in[0,\infty)\big\}$ be its local time at the level a. Then for all $t\in[0,\infty)$,*

$$\begin{aligned} \big(X(t)-a\big)^+ &= \big(X(0)-a\big)^+ + \int_{0+}^{t} \mathbb{1}_{\{X(s-)>a\}}\, dX(s) + \Lambda_X^a(t) \\ &\quad + \sum_{s\le t}\mathbb{1}_{\{X(s-)>a\}}\big(X(s)-a\big)^- + \sum_{s\le t}\mathbb{1}_{\{X(s-)\le a\}}\big(X(s)-a\big)^+, \end{aligned} \tag{6.3.59}$$

and

$$\begin{aligned} \big(X(t)-a\big)^- &= \big(X(0)-a\big)^- - \int_{0+}^{t} \mathbb{1}_{\{X(s-)\le a\}}\, dX(s) + \Lambda_X^a(t) \\ &\quad + \sum_{s\le t}\mathbb{1}_{\{X(s-)>a\}}\big(X(s)-a\big)^- + \sum_{s\le t}\mathbb{1}_{\{X(s-)\le a\}}\big(X(s)-a\big)^+. \end{aligned} \tag{6.3.60}$$

Proof. Applying Theorem 6.3.9 to the convex functions

$$g_a(x) = (x-a)^+ = \begin{cases} x-a & \text{if } x>a, \\ 0 & \text{if } x\le a, \end{cases} \tag{6.3.61}$$

and

$$h_a(x) = (x-a)^- = \begin{cases} 0 & \text{if } x>a, \\ a-x & \text{if } x\le a, \end{cases} \tag{6.3.62}$$

there exist continuous increasing processes $A^{g_a} = \big\{A^{g_a}(t), \mathcal{F}_t, t\in[0,\infty)\big\}$ and $A^{h_a} = \big\{A^{h_a}(t), \mathcal{F}_t, t\in[0,\infty)\big\}$, such that we get

$$g_a\big(X(t)\big) = g_a\big(X(0)\big) + \int_{0+}^{t} D^- g_a\big(X(s-)\big)\, dX(s) + A^{g_a}(t) \tag{6.3.63}$$

$$h_a\big(X(t)\big) = h_a\big(X(0)\big) + \int_{0+}^{t} D^- h_a\big(X(s-)\big)\, dX(s) + A^{h_a}(t), \tag{6.3.64}$$

or by setting $A^{g_a} := A_a^+ = \big\{A_a^+(t), \mathcal{F}_t, t\in 0,\infty)\big\}$ and $A^{h_a} \equiv A_a^- = \big\{A_a^-(t), \mathcal{F}_t, t\in 0,\infty)\big\}$, we have

$$g_a\big(X(t)\big) = g_a\big(X(0)\big) + \int_{0+}^{t} D^- g_a\big(X(s-)\big)\, dX(s) + A_a^+(t) \tag{6.3.65}$$

$$h_a\big(X(t)\big) = h_a\big(X(0)\big) + \int_{0+}^{t} D^- h_a\big(X(s-)\big)\, dX(s) + A_a^-(t), \tag{6.3.66}$$

alternatively,

$$\big(X(t)-a\big)^+ = \big(X(0)-a\big)^+ + \int_{0+}^{t} D^-\big(X(s-)-a\big)^+\, dX(s) + A_a^+(t) \tag{6.3.67}$$

$$\big(X(t)-a\big)^- = \big(X(0)-a\big)^- + \int_{0+}^{t} D^-\big(X(s-)-a\big)^-\, dX(s) + A_a^-(t). \tag{6.3.68}$$

It is clear from (6.3.61) that the indicator function $\mathbb{1}_{\{x>a\}}$ is the left derivative of $g_a(x)$, i.e.,

$$D^- g_a\big(X(s-)\big) \;=\; D^-\big(X(s-)-a\big)^+ \;=\; \mathbb{1}_{\{X(s-)>a\}}. \tag{6.3.69}$$

Similarly, from (6.3.62) we can see that the negative of the indicator function $\mathbb{1}_{\{x\le a\}}$ is the left derivative of $h_a(x)$, i.e.,

$$D^- h_a\big(X(s-)\big) \;=\; D^-\big(X(s-)-a\big)^- \;=\; -\mathbb{1}_{\{X(s-)\le a\}}. \tag{6.3.70}$$

So that, (6.3.67) and (6.3.68), respectively become

$$\big(X(t)-a\big)^+ \;=\; \big(X(0)-a\big)^+ + \int_{0+}^t \mathbb{1}_{\{X(s-)>a\}}\,dX(s) + A_a^+(t) \tag{6.3.71}$$

$$\big(X(t)-a\big)^- \;=\; \big(X(0)-a\big)^- - \int_{0+}^t \mathbb{1}_{\{X(s-)\le a\}}\,dX(s) + A_a^-(t). \tag{6.3.72}$$

Subtracting the formulas above, (6.3.71) and (6.3.72), we arrive at the following

$$\begin{aligned}\big(X(t)-a\big)^+ - \big(X(t)-a\big)^- &= \big(X(0)-a\big)^+ - \big(X(0)-a\big)^- + A_a^+(t) - A_a^-(t)\\ &\quad + \int_{0+}^t \mathbb{1}_{\{X(s-)>a\}}\,dX(s) + \int_{0+}^t \mathbb{1}_{\{X(s-)\le a\}}\,dX(s)\\ &= \big(X(0)-a\big)^+ - \big(X(0)-a\big)^- + A_a^+(t) - A_a^-(t)\\ &\quad + \int_{0+}^t \Big(\mathbb{1}_{\{X(s-)>a\}} + \mathbb{1}_{\{X(s-)\le a\}}\Big)\,dX(s).\end{aligned} \tag{6.3.73}$$

Now, since $x \equiv x^+ - x^-$ (similarly $x-a \equiv (x-a)^+ - (x-a)^-$), and $\mathbb{1}_{\{x>a\}} + \mathbb{1}_{\{x\le a\}} = 1$ for all $x \in \mathbb{R}$, we obtain

$$X(t)-a \;=\; X(0)-a + \int_{0+}^t dX(s) + A_a^+(t) - A_a^-(t),$$

which implies

$$X(t)-a \;=\; X(0)-a + X(t) - 2X(0) + A_a^+(t) - A_a^-(t).$$

Thus, we get $A_a^+(t) - A_a^-(t) = 0$ and hence $A_a^+(t) = A_a^-(t)$, since $X(0)=0$. Next, let

$$\begin{aligned}G_a^+(t) &= \frac{1}{2}\left(A_a^+(t) - \sum_{s\le t}\Big[g_a\big(X(s)\big) - g_a\big(X(s-)\big) - D^- g_a\big(X(s-)\big)\,\Delta X(s)\Big]\right)\\ &= \frac{1}{2}\left(A_a^+(t) - \sum_{s\le t}\Big[\big(X(s)-a\big)^+ - \big(X(s-)-a\big)^+ - D^-\big(X(s-)-a\big)^+\,\Delta X(s)\Big]\right),\end{aligned} \tag{6.3.74}$$

$$\begin{aligned}G_a^-(t) &= \frac{1}{2}\left(A_a^-(t) - \sum_{s\le t}\Big[h_a\big(X(s)\big) - h_a\big(X(s-)\big) - D^- h_a\big(X(s-)\big)\,\Delta X(s)\Big]\right)\\ &= \frac{1}{2}\left(A_a^-(t) - \sum_{s\le t}\Big[\big(X(s)-a\big)^- - \big(X(s-)-a\big)^- - D^-\big(X(s-)-a\big)^-\,\Delta X(s)\Big]\right).\end{aligned} \tag{6.3.75}$$

Then equations (6.3.69) and (6.3.70), imply

$$G_a^+(t) \;=\; \frac{1}{2}\left(A_a^+(t) - \sum_{s\le t}\Big[\big(X(s)-a\big)^+ - \big(X(s-)-a\big)^+ - \mathbb{1}_{\{X(s-)>a\}}\,\Delta X(s)\Big]\right), \tag{6.3.76}$$

$$G_a^-(t) \;=\; \frac{1}{2}\left(A_a^-(t) - \sum_{s\le t}\Big[\big(X(s)-a\big)^- - \big(X(s-)-a\big)^- + \mathbb{1}_{\{X(s-)\le a\}}\,\Delta X(s)\Big]\right). \tag{6.3.77}$$

So that, in turn, we obtain

$$A_a^+(t) = 2G_a^+(t) + \sum_{s\le t}\Big[\big(X(s)-a\big)^+ - \big(X(s-)-a\big)^+ - \mathbb{1}_{\{X(s-)>a\}}\,\Delta X(s)\Big], \tag{6.3.78}$$

$$A_a^-(t) = 2G_a^-(t) + \sum_{s\le t}\Big[\big(X(s)-a\big)^- - \big(X(s-)-a\big)^- + \mathbb{1}_{\{X(s-)\le a\}}\,\Delta X(s)\Big]. \tag{6.3.79}$$

Once again, subtracting the formulas above, (6.3.78) and (6.3.79), we deduce the following

$$\begin{aligned}
A_a^+(t) - A_a^-(t) &= 2G_a^+(t) - 2G_a^-(t) \\
&\quad + \sum_{s\le t}\Big[\big(X(s)-a\big)^+ - \big(X(s-)-a\big)^+ - \mathbb{1}_{\{X(s-)>a\}}\,\Delta X(s)\Big] \\
&\quad - \sum_{s\le t}\Big[\big(X(s)-a\big)^- - \big(X(s-)-a\big)^- + \mathbb{1}_{\{X(s-)\le a\}}\,\Delta X(s)\Big] \\
&= 2G_a^+(t) - 2G_a^-(t) \\
&\quad + \sum_{s\le t}\Big[\big(X(s)-a\big)^+ - \big(X(s)-a\big)^- - \big(X(s-)-a\big)^+ + \big(X(s-)-a\big)^- \\
&\qquad\qquad - \mathbb{1}_{\{X(s-)>a\}}\,\Delta X(s) - \mathbb{1}_{\{X(s-)\le a\}}\,\Delta X(s)\Big] \\
&= 2G_a^+(t) - 2G_a^-(t) \\
&\quad + \sum_{s\le t}\Big[\big(X(s)-a\big)^+ - \big(X(s)-a\big)^- - \Big(\big(X(s-)-a\big)^+ - \big(X(s-)-a\big)^-\Big) \\
&\qquad\qquad - \Big(\mathbb{1}_{\{X(s-)>a\}} + \mathbb{1}_{\{X(s-)\le a\}}\Big)\,\Delta X(s)\Big].
\end{aligned}$$

Then again, since $x-a \equiv (x-a)^+ - (x-a)^-$, and $\mathbb{1}_{\{x>a\}} + \mathbb{1}_{\{x\le a\}} = 1$ for all $x\in\mathbb{R}$, we get

$$A_a^+(t) - A_a^-(t) = 2G_a^+(t) - 2G_a^-(t) + \sum_{s\le t}\Big[X(s) - a - \big(X(s-)-a\big) - \Delta X(s)\Big],$$

which when combined with $\Delta X(s) \triangleq X(s) - X(s-)$, simplifies to

$$\begin{aligned}
A_a^+(t) - A_a^-(t) &= 2G_a^+(t) - 2G_a^-(t) + \sum_{s\le t}\Big[X(s) - a - X(s-) + a - X(s) + X(s-)\Big] \\
&= 2G_a^+(t) - 2G_a^-(t).
\end{aligned}$$

Therefore, the above result in conjunction with $A_a^+(t) - A_a^-(t) = 0$, yields the following

$$A_a^+(t) - A_a^-(t) = 2G_a^+(t) - 2G_a^-(t) = 0.$$

Hence $G_a^+(t) = G_a^-(t)$. Adding the formulas (6.3.71) and (6.3.72), yields

$$\begin{aligned}
\big(X(t)-a\big)^+ + \big(X(t)-a\big)^- &= \big(X(0)-a\big)^+ + \big(X(0)-a\big)^- + A_a^+(t) + A_a^-(t) \\
&\quad + \int_{0+}^t \mathbb{1}_{\{X(s-)>a\}}\,dX(s) - \int_{0+}^t \mathbb{1}_{\{X(s-)\le a\}}\,dX(s) \\
&= \big(X(0)-a\big)^+ + \big(X(0)-a\big)^- + A_a^+(t) + A_a^-(t) \\
&\quad + \int_{0+}^t \Big(\mathbb{1}_{\{X(s-)>a\}} - \mathbb{1}_{\{X(s-)\le a\}}\Big)\,dX(s).
\end{aligned}$$

Now, since $|x| \equiv x^+ + x^-$ (similarly, $|x-a| \equiv (x-a)^+ + (x-a)^-$), and from (6.3.10) we have $\operatorname{sgn}(x-a) \triangleq -\mathbb{1}_{\{x\le a\}} + \mathbb{1}_{\{x>a\}}$, we get

$$\big|X(t)-a\big| = \big|X(0)-a\big| + \int_{0+}^t \operatorname{sgn}\big(X(s-)-a\big)\,dX(s) + A_a^+(t) + A_a^-(t), \tag{6.3.80}$$

which upon comparison with (6.3.53) yields $A_a^+(t) + A_a^-(t) = A^{f_a}(t) = A_a(t)$. Further, we have

$$\begin{aligned}
G_a^+(t) + G_a^-(t) &= \frac{1}{2}\Bigg(A_a^+(t) + A_a^-(t) \\
&\quad - \sum_{s\le t}\Big[\big(X(s)-a\big)^+ - \big(X(s-)-a\big)^+ - \mathbb{1}_{\{X(s-)>a\}}\,\Delta X(s)\Big] \\
&\quad - \sum_{s\le t}\Big[\big(X(s)-a\big)^- - \big(X(s-)-a\big)^- + \mathbb{1}_{\{X(s-)\le a\}}\,\Delta X(s)\Big]\Bigg) \\
&= \frac{1}{2}\Bigg(A_a^+(t) + A_a^-(t) \\
&\quad - \sum_{s\le t}\Big[\big(X(s)-a\big)^+ + \big(X(s)-a\big)^- - \Big(\big(X(s-)-a\big)^+ + \big(X(s-)-a\big)^-\Big) \\
&\qquad - \Big(\mathbb{1}_{\{X(s-)>a\}} - \mathbb{1}_{\{X(s-)\le a\}}\Big)\,\Delta X(s)\Big]\Bigg).
\end{aligned}$$

Then again, since $|x-a| \equiv (x-a)^+ + (x-a)^-$, and $\operatorname{sgn}(x-a) \triangleq -\mathbb{1}_{\{x\le a\}} + \mathbb{1}_{\{x>a\}}$, we get

$$G_a^+(t) + G_a^-(t) = \frac{1}{2}\Bigg(A_a^+(t) + A_a^-(t) - \sum_{s\le t}\Big[\big|X(s)-a\big| - \big|X(s-)-a\big| - \operatorname{sgn}\big(X(s-)-a\big)\,\Delta X(s)\Big]\Bigg). \tag{6.3.81}$$

Thus, inserting the result $A_a^+(t) + A_a^-(t) = A^{f_a}(t) = A_a(t)$ into (6.3.81), we get

$$G_a^+(t) + G_a^-(t) = \frac{1}{2}\Bigg(A_a(t) - \sum_{s\le t}\Big[\big|X(s)-a\big| - \big|X(s-)-a\big| - \operatorname{sgn}\big(X(s-)-a\big)\,\Delta X(s)\Big]\Bigg).$$

Consequently, by Definition 6.3.12 of the local time for a general semimartingale, (6.3.56), we get $G_a^+(t)+G_a^-(t) = \Lambda_X^a(t)$, so that the result $G_a^+(t) = G_a^-(t)$ yields $G_a^+(t) = G_a^-(t) = \frac{1}{2}\,\Lambda_X^a(t)$. Hence, (6.3.71) together with (6.3.78), and the fact that $2G_a^+(t) = \Lambda_X^a(t)$, gives

$$\begin{aligned}
\big(X(t)-a\big)^+ &= \big(X(0)-a\big)^+ + \int_{0+}^t \mathbb{1}_{\{X(s-)>a\}}\,dX(s) + 2G_a^+(t) \\
&\quad + \sum_{s\le t}\Big[\big(X(s)-a\big)^+ - \big(X(s-)-a\big)^+ - \mathbb{1}_{\{X(s-)>a\}}\,\Delta X(s)\Big] \\
&= \big(X(0)-a\big)^+ + \int_{0+}^t \mathbb{1}_{\{X(s-)>a\}}\,dX(s) + \Lambda_X^a(t) \\
&\quad + \sum_{s\le t}\Big[\big(X(s)-a\big)^+ - \big(X(s-)-a\big)^+ - \mathbb{1}_{\{X(s-)>a\}}\,\Delta X(s)\Big],
\end{aligned} \tag{6.3.82}$$

where

$$\begin{aligned}
&\big(X(s)-a\big)^+ - \big(X(s-)-a\big)^+ - \mathbb{1}_{\{X(s-)>a\}}\,\Delta X(s) \\
&\qquad = \big(X(s)-a\big)^+ - \mathbb{1}_{\{X(s-)>a\}}\big(X(s-)-a\big) - \mathbb{1}_{\{X(s-)>a\}}\big(X(s)-X(s-)\big) \\
&\qquad = \big(X(s)-a\big)^+ - \mathbb{1}_{\{X(s-)>a\}}\Big(X(s-)-a+X(s)-X(s-)\Big) \\
&\qquad = \big(X(s)-a\big)^+ - \mathbb{1}_{\{X(s-)>a\}}\big(X(s)-a\big) \\
&\qquad = \big(X(s)-a\big)^+ - \mathbb{1}_{\{X(s-)>a\}}\Big(\big(X(s)-a\big)^+ - \big(X(s)-a\big)^-\Big) \\
&\qquad = \Big(1-\mathbb{1}_{\{X(s-)>a\}}\Big)\big(X(s)-a\big)^+ + \mathbb{1}_{\{X(s-)>a\}}\big(X(s)-a\big)^- \\
&\qquad = \mathbb{1}_{\{X(s-)\le a\}}\big(X(s)-a\big)^+ + \mathbb{1}_{\{X(s-)>a\}}\big(X(s)-a\big)^-.
\end{aligned}$$

Thus, upon substituting, we obtain

$$\begin{aligned}\big(X(t)-a\big)^+ &= \big(X(0)-a\big)^+ + \int_{0+}^t \mathbb{1}_{\{X(s-)>a\}}\,dX(s) + \Lambda_X^a(t)\\ &\quad + \sum_{s\le t}\Big[\mathbb{1}_{\{X(s-)>a\}}\big(X(s)-a\big)^- + \mathbb{1}_{\{X(s-)\le a\}}\big(X(s)-a\big)^+\Big].\end{aligned} \tag{6.3.83}$$

Similarly, (6.3.72) together with (6.3.79), and the fact that $2G_a^-(t) = \Lambda_X^a(t)$, gives

$$\begin{aligned}\big(X(t)-a\big)^- &= \big(X(0)-a\big)^- - \int_{0+}^t \mathbb{1}_{\{X(s-)\le a\}}\,dX(s) + 2G_a^-(t)\\ &\quad + \sum_{s\le t}\Big[\big(X(s)-a\big)^- - \big(X(s-)-a\big)^- + \mathbb{1}_{\{X(s-)\le a\}}\,\Delta X(s)\Big]\\ &= \big(X(0)-a\big)^- - \int_{0+}^t \mathbb{1}_{\{X(s-)\le a\}}\,dX(s) + \Lambda_X^a(t)\\ &\quad + \sum_{s\le t}\Big[\big(X(s)-a\big)^- - \big(X(s-)-a\big)^- + \mathbb{1}_{\{X(s-)\le a\}}\,\Delta X(s)\Big],\end{aligned} \tag{6.3.84}$$

where

$$\begin{aligned}&\big(X(s)-a\big)^- - \big(X(s-)-a\big)^- + \mathbb{1}_{\{X(s-)\le a\}}\,\Delta X(s)\\ &\quad= \big(X(s)-a\big)^- + \mathbb{1}_{\{X(s-)\le a\}}\big(X(s-)-a\big) + \mathbb{1}_{\{X(s-)\le a\}}\big(X(s)-X(s-)\big)\\ &\quad= \big(X(s)-a\big)^- + \mathbb{1}_{\{X(s-)\le a\}}\Big(X(s-)-a+X(s)-X(s-)\Big)\\ &\quad= \big(X(s)-a\big)^- + \mathbb{1}_{\{X(s-)\le a\}}\big(X(s)-a\big)\\ &\quad= \big(X(s)-a\big)^- + \mathbb{1}_{\{X(s-)\le a\}}\Big(\big(X(s)-a\big)^+ - \big(X(s)-a\big)^-\Big)\\ &\quad= \Big(1-\mathbb{1}_{\{X(s-)\le a\}}\Big)\big(X(s)-a\big)^- + \mathbb{1}_{\{X(s-)\le a\}}\big(X(s)-a\big)^+\\ &\quad= \mathbb{1}_{\{X(s-)>a\}}\big(X(s)-a\big)^- + \mathbb{1}_{\{X(s-)\le a\}}\big(X(s)-a\big)^+.\end{aligned}$$

Thus, upon substituting, we obtain

$$\begin{aligned}\big(X(t)-a\big)^- &= \big(X(0)-a\big)^- - \int_{0+}^t \mathbb{1}_{\{X(s-)\le a\}}\,dX(s) + \Lambda_X^a(t)\\ &\quad + \sum_{s\le t}\Big[\mathbb{1}_{\{X(s-)>a\}}\big(X(s)-a\big)^- + \mathbb{1}_{\{X(s-)\le a\}}\big(X(s)-a\big)^+\Big],\end{aligned} \tag{6.3.85}$$

and the proof is complete. ■

From (6.3.81) and using the fact that $G_a^+(t) + G_a^-(t) = \Lambda_X^a(t)$, we have

$$\begin{aligned}A_a^+(t) + A_a^-(t) &= 2\left(G_a^+(t) + G_a^-(t)\right) + \sum_{s\le t}\Big[\big|X(s)-a\big| - \big|X(s-)-a\big| - \operatorname{sgn}\big(X(s-)-a\big)\,\Delta X(s)\Big]\\ &= 2\Lambda_X^a(t) + \sum_{s\le t}\Big[\big|X(s)-a\big| - \big|X(s-)-a\big| - \operatorname{sgn}\big(X(s-)-a\big)\,\Delta X(s)\Big],\end{aligned}$$

which when substituted into (6.3.80) yields the Tanaka-Meyer formula provided in (6.3.58). Moreover, adding (6.3.59) and (6.3.60), gives

$$\begin{aligned}\big|X(t)-a\big| &= \big|X(0)-a\big| + \int_{0+}^t \operatorname{sgn}\big(X(s-)-a\big)\,dX(s) + 2\Lambda_X^a(t)\\ &\quad + 2\sum_{s\le t}\mathbb{1}_{\{X(s-)>a\}}\big(X(s)-a\big)^- + 2\sum_{s\le t}\mathbb{1}_{\{X(s-)\le a\}}\big(X(s)-a\big)^+.\end{aligned} \tag{6.3.86}$$

Furthermore, from equations (6.3.71), (6.3.72) and (6.3.54), we have the following

$$A_a^+(t) = \big(X(t)-a\big)^+ - \big(X(0)-a\big)^+ - \int_{0+}^t \mathbb{1}_{\{X(s-)>a\}}\, dX(s) \tag{6.3.87}$$

$$A_a^-(t) = \big(X(t)-a\big)^- - \big(X(0)-a\big)^- + \int_{0+}^t \mathbb{1}_{\{X(s-)\le a\}}\, dX(s) \tag{6.3.88}$$

$$A_a(t) = \big|X(t)-a\big| - \big|X(0)-a\big| - \int_{0+}^t \operatorname{sgn}\big(X(s-)-a\big)\, dX(s). \tag{6.3.89}$$

Consequently, by inserting the last equation above into equation (6.3.56) of Definition 6.3.12 or by inspecting equation (6.3.58), we arrive at the following definition for the local time for a general, arbitrary semimartingale.

Definition 6.3.14 (Local Time for a General Semimartingale). *Let $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ be a* ***semimartingale****, and let $a \in \mathbb{R}$. The* ***local time for the semimartingale*** *X at a, denoted $\Lambda_X^a \equiv \Lambda_X(a) = \big\{\Lambda_X^a(t) \equiv \Lambda_X(a,t), \mathcal{F}_t, t \in [0,\infty)\big\}$, is defined to be the process given by*

$$\begin{aligned}\Lambda_X^a(t) = \frac{1}{2}\Bigg(&\big|X(t)-a\big| - \big|X(0)-a\big| - \int_{0+}^t \operatorname{sgn}\big(X(s-)-a\big)\, dX(s)\\ &- \sum_{s\le t}\Big[\big|X(s)-a\big| - \big|X(s-)-a\big| - \operatorname{sgn}\big(X(s-)-a\big)\,\Delta X(s)\Big]\Bigg).\end{aligned} \tag{6.3.90}$$

Equivalently, the ***local time for the semimartingale*** *X at the level 0 (i.e., at the origin), denoted $\Lambda_X^0 \equiv \Lambda_X(0) = \big\{\Lambda_X^0(t) \equiv \Lambda_X(0,t), \mathcal{F}_t, t \in [0,\infty)\big\}$, is defined to be the process given by*

$$\begin{aligned}\Lambda_X(t) = \frac{1}{2}\Bigg(&\big|X(t)\big| - \big|X(0)\big| - \int_{0+}^t \operatorname{sgn}\big(X(s-)\big)\, dX(s)\\ &- \sum_{s\le t}\Big[\big|X(s)\big| - \big|X(s-)\big| - \operatorname{sgn}\big(X(s-)\big)\,\Delta X(s)\Big]\Bigg).\end{aligned} \tag{6.3.91}$$

The semimartingale local time Λ_X is a measure of the amount of time a process X spends near or at the origin.

Using the fact that $\Delta X(s) \triangleq X(s) - X(s-)$ and $X(0) = 0$, note that for a *nonnegative semimartingale* X, the local time at 0 is given by

$$\begin{aligned}\Lambda_X(t) &= \frac{1}{2}\left(X(t) - X(0) - \int_{0+}^t \operatorname{sgn}\big(X(s-)\big)\, dX(s) - \sum_{s\le t}\Big[X(s) - X(s-) - \operatorname{sgn}\big(X(s-)\big)\,\Delta X(s)\Big]\right)\\
&= \frac{1}{2}\left(\int_{0+}^t dX(s) - \int_{0+}^t \operatorname{sgn}\big(X(s-)\big)\, dX(s) - \sum_{s\le t}\Big[\Delta X(s) - \operatorname{sgn}\big(X(s-)\big)\,\Delta X(s)\Big]\right)\\
&= \frac{1}{2}\left(\int_{0+}^t \Big(1 - \operatorname{sgn}\big(X(s-)\big)\Big)\, dX(s) - \sum_{s\le t}\Big[\Big(1 - \operatorname{sgn}\big(X(s-)\big)\Big)\,\Delta X(s)\Big]\right)\\
&= \frac{1}{2}\left(\int_{0+}^t 2\Big(1 - \mathbb{1}_{(0,\infty)}\big(X(s-)\big)\Big)\, dX(s) - \sum_{s\le t}\Big[2\Big(1 - \mathbb{1}_{(0,\infty)}\big(X(s-)\big)\Big)\,\Delta X(s)\Big]\right)\\
&= \frac{1}{2}\left(\int_{0+}^t 2\,\mathbb{1}_{(-\infty,0]}\big(X(s-)\big)\, dX(s) - \sum_{s\le t}\Big[2\,\mathbb{1}_{(-\infty,0]}\big(X(s-)\big)\,\Delta X(s)\Big]\right)\\
&= \frac{1}{2}\left(\int_{0+}^t 2\,\mathbb{1}_{\{0\}}\big(X(s-)\big)\, dX(s) - \sum_{s\le t}\Big[2\,\mathbb{1}_{\{0\}}\big(X(s-)\big)\,\Delta X(s)\Big]\right)\\
&= \int_{0+}^t \mathbb{1}_{\{0\}}\big(X(s-)\big)\, dX(s) - \sum_{s\le t}\mathbb{1}_{\{0\}}\big(X(s-)\big)\,\Delta X(s).\end{aligned}$$

Therefore for a nonnegative semimartingale X, we have

$$\begin{aligned}\Lambda_X(t) &= \int_{0+}^t \mathbb{1}_{\{X(s-)\le 0\}}\, dX(s) - \sum_{s\le t} \mathbb{1}_{\{X(s-)\le 0\}}\, \Delta X(s) \\ &= \int_{0+}^t \mathbb{1}_{\{X(s-)= 0\}}\, dX(s) - \sum_{s\le t} \mathbb{1}_{\{X(s-)= 0\}}\, \Delta X(s), \end{aligned} \tag{6.3.92}$$

and for any semimartingale X, we have

$$\Lambda_X(t) = \int_{0+}^t \mathbb{1}_{\{X(s-)=0\}}\, dX^+(s) - \sum_{s\le t} \mathbb{1}_{\{X(s-)=0\}}\, \Delta X^+(s), \tag{6.3.93}$$

since $X = X^+$ when X is a nonnegative semimartingale. Analogously, since $\operatorname{sgn}(x) = \operatorname{sgn}(x^+)$, we have for the nonnegative semimartingale X^+,

$$\begin{aligned}\Lambda_{X^+}(t) &= \int_{0+}^t \mathbb{1}_{\{X^+(s-)=0\}}\, dX^+(s) - \sum_{s\le t} \mathbb{1}_{\{X^+(s-)=0\}}\, \Delta X^+(s) \\ &= \int_{0+}^t \mathbb{1}_{\{X(s-)\le 0\}}\, dX^+(s) - \sum_{s\le t} \mathbb{1}_{\{X(s-)\le 0\}}\, \Delta X^+(s). \end{aligned} \tag{6.3.94}$$

Since the definition of local time ensures that $\Lambda_{X^+}(t) \equiv \Lambda_X(t)$ for all $t \in [0,\infty)$, it follows from (6.3.93) that for any semimartingale X,

$$\Lambda_{X^+}(t) = \int_{0+}^t \mathbb{1}_{\{X(s-)=0\}}\, dX^+(s) - \sum_{s\le t} \mathbb{1}_{\{X(s-)=0\}}\, \Delta X^+(s). \tag{6.3.95}$$

Alternatively, we can show (6.3.95) by considering the maximum of two semimartingales. Consider two semimartingales X and Y. We shall follow the same idea employed in the proof of the second theorem in Ouknine (1990) or in the proof of Ouknine & Rutkowski (1995, Proposition 2.3). Let us denote the maximum of these two semimartingales by $Z = X \vee Y$. Then

$$\begin{aligned}\big\{Z(t-)=0\big\} &= \big\{\big(X(t-)\vee Y(t-)\big)=0\big\} \\ &= \big\{X(t-)<Y(t-)=0\big\}\cup\big\{Y(t-)<X(t-)=0\big\}\cup\big\{X(t-)=Y(t-)=0\big\}.\end{aligned}$$

Therefore, we deduce

$$\mathbb{1}_{\{Z(t-)=0\}}\, dZ^+(t) = \mathbb{1}_{\{X(t-)<Y(t-)=0\}}\, dZ^+(t) + \mathbb{1}_{\{Y(t-)<X(t-)=0\}}\, dZ^+(t) + \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\, dZ^+(t). \tag{6.3.96}$$

By Zheng (1982a), the following authors Ouknine (1990), Pamen (2009) and Ghomrasni & Pamen (2010) remark that the predictable set $\mathcal{H} = \big\{X(t-)<Y(t-)\big\} = \big\{X(t-)<Y(t-)=Z(t-)\big\}$ is not a random open set so that the theory developed in Zheng (1982a) cannot be applied directly to replace the semimartingale $Z = X\vee Y$ by the semimartingale Y (or, to replace the semimartingale $Z^+ = \big(X\vee Y\big)^+$ by the semimartingale Y^+), which are equal in the set $\mathcal{H}$. However, the semimartingale $V = \big(X\vee Y\big) - Y = Z - Y$ is such that $V_- = 0$,[6] in the set $\mathcal{H}$. Thus, $\mathbb{1}_{\mathcal{H}}\, dV(t) = \mathbb{1}_{\mathcal{H}}\big(\mathbb{1}_{\{V(t-)=0\}}\, dV(t)\big)$, and the latter is of finite variation, and null in any open interval where V is constant, thus in the interior of $\mathcal{H}$. Therefore, since $V = Z - Y$, we have

$$\begin{aligned}&\mathbb{1}_{\mathcal{H}}\, d\big(Z(t)-Y(t)\big) \\ &\quad= \mathbb{1}_{\{X(t-)<Y(t-)=Z(t-)\}}\, d\big(Z(t)-Y(t)\big) = \mathbb{1}_{\{X(t-)<Y(t-)\}}\mathbb{1}_{\{Y(t-)=Z(t-)\}}\, d\big(Z(t)-Y(t)\big) = 0.\end{aligned}$$

[6]The notation X_- denotes the left-continuous version of X, and means the process whose value at t is given by

$$(X_-)_t = X(t-) = \lim_{s\to t-} X(s) = \lim_{s\uparrow t} X(s);$$

also, $(X_-)_0 = X(0-) = 0$, by convention.

Hence, we get

$$\mathbb{1}_{\{X(t-)<Y(t-)=Z(t-)\}}\,dZ(t) \;=\; \mathbb{1}_{\{X(t-)<Y(t-)=Z(t-)\}}\,dY(t) \;=\; \mathbb{1}_{\{X(t-)<Y(t-)\}}\mathbb{1}_{\{Y(t-)=Z(t-)\}}\,dY(t).$$

Then, the replacement of Z by Y is indeed permitted. For the case where $Z(t-) = 0$ for all $t \in [0,\infty)$, we similarly have

$$\mathbb{1}_{\{X(t-)<Y(t-)=0\}}\,dZ(t) \;=\; \mathbb{1}_{\{X(t-)<Y(t-)=0\}}\,dY(t) \;=\; \mathbb{1}_{\{X(t-)<0\}}\mathbb{1}_{\{Y(t-)=0\}}\,dY(t).$$

Hence, by replacing Z by Y, we can write the first term on the right-hand side of equation (6.3.96) as

$$\mathbb{1}_{\{X(t-)<Y(t-)=0\}}\,dZ^+(t) \;=\; \mathbb{1}_{\{X(t-)<Y(t-)=0\}}\,dY^+(t) \;=\; \mathbb{1}_{\{X(t-)<0\}}\mathbb{1}_{\{Y(t-)=0\}}\,dY^+(t). \tag{6.3.97}$$

Similarly, the second term on the right-hand side of (6.3.96) can be written as follows

$$\mathbb{1}_{\{Y(t-)<X(t-)=0\}}\,dZ^+(t) \;=\; \mathbb{1}_{\{Y(t-)<X(t-)=0\}}\,dX^+(t) \;=\; \mathbb{1}_{\{Y(t-)<0\}}\mathbb{1}_{\{X(t-)=0\}}\,dX^+(t). \tag{6.3.98}$$

Employing the fact that $Z = X \vee Y = X + \big(Y - X\big)^+ = Y + \big(X - Y\big)^+$, we shall evaluate the last term on the right-hand side of (6.3.96), by making use of the following representation of Z^+

$$Z^+ \;=\; \big(X \vee Y\big)^+ \;=\; X^+ \vee Y^+ \;=\; X^+ + \big(Y^+ - X^+\big)^+ \;\equiv\; Y^+ + \big(X^+ - Y^+\big)^+.$$

Thus, the last term becomes

$$\begin{aligned}
\mathbb{1}_{\{X(t-)=Y(t-)=0\}}\,dZ^+(t) &= \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\,dY^+(t) + \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\,d\Big(\big(X^+(t) - Y^+(t)\big)^+\Big) && (6.3.99)\\
&= \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\,dX^+(t) + \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\,d\Big(\big(Y^+(t) - X^+(t)\big)^+\Big). && (6.3.100)
\end{aligned}$$

Using (6.3.97), (6.3.98) and (6.3.99), we may cast (6.3.96) in the following form

$$\begin{aligned}
\mathbb{1}_{\{Z(t-)=0\}}\,dZ^+(t) &= \mathbb{1}_{\{X(t-)<0\}}\mathbb{1}_{\{Y(t-)=0\}}\,dY^+(t) + \mathbb{1}_{\{Y(t-)<0\}}\mathbb{1}_{\{X(t-)=0\}}\,dX^+(t)\\
&\quad + \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\,dY^+(t) + \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\,d\Big(\big(X^+(t) - Y^+(t)\big)^+\Big)\\
&= \mathbb{1}_{\{X(t-)<0\}}\mathbb{1}_{\{Y(t-)=0\}}\,dY^+(t) + \mathbb{1}_{\{Y(t-)<0\}}\mathbb{1}_{\{X(t-)=0\}}\,dX^+(t)\\
&\quad + \mathbb{1}_{\{X(t-)=0\}}\mathbb{1}_{\{Y(t-)=0\}}\,dY^+(t) + \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\,d\Big(\big(X^+(t) - Y^+(t)\big)^+\Big)\\
&= \Big(\mathbb{1}_{\{X(t-)<0\}} + \mathbb{1}_{\{X(t-)=0\}}\Big)\mathbb{1}_{\{Y(t-)=0\}}\,dY^+(t) + \mathbb{1}_{\{Y(t-)<0\}}\mathbb{1}_{\{X(t-)=0\}}\,dX^+(t)\\
&\quad + \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\,d\Big(\big(X^+(t) - Y^+(t)\big)^+\Big)\\
&= \mathbb{1}_{\{X(t-)\le 0\}}\mathbb{1}_{\{Y(t-)=0\}}\,dY^+(t) + \mathbb{1}_{\{Y(t-)<0\}}\mathbb{1}_{\{X(t-)=0\}}\,dX^+(t)\\
&\quad + \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\,d\Big(\big(X^+(t) - Y^+(t)\big)^+\Big).
\end{aligned} \tag{6.3.101}$$

Now, let X be a semimartingale and let $Z \equiv X^+ = \max(X,0) = X \vee 0$, so that $X \equiv X$ and $Y \equiv 0$, then we have

$$\mathbb{1}_{\{Z(t-)=0\}}\,dZ^+(t) \;=\; \mathbb{1}_{\{X^+(t-)=0\}}\,d\big(X^+\big)^+(t) \;=\; \mathbb{1}_{\{X^+(t-)=0\}}\,dX^+(t).$$

Hence, by applying (6.3.101), we obtain

$$\mathbb{1}_{\{X^+(t-)=0\}}\,dX^+(t) \;=\; \mathbb{1}_{\{0<0\}}\mathbb{1}_{\{X(t-)=0\}}\,dX^+(t) + \mathbb{1}_{\{X(t-)=0=0\}}\,d\big(X^+(t)\big)^+.$$

Therefore, we have for any semimartingale X,

$$\mathbb{1}_{\{X^+(t-)=0\}}\,dX^+(t) \;=\; \mathbb{1}_{\{X(t-)=0\}}\,dX^+(t). \tag{6.3.102}$$

Similarly, for any semimartingale X, we have

$$\mathbb{1}_{\{X^+(t-)=0\}}\,\Delta X^+(t) \;=\; \mathbb{1}_{\{X(t-)=0\}}\,\Delta X^+(t). \tag{6.3.103}$$

Hence, (6.3.102) in conjunction with (6.3.103) yields

$$\int_{0+}^{t} \mathbb{1}_{\{X^+(s-)=0\}}\, dX^+(s) - \sum_{s\le t} \mathbb{1}_{\{X^+(s-)=0\}}\, \Delta X^+(s) \;=\; \int_{0+}^{t} \mathbb{1}_{\{X(s-)=0\}}\, dX^+(s) - \sum_{s\le t} \mathbb{1}_{\{X(s-)=0\}}\, \Delta X^+(s), \tag{6.3.104}$$

which, by (6.3.94), in turn yields (6.3.95). We could also consider the minimum of two semimartingales. Consider two semimartingales X and Y. Let us denote the minimum of these two semimartingales by $V = X \wedge Y$. Then

$$\begin{aligned} \big\{V(t-)=0\big\} &= \big\{\big(X(t-)\wedge Y(t-)\big)=0\big\} \\ &= \big\{X(t-)>Y(t-)=0\big\}\cup\big\{Y(t-)>X(t-)=0\big\}\cup\big\{X(t-)=Y(t-)=0\big\}. \end{aligned}$$

Therefore, we deduce

$$\mathbb{1}_{\{V(t-)=0\}}\, dV^-(t) \;=\; \mathbb{1}_{\{X(t-)>Y(t-)=0\}}\, dV^-(t) + \mathbb{1}_{\{Y(t-)>X(t-)=0\}}\, dV^-(t) + \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\, dV^-(t). \tag{6.3.105}$$

Following a similar argument to that used to arrive at (6.3.97) for the minimum process, the replacement of V by Y is permitted, and we can write the first term on the right-hand side of equation (6.3.105) as

$$\mathbb{1}_{\{X(t-)>Y(t-)=0\}}\, dV^-(t) \;=\; \mathbb{1}_{\{X(t-)>Y(t-)=0\}}\, dY^-(t) \;=\; \mathbb{1}_{\{X(t-)>0\}}\mathbb{1}_{\{Y(t-)=0\}}\, dY^-(t). \tag{6.3.106}$$

Similarly, the second term on the right-hand side of (6.3.105) can be written as follows

$$\mathbb{1}_{\{Y(t-)>X(t-)=0\}}\, dV^-(t) \;=\; \mathbb{1}_{\{Y(t-)>X(t-)=0\}}\, dX^-(t) \;=\; \mathbb{1}_{\{Y(t-)>0\}}\mathbb{1}_{\{X(t-)=0\}}\, dX^-(t). \tag{6.3.107}$$

We shall evaluate the last term on the right-hand side of (6.3.105), by making use of the following representation of V^-

$$V^- \;=\; \big(X\wedge Y\big)^- \;=\; X^- + \big(Y^- - X^-\big)^+ \;\equiv\; Y^- + \big(X^- - Y^-\big)^+.$$

Thus, the last term becomes

$$\begin{aligned} \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\, dV^-(t) &= \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\, dX^-(t) + \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\, d\Big(\big(Y^-(t)-X^-(t)\big)^+\Big) && (6.3.108)\\ &= \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\, dY^-(t) + \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\, d\Big(\big(X^-(t)-Y^-(t)\big)^+\Big). && (6.3.109) \end{aligned}$$

Using (6.3.106), (6.3.107) and (6.3.109), we may cast (6.3.105) in the following form

$$\begin{aligned} \mathbb{1}_{\{V(t-)=0\}}\, dV^-(t) &= \mathbb{1}_{\{X(t-)>0\}}\mathbb{1}_{\{Y(t-)=0\}}\, dY^-(t) + \mathbb{1}_{\{Y(t-)>0\}}\mathbb{1}_{\{X(t-)=0\}}\, dX^-(t) \\ &\quad + \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\, dY^-(t) + \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\, d\Big(\big(X^-(t)-Y^-(t)\big)^+\Big) \\ &= \mathbb{1}_{\{X(t-)>0\}}\mathbb{1}_{\{Y(t-)=0\}}\, dY^-(t) + \mathbb{1}_{\{Y(t-)>0\}}\mathbb{1}_{\{X(t-)=0\}}\, dX^-(t) \\ &\quad + \mathbb{1}_{\{X(t-)=0\}}\mathbb{1}_{\{Y(t-)=0\}}\, dY^-(t) + \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\, d\Big(\big(X^-(t)-Y^-(t)\big)^+\Big) \\ &= \Big(\mathbb{1}_{\{X(t-)>0\}} + \mathbb{1}_{\{X(t-)=0\}}\Big)\mathbb{1}_{\{Y(t-)=0\}}\, dY^-(t) + \mathbb{1}_{\{Y(t-)>0\}}\mathbb{1}_{\{X(t-)=0\}}\, dX^-(t) \\ &\quad + \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\, d\Big(\big(X^-(t)-Y^-(t)\big)^+\Big) \\ &= \mathbb{1}_{\{X(t-)\ge 0\}}\mathbb{1}_{\{Y(t-)=0\}}\, dY^-(t) + \mathbb{1}_{\{Y(t-)>0\}}\mathbb{1}_{\{X(t-)=0\}}\, dX^-(t) \\ &\quad + \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\, d\Big(\big(X^-(t)-Y^-(t)\big)^+\Big). \end{aligned} \tag{6.3.110}$$

Now, let X be a semimartingale and let $V \equiv -X^+ = \min(-X,0) = (-X)\wedge 0$, so that $X \equiv -X$ and $Y \equiv 0$, then we have

$$\mathbb{1}_{\{V(t-)=0\}}\, dV^-(t) \;=\; \mathbb{1}_{\{-X^+(t-)=0\}}\, d\big(-X^+\big)^-(t) \;=\; \mathbb{1}_{\{X^+(t-)=0\}}\, dX^+(t).$$

Hence, by applying (6.3.110), we obtain

$$\mathbb{1}_{\{X^+(t-)=0\}}\, dX^+(t) \quad = \quad \mathbb{1}_{\{0>0\}}\mathbb{1}_{\{-X(t-)=0\}}\, d\big(-X\big)^-(t) + \mathbb{1}_{\{X(t-)=0=0\}}\, d\big(\big(-X\big)^-(t)\big)^+.$$

Therefore, for any semimartingale X, we have the result (6.3.102)

$$\mathbb{1}_{\{X^+(t-)=0\}}\, dX^+(t) \quad = \quad \mathbb{1}_{\{X(t-)=0\}}\, dX^+(t),$$

and from this the same conclusions may be drawn.

Moreover, integrating $\mathbb{1}_{\{X(s-)=a\}}$ and $\mathbb{1}_{\{X(s-)\le a\}}$ with respect to the two sides of (6.3.59), we obtain the following two formulas for local times for general semimartingales.

Corollary 6.3.15 ([**Yan (2002, Corollary 2.2.53)**]). *Let $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ be a **semimartingale** and $a \in \mathbb{R}$. Then*

$$\Lambda_X^a(t) \quad = \quad \Lambda_{X-a}(t) \quad = \quad \int_{0+}^t \mathbb{1}_{\{X(s-)=a\}}\, d\big(X(s)-a\big)^+ - \sum_{s\le t} \mathbb{1}_{\{X(s-)=a\}}\, \Delta\big(X(s)-a\big)^+, \tag{6.3.111}$$

$$\Lambda_X^a(t) \quad = \quad \Lambda_{X-a}(t) \quad = \quad \int_{0+}^t \mathbb{1}_{\{X(s-)\le a\}}\, d\big(X(s)-a\big)^+ - \sum_{s\le t} \mathbb{1}_{\{X(s-)\le a\}}\, \Delta\big(X(s)-a\big)^+. \tag{6.3.112}$$

Therefore, from the above corollary and the results above, we have $\Lambda_X^a(t) = \Lambda_{X-a}(t) = \Lambda_{(X-a)^+}(t)$, for all $t \in [0,\infty)$. Thus, we have the following

$$\begin{aligned}
\Lambda_{(X-a)^+} \quad &= \quad \int_{0+}^t \mathbb{1}_{\{(X(s-)-a)^+=0\}}\, d\big(X(s)-a\big)^+ - \sum_{s\le t} \mathbb{1}_{\{(X(s-)-a)^+=0\}}\, \Delta\big(X(s)-a\big)^+ \\
&= \quad \int_{0+}^t \mathbb{1}_{\{X(s-)\le a\}}\, d\big(X(s)-a\big)^+ - \sum_{s\le t} \mathbb{1}_{\{X(s-)\le a\}}\, \Delta\big(X(s)-a\big)^+ \\
&= \quad \int_{0+}^t \mathbb{1}_{\{X(s-)=a\}}\, d\big(X(s)-a\big)^+ - \sum_{s\le t} \mathbb{1}_{\{X(s-)=a\}}\, \Delta\big(X(s)-a\big)^+.
\end{aligned}$$

Local times for semimartingales appear in the theorem below which offers an extension of Itô's formula from C^2 functions to convex functions.

Theorem 6.3.16 (Meyer-Itô Formula [**Protter (2004, Chap. IV, §7, Theorem 70, pp. 214–215)**]). *Let $f : \mathbb{R} \to \mathbb{R}$ be the difference of two convex functions, let $D^- f$ be its left derivative, and let μ be the signed measure (when restricted to compacts) which is the second derivative of f in the generalised function sense (i.e., in the distributional sense). Then the following equation holds for all $t \in [0,\infty)$,*

$$\begin{aligned}
f\big(X(t)\big) \quad &= \quad f\big(X(0)\big) + \int_{0+}^t D^- f\big(X(s-)\big)\, dX(s) + \int_{-\infty}^{\infty} \Lambda_X^a(t)\, \mu(da) \\
&\quad + \sum_{s\le t} \Big[f\big(X(s)\big) - f\big(X(s-)\big) - D^- f\big(X(s-)\big)\, \Delta X(s) \Big],
\end{aligned} \tag{6.3.113}$$

*where $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ is a **semimartingale** and $\Lambda_X^a = \big\{\Lambda_X^a(t), \mathcal{F}_t, t \in [0,\infty)\big\}$ is its local time at the level a.*

Proof. See Protter (2004, Chap. IV, §7, Proof of Theorem 70, pp. 215–216). ■

6.3.3 Local Time for Continuous Semimartingales

The concept of local time and its applications to obtain a generalised Itô formula can be extended from the case of Brownian motion to that of continuous semimartingales, i.e., semimartingale local time is a generalisation of Brownian local time. The significant difference is that time-integrals now become integrals with respect to

quadratic variation. In this section, we introduce the notion of a semimartingale local time in the continuous-time context, in order to be able to represent the rank processes and ranked market weight processes derived from pathwise mutually nondegenerate absolutely continuous semimartingales and those derived from continuous semimartingales.

Theorem 6.3.17 ([**Karatzas & Shreve (1991, §3.7, p. 220), Revuz & Yor (1999, Chap. VI, §1, Theorem 1.1, p. 221)**]). *Let $f : \mathbb{R} \to \mathbb{R}$ be convex and let $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ be a* ***continuous semimartingale****. Then there exists a continuous increasing process $A^f = \big\{A^f(t), \mathcal{F}_t, t \in [0,\infty)\big\}$, such that $f(X)$ is a* ***continuous semimartingale*** *and one has*

$$f\big(X(t)\big) = f\big(X(0)\big) + \int_0^t D^- f\big(X(s)\big)\, dM(s) + \int_0^t D^- f\big(X(s)\big)\, dV(s) + A^f(t) \tag{6.3.114}$$

$$= f\big(X(0)\big) + \int_0^t D^- f\big(X(s)\big)\, dX(s) + A^f(t), \tag{6.3.115}$$

for $t \in [0,\infty)$, where $D^- f$ denotes the left-hand derivative of f and A^f is an adapted, continuous, increasing process.

Proof. See Revuz & Yor (1999, Chap. VI, §1, Proof of Theorem 1.1, p. 221). ■

Since $f_a(x)$, given by (6.3.50), is convex with left derivative $D^- f_a(x) = \operatorname{sgn}(x-a)$, by Theorem 6.3.17 we have for a continuous semimartingale X and process $A^{f_a} = \big\{A^{f_a}(t), \mathcal{F}_t, t \in [0,\infty)\big\}$,

$$f_a\big(X(t)\big) \equiv \big|X(t)-a\big| = \big|X(0)-a\big| + \int_0^t \operatorname{sgn}\big(X(s)-a\big)\, dX(s) + A^{f_a}(t) \tag{6.3.116}$$

$$= \big|X(0)-a\big| + \int_0^t \operatorname{sgn}\big(X(s)-a\big)\, dX(s) + A_a(t), \tag{6.3.117}$$

where $A^{f_a} = A_a = \big\{A_a(t), \mathcal{F}_t, t \in [0,\infty)\big\}$ is the continuous, increasing process of Theorem 6.3.17. Employing (6.3.50) and (6.3.116) as defined above we can define the concept of a *local time* for a *continuous semimartingale* [see Karatzas & Shreve (1991)].

Definition 6.3.18 (Local Time for a Continuous Semimartingale). *Let $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ be a* ***continuous semimartingale****, and let f_a and $A^{f_a} = A_a$ be defined as in* (6.3.50) *and* (6.3.116) *above, where $a \in \mathbb{R}$. The* ***local time for the continuous semimartingale*** *X at a, denoted $\Lambda_X^a \equiv \Lambda_X(a) = \big\{\Lambda_X^a(t) \equiv \Lambda_X(a,t), \mathcal{F}_t, t \in [0,\infty)\big\}$, is defined to be the increasing continuous process given by*

$$\Lambda_X^a(t) \equiv \Lambda_X(a,t) \triangleq \frac{1}{2} A^{f_a}(t) = \frac{1}{2} A_a(t). \tag{6.3.118}$$

Theorem 6.3.19 (Tanaka-Meyer Formulae [**Karatzas & Shreve (1991, §3.7, p. 220), Revuz & Yor (1999, Chap. VI, §1, Theorem 1.2, p. 222)**]). *Let $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ be a* ***continuous semimartingale*** *and let $\Lambda_X^a = \big\{\Lambda_X^a(t), \mathcal{F}_t, t \in [0,\infty)\big\}$ be its local time at the level a, for $a \in \mathbb{R}$. Then for all $t \in [0,\infty)$, we have the* ***Tanaka-Meyer formulae****,*

$$\big(X(t)-a\big)^+ = \big(X(0)-a\big)^+ + \int_0^t \mathbb{1}_{(a,\infty)}\big(X(s)\big)\, dX(s) + \Lambda_X^a(t), \tag{6.3.119}$$

$$\big(X(t)-a\big)^- = \big(X(0)-a\big)^- - \int_0^t \mathbb{1}_{(-\infty,a]}\big(X(s)\big)\, dX(s) + \Lambda_X^a(t), \tag{6.3.120}$$

$$\big|X(t)-a\big| = \big|X(0)-a\big| + \int_0^t \operatorname{sgn}\big(X(s)-a\big)\, dX(s) + 2\Lambda_X^a(t). \tag{6.3.121}$$

Proof. With $a \in \mathbb{R}$ fixed, we apply Theorem 6.3.17 to the convex functions

$$g_a(x) = (x-a)^+ = \begin{cases} x-a & \text{if } x > a, \\ 0 & \text{if } x \le a, \end{cases}$$

$$h_a(x) \;=\; (x-a)^- \;=\; \begin{cases} 0 & \text{if} \quad x > a, \\ a-x & \text{if} \quad x \le a, \end{cases}$$

and

$$f_a(x) \;=\; |x-a|,$$

we get, for all $t \in [0,\infty)$,

$$g_a\big(X(t)\big) \;=\; g_a\big(X(0)\big) + \int_0^t D^- g_a\big(X(s)\big)\, dX(s) + A^{g_a}(t) \tag{6.3.122}$$

$$h_a\big(X(t)\big) \;=\; h_a\big(X(0)\big) + \int_0^t D^- h_a\big(X(s)\big)\, dX(s) + A^{h_a}(t) \tag{6.3.123}$$

$$f_a\big(X(t)\big) \;=\; f_a\big(X(0)\big) + \int_0^t D^- f_a\big(X(s)\big)\, dX(s) + A^{f_a}(t), \tag{6.3.124}$$

and with the conventions $A_a^+(t) \triangleq A^{g_a}(t)$, $A_a^-(t) \triangleq A^{h_a}(t)$ and $A_a(t) \triangleq A^{f_a}(t)$, for all $t \in [0,\infty)$, we obtain

$$g_a\big(X(t)\big) \;=\; g_a\big(X(0)\big) + \int_0^t D^- g_a\big(X(s)\big)\, dX(s) + A_a^+(t) \tag{6.3.125}$$

$$h_a\big(X(t)\big) \;=\; h_a\big(X(0)\big) + \int_0^t D^- h_a\big(X(s)\big)\, dX(s) + A_a^-(t) \tag{6.3.126}$$

$$f_a\big(X(t)\big) \;=\; f_a\big(X(0)\big) + \int_0^t D^- f_a\big(X(s)\big)\, dX(s) + A_a(t), \tag{6.3.127}$$

alternatively,

$$\big(X(t)-a\big)^+ \;=\; \big(X(0)-a\big)^+ + \int_0^t D^-\big(X(s)-a\big)^+\, dX(s) + A_a^+(t) \tag{6.3.128}$$

$$\big(X(t)-a\big)^- \;=\; \big(X(0)-a\big)^- + \int_0^t D^-\big(X(s)-a\big)^-\, dX(s) + A_a^-(t) \tag{6.3.129}$$

$$\big|X(t)-a\big| \;=\; \big|X(0)-a\big| + \int_0^t D^-\big|X(s)-a\big|\, dX(s) + A_a(t). \tag{6.3.130}$$

As in the proof of Theorem 6.3.13, it is clear from (6.3.61) that the indicator function $\mathbb{1}_{\{x>a\}}$ is the left derivative of $g_a(x)$, i.e.,

$$D^- g_a\big(X(s)\big) \;=\; D^-\big(X(s)-a\big)^+ \;=\; \mathbb{1}_{\{X(s)>a\}} \;\equiv\; \mathbb{1}_{(a,\infty)}\big(X(s)\big). \tag{6.3.131}$$

Similarly, from (6.3.62) we can see that the negative of the indicator function $\mathbb{1}_{\{x\le a\}}$ is the left derivative of $h_a(x)$, i.e.,

$$D^- h_a\big(X(s)\big) \;=\; D^-\big(X(s)-a\big)^- \;=\; -\mathbb{1}_{\{X(s)\le a\}} \;\equiv\; -\mathbb{1}_{(-\infty,a]}\big(X(s)\big). \tag{6.3.132}$$

Also, the $\operatorname{sgn}(x-a)$ is the left derivative of $f_a(x)$, and we have

$$D^- f_a\big(X(s)\big) \;=\; D^-\big|X(s)-a\big| \;=\; \operatorname{sgn}\big(X(s)-a\big). \tag{6.3.133}$$

So that, (6.3.128), (6.3.129) and (6.3.130), respectively become

$$\big(X(t)-a\big)^+ \;=\; \big(X(0)-a\big)^+ + \int_0^t \mathbb{1}_{(a,\infty)}\big(X(s)\big)\, dX(s) + A_a^+(t) \tag{6.3.134}$$

$$\big(X(t)-a\big)^- \;=\; \big(X(0)-a\big)^- - \int_0^t \mathbb{1}_{(-\infty,a]}\big(X(s)\big)\, dX(s) + A_a^-(t) \tag{6.3.135}$$

$$\big|X(t)-a\big| \;=\; \big|X(0)-a\big| + \int_0^t \operatorname{sgn}\big(X(s)-a\big)\, dX(s) + A_a(t). \tag{6.3.136}$$

Upon subtraction of the above formulas, (6.3.134) and (6.3.135), we arrive at the following

$$\begin{aligned}\big(X(t)-a\big)^+ - \big(X(t)-a\big)^- &= \big(X(0)-a\big)^+ - \big(X(0)-a\big)^- + A_a^+(t) - A_a^-(t) \\ &\quad + \int_0^t \mathbb{1}_{(a,\infty)}\big(X(s)\big)\,dX(s) + \int_0^t \mathbb{1}_{(-\infty,a]}\big(X(s)\big)\,dX(s) \\ &= \big(X(0)-a\big)^+ - \big(X(0)-a\big)^- + A_a^+(t) - A_a^-(t) \\ &\quad + \int_0^t \Big(\mathbb{1}_{(a,\infty)}\big(X(s)\big) + \mathbb{1}_{(-\infty,a]}\big(X(s)\big)\Big)\,dX(s).\end{aligned} \tag{6.3.137}$$

Again, since $x \equiv x^+ - x^-$, and $\mathbb{1}_{(a,\infty)}(x) + \mathbb{1}_{(-\infty,a]}(x) = 1$ for all $x \in \mathbb{R}$, we obtain

$$X(t) - a = X(0) - a + \int_0^t dX(s) + A_a^+(t) - A_a^-(t),$$

which implies

$$X(t) - a = X(0) - a + X(t) - X(0) + A_a^+(t) - A_a^-(t).$$

Thus, we get $A_a^+(t) - A_a^-(t) = 0$, and hence $A_a^+(t) = A_a^-(t)$. Now, since $|x| \equiv x^+ + x^-$, and from (6.3.9) we have $\operatorname{sgn}(x-a) \triangleq -\mathbb{1}_{(-\infty,a]}(x) + \mathbb{1}_{(a,\infty)}(x)$, we obtain upon addition of the formulas (6.3.134) and (6.3.135),

$$\begin{aligned}\big(X(t)-a\big)^+ + \big(X(t)-a\big)^- &= \big(X(0)-a\big)^+ + \big(X(0)-a\big)^- + A_a^+(t) + A_a^-(t) \\ &\quad + \int_0^t \mathbb{1}_{(a,\infty)}\big(X(s)\big)\,dX(s) - \int_0^t \mathbb{1}_{(-\infty,a]}\big(X(s)\big)\,dX(s) \\ &= \big(X(0)-a\big)^+ + \big(X(0)-a\big)^- + A_a^+(t) + A_a^-(t) \\ &\quad + \int_0^t \Big(\mathbb{1}_{(a,\infty)}\big(X(s)\big) - \mathbb{1}_{(-\infty,a]}\big(X(s)\big)\Big)\,dX(s).\end{aligned}$$

Hence, we get

$$\big|X(t)-a\big| = \big|X(0)-a\big| + \int_0^t \operatorname{sgn}\big(X(s)-a\big)\,dX(s) + A_a^+(t) + A_a^-(t), \tag{6.3.138}$$

which upon comparison with (6.3.136) yields $A_a^+(t) + A_a^-(t) = A_a(t)$. So that $A_a^\pm(t) = \frac{1}{2}\,A_a(t)$. Consequently, by Definition 6.3.18 of the local time for a continuous semimartingale, (6.3.118) (i.e., $A_a(t) \triangleq 2\Lambda_X^a(t)$), we get $A_a^+(t) + A_a^-(t) = 2\Lambda_X^a(t)$, so that $A_a^\pm(t) = \Lambda_X^a(t)$. The processes $A_a^\pm(t)$, $A_a(t)$ and $\Lambda_X^a(t)$, are adapted, continuous, and nondecreasing in t. Hence, (6.3.134) together with the fact that $A_a^+(t) = \Lambda_X^a(t)$, gives

$$\big(X(t)-a\big)^+ = \big(X(0)-a\big)^+ + \int_0^t \mathbb{1}_{(a,\infty)}\big(X(s)\big)\,dX(s) + \Lambda_X^a(t).$$

Similarly, (6.3.135) together with the fact that $A_a^-(t) = \Lambda_X^a(t)$, gives

$$\big(X(t)-a\big)^- = \big(X(0)-a\big)^- - \int_0^t \mathbb{1}_{(-\infty,a]}\big(X(s)\big)\,dX(s) + \Lambda_X^a(t).$$

Finally, since $A_a(t) = 2\Lambda_X^a(t)$ by definition, (6.3.136) becomes the definitive *Tanaka-Meyer formula*

$$\big|X(t)-a\big| = \big|X(0)-a\big| + \int_0^t \operatorname{sgn}\big(X(s)-a\big)\,dX(s) + 2\Lambda_X^a(t).$$

■

The lack of symmetry in (6.3.119) and (6.3.120) of Theorem 6.3.19 is due to the fact that we have chosen to work with *left* derivatives. This is also the reason for the choice of the function sgn. Thus, for a continuous semimartingale X and $a = 0$, the *Tanaka-Meyer formula* in (6.3.121) becomes

$$\big|X(t)\big| = \big|X(0)\big| + \int_0^t \operatorname{sgn}\big(X(s)\big)\,dX(s) + 2\Lambda_X^0(t) \tag{6.3.139}$$

$$= \big|X(0)\big| + \int_0^t \operatorname{sgn}\big(X(s)\big)\,dX(s) + 2\Lambda_X(t), \tag{6.3.140}$$

where $\Lambda_X(t) = \Lambda_X^0(t)$ denotes the local time for the continuous semimartingale X at the origin. Note that the asymmetry in sgn induces an asymmetry in the local time, i.e., in general, Λ_X^a differs from Λ_{-X}^a.

Furthermore, from equations (6.3.134), (6.3.135) and (6.3.136), we have the following

$$A_a^+(t) = \big(X(t)-a\big)^+ - \big(X(0)-a\big)^+ - \int_0^t \mathbb{1}_{(a,\infty)}\big(X(s)\big)\,dX(s) \tag{6.3.141}$$

$$A_a^-(t) = \big(X(t)-a\big)^- - \big(X(0)-a\big)^- + \int_0^t \mathbb{1}_{(-\infty,a]}\big(X(s)\big)\,dX(s) \tag{6.3.142}$$

$$A_a(t) = \big|X(t)-a\big| - \big|X(0)-a\big| - \int_0^t \operatorname{sgn}\big(X(s)-a\big)\,dX(s). \tag{6.3.143}$$

Consequently, by inserting the last equation above into equation (6.3.118) of Definition 6.3.18 or by inspecting equation (6.3.121), we arrive at the following definition for the local time for a continuous semimartingale.

Definition 6.3.20 (Local Time for a Continuous Semimartingale). *Let $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ be a **continuous semimartingale**, and let $a \in \mathbb{R}$. The **local time for the continuous semimartingale** X at a, denoted $\Lambda_X^a \equiv \Lambda_X(a) = \big\{\Lambda_X^a(t) \equiv \Lambda_X(a,t), \mathcal{F}_t, t \in [0,\infty)\big\}$, is defined to be the process given by*

$$\Lambda_X^a(t) = \frac{1}{2}\left(\big|X(t)-a\big| - \big|X(0)-a\big| - \int_0^t \operatorname{sgn}\big(X(s)-a\big)\,dX(s)\right). \tag{6.3.144}$$

*Equivalently, the **local time for the continuous semimartingale** X at the level 0 (i.e., at the origin), denoted $\Lambda_X^0 \equiv \Lambda_X(0) = \big\{\Lambda_X^0(t) \equiv \Lambda_X(0,t), \mathcal{F}_t, t \in [0,\infty)\big\}$, is defined to be the process given by*

$$\Lambda_X^0(t) = \Lambda_X(t) = \frac{1}{2}\left(\big|X(t)\big| - \big|X(0)\big| - \int_0^t \operatorname{sgn}\big(X(s)\big)\,dX(s)\right). \tag{6.3.145}$$

The semimartingale local time Λ_X measures the amount of time the process X spends near or at the origin, so $d\Lambda_X(t)$ defines a positive measure on $[0,T]$ that is concentrated on the set of points $\big\{t \in [0,T] \mid X(t) = 0\big\}$.

Note that for a *nonnegative continuous semimartingale* X, the local time at 0 is given by

$$\begin{aligned}
\Lambda_X^0(t) = \Lambda_X(t) &= \frac{1}{2}\left(X(t) - X(0) - \int_0^t \operatorname{sgn}\big(X(s)\big)\,dX(s)\right) \\
&= \frac{1}{2}\left(\int_0^t dX(s) - \int_0^t \operatorname{sgn}\big(X(s)\big)\,dX(s)\right) \\
&= \frac{1}{2}\left(\int_0^t \Big(1 - \operatorname{sgn}\big(X(s)\big)\Big)\,dX(s)\right) \\
&= \frac{1}{2}\left(\int_0^t 2\Big(1 - \mathbb{1}_{(0,\infty)}\big(X(s)\big)\Big)\,dX(s)\right) \\
&= \frac{1}{2}\left(\int_0^t 2\,\mathbb{1}_{(-\infty,0]}\big(X(s)\big)\,dX(s)\right) \\
&= \frac{1}{2}\left(\int_0^t 2\,\mathbb{1}_{\{0\}}\big(X(s)\big)\,dX(s)\right) \\
&= \int_0^t \mathbb{1}_{\{0\}}\big(X(s)\big)\,dX(s).
\end{aligned}$$

Therefore, we have for a nonnegative continuous semimartingale X

$$\Lambda_X(t) = \int_0^t \mathbb{1}_{\{X(s)\le 0\}}\,dX(s) \tag{6.3.146}$$

$$= \int_0^t \mathbb{1}_{\{X(s)=0\}}\,dX(s). \tag{6.3.147}$$

Moreover, $X \geq 0$ implies that $X^+(t) = X(t)$ for all $t \in [0, \infty)$, and we have for any continuous semimartingale X

$$\Lambda_X(t) = \int_0^t \mathbb{1}_{\{X(s)=0\}}\, dX^+(s). \tag{6.3.148}$$

Analogously, since $\operatorname{sgn}(x) = \operatorname{sgn}(x^+)$, we have for the nonnegative continuous semimartingale X^+,

$$\begin{aligned} \Lambda_{X^+}(t) &= \int_0^t \mathbb{1}_{\{X^+(s)=0\}}\, dX^+(s) \\ &= \int_0^t \mathbb{1}_{\{X(s)\leq 0\}}\, dX^+(s). \end{aligned} \tag{6.3.149}$$

Since $\Lambda_{X^+}(t) = \Lambda_X(t)$ for any continuous semimartingale X for all $t \in [0, \infty)$, from (6.3.148) we have for any continuous semimartingale X,

$$\Lambda_{X^+}(t) = \int_0^t \mathbb{1}_{\{X(s)=0\}}\, dX^+(s). \tag{6.3.150}$$

Alternatively, we can show (6.3.150) by considering the maximum of two continuous semimartingales. Consider two continuous semimartingales X and Y. Let us denote the maximum of these two continuous semimartingales by $Z = X \vee Y$. Then

$$\big\{Z(t) = 0\big\} = \big\{\big(X(t) \vee Y(t)\big) = 0\big\} = \big\{X(t) < Y(t) = 0\big\} \cup \big\{Y(t) < X(t) = 0\big\} \cup \big\{X(t) = Y(t) = 0\big\}.$$

Therefore, we deduce

$$\mathbb{1}_{\{Z(t)=0\}}\, dZ^+(t) = \mathbb{1}_{\{X(t)<Y(t)=0\}}\, dZ^+(t) + \mathbb{1}_{\{Y(t)<X(t)=0\}}\, dZ^+(t) + \mathbb{1}_{\{X(t)=Y(t)=0\}}\, dZ^+(t). \tag{6.3.151}$$

Since the replacement of Z by Y is permitted, we can write the first term on the right-hand side of equation (6.3.151) as

$$\mathbb{1}_{\{X(t)<Y(t)=0\}}\, dZ^+(t) = \mathbb{1}_{\{X(t)<Y(t)=0\}}\, dY^+(t) = \mathbb{1}_{\{X(t)<0\}}\mathbb{1}_{\{Y(t)=0\}}\, dY^+(t). \tag{6.3.152}$$

Similarly, the second term on the right-hand side of (6.3.151) can be written as follows

$$\mathbb{1}_{\{Y(t)<X(t)=0\}}\, dZ^+(t) = \mathbb{1}_{\{Y(t)<X(t)=0\}}\, dX^+(t) = \mathbb{1}_{\{Y(t)<0\}}\mathbb{1}_{\{X(t)=0\}}\, dX^+(t). \tag{6.3.153}$$

Employing the fact that $Z = X \vee Y = X + \big(Y - X\big)^+ = Y + \big(X - Y\big)^+$, we shall evaluate the last term on the right-hand side of (6.3.151), by making use of the following representation of Z^+

$$Z^+ = \big(X \vee Y\big)^+ = X^+ \vee Y^+ = X^+ + \big(Y^+ - X^+\big)^+ \equiv Y^+ + \big(X^+ - Y^+\big)^+.$$

Thus, the last term becomes

$$\mathbb{1}_{\{X(t)=Y(t)=0\}}\, dZ^+(t) = \mathbb{1}_{\{X(t)=Y(t)=0\}}\, dY^+(t) + \mathbb{1}_{\{X(t)=Y(t)=0\}}\, d\Big(\big(X^+(t) - Y^+(t)\big)^+\Big) \tag{6.3.154}$$

$$= \mathbb{1}_{\{X(t)=Y(t)=0\}}\, dX^+(t) + \mathbb{1}_{\{X(t)=Y(t)=0\}}\, d\Big(\big(Y^+(t) - X^+(t)\big)^+\Big). \tag{6.3.155}$$

Using (6.3.152), (6.3.153) and (6.3.154), we may cast (6.3.151) in the following form

$$\begin{aligned} \mathbb{1}_{\{Z(t)=0\}}\, dZ^+(t) &= \mathbb{1}_{\{X(t)<0\}}\mathbb{1}_{\{Y(t)=0\}}\, dY^+(t) + \mathbb{1}_{\{Y(t)<0\}}\mathbb{1}_{\{X(t)=0\}}\, dX^+(t) \\ &\quad + \mathbb{1}_{\{X(t)=Y(t)=0\}}\, dY^+(t) + \mathbb{1}_{\{X(t)=Y(t)=0\}}\, d\Big(\big(X^+(t) - Y^+(t)\big)^+\Big) \\ &= \mathbb{1}_{\{X(t)<0\}}\mathbb{1}_{\{Y(t)=0\}}\, dY^+(t) + \mathbb{1}_{\{Y(t)<0\}}\mathbb{1}_{\{X(t)=0\}}\, dX^+(t) \\ &\quad + \mathbb{1}_{\{X(t)=0\}}\mathbb{1}_{\{Y(t)=0\}}\, dY^+(t) + \mathbb{1}_{\{X(t)=Y(t)=0\}}\, d\Big(\big(X^+(t) - Y^+(t)\big)^+\Big) \\ &= \Big(\mathbb{1}_{\{X(t)<0\}} + \mathbb{1}_{\{X(t)=0\}}\Big)\mathbb{1}_{\{Y(t)=0\}}\, dY^+(t) + \mathbb{1}_{\{Y(t)<0\}}\mathbb{1}_{\{X(t)=0\}}\, dX^+(t) \\ &\quad + \mathbb{1}_{\{X(t)=Y(t)=0\}}\, d\Big(\big(X^+(t) - Y^+(t)\big)^+\Big) \\ &= \mathbb{1}_{\{X(t)\leq 0\}}\mathbb{1}_{\{Y(t)=0\}}\, dY^+(t) + \mathbb{1}_{\{Y(t)<0\}}\mathbb{1}_{\{X(t)=0\}}\, dX^+(t) \\ &\quad + \mathbb{1}_{\{X(t)=Y(t)=0\}}\, d\Big(\big(X^+(t) - Y^+(t)\big)^+\Big). \end{aligned} \tag{6.3.156}$$

Now, let X be a continuous semimartingale and let $Z \equiv X^+ = \max(X,0) = X \vee 0$, so that $X \equiv X$ and $Y \equiv 0$, then we have

$$\mathbb{1}_{\{Z(t)=0\}}\, dZ^+(t) \quad = \quad \mathbb{1}_{\{X^+(t)=0\}}\, d\left(X^+\right)^+(t) \quad = \quad \mathbb{1}_{\{X^+(t)=0\}}\, dX^+(t).$$

Hence, by applying (6.3.156), we obtain

$$\mathbb{1}_{\{X^+(t)=0\}}\, dX^+(t) \quad = \quad \mathbb{1}_{\{0<0\}}\mathbb{1}_{\{X(t)=0\}}\, dX^+(t) + \mathbb{1}_{\{X(t)=0=0\}}\, d\left(X^+(t)\right)^+.$$

Therefore, for any continuous semimartingale X, we have

$$\mathbb{1}_{\{X^+(t)=0\}}\, dX^+(t) \quad = \quad \mathbb{1}_{\{X(t)=0\}}\, dX^+(t). \tag{6.3.157}$$

Hence, (6.3.149) in conjunction with (6.3.157) yields (6.3.150)

$$\Lambda_{X^+}(t) \quad = \quad \int_0^t \mathbb{1}_{\{X^+(s)=0\}}\, dX^+(s) \quad = \quad \int_0^t \mathbb{1}_{\{X(s)=0\}}\, dX^+(s).$$

We could also consider the minimum of two continuous semimartingales. Consider two continuous semimartingales X and Y. Let us denote the minimum of these two continuous semimartingales by $V = X \wedge Y$. Then

$$\left\{V(t)=0\right\} \quad = \quad \left\{\left(X(t) \wedge Y(t)\right)=0\right\} \quad = \quad \left\{X(t) > Y(t) = 0\right\} \cup \left\{Y(t) > X(t) = 0\right\} \cup \left\{X(t) = Y(t) = 0\right\}.$$

Therefore, we deduce

$$\mathbb{1}_{\{V(t)=0\}}\, dV^-(t) \quad = \quad \mathbb{1}_{\{X(t)>Y(t)=0\}}\, dV^-(t) + \mathbb{1}_{\{Y(t)>X(t)=0\}}\, dV^-(t) + \mathbb{1}_{\{X(t)=Y(t)=0\}}\, dV^-(t). \tag{6.3.158}$$

Following a similar argument to that used to arrive at (6.3.97) for the minimum process, the replacement of V by Y is permitted, and we can write the first term on the right-hand side of equation (6.3.158) as

$$\mathbb{1}_{\{X(t)>Y(t)=0\}}\, dV^-(t) \quad = \quad \mathbb{1}_{\{X(t)>Y(t)=0\}}\, dY^-(t) \quad = \quad \mathbb{1}_{\{X(t)>0\}}\mathbb{1}_{\{Y(t)=0\}}\, dY^-(t). \tag{6.3.159}$$

Similarly, the second term on the right-hand side of (6.3.158) can be written as follows

$$\mathbb{1}_{\{Y(t)>X(t)=0\}}\, dV^-(t) \quad = \quad \mathbb{1}_{\{Y(t)>X(t)=0\}}\, dX^-(t) \quad = \quad \mathbb{1}_{\{Y(t)>0\}}\mathbb{1}_{\{X(t)=0\}}\, dX^-(t). \tag{6.3.160}$$

We shall evaluate the last term on the right-hand side of (6.3.158), by making use of the following representation of V^-

$$V^- \quad = \quad \left(X \wedge Y\right)^- \quad = \quad X^- + \left(Y^- - X^-\right)^+ \quad \equiv \quad Y^- + \left(X^- - Y^-\right)^+.$$

Thus, the last term becomes

$$\mathbb{1}_{\{X(t)=Y(t)=0\}}\, dV^-(t) \quad = \quad \mathbb{1}_{\{X(t)=Y(t)=0\}}\, dX^-(t) + \mathbb{1}_{\{X(t)=Y(t)=0\}}\, d\left(\left(Y^-(t) - X^-(t)\right)^+\right) \tag{6.3.161}$$

$$= \quad \mathbb{1}_{\{X(t)=Y(t)=0\}}\, dY^-(t) + \mathbb{1}_{\{X(t)=Y(t)=0\}}\, d\left(\left(X^-(t) - Y^-(t)\right)^+\right). \tag{6.3.162}$$

Using (6.3.159), (6.3.160) and (6.3.162), we may cast (6.3.158) in the following form

$$\begin{aligned}
\mathbb{1}_{\{V(t)=0\}}\, dV^-(t) \quad &= \quad \mathbb{1}_{\{X(t)>0\}}\mathbb{1}_{\{Y(t)=0\}}\, dY^-(t) + \mathbb{1}_{\{Y(t)>0\}}\mathbb{1}_{\{X(t)=0\}}\, dX^-(t) \\
&\quad + \mathbb{1}_{\{X(t)=Y(t)=0\}}\, dY^-(t) + \mathbb{1}_{\{X(t)=Y(t)=0\}}\, d\left(\left(X^-(t) - Y^-(t)\right)^+\right) \\
&= \quad \mathbb{1}_{\{X(t)>0\}}\mathbb{1}_{\{Y(t)=0\}}\, dY^-(t) + \mathbb{1}_{\{Y(t)>0\}}\mathbb{1}_{\{X(t)=0\}}\, dX^-(t) \\
&\quad + \mathbb{1}_{\{X(t)=0\}}\mathbb{1}_{\{Y(t)=0\}}\, dY^-(t) + \mathbb{1}_{\{X(t)=Y(t)=0\}}\, d\left(\left(X^-(t) - Y^-(t)\right)^+\right) \\
&= \quad \left(\mathbb{1}_{\{X(t)>0\}} + \mathbb{1}_{\{X(t)=0\}}\right)\mathbb{1}_{\{Y(t)=0\}}\, dY^-(t) + \mathbb{1}_{\{Y(t)>0\}}\mathbb{1}_{\{X(t)=0\}}\, dX^-(t) \\
&\quad + \mathbb{1}_{\{X(t)=Y(t)=0\}}\, d\left(\left(X^-(t) - Y^-(t)\right)^+\right) \\
&= \quad \mathbb{1}_{\{X(t)\geq 0\}}\mathbb{1}_{\{Y(t)=0\}}\, dY^-(t) + \mathbb{1}_{\{Y(t)>0\}}\mathbb{1}_{\{X(t)=0\}}\, dX^-(t) \\
&\quad + \mathbb{1}_{\{X(t)=Y(t)=0\}}\, d\left(\left(X^-(t) - Y^-(t)\right)^+\right).
\end{aligned} \tag{6.3.163}$$

Now, let X be a continuous semimartingale and let $V \equiv -X^+ = \min(-X, 0) = (-X) \wedge 0$, so that $X \equiv -X$ and $Y \equiv 0$, then we have

$$\mathbb{1}_{\{V(t)=0\}}\, dV^-(t) \quad = \quad \mathbb{1}_{\{-X^+(t)=0\}}\, d\big(-X^+\big)^-(t) \quad = \quad \mathbb{1}_{\{X^+(t)=0\}}\, dX^+(t).$$

Hence, by applying (6.3.163), we obtain

$$\mathbb{1}_{\{X^+(t)=0\}}\, dX^+(t) \quad = \quad \mathbb{1}_{\{0>0\}}\mathbb{1}_{\{-X(t)=0\}}\, d\big(-X\big)^-(t) + \mathbb{1}_{\{X(t)=0=0\}}\, d\big(\big(-X\big)^-(t)\big)^+.$$

Therefore, for any continuous semimartingale X, we have the result (6.3.157)

$$\mathbb{1}_{\{X^+(t)=0\}}\, dX^+(t) \quad = \quad \mathbb{1}_{\{X(t)=0\}}\, dX^+(t),$$

and from this the same conclusions may be drawn.

Corollary 6.3.21. *Let $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ be a **continuous semimartingale** and $a \in \mathbb{R}$. Then*

$$\Lambda_X^a(t) \quad = \quad \Lambda_{X-a}(t) \quad = \quad \int_0^t \mathbb{1}_{\{X(s)=a\}}\, d\big(X(s)-a\big)^+, \tag{6.3.164}$$

$$\Lambda_X^a(t) \quad = \quad \Lambda_{X-a}(t) \quad = \quad \int_0^t \mathbb{1}_{\{X(s)\le a\}}\, d\big(X(s)-a\big)^+. \tag{6.3.165}$$

Therefore, from the above corollary and the results above, we have $\Lambda_X^a(t) = \Lambda_{X-a}(t) = \Lambda_{(X-a)^+}(t)$, for all $t \in [0,\infty)$. Thus, we have the following

$$\begin{aligned}
\Lambda_{(X-a)^+} &= \int_0^t \mathbb{1}_{\{(X(s)-a)^+=0\}}\, d\big(X(s)-a\big)^+ \\
&= \int_0^t \mathbb{1}_{\{X(s)\le a\}}\, d\big(X(s)-a\big)^+ \\
&= \int_0^t \mathbb{1}_{\{X(s)=a\}}\, d\big(X(s)-a\big)^+.
\end{aligned} \tag{6.3.166}$$

The following theorem demonstrates that convex functions of continuous semimartingales are themselves continuous semimartingales, and the requisite decomposition is provided. Essentially, the generalised formula provided below extends the version of Itô's formula to a function f that is the difference of two convex functions.

Theorem 6.3.22 (A Generalised Itô Formula for Convex Functions: Itô-Tanaka Formula **[Karatzas & Shreve (1991, §3.7, Theorem 3.7.1, pp. 218–219)]**). *Let $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ be a **continuous semimartingale**. Then there exists a **semimartingale local time** for X at the level a, $\Lambda_X^a = \big\{\Lambda_X^a(t), \mathcal{F}_t, t \in [0,\infty)\big\}$, such that the following hold:*

(*i*) (**[Karatzas & Shreve (1991, §3.7, Statement (ii) of Theorem 3.7.1, p. 218)]**). *For every $a \in \mathbb{R}$, the mapping $t \mapsto \Lambda_X^a(t,\omega)$ is continuous and nondecreasing with $\Lambda_X^a(0,\omega) = 0$ where $\omega \in \Omega$, and*

$$\int_0^\infty \mathbb{1}_{\mathbb{R}\setminus\{a\}}\big(X(t)\big)\, d\Lambda_X^a(t) \quad = \quad 0, \qquad a.s. \tag{6.3.167}$$

(*ii*) (Occupation Time Density Formula **[Karatzas & Shreve (1991, §3.7, Statement (iii) of Theorem 3.7.1, p. 218), Revuz & Yor (1999, Chap. VI, §1, Corollary 1.6, p. 224)]**). *For every positive Borel-measurable function $k : \mathbb{R} \to [0,\infty)$, the identity (known as the **occupation time density formula**) holds*

$$\int_0^t k\big(X(s)\big)\, d\langle X\rangle_s \quad = \quad 2\int_{-\infty}^\infty k(a)\Lambda_X^a(t)\, da, \qquad t \in [0,\infty), \qquad a.s. \tag{6.3.168}$$

(*iii*) ([**Karatzas & Shreve (1991, §3.7, Statement (iv) of Theorem 3.7.1, p. 218)**]). *For $\omega \in \Omega$, the limits*

$$\Lambda_X(a+,t,\omega) = \Lambda_X^{a+}(t,\omega) \triangleq \lim_{b\downarrow a} \Lambda_X^b(t,\omega) = \lim_{b\to a+} \Lambda_X^b(t,\omega) = \Lambda_X^a(t,\omega), \tag{6.3.169}$$

$$\Lambda_X(a-,t,\omega) = \Lambda_X^{a-}(t,\omega) \triangleq \lim_{b\uparrow a} \Lambda_X^b(t,\omega) = \lim_{b\to a-} \Lambda_X^b(t,\omega), \tag{6.3.170}$$

exist for all $(t,a) \in [0,\infty) \times \mathbb{R}$. We express this property by saying that Λ is a.s. jointly continuous in t and càdlàg in a.

(*iv*) (Itô-Tanaka Formula [**Karatzas & Shreve (1991, §3.7, Statement (v) of Theorem 3.7.1, pp. 218–219), Revuz & Yor (1999, Chap. VI, §1, Theorem 1.5, p. 223)**]). *For every convex function (or a linear combination of convex functions) $f : \mathbb{R} \to \mathbb{R}$ (which can be expressed as the difference of two convex functions), we get the* ***generalised change of variable formula****, also known as the* ***Meyer-Itô formula*** *or the* ***Itô-Tanaka formula****,*

$$f\big(X(t)\big) = f\big(X(0)\big) + \int_0^t D^- f\big(X(s)\big)\, dM(s) + \int_0^t D^- f\big(X(s)\big)\, dV(s) + \int_{-\infty}^{\infty} \Lambda_X^a(t)\, \mu(da) \tag{6.3.171}$$

$$= f\big(X(0)\big) + \int_0^t D^- f\big(X(s)\big)\, dX(s) + \int_{-\infty}^{\infty} \Lambda_X^a(t)\, \mu(da), \tag{6.3.172}$$

for $t \in [0,\infty)$, a.s., where $D^- f$ denotes the left-hand derivative and μ denotes the second derivative measure (i.e., the second distributional derivative).

Proof. See Karatzas & Shreve (1991, §3.7, Proof of Theorem 3.7.1, pp. 223–225), or Revuz & Yor (1999, Chap. VI, §1, Proof of Theorem 1.5, p. 224) and Revuz & Yor (1999, Chap. VI, §1, Proof of Corollary 1.6, p. 224). ■

Statement (ii) in Theorem 6.3.22 above is also offered in Protter (2004) in the following corollary which gives an interpretation of the semimartingale local time as an occupation density.

Corollary 6.3.23 ([**Protter (2004, Chap. IV, §7, Corollary 1 of Theorem 70, p. 216)**]). *Let $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ be a* ***continuous semimartingale*** *with local time $\Lambda_X^a = \big\{\Lambda_X^a(t), \mathcal{F}_t, t \in [0,\infty)\big\}$, with $a \in \mathbb{R}$. Let $k : \mathbb{R} \to [0,\infty)$ be a bounded Borel-measurable function. Then, a.s.,*

$$2\int_{-\infty}^{\infty} \Lambda_X^a(t) k(a)\, da = \int_0^t k\big(X(s)\big)\, d\langle X\rangle_s. \tag{6.3.173}$$

Proof. See Protter (2004, Chap. IV, §7, Proof of Corollary 1 of Theorem 70, p. 216). Let k be a twice continuously differentiable, convex function. Comparing (6.3.172) with Itô's formula shows that

$$\int_{-\infty}^{\infty} \Lambda_X^a(t)\, \mu(da) = \frac{1}{2}\int_0^t f''\big(X(s)\big)\, d\langle X\rangle_s, \tag{6.3.174}$$

$$\int_{-\infty}^{\infty} \Lambda_X^a(t) f''(a)\, da = \frac{1}{2}\int_0^t f''\big(X(s)\big)\, d\langle X\rangle_s, \tag{6.3.175}$$

$$2\int_{-\infty}^{\infty} \Lambda_X^a(t) f''(a)\, da = \int_0^t f''\big(X(s)\big)\, d\langle X\rangle_s, \tag{6.3.176}$$

where $\mu(da) = f''(a)\, da$ for a twice continuously differentiable function. Since the above holds for any continuous and positive function f'', a monotone class argument shows that it must hold for any bounded, Borel-measurable function k. ■

Notice that from (6.3.167) we also have

$$\int_0^{\infty} \mathbb{1}_{(-\infty,a)}\big(X(t)\big)\, d\Lambda_X^a(t) = 0, \quad \text{a.s.}, \tag{6.3.177}$$

$$\int_0^{\infty} \mathbb{1}_{(a,\infty)}\big(X(t)\big)\, d\Lambda_X^a(t) = 0, \quad \text{a.s.}, \tag{6.3.178}$$

$$\mathbb{1}_{\mathbb{R}\setminus\{a\}}\big(X(t)\big)\, d\Lambda_X^a(t) = 0, \quad \text{a.s.} \tag{6.3.179}$$

In the following corollary, we record an important special case of Corollary 6.3.23, (6.3.175) or (6.3.168) of Theorem 6.3.22.

Corollary 6.3.24 ([**Protter (2004, Chap. IV, §7, Corollary 2 of Theorem 70, p. 216)**]). *Let* $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ *be a* ***continuous semimartingale*** *with local time* $\Lambda_X^a = \big\{\Lambda_X^a(t), \mathcal{F}_t, t \in [0,\infty)\big\}$, *with* $a \in \mathbb{R}$. *Then, a.s.,*

$$\langle M \rangle_t \quad = \quad \langle X \rangle_t \quad = \quad 2\int_{-\infty}^{\infty} \Lambda_X^a(t)\, da, \qquad t \in [0,\infty). \tag{6.3.180}$$

Proof. Refer to Karatzas & Shreve (1991, §3.7, Proof of Theorem 3.7.1, pp. 223–225). Comparison of (6.3.172) with the result from the usual Itô's formula reveals that

$$\int_{-\infty}^{\infty} \Lambda_X^a(t) f''(a)\, da \quad = \quad \frac{1}{2}\int_0^t f''\big(X(s)\big)\, d\langle X \rangle_s.$$

Now, consider (6.3.172) in the special case of the following twice continuously differentiable, convex function $f(x) = x^2$. Then $\mu(da) \equiv f''(a)\, da = 2\, da$, and we get

$$2\int_{-\infty}^{\infty} \Lambda_X^a(t)\, da \quad = \quad \int_0^t d\langle X \rangle_s.$$

Thus, we get the desired result

$$2\int_{-\infty}^{\infty} \Lambda_X^a(t)\, da \quad = \quad \langle X \rangle_t.$$

■

Corollary 6.3.25 (Extension of the Occupation Times Formula [**Revuz & Yor (1999, Chap. VI, §1, Exercise 1.15, p. 232)**]). *Let* $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ *be a* ***continuous semimartingale*** *with local time* $\Lambda_X^a = \big\{\Lambda_X^a(t), \mathcal{F}_t, t \in [0,\infty)\big\}$, *with* $a \in \mathbb{R}$. *Then, a.s., for every positive Borel function* $k : [0,\infty) \to [0,\infty)$

$$\int_0^t k\big(X(s)\big)\, d\langle X \rangle_s \quad = \quad 2\int_{-\infty}^{\infty} da \int_0^t k(a)\, d\Lambda_X^a(s) \quad = \quad 2\int_{-\infty}^{\infty}\int_0^t k(a)\, d\Lambda_X^a(s)\, da, \tag{6.3.181}$$

for $t \in [0,\infty)$.

Proof. Also the proof of statement (ii) of Theorem 6.3.22.

Employing the result of Corollary 6.3.24, we have for any measurable function $h : [0,\infty) \to [0,\infty)$,

$$\int_0^{\infty} h(s)\, d\langle X \rangle_s \quad = \quad 2\int_0^{\infty} h(s) \int_{-\infty}^{\infty} d\Lambda_X^a(s)\, da \quad = \quad 2\int_{-\infty}^{\infty}\int_0^{\infty} h(s)\, d\Lambda_X^a(s)\, da, \qquad \text{a.s.}$$

Now, if $k : \mathbb{R} \to [0,\infty)$ is measurable, we may take $h(s) = \mathbb{1}_{[0,t]}(s) \cdot k\big(X(s)\big)$ and obtain a.s.

$$\begin{aligned}
&\int_0^t k\big(X(s)\big)\, d\langle X \rangle_s \\
&\qquad = \; 2\int_{-\infty}^{\infty}\int_0^{\infty} \mathbb{1}_{[0,t]}(s) \cdot k\big(X(s)\big)\, d\Lambda_X^a(s)\, da \\
&\qquad = \; 2\int_{-\infty}^{\infty}\int_0^{t} k\big(X(s)\big)\, d\Lambda_X^a(s)\, da \\
&\qquad = \; 2\int_{-\infty}^{\infty}\int_0^{t} k\big(X(s)\big)\Big(\mathbb{1}_{\{a\}}\big(X(s)\big) + \mathbb{1}_{\mathbb{R}\setminus\{a\}}\big(X(s)\big)\Big)\, d\Lambda_X^a(s)\, da \\
&\qquad = \; 2\int_{-\infty}^{\infty}\int_0^{t} k\big(X(s)\big)\mathbb{1}_{\{a\}}\big(X(s)\big)\, d\Lambda_X^a(s)\, da + 2\int_{-\infty}^{\infty}\int_0^{t} k\big(X(s)\big)\mathbb{1}_{\mathbb{R}\setminus\{a\}}\big(X(s)\big)\, d\Lambda_X^a(s)\, da,
\end{aligned} \tag{6.3.182}$$

which by (6.3.179), a consequence of (6.3.167) of Theorem 6.3.22, simplifies to

$$\begin{aligned}
\int_0^t k\big(X(s)\big)\, d\langle X\rangle_s &= 2\int_{-\infty}^{\infty}\int_0^t k\big(X(s)\big)\mathbb{1}_{\{a\}}\big(X(s)\big)\, d\Lambda_X^a(s)\, da \\
&= 2\int_{-\infty}^{\infty}\int_0^t k(a)\, d\Lambda_X^a(s)\, da \\
&= 2\int_{-\infty}^{\infty} k(a)\int_0^t d\Lambda_X^a(s)\, da \\
&= 2\int_{-\infty}^{\infty} k(a)\Lambda_X^a(t)\, da,
\end{aligned}$$

for all $t \in [0,\infty)$, since $\Lambda_X^a(0) = 0$. ■

Corollary 6.3.26 ([**Karatzas & Shreve (1991), Fernholz (2002)**]). *Let $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ be a* ***continuous semimartingale*** *with local time $\Lambda_X^a = \big\{\Lambda_X^a(t), \mathcal{F}_t, t \in [0,\infty)\big\}$, with $a \in \mathbb{R}$. Then, a.s.,*

$$\mathbb{1}_{\{a\}}\big(X(t)\big)\, d\Lambda_X^a(t) = d\Lambda_X^a(t). \tag{6.3.183}$$

Proof. From (6.3.179), we can write the left-hand side of (6.3.183) as follows

$$\begin{aligned}
\mathbb{1}_{\{a\}}\big(X(t)\big)\, d\Lambda_X^a(t) &= \mathbb{1}_{\{a\}}\big(X(t)\big)\, d\Lambda_X^a(t) + \mathbb{1}_{\mathbb{R}\setminus\{a\}}\big(X(t)\big)\, d\Lambda_X^a(t) \\
&= \Big(\mathbb{1}_{\{a\}}\big(X(t)\big) + \mathbb{1}_{\mathbb{R}\setminus\{a\}}\big(X(t)\big)\Big)\, d\Lambda_X^a(t) \\
&= d\Lambda_X^a(t).
\end{aligned}$$

Alternatively, in integral form, following from (6.3.167) of Theorem 6.3.22, we have

$$\begin{aligned}
\int_0^t \mathbb{1}_{\{a\}}\big(X(s)\big)\, d\Lambda_X^a(s) &= \int_0^t \mathbb{1}_{\{a\}}\big(X(s)\big)\, d\Lambda_X^a(s) + \int_0^t \mathbb{1}_{\mathbb{R}\setminus\{a\}}\big(X(s)\big)\, d\Lambda_X^a(s) \\
&= \int_0^t \Big(\mathbb{1}_{\{a\}}\big(X(s)\big) + \mathbb{1}_{\mathbb{R}\setminus\{a\}}\big(X(s)\big)\Big)\, d\Lambda_X^a(s) \\
&= \int_0^t d\Lambda_X^a(s) \\
&= \Lambda_X^a(t) - \Lambda_X^a(0) \\
&= \Lambda_X^a(t),
\end{aligned}$$

since $\Lambda_X^a(0) = 0$. ■

This implies, for example, that for one-dimensional Brownian motion W, L^a is a nonnegative random measure on $[0,\infty)$ that almost surely has support contained in the set $\{t : W(t) = a\}$, and hence is singular with respect to Lebesgue measure [Fernholz (2002)].

Corollary 6.3.27. *Let $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ be a* ***continuous semimartingale*** *with local time $\Lambda_X^a = \big\{\Lambda_X^a(t), \mathcal{F}_t, t \in [0,\infty)\big\}$, with $a \in \mathbb{R}$. Then, for every Borel set $B \in \mathcal{B}(\mathbb{R})$, we have for all $t \in [0,\infty)$, a.s.,*

$$\int_0^t \mathbb{1}_B\big(X(s)\big)\, d\langle X\rangle_s = 2\int_B \Lambda_X^a(t)\, da. \tag{6.3.184}$$

Proof. The result follows easily from (6.3.168) of Theorem 6.3.22, where we let $k(x) = \mathbb{1}_B(x)$,

$$\begin{aligned}
\int_0^t \mathbb{1}_B\big(X(s)\big)\, d\langle X\rangle_s &= 2\int_{-\infty}^{\infty} \mathbb{1}_B(a)\Lambda_X^a(t)\, da \\
&= 2\int_B \Lambda_X^a(t)\, da.
\end{aligned}$$

■

Corollary 6.3.28 ([**Karatzas & Shreve (1991, §3.7, Exercise 3.7.10, p. 225)**]). *Let $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ be a* ***continuous semimartingale*** *with local time $\Lambda_X^a = \big\{\Lambda_X^a(t), \mathcal{F}_t, t \in [0,\infty)\big\}$, with $a \in \mathbb{R}$. Then, a.s.,*

$$\int_0^t \mathbb{1}_{\{a\}}\big(X(s)\big)\, d\langle M\rangle_s \;=\; \int_0^t \mathbb{1}_{\{a\}}\big(X(s)\big)\, d\langle X\rangle_s \;=\; 0, \qquad a.s. \tag{6.3.185}$$

Proof. The result follows easily from (6.3.168) of Theorem 6.3.22, where we let $k(x) = \mathbb{1}_{\{a\}}(x)$,

$$\begin{aligned}\int_0^t \mathbb{1}_{\{a\}}\big(X(s)\big)\, d\langle X\rangle_s &= 2\int_{-\infty}^{\infty} \mathbb{1}_{\{a\}}(a)\Lambda_X^a(t)\, da \\ &= 2\int_{\{a\}} \Lambda_X^a(t)\, da \\ &= 0.\end{aligned}$$

■

Corollary 6.3.29 ([**Revuz & Yor (1999, Chap. VI, §1, Corollary 1.9, p. 227), Protter (2004, Chap. IV, §7, Corollary 3 of Theorem 75, p. 225)**]). *Let $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ be a* ***continuous semimartingale****. Then, for all $a \in \mathbb{R}$ and $t \in [0,\infty)$, we have a.s.,*

$$\Lambda_X^a(t) \;=\; \Lambda_X^{a+}(t) \;=\; \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{\{a\le X(s)<a+\varepsilon\}}\, d\langle X\rangle_s \;=\; \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{[a,a+\varepsilon)}\big(X(s)\big)\, d\langle X\rangle_s, \tag{6.3.186}$$

where

$$\Lambda_X^{a+}(t) \;=\; \Lambda_X(a+,t) \;\triangleq\; \lim_{b\downarrow a}\Lambda_X^b(t) \;=\; \lim_{b\to a+}\Lambda_X^b(t) \;\equiv\; \Lambda_X^a(t), \tag{6.3.187}$$

and,

$$\Lambda_X^{a-}(t) \;=\; \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{\{a-\varepsilon<X(s)\le a\}}\, d\langle X\rangle_s \;=\; \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{(a-\varepsilon,a]}\big(X(s)\big)\, d\langle X\rangle_s, \tag{6.3.188}$$

where sgn *is right-continuous and the left-hand derivative of $f_0(x) = |x|$ is replaced by the right-hand derivative,*

$$\Lambda_X^{a-}(t) \;=\; \Lambda_X(a-,t) \;\triangleq\; \lim_{b\uparrow a}\Lambda_X^b(t) \;=\; \lim_{b\to a-}\Lambda_X^b(t). \tag{6.3.189}$$

If $M = \{M(t), \mathcal{F}_t, t \in [0,\infty)\}$ is a ***continuous local martingale****, we have*

$$\Lambda_M^a(t) \;=\; \lim_{\varepsilon\downarrow 0}\frac{1}{4\varepsilon}\int_0^t \mathbb{1}_{\{|M(s)-a|\le\varepsilon\}}\, d\langle M\rangle_s \;=\; \lim_{\varepsilon\downarrow 0}\frac{1}{4\varepsilon}\int_0^t \mathbb{1}_{(a-\varepsilon,a+\varepsilon)}\big(M(s)\big)\, d\langle M\rangle_s. \tag{6.3.190}$$

Proof. This is a straightforward consequence of the occupation times formula and the right-continuity in a of $\Lambda_X^a(t)$. ■

Corollary 6.3.30 ([**Revuz & Yor (1999, Chap. VI, §1, Exercise 1.25, p. 234), Protter (2004, Chap. IV, §7, Corollary 3 of Theorem 75, p. 225)**]). *Let $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ be a* ***continuous semimartingale****. Then, for all $a \in \mathbb{R}$ and $t \in [0,\infty)$, if we take $\widetilde{\mathrm{sgn}}(x)$ to be 1 for $x > 0$, -1 for $x < 0$ and 0 for $x = 0$ (i.e., define the* sgn *function to be symmetric), then there is a unique increasing process $\widetilde{\Lambda}_X^a = \big\{\widetilde{\Lambda}_X^a(t), \mathcal{F}_t, t \in [0,\infty)\big\}$ (the symmetric local time) such that*

$$\big|X(t)-a\big| \;=\; \big|X(0)-a\big| + \int_0^t \widetilde{\mathrm{sgn}}\big(X(s)-a\big)\, dX(s) + 2\widetilde{\Lambda}_X^a(t). \tag{6.3.191}$$

A symmetrised result follows trivially, namely

$$\widetilde{\Lambda}_X^a(t) \;=\; \frac{\Lambda_X^{a+}(t)+\Lambda_X^{a-}(t)}{2} \;=\; \frac{\Lambda_X^a(t)+\Lambda_X^{a-}(t)}{2},$$

such that

$$\begin{aligned}\widetilde{\Lambda}_X^a(t) &= \frac{1}{2}\left(\lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{[a,a+\varepsilon)}\big(X(s)\big)\,d\langle X\rangle_s + \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{(a-\varepsilon,a]}\big(X(s)\big)\,d\langle X\rangle_s\right)\\ &= \lim_{\varepsilon\downarrow 0}\frac{1}{4\varepsilon}\left(\int_0^t \mathbb{1}_{[a,a+\varepsilon)}\big(X(s)\big)\,d\langle X\rangle_s + \int_0^t \mathbb{1}_{(a-\varepsilon,a]}\big(X(s)\big)\,d\langle X\rangle_s\right)\\ &= \lim_{\varepsilon\downarrow 0}\frac{1}{4\varepsilon}\int_0^t \Big(\mathbb{1}_{[a,a+\varepsilon)}\big(X(s)\big) + \mathbb{1}_{(a-\varepsilon,a]}\big(X(s)\big)\Big)\,d\langle X\rangle_s.\end{aligned}$$

Therefore, we have a.s.,

$$\widetilde{\Lambda}_X^a(t) = \lim_{\varepsilon\downarrow 0}\frac{1}{4\varepsilon}\int_0^t \mathbb{1}_{\{|X(s)-a|\le\varepsilon\}}\,d\langle X\rangle_s = \lim_{\varepsilon\downarrow 0}\frac{1}{4\varepsilon}\int_0^t \mathbb{1}_{(a-\varepsilon,a+\varepsilon)}\big(X(s)\big)\,d\langle X\rangle_s. \tag{6.3.192}$$

Corollary 6.3.29 is intuitively appealing as it justifies thinking of local time as an *occupation time density*.

Corollary 6.3.31 ([**Revuz & Yor (1999, Chap. VI, §1, Exercise 1.25, p. 234)**]). *Let $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ be a* ***continuous semimartingale****. Then there exists a* ***semimartingale local time*** *for X at the level a, $\Lambda_X^a = \big\{\Lambda_X^a(t), \mathcal{F}_t, t \in [0,\infty)\big\}$, such that for every convex function $f : \mathbb{R} \to \mathbb{R}$ (which can be expressed as the difference of two convex functions), we get the* ***generalised change of variable formula****,*

$$f\big(X(t)\big) = f\big(X(0)\big) + \int_0^t \frac{1}{2}\Big(D^+f\big(X(s)\big) + D^-f\big(X(s)\big)\Big)\,dX(s) + \int_{-\infty}^{\infty}\widetilde{\Lambda}_X^a(t)\,\mu(da), \tag{6.3.193}$$

where D^+f denotes the right-hand derivative, D^-f denotes the left-hand derivative and μ denotes the second derivative measure (i.e., the second distributional derivative).

Corollary 6.3.32 ([**Karatzas & Shreve (1991, §3.7, Proof of Theorem 3.7.1, pp. 223–225), Revuz & Yor (1999, Chap. VI, §1, Theorem 1.7, p. 225), Protter (2004, Chap. IV, §7, Corollary 1 of Theorem 75, p. 224)**]). *Let $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ be a* ***continuous semimartingale*** *and let $\Lambda_X^a = \big\{\Lambda_X^a(t), \mathcal{F}_t, t \in [0,\infty)\big\}$ be its* ***semimartingale local time*** *at the level a, for $a \in \mathbb{R}$. If X has the following decomposition $X = M + V$, then*

$$\Lambda_X^a(t) - \Lambda_X^{a-}(t) = \int_0^t \mathbb{1}_{\{X(s)=a\}}\,dV(s) = \int_0^t \mathbb{1}_{\{X(s)=a\}}\,dX(s). \tag{6.3.194}$$

Proof. Define the following stochastic integrals for $t \in [0,\infty)$ and $a \in \mathbb{R}$,

$$\widehat{M}^a(t) \triangleq \int_0^t \mathbb{1}_{(a,\infty)}\big(X(s)\big)\,dM(s) = \int_0^t \mathbb{1}_{\{X(s)>a\}}\,dM(s), \tag{6.3.195}$$

$$\widehat{V}^a(t) \triangleq \int_0^t \mathbb{1}_{(a,\infty)}\big(X(s)\big)\,dV(s) = \int_0^t \mathbb{1}_{\{X(s)>a\}}\,dV(s). \tag{6.3.196}$$

Moreover, by Lebesgue's theorem, we have for all $(t,a) \in [0,\infty)\times\mathbb{R}$ [see Karatzas & Shreve (1991, §3.7, Problem 3.7.6, p. 223)],

$$\widehat{M}^{a-}(t) \triangleq \lim_{b\uparrow a}\widehat{M}^b(t) = \lim_{b\uparrow a}\int_0^t \mathbb{1}_{\{X(s)>b\}}\,dM(s) = \int_0^t \mathbb{1}_{\{X(s)\ge a\}}\,dM(s), \tag{6.3.197}$$

$$\widehat{V}^{a-}(t) \triangleq \lim_{b\uparrow a}\widehat{V}^b(t) = \lim_{b\uparrow a}\int_0^t \mathbb{1}_{\{X(s)>b\}}\,dV(s) = \int_0^t \mathbb{1}_{\{X(s)\ge a\}}\,dV(s). \tag{6.3.198}$$

By the Tanaka-Meyer formula (6.3.119) and equations (6.3.195) and (6.3.196) above, we have

$$\begin{aligned}\Lambda_X^a(t) &= \big(X(t)-a\big)^+ - \big(X(0)-a\big)^+ - \int_0^t \mathbb{1}_{(a,\infty)}\big(X(s)\big)\,dX(s)\\ &= \big(X(t)-a\big)^+ - \big(X(0)-a\big)^+ - \int_0^t \mathbb{1}_{\{X(s)>a\}}\,dX(s)\\ &= \big(X(t)-a\big)^+ - \big(X(0)-a\big)^+ - \int_0^t \mathbb{1}_{\{X(s)>a\}}\,dM(s) - \int_0^t \mathbb{1}_{\{X(s)>a\}}\,dV(s)\\ &= \big(X(t)-a\big)^+ - \big(X(0)-a\big)^+ - \widehat{M}^a(t) - \widehat{V}^a(t).\end{aligned} \tag{6.3.199}$$

Thus, by (6.3.170) of Theorem 6.3.22, (6.3.197), (6.3.198) and (6.3.199), it follows that

$$\begin{aligned}
\Lambda_X^a(t) - \Lambda_X^{a-}(t) &= \Lambda_X^a(t) - \lim_{b\uparrow a} \Lambda_X^b(t) \\
&= \Big(\big(X(t)-a\big)^+ - \big(X(0)-a\big)^+ - \widehat{M}^a(t) - \widehat{V}^a(t) \Big) - \lim_{b\uparrow a} \Big(\big(X(t)-b\big)^+ - \big(X(0)-b\big)^+ - \widehat{M}^b(t) - \widehat{V}^b(t) \Big) \\
&= \Big(\big(X(t)-a\big)^+ - \big(X(0)-a\big)^+ - \widehat{M}^a(t) - \widehat{V}^a(t) \Big) - \Big(\big(X(t)-a\big)^+ - \big(X(0)-a\big)^+ - \lim_{b\uparrow a} \widehat{M}^b(t) - \lim_{b\uparrow a} \widehat{V}^b(t) \Big) \\
&= \Big(\big(X(t)-a\big)^+ - \big(X(0)-a\big)^+ - \widehat{M}^a(t) - \widehat{V}^a(t) \Big) - \Big(\big(X(t)-a\big)^+ - \big(X(0)-a\big)^+ - \widehat{M}^{a-}(t) - \widehat{V}^{a-}(t) \Big) \\
&= \widehat{M}^{a-}(t) - \widehat{M}^a(t) + \widehat{V}^{a-}(t) - \widehat{V}^a(t) \\
&= \int_0^t \mathbb{1}_{\{X(s)\geq a\}}\, dM(s) - \int_0^t \mathbb{1}_{\{X(s)>a\}}\, dM(s) + \int_0^t \mathbb{1}_{\{X(s)\geq a\}}\, dV(s) - \int_0^t \mathbb{1}_{\{X(s)>a\}}\, dV(s) \\
&= \int_0^t \mathbb{1}_{\{X(s)=a\}}\, dM(s) + \int_0^t \mathbb{1}_{\{X(s)=a\}}\, dV(s) \\
&= \int_0^t \mathbb{1}_{\{X(s)=a\}}\, dX(s).
\end{aligned}$$

Finally, by (6.3.185) of Corollary 6.3.28, we have

$$\int_0^t \mathbb{1}_{\{a\}}\big(X(s)\big)\, d\langle X\rangle_s = \int_0^t \mathbb{1}_{\{X(s)=a\}}\, d\langle X\rangle_s = \int_0^t \mathbb{1}_{\{X(s)=a\}}\, d\langle M\rangle_s = 0,$$

so that

$$\int_0^t \mathbb{1}_{\{X(s)=a\}}\, dM(s) = 0.$$

Therefore, we obtain the desired result

$$\Lambda_X^a(t) - \Lambda_X^{a-}(t) = \int_0^t \mathbb{1}_{\{X(s)=a\}}\, dV(s).$$

In the same way

$$\widehat{M}^{a+}(t) \triangleq \lim_{b\downarrow a} \widehat{M}^b(t) = \lim_{b\downarrow a} \int_0^t \mathbb{1}_{\{X(s)>b\}}\, dM(s) = \int_0^t \mathbb{1}_{\{X(s)>a\}}\, dM(s) = \widehat{M}^a(t), \tag{6.3.200}$$

$$\widehat{V}^{a+}(t) \triangleq \lim_{b\downarrow a} \widehat{V}^b(t) = \lim_{b\downarrow a} \int_0^t \mathbb{1}_{\{X(s)>b\}}\, dV(s) = \int_0^t \mathbb{1}_{\{X(s)>a\}}\, dV(s) = \widehat{V}^a(t), \tag{6.3.201}$$

so that $\Lambda_X^a(t) = \Lambda_X^{a+}(t)$. ■

Corollary 6.3.33. *Let $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ be a **continuous semimartingale** and $a \in \mathbb{R}$. Then*

$$\Lambda_X^{a-}(t) = \int_0^t \mathbb{1}_{\{X(s)=a\}}\, d\big(X(s)-a\big)^-. \tag{6.3.202}$$

Proof. By virtue of Corollary 6.3.32 and Corollary 6.3.21, we get

$$\begin{aligned}
\Lambda_X^{a-}(t) &= \Lambda_X^a(t) - \int_0^t \mathbb{1}_{\{X(s)=a\}}\, dX(s) \\
&= \int_0^t \mathbb{1}_{\{X(s)=a\}}\, d\big(X(s)-a\big)^+ - \int_0^t \mathbb{1}_{\{X(s)=a\}}\, dX(s) \\
&= \int_0^t \mathbb{1}_{\{X(s)=a\}} \Big(d\big(X(s)-a\big)^+ - dX(s) \Big) \\
&= \int_0^t \mathbb{1}_{\{X(s)=a\}}\, d\Big(\big(X(s)-a\big)^+ - X(s) \Big),
\end{aligned}$$

hence using the fact that $(x-a)^+ - (x-a)^- = x-a$, we obtain the required result

$$\Lambda_X^{a-}(t) = \int_0^t \mathbb{1}_{\{X(s)=a\}}\, d\big(X(s)-a\big)^-.$$

■

Therefore, from the above corollary and the results above, it follows that $\Lambda_{(X-a)^-}(t) = \Lambda_X^{a-}(t)$, for all $t \in [0,\infty)$ and the following formula holds

$$\Lambda_{(X-a)^-}(t) = \int_0^t \mathbb{1}_{\{X(s)=a\}}\, d\big(X(s)-a\big)^-. \tag{6.3.203}$$

Setting $a=0$ in (6.3.202), we have for any continuous semimartingale X,

$$\Lambda_X^{0-}(t) = \int_0^t \mathbb{1}_{\{X(s)=0\}}\, dX^-(s),$$

and since $\Lambda_{X^-}(t) = \Lambda_X^{0-}(t)$, for all $t \in [0,\infty)$, we subsequently deduce for any continuous semimartingale X,

$$\Lambda_{X^-}(t) = \int_0^t \mathbb{1}_{\{X(s)=0\}}\, dX^-(s). \tag{6.3.204}$$

Alternatively, we can show (6.3.204) by considering the maximum of two continuous semimartingales as was done earlier for (6.3.150). First notice from (6.3.146), we have for the nonnegative continuous semimartingale X^-

$$\begin{aligned}\Lambda_{X^-}(t) &= \int_0^t \mathbb{1}_{\{X^-(s)=0\}}\, dX^-(s) \\ &= \int_0^t \mathbb{1}_{\{X(s)>0\}}\, dX^-(s).\end{aligned} \tag{6.3.205}$$

Similarly, for $a \in \mathbb{R}$, we have the following

$$\begin{aligned}\Lambda_{(X-a)^-}(t) &= \int_0^t \mathbb{1}_{\{(X(s)-a)^-=0\}}\, d\big(X(s)-a\big)^- \\ &= \int_0^t \mathbb{1}_{\{X(s)>a\}}\, d\big(X(s)-a\big)^-.\end{aligned}$$

Let X be a continuous semimartingale and let $Z \equiv X^- = \max(-X,0) = (-X) \vee 0$, so that $X \equiv -X$ and $Y \equiv 0$, then we have

$$\mathbb{1}_{\{Z(t)=0\}}\, dZ^+(t) = \mathbb{1}_{\{X^-(t)=0\}}\, d\big(X^-\big)^+(t) = \mathbb{1}_{\{X^-(t)=0\}}\, dX^-(t).$$

Hence, by applying (6.3.156), we obtain

$$\begin{aligned}\mathbb{1}_{\{X^-(t)=0\}}\, dX^-(t) &= \mathbb{1}_{\{0<0\}}\mathbb{1}_{\{-X(t)=0\}}\, d\big(-X\big)^+(t) + \mathbb{1}_{\{-X(t)=0=0\}}\, d\big(\big(-X\big)^+(t)\big)^+ \\ &= \mathbb{1}_{\{X(t)=0\}}\, d\big(-X\big)^+(t).\end{aligned}$$

Therefore, since $\big(-X\big)^+ = X^-$, we have for any continuous semimartingale X,

$$\mathbb{1}_{\{X^-(t)=0\}}\, dX^-(t) = \mathbb{1}_{\{X(t)=0\}}\, dX^-(t). \tag{6.3.206}$$

Hence, (6.3.205) in conjunction with (6.3.206) yields (6.3.204)

$$\Lambda_{X^-}(t) = \int_0^t \mathbb{1}_{\{X^-(s)=0\}}\, dX^-(s) = \int_0^t \mathbb{1}_{\{X(s)=0\}}\, dX^-(s).$$

Alternatively considering the minimum process we let X be a continuous semimartingale and let $V \equiv -X^- = \min(X,0) = X \wedge 0$, so that $X \equiv X$ and $Y \equiv 0$, then we have

$$\mathbb{1}_{\{V(t)=0\}}\, dV^-(t) = \mathbb{1}_{\{-X^-(t)=0\}}\, d\big(-X^-\big)^-(t) = \mathbb{1}_{\{X^-(t)=0\}}\, dX^-(t).$$

Hence, by applying (6.3.163), we obtain

$$\mathbb{1}_{\{X^-(t)=0\}}\, dX^-(t) \quad = \quad \mathbb{1}_{\{0>0\}}\mathbb{1}_{\{X(t)=0\}}\, dX^-(t) + \mathbb{1}_{\{X(t)=0=0\}}\, d\big(X^-(t)\big)^+.$$

Therefore, for any continuous semimartingale X, we have the result (6.3.206),

$$\mathbb{1}_{\{X^-(t)=0\}}\, dX^-(t) \quad = \quad \mathbb{1}_{\{X(t)=0\}}\, dX^-(t).$$

Thus, to summarise we have the following important identities that will be employed hereafter. For any continuous semimartingale X, the following formulae hold

$$\Lambda_{(X-a)^+}(t) \quad = \quad \Lambda_X^{a+}(t) \quad = \quad \int_0^t \mathbb{1}_{\{X(s)=a\}}\, d\big(X(s)-a\big)^+, \tag{6.3.207}$$

$$\Lambda_{(X-a)^-}(t) \quad = \quad \Lambda_X^{a-}(t) \quad = \quad \int_0^t \mathbb{1}_{\{X(s)=a\}}\, d\big(X(s)-a\big)^-, \tag{6.3.208}$$

where $\Lambda_X^{a+}(t) = \Lambda_{(X-a)^+}(t) = \Lambda_{X-a}(t) = \Lambda_X^a(t)$ and $\Lambda_X^{a-}(t) = \Lambda_{(X-a)^-}(t)$ for any continuous semimartingale X and every $a \in \mathbb{R}$. For $a = 0$, we have the following identities for any continuous semimartingale X, where $\Lambda_X^{0+}(t) = \Lambda_{X^+}(t) = \Lambda_X(t)$ and $\Lambda_X^{0-}(t) = \Lambda_{X^-}(t)$ for any continuous semimartingale X,

$$\Lambda_{X^+}(t) \quad = \quad \Lambda_X^{0+}(t) \quad = \quad \int_0^t \mathbb{1}_{\{X(s)=0\}}\, dX^+(s), \tag{6.3.209}$$

$$\Lambda_{X^-}(t) \quad = \quad \Lambda_X^{0-}(t) \quad = \quad \int_0^t \mathbb{1}_{\{X(s)=0\}}\, dX^-(s). \tag{6.3.210}$$

The local time for a continuous semimartingale, $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$, can also be expressed as

$$\Lambda_X^a(t) \quad = \quad \frac{1}{2}\int_0^t \delta_a\big(X(s)\big)\, d\langle X\rangle_s \quad = \quad \frac{1}{2}\int_0^t \delta\big(X(s)-a\big)\, d\langle X\rangle_s. \tag{6.3.211}$$

Now, consider the nondecreasing, convex function $g_a(x) = (x-a)^+$, which is continuously differentiable on $\mathbb{R}\setminus\{a\}$. The left-hand derivative of g_a is $D^-g_a(x) = \mathbb{1}_{\{x>a\}} = \mathbb{1}_{(a,\infty)}(x)$, so that the second derivative of g_a in the distributional sense is $g_a''(x) = \delta_a(x) = \delta(x-a)$. Thus, by letting $k(x) = g_a''(x)$, (6.3.168) of Theorem 6.3.22 gives the intuitive interpretation of local time as (6.3.211). Then, by (6.3.119) of Theorem 6.3.19, we can write

$$\big(X(t)-a\big)^+ \quad = \quad \big(X(0)-a\big)^+ + \int_0^t \mathbb{1}_{(a,\infty)}\big(X(s)\big)\, dX(s) + \frac{1}{2}\int_0^t \delta\big(X(s)-a\big)\, d\langle X\rangle_s. \tag{6.3.212}$$

The above result also follows from (6.3.172) of Theorem 6.3.22 with $f(x) = g_a(x) = (x-a)^+$, where $D^-g_a(x) = \mathbb{1}_{(a,\infty)}(x)$ and $\mu(da) = g_a''(a)\, da$ and by setting $k(x) := g_a''(x) = \delta(x-a)$ in (6.3.168). Similarly, we could consider the nonincreasing, convex function $h_a(x) = (x-a)^-$, which is continuously differentiable on $\mathbb{R}\setminus\{a\}$. The left-hand derivative of h_a is $D^-h_a(x) = -\mathbb{1}_{\{x\le a\}} = -\mathbb{1}_{(-\infty,a]}(x)$, so that the second derivative of h_a in the distributional sense is $h_a''(x) = \delta_a(x) = \delta(x-a)$. Then, by (6.3.120) of Theorem 6.3.19, we can write

$$\big(X(t)-a\big)^- \quad = \quad \big(X(0)-a\big)^- - \int_0^t \mathbb{1}_{(-\infty,a]}\big(X(s)\big)\, dX(s) + \frac{1}{2}\int_0^t \delta\big(X(s)-a\big)\, d\langle X\rangle_s. \tag{6.3.213}$$

The above result also follows from (6.3.172) of Theorem 6.3.22 with $f(x) = h_a(x) = (x-a)^-$, where $D^-h_a(x) = -\mathbb{1}_{(-\infty,a]}(x)$ and $\mu(da) = h_a''(a)\, da$ and by setting $k(x) := h_a''(x) = \delta(x-a)$ in (6.3.168). Moreover, we could consider the convex function $f_a(x) = |x-a|$, which is continuously differentiable on $\mathbb{R}\setminus\{a\}$. The left-hand derivative of f_a is $D^-f_a(x) = \operatorname{sgn}(x-a)$, so that the second derivative of f_a in the distributional sense is $f_a''(x) = 2\delta(x-a)$. So that, once again, (6.3.168) of Theorem 6.3.22 gives the intuitive interpretation of local time as (6.3.211). Then, by (6.3.121) of Theorem 6.3.19, we can write

$$\big|X(t)-a\big| \quad = \quad \big|X(0)-a\big| + \int_0^t \operatorname{sgn}\big(X(s)-a\big)\, dX(s) + \int_0^t \delta\big(X(s)-a\big)\, d\langle X\rangle_s. \tag{6.3.214}$$

The above result also follows from (6.3.172) of Theorem 6.3.22 with $f(x) = f_a(x) = |x-a|$, where $D^-f_a(x) = \operatorname{sgn}(x-a)$ and $\mu(da) = f_a''(a)\, da$ and by setting $k(x) := f_a''(x) = 2\delta(x-a)$ in (6.3.168).

6.4 Fundamental Nondegeneracy Conditions

We shall soon see that in order to effectively use local times for the purposes of obtaining a decomposition of the ranked stock price processes, we shall require an assumption regarding the *level of nondegeneracy* of the original stock price processes. In particular, we shall assume that the stock price processes we consider, exhibit a certain level of nondegeneracy.

6.4.1 Pathwise Mutual Nondegeneracy

Definition 6.4.1 (Pathwise Mutual Nondegeneracy). *The processes $X_1, \ldots, X_n$ are* ***pathwise mutually nondegenerate*** *if for all $i, j, k = 1, 2, \ldots, n$, the following conditions hold:*

(*i*) *for all $i \neq j$,*

$$\text{Leb}\left\{t \in [0, \infty) \;\middle|\; X_i(t) = X_j(t)\right\} \quad = \quad 0, \qquad a.s.; \tag{6.4.1}$$

(*ii*) *for all $i < j < k$, the set*

$$\left\{t \in [0, \infty) \;\middle|\; X_i(t) = X_j(t) = X_k(t)\right\} \quad = \quad \varnothing, \qquad a.s. \tag{6.4.2}$$

- **Condition (i)** states that for any two indices $i < j$, the Lebesgue measure of the set $\{t \in [0, \infty) \mid X_i(t) = X_j(t)\}$ is zero, or the set $\{t \in [0, \infty) \mid X_i(t) = X_j(t)\}$ is Lebesgue-null, a.s., for all $i \neq j$. For Brownian motion paths, condition (i) follows from the fact that, with probability one, the (zero) set of one-dimensional Brownian motion has Lebesgue measure zero.

- **Condition (ii)** ensures that *triple points* or *"triple collisions"* do not exist, i.e., no three stock price processes ever coincide at the same time or not more than two stock price processes coincide at the same time, almost surely. Thus, we are only interested in stock markets which exhibit an absence of "triple collisions" for a system of stock price processes that interact through their ranks, and we assume that such "triple collisions" can never occur in the stock markets that are of concern to us. For Brownian motion paths, condition (ii) follows from the fact that, with probability one, two-dimensional Brownian motion never returns to the origin [see Karatzas & Shreve (1991)]. Ichiba & Karatzas (2010) study the collisions and examine the behaviour of Brownian particles and establish sufficient conditions for the absence, and for the presence, of "triple collisions" among n Brownian particles. Put differently, they study conditions on the drift and diffusion coefficients of Brownian motion, under which three Brownian particles moving on the real line can collide at the same time, and conditions under which such "triple collisions" can never occur. Refer to Ichiba & Karatzas (2010) for a detailed account of their study. The reason for imposing this condition will be made apparent when obtaining the requisite decomposition of the ranked stock price processes.

6.5 A Fundamental Property of Continuous Semimartingales

Stock price processes possess certain properties that facilitate the application of local times. In order to apply local times, we must restrict our consideration to a limited class of continuous semimartingales. Fortunately, this restricted class of continuous semimartingales is broad enough to include the market weights that are of interest to us. We shall consider one such property of continuous semimartingales (i.e., the stock price processes) that will enable us to do just that, namely *absolute continuity* and the corresponding restricted class of continuous semimartingales that we shall take into consideration are referred to as the *absolutely continuous semimartingales*.

6.5.1 Absolute Continuity and Absolutely Continuous Semimartingales

Definition 6.5.1 (Absolute Continuity). *Let $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ be a* ***continuous semimartingale*** *with canonical decomposition*

$$X(t) \quad = \quad X(0) + M_X(t) + V_X(t), \qquad t \in [0,T], \qquad a.s., \tag{6.5.1}$$

where $M_X = \{M_X(t), \mathcal{F}_t, t \in [0,\infty)\}$ is a continuous local martingale and $V_X = \{V_X(t), \mathcal{F}_t, t \in [0,\infty)\}$ is a continuous process of locally bounded variation. Then $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ is ***absolutely continuous*** *(i.e., X is an* ***absolutely continuous semimartingale****) if the random signed measure dV_X and $d\langle X\rangle = d\langle M_X\rangle$ are both almost surely absolutely continuous*[7] *with respect to Lebesgue measure on $[0,T]$, i.e., $dV_X \ll \text{Leb}$ a.s., and $d\langle X\rangle \ll \text{Leb}$ a.s.*

6.5.2 Properties of Functions of Absolutely Continuous Semimartingales

The next two lemmas show that twice continuously differentiable functions of *absolutely continuous semimartingales* are themselves *absolutely continuous semimartingales*.

Lemma 6.5.2 ([**Fernholz (2002)**]). *Suppose $X = \{X(t), \mathcal{F}_t, t \in [0,T]\}$ and $Y = \{Y(t), \mathcal{F}_t, t \in [0,T]\}$ are* ***absolutely continuous semimartingales****. Then, the random signed measure $d\langle X, Y\rangle$ is almost surely* ***absolutely continuous*** *with respect to Lebesgue measure on $[0,T]$ (i.e., $d\langle X, Y\rangle \ll \text{Leb}$ a.s.).*

Proof. Consider the quadratic variation processes $\langle X+Y\rangle$ and $\langle X-Y\rangle$, that are almost surely increasing on $[0,T]$, with

$$\langle X \pm Y\rangle_t \quad = \quad \langle X\rangle_t \pm 2\langle X, Y\rangle_t + \langle Y\rangle_t, \qquad t \in [0,T], \qquad \text{a.s.}$$

Let A be a Lebesgue-measurable subset of $[0,T]$, then, we have a.s.,

$$\int_A d\langle X \pm Y\rangle_t \quad = \quad \int_A d\langle X\rangle_t \pm 2\int_A d\langle X, Y\rangle_t + \int_A d\langle Y\rangle_t.$$

Since the quadratic variation processes $\langle X \pm Y\rangle_t$ are almost surely increasing on $[0,T]$, we have a.s.,

$$\int_A d\langle X \pm Y\rangle_t \quad \geq \quad 0.$$

So that, we have a.s.,

$$\int_A d\langle X\rangle_t \pm 2\int_A d\langle X, Y\rangle_t + \int_A d\langle Y\rangle_t \quad \geq \quad 0.$$

Therefore, we have a.s.,

$$2\int_A d\langle X, Y\rangle_t \quad \leq \quad \int_A d\langle X\rangle_t + \int_A d\langle Y\rangle_t,$$

and

$$-2\int_A d\langle X, Y\rangle_t \quad \leq \quad \int_A d\langle X\rangle_t + \int_A d\langle Y\rangle_t.$$

Hence, a.s.,

$$2\left|\int_A d\langle X, Y\rangle_t\right| \quad \leq \quad \int_A d\langle X\rangle_t + \int_A d\langle Y\rangle_t.$$

[7] Let μ and ν be two given measures on the same measurable space. Then the measure μ is said to be *absolutely continuous* with respect to ν (i.e., $\mu \ll \nu$), if $\mu(A) = 0$ for every measurable set A for which $\nu(A) = 0$.

Since X and Y are assumed to be absolutely continuous semimartingales and are thus absolutely continuous with respect to Lebesgue measure (i.e., $d\langle X\rangle \ll$ Leb a.s., and $d\langle Y\rangle \ll$ Leb a.s.), if A has Lebesgue measure zero, then the right-hand side of the above inequality is zero, and we deduce a.s. that

$$\int_A d\langle X,Y\rangle_t \quad = \quad 0.$$

Since this holds for any set A of Lebesgue measure zero, the random signed measure $d\langle X,Y\rangle$ is a.s. absolutely continuous with respect to Lebesgue measure on $[0,T]$. ■

Consequently, the above lemma implies that stock price processes, the market value process, and the market weight processes, are all *absolutely continuous semimartingales*. The next lemma shows that when Itô's formula is applied to absolutely continuous semimartingales, the result is an absolutely continuous semimartingale.

Lemma 6.5.3 ([**Fernholz (2002)**]). *Suppose $X_1,\ldots,X_n$ are **absolutely continuous semimartingales**, and f is a real-valued, twice continuously differentiable function defined on the range of $(X_1,\ldots,X_n)$ on $\mathbb{R}^n$. Then $f(X_1,\ldots,X_n)$ is an **absolutely continuous semimartingale**.*

Proof. Define the process $Y=\{Y(t),\mathcal{F}_t,t\in[0,\infty)\}$ by $Y(t)\triangleq f\big(X_1(t),\ldots,X_n(t)\big)$, then Itô's formula implies that Y is a continuous semimartingale and an application of Itô's formula to $Y(t)$ yields for $t\in[0,T]$, a.s.,

$$\begin{aligned} dY(t) &= \sum_{i=1}^n D_iY(t)\,dX_i(t)+\frac{1}{2}\sum_{i,j=1}^n D_{ij}Y(t)\,d\langle X_i,X_j\rangle_t \\ &= \sum_{i=1}^n D_if\big(X_1(t),\ldots,X_n(t)\big)\,dX_i(t)+\frac{1}{2}\sum_{i,j=1}^n D_{ij}f\big(X_1(t),\ldots,X_n(t)\big)\,d\langle X_i,X_j\rangle_t. \end{aligned} \tag{6.5.2}$$

Let

$$dY(t) \quad = \quad dM_Y(t)+dV_Y(t),$$

be the decomposition for the continuous semimartingale Y. Then for all $t\in[0,T]$, we have a.s.,

$$\begin{aligned} dV_Y(t) &= \sum_{i=1}^n D_iY(t)\,dV_{X_i}(t)+\frac{1}{2}\sum_{i,j=1}^n D_{ij}Y(t)\,d\langle X_i,X_j\rangle_t \\ &= \sum_{i=1}^n D_if\big(X_1(t),\ldots,X_n(t)\big)\,dV_{X_i}(t)+\frac{1}{2}\sum_{i,j=1}^n D_{ij}f\big(X_1(t),\ldots,X_n(t)\big)\,d\langle X_i,X_j\rangle_t. \end{aligned} \tag{6.5.3}$$

Since X_i, for all $i=1,2,\ldots,n$, are by assumption absolutely continuous semimartingales, with decomposition

$$X_i(t) \quad = \quad X_i(0)+M_{X_i}(t)+V_{X_i}(t), \qquad t\in[0,T], \qquad \text{a.s.},$$

for all $i=1,2,\ldots,n$, we have by Definition 6.5.1 that the random signed measure dV_{X_i} is almost surely absolutely continuous with respect to Lebesgue measure on $[0,T]$, i.e., $dV_{X_i}\ll$ Leb a.s. Thus, the first term on the right-hand side of (6.5.3) is absolutely continuous, since $D_if\big(X_1(t),\ldots,X_n(t)\big)$ is almost surely continuous in t, for $i=1,2,\ldots,n$, and dV_{X_i} is absolutely continuous. Since X_i and X_j are absolutely continuous semimartingales for all $i,j=1,2,\ldots,n$, Lemma 6.5.2 implies that the random signed measure $d\langle X_i,X_j\rangle$ is almost surely absolutely continuous with respect to Lebesgue measure on $[0,T]$ for all $i,j=1,2,\ldots,n$, i.e., $d\langle X_i,X_j\rangle\ll$ Leb a.s., for all $i,j=1,2,\ldots,n$. Thus, the second term on the right-hand side of (6.5.3) is also absolutely continuous, since $D_{ij}f\big(X_1(t),\ldots,X_n(t)\big)$ is almost surely continuous in t for $i,j=1,2,\ldots,n$, and $d\langle X_i,X_j\rangle$ is absolutely continuous. Hence, from (6.5.3), we can conclude that the random signed measure dV_Y is almost surely absolutely continuous with respect to Lebesgue measure on $[0,T]$, i.e., $dV_Y\ll$ Leb a.s. Now, for all

$t \in [0,T]$, we have a.s.

$$\begin{aligned} d\langle Y\rangle_t &= d\langle M_Y\rangle_t = d\left\langle \int_0^{\cdot} \sum_{i=1}^n D_i Y_s \, dM_{X_i,s}, \int_0^{\cdot} \sum_{j=1}^n D_j Y_s \, dM_{X_j,s} \right\rangle_t \\ &= \sum_{i,j=1}^n D_i Y(t) \, D_j Y(t) \, d\left\langle M_{X_i}, M_{X_j}\right\rangle_t \\ &= \sum_{i,j=1}^n D_i Y(t) \, D_j Y(t) \, d\left\langle X_i, X_j\right\rangle_t \\ &= \sum_{i,j=1}^n D_i f\big(X_1(t),\ldots,X_n(t)\big) \, D_j f\big(X_1(t),\ldots,X_n(t)\big) \, d\left\langle X_i, X_j\right\rangle_t. \end{aligned}$$

Thus, similar reasoning to that employed above establishes that the random signed measure $d\langle Y\rangle$ is almost surely absolutely continuous with respect to Lebesgue measure on $[0,T]$, i.e., $d\langle Y\rangle \ll \text{Leb}$ a.s. To conclude, since both the random signed measures dV_Y and $d\langle Y\rangle$ are almost surely absolutely continuous with respect to Lebesgue measure on $[0,T]$, the process $Y = f(X_1,\ldots,X_n)$ is an absolutely continuous semimartingale by Definition 6.5.1. ■

6.5.3 Local Time for Absolutely Continuous Semimartingales

The next two lemmas consider the local time for *absolutely continuous semimartingales*. The second lemma determines the relationship between the local times $\Lambda_{|X|}$ and Λ_X for an absolutely continuous semimartingale. The relationship will be a necessary requisite in what remains in this chapter.

Lemma 6.5.4 ([**Fernholz (2002)**]). *Let $X = \{X(t), \mathcal{F}_t, t \in [0,T]\}$ be an* ***absolutely continuous semimartingale*** *and let $Y = \{Y(t), \mathcal{F}_t, t \in [0,\infty)\}$ be a* ***continuous semimartingale****. Suppose that the set $\big\{t \in [0,T] \,\big|\, Y(t) = 0\big\}$ has Lebesgue measure zero, a.s. Then*

$$\int_0^t \mathbb{1}_{\{0\}}\big(Y(s)\big) \, dX(s) = 0, \qquad t \in [0,T], \qquad a.s. \tag{6.5.4}$$

Proof. The absolutely continuous semimartingale X has the following decomposition,

$$dX(t) = dM_X(t) + dV_X(t), \qquad t \in [0,T]. \tag{6.5.5}$$

Since X is an absolutely continuous semimartingale, it follows from Definition 6.5.1 that dV_X is almost surely absolutely continuous with respect to Lebesgue measure on $[0,T]$, i.e., $dV_X \ll \text{Leb}$ a.s., and $d\langle M_X\rangle = d\langle X\rangle$ is almost surely absolutely continuous with respect to Lebesgue measure on $[0,T]$, i.e., $d\langle M_X\rangle \ll \text{Leb}$ a.s., which when combined with the assumption that the Lebesgue measure of the set $\big\{t \in [0,T] \,\big|\, Y(t) = 0\big\}$ is zero, a.s., yields the following

$$\int_0^t \mathbb{1}_{\{0\}}\big(Y(s)\big) \, dV_X(s) = 0, \qquad t \in [0,T], \qquad \text{a.s.}, \tag{6.5.6}$$

and

$$\int_0^t \mathbb{1}_{\{0\}}\big(Y(s)\big) \, d\langle M_X\rangle_s = \int_0^t \mathbb{1}_{\{0\}}\big(Y(s)\big) \, d\langle X\rangle_s = 0, \qquad t \in [0,T], \qquad \text{a.s.} \tag{6.5.7}$$

Therefore, from (6.5.5), we have

$$\int_0^t \mathbb{1}_{\{0\}}\big(Y(s)\big) \, dX(s) = \int_0^t \mathbb{1}_{\{0\}}\big(Y(s)\big) \, dM_X(s) + \int_0^t \mathbb{1}_{\{0\}}\big(Y(s)\big) \, dV_X(s),$$

which, in conjunction with (6.5.6), reduces to

$$\int_0^t \mathbb{1}_{\{0\}}\big(Y(s)\big) \, dX(s) = \int_0^t \mathbb{1}_{\{0\}}\big(Y(s)\big) \, dM_X(s), \qquad t \in [0,T], \qquad \text{a.s.}$$

Define the following process $U = \{U(t), \mathcal{F}_t, t \in [0,T]\}$ by

$$U(t) \triangleq \int_0^t \mathbb{1}_{\{0\}}\big(Y(s)\big)\, dM_X(s), \qquad t \in [0,T],$$

which is a square-integrable continuous local martingale. Then, for all $t \in [0,T]$, we have a.s.

$$\langle U \rangle_t = \int_0^t \Big(\mathbb{1}_{\{0\}}\big(Y(s)\big)\Big)^2 d\langle M_X \rangle_s = \int_0^t \mathbb{1}_{\{0\}}\big(Y(s)\big)\, d\langle M_X \rangle_s = \int_0^t \mathbb{1}_{\{0\}}\big(Y(s)\big)\, d\langle X \rangle_s.$$

It thus follows from (6.5.7) that $\langle U \rangle_t = 0$ for $t \in [0,T]$, a.s. Appealing to Karatzas & Shreve (1991, §1.5, Problem 1.5.12 and Exercise 1.5.21, pp. 35–37), we deduce that $U(t) \equiv 0$ for $t \in [0,T]$, a.s. Hence,

$$\int_0^t \mathbb{1}_{\{0\}}\big(Y(s)\big)\, dM_X(s) = \int_0^t \mathbb{1}_{\{0\}}\big(Y(s)\big)\, dX(s) = 0, \qquad t \in [0,T], \qquad \text{a.s.}$$

■

Lemma 6.5.5 ([**Fernholz (2002)**]). *Let $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ be an* ***absolutely continuous semimartingale*** *such that the set $\big\{t \in [0,T] \,\big|\, X(t) = 0\big\}$ has Lebesgue measure zero, almost surely. Then*

$$\Lambda_{|X|}(t) = 2\Lambda_X(t), \qquad t \in [0,T], \qquad a.s. \tag{6.5.8}$$

Proof. By (6.3.145) of Definition 6.3.20, the *local time for the* (*absolutely*) *continuous semimartingale* X at the level 0 is given by

$$\Lambda_X(t) = \frac{1}{2}\left(\big|X(t)\big| - \big|X(0)\big| - \int_0^t \operatorname{sgn}\big(X(s)\big)\, dX(s)\right), \qquad t \in [0,T].$$

Therefore, for $t \in [0,T]$, we have

$$2\Lambda_X(t) = \big|X(t)\big| - \big|X(0)\big| - \int_0^t \operatorname{sgn}\big(X(s)\big)\, dX(s), \tag{6.5.9}$$

which implies that for $t \in [0,T]$, a.s.,

$$2\, d\Lambda_X(t) = d\big|X(t)\big| - \operatorname{sgn}\big(X(t)\big)\, dX(t). \tag{6.5.10}$$

Analogously, by (6.3.145) of Definition 6.3.20, the *local time for the continuous semimartingale* $|X|$ at the level 0 is given by

$$\Lambda_{|X|}(t) = \frac{1}{2}\left(\big|X(t)\big| - \big|X(0)\big| - \int_0^t \operatorname{sgn}\big(\big|X(s)\big|\big)\, d\big|X(s)\big|\right), \qquad t \in [0,T].$$

Therefore, for $t \in [0,T]$, we have

$$2\Lambda_{|X|}(t) = \big|X(t)\big| - \big|X(0)\big| - \int_0^t \operatorname{sgn}\big(\big|X(s)\big|\big)\, d\big|X(s)\big|. \tag{6.5.11}$$

Now, from (6.5.9), we have for $t \in [0,T]$,

$$\big|X(t)\big| - \big|X(0)\big| = 2\Lambda_X(t) + \int_0^t \operatorname{sgn}\big(X(s)\big)\, dX(s). \tag{6.5.12}$$

In addition, from (6.5.10), we have a.s., for $t \in [0,T]$,

$$d\big|X(t)\big| = 2\, d\Lambda_X(t) + \operatorname{sgn}\big(X(t)\big)\, dX(t). \tag{6.5.13}$$

Thus, substituting (6.5.12) and (6.5.13) into (6.5.11), we obtain a.s., for $t \in [0, T]$,

$$
\begin{aligned}
&2\Lambda_{|X|}(t) \\
&= 2\Lambda_X(t) + \int_0^t \operatorname{sgn}\big(X(s)\big)\, dX(s) - \int_0^t \operatorname{sgn}\big(|X(s)|\big) \Big(2\, d\Lambda_X(s) + \operatorname{sgn}\big(X(s)\big)\, dX(s)\Big) \\
&= 2\Lambda_X(t) + \int_0^t \operatorname{sgn}\big(X(s)\big)\, dX(s) - 2\int_0^t \operatorname{sgn}\big(|X(s)|\big)\, d\Lambda_X(s) - \int_0^t \operatorname{sgn}\big(|X(s)|\big) \operatorname{sgn}\big(X(s)\big)\, dX(s) \\
&= 2\Lambda_X(t) + \int_0^t \operatorname{sgn}\big(X(s)\big)\, dX(s) - 2\int_0^t \operatorname{sgn}\big(|X(s)|\big) \mathbb{1}_{\{0\}}\big(X(s)\big)\, d\Lambda_X(s) - \int_0^t \operatorname{sgn}\big(|X(s)|\big) \operatorname{sgn}\big(X(s)\big)\, dX(s),
\end{aligned}
$$

which follows from Corollary 6.3.26, by setting $a := 0$ in (6.3.183). Rearranging the last equation produces the following

$$
2\Lambda_{|X|}(t) = 2\Lambda_X(t) + \int_0^t \operatorname{sgn}\big(X(s)\big)\Big(1 - \operatorname{sgn}\big(|X(s)|\big)\Big)\, dX(s) - 2\int_0^t \operatorname{sgn}\big(|X(s)|\big) \mathbb{1}_{\{0\}}\big(X(s)\big)\, d\Lambda_X(s).
$$

From Definition 6.3.1, (6.3.8) of the left-continuous signum function, we have for $x \in \mathbb{R}$,

$$
\operatorname{sgn}(-x) \triangleq \begin{cases} 1 & \text{if} \quad x < 0, \\ -1 & \text{if} \quad x \geq 0, \end{cases} \tag{6.5.14}
$$

so that, the above equation together with (6.3.8) implies that

$$
\operatorname{sgn}(|x|) = \begin{cases} \operatorname{sgn}(x) & \text{if} \quad x > 0 \\ \operatorname{sgn}(-x) & \text{if} \quad x \leq 0 \end{cases} = \begin{cases} 1 & \text{if} \quad x \neq 0, \\ -1 & \text{if} \quad x = 0. \end{cases} \tag{6.5.15}
$$

Therefore, since $\operatorname{sgn}(0) = -1$, we have for $x \in \mathbb{R}$,

$$
\operatorname{sgn}(x)\big(1 - \operatorname{sgn}(|x|)\big) = \begin{cases} 0 & \text{if} \quad x \neq 0 \\ -2 & \text{if} \quad x = 0 \end{cases} = -2\,\mathbb{1}_{\{0\}}(x). \tag{6.5.16}
$$

Furthermore, for $x \in \mathbb{R}$, we have

$$
\operatorname{sgn}(|x|)\mathbb{1}_{\{0\}}(x) = \begin{cases} 0 & \text{if} \quad x \neq 0 \\ -1 & \text{if} \quad x = 0 \end{cases} = -\,\mathbb{1}_{\{0\}}(x). \tag{6.5.17}
$$

Thus, employing the results (6.5.16) and (6.5.17) above, we obtain

$$
\begin{aligned}
2\Lambda_{|X|}(t) &= 2\Lambda_X(t) - 2\int_0^t \mathbb{1}_{\{0\}}\big(X(s)\big)\, dX(s) + 2\int_0^t \mathbb{1}_{\{0\}}\big(X(s)\big)\, d\Lambda_X(s) \\
&= 2\Lambda_X(t) - 2\int_0^t \mathbb{1}_{\{0\}}\big(X(s)\big)\, dX(s) + 2\int_0^t d\Lambda_X(s) \qquad (6.5.18) \\
&= 2\Lambda_X(t) - 2\int_0^t \mathbb{1}_{\{0\}}\big(X(s)\big)\, dX(s) + 2\big(\Lambda_X(t) - \Lambda_X(0)\big) \\
&= 2\Lambda_X(t) - 2\int_0^t \mathbb{1}_{\{0\}}\big(X(s)\big)\, dX(s) + 2\Lambda_X(t) \qquad (6.5.19) \\
&= 4\Lambda_X(t) - 2\int_0^t \mathbb{1}_{\{0\}}\big(X(s)\big)\, dX(s) \\
&= 4\Lambda_X(t), \qquad (6.5.20)
\end{aligned}
$$

where (6.5.18) follows from (6.3.183) of Corollary 6.3.26 with $a = 0$, (6.5.19) follows from the fact that $\Lambda_X(0) \equiv 0$, and (6.5.20) follows from (6.5.4) of Lemma 6.5.4, since X is an absolutely continuous semimartingale and $X(s) = 0$ only on a set of Lebesgue measure zero, almost surely. Hence, we have the desired result $\Lambda_{|X|}(t) = 2\Lambda_X(t)$. ■

6.6 Representation of the Rank Processes

6.6.1 Decomposition of the Rank Processes of Pathwise Mutually Nondegenerate Absolutely Continuous Semimartingales

Recall, that the rank processes are constructed through the use of maximum and minimum operations. It thus makes sense to first consider the maximum and minimum processes. The lemma below is crucial for the proposition that follows, as it provides a representation of the dynamics of the maximum and minimum processes of *pathwise mutually nondegenerate absolutely continuous semimartingales*.

Lemma 6.6.1 ([**Fernholz (2002)**]). *Let $X = \{X(t), \mathcal{F}_t, t \in [0,T]\}$ and $Y = \{Y(t), \mathcal{F}_t, t \in [0,T]\}$ be* ***pathwise mutually nondegenerate absolutely continuous semimartingales****. Then, a.s, for $t \in [0,T]$,*

$$d\max\big(X(t),Y(t)\big) = \mathbb{1}_{(0,\infty)}\big(X(t)-Y(t)\big)\,dX(t) + \mathbb{1}_{(0,\infty)}\big(Y(t)-X(t)\big)\,dY(t) + d\Lambda_{X-Y}(t), \tag{6.6.1}$$

$$d\min\big(X(t),Y(t)\big) = \mathbb{1}_{(0,\infty)}\big(Y(t)-X(t)\big)\,dX(t) + \mathbb{1}_{(0,\infty)}\big(X(t)-Y(t)\big)\,dY(t) - d\Lambda_{X-Y}(t). \tag{6.6.2}$$

Proof. For $x, y \in \mathbb{R}$, we have $x \vee y = \frac{1}{2}\big(x+y+|x-y|\big)$, so that for $t \in [0,T]$ we have a.s.,

$$\begin{aligned} d\max\big(X(t),Y(t)\big) \equiv d\big(X(t)\vee Y(t)\big) &= \frac{1}{2}\,d\Big(X(t)+Y(t)+\big|X(t)-Y(t)\big|\Big) \\ &= \frac{1}{2}\Big(dX(t)+dY(t)+d\big|X(t)-Y(t)\big|\Big). \end{aligned} \tag{6.6.3}$$

For the continuous semimartingale $X-Y$, from (6.3.121) of Theorem 6.3.19, we have the following Tanaka-Meyer formula for $a=0$,

$$\big|X(t)-Y(t)\big| = \big|X(0)-Y(0)\big| + \int_0^t \operatorname{sgn}\big(X(s)-Y(s)\big)\,d\big(X(s)-Y(s)\big) + 2\Lambda_{X-Y}(t).$$

Therefore, from the above equation or from (6.5.13), we have the result for the continuous semimartingale $X-Y$,

$$d\big|X(t)-Y(t)\big| = \operatorname{sgn}\big(X(t)-Y(t)\big)\,d\big(X(t)-Y(t)\big) + 2\,d\Lambda_{X-Y}(t). \tag{6.6.4}$$

Substituting the above result into (6.6.3) yields,

$$\begin{aligned} d\max\big(X(t),Y(t)\big) &= \frac{1}{2}\Big(dX(t)+dY(t)+\operatorname{sgn}\big(X(t)-Y(t)\big)\,d\big(X(t)-Y(t)\big)+2\,d\Lambda_{X-Y}(t)\Big) \\ &= \frac{1}{2}\Big(dX(t)+dY(t)+\operatorname{sgn}\big(X(t)-Y(t)\big)\,dX(t)-\operatorname{sgn}\big(X(t)-Y(t)\big)\,dY(t)\Big)+d\Lambda_{X-Y}(t) \\ &= \frac{1}{2}\Big(\Big[1+\operatorname{sgn}\big(X(t)-Y(t)\big)\Big]\,dX(t)+\Big[1-\operatorname{sgn}\big(X(t)-Y(t)\big)\Big]\,dY(t)\Big)+d\Lambda_{X-Y}(t). \end{aligned} \tag{6.6.5}$$

From Definition 6.3.1, (6.3.8) of the left-continuous signum function, we have for $x \in \mathbb{R}$,

$$\operatorname{sgn}(x-y) = \begin{cases} 1 & \text{if} \quad x-y>0 \\ -1 & \text{if} \quad x-y\le 0 \end{cases} = \begin{cases} 1 & \text{if} \quad x>y, \\ -1 & \text{if} \quad x\le y. \end{cases} \tag{6.6.6}$$

Therefore, we obtain

$$1+\operatorname{sgn}(x-y) = \begin{cases} 2 & \text{if} \quad x-y>0 \\ 0 & \text{if} \quad x-y\le 0 \end{cases} \equiv 2\,\mathbb{1}_{(0,\infty)}(x-y), \tag{6.6.7}$$

and

$$1-\operatorname{sgn}(x-y) = \begin{cases} 0 & \text{if} \quad x-y>0 \\ 2 & \text{if} \quad x-y\le 0 \end{cases} = \begin{cases} 0 & \text{if} \quad y-x<0 \\ 2 & \text{if} \quad y-x\ge 0 \end{cases} \equiv 2\,\mathbb{1}_{[0,\infty)}(y-x). \tag{6.6.8}$$

Inserting (6.6.7) and (6.6.8) into (6.6.5), yields

$$\begin{aligned}
&d\max\big(X(t),Y(t)\big)\\
&\quad= \mathbb{1}_{(0,\infty)}\big(X(t)-Y(t)\big)\,dX(t)+\mathbb{1}_{[0,\infty)}\big(Y(t)-X(t)\big)\,dY(t)+d\Lambda_{X-Y}(t)\\
&\quad= \mathbb{1}_{(0,\infty)}\big(X(t)-Y(t)\big)\,dX(t)+\Big(\mathbb{1}_{\{0\}}\big(Y(t)-X(t)\big)+\mathbb{1}_{(0,\infty)}\big(Y(t)-X(t)\big)\Big)\,dY(t)+d\Lambda_{X-Y}(t)\\
&\quad= \mathbb{1}_{(0,\infty)}\big(X(t)-Y(t)\big)\,dX(t)+\mathbb{1}_{\{0\}}\big(Y(t)-X(t)\big)\,dY(t)+\mathbb{1}_{(0,\infty)}\big(Y(t)-X(t)\big)\,dY(t)+d\Lambda_{X-Y}(t)\\
&\quad= \mathbb{1}_{(0,\infty)}\big(X(t)-Y(t)\big)\,dX(t)+\mathbb{1}_{(0,\infty)}\big(Y(t)-X(t)\big)\,dY(t)+d\Lambda_{X-Y}(t),
\end{aligned}$$

where the last equality follows from (6.4.1) of Definition 6.4.1 in conjunction with (6.5.4) of Lemma 6.5.4. Since X and Y are assumed to be pathwise mutually nondegenerate, by condition (i) of Definition 6.4.1, we have $\text{Leb}\big\{t\in[0,T]\;\big|\;X(t)=Y(t)\big\}=0$, a.s. Thus, the set $\big\{t\in[0,T]\;\big|\;X(t)=Y(t)\big\}$ has Lebesgue measure zero, almost surely. Moreover, since X and Y are assumed to be absolutely continuous, this implies that $Y-X$ is a continuous semimartingale. Thus, this combined with the pathwise mutual nondegeneracy result implies that the set $\big\{t\in[0,T]\;\big|\;Y(t)-X(t)=0\big\}$ has Lebesgue measure zero, almost surely, it follows from Lemma 6.5.4 that $\mathbb{1}_{\{0\}}\big(Y(t)-X(t)\big)\,dY(t)=0$ for $t\in[0,T]$, a.s., since

$$\int_0^t \mathbb{1}_{\{0\}}\big(Y(s)-X(s)\big)\,dY(s) \;=\; 0, \qquad t\in[0,T], \qquad \text{a.s.},$$

which completes the first part of the proof. Now, for the second part of the proof, let $x,y\in\mathbb{R}$, then we have $x\wedge y=\frac{1}{2}\big(x+y-|x-y|\big)$, so that for $t\in[0,T]$ we have a.s.,

$$\begin{aligned}
d\min\big(X(t),Y(t)\big) \;\equiv\; d\big(X(t)\wedge Y(t)\big) &= \frac{1}{2}\,d\Big(X(t)+Y(t)-\big|X(t)-Y(t)\big|\Big)\\
&= \frac{1}{2}\Big(dX(t)+dY(t)-d\big|X(t)-Y(t)\big|\Big). \qquad (6.6.9)
\end{aligned}$$

Substituting (6.6.4) into (6.6.3) yields,

$$\begin{aligned}
d\min\big(X(t),Y(t)\big) &= \frac{1}{2}\Big(dX(t)+dY(t)-\text{sgn}\big(X(t)-Y(t)\big)\,d\big(X(t)-Y(t)\big)-2\,d\Lambda_{X-Y}(t)\Big)\\
&= \frac{1}{2}\Big(dX(t)+dY(t)-\text{sgn}\big(X(t)-Y(t)\big)\,dX(t)+\text{sgn}\big(X(t)-Y(t)\big)\,dY(t)\Big)-d\Lambda_{X-Y}(t)\\
&= \frac{1}{2}\Big(\Big[1-\text{sgn}\big(X(t)-Y(t)\big)\Big]\,dX(t)+\Big[1+\text{sgn}\big(X(t)-Y(t)\big)\Big]\,dY(t)\Big)+d\Lambda_{X-Y}(t).
\end{aligned} \qquad (6.6.10)$$

Inserting (6.6.7) and (6.6.8) into (6.6.10), yields

$$\begin{aligned}
&d\min\big(X(t),Y(t)\big)\\
&\quad= \mathbb{1}_{[0,\infty)}\big(Y(t)-X(t)\big)\,dX(t)+\mathbb{1}_{(0,\infty)}\big(X(t)-Y(t)\big)\,dY(t)+d\Lambda_{X-Y}(t)\\
&\quad= \Big(\mathbb{1}_{\{0\}}\big(Y(t)-X(t)\big)+\mathbb{1}_{(0,\infty)}\big(Y(t)-X(t)\big)\Big)\,dX(t)+\mathbb{1}_{(0,\infty)}\big(X(t)-Y(t)\big)\,dY(t)+d\Lambda_{X-Y}(t)\\
&\quad= \mathbb{1}_{\{0\}}\big(Y(t)-X(t)\big)\,dX(t)+\mathbb{1}_{(0,\infty)}\big(Y(t)-X(t)\big)\,dX(t)+\mathbb{1}_{(0,\infty)}\big(X(t)-Y(t)\big)\,dY(t)+d\Lambda_{X-Y}(t)\\
&\quad= \mathbb{1}_{(0,\infty)}\big(Y(t)-X(t)\big)\,dX(t)+\mathbb{1}_{(0,\infty)}\big(X(t)-Y(t)\big)\,dY(t)+d\Lambda_{X-Y}(t),
\end{aligned}$$

where the last equality follows from (6.4.1) of Definition 6.4.1 in conjunction with (6.5.4) of Lemma 6.5.4. Since X and Y are assumed to be pathwise mutually nondegenerate, by condition (i) of Definition 6.4.1, we have $\text{Leb}\big\{t\in[0,T]\;\big|\;X(t)=Y(t)\big\}=0$, a.s. Thus, the set $\big\{t\in[0,T]\;\big|\;X(t)=Y(t)\big\}$ has Lebesgue measure zero, almost surely. Moreover, since X and Y are assumed to be absolutely continuous, this implies that $Y-X$ is a continuous semimartingale. Thus, this combined with the pathwise mutual nondegeneracy result that the set $\big\{t\in[0,T]\;\big|\;Y(t)-X(t)=0\big\}$ has Lebesgue measure zero, almost surely, it follows from Lemma 6.5.4 that $\mathbb{1}_{\{0\}}\big(Y(t)-X(t)\big)\,dX(t)=0$ for $t\in[0,T]$, a.s., since

$$\int_0^t \mathbb{1}_{\{0\}}\big(Y(s)-X(s)\big)\,dX(s) \;=\; 0, \qquad t\in[0,T], \qquad \text{a.s.},$$

which completes the proof. Alternatively, we could use the fact that $\min(x,y) = -\max(-x,-y)$ and the result then follows from (6.6.1). ■

A crucial result that is required to prove the central theorem of this chapter, is provided in the next proposition, i.e., the fact that the ranked stock price processes $X_{(1)},\ldots,X_{(n)}$ cannot alone be represented purely by individual stocks or by portfolios of stocks. Thus, the following proposition is the result that we precisely need in order to prove the main crux of this chapter. The proposition shows that rank processes derived from pathwise mutually nondegenerate absolutely continuous semimartingales can be expressed in terms of the original processes, adjusted by local times.

Proposition 6.6.2 ([**Fernholz (2002)**]). *Let $X_1,\ldots,X_n$ be **pathwise mutually nondegenerate absolutely continuous semimartingales**, and for $t\in[0,T]$, let $\boldsymbol{p}_t$ be the random permutation defined in Definition 6.2.2 such that (6.2.3) and (6.2.4) hold. Then the **rank processes** $X_{(k)}$, for all $k=1,2,\ldots,n$, are **continuous semimartingales** such that a.s., for $t\in[0,T]$,*

$$dX_{(k)}(t) = \sum_{i=1}^{n} \mathbb{1}_{\{p_t(k)=i\}}\, dX_i(t) + \frac{1}{2}\, d\Lambda_{X_{(k)}-X_{(k+1)}}(t) - \frac{1}{2}\, d\Lambda_{X_{(k-1)}-X_{(k)}}(t) \tag{6.6.11}$$

$$= \sum_{i=1}^{n} \mathbb{1}_{\{i\}}\big(p_t(k)\big)\, dX_i(t) + \frac{1}{2}\, d\Lambda_{X_{(k)}-X_{(k+1)}}(t) - \frac{1}{2}\, d\Lambda_{X_{(k-1)}-X_{(k)}}(t), \tag{6.6.12}$$

where $\mathbb{1}_{\{p_t(k)=i\}} = \mathbb{1}_{\{i\}}\big(p_t(k)\big) = \mathbb{1}_{\{0\}}\big(p_t(k)-i\big)$, or in integral form

$$X_{(k)}(t) = X_{(k)}(0) + \sum_{i=1}^{n}\int_0^t \mathbb{1}_{\{i\}}\big(p_s(k)\big)\, dX_i(s) + \frac{1}{2}\,\Lambda_{X_{(k)}-X_{(k+1)}}(t) - \frac{1}{2}\,\Lambda_{X_{(k-1)}-X_{(k)}}(t). \tag{6.6.13}$$

By convention, we have $\Lambda_{X_{(0)}-X_{(1)}}(t) \triangleq 0$ and $\Lambda_{X_{(n)}-X_{(n+1)}}(t) \triangleq 0$, for all $t\in[0,T]$.

Proof. In this rather technical proof, it is convenient to explicitly show the dependence of all random variables and processes on $\omega\in\Omega$. Choose a subset $\Omega'\subset\Omega$ with $\mathbb{P}(\Omega')=1$, such that for $\omega\in\Omega'$, and $i,j,k\in\{1,2,\ldots,n\}$, the following conditions will apply:

(i) for $i=1,2,\ldots,n$, $X_i(t,\omega)$ is continuous in t;

(ii) for $k=1,2,\ldots,n-1$, and for $X_{(k-1)}$, $X_{(k)}$ and $X_{(k+1)}$ **continuous semimartingales**, and $t\in[0,T]$,

$$d\Lambda_{X_{(k)}-X_{(k+1)}}(t,\omega) = \mathbb{1}_{\{0\}}\big(X_{(k)}(t,\omega)-X_{(k+1)}(t,\omega)\big)\, d\Lambda_{X_{(k)}-X_{(k+1)}}(t,\omega), \tag{6.6.14}$$

$$d\Lambda_{X_{(k-1)}-X_{(k)}}(t,\omega) = \mathbb{1}_{\{0\}}\big(X_{(k-1)}(t,\omega)-X_{(k)}(t,\omega)\big)\, d\Lambda_{X_{(k-1)}-X_{(k)}}(t,\omega); \tag{6.6.15}$$

(iii) for all $i\neq j$ and X_i and X_j **pathwise mutually nondegenerate**, the set

$$\mathrm{Leb}\Big\{t\in[0,T]\;\Big|\; X_i(t,\omega)=X_j(t,\omega)\Big\} = 0, \qquad \text{a.s.}; \tag{6.6.16}$$

(iv) for all $i<j<k$ and X_i, X_j and X_k **pathwise mutually nondegenerate**, the set

$$\Big\{t\in[0,T]\;\Big|\; X_i(t,\omega)=X_j(t,\omega)=X_k(t,\omega)\Big\} = \varnothing, \qquad \text{a.s.}; \tag{6.6.17}$$

(v) for all $i\neq j$, and for X_i and X_j **absolutely continuous semimartingales**, and $t\in[0,T]$,

$$d\Lambda_{X_i-X_j}(t,\omega) = \frac{1}{2}\, d\Lambda_{|X_i-X_j|}(t,\omega); \tag{6.6.18}$$

(vi) for all $i\neq j$, and for X_i and X_j **pathwise mutually nondegenerate absolutely continuous semimartingales**, and $t\in[0,T]$,

$$\begin{aligned} &d\max\big(X_i(t,\omega),X_j(t,\omega)\big) \\ &\quad = \mathbb{1}_{(0,\infty)}\big(X_i(t,\omega)-X_j(t,\omega)\big)\, dX_i(t,\omega) + \mathbb{1}_{(0,\infty)}\big(X_j(t,\omega)-X_i(t,\omega)\big)\, dX_j(t,\omega) + d\Lambda_{X_i-X_j}(t,\omega); \end{aligned} \tag{6.6.19}$$

(vii) for all $i \neq j$, and for X_i and X_j **pathwise mutually nondegenerate absolutely continuous semimartingales**, and $t \in [0,T]$,

$$\begin{aligned} & d\min\big(X_i(t,\omega), X_j(t,\omega)\big) \\ & \quad = \mathbb{1}_{(0,\infty)}\big(X_j(t,\omega) - X_i(t,\omega)\big)\, dX_i(t,\omega) + \mathbb{1}_{(0,\infty)}\big(X_i(t,\omega) - X_j(t,\omega)\big)\, dX_j(t,\omega) - d\Lambda_{X_i - X_j}(t,\omega). \end{aligned} \tag{6.6.20}$$

- **Condition (i)** follows from Definition 6.5.1.
- **Condition (ii)** follows from (6.3.183) of Corollary 6.3.26 with $a = 0$, if $X_{(k-1)}$, $X_{(k)}$ and $X_{(k+1)}$ are **continuous semimartingales**.
- **Conditions (iii) and (iv)** respectively follow directly from conditions (i) and (ii) of Definition 6.4.1 for **pathwise mutually nondegenerate** processes.
- **Condition (v)** follows from (6.5.8) of Lemma 6.5.5, if X_i and X_j are **absolutely continuous semimartingales**.
- **Conditions (vi) and (vii)** follow respectively from (6.6.1) and (6.6.2), of Lemma 6.6.1, if X_i and X_j are **pathwise mutually nondegenerate absolutely continuous semimartingales**.

Suppose that $\omega \in \Omega'$, $t_0 \in [0,T]$, and $k \in \{1,2,\ldots,n\}$, and let $m(\omega) \triangleq p_{t_0}(k,\omega)$ (thus, $\mathbb{1}_{\{m(\omega)\}}\big(p_{t_0}(k,\omega)\big) = 1$), so that $X_{m(\omega)}(t_0,\omega) \triangleq X_{p_{t_0}(k,\omega)}(t_0,\omega)$. Then from (6.2.3) of Definition 6.2.2, where $X_{p_{t_0}(k,\omega)}(t_0,\omega) = X_{(k)}(t_0,\omega)$, we have

$$X_{(k)}(t_0,\omega) \quad = \quad X_{m(\omega)}(t_0,\omega). \tag{6.6.21}$$

There are two cases we shall consider:

- The first case considers the situation where the stock price processes all assume different values for all $t_0 \in [0,T]$, i.e., for all $i \neq m(\omega)$,

$$X_i(t_0,\omega) \quad \neq \quad X_{m(\omega)}(t_0,\omega). \tag{6.6.22}$$

  Condition (i) implies that (6.6.22) continues to hold for all t in some neighbourhood U of t_0. Hence, for all $t \in U$, we have from (6.6.22),

$$X_{(k-1)}(t,\omega) \quad > \quad X_{(k)}(t,\omega) \quad > \quad X_{(k+1)}(t,\omega). \tag{6.6.23}$$

  Thus, for $t \in U$, the permutation vector $\boldsymbol{p}_t$ is constant for this case. In addition, for all $t \in U$, by assumption we have $p_t(k,\omega) \triangleq m(\omega)$, which implies that $\mathbb{1}_{\{m(\omega)\}}\big(p_t(k,\omega)\big) = 1$ and $X_{p_t(k,\omega)}(t,\omega) \triangleq X_{m(\omega)}(t,\omega)$. Thus, from (6.6.22), for all $i \neq m(\omega)$, $\mathbb{1}_{\{i\}}\big(p_t(k,\omega)\big) = 0$ holds. In precise terms, we have

$$\mathbb{1}_{\{i\}}\big(p_t(k,\omega)\big) \quad = \quad \begin{cases} 1 & \text{if} \quad i = m(\omega) \equiv p_t(k,\omega), \\ 0 & \text{if} \quad i \neq m(\omega) \equiv p_t(k,\omega). \end{cases} \tag{6.6.24}$$

  Moreover, from (6.6.23), we deduce the following

$$\mathbb{1}_{\{0\}}\big(X_{(k)}(t,\omega) - X_{(k+1)}(t,\omega)\big) \quad = \quad 0, \tag{6.6.25}$$

$$\mathbb{1}_{\{0\}}\big(X_{(k-1)}(t,\omega) - X_{(k)}(t,\omega)\big) \quad = \quad 0. \tag{6.6.26}$$

Hence,

$$
\begin{aligned}
dX_{(k)}(t,\omega) & \\
&= dX_{m(\omega)}(t,\omega) && (6.6.27)\\
&= \mathbb{1}_{\{m(\omega)\}}\big(p_t(k,\omega)\big)\, dX_{m(\omega)}(t,\omega) + \sum_{i\neq m(\omega)}^{n} \mathbb{1}_{\{i\}}\big(p_t(k,\omega)\big)\, dX_i(t,\omega) && (6.6.28)\\
&= \sum_{i=1}^{n} \mathbb{1}_{\{i\}}\big(p_t(k,\omega)\big)\, dX_i(t,\omega) \\
&= \sum_{i=1}^{n} \mathbb{1}_{\{i\}}\big(p_t(k,\omega)\big)\, dX_i(t,\omega) + \frac{1}{2}\, \mathbb{1}_{\{0\}}\big(X_{(k)}(t,\omega) - X_{(k+1)}(t,\omega)\big)\, d\Lambda_{X_{(k)}-X_{(k+1)}}(t,\omega) \\
&\quad - \frac{1}{2}\, \mathbb{1}_{\{0\}}\big(X_{(k-1)}(t,\omega) - X_{(k)}(t,\omega)\big)\, d\Lambda_{X_{(k-1)}-X_{(k)}}(t,\omega) && (6.6.29)\\
&= \sum_{i=1}^{n} \mathbb{1}_{\{i\}}\big(p_t(k,\omega)\big)\, dX_i(t,\omega) + \frac{1}{2}\, d\Lambda_{X_{(k)}-X_{(k+1)}}(t,\omega) - \frac{1}{2}\, d\Lambda_{X_{(k-1)}-X_{(k)}}(t,\omega), && (6.6.30)
\end{aligned}
$$

where (6.6.27) follows from (6.6.21), (6.6.28) follows from (6.6.24), (6.6.29) follows from (6.6.25) together with (6.6.26), and finally (6.6.30) follows from (6.6.14) and (6.6.15) of condition (ii). Thus, we have derived the desired result (6.6.11) for the first case.

- The second case occurs at some time $t_0 \in [0,T]$, when there are precisely two stock price processes identical in value, i.e., there is an $r(\omega) \neq m(\omega)$, with $1 \leq r(\omega) \leq n$, such that

$$X_{r(\omega)}(t_0,\omega) \quad = \quad X_{m(\omega)}(t_0,\omega). \tag{6.6.31}$$

Conditions (i) and (iv) (which ensure that there are no "triple collisions" of the stock price processes, i.e., at most two stocks at time $t_0 \in [0,T]$ can coincide and have identical value, as is the requirement of the second case) imply that there is a neighbourhood U of t_0 such that for all $t \in U$, either the kth ranked stock is possibly identical in value to the $(k+1)$th ranked stock,

$$X_{(k-1)}(t,\omega) \quad > \quad X_{(k)}(t,\omega) \quad \geq \quad X_{(k+1)}(t,\omega) \quad > \quad X_{(k+2)}(t,\omega), \tag{6.6.32}$$

in which case for $t \in U$,

$$X_{(k)}(t,\omega) \quad = \quad \max\big(X_{m(\omega)}(t,\omega), X_{r(\omega)}(t,\omega)\big), \tag{6.6.33}$$

which follows from (6.6.31) and the equality between the kth and $(k+1)$th ranked stocks as specified in (6.6.32), thus the kth ranked stock, $X_{(k)}(t,\omega)$, corresponds to either $X_{m(\omega)}(t,\omega)$ or $X_{r(\omega)}(t,\omega)$, likewise for the $(k+1)$th ranked stock, $X_{(k+1)}(t,\omega)$. Clearly, if $X_{(k)}(t,\omega) = X_{m(\omega)}(t,\omega)$ then $X_{(k+1)}(t,\omega) = X_{r(\omega)}(t,\omega)$ and if $X_{(k)}(t,\omega) = X_{r(\omega)}(t,\omega)$ then $X_{(k+1)}(t,\omega) = X_{m(\omega)}(t,\omega)$. Then, the inequality between the kth and $(k+1)$th ranked stocks in (6.6.32), implies that $X_{(k)}(t,\omega)$ must be the larger of $X_{m(\omega)}(t,\omega)$ and $X_{r(\omega)}(t,\omega)$, hence the maximum. Furthermore, we have

$$\big|X_{r(\omega)}(t,\omega) - X_{m(\omega)}(t,\omega)\big| \quad = \quad X_{(k)}(t,\omega) - X_{(k+1)}(t,\omega), \tag{6.6.34}$$

since $X_{(k)}(t,\omega) - X_{(k+1)}(t,\omega) \geq 0$, or the kth ranked stock is possibly identical in value to the $(k-1)$th ranked stock,

$$X_{(k-2)}(t,\omega) \quad > \quad X_{(k-1)}(t,\omega) \quad \geq \quad X_{(k)}(t,\omega) \quad > \quad X_{(k+1)}(t,\omega), \tag{6.6.35}$$

in which case for $t \in U$,

$$X_{(k)}(t,\omega) \quad = \quad \min\big(X_{m(\omega)}(t,\omega), X_{r(\omega)}(t,\omega)\big), \tag{6.6.36}$$

which follows from (6.6.31) and the equality between the kth and $(k-1)$th ranked stocks as specified in (6.6.35), thus the kth ranked stock, $X_{(k)}(t,\omega)$, corresponds to either $X_{m(\omega)}(t,\omega)$ or $X_{r(\omega)}(t,\omega)$, likewise for

the $(k-1)$th ranked stock, $X_{(k-1)}(t,\omega)$. Clearly, if $X_{(k)}(t,\omega) = X_{m(\omega)}(t,\omega)$ then $X_{(k-1)}(t,\omega) = X_{r(\omega)}(t,\omega)$ and if $X_{(k)}(t,\omega) = X_{r(\omega)}(t,\omega)$ then $X_{(k+1)}(t,\omega) = X_{m(\omega)}(t,\omega)$. Then, the inequality between the kth and $(k-1)$th ranked stocks in (6.6.35), implies that $X_{(k)}(t,\omega)$ is the smaller of $X_{m(\omega)}(t,\omega)$ and $X_{r(\omega)}(t,\omega)$, hence the minimum. Furthermore, we have

$$\left|X_{r(\omega)}(t,\omega) - X_{m(\omega)}(t,\omega)\right| = X_{(k-1)}(t,\omega) - X_{(k)}(t,\omega), \tag{6.6.37}$$

since $X_{(k-1)}(t,\omega) - X_{(k)}(t,\omega) \geq 0$. Moreover, from (6.6.31), for all $i \notin \{m(\omega), r(\omega)\}$, $\mathbb{1}_{\{i\}}\big(p_t(k,\omega)\big) = 0$ holds. Now, suppose that (6.6.32) and (6.6.33) hold. Then for $t \in U$,

$$\begin{aligned}
dX_{(k)}(t,\omega) &= d\max\big(X_{m(\omega)}(t,\omega), X_{r(\omega)}(t,\omega)\big) && (6.6.38)\\
&= \mathbb{1}_{(0,\infty)}\big(X_{m(\omega)}(t,\omega) - X_{r(\omega)}(t,\omega)\big)\, dX_{m(\omega)}(t,\omega) + d\Lambda_{X_{m(\omega)}-X_{r(\omega)}}(t,\omega) \\
&\quad + \mathbb{1}_{(0,\infty)}\big(X_{r(\omega)}(t,\omega) - X_{m(\omega)}(t,\omega)\big)\, dX_{r(\omega)}(t,\omega) && (6.6.39)\\
&= \mathbb{1}_{(0,\infty)}\big(X_{m(\omega)}(t,\omega) - X_{r(\omega)}(t,\omega)\big)\, dX_{m(\omega)}(t,\omega) + d\Lambda_{X_{r(\omega)}-X_{m(\omega)}}(t,\omega) \\
&\quad + \mathbb{1}_{(0,\infty)}\big(X_{r(\omega)}(t,\omega) - X_{m(\omega)}(t,\omega)\big)\, dX_{r(\omega)}(t,\omega) \\
&= \mathbb{1}_{(0,\infty)}\big(X_{m(\omega)}(t,\omega) - X_{r(\omega)}(t,\omega)\big)\, dX_{m(\omega)}(t,\omega) + \frac{1}{2}\, d\Lambda_{|X_{r(\omega)}-X_{m(\omega)}|}(t,\omega) \\
&\quad + \mathbb{1}_{(0,\infty)}\big(X_{r(\omega)}(t,\omega) - X_{m(\omega)}(t,\omega)\big)\, dX_{r(\omega)}(t,\omega) && (6.6.40)\\
&= \mathbb{1}_{(0,\infty)}\big(X_{m(\omega)}(t,\omega) - X_{r(\omega)}(t,\omega)\big)\, dX_{m(\omega)}(t,\omega) + \frac{1}{2}\, d\Lambda_{X_{(k)}-X_{(k+1)}}(t,\omega) \\
&\quad + \mathbb{1}_{(0,\infty)}\big(X_{r(\omega)}(t,\omega) - X_{m(\omega)}(t,\omega)\big)\, dX_{r(\omega)}(t,\omega), && (6.6.41)
\end{aligned}$$

where (6.6.38) follows from (6.6.33), condition (vi) given by (6.6.19) implies (6.6.39), condition (v) given by (6.6.18) implies (6.6.40) and (6.6.41) follows from (6.6.34). Now, by appealing to (6.6.33) we deduce that

$$\begin{aligned}
\mathbb{1}_{(0,\infty)}\big(X_{m(\omega)}(t,\omega) - X_{r(\omega)}(t,\omega)\big) &= \begin{cases} 1 & \text{if} \quad X_{m(\omega)}(t,\omega) > X_{r(\omega)}(t,\omega), \\ 0 & \text{if} \quad X_{m(\omega)}(t,\omega) \leq X_{r(\omega)}(t,\omega), \end{cases} \\
&= \begin{cases} 1 & \text{if} \quad \max\big(X_{m(\omega)}(t,\omega), X_{r(\omega)}(t,\omega)\big) = X_{m(\omega)}(t,\omega), \\ 0 & \text{if} \quad \max\big(X_{m(\omega)}(t,\omega), X_{r(\omega)}(t,\omega)\big) = X_{r(\omega)}(t,\omega), \end{cases} \\
&= \begin{cases} 1 & \text{if} \quad X_{(k)}(t,\omega) = X_{m(\omega)}(t,\omega), \\ 0 & \text{if} \quad X_{(k)}(t,\omega) = X_{r(\omega)}(t,\omega). \end{cases}
\end{aligned}$$

Thus, since $p_t(k,\omega)$ is the associated index of the kth ranked stock, we deduce that

$$\begin{aligned}
\mathbb{1}_{(0,\infty)}\big(X_{m(\omega)}(t,\omega) - X_{r(\omega)}(t,\omega)\big) &= \begin{cases} 1 & \text{if} \quad p_t(k,\omega) = m(\omega), \\ 0 & \text{if} \quad p_t(k,\omega) = r(\omega), \end{cases} \\
&= \begin{cases} 1 & \text{if} \quad p_t(k,\omega) = m(\omega), \\ 0 & \text{if} \quad p_t(k,\omega) \neq m(\omega). \end{cases}
\end{aligned}$$

Therefore, for $t \in U$, we have

$$\mathbb{1}_{(0,\infty)}\big(X_{m(\omega)}(t,\omega) - X_{r(\omega)}(t,\omega)\big) = \mathbb{1}_{\{m(\omega)\}}\big(p_t(k,\omega)\big). \tag{6.6.42}$$

Similarly, for $t \in U$, we have

$$\mathbb{1}_{(0,\infty)}\big(X_{r(\omega)}(t,\omega) - X_{m(\omega)}(t,\omega)\big) = \mathbb{1}_{\{r(\omega)\}}\big(p_t(k,\omega)\big). \tag{6.6.43}$$

Substituting (6.6.42) and (6.6.43) into (6.6.41), yields

$$\begin{aligned}
dX_{(k)}(t,\omega) &= \mathbb{1}_{\{m(\omega)\}}\big(p_t(k,\omega)\big)\, dX_{m(\omega)}(t,\omega) + \mathbb{1}_{\{r(\omega)\}}\big(p_t(k,\omega)\big)\, dX_{r(\omega)}(t,\omega) \\
&\quad + \frac{1}{2}\, d\Lambda_{X_{(k)}-X_{(k+1)}}(t,\omega).
\end{aligned}$$

Recall that for $i \notin \{m(\omega), r(\omega)\}$, $\mathbb{1}_{\{i\}}\big(p_t(k,\omega)\big) = 0$, further from (6.6.32), we deduce the following

$$\mathbb{1}_{\{0\}}\big(X_{(k-1)}(t,\omega) - X_{(k)}(t,\omega)\big) = 0, \tag{6.6.44}$$

$$\mathbb{1}_{\{0\}}\big(X_{(k+1)}(t,\omega) - X_{(k+2)}(t,\omega)\big) = 0. \tag{6.6.45}$$

Therefore, from (6.6.44), we obtain

$$\begin{aligned}
&dX_{(k)}(t,\omega) \\
&= \mathbb{1}_{\{m(\omega)\}}\big(p_t(k,\omega)\big)\, dX_{m(\omega)}(t,\omega) + \mathbb{1}_{\{r(\omega)\}}\big(p_t(k,\omega)\big)\, dX_{r(\omega)}(t,\omega) + \sum_{i \neq m(\omega), r(\omega)}^{n} \mathbb{1}_{\{i\}}\big(p_t(k,\omega)\big)\, dX_i(t,\omega) \\
&\quad + \frac{1}{2}\, d\Lambda_{X_{(k)}-X_{(k+1)}}(t,\omega) - \frac{1}{2}\, \mathbb{1}_{\{0\}}\big(X_{(k-1)}(t,\omega) - X_{(k)}(t,\omega)\big)\, d\Lambda_{X_{(k-1)}-X_{(k)}}(t,\omega) \\
&= \sum_{i=1}^{n} \mathbb{1}_{\{i\}}\big(p_t(k,\omega)\big)\, dX_i(t,\omega) + \frac{1}{2}\, d\Lambda_{X_{(k)}-X_{(k+1)}}(t,\omega) - \frac{1}{2}\, d\Lambda_{X_{(k-1)}-X_{(k)}}(t,\omega),
\end{aligned}$$

where the last expression follows from (6.6.15) of condition (ii). Hence, if (6.6.32) and (6.6.33) hold, then (6.6.11) is valid. Now, suppose that (6.6.35) and (6.6.36) hold. Then for $t \in U$,

$$dX_{(k)}(t,\omega) = d\min\big(X_{m(\omega)}(t,\omega), X_{r(\omega)}(t,\omega)\big) \tag{6.6.46}$$

$$\begin{aligned}
&= \mathbb{1}_{(0,\infty)}\big(X_{r(\omega)}(t,\omega) - X_{m(\omega)}(t,\omega)\big)\, dX_{m(\omega)}(t,\omega) - d\Lambda_{X_{m(\omega)}-X_{r(\omega)}}(t,\omega) \\
&\quad + \mathbb{1}_{(0,\infty)}\big(X_{m(\omega)}(t,\omega) - X_{r(\omega)}(t,\omega)\big)\, dX_{r(\omega)}(t,\omega)
\end{aligned} \tag{6.6.47}$$

$$\begin{aligned}
&= \mathbb{1}_{(0,\infty)}\big(X_{r(\omega)}(t,\omega) - X_{m(\omega)}(t,\omega)\big)\, dX_{m(\omega)}(t,\omega) - d\Lambda_{X_{r(\omega)}-X_{m(\omega)}}(t,\omega) \\
&\quad + \mathbb{1}_{(0,\infty)}\big(X_{m(\omega)}(t,\omega) - X_{r(\omega)}(t,\omega)\big)\, dX_{r(\omega)}(t,\omega)
\end{aligned}$$

$$\begin{aligned}
&= \mathbb{1}_{(0,\infty)}\big(X_{r(\omega)}(t,\omega) - X_{m(\omega)}(t,\omega)\big)\, dX_{m(\omega)}(t,\omega) - \frac{1}{2}\, d\Lambda_{|X_{r(\omega)}-X_{m(\omega)}|}(t,\omega) \\
&\quad + \mathbb{1}_{(0,\infty)}\big(X_{m(\omega)}(t,\omega) - X_{r(\omega)}(t,\omega)\big)\, dX_{r(\omega)}(t,\omega)
\end{aligned} \tag{6.6.48}$$

$$\begin{aligned}
&= \mathbb{1}_{(0,\infty)}\big(X_{r(\omega)}(t,\omega) - X_{m(\omega)}(t,\omega)\big)\, dX_{m(\omega)}(t,\omega) - \frac{1}{2}\, d\Lambda_{X_{(k-1)}-X_{(k)}}(t,\omega) \\
&\quad + \mathbb{1}_{(0,\infty)}\big(X_{m(\omega)}(t,\omega) - X_{r(\omega)}(t,\omega)\big)\, dX_{r(\omega)}(t,\omega),
\end{aligned} \tag{6.6.49}$$

where (6.6.46) follows from (6.6.36), condition (vii) given by (6.6.20) implies (6.6.47), condition (v) given by (6.6.18) implies (6.6.48) and (6.6.49) follows from (6.6.37). Now, by appealing to (6.6.36) we deduce that

$$\begin{aligned}
\mathbb{1}_{(0,\infty)}\big(X_{r(\omega)}(t,\omega) - X_{m(\omega)}(t,\omega)\big) &= \begin{cases} 1 & \text{if} \quad X_{r(\omega)}(t,\omega) > X_{m(\omega)}(t,\omega), \\ 0 & \text{if} \quad X_{r(\omega)}(t,\omega) \leq X_{m(\omega)}(t,\omega), \end{cases} \\
&= \begin{cases} 1 & \text{if} \quad \min\big(X_{m(\omega)}(t,\omega), X_{r(\omega)}(t,\omega)\big) = X_{m(\omega)}(t,\omega), \\ 0 & \text{if} \quad \min\big(X_{m(\omega)}(t,\omega), X_{r(\omega)}(t,\omega)\big) = X_{r(\omega)}(t,\omega), \end{cases} \\
&= \begin{cases} 1 & \text{if} \quad X_{(k)}(t,\omega) = X_{m(\omega)}(t,\omega), \\ 0 & \text{if} \quad X_{(k)}(t,\omega) = X_{r(\omega)}(t,\omega). \end{cases}
\end{aligned}$$

Thus, since $p_t(k,\omega)$ is the associated index of the kth ranked stock, we deduce that

$$\begin{aligned}
\mathbb{1}_{(0,\infty)}\big(X_{r(\omega)}(t,\omega) - X_{m(\omega)}(t,\omega)\big) &= \begin{cases} 1 & \text{if} \quad p_t(k,\omega) = m(\omega), \\ 0 & \text{if} \quad p_t(k,\omega) = r(\omega), \end{cases} \\
&= \begin{cases} 1 & \text{if} \quad p_t(k,\omega) = m(\omega), \\ 0 & \text{if} \quad p_t(k,\omega) \neq m(\omega). \end{cases}
\end{aligned}$$

Therefore, for $t \in U$, we have

$$\mathbb{1}_{(0,\infty)}\big(X_{r(\omega)}(t,\omega) - X_{m(\omega)}(t,\omega)\big) = \mathbb{1}_{\{m(\omega)\}}\big(p_t(k,\omega)\big). \tag{6.6.50}$$

Similarly, for $t \in U$, we have

$$\mathbb{1}_{(0,\infty)}\big(X_{m(\omega)}(t,\omega) - X_{r(\omega)}(t,\omega)\big) \quad = \quad \mathbb{1}_{\{r(\omega)\}}\big(p_t(k,\omega)\big). \tag{6.6.51}$$

Substituting (6.6.50) and (6.6.51) into (6.6.49), yields

$$\begin{aligned} dX_{(k)}(t,\omega) \quad &= \quad \mathbb{1}_{\{m(\omega)\}}\big(p_t(k,\omega)\big)\, dX_{m(\omega)}(t,\omega) + \mathbb{1}_{\{r(\omega)\}}\big(p_t(k,\omega)\big)\, dX_{r(\omega)}(t,\omega) \\ &\quad - \frac{1}{2}\, d\Lambda_{X_{(k-1)}-X_{(k)}}(t,\omega). \end{aligned}$$

Recall that for $i \notin \big\{m(\omega), r(\omega)\big\}$, $\mathbb{1}_{\{i\}}\big(p_t(k,\omega)\big) = 0$, further from (6.6.35), we deduce the following

$$\mathbb{1}_{\{0\}}\big(X_{(k-2)}(t,\omega) - X_{(k-1)}(t,\omega)\big) \quad = \quad 0, \tag{6.6.52}$$

$$\mathbb{1}_{\{0\}}\big(X_{(k)}(t,\omega) - X_{(k+1)}(t,\omega)\big) \quad = \quad 0. \tag{6.6.53}$$

Therefore, from (6.6.53), we obtain

$$\begin{aligned} &dX_{(k)}(t,\omega) \\ &= \quad \mathbb{1}_{\{m(\omega)\}}\big(p_t(k,\omega)\big)\, dX_{m(\omega)}(t,\omega) + \mathbb{1}_{\{r(\omega)\}}\big(p_t(k,\omega)\big)\, dX_{r(\omega)}(t,\omega) + \sum_{i \neq m(\omega), r(\omega)}^{n} \mathbb{1}_{\{i\}}\big(p_t(k,\omega)\big)\, dX_i(t,\omega) \\ &\quad - \frac{1}{2}\, d\Lambda_{X_{(k-1)}-X_{(k)}}(t,\omega) + \frac{1}{2}\, \mathbb{1}_{\{0\}}\big(X_{(k)}(t,\omega) - X_{(k+1)}(t,\omega)\big)\, d\Lambda_{X_{(k)}-X_{(k+1)}}(t,\omega) \\ &= \quad \sum_{i=1}^{n} \mathbb{1}_{\{i\}}\big(p_t(k,\omega)\big)\, dX_i(t,\omega) - \frac{1}{2}\, d\Lambda_{X_{(k-1)}-X_{(k)}}(t,\omega) + \frac{1}{2}\, d\Lambda_{X_{(k)}-X_{(k+1)}}(t,\omega), \end{aligned}$$

where the last expression follows from (6.6.14) of condition (ii). Hence, if (6.6.35) and (6.6.36) hold, then (6.6.11) is valid. Thus (6.6.11) is valid for all $\omega \in \Omega'$. Since $\mathbb{P}(\Omega') = 1$, the proposition is proved.

- The possible third case, where more than two stock price processes assume identical values, is precluded by the pathwise mutual nondegeneracy condition of Definition 6.4.1 that the stock price processes obey.

■

This proposition provides a representation of the rank processes of pathwise mutually nondegenerate absolutely continuous semimartingales, and it demonstrates that the rank processes which are derived from pathwise mutually nondegenerate absolutely continuous semimartingales can be expressed in terms of the *original processes* $X_1, X_2, \ldots, X_n$, adjusted by semimartingale local times. Let us interpret the above proposition for $X_{(1)}(t)$, the largest stock at time $t \in [0,\infty)$. In this case, when the two largest stocks are momentarily equal in value, the sum on the right-hand side in the proposition cannot anticipate which one of them will be the maximum or the largest stock in the near future. The local time component in this scenario $\frac{1}{2}\, d\Lambda_{X_{(1)}-X_{(2)}}$ is there to adjust for this difference.

6.6.2 Decomposition of the Rank Processes of Absolutely Continuous Semimartingales

We shall need the following definitions:

$$S_t(k) \quad \triangleq \quad \Big\{i \;:\; X_i(t) = X_{(k)}(t)\Big\}, \qquad \text{and} \qquad N_t(k) \quad \triangleq \quad \big|S_t(k)\big|, \tag{6.6.54}$$

in particular, $S_t(k)$ is the set of indices (subscripts) of processes which are kth ranked, and its associated cardinality $N_t(k)$ denotes the number of indices (subscripts) such that $X_i(t) = X_{(k)}(t)$, in other words $N_t(k)$ is the number of processes that are at (equal) rank at time $t \in [0,T]$. As such, $N_t(k)$ can alternatively be represented as

$$N_t(k) \quad = \quad \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t) = X_i(t)\}}. \tag{6.6.55}$$

For $k = 1, 2, \ldots, n$, let $u(k) = \{u_t(k), t \in [0,T]\} : [0,T] \times \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be any predictable process with the property:

$$X_{(k)}(t) \quad = \quad X_{u_t(k)}(t). \tag{6.6.56}$$

Hence, define the following set

$$\begin{aligned} \mathcal{U} \quad \triangleq \quad & \Big\{ u_\cdot(\cdot) : [0,T] \times \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\} \;\Big|\; u_\cdot(k) \text{ is predictable} \\ & \text{and } X_{(k)}(t) = X_{u_t(k)}(t) \text{ for all } t \in [0,T],\ k = 1, 2, \ldots, n \Big\}. \end{aligned} \tag{6.6.57}$$

We now extend Proposition 6.6.2 to the case where three or more processes may be equal at a given time $t \in [0,T]$, i.e., we shall consider the case in which there exists $\Omega' \subset \Omega$, $\mathbb{P}(\Omega') = 1$, such that for $\omega \in \Omega'$, the set $\{t \in [0,T] \mid X_i(t,\omega) = X_j(t,\omega) = X_k(t,\omega)\}$ is finite for all $i < j < k$.

Theorem 6.6.3. *Let $X_1, \ldots, X_n$ be **absolutely continuous semimartingales** satisfying condition (i) of the pathwise mutual nondegeneracy conditions provided in Definition 6.4.1 (i.e., for all $i \neq j$, the set $\{t \in [0,T] \mid X_i(t) = X_j(t)\}$ has Lebesgue measure zero, almost surely). Then the kth rank processes $X_{(k)}$, for all $k = 1, 2, \ldots, n$, are **continuous semimartingales** such that a.s., for $t \in [0,T]$,*

$$\begin{aligned} dX_{(k)}(t) \quad &= \quad \sum_{i=1}^n \mathbb{1}_{\{u_t(k)=i\}}\, dX_i(t) + \sum_{i=1}^n \frac{1}{N_t(k)}\, d\Lambda_{(X_{(k)}-X_i)^+}(t) - \sum_{i=1}^n \frac{1}{N_t(k)}\, d\Lambda_{(X_{(k)}-X_i)^-}(t) && (6.6.58) \\ &= \quad \sum_{i=1}^n \mathbb{1}_{\{u_t(k)=i\}}\, dX_i(t) + \frac{1}{N_t(k)} \sum_{i=1}^n d\Lambda_{(X_{(k)}-X_i)^+}(t) - \frac{1}{N_t(k)} \sum_{i=1}^n d\Lambda_{(X_{(k)}-X_i)^-}(t), && (6.6.59) \end{aligned}$$

for any u in the set $\mathcal{U}$ of (6.6.57) and $N_\cdot(k)$ is defined as in (6.6.54) above.

Proof. This proof makes use of the following result which can be found in Banner & Ghomrasni (2008, Proposition 2.4): under the assumptions laid out in the theorem above (i.e., the semimartingales $X_1, \ldots, X_n$ are assumed to be absolutely continuous such that for all $i \neq j$, the set $\{t \in [0,T] \mid X_i(t) = X_j(t)\}$ has Lebesgue measure zero, almost surely), for any k,

$$\sum_{i=1}^n \int_0^t \mathbb{1}_{\{u_s(k)=i\}}\, dX_i(s) \quad = \quad \int_0^t \sum_{i=1}^n \mathbb{1}_{\{u_s(k)=i\}}\, dX_i(s),$$

is independent of the choice of $u \in \mathcal{U}$, for all $t \in [0,T]$, a.s. For the proof of this result, see Banner & Ghomrasni (2008, Proof of Proposition 2.4). Now, let

$$\mathcal{J} \quad \triangleq \quad \Big\{ \boldsymbol{j} = (j_1, j_2, \ldots, j_n) \;\Big|\; j_i \in \{1, \ldots, i\} \text{ for all } i = 1, 2, \ldots, n \Big\}.$$

For any $\boldsymbol{j} \in \mathcal{J}$, we define a member $u^{\boldsymbol{j}}$ of the set $\mathcal{U}$ (6.6.57) by setting

$$u_t^{\boldsymbol{j}}(k) \quad \triangleq \quad \boldsymbol{j}_{N_t(k)}\text{th smallest element of } S_t(k).$$

In other words, suppose that at time $t \in [0,T]$, precisely m of the processes $X_1, \ldots, X_n$ have rank k. If these processes are $X_{i_1}, \ldots, X_{i_m}$, then $u_t^{\boldsymbol{j}}(k)$ is simply the $\boldsymbol{j}_m$th smallest among the indices $i_1, \ldots, i_m$. In the case where $\boldsymbol{j} = (1, 1, \ldots, 1)$, $u_t^{\boldsymbol{j}}(k)$ picks out the lowest index among all the processes of rank k at time $t \in [0,T]$, regardless of how many such processes there are, i.e., $u_t^{\boldsymbol{j}} = \boldsymbol{p}_t$ for the random permutation $\boldsymbol{p}_t$. For any $u \in \mathcal{U}$, we have for $k = 1, 2, \ldots, n$,

$$\begin{aligned} \sum_{i=1}^n \mathbb{1}_{\{u_t(k)=i\}}\, dX_{(k)}(t) \quad &= \quad \sum_{i=1}^n \mathbb{1}_{\{u_t(k)=i\}}\, dX_i(t) + \sum_{i=1}^n \mathbb{1}_{\{u_t(k)=i\}} \Big[dX_{(k)}(t) - dX_i(t) \Big] \\ &= \quad \sum_{i=1}^n \mathbb{1}_{\{u_t(k)=i\}}\, dX_i(t) + \sum_{i=1}^n \mathbb{1}_{\{u_t(k)=i\}}\, d\big(X_{(k)}(t) - X_i(t)\big). \end{aligned}$$

Since $\left\{\mathbb{1}_{\{u_t(k)=i\}}\right\}_{i=1}^n$ is a partition of unity, i.e., we have the following property

$$\sum_{i=1}^n \mathbb{1}_{\{u_t(k)=i\}} = 1,$$

we therefore have

$$dX_{(k)}(t) = \sum_{i=1}^n \mathbb{1}_{\{u_t(k)=i\}}\, dX_i(t) + \sum_{i=1}^n \mathbb{1}_{\{u_t(k)=i\}}\, d\left(X_{(k)}(t) - X_i(t)\right). \tag{6.6.60}$$

Notice from the set $\mathcal{J}$ that $|\mathcal{J}| = n!$ since $j_1 = 1, j_2 \in \{1,2\}, \ldots, j_n \in \{1,2,\ldots,n\}$ (i.e., there is one element for j_1, there are two possible values for j_2, etc., and there are n possible values for j_n). Therefore, there are $1 \times 2 \times 3 \times \cdots \times n = n!$ possible ways of representing the vector $\boldsymbol{j}$ and thus the set $\mathcal{J}$ consists of $n!$ elements. Replacing u by $u^{\boldsymbol{j}}$ for $\boldsymbol{j} \in \mathcal{J}$, then summing over all such $\boldsymbol{j}$, and using the fact that $|\mathcal{J}| = n!$, we get for any $u \in \mathcal{U}$

$$\begin{aligned}
\sum_{\boldsymbol{j}\in\mathcal{J}} dX_{(k)}(t) &= \sum_{\boldsymbol{j}\in\mathcal{J}}\sum_{i=1}^n \mathbb{1}_{\{u_t^{\boldsymbol{j}}(k)=i\}}\, dX_i(t) + \sum_{\boldsymbol{j}\in\mathcal{J}}\sum_{i=1}^n \mathbb{1}_{\{u_t^{\boldsymbol{j}}(k)=i\}}\, d\left(X_{(k)}(t) - X_i(t)\right),\\
n!\, dX_{(k)}(t) &= \sum_{\boldsymbol{j}\in\mathcal{J}}\sum_{i=1}^n \mathbb{1}_{\{u_t^{\boldsymbol{j}}(k)=i\}}\, dX_i(t) + \sum_{\boldsymbol{j}\in\mathcal{J}}\sum_{i=1}^n \mathbb{1}_{\{u_t^{\boldsymbol{j}}(k)=i\}}\, d\left(X_{(k)}(t) - X_i(t)\right), \qquad (6.6.61)\\
n!\, dX_{(k)}(t) &= \sum_{i=1}^n\sum_{\boldsymbol{j}\in\mathcal{J}} \mathbb{1}_{\{u_t^{\boldsymbol{j}}(k)=i\}}\, dX_i(t) + \sum_{i=1}^n\sum_{\boldsymbol{j}\in\mathcal{J}} \mathbb{1}_{\{u_t^{\boldsymbol{j}}(k)=i\}}\, d\left(X_{(k)}(t) - X_i(t)\right). \qquad (6.6.62)
\end{aligned}$$

Now under the above assumptions, the result stated at the start of the proof allows us to keep the first term on the right-hand side of equation (6.6.60) as

$$\sum_{i=1}^n \mathbb{1}_{\{u_t(k)=i\}}\, dX_i(t),$$

since the above expression is independent of the choice of u, so that

$$\sum_{i=1}^n \mathbb{1}_{\{u_t^{\boldsymbol{j}}(k)=i\}}\, dX_i(t) = \sum_{i=1}^n \mathbb{1}_{\{u_t(k)=i\}}\, dX_i(t).$$

Thus, we may rewrite (6.6.61) as

$$\begin{aligned}
n!\, dX_{(k)}(t) &= \sum_{\boldsymbol{j}\in\mathcal{J}}\sum_{i=1}^n \mathbb{1}_{\{u_t(k)=i\}}\, dX_i(t) + \sum_{\boldsymbol{j}\in\mathcal{J}}\sum_{i=1}^n \mathbb{1}_{\{u_t^{\boldsymbol{j}}(k)=i\}}\, d\left(X_{(k)}(t) - X_i(t)\right)\\
&= n!\sum_{i=1}^n \mathbb{1}_{\{u_t(k)=i\}}\, dX_i(t) + \sum_{i=1}^n\sum_{\boldsymbol{j}\in\mathcal{J}} \mathbb{1}_{\{u_t^{\boldsymbol{j}}(k)=i\}}\, d\left(X_{(k)}(t) - X_i(t)\right)\\
&= n!\sum_{i=1}^n \mathbb{1}_{\{u_t(k)=i\}}\, dX_i(t) + \sum_{i=1}^n d\left(X_{(k)}(t) - X_i(t)\right)\left[\sum_{\boldsymbol{j}\in\mathcal{J}} \mathbb{1}_{\{u_t^{\boldsymbol{j}}(k)=i\}}\right]. \qquad (6.6.63)
\end{aligned}$$

If $m = N_t(k)$ and $\boldsymbol{j} = (j_1, \ldots, j_n)$, then by definition we have for any fixed $j_1, j_2, \ldots, j_{m-1}, j_{m+1}, \ldots, j_{n-1}, j_n$,

$$\sum_{j_m=1}^m \mathbb{1}_{\{u_t^{\boldsymbol{j}}(k)=i\}} = \sum_{j_m=1}^m \mathbb{1}_{\left\{u_t^{(j_1,\ldots,j_n)}(k)=i\right\}} = \mathbb{1}_{\left\{X_{(k)}(t)=X_i(t)\right\}}.$$

Therefore, we have for all $i, k = 1, 2, \ldots, n$ and for all $t \in [0, T]$,

$$\begin{aligned}
\sum_{\boldsymbol{j} \in \mathcal{J}} \mathbb{1}_{\{u_t^{\boldsymbol{j}}(k)=i\}} &= \sum_{j_1=1}^{1} \sum_{j_2=1}^{2} \cdots \sum_{j_{m-1}=1}^{m-1} \sum_{j_{m+1}=1}^{m+1} \cdots \sum_{j_n=1}^{n} \left[\sum_{j_m=1}^{m} \mathbb{1}_{\{u_t^{\boldsymbol{j}}(k)=i\}} \right] \\
&= \sum_{j_1=1}^{1} \sum_{j_2=1}^{2} \cdots \sum_{j_{m-1}=1}^{m-1} \sum_{j_{m+1}=1}^{m+1} \cdots \sum_{j_n=1}^{n} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}} \\
&= \frac{n!}{m} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}} \\
&= \frac{n!}{N_t(k)} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}.
\end{aligned}$$

Thus, we have the following formula

$$\sum_{\boldsymbol{j} \in \mathcal{J}} \mathbb{1}_{\{u_t^{\boldsymbol{j}}(k)=i\}} = \frac{n!}{N_t(k)} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}. \tag{6.6.64}$$

Using (6.6.64) in (6.6.63), we obtain

$$\begin{aligned}
n!\, dX_{(k)}(t) &= n! \sum_{i=1}^{n} \mathbb{1}_{\{u_t(k)=i\}}\, dX_i(t) + \sum_{i=1}^{n} \frac{n!}{N_t(k)} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, d\big(X_{(k)}(t) - X_i(t)\big) \\
&= n! \sum_{i=1}^{n} \mathbb{1}_{\{u_t(k)=i\}}\, dX_i(t) + n! \sum_{i=1}^{n} \frac{1}{N_t(k)} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, d\big(X_{(k)}(t) - X_i(t)\big).
\end{aligned}$$

Hence, we have

$$dX_{(k)}(t) = \sum_{i=1}^{n} \mathbb{1}_{\{u_t(k)=i\}}\, dX_i(t) + \sum_{i=1}^{n} \frac{1}{N_t(k)} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, d\big(X_{(k)}(t) - X_i(t)\big). \tag{6.6.65}$$

Now, since $x \equiv x^+ - x^-$ (similarly, $x - y \equiv (x-y)^+ - (x-y)^-$), we obtain

$$\begin{aligned}
dX_{(k)}(t) &= \sum_{i=1}^{n} \mathbb{1}_{\{u_t(k)=i\}}\, dX_i(t) + \sum_{i=1}^{n} \frac{1}{N_t(k)} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, d\Big(\big(X_{(k)}(t) - X_i(t)\big)^+ - \big(X_{(k)}(t) - X_i(t)\big)^-\Big) \\
&= \sum_{i=1}^{n} \mathbb{1}_{\{u_t(k)=i\}}\, dX_i(t) + \sum_{i=1}^{n} \frac{1}{N_t(k)} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, d\Big(\big(X_{(k)}(t) - X_i(t)\big)^+\Big) \\
&\quad - \sum_{i=1}^{n} \frac{1}{N_t(k)} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, d\Big(\big(X_{(k)}(t) - X_i(t)\big)^-\Big) \\
&= \sum_{i=1}^{n} \mathbb{1}_{\{u_t(k)=i\}}\, dX_i(t) + \sum_{i=1}^{n} \frac{1}{N_t(k)} \mathbb{1}_{\{X_{(k)}(t)-X_i(t)=0\}}\, d\Big(\big(X_{(k)}(t) - X_i(t)\big)^+\Big) \\
&\quad - \sum_{i=1}^{n} \frac{1}{N_t(k)} \mathbb{1}_{\{X_{(k)}(t)-X_i(t)=0\}}\, d\Big(\big(X_{(k)}(t) - X_i(t)\big)^-\Big).
\end{aligned}$$

Now, recall (6.3.146)

$$\Lambda_X(t) = \int_0^t \mathbb{1}_{\{X(s)=0\}}\, dX(s),$$

which is valid for all *continuous nonnegative semimartingales* X, and therefore the following also holds for continuous nonnegative semimartingales

$$d\Lambda_X(t) = \mathbb{1}_{\{X(t)=0\}}\, dX(t). \tag{6.6.66}$$

In addition, from (6.3.150) or (6.3.209), for the continuous semimartingale X, we have

$$\Lambda_{X^+}(t) = \int_0^t \mathbb{1}_{\{X(s)=0\}}\, dX^+(s).$$

Thus, the following formula also holds

$$d\Lambda_{X^+}(t) \quad = \quad \mathbb{1}_{\{X(t)=0\}}\, dX^+(t). \tag{6.6.67}$$

Similarly, from (6.3.210), we have for any continuous semimartingale X,

$$\Lambda_{X^-}(t) \quad = \quad \int_0^t \mathbb{1}_{\{X(s)=0\}}\, dX^-(s),$$

so that

$$d\Lambda_{X^-}(t) \quad = \quad \mathbb{1}_{\{X(t)=0\}}\, dX^-(t). \tag{6.6.68}$$

Since $\big(X_{(k)}(t) - X_i(t)\big)^+$ and $\big(X_{(k)}(t) - X_i(t)\big)^-$ are *continuous nonnegative semimartingales*, applying (6.6.66) (or (6.6.67) and (6.6.68)) to these continuous nonnegative semimartingales yields

$$\begin{aligned}
\mathbb{1}_{\{X_{(k)}(t)-X_i(t)=0\}}\, d\Big(\big(X_{(k)}(t) - X_i(t)\big)^+\Big) &= d\Lambda_{\left(X_{(k)}-X_i\right)^+}(t), \\
\mathbb{1}_{\{X_{(k)}(t)-X_i(t)=0\}}\, d\Big(\big(X_{(k)}(t) - X_i(t)\big)^-\Big) &= d\Lambda_{\left(X_{(k)}-X_i\right)^-}(t).
\end{aligned}$$

Employing the above equations, we arrive at the following

$$dX_{(k)}(t) \quad = \quad \sum_{i=1}^n \mathbb{1}_{\{u_t(k)=i\}}\, dX_i(t) + \sum_{i=1}^n \frac{1}{N_t(k)}\, d\Lambda_{\left(X_{(k)}-X_i\right)^+}(t) - \sum_{i=1}^n \frac{1}{N_t(k)}\, d\Lambda_{\left(X_{(k)}-X_i\right)^-}(t).$$

■

Theorem 6.6.4 ([**Yan (1985, 1989), Ouknine (1990)**]). *For **absolutely continuous semimartingales** X and Y, the processes $X \vee Y$ and $X \wedge Y$ are **continuous semimartingales**, and we have for all $t \in [0,T]$,*

$$\Lambda_{X \vee Y}(t) + \Lambda_{X \wedge Y}(t) \quad = \quad \Lambda_X(t) + \Lambda_Y(t). \tag{6.6.69}$$

This result can be extended to the case of three or more absolutely continuous semimartingales.

Theorem 6.6.5 ([**Banner & Ghomrasni (2008)**]). *For **absolutely continuous semimartingales** $X_1, \ldots, X_n$, the rank processes $X_{(1)}, \ldots, X_{(n)}$ are **continuous semimartingales**, and we have for all $t \in [0,T]$,*

$$\sum_{i=1}^n \Lambda_{X_{(i)}}(t) \quad = \quad \sum_{i=1}^n \Lambda_{X_i}(t). \tag{6.6.70}$$

Proof. Refer to Banner & Ghomrasni (2008, Proof of Theorem 2.2), where the formula above is provided in the following form,

$$\sum_{k=1}^n \Lambda_{X_{(k)}}(t) \quad = \quad \sum_{i=1}^n \Lambda_{X_i}(t). \tag{6.6.71}$$

■

Theorem 6.6.6 ([**Banner & Ghomrasni (2008)**]). *Let $X_1, \ldots, X_n$ be **absolutely continuous semimartingales** satisfying condition (i) of the pathwise mutual nondegeneracy conditions provided in Definition* 6.4.1 *(i.e., for all $i \neq j$, the set $\big\{t \in [0,T] \;\big|\; X_i(t) = X_j(t)\big\}$ has Lebesgue measure zero, almost surely). Then the kth rank processes $X_{(k)}$, for all $k = 1, 2, \ldots, n$, are **continuous semimartingales** such that a.s., for $t \in [0,T]$,*

$$dX_{(k)}(t) \quad = \quad \sum_{i=1}^n \mathbb{1}_{\{u_t(k)=i\}}\, dX_i(t) + \sum_{i=1}^n \frac{1}{N_t(k)}\, d\Lambda_{\left(X_{(k)}-X_{(i)}\right)^+}(t) - \sum_{i=1}^n \frac{1}{N_t(k)}\, d\Lambda_{\left(X_{(k)}-X_{(i)}\right)^-}(t) \tag{6.6.72}$$

$$= \quad \sum_{i=1}^n \mathbb{1}_{\{u_t(k)=i\}}\, dX_i(t) + \sum_{\ell=k+1}^n \frac{1}{N_t(k)}\, d\Lambda_{X_{(k)}-X_{(\ell)}}(t) - \sum_{\ell=1}^{k-1} \frac{1}{N_t(k)}\, d\Lambda_{X_{(\ell)}-X_{(k)}}(t), \tag{6.6.73}$$

for any u in the set $\mathcal{U}$ of (6.6.57) *and $N_\cdot(k)$ is defined as in* (6.6.54) *above.*

Proof. Apply (6.6.70) of Theorem 6.6.5 to the last two sums on the right-hand side of (6.6.58) of Theorem 6.6.3, to obtain

$$\begin{aligned} dX_{(k)}(t) &= \sum_{i=1}^{n} \mathbb{1}_{\{u_t(k)=i\}}\, dX_i(t) + \sum_{i=1}^{n} \frac{1}{N_t(k)}\, d\Lambda_{(X_{(k)}-X_i)^+}(t) - \sum_{i=1}^{n} \frac{1}{N_t(k)}\, d\Lambda_{(X_{(k)}-X_i)^-}(t) \\ &= \sum_{i=1}^{n} \mathbb{1}_{\{u_t(k)=i\}}\, dX_i(t) + \sum_{i=1}^{n} \frac{1}{N_t(k)}\, d\Lambda_{(X_{(k)}-X_{(i)})^+}(t) - \sum_{i=1}^{n} \frac{1}{N_t(k)}\, d\Lambda_{(X_{(k)}-X_{(i)})^-}(t). \end{aligned}$$

Now, note that $\big(X_{(k)}(t)-X_{(\ell)}(t)\big)^+$ (the positive part) and $\big(X_{(k)}(t)-X_{(\ell)}(t)\big)^-$ (the negative part) are continuous nonnegative semimartingales given respectively by

$$\begin{aligned} &\big(X_{(k)}(t) - X_{(\ell)}(t)\big)^+ \\ &\quad \equiv \begin{cases} X_{(k)}(t) - X_{(\ell)}(t) & \text{if} \quad X_{(k)}(t) > X_{(\ell)}(t) \\ 0 & \text{if} \quad X_{(k)}(t) \le X_{(\ell)}(t) \end{cases} \equiv \begin{cases} X_{(k)}(t) - X_{(\ell)}(t) & \text{if} \quad \ell > k, \\ 0 & \text{if} \quad \ell \le k, \end{cases} \end{aligned} \tag{6.6.74}$$

and

$$\begin{aligned} &\big(X_{(k)}(t) - X_{(\ell)}(t)\big)^- \\ &\quad \equiv \begin{cases} X_{(\ell)}(t) - X_{(k)}(t) & \text{if} \quad X_{(k)}(t) < X_{(\ell)}(t) \\ 0 & \text{if} \quad X_{(k)}(t) \ge X_{(\ell)}(t) \end{cases} \equiv \begin{cases} X_{(\ell)}(t) - X_{(k)}(t) & \text{if} \quad \ell < k, \\ 0 & \text{if} \quad \ell \ge k. \end{cases} \end{aligned} \tag{6.6.75}$$

Therefore, by the above information, we arrive at the following

$$dX_{(k)}(t) = \sum_{i=1}^{n} \mathbb{1}_{\{u_t(k)=i\}}\, dX_i(t) + \sum_{\ell=k+1}^{n} \frac{1}{N_t(k)}\, d\Lambda_{X_{(k)}-X_{(\ell)}}(t) - \sum_{\ell=1}^{k-1} \frac{1}{N_t(k)}\, d\Lambda_{X_{(\ell)}-X_{(k)}}(t).$$

■

Corollary 6.6.7 ([**Banner & Ghomrasni (2008)**]). *Let $X_1, \ldots, X_n$ be **pathwise mutually nondegenerate absolutely continuous semimartingales**, i.e., suppose that $X_1, \ldots, X_n$ satisfy the hypotheses of Theorem 6.6.6, and in addition, for all $i < j < k$ we have $\big\{t \in [0,T] \;\big|\; X_i(t) = X_j(t) = X_k(t)\big\} = \varnothing$, almost surely. Then the kth rank processes $X_{(k)}$, for all $k = 1, 2, \ldots, n$, are **continuous semimartingales** such that a.s., for $t \in [0,T]$,*

$$dX_{(k)}(t) = \sum_{i=1}^{n} \mathbb{1}_{\{u_t(k)=i\}}\, dX_i(t) + \frac{1}{2}\, d\Lambda_{X_{(k)}-X_{(k+1)}}(t) - \frac{1}{2}\, d\Lambda_{X_{(k-1)}-X_{(k)}}(t), \tag{6.6.76}$$

for any u in the set $\mathcal{U}$ of (6.6.57).

Proof. Indeed, in Corollary 6.6.7, the added assumption that for all $i < j < k$ we have $\big\{t \in [0,T] \;\big|\; X_i(t) = X_j(t) = X_k(t)\big\} = \varnothing$, a.s., i.e., there are no triple points, implies that at any point in time $t \in [0,T]$, at most two stocks can coincide. Hence, there could possibly be one stock or two stocks that are at rank k at time $t \in [0,T]$. So, we must have $N_t(k) = 1$ or $N_t(k) = 2$ for all $t \in [0,T]$. When $N_t(k) = 1$, the local time of the continuous semimartingales $X_{(k)} - X_{(\ell)}$ for $\ell = k+1, \ldots, n$, at the level zero at time $t \in [0,T]$, is zero, since $X_{(k)}(t) \neq X_{(\ell)}(t)$ for all $\ell = k+1, \ldots, n$, $t \in [0,T]$. Similarly, for the local time of the continuous semimartingales $X_{(\ell)} - X_{(k)}$ for $\ell = 1, \ldots, k-1$, at the level zero at time $t \in [0,T]$. Thus the associated local times in (6.6.73) vanish. Whereas, when $N_t(k) = 2$, we could have either $X_{(k)}(t) = X_{(k+1)}(t)$ or $X_{(k)}(t) = X_{(k-1)}(t)$, for $t \in [0,T]$. Thus, only the two associated local times make a nonzero contribution. Note that in the case where $u_t(\cdot)$, $\boldsymbol{u}_t$, is the random permutation $p_t(\cdot)$, $\boldsymbol{p}_t$, Corollary 6.6.7 reduces to (6.6.11) of Proposition 6.6.2. ■

6.6.3 Decomposition of the Rank Processes of Continuous Semimartingales

Theorem 6.6.8. *Let $X_1, \ldots, X_n$ be **continuous semimartingales**, then the kth rank processes $X_{(k)}$, for all $k = 1, 2, \ldots, n$, are **continuous semimartingales** such that a.s., for $t \in [0, T]$,*

$$dX_{(k)}(t) = \sum_{i=1}^{n} \frac{1}{N_t(k)} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, dX_i(t) + \sum_{i=1}^{n} \frac{1}{N_t(k)}\, d\Lambda_{(X_{(k)}-X_i)^+}(t) - \sum_{i=1}^{n} \frac{1}{N_t(k)}\, d\Lambda_{(X_{(k)}-X_i)^-}(t). \tag{6.6.77}$$

Proof. From (6.6.55), we obtain for $t \in [0, T]$,

$$N_t(k)\, dX_{(k)}(t) = \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, dX_{(k)}(t), \tag{6.6.78}$$

which can be expressed as

$$\begin{aligned} N_t(k)\, dX_{(k)}(t) &= \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, dX_i(t) + \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}} \Big[dX_{(k)}(t) - dX_i(t)\Big] \\ &= \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, dX_i(t) + \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, d\big(X_{(k)}(t) - X_i(t)\big). \end{aligned}$$

Again, since $x \equiv x^+ - x^-$, following the same idea as that employed in the proof of Theorem 6.6.3, we obtain

$$\begin{aligned} N_t(k)\, dX_{(k)}(t) &= \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, dX_i(t) + \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t)-X_i(t)=0\}}\, d\Big(\big(X_{(k)}(t) - X_i(t)\big)^+\Big) \\ &\quad - \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t)-X_i(t)=0\}}\, d\Big(\big(X_{(k)}(t) - X_i(t)\big)^-\Big). \end{aligned}$$

Since $\big(X_{(k)}(t) - X_i(t)\big)^+$ and $\big(X_{(k)}(t) - X_i(t)\big)^-$ are *continuous nonnegative semimartingales*, applying (6.6.66) (or (6.6.67) and (6.6.68)) to these continuous nonnegative semimartingales yields

$$\begin{aligned} \mathbb{1}_{\{X_{(k)}(t)-X_i(t)=0\}}\, d\Big(\big(X_{(k)}(t) - X_i(t)\big)^+\Big) &= d\Lambda_{(X_{(k)}-X_i)^+}(t), \\ \mathbb{1}_{\{X_{(k)}(t)-X_i(t)=0\}}\, d\Big(\big(X_{(k)}(t) - X_i(t)\big)^-\Big) &= d\Lambda_{(X_{(k)}-X_i)^-}(t). \end{aligned}$$

Employing the above equations, we arrive at the following

$$N_t(k)\, dX_{(k)}(t) = \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, dX_i(t) + \sum_{i=1}^{n} d\Lambda_{(X_{(k)}-X_i)^+}(t) - \sum_{i=1}^{n} d\Lambda_{(X_{(k)}-X_i)^-}(t).$$

Hence, we have for $t \in [0, T]$,

$$\begin{aligned} dX_{(k)}(t) &= \frac{1}{N_t(k)} \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, dX_i(t) + \frac{1}{N_t(k)} \sum_{i=1}^{n} d\Lambda_{(X_{(k)}-X_i)^+}(t) - \frac{1}{N_t(k)} \sum_{i=1}^{n} d\Lambda_{(X_{(k)}-X_i)^-}(t) \\ &= \sum_{i=1}^{n} \frac{1}{N_t(k)} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, dX_i(t) + \sum_{i=1}^{n} \frac{1}{N_t(k)}\, d\Lambda_{(X_{(k)}-X_i)^+}(t) - \sum_{i=1}^{n} \frac{1}{N_t(k)}\, d\Lambda_{(X_{(k)}-X_i)^-}(t). \end{aligned}$$

■

Theorem 6.6.9. *Let $X_1, \ldots, X_n$ be **continuous semimartingales**, then the rank processes $X_{(1)}, \ldots, X_{(n)}$ are **continuous semimartingales**, and the following equality holds for $t \in [0, T]$:*

$$\sum_{i=1}^{n} \mathbb{1}_{\{X_{(i)}(t)=0\}}\, d\big(X_{(i)}(t)\big)^+ = \sum_{i=1}^{n} \mathbb{1}_{\{X_i(t)=0\}}\, d\big(X_i(t)\big)^+, \tag{6.6.79}$$

alternatively expressed as,

$$\sum_{i=1}^{n} \mathbb{1}_{\{X_{(i)}(t)=0\}}\, dX_{(i)}^{+}(t) \quad = \quad \sum_{i=1}^{n} \mathbb{1}_{\{X_i(t)=0\}}\, dX_i^{+}(t). \tag{6.6.80}$$

Proof. Refer to the proof of Theorem 6.6.18 for general semimartingales. ■

Corollary 6.6.10. *Let $X_1, X_2, \ldots, X_n$ be* ***continuous nonnegative semimartingales****, then the rank processes $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ are* ***continuous nonnegative semimartingales****, and the following equality holds for $t \in [0, T]$:*

$$\sum_{i=1}^{n} \mathbb{1}_{\{X_{(i)}(t)=0\}}\, dX_{(i)}(t) \quad = \quad \sum_{i=1}^{n} \mathbb{1}_{\{X_i(t)=0\}}\, dX_i(t). \tag{6.6.81}$$

Proof. Refer to Ghomrasni (2005, Proof of Theorem 2.1). ■

Corollary 6.6.11. *Let $X_1, \ldots, X_n$ be* ***continuous semimartingales****. Then the kth ranked processes $X_{(k)}$, for all $k = 1, 2, \ldots, n$, are* ***continuous semimartingales****, and we have*

$$\sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t)=X_{(i)}(t)\}}\, d\Big(\big(X_{(i)}(t) - X_{(k)}(t)\big)^{+}\Big) \quad = \quad \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, d\Big(\big(X_i(t) - X_{(k)}(t)\big)^{+}\Big). \tag{6.6.82}$$

Proof. Define the processes $Y_1, Y_2, \ldots, Y_n$ by

$$Y_i(t) \quad \triangleq \quad X_i(t) - X_{(k)}(t), \tag{6.6.83}$$

for all $i = 1, 2, \ldots, n$, and fixed k, and $t \in [0, T]$. Then, the assumption that $X_1, X_2, \ldots, X_n$ are continuous semimartingales, implies that $Y_1, Y_2, \ldots, Y_n$ are continuous semimartingales. Moreover, we also define the processes $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ by

$$Y_{(i)}(t) \quad \triangleq \quad X_{(i)}(t) - X_{(k)}(t), \tag{6.6.84}$$

for all $i = 1, 2, \ldots, n$, and $t \in [0, T]$. These processes are then the ranked processes of $Y_i(t)$, for $i = 1, 2, \ldots, n$ and satisfy the property $Y_{(1)}(t) \geq Y_{(2)}(t) \geq \cdots \geq Y_{(n-1)}(t) \geq Y_{(n)}(t)$. These ranked processes are also continuous semimartingales. Thus, for the continuous semimartingales $Y_1, Y_2, \ldots, Y_n$, we have by Theorem 6.6.9

$$\sum_{i=1}^{n} \mathbb{1}_{\{Y_{(i)}(t)=0\}}\, dY_{(i)}^{+}(t) \quad = \quad \sum_{i=1}^{n} \mathbb{1}_{\{Y_i(t)=0\}}\, dY_i^{+}(t).$$

Hence, equations (6.6.83) and (6.6.84) imply that

$$\sum_{i=1}^{n} \mathbb{1}_{\{X_{(i)}(t)-X_{(k)}(t)=0\}}\, d\Big(\big(X_{(i)}(t) - X_{(k)}(t)\big)^{+}\Big) \quad = \quad \sum_{i=1}^{n} \mathbb{1}_{\{X_i(t)-X_{(k)}(t)=0\}}\, d\Big(\big(X_i(t) - X_{(k)}(t)\big)^{+}\Big),$$

and the desired result follows. ■

Corollary 6.6.12. *Let $X_1, \ldots, X_n$ be* ***continuous semimartingales****. Then the kth ranked processes $X_{(k)}$, for all $k = 1, 2, \ldots, n$, are* ***continuous semimartingales****, and we have*

$$\sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t)=X_{(i)}(t)\}}\, d\Big(\big(X_{(k)}(t) - X_{(i)}(t)\big)^{+}\Big) \quad = \quad \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, d\Big(\big(X_{(k)}(t) - X_i(t)\big)^{+}\Big). \tag{6.6.85}$$

Proof. Define the processes $Z_1, Z_2, \ldots, Z_n$ by

$$Z_i(t) \triangleq X_{(k)}(t) - X_i(t), \tag{6.6.86}$$

for all $i = 1, 2, \ldots, n$, and fixed k, and $t \in [0, T]$. Then, the assumption that $X_1, X_2, \ldots, X_n$ are continuous semimartingales, implies that $Z_1, Z_2, \ldots, Z_n$ are continuous semimartingales. Moreover, we also define the processes $Z_{(1)}, Z_{(2)}, \ldots, Z_{(n)}$ by

$$Z_{(n-i+1)}(t) \triangleq X_{(k)}(t) - X_{(i)}(t), \tag{6.6.87}$$

for all $i = 1, 2, \ldots, n$, and $t \in [0, T]$. These processes are then the ranked processes of $Z_i(t)$, for $i = 1, 2, \ldots, n$ and satisfy the property $Z_{(1)}(t) \geq Z_{(2)}(t) \geq \cdots \geq Z_{(n-1)}(t) \geq Z_{(n)}(t)$. These ranked processes are also continuous semimartingales. Thus, for the continuous semimartingales $Z_1, Z_2, \ldots, Z_n$, we have by Theorem 6.6.9

$$\sum_{i=1}^{n} \mathbb{1}_{\{Z_{(i)}(t)=0\}}\, dZ_{(i)}^{+}(t) = \sum_{i=1}^{n} \mathbb{1}_{\{Z_i(t)=0\}}\, dZ_i^{+}(t).$$

Hence, equations (6.6.86) and (6.6.87) imply that

$$\sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t)-X_{(i)}(t)=0\}}\, d\Big(\big(X_{(k)}(t) - X_{(i)}(t)\big)^{+}\Big) = \sum_{i=1}^{n} \mathbb{1}_{\{X_k(t)-X_{(i)}(t)=0\}}\, d\Big(\big(X_{(k)}(t) - X_i(t)\big)^{+}\Big),$$

and the desired result follows. ■

Theorem 6.6.13 ([**Yan (1985, 1989), Ouknine (1990)**]). *For **continuous semimartingales** X and Y, the processes $X \vee Y$ and $X \wedge Y$ are **continuous semimartingales**, and we have for all $t \in [0, T]$,*

$$\Lambda_{X \vee Y}(t) + \Lambda_{X \wedge Y}(t) = \Lambda_X(t) + \Lambda_Y(t). \tag{6.6.88}$$

This result can be extended to the case of three or more continuous semimartingales.

Theorem 6.6.14 ([**Banner & Ghomrasni (2008)**]). *For **continuous semimartingales** $X_1, \ldots, X_n$, the rank processes $X_{(1)}, \ldots, X_{(n)}$ are **continuous semimartingales**, and we have for all $t \in [0, T]$,*

$$\sum_{i=1}^{n} \Lambda_{X_{(i)}}(t) = \sum_{i=1}^{n} \Lambda_{X_i}(t), \tag{6.6.89}$$

alternatively,

$$\sum_{k=1}^{n} \Lambda_{X_{(k)}}(t) = \sum_{i=1}^{n} \Lambda_{X_i}(t). \tag{6.6.90}$$

Proof. We proceed by induction. The case $n = 1$ is trivial. The case $n = 2$ is precisely Theorem 6.6.13. Now assume the result (6.6.89) holds for some n. Given continuous semimartingales $X_1, \ldots, X_n, X_{n+1}$, the kth rank process $X_{(k)}$, $k = 1, 2, \ldots, n$ is defined by (6.2.1) in Definition 6.2.1. Moreover, set

$$X_{[k]}(t) \triangleq \max_{1 \leq i_1 < \cdots < i_k \leq n+1} \min\Big(X_{i_1}(t), \ldots, X_{i_k}(t)\Big), \qquad t \in [0, T], \tag{6.6.91}$$

where $i_1 \geq 1$ and $i_k \leq n + 1$. The process $X_{[k]} = \big\{X_{[k]}(t), t \in [0, T]\big\}$ is the kth rank process with respect to all $n + 1$ semimartingales $X_1, \ldots, X_n, X_{n+1}$. It will be convenient to set $X_{(0)}(t) \equiv \infty$. Suppose first that $k > 1$. By Theorem 6.6.13, we have

$$\Lambda_{\left(X_{(k-1)} \wedge X_{n+1}\right) \vee X_{(k)}}(t) + \Lambda_{\left(X_{(k-1)} \wedge X_{n+1}\right) \wedge X_{(k)}}(t) = \Lambda_{X_{(k-1)} \wedge X_{n+1}}(t) + \Lambda_{X_{(k)}}(t). \tag{6.6.92}$$

Since $X_{(k-1)}(t) \geq X_{(k)}(t)$ for all $t \in [0,T]$, the second term on the left-hand side of the above equation is simply $\Lambda_{X_{(k)} \wedge X_{n+1}}(t)$. In addition, for the first term on the left-hand side of the above equation, we have

$$\Big(\big(X_{(k-1)} \wedge X_{n+1}\big) \vee X_{(k)}\Big)(t) = \begin{cases} X_{(k-1)}(t) & \text{if} \quad X_{n+1}(t) \geq X_{(k-1)}(t) \geq X_{(k)}(t), \\ X_{n+1}(t) & \text{if} \quad X_{(k-1)}(t) \geq X_{n+1}(t) \geq X_{(k)}(t), \\ X_{(k)}(t) & \text{if} \quad X_{(k-1)}(t) \geq X_{(k)}(t) \geq X_{n+1}(t). \end{cases} \tag{6.6.93}$$

For the case where $X_{n+1}(t) \geq X_{(k-1)}(t) \geq X_{(k)}(t)$, it is clear that $X_{n+1}(t)$ could be either of the following $X_{[1]}(t), \ldots, X_{[k-1]}(t)$, so that the $(k-1)$th rank process of the n semimartingales now becomes the kth rank process of all the $n+1$ semimartingales $X_{(k-1)}(t) \equiv X_{[k]}(t)$, i.e., it moves one down in rank since X_{n+1} is larger than $X_{(k-1)}$ and thus results in shifts of rank. For the case where $X_{(k-1)}(t) \geq X_{n+1}(t) \geq X_{(k)}(t)$, the $(k-1)$th rank process of the n semimartingales is the $(k-1)$th rank process of all $n+1$ semimartingales $X_{(k-1)}(t) \equiv X_{[k-1]}(t)$, since the addition of X_{n+1} causes no shifts in rank for ranks above $k-1$. However, since $X_{n+1}(t) \geq X_{(k)}(t)$, this causes shifts in rank for ranks below $k-1$, and we have $X_{n+1}(t) \equiv X_{[k]}(t)$ and $X_{(k)}(t) \equiv X_{[k+1]}(t)$. Finally, for the case where $X_{(k-1)}(t) \geq X_{(k)}(t) \geq X_{n+1}(t)$, the $(k-1)$th rank process of the n semimartingales is the $(k-1)$th rank process of all $n+1$ semimartingales $X_{(k-1)}(t) \equiv X_{[k-1]}(t)$ and the kth rank process of the n semimartingales is the kth rank process of all $n+1$ semimartingales $X_{(k)}(t) \equiv X_{[k]}(t)$, since the addition of X_{n+1} causes no shifts in rank for ranks above k. Thus, $X_{n+1}(t)$ could be either of the following $X_{[k+1]}(t), \ldots, X_{[n+1]}(t)$. Consequently, for each case we have

$$\Big(\big(X_{(k-1)} \wedge X_{n+1}\big) \vee X_{(k)}\Big)(t) = \begin{cases} X_{(k-1)}(t) \equiv X_{[k]}(t) & \text{if} \quad X_{n+1}(t) \geq X_{(k-1)}(t) \geq X_{(k)}(t), \\ X_{n+1}(t) \equiv X_{[k]}(t) & \text{if} \quad X_{(k-1)}(t) \geq X_{n+1}(t) \geq X_{(k)}(t), \\ X_{(k)}(t) \equiv X_{[k]}(t) & \text{if} \quad X_{(k-1)}(t) \geq X_{(k)}(t) \geq X_{n+1}(t). \end{cases}$$

Thus, $\big(\big(X_{(k-1)} \wedge X_{n+1}\big) \vee X_{(k)}\big)(t)$ is the kth smallest of the processes $X_1(t), \ldots, X_{n+1}(t)$, so that $\big(\big(X_{(k-1)} \wedge X_{n+1}\big) \vee X_{(k)}\big)(t) \equiv X_{[k]}(t)$ for all $t \in [0,T]$. This shows that $X_{[k]}$ is a continuous semimartingale for $k = 1, 2, \ldots, n$. Therefore, for $k = 2, \ldots, n$, (6.6.92) becomes

$$\Lambda_{X_{[k]}}(t) + \Lambda_{X_{(k)} \wedge X_{n+1}}(t) = \Lambda_{X_{(k-1)} \wedge X_{n+1}}(t) + \Lambda_{X_{(k)}}(t). \tag{6.6.94}$$

If $k = 1$, then since $X_{(0)}(t) \equiv \infty$, the above equation reduces to

$$\Lambda_{X_{[1]}}(t) + \Lambda_{X_{(1)} \wedge X_{n+1}}(t) = \Lambda_{X_{n+1}}(t) + \Lambda_{X_{(1)}}(t). \tag{6.6.95}$$

This equation also follows from Theorem 6.6.13, where we have

$$\Lambda_{X_{(1)} \vee X_{n+1}}(t) + \Lambda_{X_{(1)} \wedge X_{n+1}}(t) = \Lambda_{X_{n+1}}(t) + \Lambda_{X_{(1)}}(t).$$

Observe that $\big(X_{(1)} \vee X_{n+1}\big)(t) \equiv X_{[1]}(t)$ for all $t \in [0,T]$. Then we get the result (6.6.95) which is precisely (6.6.94) for $k = 1$. Thus, we have the following formula which holds for all $k = 1, 2, \ldots, n$ and $t \in [0,T]$

$$\Lambda_{X_{[k]}}(t) + \Lambda_{X_{(k)} \wedge X_{n+1}}(t) = \Lambda_{X_{(k-1)} \wedge X_{n+1}}(t) + \Lambda_{X_{(k)}}(t), \tag{6.6.96}$$

and, we have

$$\Lambda_{X_{(k)}}(t) = \Lambda_{X_{[k]}}(t) + \Lambda_{X_{(k)} \wedge X_{n+1}}(t) - \Lambda_{X_{(k-1)} \wedge X_{n+1}}(t). \tag{6.6.97}$$

Finally, by the induction hypothesis, equation (6.6.97) and the facts that $\big(X_{(n)} \wedge X_{n+1}\big)(t) \equiv X_{[n+1]}(t)$ and

$\big(X_{(0)} \wedge X_{n+1}\big)(t) \equiv X_{n+1}(t)$, we see that $X_{[n+1]}$ is a continuous semimartingale, and we have

$$\begin{aligned}
\sum_{i=1}^{n+1} \Lambda_{X_i}(t) &= \sum_{i=1}^{n} \Lambda_{X_i}(t) + \Lambda_{X_{n+1}}(t) \\
&= \sum_{i=1}^{n} \Lambda_{X_{(i)}}(t) + \Lambda_{X_{n+1}}(t) \\
&= \sum_{k=1}^{n} \Lambda_{X_{(k)}}(t) + \Lambda_{X_{n+1}}(t) \\
&= \sum_{k=1}^{n} \Big[\Lambda_{X_{[k]}}(t) + \Lambda_{X_{(k)} \wedge X_{n+1}}(t) - \Lambda_{X_{(k-1)} \wedge X_{n+1}}(t)\Big] + \Lambda_{X_{n+1}}(t) \\
&= \sum_{k=1}^{n} \Lambda_{X_{[k]}}(t) + \sum_{k=1}^{n} \Big[\Lambda_{X_{(k)} \wedge X_{n+1}}(t) - \Lambda_{X_{(k-1)} \wedge X_{n+1}}(t)\Big] + \Lambda_{X_{n+1}}(t) \\
&= \sum_{k=1}^{n} \Lambda_{X_{[k]}}(t) + \Lambda_{X_{(n)} \wedge X_{n+1}}(t) - \Lambda_{X_{(0)} \wedge X_{n+1}}(t) + \Lambda_{X_{n+1}}(t) \\
&= \sum_{k=1}^{n} \Lambda_{X_{[k]}}(t) + \Lambda_{X_{[n+1]}}(t) - \Lambda_{X_{n+1}}(t) + \Lambda_{X_{n+1}}(t) \\
&= \sum_{k=1}^{n+1} \Lambda_{X_{[k]}}(t) \\
&= \sum_{i=1}^{n+1} \Lambda_{X_{[i]}}(t),
\end{aligned}$$

for all $t \in [0, T]$, the desired result follows by induction. The result is also a consequence of Theorem 6.6.9 and (6.3.148). ■

Consequently, from the foregoing theorem, we have the following results

$$\sum_{i=1}^{n} \Lambda_{X^+_{(i)}}(t) = \sum_{i=1}^{n} \Lambda_{X^+_i}(t), \tag{6.6.98}$$

$$\sum_{i=1}^{n} \Lambda_{X^-_{(i)}}(t) = \sum_{i=1}^{n} \Lambda_{X^-_i}(t). \tag{6.6.99}$$

Corollary 6.6.15. *For **continuous semimartingales** $X_1, \ldots, X_n$, the rank processes $X_{(1)}, \ldots, X_{(n)}$ are **continuous semimartingales**, and we have for all $t \in [0, T]$,*

$$\sum_{i=1}^{n} \Lambda_{X_{(k)} - X_{(i)}}(t) = \sum_{i=1}^{n} \Lambda_{X_{(k)} - X_i}(t). \tag{6.6.100}$$

Proof. The result follows directly from Corollary 6.6.12 and (6.3.148), or is a direct consequence of Theorem 6.6.14. ■

Consequently, from the foregoing corollary, we have the following results

$$\sum_{i=1}^{n} \Lambda_{\left(X_{(k)} - X_{(i)}\right)^+}(t) = \sum_{i=1}^{n} \Lambda_{\left(X_{(k)} - X_i\right)^+}(t), \tag{6.6.101}$$

$$\sum_{i=1}^{n} \Lambda_{\left(X_{(k)} - X_{(i)}\right)^-}(t) = \sum_{i=1}^{n} \Lambda_{\left(X_{(k)} - X_i\right)^-}(t). \tag{6.6.102}$$

Theorem 6.6.16 ([**Banner & Ghomrasni (2008)**]). *Let $X_1, \ldots, X_n$ be **continuous semimartingales**, then the kth rank processes $X_{(k)}$, for all $k = 1, 2, \ldots, n$, are **continuous semimartingales** such that a.s., for $t \in [0, T]$,*

$$\begin{aligned} & dX_{(k)}(t) \\ &= \sum_{i=1}^{n} \frac{1}{N_t(k)} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, dX_i(t) + \sum_{i=1}^{n} \frac{1}{N_t(k)}\, d\Lambda_{(X_{(k)}-X_{(i)})^+}(t) - \sum_{i=1}^{n} \frac{1}{N_t(k)}\, d\Lambda_{(X_{(k)}-X_{(i)})^-}(t), \quad (6.6.103) \\ &= \sum_{i=1}^{n} \frac{1}{N_t(k)} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, dX_i(t) + \sum_{\ell=k+1}^{n} \frac{1}{N_t(k)}\, d\Lambda_{X_{(k)}-X_{(\ell)}}(t) - \sum_{\ell=1}^{k-1} \frac{1}{N_t(k)}\, d\Lambda_{X_{(\ell)}-X_{(k)}}(t). \quad (6.6.104) \end{aligned}$$

Proof. Apply either (6.6.89) of Theorem 6.6.14 (alternatively, (6.6.98) in combination with (6.6.99)), or (6.6.100) of Corollary 6.6.15 (alternatively, (6.6.101) in combination with (6.6.102)), to the last two sums on the right-hand side of (6.6.77) of Theorem 6.6.8, to obtain

$$\begin{aligned} dX_{(k)}(t) &= \sum_{i=1}^{n} \frac{1}{N_t(k)} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, dX_i(t) + \sum_{i=1}^{n} \frac{1}{N_t(k)}\, d\Lambda_{(X_{(k)}-X_i)^+}(t) - \sum_{i=1}^{n} \frac{1}{N_t(k)}\, d\Lambda_{(X_{(k)}-X_i)^-}(t) \\ &= \sum_{i=1}^{n} \frac{1}{N_t(k)} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, dX_i(t) + \sum_{i=1}^{n} \frac{1}{N_t(k)}\, d\Lambda_{(X_{(k)}-X_{(i)})^+}(t) - \sum_{i=1}^{n} \frac{1}{N_t(k)}\, d\Lambda_{(X_{(k)}-X_{(i)})^-}(t). \end{aligned}$$

Therefore, by (6.6.74) and (6.6.75), we arrive at the following

$$dX_{(k)}(t) = \sum_{i=1}^{n} \frac{1}{N_t(k)} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, dX_i(t) + \sum_{\ell=k+1}^{n} \frac{1}{N_t(k)}\, d\Lambda_{X_{(k)}-X_{(\ell)}}(t) - \sum_{\ell=1}^{k-1} \frac{1}{N_t(k)}\, d\Lambda_{X_{(\ell)}-X_{(k)}}(t).$$

An alternative proof is offered in Banner & Ghomrasni (2008, Proof of Theorem 2.3), where the proof is almost identical to that of Theorem 6.6.3 and Theorem 6.6.6, combined. Since, we may rewrite (6.6.62) as

$$n!\, dX_{(k)}(t) = \sum_{i=1}^{n} dX_i(t) \Big[\sum_{\boldsymbol{j} \in \mathcal{J}} \mathbb{1}_{\{u_t^{\boldsymbol{j}}(k)=i\}} \Big] + \sum_{i=1}^{n} d\big(X_{(k)}(t) - X_i(t)\big) \Big[\sum_{\boldsymbol{j} \in \mathcal{J}} \mathbb{1}_{\{u_t^{\boldsymbol{j}}(k)=i\}} \Big].$$

Using (6.6.64) in the above equation, we get

$$\begin{aligned} n!\, dX_{(k)}(t) &= \sum_{i=1}^{n} \frac{n!}{N_t(k)} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, dX_i(t) + \sum_{i=1}^{n} \frac{n!}{N_t(k)} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, d\big(X_{(k)}(t) - X_i(t)\big) \\ &= n! \sum_{i=1}^{n} \frac{1}{N_t(k)} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, dX_i(t) + n! \sum_{i=1}^{n} \frac{1}{N_t(k)} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, d\big(X_{(k)}(t) - X_i(t)\big). \end{aligned}$$

Hence, we have

$$dX_{(k)}(t) = \sum_{i=1}^{n} \frac{1}{N_t(k)} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, dX_i(t) + \sum_{i=1}^{n} \frac{1}{N_t(k)} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, d\big(X_{(k)}(t) - X_i(t)\big).$$

The second term on the right-hand side is handled as before. ■

6.6.4 Decomposition of the Rank Processes of General Semimartingales

We shall need the following definitions:

$$S_{t-}(k) \triangleq \Big\{ i \,:\, X_i(t-) = X_{(k)}(t-) \Big\}, \qquad \text{and} \qquad N_{t-}(k) \triangleq \big| S_{t-}(k) \big|, \quad (6.6.105)$$

in particular, $S_{t-}(k)$ is the set of indices (subscripts) of processes which are kth ranked, and its associated cardinality $N_{t-}(k)$ denotes the number of indices (subscripts) such that $X_i(t-) = X_{(k)}(t-)$, in other words

$N_{t-}(k)$ is the number of processes that are at (equal) rank at time $t \in [0,T]$. As such, $N_{t-}(k)$ can alternatively be represented as

$$N_{t-}(k) \quad = \quad \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t-)=X_i(t-)\}}. \tag{6.6.106}$$

Theorem 6.6.17 ([**Pamen (2009), Ghomrasni & Pamen (2010)**]). *Let $X_1, \ldots, X_n$ be **semimartingales**. Then the kth ranked processes $X_{(k)}$, for all $k = 1, 2, \ldots, n$, are **semimartingales** such that a.s., for $t \in [0,T]$,*

$$\begin{aligned} & dX_{(k)}(t) \\ & = \sum_{i=1}^{n} \frac{1}{N_{t-}(k)} \mathbb{1}_{\{X_{(k)}(t-)=X_i(t-)\}}\, dX_i(t) + \sum_{i=1}^{n} \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{(X_{(k)}-X_i)^+}(t) - \sum_{i=1}^{n} \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{(X_{(k)}-X_i)^-}(t), \end{aligned} \tag{6.6.107}$$

or,

$$\begin{aligned} & dX_{(k)}(t) \\ & = \sum_{i=1}^{n} \frac{1}{N_{t-}(k)} \mathbb{1}_{\{X_{(k)}(t-)=X_{(i)}(t-)\}}\, dX_{(i)}(t) + \sum_{i=1}^{n} \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{(X_{(k)}-X_{(i)})^+}(t) - \sum_{i=1}^{n} \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{(X_{(k)}-X_{(i)})^-}(t) \end{aligned} \tag{6.6.108}$$

$$= \sum_{i=1}^{n} \frac{1}{N_{t-}(k)} \mathbb{1}_{\{X_{(k)}(t-)=X_{(i)}(t-)\}}\, dX_{(i)}(t) + \sum_{\ell=k+1}^{n} \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{X_{(k)}-X_{(\ell)}}(t) - \sum_{\ell=1}^{k-1} \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{X_{(\ell)}-X_{(k)}}(t), \tag{6.6.109}$$

where

$$\mathcal{L}_X(t) \quad \triangleq \quad \Lambda_X(t) + \sum_{s \le t} \mathbb{1}_{\{X(s-)=0\}}\, \Delta X(s), \tag{6.6.110}$$

and $\Lambda_X(t)$ is the local time for the semimartingale X at the level zero.

Proof. Firstly, recall (6.3.92) for a *nonnegative semimartingale* X,

$$\Lambda_X(t) \quad = \quad \int_{0+}^{t} \mathbb{1}_{\{X(s-)=0\}}\, dX(s) - \sum_{s \le t} \mathbb{1}_{\{X(s-)=0\}}\, \Delta X(s).$$

Therefore,

$$\int_{0+}^{t} \mathbb{1}_{\{X(s-)=0\}}\, dX(s) \quad = \quad \Lambda_X(t) + \sum_{s \le t} \mathbb{1}_{\{X(s-)=0\}}\, \Delta X(s).$$

Thus, by the manner in which we defined the process $\mathcal{L}_X = \big\{\mathcal{L}_X(t), \mathcal{F}_t, t \in [0,T]\big\}$, as in (6.6.110), we have the following formula for any semimartingale X,

$$\mathcal{L}_X(t) \quad = \quad \int_{0+}^{t} \mathbb{1}_{\{X(s-)=0\}}\, dX(s), \tag{6.6.111}$$

which is then valid for all *nonnegative semimartingales* X, and therefore the following also holds for nonnegative semimartingales

$$d\mathcal{L}_X(t) \quad = \quad \mathbb{1}_{\{X(t-)=0\}}\, dX(t). \tag{6.6.112}$$

We can also define $\mathcal{L}_{X^+}$, in much the same way we defined $\mathcal{L}_X$ in (6.6.110), as follows

$$\mathcal{L}_{X^+}(t) \quad \triangleq \quad \Lambda_{X^+}(t) + \sum_{s \le t} \mathbb{1}_{\{X^+(s-)=0\}}\, \Delta X^+(s) \quad = \quad \int_{0+}^{t} \mathbb{1}_{X^+(s-)=0}\, dX^+(s). \tag{6.6.113}$$

By (6.3.103), we can express the above as follows

$$\mathcal{L}_{X^+}(t) = \Lambda_{X^+}(t) + \sum_{s \le t} \mathbb{1}_{\{X(s-)=0\}} \, \Delta X^+(s). \tag{6.6.114}$$

From (6.3.95), for the nonnegative semimartingale X^+, we have

$$\Lambda_{X^+}(t) = \int_{0+}^{t} \mathbb{1}_{\{X(s-)=0\}} \, dX^+(s) - \sum_{s \le t} \mathbb{1}_{\{X(s-)=0\}} \, \Delta X^+(s).$$

Therefore,

$$\int_{0+}^{t} \mathbb{1}_{\{X(s-)=0\}} \, dX^+(s) = \Lambda_{X^+}(t) + \sum_{s \le t} \mathbb{1}_{\{X(s-)=0\}} \, \Delta X^+(s).$$

Thus, we have the following formula for any semimartingale X,

$$\mathcal{L}_{X^+}(t) = \int_{0+}^{t} \mathbb{1}_{\{X(s-)=0\}} \, dX^+(s), \tag{6.6.115}$$

and therefore the following also holds

$$d\mathcal{L}_{X^+}(t) = \mathbb{1}_{\{X(t-)=0\}} \, dX^+(t). \tag{6.6.116}$$

In a similar fashion we can show for any semimartingale X,

$$\mathcal{L}_{X^-}(t) = \int_{0+}^{t} \mathbb{1}_{\{X(s-)=0\}} \, dX^-(s),$$

so that

$$d\mathcal{L}_{X^-}(t) = \mathbb{1}_{\{X(t-)=0\}} \, dX^-(t). \tag{6.6.117}$$

Now, from (6.6.106), we have for $t \in [0, T]$,

$$N_{t-}(k) = \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t-)=X_i(t-)\}} \equiv \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t-)=X_{(i)}(t-)\}}.$$

Therefore, for $t \in [0, T]$, we obtain

$$N_{t-}(k) \, dX_{(k)}(t) = \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t-)=X_i(t-)\}} \, dX_{(k)}(t), \tag{6.6.118}$$

and

$$N_{t-}(k) \, dX_{(k)}(t) = \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t-)=X_{(i)}(t-)\}} \, dX_{(k)}(t). \tag{6.6.119}$$

First consider (6.6.119), which we can express as

$$\begin{aligned} N_{t-}(k) \, dX_{(k)}(t) &= \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t-)=X_{(i)}(t-)\}} \, dX_{(i)}(t) + \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t-)=X_{(i)}(t-)\}} \left[dX_{(k)}(t) - dX_{(i)}(t) \right] \\ &= \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t-)=X_{(i)}(t-)\}} \, dX_{(i)}(t) + \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t-)=X_{(i)}(t-)\}} \, d\left(X_{(k)}(t) - X_{(i)}(t) \right). \end{aligned}$$

Again, since $x \equiv x^+ - x^-$, following the same idea as that employed in the proof of Theorem 6.6.3, we obtain

$$\begin{aligned} N_{t-}(k) \, dX_{(k)}(t) &= \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t-)=X_{(i)}(t-)\}} \, dX_{(i)}(t) + \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t-)-X_{(i)}(t-)=0\}} \, d\left(\left(X_{(k)}(t) - X_{(i)}(t) \right)^+ \right) \\ &\quad - \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t-)-X_{(i)}(t-)=0\}} \, d\left(\left(X_{(k)}(t) - X_{(i)}(t) \right)^- \right). \end{aligned}$$

Since $\big(X_{(k)}(t) - X_{(i)}(t)\big)^+$ and $\big(X_{(k)}(t) - X_{(i)}(t)\big)^-$ are *nonnegative semimartingales*, applying (6.6.112) (or (6.6.116) and (6.6.117)) to these nonnegative semimartingales yields

$$\begin{aligned}
\mathbb{1}_{\{X_{(k)}(t-)-X_{(i)}(t-)=0\}}\, d\Big(\big(X_{(k)}(t) - X_{(i)}(t)\big)^+\Big) &= d\mathcal{L}_{\left(X_{(k)}-X_{(i)}\right)^+}(t),\\
\mathbb{1}_{\{X_{(k)}(t-)-X_{(i)}(t-)=0\}}\, d\Big(\big(X_{(k)}(t) - X_{(i)}(t)\big)^-\Big) &= d\mathcal{L}_{\left(X_{(k)}-X_{(i)}\right)^-}(t).
\end{aligned}$$

Employing the above equations, we arrive at the following

$$N_{t-}(k)\, dX_{(k)}(t) = \sum_{i=1}^n \mathbb{1}_{\{X_{(k)}(t-)=X_{(i)}(t-)\}}\, dX_{(i)}(t) + \sum_{i=1}^n d\mathcal{L}_{\left(X_{(k)}-X_{(i)}\right)^+}(t) - \sum_{i=1}^n d\mathcal{L}_{\left(X_{(k)}-X_{(i)}\right)^-}(t).$$

Hence, we have for $t \in [0, T]$,

$$\begin{aligned}
&dX_{(k)}(t)\\
&= \frac{1}{N_{t-}(k)} \sum_{i=1}^n \mathbb{1}_{\{X_{(k)}(t-)=X_{(i)}(t-)\}}\, dX_{(i)}(t) + \frac{1}{N_{t-}(k)} \sum_{i=1}^n d\mathcal{L}_{\left(X_{(k)}-X_{(i)}\right)^+}(t) - \frac{1}{N_{t-}(k)} \sum_{i=1}^n d\mathcal{L}_{\left(X_{(k)}-X_{(i)}\right)^-}(t)\\
&= \sum_{i=1}^n \frac{1}{N_{t-}(k)} \mathbb{1}_{\{X_{(k)}(t-)=X_{(i)}(t-)\}}\, dX_{(i)}(t) + \sum_{i=1}^n \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{\left(X_{(k)}-X_{(i)}\right)^+}(t) - \sum_{i=1}^n \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{\left(X_{(k)}-X_{(i)}\right)^-}(t),
\end{aligned} \tag{6.6.120}$$

which proves (6.6.108). Thus, from equations (6.6.74) and (6.6.75), it follows that (6.6.120) becomes (6.6.109)

$$\begin{aligned}
&dX_{(k)}(t)\\
&= \sum_{i=1}^n \frac{1}{N_{t-}(k)} \mathbb{1}_{\{X_{(k)}(t-)=X_{(i)}(t-)\}}\, dX_{(i)}(t) + \sum_{\ell=k+1}^n \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{X_{(k)}-X_{(\ell)}}(t) - \sum_{\ell=1}^{k-1} \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{X_{(\ell)}-X_{(k)}}(t).
\end{aligned}$$

Now consider (6.6.118), which we can express as

$$\begin{aligned}
N_{t-}(k)\, dX_{(k)}(t) &= \sum_{i=1}^n \mathbb{1}_{\{X_{(k)}(t-)=X_i(t-)\}}\, dX_i(t) + \sum_{i=1}^n \mathbb{1}_{\{X_{(k)}(t-)=X_i(t-)\}} \Big[dX_{(k)}(t) - dX_i(t)\Big]\\
&= \sum_{i=1}^n \mathbb{1}_{\{X_{(k)}(t-)=X_i(t-)\}}\, dX_i(t) + \sum_{i=1}^n \mathbb{1}_{\{X_{(k)}(t-)=X_i(t-)\}}\, d\big(X_{(k)}(t) - X_i(t)\big).
\end{aligned}$$

Again, since $x \equiv x^+ - x^-$, following the same idea as that employed in the proof of Theorem 6.6.3, we obtain

$$\begin{aligned}
N_{t-}(k)\, dX_{(k)}(t) &= \sum_{i=1}^n \mathbb{1}_{\{X_{(k)}(t-)=X_i(t-)\}}\, dX_i(t) + \sum_{i=1}^n \mathbb{1}_{\{X_{(k)}(t-)-X_i(t-)=0\}}\, d\Big(\big(X_{(k)}(t) - X_i(t)\big)^+\Big)\\
&\quad - \sum_{i=1}^n \mathbb{1}_{\{X_{(k)}(t-)-X_i(t-)=0\}}\, d\Big(\big(X_{(k)}(t) - X_i(t)\big)^-\Big).
\end{aligned}$$

Since $\big(X_{(k)}(t)-X_i(t)\big)^+$ and $\big(X_{(k)}(t)-X_i(t)\big)^-$ are *nonnegative semimartingales*, applying (6.6.112) (or (6.6.116) and (6.6.117)) to these nonnegative semimartingales yields

$$\begin{aligned}
\mathbb{1}_{\{X_{(k)}(t-)-X_i(t-)=0\}}\, d\Big(\big(X_{(k)}(t) - X_i(t)\big)^+\Big) &= d\mathcal{L}_{\left(X_{(k)}-X_i\right)^+}(t),\\
\mathbb{1}_{\{X_{(k)}(t-)-X_i(t-)=0\}}\, d\Big(\big(X_{(k)}(t) - X_i(t)\big)^-\Big) &= d\mathcal{L}_{\left(X_{(k)}-X_i\right)^-}(t).
\end{aligned}$$

Employing the above equations, we arrive at the following

$$N_{t-}(k)\, dX_{(k)}(t) = \sum_{i=1}^n \mathbb{1}_{\{X_{(k)}(t-)=X_i(t-)\}}\, dX_i(t) + \sum_{i=1}^n d\mathcal{L}_{\left(X_{(k)}-X_i\right)^+}(t) - \sum_{i=1}^n d\mathcal{L}_{\left(X_{(k)}-X_i\right)^-}(t).$$

Hence, we have for $t \in [0,T]$,

$$\begin{aligned} & dX_{(k)}(t) \\ &= \frac{1}{N_{t-}(k)} \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t-)=X_i(t-)\}}\, dX_i(t) + \frac{1}{N_{t-}(k)} \sum_{i=1}^{n} d\mathcal{L}_{(X_{(k)}-X_i)^+}(t) - \frac{1}{N_{t-}(k)} \sum_{i=1}^{n} d\mathcal{L}_{(X_{(k)}-X_i)^-}(t) \\ &= \sum_{i=1}^{n} \frac{1}{N_{t-}(k)} \mathbb{1}_{\{X_{(k)}(t-)=X_i(t-)\}}\, dX_i(t) + \sum_{i=1}^{n} \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{(X_{(k)}-X_i)^+}(t) - \sum_{i=1}^{n} \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{(X_{(k)}-X_i)^-}(t), \end{aligned} \tag{6.6.121}$$

which proves (6.6.107). ■

Theorem 6.6.18 ([**Pamen (2009), Ghomrasni & Pamen (2010)**]). *Let $X_1, \ldots, X_n$ be **semimartingales**, then the rank processes $X_{(1)}, \ldots, X_{(n)}$ are **semimartingales**, and the following equality holds for $t \in [0,T]$:*

$$\sum_{i=1}^{n} \mathbb{1}_{\{X_{(i)}(t-)=0\}}\, d\left(X_{(i)}(t)\right)^+ = \sum_{i=1}^{n} \mathbb{1}_{\{X_i(t-)=0\}}\, d\left(X_i(t)\right)^+, \tag{6.6.122}$$

alternatively expressed as,

$$\sum_{i=1}^{n} \mathbb{1}_{\{X_{(i)}(t-)=0\}}\, dX_{(i)}^+(t) = \sum_{i=1}^{n} \mathbb{1}_{\{X_i(t-)=0\}}\, dX_i^+(t). \tag{6.6.123}$$

Proof. We shall proceed by induction. The case $n = 1$ is trivial. For $n = 2$, we must show that

$$\mathbb{1}_{\{X_{(1)}(t-)=0\}}\, dX_{(1)}^+(t) + \mathbb{1}_{\{X_{(2)}(t-)=0\}}\, dX_{(2)}^+(t) = \mathbb{1}_{\{X_1(t-)=0\}}\, dX_1^+(t) + \mathbb{1}_{\{X_2(t-)=0\}}\, dX_2^+(t). \tag{6.6.124}$$

To do this, we shall consider the maximum and the minimum of two semimartingales X_1 and X_2. Let us denote the maximum of these two semimartingales by $X_{(1)} = X_1 \vee X_2$ and the minimum of these two semimartingales by $X_{(2)} = X_1 \wedge X_2$. Then, setting $Z := X_{(1)}$, $X := X_1$ and $Y := X_2$, in (6.3.101), yields the following expression for the maximum of these two semimartingales

$$\begin{aligned} \mathbb{1}_{\{X_{(1)}(t-)=0\}}\, dX_{(1)}^+(t) &= \mathbb{1}_{\{X_1(t-)\le 0\}} \mathbb{1}_{\{X_2(t-)=0\}}\, dX_2^+(t) + \mathbb{1}_{\{X_2(t-)<0\}} \mathbb{1}_{\{X_1(t-)=0\}}\, dX_1^+(t) \\ &\quad + \mathbb{1}_{\{X_1(t-)=X_2(t-)=0\}}\, d\left(\left(X_1^+(t) - X_2^+(t)\right)^+\right). \end{aligned} \tag{6.6.125}$$

Now consider the minimum of two semimartingales. Consider two semimartingales X and Y. We shall follow the same idea employed in the proof of the second theorem in Ouknine (1990) or in the proof of Ouknine & Rutkowski (1995, Proposition 2.3). Let us denote the minimum of these two semimartingales by $V = X \wedge Y$. Then

$$\begin{aligned} \{V(t-) = 0\} &= \{(X(t-) \wedge Y(t-)) = 0\} \\ &= \{X(t-) > Y(t-) = 0\} \cup \{Y(t-) > X(t-) = 0\} \cup \{X(t-) = Y(t-) = 0\}. \end{aligned}$$

Therefore, we deduce

$$\mathbb{1}_{\{V(t-)=0\}}\, dV^+(t) = \mathbb{1}_{\{X(t-)>Y(t-)=0\}}\, dV^+(t) + \mathbb{1}_{\{Y(t-)>X(t-)=0\}}\, dV^+(t) + \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\, dV^+(t). \tag{6.6.126}$$

Following a similar argument to that used to arrive at (6.3.97) for the minimum process, the replacement of V by Y is permitted, and we can write the first term on the right-hand side of equation (6.6.126) as

$$\mathbb{1}_{\{X(t-)>Y(t-)=0\}}\, dV^+(t) = \mathbb{1}_{\{X(t-)>Y(t-)=0\}}\, dY^+(t) = \mathbb{1}_{\{X(t-)>0\}} \mathbb{1}_{\{Y(t-)=0\}}\, dY^+(t). \tag{6.6.127}$$

Similarly, the second term on the right-hand side of (6.6.126) can be written as follows

$$\mathbb{1}_{\{Y(t-)>X(t-)=0\}}\, dV^+(t) = \mathbb{1}_{\{Y(t-)>X(t-)=0\}}\, dX^+(t) = \mathbb{1}_{\{Y(t-)>0\}} \mathbb{1}_{\{X(t-)=0\}}\, dX^+(t). \tag{6.6.128}$$

Employing the fact that $V = X \wedge Y = X - \big(X - Y\big)^+ = Y - \big(Y - X\big)^+$, we shall evaluate the last term on the right-hand side of (6.6.126), by making use of the following representation of V^+

$$V^+ \;=\; \big(X \wedge Y\big)^+ \;=\; X^+ \wedge Y^+ \;=\; X^+ - \big(X^+ - Y^+\big)^+ \;\equiv\; Y^+ - \big(Y^+ - X^+\big)^+.$$

Thus, the last term becomes

$$\mathbb{1}_{\{X(t-)=Y(t-)=0\}}\, dV^+(t) \;=\; \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\, dX^+(t) - \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\, d\Big(\big(X^+(t) - Y^+(t)\big)^+\Big) \tag{6.6.129}$$

$$= \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\, dY^+(t) - \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\, d\Big(\big(Y^+(t) - X^+(t)\big)^+\Big). \tag{6.6.130}$$

Using (6.6.127), (6.6.128) and (6.6.129), we may cast (6.6.126) in the following form

$$\begin{aligned}
\mathbb{1}_{\{V(t-)=0\}}\, dV^+(t) &= \mathbb{1}_{\{X(t-)>0\}}\mathbb{1}_{\{Y(t-)=0\}}\, dY^+(t) + \mathbb{1}_{\{Y(t-)>0\}}\mathbb{1}_{\{X(t-)=0\}}\, dX^+(t) \\
&\quad + \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\, dX^+(t) - \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\, d\Big(\big(X^+(t) - Y^+(t)\big)^+\Big) \\
&= \mathbb{1}_{\{X(t-)>0\}}\mathbb{1}_{\{Y(t-)=0\}}\, dY^+(t) + \mathbb{1}_{\{Y(t-)>0\}}\mathbb{1}_{\{X(t-)=0\}}\, dX^+(t) \\
&\quad + \mathbb{1}_{\{X(t-)=0\}}\mathbb{1}_{\{Y(t-)=0\}}\, dX^+(t) - \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\, d\Big(\big(X^+(t) - Y^+(t)\big)^+\Big) \\
&= \mathbb{1}_{\{X(t-)>0\}}\mathbb{1}_{\{Y(t-)=0\}}\, dY^+(t) + \Big(\mathbb{1}_{\{Y(t-)>0\}} + \mathbb{1}_{\{Y(t-)=0\}}\Big)\mathbb{1}_{\{X(t-)=0\}}\, dX^+(t) \\
&\quad - \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\, d\Big(\big(X^+(t) - Y^+(t)\big)^+\Big) \\
&= \mathbb{1}_{\{X(t-)>0\}}\mathbb{1}_{\{Y(t-)=0\}}\, dY^+(t) + \mathbb{1}_{\{Y(t-)\geq 0\}}\mathbb{1}_{\{X(t-)=0\}}\, dX^+(t) \\
&\quad - \mathbb{1}_{\{X(t-)=Y(t-)=0\}}\, d\Big(\big(X^+(t) - Y^+(t)\big)^+\Big).
\end{aligned} \tag{6.6.131}$$

Now, setting $V := X_{(2)}$, $X := X_1$ and $Y := X_2$, in (6.6.131), yields the following expression for the minimum of these two semimartingales

$$\begin{aligned}
\mathbb{1}_{\{X_{(2)}(t-)=0\}}\, dX^+_{(2)}(t) &= \mathbb{1}_{\{X_1(t-)>0\}}\mathbb{1}_{\{X_2(t-)=0\}}\, dX_2^+(t) + \mathbb{1}_{\{X_2(t-)\geq 0\}}\mathbb{1}_{\{X_1(t-)=0\}}\, dX_1^+(t) \\
&\quad - \mathbb{1}_{\{X_1(t-)=X_2(t-)=0\}}\, d\Big(\big(X_1^+(t) - X_2^+(t)\big)^+\Big).
\end{aligned} \tag{6.6.132}$$

Adding the left-hand side of (6.6.125) to the left-hand side of (6.6.132), yields the desired result (6.6.124) for $n = 2$. Now assume the result holds for some n. Given semimartingales $X_1, \ldots, X_n, X_{n+1}$, the kth rank process $X_{(k)}$, $k = 1, 2, \ldots, n$ is defined by (6.2.1) in Definition 6.2.1 and the kth rank process with respect to all $n+1$ semimartingales, $X_{[k]} = \big\{X_{[k]}(t), t \in [0,T]\big\}$, is given by (6.6.91). It will be convenient to set $X_{(0)}(t) \equiv \infty$. Suppose first that $k > 1$. By (6.6.124), we have

$$\begin{aligned}
&\mathbb{1}_{\left\{\left(X_{(k-1)}(t-)\,\wedge\, X_{n+1}(t-)\right)\,\vee\, X_{(k)}(t-)=0\right\}}\, d\Big(\Big(X^+_{(k-1)}(t) \,\wedge\, X^+_{n+1}(t)\Big) \,\vee\, X^+_{(k)}(t)\Big) \\
&+ \mathbb{1}_{\left\{\left(X_{(k-1)}(t-)\,\wedge\, X_{n+1}(t-)\right)\,\wedge\, X_{(k)}(t-)=0\right\}}\, d\Big(\Big(X^+_{(k-1)}(t) \,\wedge\, X^+_{n+1}(t)\Big) \,\wedge\, X^+_{(k)}(t)\Big) \\
&\qquad = \mathbb{1}_{\{X_{(k-1)}(t-)\,\wedge\, X_{n+1}(t-)=0\}}\, d\Big(X^+_{(k-1)}(t) \,\wedge\, X^+_{n+1}(t)\Big) + \mathbb{1}_{\{X_{(k)}(t-)=0\}}\, dX^+_{(k)}(t).
\end{aligned} \tag{6.6.133}$$

Since $X_{(k-1)}(t) \geq X_{(k)}(t)$ for all $t \in [0,T]$, the second term on the left-hand side of the above equation is simply $\mathbb{1}_{\{X_{(k)}(t-)\,\wedge\, X_{n+1}(t-)=0\}}\, d\big(X^+_{(k)}(t) \,\wedge\, X^+_{n+1}(t)\big)$. Using (6.6.93) and the same argument used to arrive at (6.6.94), the first term on the left-hand side of the above equation becomes $\mathbb{1}_{\{X_{[k]}(t-)=0\}}\, dX^+_{[k]}(t)$, so that (6.6.133) becomes

$$\begin{aligned}
&\mathbb{1}_{\{X_{[k]}(t-)=0\}}\, dX^+_{[k]}(t) + \mathbb{1}_{\{X_{(k)}(t-)\,\wedge\, X_{n+1}(t-)=0\}}\, d\Big(X^+_{(k)}(t) \,\wedge\, X^+_{n+1}(t)\Big) \\
&\qquad = \mathbb{1}_{\{X_{(k-1)}(t-)\,\wedge\, X_{n+1}(t-)=0\}}\, d\Big(X^+_{(k-1)}(t) \,\wedge\, X^+_{n+1}(t)\Big) + \mathbb{1}_{\{X_{(k)}(t-)=0\}}\, dX^+_{(k)}(t),
\end{aligned} \tag{6.6.134}$$

for $k = 2, \ldots, n$. If $k = 1$, then since $X_{(0)}(t) \equiv \infty$, the above equation reduces to

$$\begin{aligned}
&\mathbb{1}_{\{X_{[1]}(t-)=0\}}\, dX^+_{[1]}(t) + \mathbb{1}_{\{X_{(1)}(t-)\,\wedge\, X_{n+1}(t-)=0\}}\, d\Big(X^+_{(1)}(t) \,\wedge\, X^+_{n+1}(t)\Big) \\
&\qquad = \mathbb{1}_{\{X_{n+1}(t-)=0\}}\, dX^+_{n+1}(t) + \mathbb{1}_{\{X_{(1)}(t-)=0\}}\, dX^+_{(1)}(t).
\end{aligned} \tag{6.6.135}$$

This equation also follows from (6.6.124), where we have

$$\begin{aligned}\mathbb{1}_{\{X_{(1)}(t-)\vee X_{n+1}(t-)=0\}}\, d\left(X^+_{(1)}(t)\vee X^+_{n+1}(t)\right)+\mathbb{1}_{\{X_{(1)}(t-)\wedge X_{n+1}(t-)=0\}}\, d\left(X^+_{(1)}(t)\wedge X^+_{n+1}(t)\right)\\ = \mathbb{1}_{\{X_{n+1}(t-)=0\}}\, dX^+_{n+1}(t)+\mathbb{1}_{\{X_{(1)}(t-)=0\}}\, dX^+_{(1)}(t).\end{aligned}$$

Observe that $\big(X_{(1)}\vee X_{n+1}\big)(t)\equiv X_{[1]}(t)$ for all $t\in[0,T]$. Then we get the result (6.6.135) which is precisely (6.6.134) for $k=1$. Thus, we have the following formula which holds for all $k=1,2,\ldots,n$ and $t\in[0,T]$

$$\begin{aligned}\mathbb{1}_{\{X_{[k]}(t-)=0\}}\, dX^+_{[k]}(t)+\mathbb{1}_{\{X_{(k)}(t-)\wedge X_{n+1}(t-)=0\}}\, d\left(X^+_{(k)}(t)\wedge X^+_{n+1}(t)\right)\\ = \mathbb{1}_{\{X_{(k-1)}(t-)\wedge X_{n+1}(t-)=0\}}\, d\left(X^+_{(k-1)}(t)\wedge X^+_{n+1}(t)\right)+\mathbb{1}_{\{X_{(k)}(t-)=0\}}\, dX^+_{(k)}(t),\end{aligned}\tag{6.6.136}$$

and, we have

$$\begin{aligned}\mathbb{1}_{\{X_{(k)}(t-)=0\}}\, dX^+_{(k)}(t) &= \mathbb{1}_{\{X_{[k]}(t-)=0\}}\, dX^+_{[k]}(t)+\mathbb{1}_{\{X_{(k)}(t-)\wedge X_{n+1}(t-)=0\}}\, d\left(X^+_{(k)}(t)\wedge X^+_{n+1}(t)\right)\\ &\quad -\mathbb{1}_{\{X_{(k-1)}(t-)\wedge X_{n+1}(t-)=0\}}\, d\left(X^+_{(k-1)}(t)\wedge X^+_{n+1}(t)\right).\end{aligned}\tag{6.6.137}$$

Finally, by the induction hypothesis, equation (6.6.137) and the facts that $\big(X_{(n)}\wedge X_{n+1}\big)(t)\equiv X_{[n+1]}(t)$ and $\big(X_{(0)}\wedge X_{n+1}\big)(t)\equiv X_{n+1}(t)$, we see that $X_{[n+1]}$ is a semimartingale, and we have

$$\begin{aligned}\sum_{i=1}^{n+1}\mathbb{1}_{\{X_i(t-)=0\}}\, dX^+_i(t) &= \sum_{i=1}^{n}\mathbb{1}_{\{X_i(t-)=0\}}\, dX^+_i(t)+\mathbb{1}_{\{X_{n+1}(t-)=0\}}\, dX^+_{n+1}(t)\\ &= \sum_{i=1}^{n}\mathbb{1}_{\{X_{(i)}(t-)=0\}}\, dX^+_{(i)}(t)+\mathbb{1}_{\{X_{n+1}(t-)=0\}}\, dX^+_{n+1}(t)\\ &= \sum_{k=1}^{n}\mathbb{1}_{\{X_{(k)}(t-)=0\}}\, dX^+_{(k)}(t)+\mathbb{1}_{\{X_{n+1}(t-)=0\}}\, dX^+_{n+1}(t)\\ &= \sum_{k=1}^{n}\mathbb{1}_{\{X_{[k]}(t-)=0\}}\, dX^+_{[k]}(t)+\mathbb{1}_{\{X_{n+1}(t-)=0\}}\, dX^+_{n+1}(t)\\ &\quad +\sum_{k=1}^{n}\Big[\mathbb{1}_{\{X_{(k)}(t-)\wedge X_{n+1}(t-)=0\}}\, d\left(X^+_{(k)}(t)\wedge X^+_{n+1}(t)\right)\\ &\qquad -\mathbb{1}_{\{X_{(k-1)}(t-)\wedge X_{n+1}(t-)=0\}}\, d\left(X^+_{(k-1)}(t)\wedge X^+_{n+1}(t)\right)\Big]\\ &= \sum_{k=1}^{n}\mathbb{1}_{\{X_{[k]}(t-)=0\}}\, dX^+_{[k]}(t)+\mathbb{1}_{\{X_{n+1}(t-)=0\}}\, dX^+_{n+1}(t)\\ &\quad +\mathbb{1}_{\{X_{(n)}(t-)\wedge X_{n+1}(t-)=0\}}\, d\left(X^+_{(n)}(t)\wedge X^+_{n+1}(t)\right)\\ &\quad -\mathbb{1}_{\{X_{(0)}(t-)\wedge X_{n+1}(t-)=0\}}\, d\left(X^+_{(0)}(t)\wedge X^+_{n+1}(t)\right),\end{aligned}$$

since $(X \wedge Y)^+ = X^+ \wedge Y^+$, we obtain

$$\begin{aligned}
\sum_{i=1}^{n+1} \mathbb{1}_{\{X_i(t-)=0\}}\, dX_i^+(t) &= \sum_{k=1}^{n} \mathbb{1}_{\{X_{[k]}(t-)=0\}}\, dX_{[k]}^+(t) + \mathbb{1}_{\{X_{n+1}(t-)=0\}}\, dX_{n+1}^+(t) \\
&\quad + \mathbb{1}_{\{X_{(n)}(t-)\wedge X_{n+1}(t-)=0\}}\, d\Big(\big(X_{(n)}(t) \wedge X_{n+1}(t)\big)^+\Big) \\
&\quad - \mathbb{1}_{\{X_{(0)}(t-)\wedge X_{n+1}(t-)=0\}}\, d\Big(\big(X_{(0)}(t) \wedge X_{n+1}(t)\big)^+\Big) \\
&= \sum_{k=1}^{n} \mathbb{1}_{\{X_{[k]}(t-)=0\}}\, dX_{[k]}^+(t) + \mathbb{1}_{\{X_{n+1}(t-)=0\}}\, dX_{n+1}^+(t) \\
&\quad + \mathbb{1}_{\{X_{[n+1]}(t-)=0\}}\, dX_{[n+1]}^+(t) - \mathbb{1}_{\{X_{n+1}(t-)=0\}}\, dX_{n+1}^+(t) \\
&= \sum_{k=1}^{n+1} \mathbb{1}_{\{X_{[k]}(t-)=0\}}\, dX_{[k]}^+(t) \\
&= \sum_{i=1}^{n+1} \mathbb{1}_{\{X_{[i]}(t-)=0\}}\, dX_{[i]}^+(t),
\end{aligned}$$

for all $t \in [0,T]$, the desired result follows by induction. ■

Corollary 6.6.19 ([**Pamen (2009), Ghomrasni & Pamen (2010)**]). *Let $X_1, \ldots, X_n$ be **nonnegative semimartingales**, then the rank processes $X_{(1)}, \ldots, X_{(n)}$ are **nonnegative semimartingales**, and the following equality holds for $t \in [0,T]$:*

$$\sum_{i=1}^{n} \mathbb{1}_{\{X_{(i)}(t-)=0\}}\, dX_{(i)}(t) = \sum_{i=1}^{n} \mathbb{1}_{\{X_i(t-)=0\}}\, dX_i(t). \tag{6.6.138}$$

Proof. The proof follows directly from Theorem 6.6.18, and the fact that for all $i = 1, 2, \ldots, n$,

$$X_i^+(t) = \begin{cases} X_i(t) & \text{if} \quad X_i(t) > 0, \\ 0 & \text{if} \quad X_i(t) \le 0. \end{cases} \tag{6.6.139}$$

The above result coupled with the assumption that X_i is a nonnegative semimartingale (i.e., $X_i(t) \ge 0$ for all $t \in [0,T]$), yields the desired result. ■

For the nonnegative semimartingale X^+, it follows from (6.6.138)

$$\sum_{i=1}^{n} \mathbb{1}_{\{X_{(i)}^+(t-)=0\}}\, dX_{(i)}^+(t) = \sum_{i=1}^{n} \mathbb{1}_{\{X_i^+(t-)=0\}}\, dX_i^+(t). \tag{6.6.140}$$

From (6.6.139), for a general semimartingale X, the following equality holds

$$\mathbb{1}_{\{X^+(t-)=0\}} = \mathbb{1}_{\{X(t-)\le 0\}}, \tag{6.6.141}$$

which when substituted into (6.6.140), yields

$$\sum_{i=1}^{n} \mathbb{1}_{\{X_{(i)}(t-)\le 0\}}\, dX_{(i)}^+(t) = \sum_{i=1}^{n} \mathbb{1}_{\{X_i(t-)\le 0\}}\, dX_i^+(t). \tag{6.6.142}$$

Corollary 6.6.20 ([**Pamen (2009), Ghomrasni & Pamen (2010)**]). *Let $X_1, \ldots, X_n$ be **semimartingales**. Then the kth ranked processes $X_{(k)}$, for all $k = 1, 2, \ldots, n$, are **semimartingales**, and we have*

$$\sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t-)=X_{(i)}(t-)\}}\, d\Big(\big(X_{(i)}(t) - X_{(k)}(t)\big)^+\Big) = \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t-)=X_i(t-)\}}\, d\Big(\big(X_i(t) - X_{(k)}(t)\big)^+\Big). \tag{6.6.143}$$

Proof. Define the processes $Y_1, Y_2, \ldots, Y_n$ by

$$Y_i(t) \triangleq X_i(t) - X_{(k)}(t), \tag{6.6.144}$$

for all $i = 1, 2, \ldots, n$, and fixed k, and $t \in [0, T]$. Then, the assumption that $X_1, X_2, \ldots, X_n$ are semimartingales, implies that $Y_1, Y_2, \ldots, Y_n$ are semimartingales. Moreover, we also define the processes $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ by

$$Y_{(i)}(t) \triangleq X_{(i)}(t) - X_{(k)}(t), \tag{6.6.145}$$

for all $i = 1, 2, \ldots, n$, and $t \in [0, T]$. These processes are then the ranked processes of $Y_i(t)$, for $i = 1, 2, \ldots, n$ and satisfy the property $Y_{(1)}(t) \geq Y_{(2)}(t) \geq \cdots \geq Y_{(n-1)}(t) \geq Y_{(n)}(t)$. These ranked processes are also semimartingales. Thus, for the semimartingales $Y_1, Y_2, \ldots, Y_n$, we have by Theorem 6.6.18

$$\sum_{i=1}^{n} \mathbb{1}_{\{Y_{(i)}(t-)=0\}}\, dY_{(i)}^{+}(t) = \sum_{i=1}^{n} \mathbb{1}_{\{Y_i(t-)=0\}}\, dY_i^{+}(t).$$

Hence, equations (6.6.144) and (6.6.145) imply that

$$\sum_{i=1}^{n} \mathbb{1}_{\{X_{(i)}(t-)-X_{(k)}(t-)=0\}}\, d\Big(\big(X_{(i)}(t) - X_{(k)}(t)\big)^{+}\Big) = \sum_{i=1}^{n} \mathbb{1}_{\{X_i(t-)-X_{(k)}(t-)=0\}}\, d\Big(\big(X_i(t) - X_{(k)}(t)\big)^{+}\Big),$$

and the desired result follows. ■

Corollary 6.6.21. *Let $X_1, \ldots, X_n$ be **semimartingales**. Then the kth ranked processes $X_{(k)}$, for all $k = 1, 2, \ldots, n$, are **semimartingales**, and we have*

$$\sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t-)=X_{(i)}(t-)\}}\, d\Big(\big(X_{(k)}(t) - X_{(i)}(t)\big)^{+}\Big) = \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t-)=X_i(t-)\}}\, d\Big(\big(X_{(k)}(t) - X_i(t)\big)^{+}\Big). \tag{6.6.146}$$

Proof. Define the processes $Z_1, Z_2, \ldots, Z_n$ by

$$Z_i(t) \triangleq X_{(k)}(t) - X_i(t), \tag{6.6.147}$$

for all $i = 1, 2, \ldots, n$, and fixed k, and $t \in [0, T]$. Then, the assumption that $X_1, X_2, \ldots, X_n$ are semimartingales, implies that $Z_1, Z_2, \ldots, Z_n$ are semimartingales. Moreover, we also define the processes $Z_{(1)}, Z_{(2)}, \ldots, Z_{(n)}$ by

$$Z_{(n-i+1)}(t) \triangleq X_{(k)}(t) - X_{(i)}(t), \tag{6.6.148}$$

for all $i = 1, 2, \ldots, n$, and $t \in [0, T]$. These processes are then the ranked processes of $Z_i(t)$, for $i = 1, 2, \ldots, n$ and satisfy the property $Z_{(1)}(t) \geq Z_{(2)}(t) \geq \cdots \geq Z_{(n-1)}(t) \geq Z_{(n)}(t)$. These ranked processes are also semimartingales. Thus, for the semimartingales $Z_1, Z_2, \ldots, Z_n$, we have by Theorem 6.6.18

$$\sum_{i=1}^{n} \mathbb{1}_{\{Z_{(i)}(t-)=0\}}\, dZ_{(i)}^{+}(t) = \sum_{i=1}^{n} \mathbb{1}_{\{Z_i(t-)=0\}}\, dZ_i^{+}(t).$$

Hence, equations (6.6.147) and (6.6.148) imply that

$$\sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t-)-X_{(i)}(t-)=0\}}\, d\Big(\big(X_{(k)}(t) - X_{(i)}(t)\big)^{+}\Big) = \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t-)-X_i(t-)=0\}}\, d\Big(\big(X_{(k)}(t) - X_i(t)\big)^{+}\Big),$$

and the desired result follows. ■

Theorem 6.6.22. *For **semimartingales** $X_1, \ldots, X_n$, the rank processes $X_{(1)}, \ldots, X_{(n)}$ are **semimartingales**, and we have for all $t \in [0, T]$,*

$$\sum_{i=1}^{n} \mathcal{L}_{X_{(i)}}(t) = \sum_{i=1}^{n} \mathcal{L}_{X_i}(t). \tag{6.6.149}$$

Proof. By (6.6.110) for a nonnegative semimartingale X, we have

$$\mathcal{L}_X(t) \quad = \quad \Lambda_X(t) + \sum_{s \le t} \mathbb{1}_{\{X(s-)=0\}} \, \Delta X(s) \quad = \quad \int_{0+}^{t} \mathbb{1}_{\{X(s-)=0\}} \, dX(s).$$

Therefore, for any semimartingale X, we get

$$\mathcal{L}_X(t) \quad = \quad \Lambda_X(t) + \sum_{s \le t} \mathbb{1}_{\{X(s-)=0\}} \, \Delta X^+(s) \quad = \quad \int_{0+}^{t} \mathbb{1}_{\{X(s-)=0\}} \, dX^+(s), \tag{6.6.150}$$

which follows from (6.3.93). Hence, for any semimartingale X, the following holds

$$d\mathcal{L}_X(t) \quad = \quad \mathbb{1}_{\{X(t-)=0\}} \, dX^+(t).$$

Combining this result with that of Theorem 6.6.18,

$$\sum_{i=1}^{n} \mathbb{1}_{\left\{X_{(i)}(t-)=0\right\}} \, dX^+_{(i)}(t) \quad = \quad \sum_{i=1}^{n} \mathbb{1}_{\left\{X_i(t-)=0\right\}} \, dX^+_i(t),$$

gives

$$\sum_{i=1}^{n} d\mathcal{L}_{X_{(i)}}(t) \quad = \quad \sum_{i=1}^{n} d\mathcal{L}_{X_i}(t), \tag{6.6.151}$$

and the required result follows. ■

Corollary 6.6.23. *For **semimartingales** $X_1, \ldots, X_n$, the rank processes $X_{(1)}, \ldots, X_{(n)}$ are **semimartingales**, and we have for all $t \in [0, T]$,*

$$\sum_{i=1}^{n} \mathcal{L}_{X^+_{(i)}}(t) \quad = \quad \sum_{i=1}^{n} \mathcal{L}_{X^+_i}(t). \tag{6.6.152}$$

Proof. For all semimartingales $X_1, \ldots, X_n$, this follows directly from Theorem 6.6.22 since X_i^+ are semimartingales, for all $i = 1, 2, \ldots, n$. Alternatively, by Theorem 6.6.18, the following equality holds,

$$\sum_{i=1}^{n} \mathbb{1}_{\left\{X_{(i)}(t-)=0\right\}} \, dX^+_{(i)}(t) \quad = \quad \sum_{i=1}^{n} \mathbb{1}_{\left\{X_i(t-)=0\right\}} \, dX^+_i(t).$$

Moreover, from (6.6.115), we know that for all semimartingales X, the following holds

$$\mathcal{L}_{X^+}(t) \quad = \quad \int_{0+}^{t} \mathbb{1}_{\{X(s-)=0\}} \, dX^+(s).$$

Equivalently, by (6.6.116), we have

$$d\mathcal{L}_{X^+}(t) \quad = \quad \mathbb{1}_{\{X(t-)=0\}} \, dX^+(t).$$

Hence, the preceding equation coupled with (6.6.122) of Theorem 6.6.18, gives

$$\sum_{i=1}^{n} d\mathcal{L}_{X^+_{(i)}}(t) \quad = \quad \sum_{i=1}^{n} d\mathcal{L}_{X^+_i}(t), \tag{6.6.153}$$

and the result follows. This result can also be achieved by noticing that (6.6.115) and (6.6.150) yields $\mathcal{L}_{X^+}(t) \equiv \mathcal{L}_X(t)$ for all $t \in [0, T]$. Inserting this into (6.6.149) of Theorem 6.6.22 gives the result. ■

Corollary 6.6.24. *For **semimartingales** $X_1, \ldots, X_n$, the rank processes $X_{(1)}, \ldots, X_{(n)}$ are **semimartingales**, and we have for all $t \in [0, T]$,*

$$\sum_{i=1}^{n} \mathcal{L}_{X^-_{(i)}}(t) \quad = \quad \sum_{i=1}^{n} \mathcal{L}_{X^-_i}(t). \tag{6.6.154}$$

Proof. For all semimartingales $X_1, \ldots, X_n$, the result follows directly from Theorem 6.6.22 since X_i^- are semimartingales, for all $i = 1, 2, \ldots, n$. Alternatively, consider the following family of semimartingales defined by $Z_i \triangleq -X_i$, for all $i = 1, 2, \ldots, n$. Then for $t \in [0, T]$, the corresponding rank processes are given by

$$Z_{(n)}(t) = -X_{(1)}(t) \leq Z_{(n-1)}(t) = -X_{(2)}(t) \leq \cdots \leq Z_{(n-i+1)}(t) = -X_{(i)}(t) \leq \cdots \leq Z_{(1)}(t) = -X_{(n)}(t),$$

or, equivalently

$$Z_{(1)}(t) = -X_{(n)}(t) \geq Z_{(2)}(t) = -X_{(n-1)}(t) \geq \cdots \geq Z_{(i)}(t) = -X_{(n-i+1)}(t) \geq \cdots \geq Z_{(n)}(t) = -X_{(1)}(t).$$

Thus, from Theorem 6.6.18, we have

$$\sum_{i=1}^{n} \mathbb{1}_{\{Z_{(n-i+1)}(t-)=0\}} \, dZ^+_{(n-i+1)}(t) \quad = \quad \sum_{i=1}^{n} \mathbb{1}_{\{Z_{n-i+1}(t-)=0\}} \, dZ^+_{n-i+1}(t),$$

and we equivalently obtain

$$\sum_{i=1}^{n} \mathbb{1}_{\{Z_{(i)}(t-)=0\}} \, dZ^+_{(i)}(t) \quad = \quad \sum_{i=1}^{n} \mathbb{1}_{\{Z_i(t-)=0\}} \, dZ^+_i(t).$$

Thus, for the nonnegative semimartingales Z_i^+, for all $i = 1, 2, \ldots, n$, we have by Theorem 6.6.22 or Corollary 6.6.23,

$$\sum_{i=1}^{n} \mathcal{L}_{Z^+_{(i)}}(t) \quad = \quad \sum_{i=1}^{n} \mathcal{L}_{Z^+_i}(t). \tag{6.6.155}$$

Since $Z_i = -X_i$, we ascertain that $Z_i^+ \equiv X_i^-$, so that

$$\sum_{i=1}^{n} \mathcal{L}_{Z^+_i}(t) \quad = \quad \sum_{i=1}^{n} \mathcal{L}_{X^-_i}(t),$$

also from the above inequalities we have for all $t \in [0, T]$, $Z^+_{(i)}(t) = X^-_{(n-i+1)}(t)$ or $Z^+_{(n-i+1)}(t) = X^-_{(i)}(t)$, so that

$$\sum_{i=1}^{n} \mathcal{L}_{Z^+_{(i)}}(t) \quad = \quad \sum_{i=1}^{n} \mathcal{L}_{Z^+_{(n-i+1)}}(t) \quad = \quad \sum_{i=1}^{n} \mathcal{L}_{X^-_{(i)}}(t).$$

Using these results, as the right-hand side and left-hand side of (6.6.155) respectively, we obtain

$$\sum_{i=1}^{n} \mathcal{L}_{X^-_{(i)}}(t) \quad = \quad \sum_{i=1}^{n} \mathcal{L}_{X^-_i}(t).$$

■

Corollary 6.6.25. *For **semimartingales** $X_1, \ldots, X_n$, the rank processes $X_{(1)}, \ldots, X_{(n)}$ are **semimartingales**, and we have for all $t \in [0, T]$,*

$$\sum_{i=1}^{n} \mathcal{L}_{X_{(k)} - X_{(i)}}(t) \quad = \quad \sum_{i=1}^{n} \mathcal{L}_{X_{(k)} - X_i}(t). \tag{6.6.156}$$

Proof. The result follows directly from Corollary 6.6.21 and (6.3.148), or is a direct consequence of Theorem 6.6.22. ■

Consequently, from the foregoing corollary, we have the following results

$$\sum_{i=1}^{n} \mathcal{L}_{\left(X_{(k)}-X_{(i)}\right)^+}(t) = \sum_{i=1}^{n} \mathcal{L}_{\left(X_{(k)}-X_i\right)^+}(t), \tag{6.6.157}$$

$$\sum_{i=1}^{n} \mathcal{L}_{\left(X_{(k)}-X_{(i)}\right)^-}(t) = \sum_{i=1}^{n} \mathcal{L}_{\left(X_{(k)}-X_i\right)^-}(t). \tag{6.6.158}$$

Theorem 6.6.26 ([**Pamen (2009), Ghomrasni & Pamen (2010)**]). *Let $X_1, \ldots, X_n$ be **semimartingales**. Then the kth ranked processes $X_{(k)}$, for all $k = 1, 2, \ldots, n$, are **semimartingales** such that a.s., for $t \in [0, T]$,*

$$\begin{aligned} & dX_{(k)}(t) \\ &= \sum_{i=1}^{n} \frac{1}{N_{t-}(k)} \mathbb{1}_{\{X_{(k)}(t-)=X_i(t-)\}}\, dX_i(t) + \sum_{i=1}^{n} \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{\left(X_{(k)}-X_{(i)}\right)^+}(t) - \sum_{i=1}^{n} \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{\left(X_{(k)}-X_{(i)}\right)^-}(t), \end{aligned} \tag{6.6.159}$$

$$= \sum_{i=1}^{n} \frac{1}{N_{t-}(k)} \mathbb{1}_{\{X_{(k)}(t-)=X_i(t-)\}}\, dX_i(t) + \sum_{\ell=k+1}^{n} \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{X_{(k)}-X_{(\ell)}}(t) - \sum_{\ell=1}^{k-1} \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{X_{(\ell)}-X_{(k)}}(t). \tag{6.6.160}$$

Proof. Equation (6.6.149) of Theorem 6.6.22 (alternatively, equation (6.6.152) of Corollary 6.6.23 in conjunction with equation (6.6.154) of Corollary 6.6.24), or equation (6.6.156) of Corollary 6.6.25 (alternatively, equation (6.6.157) in combination with equation (6.6.158)), imply that (6.6.107) of Theorem 6.6.17 can be written as

$$\begin{aligned} & dX_{(k)}(t) \\ &= \sum_{i=1}^{n} \frac{1}{N_{t-}(k)} \mathbb{1}_{\{X_{(k)}(t-)=X_i(t-)\}}\, dX_i(t) + \sum_{i=1}^{n} \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{\left(X_{(k)}-X_i\right)^+}(t) - \sum_{i=1}^{n} \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{\left(X_{(k)}-X_i\right)^-}(t) \\ &= \sum_{i=1}^{n} \frac{1}{N_{t-}(k)} \mathbb{1}_{\{X_{(k)}(t-)=X_i(t-)\}}\, dX_i(t) + \sum_{i=1}^{n} \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{\left(X_{(k)}-X_{(i)}\right)^+}(t) - \sum_{i=1}^{n} \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{\left(X_{(k)}-X_{(i)}\right)^-}(t). \end{aligned}$$

Therefore, by (6.6.74) and (6.6.75), we arrive at the following

$$\begin{aligned} & dX_{(k)}(t) \\ &= \sum_{i=1}^{n} \frac{1}{N_{t-}(k)} \mathbb{1}_{\{X_{(k)}(t-)=X_i(t-)\}}\, dX_i(t) + \sum_{\ell=k+1}^{n} \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{X_{(k)}-X_{(\ell)}}(t) - \sum_{\ell=1}^{k-1} \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{X_{(\ell)}-X_{(k)}}(t). \end{aligned}$$

■

Theorem 6.6.27. *Let $X_1, \ldots, X_n$ be **semimartingales**. Then the kth ranked processes $X_{(k)}$, for all $k = 1, 2, \ldots, n$, are **semimartingales** such that a.s., for $t \in [0, T]$,*

$$\begin{aligned} & dX_{(k)}(t) \\ &= \sum_{i=1}^{n} \frac{1}{N_{t-}(k)} \mathbb{1}_{\{X_{(k)}(t-)=X_{(i)}(t-)\}}\, dX_{(i)}(t) + \sum_{i=1}^{n} \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{\left(X_{(k)}-X_i\right)^+}(t) - \sum_{i=1}^{n} \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{\left(X_{(k)}-X_i\right)^-}(t). \end{aligned} \tag{6.6.161}$$

Proof. Equation (6.6.149) of Theorem 6.6.22 (alternatively, equation (6.6.152) of Corollary 6.6.23 in conjunction with equation (6.6.154) of Corollary 6.6.24), or equation (6.6.156) of Corollary 6.6.25 (alternatively, equation (6.6.157) in combination with equation (6.6.158)), imply that (6.6.108) of Theorem 6.6.17 can be written as

$$\begin{aligned} & dX_{(k)}(t) \\ &= \sum_{i=1}^{n} \frac{1}{N_{t-}(k)} \mathbb{1}_{\{X_{(k)}(t-)=X_{(i)}(t-)\}}\, dX_{(i)}(t) + \sum_{i=1}^{n} \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{(X_{(k)}-X_{(i)})^+}(t) - \sum_{i=1}^{n} \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{(X_{(k)}-X_{(i)})^-}(t) \\ &= \sum_{i=1}^{n} \frac{1}{N_{t-}(k)} \mathbb{1}_{\{X_{(k)}(t-)=X_{(i)}(t-)\}}\, dX_{(i)}(t) + \sum_{i=1}^{n} \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{(X_{(k)}-X_i)^+}(t) - \sum_{i=1}^{n} \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{(X_{(k)}-X_i)^-}(t). \end{aligned}$$

■

Proposition 6.6.28 ([**Pamen (2009), Ghomrasni & Pamen (2010)**]). *Let $X_1, \ldots, X_n$ be **semimartingales**. Then we have for $t \in [0, T]$,*

$$\sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t-)=X_{(i)}(t-)\}}\, dX_{(i)}(t) = \sum_{i=1}^{n} \mathbb{1}_{\{X_{(k)}(t-)=X_i(t-)\}}\, dX_i(t). \tag{6.6.162}$$

Proof. Upon comparison of (6.6.107) of Theorem 6.6.17 and (6.6.161) of Theorem 6.6.27, we get

$$\sum_{i=1}^{n} \frac{1}{N_{t-}(k)} \mathbb{1}_{\{X_{(k)}(t-)=X_{(i)}(t-)\}}\, dX_{(i)}(t) = \sum_{i=1}^{n} \frac{1}{N_{t-}(k)} \mathbb{1}_{\{X_{(k)}(t-)=X_i(t-)\}}\, dX_i(t),$$

and the result follows. We could also compare (6.6.108) of Theorem 6.6.17 to (6.6.159) of Theorem 6.6.26, or (6.6.109) of Theorem 6.6.17 to (6.6.160) of Theorem 6.6.26, to yield the same result. ■

Theorem 6.6.29 ([**Yan (1985, 1989), Ouknine (1990)**]). *For **semimartingales** X and Y, the processes $X \vee Y$ and $X \wedge Y$ are **semimartingales**, and we have for all $t \in [0, T]$,*

$$\Lambda_{X \vee Y}(t) + \Lambda_{X \wedge Y}(t) = \Lambda_X(t) + \Lambda_Y(t). \tag{6.6.163}$$

Theorem 6.6.30 ([**Pamen (2009), Ghomrasni & Pamen (2010)**]). *Let $X_1, \ldots, X_n$ be **semimartingales**. Then we have for $t \in [0, T]$,*

$$\sum_{i=1}^{n} \Lambda_{X_{(i)}}(t) = \sum_{i=1}^{n} \Lambda_{X_i}(t), \tag{6.6.164}$$

where $\Lambda_X(t)$ is the local time of the semimartingale X at the level zero.

Proof. By virtue of (6.6.152) of Corollary 6.6.23, we have

$$\sum_{i=1}^{n} \mathcal{L}_{X^+_{(i)}}(t) = \sum_{i=1}^{n} \mathcal{L}_{X^+_i}(t).$$

Employing (6.6.114) gives

$$\begin{aligned} \sum_{i=1}^{n} \mathcal{L}_{X^+_{(i)}}(t) &= \sum_{i=1}^{n} \left[\Lambda_{X^+_{(i)}}(t) + \sum_{s \le t} \mathbb{1}_{\{X_{(i)}(s-)=0\}}\, \Delta X^+_{(i)}(s) \right] \\ &= \sum_{i=1}^{n} \Lambda_{X^+_{(i)}}(t) + \sum_{s \le t} \sum_{i=1}^{n} \mathbb{1}_{\{X_{(i)}(s-)=0\}}\, \Delta X^+_{(i)}(s), \end{aligned} \tag{6.6.165}$$

and

$$\begin{aligned}\sum_{i=1}^{n}\mathcal{L}_{X_i^+}(t) &= \sum_{i=1}^{n}\left[\Lambda_{X_i^+}(t)+\sum_{s\le t}\mathbb{1}_{\{X_i(s-)=0\}}\,\Delta X_i^+(s)\right]\\ &= \sum_{i=1}^{n}\Lambda_{X_i^+}(t)+\sum_{s\le t}\sum_{i=1}^{n}\mathbb{1}_{\{X_i(s-)=0\}}\,\Delta X_i^+(s). \qquad (6.6.166)\end{aligned}$$

Thus, from (6.6.152) we can equate (6.6.165) and (6.6.166). Thus by (6.6.123) of Theorem 6.6.18 and by the continuity of local time, we get

$$\sum_{i=1}^{n}\Lambda_{X_{(i)}^+}(t) = \sum_{i=1}^{n}\Lambda_{X_i^+}(t). \qquad (6.6.167)$$

Hence, the desired result follows from the fact that $\Lambda_X(t)=\Lambda_{X^+}(t)$ for all $t\in[0,T]$, for all semimartingales X. ■

6.7 Representation of the Ranked Market Weight Processes

In this section, we shall apply the results of the previous section to the ranked market weight processes, and obtain a representation for the ranked market weight processes that involves the use of semimartingale local times.

6.7.1 Rank Market Weight Processes

Definition 6.7.1 (Rank Market Weight Process). *Let $\mathcal{M}$ be a market of stocks $X_1,X_2,\ldots,X_n$, and let $\mu_1,\mu_2,\ldots,\mu_n$ be the market weight processes. For $t\in[0,T]$, let $\boldsymbol{p}_t=\big(p_t(1),p_t(2),\ldots,p_t(n)\big)$ denote the **random permutation** of $\{1,2,\ldots,n\}$, then for $k=1,2,\ldots,n$, the following hold*

$$\mu_{p_t(k)}(t) \triangleq \mu_{(k)}(t), \qquad (6.7.1)$$

and,

$$\text{if}\quad \mu_{(k)}(t) = \mu_{(k+1)}(t) \quad \text{then}\quad p_t(k) < p_t(k+1). \qquad (6.7.2)$$

6.7.2 Decomposition of the Ranked Market Weight Processes of Pathwise Mutually Nondegenerate Absolutely Continuous Semimartingales

Here, we apply Proposition 6.6.2 to the ranked market weight processes, which deals exclusively with pathwise mutually nondegenerate absolutely continuous semimartingales.

Corollary 6.7.2 ([**Fernholz (2002)**]). *Let $\mathcal{M}$ be a market of stocks $X_1,\ldots,X_n$ that are **pathwise mutually nondegenerate**, and for $t\in[0,T]$, let $\boldsymbol{p}_t$ be the random permutation defined in Definition 6.7.1 such that (6.7.1) and (6.7.2) hold. Then the **market weight processes** $\mu_1,\mu_2,\ldots,\mu_n$, satisfy*

$$\begin{aligned}d\log\mu_{(k)}(t) &= \sum_{i=1}^{n}\mathbb{1}_{\{i\}}\big(p_t(k)\big)\,d\log\mu_i(t)+\frac{1}{2}\,d\Lambda_{\log\mu_{(k)}-\log\mu_{(k+1)}}(t)-\frac{1}{2}\,d\Lambda_{\log\mu_{(k-1)}-\log\mu_{(k)}}(t) \qquad (6.7.3)\\ &= \sum_{i=1}^{n}\mathbb{1}_{\{i\}}\big(p_t(k)\big)\,d\log\mu_i(t)+\frac{1}{2}\,d\mathfrak{L}_{k,k+1}(t)-\frac{1}{2}\,d\mathfrak{L}_{k-1,k}(t), \qquad (6.7.4)\end{aligned}$$

a.s., for $t\in[0,T]$, where $\mathbb{1}_{\{i\}}\big(p_t(k)\big)=\mathbb{1}_{\{p_t(k)=i\}}=\mathbb{1}_{\{0\}}\big(p_t(k)-i\big)$, or in integral form

$$\log\mu_{(k)}(t) = \log\mu_{(k)}(0)+\sum_{i=1}^{n}\int_0^t\mathbb{1}_{\{i\}}\big(p_s(k)\big)\,d\log\mu_i(s)+\frac{1}{2}\,\mathfrak{L}_{k,k+1}(t)-\frac{1}{2}\,\mathfrak{L}_{k-1,k}(t). \qquad (6.7.5)$$

Here the quantity $\mathfrak{L}_{k,k+1}(t) \equiv \Lambda_{\Xi_k}(t)$ is the semimartingale local time at the origin, accumulated by the nonnegative process

$$\Xi_k(t) \quad \triangleq \quad \log \mu_{(k)}(t) - \log \mu_{(k+1)}(t), \qquad t \in [0,T]. \tag{6.7.6}$$

It measures the cumulative effect of the changes that have occurred during the time interval $[0,t]$ between ranks k and $k+1$. By convention, we have $\Lambda_{\log \mu_{(0)} - \log \mu_{(1)}}(t) \equiv \mathfrak{L}_{0,1}(t) \triangleq 0$ and $\Lambda_{\log \mu_{(n)} - \log \mu_{(n+1)}}(t) \equiv \mathfrak{L}_{n,n+1}(t) \triangleq 0$, for all $t \in [0,T]$.

Proof. Note that the permutation $\boldsymbol{p}_t$ is uniquely defined by (6.7.1) and (6.7.2), and associates each rank process with one of the original market weights that has the same value at time t. That the market weights are absolutely continuous semimartingales follows from Definition 2.2.12 and Lemma 6.5.3. Hence, Proposition 6.6.2 can be applied, and the corollary follows. ■

Thus, the market weight processes are absolutely continuous semimartingales, and so if the stocks $X_1, \ldots, X_n$ in the market $\mathcal{M}$ are pathwise mutually nondegenerate, the above corollary allows us to represent the ranked market weights $\mu_{(1)}, \ldots, \mu_{(n)}$ in terms of the original market weights $\mu_1, \ldots, \mu_n$.

6.7.3 Decomposition of the Ranked Market Weight Processes of Absolutely Continuous Semimartingales

Consider the case where three or more processes may be equal at a given time $t \in [0,T]$, i.e., we shall consider the case in which there exists $\Omega' \subset \Omega$, $\mathbb{P}(\Omega') = 1$, such that for $\omega \in \Omega'$, the set $\big\{t \in [0,T] \;\big|\; X_i(t,\omega) = X_j(t,\omega) = X_k(t,\omega)\big\}$ is finite for all $i < j < k$.

Corollary 6.7.3. *Let $\mathcal{M}$ be a market of stocks $X_1, \ldots, X_n$ satisfying condition (i) of the pathwise mutual nondegeneracy conditions provided in Definition 6.4.1 (i.e., for all $i \neq j$, the set $\big\{t \in [0,T] \;\big|\; X_i(t) = X_j(t)\big\}$ has Lebesgue measure zero, almost surely). Suppose that u is an element of the set $\mathcal{U}$ defined in (6.6.57). Then the **market weight processes** $\mu_1, \mu_2, \ldots, \mu_n$, satisfy*

$$\begin{aligned} & d \log \mu_{(k)}(t) \\ &= \sum_{i=1}^{n} \mathbb{1}_{\{u_t(k)=i\}}\, d \log \mu_i(t) + \sum_{\ell=k+1}^{n} \frac{1}{N_t(k)}\, d\Lambda_{\log \mu_{(k)} - \log \mu_{(\ell)}}(t) - \sum_{\ell=1}^{k-1} \frac{1}{N_t(k)}\, d\Lambda_{\log \mu_{(\ell)} - \log \mu_{(k)}}(t) \qquad (6.7.7) \\ &= \sum_{i=1}^{n} \mathbb{1}_{\{u_t(k)=i\}}\, d \log \mu_i(t) + \sum_{\ell=k+1}^{n} \frac{1}{N_t(k)}\, d\mathfrak{L}_{k,\ell}(t) - \sum_{\ell=1}^{k-1} \frac{1}{N_t(k)}\, d\mathfrak{L}_{\ell,k}(t), \qquad (6.7.8) \end{aligned}$$

a.s., for $t \in [0,T]$, where $\mathbb{1}_{\{u_t(k)=i\}} = \mathbb{1}_{\{i\}}\big(u_t(k)\big) = \mathbb{1}_{\{0\}}\big(u_t(k) - i\big)$. Here the quantity $\mathfrak{L}_{k,\ell}(t) \equiv \Lambda_{\Xi_{k,\ell}}(t)$, for $1 \leq k < \ell \leq n$, is the semimartingale local time at the origin, accumulated by the nonnegative process

$$\Xi_{k,\ell}(t) \quad \triangleq \quad \log \mu_{(k)}(t) - \log \mu_{(\ell)}(t), \qquad t \in [0,T]. \tag{6.7.9}$$

It measures the cumulative effect of the changes that have occurred during the time interval $[0,t]$ between ranks k and ℓ. By convention, we have $\Lambda_{\log \mu_{(0)} - \log \mu_{(1)}}(t) \equiv \mathfrak{L}_{0,1}(t) \triangleq 0$ and $\Lambda_{\log \mu_{(n)} - \log \mu_{(n+1)}}(t) \equiv \mathfrak{L}_{n,n+1}(t) \triangleq 0$, for all $t \in [0,T]$.

Proof. That the market weights are absolutely continuous semimartingales follows from Definition 2.2.12 and Lemma 6.5.3. Hence, (6.6.73) of Theorem 6.6.6 can be applied, and the corollary follows. ■

6.7.4 Decomposition of the Ranked Market Weight Processes of Continuous Semimartingales

Corollary 6.7.4. *Let $\mathcal{M}$ be a market of stocks $X_1, \ldots, X_n$. Then the* ***market weight processes*** *$\mu_1, \mu_2, \ldots, \mu_n$, satisfy*

$$\begin{aligned} & d\log\mu_{(k)}(t) \\ &= \sum_{i=1}^{n} \frac{1}{N_t(k)} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, d\log\mu_i(t) + \sum_{\ell=k+1}^{n} \frac{1}{N_t(k)}\, d\Lambda_{\log\mu_{(k)}-\log\mu_{(\ell)}}(t) - \sum_{\ell=1}^{k-1} \frac{1}{N_t(k)}\, d\Lambda_{\log\mu_{(\ell)}-\log\mu_{(k)}}(t) \qquad (6.7.10) \\ &= \sum_{i=1}^{n} \frac{1}{N_t(k)} \mathbb{1}_{\{X_{(k)}(t)=X_i(t)\}}\, d\log\mu_i(t) + \sum_{\ell=k+1}^{n} \frac{1}{N_t(k)}\, d\mathfrak{L}_{k,\ell}(t) - \sum_{\ell=1}^{k-1} \frac{1}{N_t(k)}\, d\mathfrak{L}_{\ell,k}(t), \qquad (6.7.11) \end{aligned}$$

a.s., for $t \in [0, T]$.

Proof. Note that the incidence $X_{(k)}(t) = X_i(t)$ corresponds to the incidence $\mu_{(k)}(t) = \mu_i(t)$, and that the rankings of μ_i correspond to the rankings of X_i. By applying (6.6.104) of Theorem 6.6.16, the corollary follows. ■

6.7.5 Decomposition of the Ranked Market Weight Processes of General Semimartingales

Corollary 6.7.5. *Let $X_1, \ldots, X_n$ be* ***semimartingales****. Then the* ***market weight processes*** *$\mu_1, \mu_2, \ldots, \mu_n$, satisfy*

$$\begin{aligned} & d\log\mu_{(k)}(t) \\ &= \sum_{i=1}^{n} \frac{1}{N_{t-}(k)} \mathbb{1}_{\{X_{(k)}(t-)=X_i(t-)\}}\, d\log\mu_i(t) + \sum_{\ell=k+1}^{n} \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{\log\mu_{(k)}-\log\mu_{(\ell)}}(t) - \sum_{\ell=1}^{k-1} \frac{1}{N_{t-}(k)}\, d\mathcal{L}_{\log\mu_{(\ell)}-\log\mu_{(k)}}(t) \qquad (6.7.12) \\ &= \sum_{i=1}^{n} \frac{1}{N_{t-}(k)} \mathbb{1}_{\{X_{(k)}(t-)=X_i(t-)\}}\, d\log\mu_i(t) + \sum_{\ell=k+1}^{n} \frac{1}{N_{t-}(k)}\, d\,\mathrm{L}_{k,\ell}(t) - \sum_{\ell=1}^{k-1} \frac{1}{N_{t-}(k)}\, d\,\mathrm{L}_{\ell,k}(t), \qquad (6.7.13) \end{aligned}$$

a.s., for $t \in [0, T]$. Here the quantity $\mathrm{L}_{k,\ell}(t) \equiv \mathcal{L}_{\Xi_{k,\ell}}(t)$, for $1 \le k < \ell \le n$, is the semimartingale local time component at the origin, accumulated by the nonnegative process

$$\Xi_{k,\ell}(t) \triangleq \log\mu_{(k)}(t) - \log\mu_{(\ell)}(t), \qquad t \in [0, T]. \qquad (6.7.14)$$

It measures the cumulative effect of the changes that have occurred during the time interval $[0, t]$ between ranks k and ℓ. By convention, we have $\mathcal{L}_{\log\mu_{(0)}-\log\mu_{(1)}}(t) \equiv \mathrm{L}_{0,1}(t) \triangleq 0$ and $\mathcal{L}_{\log\mu_{(n)}-\log\mu_{(n+1)}}(t) \equiv \mathrm{L}_{n,n+1}(t) \triangleq 0$, for all $t \in [0, T]$.

Proof. By applying (6.6.160) of Theorem 6.6.26, the corollary follows. ■

6.8 Rank-Based Functionally Generated Portfolios

6.8.1 Equity Portfolios Generated by Functions of the Ranked Market Weight Processes

The principal result of this section is a version of Theorem 5.2.2 that holds for generating functions of the form

$$\mathbf{G}\big(\mu_1(t), \ldots, \mu_n(t)\big) = \mathfrak{G}\big(\mu_{(1)}(t), \ldots, \mu_{(n)}(t)\big), \qquad (6.8.1)$$

where $\mathfrak{G}$ is a positive twice continuously differentiable function of the ranked market weights defined on some open neighbourhood U of Δ^{n-1}. This situation is frequently encountered in practice as portfolios are often chosen exclusively from either a large-stock index or a small-stock index, and the selection of these indices depends on the rank of the stocks in the market and thus depends on the ranked market weights of the stocks. Thus, in this section, we shall consider portfolios that are generated by functions of the ranked market weights as in (6.8.1). The representation for the ranked market weight processes derived in the preceding section in Corollary 6.7.2, is employed to extend Theorem 5.2.2 to functions of the form (6.8.1). For a generating function of the form (6.8.1), $\mathbf{G}\big(\boldsymbol{\mu}(t)\big)$ or $\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)$ measures the effect that changes in the distribution of capital of the market have on some portfolio ψ. This is somewhat different to Theorem 5.2.2, where the generating function measures the dependence on individual stocks by name, i.e., by subscript or index.

It will be convenient to use the notation

$$\boldsymbol{\mu}_{(\cdot)}(t) \quad = \quad \big(\mu_{(1)}(t), \ldots, \mu_{(n)}(t)\big), \qquad t \in [0,T]. \tag{6.8.2}$$

For $k, \ell = 1, 2, \ldots, n$, we define the *relative rank covariance processes* $\tau_{(k\ell)} = \big\{\tau_{(k\ell)}(t), t \in [0,\infty)\big\}$, i.e., the ranked covariance process relative to the market, by

$$\tau_{(k\ell)}(t) \quad \triangleq \quad \tau_{p_t(k)p_t(\ell)}(t), \qquad t \in [0,T], \tag{6.8.3}$$

where $\boldsymbol{p}_t$ is the permutation defined in Definition 6.7.1. Since for all i and j, τ_{ij} is a.s. an L^1 function of t, the same is true for $\tau_{(k\ell)}$. The next theorem is the main result of this chapter. It shows that there exists a broad class of portfolio generating functions that depend on ranked market weights.

Theorem 6.8.1 ([**Fernholz (2002)**]). *Let $\mathcal{M}$ be a market of stocks $X_1, \ldots, X_n$ that are* ***pathwise mutually nondegenerate****, and let $\boldsymbol{p}_t$ be the random permutation defined by* (6.7.1) *and* (6.7.2)*. Let U be an open neighbourhood in $\mathbb{R}^n$ ($U \subset \mathbb{R}^n$) of the open positive unit $(n-1)-$simplex Δ^{n-1} and let $\mathbf{G} : U \to (0,\infty)$ be a positive twice continuously differentiable function defined on some open neighbourhood U of Δ^{n-1}. Suppose that there exists a positive twice continuously differentiable function $\mathfrak{G}$ defined on some open neighbourhood U of Δ^{n-1} such that for $\mathbf{x} = (x_1, \ldots, x_n) \in U$ and $\mathbf{x}_{(\cdot)} = \big(x_{(1)}, \ldots, x_{(n)}\big) \in U$,*

$$\mathbf{G}\big(x_1, \ldots, x_n\big) \quad = \quad \mathfrak{G}\big(x_{(1)}, \ldots, x_{(n)}\big), \tag{6.8.4}$$

and for all $k = 1, 2, \ldots, n$, $x_{(k)} D_k \log \mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big)$ is bounded for $\mathbf{x}, \mathbf{x}_{(\cdot)} \in \Delta^{n-1}$. Alternatively, such that for $\boldsymbol{\mu}(t) = \big(\mu_1(t), \ldots, \mu_n(t)\big) \in U$ and $\boldsymbol{\mu}_{(\cdot)}(t) = \big(\mu_{(1)}(t), \ldots, \mu_{(n)}(t)\big) \in U$,

$$\mathbf{G}\big(\mu_1(t), \ldots, \mu_n(t)\big) \quad = \quad \mathfrak{G}\big(\mu_{(1)}(t), \ldots, \mu_{(n)}(t)\big), \tag{6.8.5}$$

and for all $k = 1, 2, \ldots, n$, $\mu_{(k)}(t) D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)$ is bounded for $\boldsymbol{\mu}(t), \boldsymbol{\mu}_{(\cdot)}(t) \in \Delta^{n-1}$. Then for all $t \in [0,T]$, a.s., and for $k = 1, 2, \ldots, n$, the rank-dependent generating function $\mathfrak{G}$ generates the (rank-based functionally generated) portfolio ψ with ***weights***

$$\psi_{p_t(k)}(t) \quad = \quad \left(D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + 1 - \sum_{\ell=1}^{n} \mu_{(\ell)}(t)\, D_\ell \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \right) \mu_{(k)}(t), \tag{6.8.6}$$

and with ***drift process*** Υ*, that satisfies for $t \in [0,T]$, a.s.,*

$$\begin{aligned} d\Upsilon(t) \quad &= \quad \frac{-1}{2\,\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \sum_{k,\ell=1}^{n} D_{k\ell}\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\, dt \\ &\quad + \frac{1}{2}\sum_{k=1}^{n-1} \Big(\psi_{p_t(k+1)}(t) - \psi_{p_t(k)}(t)\Big)\, d\mathfrak{L}_{k,k+1}(t). \end{aligned} \tag{6.8.7}$$

Let μ be the market portfolio and ψ be the rank-based functionally generated portfolio, and let Z_μ and Z_ψ be their portfolio value processes, respectively. Then, a.s., for $t \in [0,T]$,

$$\begin{aligned} d\log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \quad &= \quad d\log\Big(Z_\psi(t)/Z_\mu(t)\Big) + \frac{1}{2\,\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \sum_{k,\ell=1}^{n} D_{k\ell}\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\, dt \\ &\quad - \frac{1}{2}\sum_{k=1}^{n-1} \Big(\psi_{p_t(k+1)}(t) - \psi_{p_t(k)}(t)\Big)\, d\mathfrak{L}_{k,k+1}(t). \end{aligned} \tag{6.8.8}$$

Proof. First, we must verify that $\boldsymbol{\psi}$ defined by (6.8.6) is indeed a portfolio, and that Υ defined by (6.8.7) is of bounded variation. Let us define

$$\phi\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \;\triangleq\; 1 - \sum_{\ell=1}^{n} \mu_{(\ell)}(t)\, D_\ell \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big), \qquad t \in [0,T]. \tag{6.8.9}$$

Then the weights in (6.8.6) are given by

$$\psi_{p_t(k)}(t) \;=\; \Big(D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + \phi\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\Big)\, \mu_{(k)}(t), \qquad t \in [0,T], \tag{6.8.10}$$

for all $k = 1, 2, \ldots, n$. Therefore if $\boldsymbol{\psi}$ satisfies (6.8.10), then

$$\begin{aligned}
\sum_{i=1}^{n} \psi_i(t) \;=\; \sum_{k=1}^{n} \psi_{p_t(k)}(t) \;&=\; \sum_{k=1}^{n} \mu_{(k)}(t)\, D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + \sum_{k=1}^{n} \mu_{(k)}(t) \phi\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \\
&=\; \sum_{k=1}^{n} \mu_{(k)}(t)\, D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + \phi\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \\
&=\; 1.
\end{aligned}$$

Thus, the weights of the rank-based generated portfolio sum to 1. In addition, the conditions on the rank-dependent generating function $\mathfrak{G}$ ensure that the weight processes ψ_i are bounded on $[0,T] \times \Omega$. Hence, $\boldsymbol{\psi}$ is a portfolio process and it follows that

$$\psi_{p_t(k)}(t) \;=\; \left(D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + 1 - \sum_{\ell=1}^{n} \mu_{(\ell)}(t)\, D_\ell \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\right) \mu_{(k)}(t),$$

is satisfied. Regarding Υ, let us consider the two expressions on the right-hand side of (6.8.7) separately. The process represented by the first expression is a.s. of bounded variation because $\tau_{(k\ell)}$ is an L^1 function of t and the rest of the terms are continuous in t. The second expression is a sum of local times multiplied by bounded functions, and hence is also of bounded variation. Therefore, Υ is a.s. of bounded variation.

We must show that the portfolio ψ defined by (6.8.6) and the drift process $\Upsilon = \{\Upsilon(t), t \in [0,T]\}$ defined by (6.8.7) satisfy (5.2.2). To accomplish this, we shall analyse the generating function term $\log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) = \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)$ in (5.2.2) and the relative return process $\log\big(Z_\psi(t)/Z_\mu(t)\big)$, and show that the difference of these two terms satisfies (6.8.7). We first need some preliminary results. For a positive twice continuously differentiable function, $\mathfrak{G}$, and for all $\mathbf{x}_{(\cdot)} \in \Delta^{n-1}$, we have

$$D_k \log \mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big) \;=\; \frac{\partial \log \mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big)}{\partial x_{(k)}} \;=\; \frac{1}{\mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big)} \frac{\partial \mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big)}{\partial x_{(k)}} \;=\; \frac{D_k \mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big)}{\mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big)}, \tag{6.8.11}$$

and employing (6.8.11), we have

$$\begin{aligned}
D_{k\ell} \log \mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big) \;=\; \frac{\partial^2 \log \mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big)}{\partial x_{(k)} \partial x_{(\ell)}} \;&=\; \frac{\partial}{\partial x_{(\ell)}} \left(\frac{1}{\mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big)} \frac{\partial \mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big)}{\partial x_{(k)}}\right) \;=\; D_\ell \left(\frac{D_k \mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big)}{\mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big)}\right) \\
&=\; \frac{1}{\mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big)} \frac{\partial^2 \mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big)}{\partial x_{(k)} \partial x_{(\ell)}} - \frac{\partial \mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big)}{\partial x_{(k)}} \frac{1}{\mathfrak{G}^2\big(\mathbf{x}_{(\cdot)}\big)} \frac{\partial \mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big)}{\partial x_{(\ell)}} \\
&=\; \frac{D_{k\ell} \mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big)}{\mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big)} - \frac{D_k \mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big)\, D_\ell \mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big)}{\mathfrak{G}^2\big(\mathbf{x}_{(\cdot)}\big)} \\
&=\; \frac{D_{k\ell} \mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big)}{\mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big)} - D_k \log \mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big)\, D_\ell \log \mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big).
\end{aligned} \tag{6.8.12}$$

Therefore for the market portfolio, $\boldsymbol{\mu}_{(\cdot)} \in \Delta^{n-1}$, we have

$$D_{k\ell} \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \;=\; \frac{D_{k\ell} \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} - D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\, D_\ell \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big). \tag{6.8.13}$$

Now, (6.7.4) of Corollary 6.7.2 together with (2.12.25) and (6.8.3), imply that, for $k, \ell = 1, 2, \ldots, n$,

$$\begin{aligned} d\left\langle \log \mu_{(k)}, \log \mu_{(\ell)} \right\rangle_t &= d\left\langle \int_0^{\cdot} \sum_{i=1}^n \mathbb{1}_{\{i\}}\big(p_s(k)\big)\, d\log \mu_{i,s}, \int_0^{\cdot} \sum_{j=1}^n \mathbb{1}_{\{j\}}\big(p_s(\ell)\big)\, d\log \mu_{j,s} \right\rangle_t \\ &= \sum_{i,j=1}^n \mathbb{1}_{\{i\}}\big(p_t(k)\big)\, \mathbb{1}_{\{j\}}\big(p_t(\ell)\big)\, d\left\langle \log \mu_i, \log \mu_j \right\rangle_t \\ &= \sum_{i,j=1}^n \mathbb{1}_{\{i\}}\big(p_t(k)\big)\, \mathbb{1}_{\{j\}}\big(p_t(\ell)\big) \tau_{ij}(t)\, dt \\ &= \tau_{p_t(k)p_t(\ell)}(t) \\ &= \tau_{(k\ell)}(t)\, dt. \end{aligned}$$

Therefore, by a similar deduction that led to (2.12.48), we have

$$d\left\langle \mu_{(k)}, \mu_{(\ell)} \right\rangle_t = \mu_{(k)}(t)\mu_{(\ell)}(t)\, \tau_{(k\ell)}(t)\, dt, \qquad t \in [0,T], \qquad \text{a.s.} \tag{6.8.14}$$

Also note that for all $t \in [0,T]$, $\sum_{k=1}^n \mu_{(k)}(t) = \sum_{i=1}^n \mu_i(t) = 1$, and hence $\sum_{k=1}^n d\mu_{(k)}(t) = d\left(\sum_{k=1}^n \mu_{(k)}(t)\right) = 0$. Consider the generating function component of the relative return, $\log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)$. Then, the above equation (6.8.14) in conjunction with an application of Itô's formula to $\log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)$, yields a.s., for $t \in [0,T]$,

$$\begin{aligned} d\log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) &= d\log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \\ &= \sum_{k=1}^n D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\, d\mu_{(k)}(t) + \frac{1}{2} \sum_{k,\ell=1}^n D_{k\ell} \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\, d\left\langle \mu_{(k)}, \mu_{(\ell)} \right\rangle_t \\ &= \sum_{k=1}^n D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\, d\mu_{(k)}(t) + \frac{1}{2} \sum_{k,\ell=1}^n D_{k\ell} \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\, dt, \end{aligned} \tag{6.8.15}$$

which when combined with (6.8.13), yields a.s., for $t \in [0,T]$,

$$\begin{aligned} & d\log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \\ &= \sum_{k=1}^n D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\, d\mu_{(k)}(t) \\ &\quad + \frac{1}{2} \sum_{k,\ell=1}^n \left[\frac{D_{k\ell}\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} - D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\, D_\ell \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \right] \mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\, dt \\ &= \sum_{k=1}^n D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\, d\mu_{(k)}(t) + \frac{1}{2\,\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \sum_{k,\ell=1}^n D_{k\ell}\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\, dt \\ &\quad - \frac{1}{2} \sum_{k,\ell=1}^n D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\, D_\ell \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\, dt. \end{aligned} \tag{6.8.16}$$

Now let us consider the relative return process $\log\big(Z_\psi(t)/Z_\mu(t)\big)$. By setting $\boldsymbol{\pi} := \boldsymbol{\psi}$ in Proposition 2.12.8 we have a.s., for $t \in [0,T]$,

$$\begin{aligned} d\log\Big(Z_\psi(t)/Z_\mu(t)\Big) &= \sum_{i=1}^n \psi_i(t)\, d\log \mu_i(t) + \gamma_\psi^*(t)\, dt \\ &= \sum_{i=1}^n \left[\sum_{k=1}^n \mathbb{1}_{\{i\}}\big(p_t(k)\big)\psi_{p_t(k)}(t) \right] d\log \mu_i(t) + \gamma_\psi^*(t)\, dt \\ &= \sum_{i=1}^n \sum_{k=1}^n \mathbb{1}_{\{i\}}\big(p_t(k)\big)\psi_{p_t(k)}(t)\, d\log \mu_i(t) + \gamma_\psi^*(t)\, dt \\ &= \sum_{k=1}^n \psi_{p_t(k)}(t) \sum_{i=1}^n \mathbb{1}_{\{i\}}\big(p_t(k)\big)\, d\log \mu_i(t) + \gamma_\psi^*(t)\, dt, \end{aligned} \tag{6.8.17}$$

since $\psi_i(t) = \sum_{k=1}^n \mathbb{1}_{\{i\}}\big(p_t(k)\big)\psi_{p_t(k)}(t)$. Notice that by (6.7.4) of Corollary 6.7.2, we have

$$\begin{aligned}\sum_{k=1}^n \psi_{p_t(k)}(t)\, d\log\mu_{(k)}(t) &= \sum_{k=1}^n \psi_{p_t(k)}(t) \sum_{i=1}^n \mathbb{1}_{\{i\}}\big(p_t(k)\big)\, d\log\mu_i(t) \\ &\quad + \frac{1}{2}\sum_{k=1}^n \psi_{p_t(k)}(t)\, d\mathfrak{L}_{k,k+1}(t) - \frac{1}{2}\sum_{k=1}^n \psi_{p_t(k)}(t)\, d\mathfrak{L}_{k-1,k}(t).\end{aligned}$$

Therefore, by considering the first term on the right-hand side of the above expression, we get

$$\begin{aligned}&\sum_{k=1}^n \psi_{p_t(k)}(t) \sum_{i=1}^n \mathbb{1}_{\{i\}}\big(p_t(k)\big)\, d\log\mu_i(t) \\ &\quad= \sum_{k=1}^n \psi_{p_t(k)}(t)\, d\log\mu_{(k)}(t) - \frac{1}{2}\sum_{k=1}^n \psi_{p_t(k)}(t)\, d\mathfrak{L}_{k,k+1}(t) + \frac{1}{2}\sum_{k=1}^n \psi_{p_t(k)}(t)\, d\mathfrak{L}_{k-1,k}(t) \\ &\quad= \sum_{k=1}^n \psi_{p_t(k)}(t)\, d\log\mu_{(k)}(t) - \frac{1}{2}\sum_{k=1}^{n-1} \psi_{p_t(k)}(t)\, d\mathfrak{L}_{k,k+1}(t) + \frac{1}{2}\sum_{k=2}^n \psi_{p_t(k)}(t)\, d\mathfrak{L}_{k-1,k}(t) \\ &\quad= \sum_{k=1}^n \psi_{p_t(k)}(t)\, d\log\mu_{(k)}(t) - \frac{1}{2}\sum_{k=1}^{n-1} \psi_{p_t(k)}(t)\, d\mathfrak{L}_{k,k+1}(t) + \frac{1}{2}\sum_{k=1}^{n-1} \psi_{p_t(k+1)}(t)\, d\mathfrak{L}_{k,k+1}(t) \\ &\quad= \sum_{k=1}^n \psi_{p_t(k)}(t)\, d\log\mu_{(k)}(t) - \frac{1}{2}\sum_{k=1}^{n-1} \Big(\psi_{p_t(k)}(t) - \psi_{p_t(k+1)}(t)\Big)\, d\mathfrak{L}_{k,k+1}(t) \\ &\quad= \sum_{k=1}^n \psi_{p_t(k)}(t)\, d\log\mu_{(k)}(t) + \frac{1}{2}\sum_{k=1}^{n-1} \Big(\psi_{p_t(k+1)}(t) - \psi_{p_t(k)}(t)\Big)\, d\mathfrak{L}_{k,k+1}(t), \qquad (6.8.18)\end{aligned}$$

since $\mathfrak{L}_{n,n+1} \equiv 0$ and $\mathfrak{L}_{0,1} \equiv 0$. From Lemma 2.4.5 we have the numéraire invariance property (2.4.26),

$$\gamma_\psi^*(t) = \frac{1}{2}\left(\sum_{k=1}^n \psi_{p_t(k)}(t)\tau_{(kk)}(t) - \sum_{k,\ell=1}^n \psi_{p_t(k)}(t)\tau_{(k\ell)}(t)\psi_{p_t(\ell)}(t)\right). \qquad (6.8.19)$$

Substituting (6.8.18) and (6.8.19) into (6.8.17) yields

$$\begin{aligned}d\log\Big(Z_\psi(t)/Z_\mu(t)\Big) &= \sum_{k=1}^n \psi_{p_t(k)}(t)\, d\log\mu_{(k)}(t) + \frac{1}{2}\sum_{k=1}^{n-1} \Big(\psi_{p_t(k+1)}(t) - \psi_{p_t(k)}(t)\Big)\, d\mathfrak{L}_{k,k+1}(t) \\ &\quad + \frac{1}{2}\sum_{k=1}^n \psi_{p_t(k)}(t)\tau_{(kk)}(t)\, dt - \frac{1}{2}\sum_{k,\ell=1}^n \psi_{p_t(k)}(t)\psi_{p_t(\ell)}(t)\tau_{(k\ell)}(t)\, dt.\end{aligned}$$

From (2.12.36), we similarly have by applying Itô's formula to $\mu_{(k)}(t) = \exp\big(\log\mu_{(k)}(t)\big)$,

$$d\log\mu_{(k)}(t) = \frac{d\mu_{(k)}(t)}{\mu_{(k)}(t)} - \frac{1}{2}\tau_{(kk)}(t)\, dt.$$

Hence, the above expression implies

$$\begin{aligned}d\log\Big(Z_\psi(t)/Z_\mu(t)\Big) &= \sum_{k=1}^n \psi_{p_t(k)}(t)\left[\frac{d\mu_{(k)}(t)}{\mu_{(k)}(t)} - \frac{1}{2}\tau_{(kk)}(t)\, dt\right] + \frac{1}{2}\sum_{k=1}^{n-1} \Big(\psi_{p_t(k+1)}(t) - \psi_{p_t(k)}(t)\Big)\, d\mathfrak{L}_{k,k+1}(t) \\ &\quad + \frac{1}{2}\sum_{k=1}^n \psi_{p_t(k)}(t)\tau_{(kk)}(t)\, dt - \frac{1}{2}\sum_{k,\ell=1}^n \psi_{p_t(k)}(t)\psi_{p_t(\ell)}(t)\tau_{(k\ell)}(t)\, dt \\ &= \sum_{k=1}^n \frac{\psi_{p_t(k)}(t)}{\mu_{(k)}(t)}\, d\mu_{(k)}(t) - \frac{1}{2}\sum_{k,\ell=1}^n \psi_{p_t(k)}(t)\psi_{p_t(\ell)}(t)\tau_{(k\ell)}(t)\, dt \\ &\quad + \frac{1}{2}\sum_{k=1}^{n-1} \Big(\psi_{p_t(k+1)}(t) - \psi_{p_t(k)}(t)\Big)\, d\mathfrak{L}_{k,k+1}(t). \qquad (6.8.20)\end{aligned}$$

With ϕ defined as in (6.8.9), we have by (6.8.10),

$$\frac{\psi_{p_t(k)}(t)}{\mu_{(k)}(t)} = D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + \phi\big(\boldsymbol{\mu}_{(\cdot)}(t)\big), \tag{6.8.21}$$

then the first term on the right-hand side of (6.8.20) becomes

$$\begin{aligned}
\sum_{k=1}^n \frac{\psi_{p_t(k)}(t)}{\mu_{(k)}(t)}\, d\mu_{(k)}(t) &= \sum_{k=1}^n \Big[D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + \phi\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\Big]\, d\mu_{(k)}(t) \\
&= \sum_{k=1}^n D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\, d\mu_{(k)}(t) + \phi\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \sum_{k=1}^n d\mu_{(k)}(t) \\
&= \sum_{k=1}^n D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\, d\mu_{(k)}(t) + \phi\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\, d\left(\sum_{k=1}^n \mu_{(k)}(t)\right) \\
&= \sum_{k=1}^n D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\, d\mu_{(k)}(t),
\end{aligned} \tag{6.8.22}$$

since $\sum_{k=1}^n d\mu_{(k)}(t) = d\big(\sum_{k=1}^n \mu_{(k)}(t)\big) = 0$. With $\psi_{p_t(k)}(t)$ defined as in (6.8.10), the second term on the right-hand side of (6.8.20) is given a.s. by, for $t \in [0,T]$,

$$\begin{aligned}
&\sum_{k,\ell=1}^n \psi_{p_t(k)}(t)\psi_{p_t(\ell)}(t)\tau_{(k\ell)}(t) \\
&\quad= \sum_{k,\ell=1}^n \Big[D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + \phi\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\Big]\Big[D_\ell \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + \phi\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\Big]\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t) \\
&\quad= \sum_{k,\ell=1}^n D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\, D_\ell \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t) \\
&\qquad + 2\,\phi\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \sum_{k,\ell=1}^n D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t) \\
&\qquad + \phi^2\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \sum_{k,\ell=1}^n \mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t) \\
&\quad= \sum_{k,\ell=1}^n D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\, D_\ell \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t) \\
&\qquad + 2\,\phi\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \sum_{k=1}^n D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\left[\sum_{\ell=1}^n \mu_{(\ell)}(t)\tau_{(k\ell)}(t)\right] \\
&\qquad + \phi^2\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \sum_{k,\ell=1}^n \mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t).
\end{aligned}$$

Thus, (2.12.54) of Lemma 2.12.4 and (2.12.56), imply that $\boldsymbol{\mu}_{(\cdot)}(t)$ is in the null space of $\big\{\tau_{(k\ell)}(t)\big\}$, and we have a.s., for $t \in [0,T]$,

$$\sum_{k,\ell=1}^n \psi_{p_t(k)}(t)\psi_{p_t(\ell)}(t)\tau_{(k\ell)}(t) = \sum_{k,\ell=1}^n D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\, D_\ell \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t). \tag{6.8.23}$$

Hence, equations (6.8.20), (6.8.22) and (6.8.23), imply that a.s., for $t \in [0,T]$, we have

$$\begin{aligned}
&d \log\Big(Z_\psi(t)/Z_\mu(t)\Big) \\
&\quad= \sum_{k=1}^n D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\, d\mu_{(k)}(t) - \frac{1}{2}\sum_{k,\ell=1}^n D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\, D_\ell \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\, dt \\
&\qquad + \frac{1}{2}\sum_{k=1}^{n-1}\Big(\psi_{p_t(k+1)}(t) - \psi_{p_t(k)}(t)\Big)\, d\mathfrak{L}_{k,k+1}(t).
\end{aligned} \tag{6.8.24}$$

Thus, by subtracting (6.8.16) from (6.8.24), we obtain

$$\begin{aligned} d\log\Big(Z_\psi(t)/Z_\mu(t)\Big) - d\log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) &= -\frac{1}{2\,\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\sum_{k,\ell=1}^{n} D_{k\ell}\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\,dt \\ &\quad + \frac{1}{2}\sum_{k=1}^{n-1}\Big(\psi_{p_t(k+1)}(t) - \psi_{p_t(k)}(t)\Big)\,d\mathfrak{L}_{k,k+1}(t). \end{aligned} \tag{6.8.25}$$

Moreover, comparing expression (6.8.24) with (6.8.16),

$$\begin{aligned} &d\log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \\ &= \sum_{k=1}^{n} D_k\log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\,d\mu_{(k)}(t) - \frac{1}{2}\sum_{k,\ell=1}^{n} D_k\log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\,D_\ell\log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\,dt \\ &\quad + \frac{1}{2\,\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\sum_{k,\ell=1}^{n} D_{k\ell}\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\,dt, \end{aligned}$$

we obtain for $t \in [0,T]$, a.s.,

$$\begin{aligned} d\log\Big(Z_\psi(t)/Z_\mu(t)\Big) &= d\log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) - \frac{1}{2\,\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\sum_{k,\ell=1}^{n} D_{k\ell}\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\,dt \\ &\quad + \frac{1}{2}\sum_{k=1}^{n-1}\Big(\psi_{p_t(k+1)}(t) - \psi_{p_t(k)}(t)\Big)\,d\mathfrak{L}_{k,k+1}(t). \end{aligned} \tag{6.8.26}$$

This expression yields (6.8.8). Now, by recalling (5.2.2) of Definition 5.2.1, it is clear from (6.8.26) that the logarithmic relative return of the portfolio ψ with respect to the market has the form provided in equation (5.2.2), where the differential of the drift process, $d\Theta(t)$, is given by the last two terms on the right-hand side of (6.8.26). Since by definition $d\log\big(Z_\psi(t)/Z_\mu(t)\big) = d\log\mathbf{G}\big(\boldsymbol{\mu}(t)\big) + d\Theta(t)$, we deduce that the drift process Υ is given by

$$\begin{aligned} d\Upsilon(t) &= \frac{-1}{2\,\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\sum_{k,\ell=1}^{n} D_{k\ell}\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\,dt \\ &\quad + \frac{1}{2}\sum_{k=1}^{n-1}\Big(\psi_{p_t(k+1)}(t) - \psi_{p_t(k)}(t)\Big)\,d\mathfrak{L}_{k,k+1}(t), \end{aligned}$$

which completes the proof. ■

6.9 Rank-Dependent Portfolio Generating Functions

Definition 6.9.1 (Rank-Dependent Generating Functions). *Let U be an open neighbourhood in $\mathbb{R}^n$ ($U \subset \mathbb{R}^n$) of the open positive unit $(n-1)-$simplex Δ^{n-1}*

$$\Delta^{n-1} = \Big\{\boldsymbol{\mu}_{(\cdot)}(t) = \big(\mu_{(1)}(t),\ldots,\mu_{(n)}(t)\big) \in \mathbb{R}^n \;\Big|\; \mu_{(1)}(t) + \cdots + \mu_{(n)}(t) = 1,\ 0 < \mu_{(k)}(t) < 1,\ k = 1,\ldots,n\Big\},$$

*and let $\mathfrak{G} : U \to (0,\infty)$ be a positive twice continuously differentiable function defined on some open neighbourhood U of Δ^{n-1}. Then $\mathfrak{G}$ **generates** a portfolio ψ if there exists a continuous, measurable and adapted process of bounded variation $\Upsilon = \{\Upsilon(t), t \in [0,\infty)\}$, such that*

$$\log\Big(Z_\psi(t)/Z_\mu(t)\Big) = \log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + \Upsilon(t), \qquad t \in [0,T], \qquad a.s., \tag{6.9.1}$$

or such that we have the equivalent differential form

$$d\log\Big(Z_\psi(t)/Z_\mu(t)\Big) = d\log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + d\Upsilon(t), \qquad t \in [0,T], \qquad a.s. \tag{6.9.2}$$

The process Υ is called the ***drift process*** *corresponding to the* ***rank-dependent generating function*** $\mathfrak{G}$*. We say that ψ is the rank-dependent portfolio generated by the function $\mathfrak{G}$. Then $\mathfrak{G}$ is called the* ***rank-dependent generating function*** *of the portfolio ψ, and the portfolio ψ is said to be the* ***rank-dependent functionally generated portfolio*** *corresponding to the* ***rank-dependent portfolio generating function*** $\mathfrak{G}$*.*

The integral form of (6.9.2) for this rank-dependent functionally generated portfolio ψ, is given by

$$\log\left(\frac{Z_\psi(T)}{Z_\mu(T)}\right) - \log\left(\frac{Z_\psi(0)}{Z_\mu(0)}\right) = \log\left(\frac{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \int_0^T d\Upsilon(t), \qquad T \in [0,\infty), \tag{6.9.3}$$

or,

$$\log\left(\frac{Z_\psi(T)}{Z_\mu(T)}\right) = \log\left(\frac{Z_\psi(0)}{Z_\mu(0)}\right) + \log\left(\frac{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \int_0^T d\Upsilon(t), \qquad T \in [0,\infty), \tag{6.9.4}$$

which can be equivalently expressed as

$$\log\left(\frac{Z_\psi(T)}{Z_\psi(0)}\right) - \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) = \log\left(\frac{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \int_0^T d\Upsilon(t), \qquad T \in [0,\infty), \tag{6.9.5}$$

or,

$$\log\left(\frac{Z_\psi(T)}{Z_\psi(0)}\right) = \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log\left(\frac{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \int_0^T d\Upsilon(t), \qquad T \in [0,\infty). \tag{6.9.6}$$

Where $Z_\psi(0) = Z_\mu(0)$, the logarithmic relative return process of this rank-dependent functionally generated portfolio ψ, with respect to the market, is given by the so-called *"master formula"* for rank-dependent portfolios, for all $T \in [0,\infty)$,

$$\log\left(\frac{Z_\psi(T)}{Z_\mu(T)}\right) = \log\left(\frac{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \int_0^T d\Upsilon(t), \tag{6.9.7}$$

$$\log\left(\frac{Z_\psi(T)}{Z_\mu(T)}\right) = \log\left(\frac{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \Upsilon(T), \tag{6.9.8}$$

alternatively, for $T \in [0,\infty)$, the above can be expressed as

$$\Upsilon(T) = \int_0^T d\Upsilon(t) = \log\left(\frac{Z_\psi(T)}{Z_\mu(T)}\right) - \log\left(\frac{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) \tag{6.9.9}$$

$$= \log\left(\frac{Z_\psi(T)\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}{Z_\mu(T)\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}\right), \tag{6.9.10}$$

with

$$\begin{aligned}\Upsilon(T) &= \int_0^T \frac{-1}{2\,\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \sum_{k,\ell=1}^{n} D_{k\ell}\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\,dt \\ &\quad + \frac{1}{2}\int_0^T \sum_{k=1}^{n-1}\Big(\psi_{p_t(k+1)}(t) - \psi_{p_t(k)}(t)\Big)\,d\mathfrak{L}_{k,k+1}(t).\end{aligned} \tag{6.9.11}$$

The stochastic differential equation, (6.9.2), associated with a rank-dependent functionally generated portfolio, decomposes the logarithmic relative return of the rank-dependent functionally generated portfolio ψ, with respect to the benchmark market portfolio, into two components. The first component is the generating function component, or more precisely the logarithmic change or variation in the value of the rank-dependent generating function $\mathfrak{G}$. The second component is the drift process Υ. Suppose that there exist continuous, measurable and adapted processes of bounded variation $\mathscr{g} = \{\mathscr{g}(t), \mathcal{F}_t, t \in [0,\infty)\}$ and $\mathscr{L} = \{\mathscr{L}(t), \mathcal{F}_t, t \in [0,\infty)\}$, then the drift process Υ provided in (6.8.7) is given by

$$d\Upsilon(t) = \mathscr{g}(t)\,dt + d\mathscr{L}(t), \tag{6.9.12}$$

for all $t \in [0,T]$, a.s., where, a.s., for $t \in [0,T]$,

$$g(t) \quad = \quad \frac{-1}{2\,\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \sum_{k,\ell=1}^{n} D_{k\ell}\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t), \tag{6.9.13}$$

and for $t \in [0,T]$,

$$d\mathscr{L}(t) \quad = \quad \frac{1}{2}\sum_{k=1}^{n-1}\Big(\psi_{p_t(k+1)}(t) - \psi_{p_t(k)}(t)\Big)\,d\mathfrak{L}_{k,k+1}(t). \tag{6.9.14}$$

Consequently, the drift process for a rank-dependent functionally generated portfolio (generated by the rank-dependent generating function $\mathfrak{G}$) has two distinct parts. The first term on the right-hand side of (6.8.7) or (6.9.12), given above by (6.9.13), is referred to as the *smooth component* of the drift process, and is similar to the drift function in (5.2.15) of Theorem 5.2.2. The second term on the right-hand side of (6.8.7) or (6.9.12), given above by (6.9.14), is the component of the drift process involving local times for changes in rank among the market weights, and is referred to as the *local time component* of the drift process. Therefore, (6.9.2) becomes

$$d\log\Big(Z_\psi(t)/Z_\mu(t)\Big) \quad = \quad d\log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + g(t)\,dt + d\mathscr{L}(t), \qquad t \in [0,T], \qquad \text{a.s.}, \tag{6.9.15}$$

and since $\Upsilon(\cdot) = \int_0^{\cdot} d\Upsilon(t) = \int_0^{\cdot} g(t)\,dt + \int_0^{\cdot} d\mathscr{L}(t) = \int_0^{\cdot} g(t)\,dt + \mathscr{L}(\cdot)$ or $\Upsilon'(t) = \frac{d\Upsilon(t)}{dt} = g(t) + \frac{d\mathscr{L}(t)}{dt}$, (6.9.7) becomes for all $t \in [0,\infty)$,

$$\log\left(\frac{Z_\psi(T)}{Z_\mu(T)}\right) \quad = \quad \log\left(\frac{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \int_0^T g(t)\,dt + \int_0^T d\mathscr{L}(t), \tag{6.9.16}$$

$$\log\left(\frac{Z_\psi(T)}{Z_\mu(T)}\right) \quad = \quad \log\left(\frac{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \int_0^T g(t)\,dt + \mathscr{L}(T). \tag{6.9.17}$$

6.10 A Generalisation of Rank-Based Functionally Generated Portfolios

6.10.1 Equity Portfolios Generated by Functions of the Ranked Market Weight Processes: A Generalisation

We now generalise Theorem 6.8.1 (where the assumption that $X_1,\ldots,X_n$ be pathwise mutually nondegenerate was required), to the case where three or more stock capitalisations may be equal at a given time $t \in [0,T]$, i.e., we shall consider the case in which there exists $\Omega' \subset \Omega$, $\mathbb{P}(\Omega') = 1$, such that for $\omega \in \Omega'$, the set $\big\{t \in [0,T] \;\big|\; X_i(t,\omega) = X_j(t,\omega) = X_k(t,\omega)\big\}$ is finite for all $i < j < k$. We shall apply the aforementioned theory on the representation of the rank processes and the representation of the ranked market weight processes in the case where the continuous semimartingales X_i in that theory are precisely the stock values in a market $\mathcal{M}$. Note that the incidence $X_i(t) = X_j(t)$ corresponds to the incidence $\mu_i(t) = \mu_j(t)$ and that the rankings of μ_i correspond to the rankings of X_i. Let us now fix $u \in \mathcal{U}$. For $k,\ell = 1,2,\ldots,n$, in terms of $u \in \mathcal{U}$, we define the *relative rank covariance processes* $\tau_{(k\ell)} = \big\{\tau_{(k\ell)}(t), t \in [0,\infty)\big\}$, i.e., the ranked covariance process relative to the market, by

$$\tau_{(k\ell)}(t) \quad \triangleq \quad \tau_{u_t(k)u_t(\ell)}(t), \qquad t \in [0,T]. \tag{6.10.1}$$

Theorem 6.10.1 ([**Banner & Ghomrasni (2008)**]). *Let $\mathcal{M}$ be a market of stocks $X_1,\ldots,X_n$ satisfying condition (i) of the pathwise mutual nondegeneracy conditions provided in Definition* 6.4.1 *(i.e., for all $i \neq j$, the set $\big\{t \in [0,T] \;\big|\; X_i(t) = X_j(t)\big\}$ has Lebesgue measure zero, almost surely). Suppose that u is an element of the set $\mathcal{U}$ defined in* (6.6.57). *Let U be an open neighbourhood in $\mathbb{R}^n$ ($U \subset \mathbb{R}^n$) of the open positive unit $(n-1)-$simplex Δ^{n-1} and let $\mathbf{G} : U \to (0,\infty)$ be a positive twice continuously differentiable function defined*

on some open neighbourhood U of Δ^{n-1}. Suppose that there exists a positive twice continuously differentiable function $\mathfrak{G} : U \to (0,\infty)$ defined on some open neighbourhood U of Δ^{n-1} such that for $\mathbf{x} = (x_1,\ldots,x_n) \in U$ and $\mathbf{x}_{(\cdot)} = \big(x_{(1)},\ldots,x_{(n)}\big) \in U$,

$$\mathbf{G}\big(x_1,\ldots,x_n\big) \;=\; \mathfrak{G}\big(x_{(1)},\ldots,x_{(n)}\big), \tag{6.10.2}$$

and for all $k = 1,2,\ldots,n$, $x_{(k)} D_k \log \mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big)$ is bounded for $\mathbf{x}, \mathbf{x}_{(\cdot)} \in \Delta^{n-1}$. Alternatively, such that for $\boldsymbol{\mu}(t) = \big(\mu_1(t),\ldots,\mu_n(t)\big) \in U$ and $\boldsymbol{\mu}_{(\cdot)}(t) = \big(\mu_{(1)}(t),\ldots,\mu_{(n)}(t)\big) \in U$,

$$\mathbf{G}\big(\mu_1(t),\ldots,\mu_n(t)\big) \;=\; \mathfrak{G}\big(\mu_{(1)}(t),\ldots,\mu_{(n)}(t)\big), \tag{6.10.3}$$

*and for all $k = 1,2,\ldots,n$, $\mu_{(k)}(t) D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)$ is bounded for $\boldsymbol{\mu}(t), \boldsymbol{\mu}_{(\cdot)}(t) \in \Delta^{n-1}$. Then for all $t \in [0,T]$, a.s., and for $k = 1,2,\ldots,n$, the rank-dependent generating function $\mathfrak{G}$ generates the (rank-based functionally generated) portfolio ψ with **weights***

$$\psi_{u_t(k)}(t) \;=\; \left(D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + 1 - \sum_{\ell=1}^{n} \mu_{(\ell)}(t)\, D_\ell \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \right) \mu_{(k)}(t), \tag{6.10.4}$$

*and with **drift process** Υ, that satisfies for $t \in [0,T]$, a.s.,*

$$\begin{aligned} d\Upsilon(t) \;&=\; \frac{-1}{2\,\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \sum_{k,\ell=1}^{n} D_{k\ell}\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\,dt \\ &\quad - \sum_{k=1}^{n} \psi_{u_t(k)}(t)\, \frac{1}{N_t(k)} \sum_{\ell=k+1}^{n} d\Lambda_{\log \mu_{(k)} - \log \mu_{(\ell)}}(t) \\ &\quad + \sum_{k=1}^{n} \psi_{u_t(k)}(t)\, \frac{1}{N_t(k)} \sum_{\ell=1}^{k-1} d\Lambda_{\log \mu_{(\ell)} - \log \mu_{(k)}}(t) \end{aligned} \tag{6.10.5}$$

$$\begin{aligned} &=\; \frac{-1}{2\,\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \sum_{k,\ell=1}^{n} D_{k\ell}\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\,dt \\ &\quad - \sum_{k=1}^{n} \psi_{u_t(k)}(t)\, \frac{1}{N_t(k)} \sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) \\ &\quad + \sum_{k=1}^{n} \psi_{u_t(k)}(t)\, \frac{1}{N_t(k)} \sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t). \end{aligned} \tag{6.10.6}$$

Let μ be the market portfolio and ψ be the rank-based functionally generated portfolio, and let Z_μ and Z_ψ be their portfolio value processes, respectively. Then, a.s., for $t \in [0,T]$,

$$\begin{aligned} d\log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \;&=\; d\log\Big(Z_\psi(t)/Z_\mu(t)\Big) + \frac{1}{2\,\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \sum_{k,\ell=1}^{n} D_{k\ell}\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\,dt \\ &\quad + \sum_{k=1}^{n} \psi_{u_t(k)}(t)\, \frac{1}{N_t(k)} \sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) \\ &\quad - \sum_{k=1}^{n} \psi_{u_t(k)}(t)\, \frac{1}{N_t(k)} \sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t) \end{aligned} \tag{6.10.7}$$

$$\begin{aligned} &=\; d\log\Big(Z_\psi(t)/Z_\mu(t)\Big) + \frac{1}{2\,\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \sum_{k,\ell=1}^{n} D_{k\ell}\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\,dt \\ &\quad + \sum_{k=1}^{n} \psi_{u_t(k)}(t)\, \frac{1}{N_t(k)} \left[\sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) - \sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t) \right]. \end{aligned} \tag{6.10.8}$$

Proof. To verify that $\boldsymbol{\psi}$ defined by (6.10.4) is indeed a portfolio and that the drift process Υ defined by (6.10.6) is of bounded variation, follow the same argument employed in the proof of Theorem 6.8.1. As in the proof of

Theorem 6.8.1, we must show that the portfolio ψ defined by (6.10.4) and the drift process $\Upsilon = \{\Upsilon(t), t \in [0,T]\}$ defined by (6.10.6) satisfy (5.2.2) or (6.9.2). To accomplish this, we shall analyse both the generating function term $\log \mathbf{G}\big(\boldsymbol{\mu}(t)\big) = \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)$ in (5.2.2) and the relative return process $\log\big(Z_\psi(t)/Z_\mu(t)\big)$, and show that the difference of these two terms satisfies (6.10.6). We shall need the results (6.8.13) and (6.8.14), that were derived in the proof of Theorem 6.8.1. First, consider the generating function component of the relative return, $\log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)$. Then, the results (6.8.14) and (6.8.13) in conjunction with an application of Itô's formula to $\log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)$, yields (6.8.16) a.s., for $t \in [0,T]$,

$$\begin{aligned} & d\log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \\ &= \sum_{k=1}^n D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\, d\mu_{(k)}(t) + \frac{1}{2\,\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \sum_{k,\ell=1}^n D_{k\ell}\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\, dt \\ &\quad - \frac{1}{2}\sum_{k,\ell=1}^n D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\, D_\ell \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\, dt. \end{aligned} \tag{6.10.9}$$

Now let us consider the relative return process $\log\big(Z_\psi(t)/Z_\mu(t)\big)$. By setting $\boldsymbol{\pi} := \boldsymbol{\psi}$ in Proposition 2.12.8 we have a.s., for $t \in [0,T]$,

$$\begin{aligned} d\log\Big(Z_\psi(t)/Z_\mu(t)\Big) &= \sum_{i=1}^n \psi_i(t)\, d\log\mu_i(t) + \gamma_\psi^*(t)\, dt \\ &= \sum_{i=1}^n \Bigg[\sum_{k=1}^n \mathbb{1}_{\{u_t(k)=i\}}\psi_{u_t(k)}(t)\Bigg]\, d\log\mu_i(t) + \gamma_\psi^*(t)\, dt \\ &= \sum_{i=1}^n\sum_{k=1}^n \mathbb{1}_{\{u_t(k)=i\}}\psi_{u_t(k)}(t)\, d\log\mu_i(t) + \gamma_\psi^*(t)\, dt \\ &= \sum_{k=1}^n \psi_{u_t(k)}(t)\sum_{i=1}^n \mathbb{1}_{\{u_t(k)=i\}}\, d\log\mu_i(t) + \gamma_\psi^*(t)\, dt, \end{aligned} \tag{6.10.10}$$

since $\psi_i(t) = \sum_{k=1}^n \mathbb{1}_{\{u_t(k)=i\}}\psi_{u_t(k)}(t)$. Now, the rest of the proof is identical to that of the proof of Theorem 6.8.1, except that the appeal to Corollary 6.7.2 should be replaced by a similar appeal to Corollary 6.7.3. We note that $X_1, \ldots, X_n$ satisfy the absolute continuity condition by virtue of Lemma 6.5.3. Since the stocks in the market $\mathcal{M}$ satisfy condition (i) of the pathwise mutual nondegeneracy conditions (i.e., for all $i \neq j$, the set $\big\{t \in [0,T] \;\big|\; X_i(t) = X_j(t)\big\}$ has Lebesgue measure zero, almost surely), we have by (6.7.8) of Corollary 6.7.3,

$$\begin{aligned} \sum_{k=1}^n \psi_{u_t(k)}(t)\, d\log\mu_{(k)}(t) &= \sum_{k=1}^n \psi_{u_t(k)}(t)\sum_{i=1}^n \mathbb{1}_{\{u_t(k)=i\}}\, d\log\mu_i(t) \\ &\quad + \sum_{k=1}^n \psi_{u_t(k)}(t)\sum_{\ell=k+1}^n \frac{1}{N_t(k)}\, d\mathfrak{L}_{k,\ell}(t) - \sum_{k=1}^n \psi_{u_t(k)}(t)\sum_{\ell=1}^{k-1}\frac{1}{N_t(k)}\, d\mathfrak{L}_{\ell,k}(t). \end{aligned}$$

The first term on the right-hand side of the preceding expression is precisely the first term on the right-hand side of (6.10.10). Therefore, by considering the first term on the right-hand side of the above expression, we get

$$\begin{aligned} & \sum_{k=1}^n \psi_{u_t(k)}(t)\sum_{i=1}^n \mathbb{1}_{\{u_t(k)=i\}}\, d\log\mu_i(t) \\ &= \sum_{k=1}^n \psi_{u_t(k)}(t)\, d\log\mu_{(k)}(t) - \sum_{k=1}^n \psi_{u_t(k)}(t)\sum_{\ell=k+1}^n \frac{1}{N_t(k)}\, d\mathfrak{L}_{k,\ell}(t) + \sum_{k=1}^n \psi_{u_t(k)}(t)\sum_{\ell=1}^{k-1}\frac{1}{N_t(k)}\, d\mathfrak{L}_{\ell,k}(t) \\ &= \sum_{k=1}^n \psi_{u_t(k)}(t)\, d\log\mu_{(k)}(t) - \sum_{k=1}^n \psi_{u_t(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=k+1}^n d\mathfrak{L}_{k,\ell}(t) + \sum_{k=1}^n \psi_{u_t(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t). \end{aligned} \tag{6.10.11}$$

Now, from Lemma 2.4.5 we have the numéraire invariance property (2.4.26),

$$\gamma_\psi^*(t) = \frac{1}{2}\left(\sum_{k=1}^n \psi_{u_t(k)}(t)\tau_{(kk)}(t) - \sum_{k,\ell=1}^n \psi_{u_t(k)}(t)\tau_{(k\ell)}(t)\psi_{u_t(\ell)}(t)\right). \tag{6.10.12}$$

Substituting (6.10.11) and (6.10.12) into (6.10.10) yields

$$\begin{aligned}
&d\log\Big(Z_\psi(t)/Z_\mu(t)\Big)\\
&= \sum_{k=1}^n \psi_{u_t(k)}(t)\,d\log\mu_{(k)}(t) - \sum_{k=1}^n \psi_{u_t(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=k+1}^n d\mathfrak{L}_{k,\ell}(t) + \sum_{k=1}^n \psi_{u_t(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t)\\
&\quad + \frac{1}{2}\sum_{k=1}^n \psi_{u_t(k)}(t)\tau_{(kk)}(t)\,dt - \frac{1}{2}\sum_{k,\ell=1}^n \psi_{u_t(k)}(t)\psi_{u_t(\ell)}(t)\tau_{(k\ell)}(t)\,dt.
\end{aligned}$$

From (2.12.36), we similarly have by applying Itô's formula to $\mu_{(k)}(t) = \exp\big(\log\mu_{(k)}(t)\big)$,

$$d\log\mu_{(k)}(t) \;=\; \frac{d\mu_{(k)}(t)}{\mu_{(k)}(t)} - \frac{1}{2}\,\tau_{(kk)}(t)\,dt.$$

Hence, the above expression implies

$$\begin{aligned}
&d\log\Big(Z_\psi(t)/Z_\mu(t)\Big)\\
&= \sum_{k=1}^n \psi_{u_t(k)}(t)\left[\frac{d\mu_{(k)}(t)}{\mu_{(k)}(t)} - \frac{1}{2}\,\tau_{(kk)}(t)\,dt\right] - \sum_{k=1}^n \psi_{u_t(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=k+1}^n d\mathfrak{L}_{k,\ell}(t) + \sum_{k=1}^n \psi_{u_t(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t)\\
&\quad + \frac{1}{2}\sum_{k=1}^n \psi_{u_t(k)}(t)\tau_{(kk)}(t)\,dt - \frac{1}{2}\sum_{k,\ell=1}^n \psi_{u_t(k)}(t)\psi_{u_t(\ell)}(t)\tau_{(k\ell)}(t)\,dt\\
&= \sum_{k=1}^n \frac{\psi_{u_t(k)}(t)}{\mu_{(k)}(t)}\,d\mu_{(k)}(t) - \frac{1}{2}\sum_{k,\ell=1}^n \psi_{u_t(k)}(t)\psi_{u_t(\ell)}(t)\tau_{(k\ell)}(t)\,dt\\
&\quad - \sum_{k=1}^n \psi_{u_t(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=k+1}^n d\mathfrak{L}_{k,\ell}(t) + \sum_{k=1}^n \psi_{u_t(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t).
\end{aligned} \tag{6.10.13}$$

Using (6.8.9) and (6.8.10), defined in the proof of Theorem 6.8.1, we can write the first term on the right-hand side of (6.10.13) as (6.8.22),

$$\sum_{k=1}^n \frac{\psi_{u_t(k)}(t)}{\mu_{(k)}(t)}\,d\mu_{(k)}(t) \;=\; \sum_{k=1}^n D_k\log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\,d\mu_{(k)}(t), \tag{6.10.14}$$

and we can write the second term on the right-hand side of (6.10.13) as (6.8.23),

$$\sum_{k,\ell=1}^n \psi_{u_t(k)}(t)\psi_{u_t(\ell)}(t)\tau_{(k\ell)}(t) \;=\; \sum_{k,\ell=1}^n D_k\log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\,D_\ell\log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t). \tag{6.10.15}$$

Hence, equations (6.10.13), (6.10.14) and (6.10.15), imply that a.s., for $t\in[0,T]$, we have

$$\begin{aligned}
&d\log\Big(Z_\psi(t)/Z_\mu(t)\Big)\\
&= \sum_{k=1}^n D_k\log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\,d\mu_{(k)}(t) - \frac{1}{2}\sum_{k,\ell=1}^n D_k\log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\,D_\ell\log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\,dt\\
&\quad - \sum_{k=1}^n \psi_{u_t(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=k+1}^n d\mathfrak{L}_{k,\ell}(t) + \sum_{k=1}^n \psi_{u_t(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t).
\end{aligned} \tag{6.10.16}$$

Thus, by subtracting (6.10.9) from (6.10.16), we obtain

$$\begin{aligned} d\log\Big(Z_\psi(t)/Z_\mu(t)\Big) - d\log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) &= -\frac{1}{2\,\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\sum_{k,\ell=1}^{n} D_{k\ell}\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\,dt \\ &\quad -\sum_{k=1}^{n}\psi_{u_t(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) \\ &\quad +\sum_{k=1}^{n}\psi_{u_t(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t). \end{aligned} \tag{6.10.17}$$

Moreover, comparing expression (6.10.16) with (6.10.9),

$$\begin{aligned} &d\log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \\ &= \sum_{k=1}^{n} D_k\log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\,d\mu_{(k)}(t) - \frac{1}{2}\sum_{k,\ell=1}^{n} D_k\log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\,D_\ell\log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\,dt \\ &\quad + \frac{1}{2\,\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\sum_{k,\ell=1}^{n} D_{k\ell}\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\,dt, \end{aligned}$$

we obtain for $t\in[0,T]$, a.s.,

$$\begin{aligned} d\log\Big(Z_\psi(t)/Z_\mu(t)\Big) &= d\log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) - \frac{1}{2\,\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\sum_{k,\ell=1}^{n} D_{k\ell}\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\,dt \\ &\quad -\sum_{k=1}^{n}\psi_{u_t(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) \\ &\quad +\sum_{k=1}^{n}\psi_{u_t(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t) \end{aligned} \tag{6.10.18}$$

$$\begin{aligned} &= d\log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) - \frac{1}{2\,\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\sum_{k,\ell=1}^{n} D_{k\ell}\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\,dt \\ &\quad -\sum_{k=1}^{n}\psi_{u_t(k)}(t)\,\frac{1}{N_t(k)}\Bigg[\sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) - \sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t)\Bigg]. \end{aligned} \tag{6.10.19}$$

This expression yields (6.10.7). Now, by recalling (5.2.2) of Definition 5.2.1, it is clear from (6.10.18) that the logarithmic relative return of the portfolio ψ with respect to the market has the form provided in equation (5.2.2), where the differential of the drift process, $d\Theta(t)$, is given by the last three terms on the right-hand side of (6.10.18). Since by definition $d\log\big(Z_\psi(t)/Z_\mu(t)\big) = d\log\mathbf{G}\big(\boldsymbol{\mu}(t)\big) + d\Theta(t)$, we deduce that the drift process Υ is given by

$$\begin{aligned} d\Upsilon(t) &= \frac{-1}{2\,\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\sum_{k,\ell=1}^{n} D_{k\ell}\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\,dt \\ &\quad -\sum_{k=1}^{n}\psi_{u_t(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) \\ &\quad +\sum_{k=1}^{n}\psi_{u_t(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t), \end{aligned}$$

which completes the proof. Note, that it is not difficult to show that condition (i) of the pathwise mutual nondegeneracy conditions is automatically satisfied if the market is nondegenerate. ■

In this more general setting, suppose that there exist continuous, measurable and adapted processes of bounded variation $\mathscr{g} = \{\mathscr{g}(t), \mathcal{F}_t, t\in[0,\infty)\}$ and $\mathscr{L} = \{\mathscr{L}(t), \mathcal{F}_t, t\in[0,\infty)\}$, then the drift process Υ provided in (6.10.6)

is given by

$$d\Upsilon(t) \quad = \quad g(t)\,dt + d\mathscr{L}(t), \tag{6.10.20}$$

for all $t \in [0,T]$, a.s., where, a.s., for $t \in [0,T]$,

$$g(t) \quad = \quad \frac{-1}{2\,\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \sum_{k,\ell=1}^{n} D_{k\ell}\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t), \tag{6.10.21}$$

as in (6.9.13), and for $t \in [0,T]$,

$$d\mathscr{L}(t) \quad = \quad -\sum_{k=1}^{n} \psi_{u_t(k)}(t)\,\frac{1}{N_t(k)} \sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) + \sum_{k=1}^{n} \psi_{u_t(k)}(t)\,\frac{1}{N_t(k)} \sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t). \tag{6.10.22}$$

Thus, more generally, the drift process for a rank-dependent functionally generated portfolio (generated by the rank-dependent generating function $\mathfrak{G}$) has two distinct parts. The first term on the right-hand side of (6.10.6) or (6.10.20), given above by (6.10.21), is referred to as the *smooth component* of the drift process, and is similar to the drift function in (5.2.15) of Theorem 5.2.2 and is the same smooth component, (6.9.13), encountered in the case where it is assumed that triple points do not exist. The second and third terms on the right-hand side of (6.10.6), given above by (6.10.22), together make up the *local time component* of the drift process in the more general setting, i.e., together they represent the component of the drift process involving local times for changes in rank among the market weights, in the general case where triple points may exist. With the results of this section, we can examine the behaviour and performance of portfolios that depend on the capital distribution of the equity market.

6.11 A General Theorem for Portfolio Generating Functions

6.11.1 A General Theorem for Time-Independent Portfolio Generating Functions: The Time-Independent Case

In this section, the main result of Pamen (2011), i.e., a general theorem for portfolio generating functions, is presented for time-independent functions.

Theorem 6.11.1 ([**Pamen (2011)**]). *Let $\mathcal{M}$ be a market of stocks $X_1, \ldots, X_n$ that are* ***pathwise mutually nondegenerate****, and let $\boldsymbol{p}_t$ be the random permutation defined by* (6.7.1) *and* (6.7.2)*. Let U be an open neighbourhood in $\mathbb{R}^n$ ($U \subset \mathbb{R}^n$) of the open positive unit $(n-1)-$simplex Δ^{n-1} and let $\mathbf{G} : U \to (0,\infty)$ be a positive twice continuously differentiable function defined on some open neighbourhood U of Δ^{n-1}. Assume that $\mu_{\alpha(k)}(t)$ is a reversible semimartingale. Suppose that there exists a positive C^1 continuous function $\mathfrak{G}$ defined on some open neighbourhood U of Δ^{n-1} such that for $\mathbf{x} = (x_1, \ldots, x_n) \in U$ and $\mathbf{x}_{(\cdot)} = \big(x_{(1)}, \ldots, x_{(n)}\big) \in U$,*

$$\mathbf{G}\big(x_1, \ldots, x_n\big) \quad = \quad \mathfrak{G}\big(x_{(1)}, \ldots, x_{(n)}\big), \tag{6.11.1}$$

and for all $k = 1, 2, \ldots, n$, $x_{(k)} D_k \log \mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big)$ is bounded for $\mathbf{x}, \mathbf{x}_{(\cdot)} \in \Delta^{n-1}$. Alternatively, such that for $\boldsymbol{\mu}(t) = \big(\mu_1(t), \ldots, \mu_n(t)\big) \in U$ and $\boldsymbol{\mu}_{(\cdot)}(t) = \big(\mu_{(1)}(t), \ldots, \mu_{(n)}(t)\big) \in U$,

$$\mathbf{G}\big(\mu_1(t), \ldots, \mu_n(t)\big) \quad = \quad \mathfrak{G}\big(\mu_{(1)}(t), \ldots, \mu_{(n)}(t)\big), \tag{6.11.2}$$

and for all $k = 1, 2, \ldots, n$, $\mu_{(k)}(t) D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)$ is bounded for $\boldsymbol{\mu}(t), \boldsymbol{\mu}_{(\cdot)}(t) \in \Delta^{n-1}$. Then for all $t \in [0,T]$, a.s., and for $k = 1, 2, \ldots, n$, the generating function $\mathfrak{G}$ generates the (functionally generated) portfolio ψ with ***weights***

$$\psi_{\alpha(k)}(t) \quad = \quad \left(D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{\alpha(\cdot)}(t)\big) + 1 - \sum_{\ell=1}^{n} \mu_{\alpha(\ell)}(t)\, D_\ell \log \mathfrak{G}\big(\boldsymbol{\mu}_{\alpha(\cdot)}(t)\big) \right) \mu_{\alpha(k)}(t), \tag{6.11.3}$$

and with ***drift process*** Υ*, that satisfies for* $t \in [0,T]$*, a.s.,*

$$d\Upsilon(t) \;=\; \frac{-1}{2\,\mathfrak{G}\big(\boldsymbol{\mu}_{\alpha(\cdot)}(t)\big)} \sum_{k=1}^{n} \int_{\mathbb{R}} D_k \mathfrak{G}\big(\boldsymbol{\mu}_{\alpha(\cdot)}(t)\big)\, d_x \Lambda_{\mu_{\alpha(k)}}(x,t), \tag{6.11.4}$$

where $\psi_{\alpha(k)}(t)$ *is either equal to* $\psi_k(t)$ *or* $\psi_{(k)}(t)$*. Let* μ *be the market portfolio and* ψ *be the functionally generated portfolio, and let* Z_μ *and* Z_ψ *be their portfolio value processes, respectively. Then, a.s., for* $t \in [0,T]$*,*

$$d\log \mathfrak{G}\big(\boldsymbol{\mu}_{\alpha(\cdot)}(t)\big) \;=\; d\log\Big(Z_\psi(t)/Z_\mu(t)\Big) + \frac{1}{2\,\mathfrak{G}\big(\boldsymbol{\mu}_{\alpha(\cdot)}(t)\big)} \sum_{k=1}^{n} \int_{\mathbb{R}} D_k \mathfrak{G}\big(\boldsymbol{\mu}_{\alpha(\cdot)}(t)\big)\, d_x \Lambda_{\mu_{\alpha(k)}}(x,t). \tag{6.11.5}$$

Proof. Refer to the main result provided in Pamen (2011, Proof of Theorem 4.1), which is similar to the proof of Theorem 6.8.1 and the proof of Theorem 6.10.1. Notice that when $\mu_{\alpha(k)}(t) \equiv \mu_k(t)$, the proof is similar to the proof of Theorem 5.2.2. ■

6.11.2 A General Theorem for Time-Dependent Portfolio Generating Functions: The Time-Dependent Case

Theorem 6.11.1 can be extended to time-dependent portfolio generating functions. Let $\mathbf{G}$ be a $C^{1,1}$ function, then we have the following theorem.

Theorem 6.11.2 ([**Pamen (2011)**]). *Let* $\mathcal{M}$ *be a market of stocks* $X_1, \ldots, X_n$ *that are* ***pathwise mutually nondegenerate****, and let* $\boldsymbol{p}_t$ *be the random permutation defined by* (6.7.1) *and* (6.7.2)*. Let* U *be an open neighbourhood in* $\mathbb{R}^n$ $(U \subset \mathbb{R}^n)$ *of the open positive unit* $(n-1)-$*simplex* Δ^{n-1} *and let* $\mathbf{G} : U \times [0,T] \to (0,\infty)$ *be a positive continuous* $C^{1,1}$ *real-valued function defined on some open neighbourhood* U *of* Δ^{n-1}*, i.e. defined on* $U \times [0,T]$ *or* $\Delta^{n-1} \times [0,T]$*. Assume that* $\mu_{\alpha(k)}(t)$ *is a reversible semimartingale. Suppose that there exists a positive* $C^{1,1}$ *continuous function* $\mathfrak{G}$ *defined on some open neighbourhood* U *of* Δ^{n-1} *such that for* $\mathbf{x} = (x_1, \ldots, x_n) \in U$ *and* $\mathbf{x}_{(\cdot)} = \big(x_{(1)}, \ldots, x_{(n)}\big) \in U$*,*

$$\mathbf{G}\big(x_1, \ldots, x_n, t\big) \;=\; \mathfrak{G}\big(x_{(1)}, \ldots, x_{(n)}, t\big), \tag{6.11.6}$$

and for all $k = 1, 2, \ldots, n$*,* $x_{(k)} D_k \log \mathfrak{G}\big(\mathbf{x}_{(\cdot)}, t\big)$ *is bounded for* $\mathbf{x}, \mathbf{x}_{(\cdot)} \in \Delta^{n-1} \times [0,T]$*. Alternatively, such that for* $\boldsymbol{\mu}(t) = \big(\mu_1(t), \ldots, \mu_n(t)\big) \in U$ *and* $\boldsymbol{\mu}_{(\cdot)}(t) = \big(\mu_{(1)}(t), \ldots, \mu_{(n)}(t)\big) \in U$*,*

$$\mathbf{G}\big(\mu_1(t), \ldots, \mu_n(t), t\big) \;=\; \mathfrak{G}\big(\mu_{(1)}(t), \ldots, \mu_{(n)}(t), t\big), \tag{6.11.7}$$

and for all $k = 1, 2, \ldots, n$*,* $\mu_{(k)}(t) D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t), t\big)$ *is bounded for* $\boldsymbol{\mu}(t), \boldsymbol{\mu}_{(\cdot)}(t) \in \Delta^{n-1} \times [0,T]$*. Then for all* $t \in [0,T]$*, a.s., and for* $k = 1, 2, \ldots, n$*, the generating function* $\mathfrak{G}$ *generates the (functionally generated) portfolio* ψ *with* ***weights***

$$\psi_{\alpha(k)}(t) \;=\; \left(D_k \log \mathfrak{G}\big(\boldsymbol{\mu}_{\alpha(\cdot)}(t), t\big) + 1 - \sum_{\ell=1}^{n} \mu_{\alpha(\ell)}(t)\, D_\ell \log \mathfrak{G}\big(\boldsymbol{\mu}_{\alpha(\cdot)}(t), t\big) \right) \mu_{\alpha(k)}(t), \tag{6.11.8}$$

and with ***drift process*** Υ*, that satisfies for* $t \in [0,T]$*, a.s.,*

$$d\Upsilon(t) \;=\; \frac{-1}{2\,\mathfrak{G}\big(\boldsymbol{\mu}_{\alpha(\cdot)}(t), t\big)} \sum_{k=1}^{n} \int_0^T \int_{\mathbb{R}} D_k \mathfrak{G}\big(\boldsymbol{\mu}_{\alpha(\cdot)}(t), t\big)\, d_x \Lambda_{\mu_{\alpha(k)}}(x,t) - D_t \log \mathfrak{G}\big(\boldsymbol{\mu}_{\alpha(\cdot)}(t), t\big)\, dt, \tag{6.11.9}$$

where D_t *represents the first partial derivative with respect to the last variable, the time variable, and* $\psi_{\alpha(k)}(t)$ *is either equal to* $\psi_k(t)$ *or* $\psi_{(k)}(t)$*. Let* μ *be the market portfolio and* ψ *be the functionally generated portfolio, and let* Z_μ *and* Z_ψ *be their portfolio value processes, respectively. Then, a.s., for* $t \in [0,T]$*,*

$$\begin{aligned} d\log \mathfrak{G}\big(\boldsymbol{\mu}_{\alpha(\cdot)}(t), t\big) \;&=\; d\log\Big(Z_\psi(t)/Z_\mu(t)\Big) + \frac{1}{2\,\mathfrak{G}\big(\boldsymbol{\mu}_{\alpha(\cdot)}(t), t\big)} \sum_{k=1}^{n} \int_0^T \int_{\mathbb{R}} D_k \mathfrak{G}\big(\boldsymbol{\mu}_{\alpha(\cdot)}(t), t\big)\, d_x \Lambda_{\mu_{\alpha(k)}}(x,t) \\ &\quad + D_t \log \mathfrak{G}\big(\boldsymbol{\mu}_{\alpha(\cdot)}(t), t\big)\, dt. \end{aligned} \tag{6.11.10}$$

Proof. See Pamen (2011, Proof of Theorem 4.6) for the proof. ■

6.12 Examples of Rank-Based Portfolio Generating Functions and their Rank-Dependent Functionally Generated Portfolios

In this section, we present some examples of rank-dependent portfolio generating functions, as well as the rank-based portfolios they generate, along with their associated drift processes comprising the smooth component and the local time component. We shall present five examples of rank-dependent functionally generated portfolios: the biggest stock, the large-stock index, the small-stock index, the $\mathbf{D}_p$-weighted large-stock index, and a portfolio with fixed weight ratios. The latter example introduces the concept of the distributional component of the relative portfolio return which measures the effect that changes in the distribution of capital have on portfolio performance.

6.12.1 The Biggest Stock

Consider the following rank-dependent generating function

$$\mathfrak{G}(\mathbf{x}_{(\cdot)}) \equiv \mathbb{G}(\mathbf{x}_{(\cdot)}) = \mathbb{G}(x_{(1)}, \ldots, x_{(n)}) \triangleq x_{(1)}, \tag{6.12.1}$$

for all $\mathbf{x}$ and $\mathbf{x}_{(\cdot)}$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\mathbf{x}, \mathbf{x}_{(\cdot)} \in \Delta^{n-1}$. Then, the corresponding market process is given by

$$\mathfrak{G}(\boldsymbol{\mu}_{(\cdot)}(t)) \equiv \mathbb{G}(\boldsymbol{\mu}_{(\cdot)}(t)) = \mathbb{G}(\mu_{(1)}(t), \ldots \mu_{(n)}(t)) \triangleq \mu_{(1)}(t), \qquad t \in [0, T]. \tag{6.12.2}$$

The above generating function generates the portfolio that invests in the biggest stock in the market only. The value of the generating function above represents the relative capitalisation of the biggest stock in the market. Subsequently, we deduce the following for all $k = 1, 2, \ldots, n$,

$$D_k \mathbb{G}(\boldsymbol{\mu}_{(\cdot)}(t)) = \frac{\partial \mathbb{G}(\boldsymbol{\mu}_{(\cdot)}(t))}{\partial \mu_{(k)}(t)} = \begin{cases} 1 & \text{if} \quad k = 1, \\ 0 & \text{if} \quad k \neq 1. \end{cases} \tag{6.12.3}$$

It follows that for all $k = 1, 2, \ldots, n$, and $t \in [0, T]$,

$$D_k \log \mathbb{G}(\boldsymbol{\mu}_{(\cdot)}(t)) = \frac{D_k \mathbb{G}(\boldsymbol{\mu}_{(\cdot)}(t))}{\mathbb{G}(\boldsymbol{\mu}_{(\cdot)}(t))} = \begin{cases} \dfrac{1}{\mu_{(1)}(t)} & \text{if} \quad k = 1, \\ 0 & \text{if} \quad k \neq 1. \end{cases} \tag{6.12.4}$$

It is clear from (6.12.1) that the rank-dependent generating function $\mathbb{G}$ is a positive twice continuously differentiable function and from (6.12.4) we can see that $\mu_{(k)}(t) D_k \log \mathbb{G}(\boldsymbol{\mu}_{(\cdot)}(t))$ is bounded on Δ^{n-1} for all $k = 1, 2, \ldots, n$, in fact $\mu_{(1)}(t) D_1 \log \mathbb{G}(\boldsymbol{\mu}_{(\cdot)}(t)) = 1$ and $\mu_{(k)}(t) D_k \log \mathbb{G}(\boldsymbol{\mu}_{(\cdot)}(t)) = 0$ for $k = 2, \ldots, n$. Therefore, from (6.8.6) of Theorem 6.8.1, with $\mathfrak{G}(\mathbf{x}_{(\cdot)}) = \mathbb{G}(\mathbf{x}_{(\cdot)})$, we obtain the following weights, for $k = 2, \ldots, n$, for the portfolio generated by (6.12.1)

$$\begin{aligned} \psi^{\chi_{(1)}}_{p_t(k)}(t) &= \left(D_k \log \mathbb{G}(\boldsymbol{\mu}_{(\cdot)}(t)) + 1 - \sum_{\ell=1}^{n} \mu_{(\ell)}(t)\, D_\ell \log \mathbb{G}(\boldsymbol{\mu}_{(\cdot)}(t)) \right) \mu_{(k)}(t) \\ &= \left(1 - \mu_{(1)}(t) \frac{1}{\mu_{(1)}(t)} \right) \mu_{(k)}(t) \\ &= 0, \end{aligned}$$

and for $k = 1$, we have

$$\begin{aligned} \psi^{\chi_{(1)}}_{p_t(1)}(t) &= \left(D_1 \log \mathbb{G}(\boldsymbol{\mu}_{(\cdot)}(t)) + 1 - \sum_{\ell=1}^{n} \mu_{(\ell)}(t)\, D_\ell \log \mathbb{G}(\boldsymbol{\mu}_{(\cdot)}(t)) \right) \mu_{(1)}(t) \\ &= \left(\frac{1}{\mu_{(1)}(t)} + 1 - \mu_{(1)}(t) \frac{1}{\mu_{(1)}(t)} \right) \mu_{(1)}(t) \\ &= \frac{\mu_{(1)}(t)}{\mu_{(1)}(t)} \\ &= 1. \end{aligned}$$

Thus, the weights corresponding to the rank-dependent generating function $\mathbb{G}$ are given by the following definition.

Definition 6.12.1 (Biggest Stock). *Let μ be the market portfolio. The portfolio process $\psi^{\chi_{(1)}} = \big\{\psi^{\chi_{(1)}}(t) = \big(\psi_1^{\chi_{(1)}}(t), \psi_2^{\chi_{(1)}}(t), \ldots, \psi_n^{\chi_{(1)}}(t)\big), t \in [0,T]\big\}$, with weights*

$$\psi_{(k)}^{\chi_{(1)}}(t) \;=\; \psi_{p_t(k)}^{\chi_{(1)}}(t) \;\triangleq\; \begin{cases} 1 & \text{if} \quad k = 1, \\ 0 & \text{if} \quad k \neq 1, \end{cases} \tag{6.12.5}$$

for $k = 1, 2, \ldots, n$, and $t \in [0,T]$, is called the ***the biggest stock portfolio (process)****.*

Here, $\psi^{\chi_{(1)}}$ represents a portfolio that holds only the largest stock in the market. It can be easily verified that the biggest stock $\psi^{\chi_{(1)}}$ satisfies the requirements of Definition 2.2.16, since the market weights are bounded on $[0,\infty)$, and thus so is the rank-dependent generating function $\mathbb{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)$. Clearly, the portfolio weights sum to 1. Now, from (6.12.3), we obtain for all $k, \ell = 1, 2, \ldots, n$,

$$D_{k\ell}\mathbb{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \;=\; \frac{\partial^2 \mathbb{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}{\partial \mu_{(k)}(t) \partial \mu_{(\ell)}(t)} \;=\; 0. \tag{6.12.6}$$

Therefore, the above result together with (6.9.13), with $\mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big) = \mathbb{G}\big(\mathbf{x}_{(\cdot)}\big)$, yields the following *smooth component of the drift process* (6.8.7) of Theorem 6.8.1

$$g^{\chi_{(1)}}(t) \;=\; g_{\psi^{\chi_{(1)}}}(t) \;=\; \frac{-1}{2\,\mathbb{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \sum_{k,\ell=1}^{n} D_{k\ell}\mathbb{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t) \;=\; 0. \tag{6.12.7}$$

Thus, the smooth component of the drift process is zero. By (6.9.14), the *local time component of the drift process* is given by

$$\begin{aligned} d\mathscr{L}^{\chi_{(1)}}(t) \;=\; d\mathscr{L}_{\psi^{\chi_{(1)}}}(t) \;&=\; \frac{1}{2}\sum_{k=1}^{n-1}\Big(\psi_{p_t(k+1)}^{\chi_{(1)}}(t) - \psi_{p_t(k)}^{\chi_{(1)}}(t)\Big)\, d\mathfrak{L}_{k,k+1}(t) \\ &=\; \frac{1}{2}\Big(-\psi_{p_t(1)}^{\chi_{(1)}}(t)\Big)\, d\mathfrak{L}_{1,2}(t) \\ &=\; -\frac{1}{2}\, d\mathfrak{L}_{1,2}(t). \end{aligned} \tag{6.12.8}$$

Hence, by (6.9.12) and (6.8.7), the drift process of the biggest stock is given by

$$d\Upsilon^{\chi_{(1)}}(t) \;=\; d\Upsilon_{\psi^{\chi_{(1)}}}(t) \;=\; g^{\chi_{(1)}}(t)\, dt + d\mathscr{L}^{\chi_{(1)}}(t) \;=\; -\frac{1}{2}\, d\mathfrak{L}_{1,2}(t). \tag{6.12.9}$$

Furthermore, by (6.9.2) or (6.9.15), the performance of this portfolio $\psi^{\chi_{(1)}}$, relative to the market portfolio satisfies, for all $t \in [0,T]$, a.s.,

$$d\log\Big(Z_{\psi^{\chi_{(1)}}}(t)/Z_\mu(t)\Big) \;=\; d\log\mathbb{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + d\Upsilon^{\chi_{(1)}}(t) \;=\; d\log\mu_{(1)}(t) - \frac{1}{2}\, d\mathfrak{L}_{1,2}(t), \tag{6.12.10}$$

or, by (6.9.8), for all $T \in [0,\infty)$, we have the relative performance of the biggest stock

$$\log\left(\frac{Z_{\psi^{\chi_{(1)}}}(T)}{Z_\mu(T)}\right) \;=\; \log\left(\frac{\mathbb{G}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathbb{G}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \Upsilon^{\chi_{(1)}}(T) \tag{6.12.11}$$

$$\;=\; \log\left(\frac{\mathbb{G}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathbb{G}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) - \frac{1}{2}\,\mathfrak{L}_{1,2}(T) \tag{6.12.12}$$

$$\;=\; \log\left(\frac{\mu_{(1)}(T)}{\mu_{(1)}(0)}\right) - \frac{1}{2}\,\mathfrak{L}_{1,2}(T), \tag{6.12.13}$$

if $Z_{\psi^{\chi_{(1)}}}(0) = Z_\mu(0)$, otherwise the integral representation is given by

$$\log\left(\frac{Z_{\psi^{\chi_{(1)}}}(T)}{Z_\mu(T)}\right) = \log\left(\frac{Z_{\psi^{\chi_{(1)}}}(0)}{Z_\mu(0)}\right) + \log\left(\frac{\mathbb{G}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathbb{G}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) - \frac{1}{2}\,\mathfrak{L}_{1,2}(T) \tag{6.12.14}$$

$$= \log\left(\frac{Z_{\psi^{\chi_{(1)}}}(0)}{Z_\mu(0)}\right) + \log\left(\frac{\mu_{(1)}(T)}{\mu_{(1)}(0)}\right) - \frac{1}{2}\,\mathfrak{L}_{1,2}(T), \tag{6.12.15}$$

or, as per equation (6.9.6), we have

$$\log\left(\frac{Z_{\psi^{\chi_{(1)}}}(T)}{Z_{\psi^{\chi_{(1)}}}(0)}\right) = \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log\left(\frac{\mathbb{G}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathbb{G}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) - \frac{1}{2}\,\mathfrak{L}_{1,2}(T) \tag{6.12.16}$$

$$= \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log\left(\frac{\mu_{(1)}(T)}{\mu_{(1)}(0)}\right) - \frac{1}{2}\,\mathfrak{L}_{1,2}(T). \tag{6.12.17}$$

Recall, that the local time process is an increasing process in time, and we have $\frac{d\mathfrak{L}_{k,k+1}(t)}{dt} \geq 0$, for $k = 1, 2, \ldots, n-1$. Thus, for the drift process of the biggest stock we have

$$\Upsilon'_{\psi^{\chi_{(1)}}}(t) = \frac{d\Upsilon^{\chi_{(1)}}(t)}{dt} = g^{\chi_{(1)}}(t) + \frac{d\mathscr{L}^{\chi_{(1)}}(t)}{dt} = -\frac{1}{2}\,\frac{d\mathfrak{L}_{1,2}(t)}{dt} \leq 0.$$

Therefore, the local time component of the drift process is decreasing and hence the drift process is itself decreasing. Moreover, since $\mu_{(1)}(t) < 1$ for all $t \in [0, T]$, we have $\log \mu_{(1)}(t) < 0$. Hence, the long-term relative performance of $\psi^{\chi_{(1)}}$, the biggest stock, will suffer if there are many changes in leadership in the market. The portfolio comprising the largest stock in the market will underperform the market if many changes between the 1st ranked stock and the 2nd ranked stock occur. Accordingly, for the biggest stock to perform well relative to the market, it must either pay dividends that are higher than the market-average dividends, or ultimately it must crush all competitors to allow for no changes in market leadership [Fernholz (2002)]. Now, consider Theorem 6.10.1. Then, by (6.10.22), the *local time component of the drift process* in the more general setting where triple points may exist, is given by

$$\begin{aligned} d\mathscr{L}^{\chi_{(1)}}(t) &= -\sum_{k=1}^{n} \psi^{\chi_{(1)}}_{p_t(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) + \sum_{k=1}^{n} \psi^{\chi_{(1)}}_{p_t(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t) \\ &= -\psi^{\chi_{(1)}}_{p_t(1)}(t)\,\frac{1}{N_t(1)}\sum_{\ell=2}^{n} d\mathfrak{L}_{1,\ell}(t) + \psi^{\chi_{(1)}}_{p_t(1)}(t)\,\frac{1}{N_t(1)}\,d\mathfrak{L}_{0,1}(t) \\ &= -\frac{1}{N_t(1)}\sum_{\ell=2}^{n} d\mathfrak{L}_{1,\ell}(t). \end{aligned} \tag{6.12.18}$$

Hence, by (6.10.20) and (6.10.6) of Theorem 6.10.1, the drift process of the biggest stock in this more general setting is given by

$$d\Upsilon^{\chi_{(1)}}(t) = g^{\chi_{(1)}}(t)\,dt + d\mathscr{L}^{\chi_{(1)}}(t) = -\frac{1}{N_t(1)}\sum_{\ell=2}^{n} d\mathfrak{L}_{1,\ell}(t). \tag{6.12.19}$$

Furthermore, by (6.9.2), the performance of this portfolio $\psi^{\chi_{(1)}}$, relative to the market portfolio satisfies, for all $t \in [0, T]$, a.s.,

$$d\log\Big(Z_{\psi^{\chi_{(1)}}}(t)/Z_\mu(t)\Big) = d\log\mu_{(1)}(t) - \frac{1}{N_t(1)}\sum_{\ell=2}^{n} d\mathfrak{L}_{1,\ell}(t). \tag{6.12.20}$$

6.12.2 The Large-Stock Index Portfolio

Fix an integer $m \in \{2, \ldots, n-1\}$ and consider the following rank-dependent generating function

$$\mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big) \equiv \mathcal{G}_L\big(\mathbf{x}_{(\cdot)}\big) = \mathcal{G}_L\big(x_{(1)}, \ldots, x_{(n)}\big) \triangleq x_{(1)} + \cdots + x_{(m)} = \sum_{k=1}^{m} x_{(k)}, \tag{6.12.21}$$

for all $\mathbf{x}$ and $\mathbf{x}_{(\cdot)}$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\mathbf{x}, \mathbf{x}_{(\cdot)} \in \Delta^{n-1}$. Then, for $t \in [0,T]$, the corresponding market process is given by

$$\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \equiv \mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) = \mathcal{G}_L\big(\mu_{(1)}(t), \ldots, \mu_{(n)}(t)\big) \triangleq \mu_{(1)}(t) + \cdots + \mu_{(m)}(t) = \sum_{k=1}^{m} \mu_{(k)}(t). \tag{6.12.22}$$

The generating function above generates the large-cap-weighted portfolio. The value of the above generating function represents the relative capitalisation of the large-stock index (i.e., the large-cap-weighted portfolio), compared to the market as a whole, composed of the m largest stocks in the equity market. Subsequently, we deduce the following for all $k = 1, 2, \ldots, n$,

$$D_k \mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) = \frac{\partial \mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}{\partial \mu_{(k)}(t)} = \begin{cases} 1 & \text{if} \quad k \le m, \\ 0 & \text{if} \quad k > m. \end{cases} \tag{6.12.23}$$

It follows that for all $k = 1, 2, \ldots, n$, and $t \in [0,T]$,

$$D_k \log \mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) = \frac{D_k \mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} = \begin{cases} \dfrac{1}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} & \text{if} \quad k \le m, \\ 0 & \text{if} \quad k > m. \end{cases} \tag{6.12.24}$$

It is clear from (6.12.21) that the rank-dependent generating function $\mathcal{G}_L$ is a positive twice continuously differentiable function and from (6.12.24) we can see that $\mu_{(k)}(t) D_k \log \mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)$ is bounded on Δ^{n-1} for all $k = 1, 2, \ldots, n$. Therefore, from (6.8.6) of Theorem 6.8.1, with $\mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big) = \mathcal{G}_L\big(\mathbf{x}_{(\cdot)}\big)$, we obtain the following weights, for $k \le m$, i.e., for $k = 1, \ldots, m$, for the portfolio generated by (6.12.21)

$$\begin{aligned}
\zeta_{p_t(k)}(t) &= \left(D_k \log \mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + 1 - \sum_{\ell=1}^{n} \mu_{(\ell)}(t)\, D_\ell \log \mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \right) \mu_{(k)}(t) \\
&= \left(\frac{1}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} + 1 - \sum_{\ell=1}^{m} \mu_{(\ell)}(t) \left[\frac{1}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \right] \right) \mu_{(k)}(t) \\
&= \left(\frac{1}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} + 1 - \sum_{\ell=1}^{m} \frac{\mu_{(\ell)}(t)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \right) \mu_{(k)}(t) \\
&= \frac{\mu_{(k)}(t)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \left(1 + \mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) - \sum_{\ell=1}^{m} \mu_{(\ell)}(t) \right) \\
&= \frac{\mu_{(k)}(t)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \left(1 + \mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) - \mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \right) \\
&= \frac{\mu_{(k)}(t)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)},
\end{aligned}$$

and for $k > m$, i.e., for $k = m+1, \ldots, n$, we have

$$\begin{aligned}
\zeta_{p_t(k)}(t) &= \left(D_k \log \mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + 1 - \sum_{\ell=1}^{n} \mu_{(\ell)}(t)\, D_\ell \log \mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \right) \mu_{(k)}(t) \\
&= \left(1 - \sum_{\ell=1}^{m} \mu_{(\ell)}(t) \left[\frac{1}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \right] \right) \mu_{(k)}(t) \\
&= \left(1 - \sum_{\ell=1}^{m} \frac{\mu_{(\ell)}(t)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \right) \mu_{(k)}(t) \\
&= \left(1 - \frac{1}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \sum_{\ell=1}^{m} \mu_{(\ell)}(t) \right) \mu_{(k)}(t) \\
&= \left(1 - \frac{1}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \right) \mu_{(k)}(t) \\
&= 0.
\end{aligned}$$

The weights $\zeta_{(k)}$, for $k = 1, 2, \ldots, n$, represent the capitalisation weights of the stocks that are members of the large-stock index. Thus, the weights corresponding to the rank-dependent generating function $\mathcal{G}_L$ for $k \leq m$ are given by

$$\zeta_{p_t(k)}(t) \;=\; \frac{\mu_{(k)}(t)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \;=\; \frac{\mu_{(k)}(t)}{\sum_{\ell=1}^m \mu_{(\ell)}(t)} \;=\; \frac{\mu_{(k)}(t)}{\mu_{(1)}(t) + \cdots + \mu_{(m)}(t)}, \tag{6.12.25}$$

and we have the following definition.

Definition 6.12.2 (Large-Stock Index Portfolio)**.** *Let μ be the market portfolio. The portfolio process $\zeta = \big\{\zeta(t) = \big(\zeta_1(t), \zeta_2(t), \ldots, \zeta_n(t)\big), t \in [0, T]\big\}$, with weights*

$$\zeta_{(k)}(t) \;=\; \zeta_{p_t(k)}(t) \;\triangleq\; \begin{cases} \dfrac{\mu_{(k)}(t)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} & \text{if} \quad k \leq m, \\ 0 & \text{if} \quad k > m, \end{cases} \tag{6.12.26}$$

for $k = 1, 2, \ldots, n$, and $t \in [0, T]$, is called the ***the large-stock index portfolio (process)****.*

Here, ζ represents a large-stock portfolio comprised of the m largest stocks in the market. It can be easily verified that the large-stock index portfolio ζ satisfies the requirements of Definition 2.2.16, since the market weights are bounded on $[0, \infty)$, and thus so is the rank-dependent generating function $\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)$. Clearly, the portfolio weights sum to 1. Now, from (6.12.23), we obtain for all $k, \ell = 1, 2, \ldots, n$,

$$D_{k\ell}\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \;=\; \frac{\partial^2 \mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}{\partial \mu_{(k)}(t) \partial \mu_{(\ell)}(t)} \;=\; 0. \tag{6.12.27}$$

Therefore, the above result together with (6.9.13), with $\mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big) = \mathcal{G}_L\big(\mathbf{x}_{(\cdot)}\big)$, yields the following *smooth component of the drift process* (6.8.7) of Theorem 6.8.1

$$g^L(t) \;=\; g_\zeta(t) \;=\; \frac{-1}{2\,\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \sum_{k,\ell=1}^n D_{k\ell}\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t) \;=\; 0. \tag{6.12.28}$$

Thus, the smooth component of the drift process is zero. By (6.9.14) and (6.12.26), the *local time component of the drift process* is given by

$$\begin{aligned} d\mathscr{L}^L(t) \;=\; d\mathscr{L}_\zeta(t) \;&=\; \frac{1}{2}\sum_{k=1}^{n-1}\Big(\zeta_{p_t(k+1)}(t) - \zeta_{p_t(k)}(t)\Big)\, d\mathfrak{L}_{k,k+1}(t) \\ &=\; \frac{1}{2}\sum_{k=1}^{m}\Big(\zeta_{p_t(k+1)}(t) - \zeta_{p_t(k)}(t)\Big)\, d\mathfrak{L}_{k,k+1}(t) \\ &=\; \frac{1}{2}\sum_{k=1}^{m-1}\Big(\zeta_{p_t(k+1)}(t) - \zeta_{p_t(k)}(t)\Big)\, d\mathfrak{L}_{k,k+1}(t) - \frac{1}{2}\,\zeta_{p_t(m)}(t)\, d\mathfrak{L}_{m,m+1}(t) \\ &=\; \frac{1}{2}\sum_{k=1}^{m-1}\left(\frac{\mu_{(k+1)}(t)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} - \frac{\mu_{(k)}(t)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\right) d\mathfrak{L}_{k,k+1}(t) - \frac{1}{2}\,\zeta_{p_t(m)}(t)\, d\mathfrak{L}_{m,m+1}(t) \\ &=\; \frac{1}{2\,\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\sum_{k=1}^{m-1}\big(\mu_{(k+1)}(t) - \mu_{(k)}(t)\big)\, d\mathfrak{L}_{k,k+1}(t) - \frac{1}{2}\,\zeta_{p_t(m)}(t)\, d\mathfrak{L}_{m,m+1}(t). \end{aligned} \tag{6.12.29}$$

From (6.3.183) of Corollary 6.3.26, with $a = 0$, we have the following standard result

$$d\Lambda_X(t) \;=\; \mathbb{1}_{\{0\}}\big(X(t)\big)\, d\Lambda_X(t).$$

Consequently, for all $k = 1, 2, \ldots, n$,

$$\begin{aligned} d\mathfrak{L}_{k,k+1}(t) \;=\; d\Lambda_{\log\mu_{(k)} - \log\mu_{(k+1)}}(t) \;&=\; \mathbb{1}_{\{0\}}\big(\log\mu_{(k)}(t) - \log\mu_{(k+1)}(t)\big)\, d\Lambda_{\log\mu_{(k)} - \log\mu_{(k+1)}}(t) \\ &=\; \mathbb{1}_{\{0\}}\big(\log\mu_{(k)}(t) - \log\mu_{(k+1)}(t)\big)\, d\mathfrak{L}_{k,k+1}(t) \\ &=\; \mathbb{1}_{\{0\}}\big(\mu_{(k)}(t) - \mu_{(k+1)}(t)\big)\, d\mathfrak{L}_{k,k+1}(t) \\ &=\; \mathbb{1}_{\{0\}}\big(\mu_{(k+1)}(t) - \mu_{(k)}(t)\big)\, d\mathfrak{L}_{k,k+1}(t). \end{aligned} \tag{6.12.30}$$

Therefore,

$$d\mathscr{L}_\zeta(t) = \frac{1}{2\,\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \sum_{k=1}^{m-1} \big(\mu_{(k+1)}(t) - \mu_{(k)}(t)\big)\, \mathbb{1}_{\{0\}}\big(\mu_{(k+1)}(t) - \mu_{(k)}(t)\big)\, d\mathfrak{L}_{k,k+1}(t) - \frac{1}{2}\,\zeta_{p_t(m)}(t)\, d\mathfrak{L}_{m,m+1}(t).$$

Thus, since $\big(\mu_{(k+1)}(t) - \mu_{(k)}(t)\big)\, \mathbb{1}_{\{0\}}\big(\mu_{(k+1)}(t) - \mu_{(k)}(t)\big) = 0$, for $k = 1, \ldots, m-1$, we get that the local time component of the drift process satisfies

$$d\mathscr{L}_\zeta(t) = -\frac{1}{2}\,\zeta_{p_t(m)}(t)\, d\mathfrak{L}_{m,m+1}(t) = -\frac{1}{2}\,\zeta_{(m)}(t)\, d\mathfrak{L}_{m,m+1}(t). \tag{6.12.31}$$

Hence, by (6.9.12) and (6.8.7), the drift process of the large-stock index portfolio is given by for $t \in [0, T]$, a.s.,

$$d\Upsilon^L(t) = d\Upsilon_\zeta(t) = g_\zeta(t)\, dt + d\mathscr{L}_\zeta(t) = -\frac{1}{2}\,\zeta_{(m)}(t)\, d\mathfrak{L}_{m,m+1}(t). \tag{6.12.32}$$

Furthermore, by (6.9.2) or (6.9.15), the performance of this portfolio ζ, relative to the market portfolio satisfies, for all $t \in [0, T]$, a.s.,

$$d\log\Big(Z_\zeta(t)/Z_\mu(t)\Big) = d\log \mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + d\Upsilon_\zeta(t) \tag{6.12.33}$$

$$= d\log \mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) - \frac{1}{2}\,\zeta_{(m)}(t)\, d\mathfrak{L}_{m,m+1}(t) \tag{6.12.34}$$

$$= d\log \mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) - \frac{1}{2}\,\frac{\mu_{(m)}(t)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\, d\mathfrak{L}_{m,m+1}(t), \tag{6.12.35}$$

or, by (6.9.8), for all $T \in [0, \infty)$, we have the relative performance of the large-stock index portfolio

$$\log\left(\frac{Z_\zeta(T)}{Z_\mu(T)}\right) = \log\left(\frac{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \Upsilon_\zeta(T) \tag{6.12.36}$$

$$= \log\left(\frac{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) - \frac{1}{2}\int_0^T \zeta_{(m)}(t)\, d\mathfrak{L}_{m,m+1}(t) \tag{6.12.37}$$

$$= \log\left(\frac{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) - \frac{1}{2}\int_0^T \frac{\mu_{(m)}(t)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\, d\mathfrak{L}_{m,m+1}(t), \tag{6.12.38}$$

if $Z_\zeta(0) = Z_\mu(0)$, otherwise the integral representation is given by

$$\log\left(\frac{Z_\zeta(T)}{Z_\mu(T)}\right) = \log\left(\frac{Z_\zeta(0)}{Z_\mu(0)}\right) + \log\left(\frac{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) - \frac{1}{2}\int_0^T \zeta_{(m)}(t)\, d\mathfrak{L}_{m,m+1}(t) \tag{6.12.39}$$

$$= \log\left(\frac{Z_\zeta(0)}{Z_\mu(0)}\right) + \log\left(\frac{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) - \frac{1}{2}\int_0^T \frac{\mu_{(m)}(t)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\, d\mathfrak{L}_{m,m+1}(t), \tag{6.12.40}$$

or, as per equation (6.9.6), we have

$$\log\left(\frac{Z_\zeta(T)}{Z_\zeta(0)}\right) = \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log\left(\frac{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) - \frac{1}{2}\int_0^T \zeta_{(m)}(t)\, d\mathfrak{L}_{m,m+1}(t) \tag{6.12.41}$$

$$= \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log\left(\frac{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) - \frac{1}{2}\int_0^T \frac{\mu_{(m)}(t)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\, d\mathfrak{L}_{m,m+1}(t). \tag{6.12.42}$$

It can be argued that, $\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)$ will be stable and mean-reverting under conditions of general market stability. In this case, over a sufficiently long period of time, the relative return will be dominated by the local time component. Since this term is a.s. decreasing, the long-term return of the large-stock index will be less than

that of the market. Now, consider Theorem 6.10.1. Then, by (6.10.22), the *local time component of the drift process* in the more general setting where triple points may exist, is given by

$$\begin{aligned} d\mathscr{L}_\zeta(t) &= -\sum_{k=1}^{n} \zeta_{p_t(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) + \sum_{k=1}^{n} \zeta_{p_t(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t) \\ &= -\sum_{k=1}^{m} \zeta_{p_t(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) + \sum_{k=1}^{m} \zeta_{p_t(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t) \qquad (6.12.43) \\ &= -\sum_{k=1}^{m} \zeta_{(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) + \sum_{k=1}^{m} \zeta_{(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t). \qquad (6.12.44) \end{aligned}$$

Hence, by (6.10.20) and (6.10.6) of Theorem 6.10.1, the drift process of the large-stock index portfolio in this more general setting is given by

$$d\Upsilon_\zeta(t) = -\sum_{k=1}^{m} \zeta_{(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) + \sum_{k=1}^{m} \zeta_{(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t). \qquad (6.12.45)$$

Furthermore, by (6.9.2), the performance of this portfolio ζ, relative to the market portfolio satisfies, for all $t \in [0,T]$, a.s.,

$$\begin{aligned} & d\log\Big(Z_\zeta(t)/Z_\mu(t)\Big) \\ &= d\log\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) - \sum_{k=1}^{m} \zeta_{(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) + \sum_{k=1}^{m} \zeta_{(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t) \qquad (6.12.46) \\ &= d\log\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) - \sum_{k=1}^{m} \frac{\mu_{(k)}(t)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\,\frac{1}{N_t(k)}\sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) + \sum_{k=1}^{m} \frac{\mu_{(k)}(t)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\,\frac{1}{N_t(k)}\sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t) \qquad (6.12.47) \\ &= d\log\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) - \frac{1}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\sum_{k=1}^{m} \mu_{(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) + \frac{1}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\sum_{k=1}^{m} \mu_{(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t) \\ &= d\log\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) - \frac{1}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\sum_{k=1}^{m} \mu_{(k)}(t)\,\frac{1}{N_t(k)}\Big[\sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) - \sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t)\Big]. \qquad (6.12.48) \end{aligned}$$

6.12.3 The Small-Stock Index Portfolio

Fix an integer $m \in \{2,\ldots,n-1\}$ and consider the following rank-dependent generating function

$$\mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big) \equiv \mathcal{G}_S\big(\mathbf{x}_{(\cdot)}\big) = \mathcal{G}_S\big(x_{(1)},\ldots,x_{(n)}\big) \triangleq x_{(m+1)} + \cdots + x_{(n)} = \sum_{k=m+1}^{n} x_{(k)}, \qquad (6.12.49)$$

for all $\mathbf{x}$ and $\mathbf{x}_{(\cdot)}$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\mathbf{x}, \mathbf{x}_{(\cdot)} \in \Delta^{n-1}$. Then, for $t \in [0,T]$, the corresponding market process is given by

$$\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \equiv \mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) = \mathcal{G}_S\big(\mu_{(1)}(t),\ldots,\mu_{(n)}(t)\big) \triangleq \mu_{(m+1)}(t) + \cdots + \mu_{(n)}(t) = \sum_{k=m+1}^{n} \mu_{(k)}(t). \qquad (6.12.50)$$

The generating function above generates the small-cap-weighted portfolio. The value of the generating function above represents the relative capitalisation of the small-stock index (i.e., the small-cap-weighted portfolio), compared to the market as a whole, composed of the $n-m$ smallest stocks in the equity market. Subsequently, we deduce the following for all $k = 1,2,\ldots,n$,

$$D_k\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) = \frac{\partial\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}{\partial\mu_{(k)}(t)} = \begin{cases} 1 & \text{if} \quad k > m, \\ 0 & \text{if} \quad k \le m. \end{cases} \qquad (6.12.51)$$

It follows that for all $k = 1, 2, \ldots, n$, and $t \in [0, T]$,

$$D_k \log \mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \;=\; \frac{D_k \mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \;=\; \begin{cases} \dfrac{1}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} & \text{if} \quad k > m, \\ 0 & \text{if} \quad k \le m. \end{cases} \tag{6.12.52}$$

It is clear from (6.12.49) that the rank-dependent generating function $\mathcal{G}_S$ is a positive twice continuously differentiable function and from (6.12.52) we can see that $\mu_{(k)}(t) D_k \log \mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)$ is bounded on Δ^{n-1} for all $k = 1, 2, \ldots, n$. Therefore, from (6.8.6) of Theorem 6.8.1, with $\mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big) = \mathcal{G}_S\big(\mathbf{x}_{(\cdot)}\big)$, we obtain the following weights, for $k > m$, i.e., for $k = m+1, \ldots, n$, for the portfolio generated by (6.12.49)

$$\begin{aligned}
\eta_{p_t(k)}(t) &= \left(D_k \log \mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + 1 - \sum_{\ell=1}^{n} \mu_{(\ell)}(t)\, D_\ell \log \mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \right) \mu_{(k)}(t) \\
&= \left(\frac{1}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} + 1 - \sum_{\ell=m+1}^{n} \mu_{(\ell)}(t) \left[\frac{1}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \right] \right) \mu_{(k)}(t) \\
&= \left(\frac{1}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} + 1 - \sum_{\ell=m+1}^{n} \frac{\mu_{(\ell)}(t)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \right) \mu_{(k)}(t) \\
&= \frac{\mu_{(k)}(t)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \left(1 + \mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) - \sum_{\ell=m+1}^{n} \mu_{(\ell)}(t) \right) \\
&= \frac{\mu_{(k)}(t)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \left(1 + \mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) - \mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \right) \\
&= \frac{\mu_{(k)}(t)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)},
\end{aligned}$$

and for $k \le m$, i.e., for $k = 1, \ldots, m$, we have

$$\begin{aligned}
\eta_{p_t(k)}(t) &= \left(D_k \log \mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + 1 - \sum_{\ell=1}^{n} \mu_{(\ell)}(t)\, D_\ell \log \mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \right) \mu_{(k)}(t) \\
&= \left(1 - \sum_{\ell=m+1}^{n} \mu_{(\ell)}(t) \left[\frac{1}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \right] \right) \mu_{(k)}(t) \\
&= \left(1 - \sum_{\ell=m+1}^{n} \frac{\mu_{(\ell)}(t)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \right) \mu_{(k)}(t) \\
&= \left(1 - \frac{1}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \sum_{\ell=m+1}^{n} \mu_{(\ell)}(t) \right) \mu_{(k)}(t) \\
&= \left(1 - \frac{1}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \right) \mu_{(k)}(t) \\
&= 0.
\end{aligned}$$

The weights $\eta_{(k)}$, for $k = 1, 2, \ldots, n$, represent the capitalisation weights of the stocks that are members of the small-stock index. Thus, the weights corresponding to the rank-dependent generating function $\mathcal{G}_S$ for $k > m$ are given by

$$\eta_{p_t(k)}(t) \;=\; \frac{\mu_{(k)}(t)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \;=\; \frac{\mu_{(k)}(t)}{\sum_{\ell=m+1}^{n} \mu_{(\ell)}(t)} \;=\; \frac{\mu_{(k)}(t)}{\mu_{(m+1)}(t) + \cdots + \mu_{(n)}(t)}, \tag{6.12.53}$$

and we have the following definition.

Definition 6.12.3 (Small-Stock Index Portfolio). *Let μ be the market portfolio. The portfolio process $\eta = \big\{\eta(t) = \big(\eta_1(t), \eta_2(t), \ldots, \eta_n(t)\big), t \in [0, T]\big\}$, with weights*

$$\eta_{(k)}(t) \;=\; \eta_{p_t(k)}(t) \;\triangleq\; \begin{cases} \dfrac{\mu_{(k)}(t)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} & \textit{if} \quad k > m, \\ 0 & \textit{if} \quad k \le m, \end{cases} \tag{6.12.54}$$

for $k = 1, 2, \ldots, n$, and $t \in [0, T]$, is called the ***the small-stock index portfolio (process)****.*

Here, η represents a small-stock portfolio comprised of the $n - m$ smallest stocks in the market. It can be easily verified that the small-stock index portfolio η satisfies the requirements of Definition 2.2.16, since the market weights are bounded on $[0, \infty)$, and thus so is the rank-dependent generating function $\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)$. Clearly, the portfolio weights sum to 1. Now, from (6.12.51), we obtain for all $k, \ell = 1, 2, \ldots, n$,

$$D_{k\ell}\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \;=\; \frac{\partial^2 \mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}{\partial \mu_{(k)}(t) \partial \mu_{(\ell)}(t)} \;=\; 0. \tag{6.12.55}$$

Therefore, the above result together with (6.9.13), with $\mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big) = \mathcal{G}_S\big(\mathbf{x}_{(\cdot)}\big)$, yields the following *smooth component of the drift process* (6.8.7) of Theorem 6.8.1

$$g^S(t) \;=\; g_\eta(t) \;=\; \frac{-1}{2\,\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \sum_{k,\ell=1}^{n} D_{k\ell}\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t) \;=\; 0. \tag{6.12.56}$$

Thus, the smooth component of the drift process is zero. By (6.9.14) and (6.12.54), the *local time component of the drift process* is given by

$$\begin{aligned}
d\mathscr{L}^S(t) \;=\; d\mathscr{L}_\eta(t) \;&=\; \frac{1}{2}\sum_{k=1}^{n-1}\Big(\eta_{p_t(k+1)}(t) - \eta_{p_t(k)}(t)\Big)\, d\mathfrak{L}_{k,k+1}(t) \\
&=\; \frac{1}{2}\sum_{k=m}^{n-1}\Big(\eta_{p_t(k+1)}(t) - \eta_{p_t(k)}(t)\Big)\, d\mathfrak{L}_{k,k+1}(t) \\
&=\; \frac{1}{2}\,\eta_{p_t(m+1)}(t)\, d\mathfrak{L}_{m,m+1}(t) + \frac{1}{2}\sum_{k=m+1}^{n-1}\Big(\eta_{p_t(k+1)}(t) - \eta_{p_t(k)}(t)\Big)\, d\mathfrak{L}_{k,k+1}(t) \\
&=\; \frac{1}{2}\,\eta_{p_t(m+1)}(t)\, d\mathfrak{L}_{m,m+1}(t) + \frac{1}{2}\sum_{k=m+1}^{n-1}\left(\frac{\mu_{(k+1)}(t)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} - \frac{\mu_{(k)}(t)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\right) d\mathfrak{L}_{k,k+1}(t) \\
&=\; \frac{1}{2}\,\eta_{p_t(m+1)}(t)\, d\mathfrak{L}_{m,m+1}(t) + \frac{1}{2\,\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\sum_{k=m+1}^{n-1}\big(\mu_{(k+1)}(t) - \mu_{(k)}(t)\big)\, d\mathfrak{L}_{k,k+1}(t).
\end{aligned} \tag{6.12.57}$$

Therefore, by (6.12.30), we have

$$\begin{aligned}
d\mathscr{L}_\eta(t) \;=\;& \frac{1}{2}\,\eta_{p_t(m+1)}(t)\, d\mathfrak{L}_{m,m+1}(t) \\
&+ \frac{1}{2\,\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\sum_{k=m+1}^{n-1}\big(\mu_{(k+1)}(t) - \mu_{(k)}(t)\big)\,\mathbb{1}_{\{0\}}\big(\mu_{(k+1)}(t) - \mu_{(k)}(t)\big)\, d\mathfrak{L}_{k,k+1}(t).
\end{aligned}$$

Thus, since $\big(\mu_{(k+1)}(t) - \mu_{(k)}(t)\big)\,\mathbb{1}_{\{0\}}\big(\mu_{(k+1)}(t) - \mu_{(k)}(t)\big) = 0$, for $k = m+1, \ldots, n-1$, we get that the local time component of the drift process satisfies

$$d\mathscr{L}_\eta(t) \;=\; \frac{1}{2}\,\eta_{p_t(m+1)}(t)\, d\mathfrak{L}_{m,m+1}(t) \;=\; \frac{1}{2}\,\eta_{(m+1)}(t)\, d\mathfrak{L}_{m,m+1}(t). \tag{6.12.58}$$

Hence, by (6.9.12) and (6.8.7), the drift process of the small-stock index portfolio is given by for $t \in [0, T]$, a.s.,

$$d\Upsilon^S(t) \;=\; d\Upsilon_\eta(t) \;=\; g_\eta(t)\, dt + d\mathscr{L}_\eta(t) \;=\; \frac{1}{2}\,\eta_{(m+1)}(t)\, d\mathfrak{L}_{m,m+1}(t). \tag{6.12.59}$$

Furthermore, by (6.9.2) or (6.9.15), the performance of this portfolio η, relative to the market portfolio satisfies, for all $t \in [0, T]$, a.s.,

$$d\log\Big(Z_\eta(t)/Z_\mu(t)\Big) \;=\; d\log\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + d\Upsilon_\eta(t) \tag{6.12.60}$$

$$\;=\; d\log\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + \frac{1}{2}\,\eta_{(m+1)}(t)\, d\mathfrak{L}_{m,m+1}(t) \tag{6.12.61}$$

$$\;=\; d\log\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + \frac{1}{2}\,\frac{\mu_{(m+1)}(t)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\, d\mathfrak{L}_{m,m+1}(t), \tag{6.12.62}$$

or, by (6.9.8), for all $T \in [0, \infty)$, we have the relative performance of the small-stock index portfolio

$$\log\left(\frac{Z_\eta(T)}{Z_\mu(T)}\right) = \log\left(\frac{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \Upsilon_\eta(T) \tag{6.12.63}$$

$$= \log\left(\frac{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \frac{1}{2}\int_0^T \eta_{(m+1)}(t)\, d\mathfrak{L}_{m,m+1}(t) \tag{6.12.64}$$

$$= \log\left(\frac{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \frac{1}{2}\int_0^T \frac{\mu_{(m+1)}(t)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\, d\mathfrak{L}_{m,m+1}(t), \tag{6.12.65}$$

if $Z_\eta(0) = Z_\mu(0)$, otherwise the integral representation is given by

$$\log\left(\frac{Z_\eta(T)}{Z_\mu(T)}\right) = \log\left(\frac{Z_\eta(0)}{Z_\mu(0)}\right) + \log\left(\frac{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \frac{1}{2}\int_0^T \eta_{(m+1)}(t)\, d\mathfrak{L}_{m,m+1}(t) \tag{6.12.66}$$

$$= \log\left(\frac{Z_\eta(0)}{Z_\mu(0)}\right) + \log\left(\frac{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \frac{1}{2}\int_0^T \frac{\mu_{(m+1)}(t)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\, d\mathfrak{L}_{m,m+1}(t), \tag{6.12.67}$$

or, as per equation (6.9.6), we have

$$\log\left(\frac{Z_\eta(T)}{Z_\eta(0)}\right) = \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log\left(\frac{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \frac{1}{2}\int_0^T \eta_{(m+1)}(t)\, d\mathfrak{L}_{m,m+1}(t) \tag{6.12.68}$$

$$= \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log\left(\frac{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \frac{1}{2}\int_0^T \frac{\mu_{(m+1)}(t)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\, d\mathfrak{L}_{m,m+1}(t). \tag{6.12.69}$$

It can be argued that, $\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)$ will be stable and mean-reverting under conditions of general market stability. In this case, over a sufficiently long period of time, the relative return will be dominated by the local time component. Since this term is a.s. increasing, the long-term return of the small-stock index will be greater than that of the market. Now, consider Theorem 6.10.1. Then, by (6.10.22), the *local time component of the drift process* in the more general setting where triple points may exist, is given by

$$d\mathscr{L}_\eta(t) = -\sum_{k=1}^n \eta_{p_t(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=k+1}^n d\mathfrak{L}_{k,\ell}(t) + \sum_{k=1}^n \eta_{p_t(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t)$$

$$= -\sum_{k=m+1}^n \eta_{p_t(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=k+1}^n d\mathfrak{L}_{k,\ell}(t) + \sum_{k=m+1}^n \eta_{p_t(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t) \tag{6.12.70}$$

$$= -\sum_{k=m+1}^n \eta_{(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=k+1}^n d\mathfrak{L}_{k,\ell}(t) + \sum_{k=m+1}^n \eta_{(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t). \tag{6.12.71}$$

Hence, by (6.10.20) and (6.10.6) of Theorem 6.10.1, the drift process of the small-stock index portfolio in this more general setting is given by

$$d\Upsilon_\eta(t) = -\sum_{k=m+1}^n \eta_{(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=k+1}^n d\mathfrak{L}_{k,\ell}(t) + \sum_{k=m+1}^n \eta_{(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t). \tag{6.12.72}$$

Furthermore, by (6.9.2), the performance of this portfolio η, relative to the market portfolio satisfies, for all

$t \in [0,T]$, a.s.,

$$
\begin{aligned}
& d\log\Big(Z_\eta(t)/Z_\mu(t)\Big) \\
&= d\log \mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) - \sum_{k=m+1}^{n} \eta_{(k)}(t)\,\frac{1}{N_t(k)} \sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) + \sum_{k=m+1}^{n} \eta_{(k)}(t)\,\frac{1}{N_t(k)} \sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t) \qquad (6.12.73)\\
&= d\log \mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) - \sum_{k=m+1}^{n} \frac{\mu_{(k)}(t)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\,\frac{1}{N_t(k)} \sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) + \sum_{k=m+1}^{n} \frac{\mu_{(k)}(t)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\,\frac{1}{N_t(k)} \sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t) \qquad (6.12.74)\\
&= d\log \mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) - \frac{1}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \sum_{k=m+1}^{n} \frac{\mu_{(k)}(t)}{N_t(k)} \sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) + \frac{1}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \sum_{k=m+1}^{n} \frac{\mu_{(k)}(t)}{N_t(k)} \sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t)\\
&= d\log \mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) - \frac{1}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \sum_{k=m+1}^{n} \mu_{(k)}(t)\,\frac{1}{N_t(k)} \Big[\sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) - \sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t)\Big]. \qquad (6.12.75)
\end{aligned}
$$

6.12.4 The $\mathbf{D}_p$-Weighted (Diversity-Weighted) Large-Stock Index Portfolio

Let ζ be the large-stock index portfolio generated by the function $\mathcal{G}_L$. Let us consider a $\mathbf{D}_p$-weighted version of the large-stock index portfolio ζ, i.e., consider a portfolio $\zeta^{(p)}$ generated by the function $\mathfrak{D}_p$. In this case, with the integer $m \in \{2,\ldots,n-1\}$, the large-stock index portfolio ζ and a fixed number $0 < p < 1$, we have the following rank-dependent generating function

$$
\mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big) \;\equiv\; \mathfrak{D}_p\big(\mathbf{x}_{(\cdot)}\big) \;=\; \mathfrak{D}_p\big(x_{(1)},\ldots,x_{(n)}\big) \;\triangleq\; \Big(\sum_{k=1}^{m} x_{(k)}^p\Big)^{\frac{1}{p}}, \qquad (6.12.76)
$$

for all $\mathbf{x}$ and $\mathbf{x}_{(\cdot)}$ in the open unit $(n-1)-$simplex Δ^{n-1}, $\mathbf{x}, \mathbf{x}_{(\cdot)} \in \Delta^{n-1}$. Then, for $t \in [0,T]$, the corresponding market process is given by

$$
\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \;\equiv\; \mathfrak{D}_p\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \;=\; \mathfrak{D}_p\big(\mu_{(1)}(t),\ldots,\mu_{(n)}(t)\big) \;\triangleq\; \Big(\sum_{k=1}^{m} \big(\mu_{(k)}(t)\big)^p\Big)^{\frac{1}{p}}. \qquad (6.12.77)
$$

By analogy with (4.6.16) and (5.6.48), Theorem 6.8.1 implies that for $k \le m$, the rank-dependent generating function $\mathfrak{D}_p$ generates a portfolio $\zeta^{(p)}$ with weights

$$
\zeta_{p_t(k)}^{(p)}(t) \;=\; \frac{\big(\mu_{(k)}(t)\big)^p}{\big(\mathfrak{D}_p\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\big)^p} \;=\; \frac{\big(\mu_{(k)}(t)\big)^p}{\sum_{\ell=1}^{m}\big(\mu_{(\ell)}(t)\big)^p} \;=\; \frac{\mu_{(k)}^p(t)}{\sum_{\ell=1}^{m}\mu_{(\ell)}^p(t)} \;=\; \frac{\mu_{(k)}^p(t)}{\mu_{(1)}^p(t)+\cdots+\mu_{(m)}^p(t)}, \qquad (6.12.78)
$$

and we have the following definition.

Definition 6.12.4 ($\mathbf{D}_p$-Weighted Large-Stock Index Portfolio)**.** *Let μ be the market portfolio. The portfolio process $\zeta^{(p)} = \big\{\zeta^{(p)}(t) = \big(\zeta_1^{(p)}(t), \zeta_2^{(p)}(t), \ldots, \zeta_n^{(p)}(t)\big), t \in [0,T]\big\}$, with weights*

$$
\zeta_{(k)}^{(p)}(t) \;=\; \zeta_{p_t(k)}^{(p)}(t) \;\triangleq\; \begin{cases} \dfrac{\big(\mu_{(k)}(t)\big)^p}{\big(\mathfrak{D}_p\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\big)^p} & \text{if} \quad k \le m \\ 0 & \text{if} \quad k > m \end{cases} \;=\; \begin{cases} \dfrac{\mu_{(k)}^p(t)}{\big(\mathfrak{D}_p\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\big)^p} & \text{if} \quad k \le m, \\ 0 & \text{if} \quad k > m, \end{cases} \qquad (6.12.79)
$$

for all $k = 1,2,\ldots,n$, $t \in [0,T]$ and $0 < p < 1$, is called the ***$\mathbf{D}_p$-weighted large-stock index portfolio (process)****, or generically, the* ***diversity-weighted large-stock index portfolio (process)****.*

Similarly, by analogy with (5.6.59) or (5.6.60), the *smooth component of the drift process* is given by

$$
g^{(p)}(t) \;=\; g_{\zeta^{(p)}}(t) \;=\; (1-p)\gamma_{\zeta^{(p)}}^{*}(t). \qquad (6.12.80)
$$

In addition, by analogy with (6.12.31) for the large-stock index portfolio, we get that the *local time component of the drift process* satisfies

$$d\mathscr{L}^{(p)}(t) \;=\; d\mathscr{L}_{\zeta^{(p)}}(t) \;=\; -\frac{1}{2}\,\zeta^{(p)}_{p_t(m)}(t)\,d\mathfrak{L}_{m,m+1}(t) \;=\; -\frac{1}{2}\,\zeta^{(p)}_{(m)}(t)\,d\mathfrak{L}_{m,m+1}(t), \tag{6.12.81}$$

since

$$\begin{aligned} d\mathscr{L}^{(p)}(t) \;=\; d\mathscr{L}_{\zeta^{(p)}}(t) \;&=\; \frac{1}{2}\sum_{k=1}^{n-1}\Big(\zeta^{(p)}_{p_t(k+1)}(t)-\zeta^{(p)}_{p_t(k)}(t)\Big)\,d\mathfrak{L}_{k,k+1}(t)\\ &=\; \frac{1}{2}\sum_{k=1}^{m}\Big(\zeta^{(p)}_{p_t(k+1)}(t)-\zeta^{(p)}_{p_t(k)}(t)\Big)\,d\mathfrak{L}_{k,k+1}(t)\\ &=\; \frac{1}{2}\sum_{k=1}^{m-1}\Big(\zeta^{(p)}_{p_t(k+1)}(t)-\zeta^{(p)}_{p_t(k)}(t)\Big)\,d\mathfrak{L}_{k,k+1}(t)-\frac{1}{2}\,\zeta^{(p)}_{p_t(m)}(t)\,d\mathfrak{L}_{m,m+1}(t)\\ &=\; \frac{1}{2}\sum_{k=1}^{m-1}\left(\frac{\mu^p_{(k+1)}(t)}{\big(\mathfrak{D}_p\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\big)^p}-\frac{\mu^p_{(k)}(t)}{\big(\mathfrak{D}_p\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\big)^p}\right)d\mathfrak{L}_{k,k+1}(t)-\frac{1}{2}\,\zeta^{(p)}_{p_t(m)}(t)\,d\mathfrak{L}_{m,m+1}(t)\\ &=\; \frac{1}{2\,\big(\mathfrak{D}_p\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\big)^p}\sum_{k=1}^{m-1}\Big(\mu^p_{(k+1)}(t)-\mu^p_{(k)}(t)\Big)\,d\mathfrak{L}_{k,k+1}(t)-\frac{1}{2}\,\zeta^{(p)}_{p_t(m)}(t)\,d\mathfrak{L}_{m,m+1}(t), \end{aligned} \tag{6.12.82}$$

where the first term on the right-hand side of the last equation is zero. Hence, by (6.9.12) and (6.8.7), the drift process of the $\mathbf{D}_p$-weighted large-stock index portfolio (i.e., the diversity-weighted large-capitalisation portfolio) is given by for $t \in [0,T]$, a.s.,

$$d\Upsilon^{(p)}(t) \;=\; d\Upsilon_{\zeta^{(p)}}(t) \;=\; g_{\zeta^{(p)}}(t)\,dt + d\mathscr{L}_{\zeta^{(p)}}(t) \;=\; (1-p)\gamma^*_{\zeta^{(p)}}(t)\,dt - \frac{1}{2}\,\zeta^{(p)}_{(m)}(t)\,d\mathfrak{L}_{m,m+1}(t). \tag{6.12.83}$$

Furthermore, by (6.9.2) or (6.9.15), the performance of this portfolio $\zeta^{(p)}$, relative to the market portfolio satisfies, for all $t \in [0,T]$, a.s.,

$$d\log\Big(Z_{\zeta^{(p)}}(t)/Z_\mu(t)\Big) \;=\; d\log\mathfrak{D}_p\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + d\Upsilon_{\zeta^{(p)}}(t) \tag{6.12.84}$$

$$\;=\; d\log\mathfrak{D}_p\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + (1-p)\gamma^*_{\zeta^{(p)}}(t)\,dt - \frac{1}{2}\,\zeta^{(p)}_{(m)}(t)\,d\mathfrak{L}_{m,m+1}(t), \tag{6.12.85}$$

or, by (6.9.8), we have the relative performance of the $\mathbf{D}_p$-weighted large-stock index portfolio for $T \in [0,\infty)$,

$$\log\left(\frac{Z_{\zeta^{(p)}}(T)}{Z_\mu(T)}\right) \;=\; \log\left(\frac{\mathfrak{D}_p\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{D}_p\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \Upsilon_{\zeta^{(p)}}(T) \tag{6.12.86}$$

$$\;=\; \log\left(\frac{\mathfrak{D}_p\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{D}_p\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + (1-p)\int_0^T \gamma^*_{\zeta^{(p)}}(t)\,dt - \frac{1}{2}\int_0^T \zeta^{(p)}_{(m)}(t)\,d\mathfrak{L}_{m,m+1}(t), \tag{6.12.87}$$

if $Z_{\zeta^{(p)}}(0) = Z_\mu(0)$, otherwise the integral representation is given by

$$\log\left(\frac{Z_{\zeta^{(p)}}(T)}{Z_\mu(T)}\right) \;=\; \log\left(\frac{Z_{\zeta^{(p)}}(0)}{Z_\mu(0)}\right) + \log\left(\frac{\mathfrak{D}_p\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{D}_p\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + (1-p)\int_0^T \gamma^*_{\zeta^{(p)}}(t)\,dt - \frac{1}{2}\int_0^T \zeta^{(p)}_{(m)}(t)\,d\mathfrak{L}_{m,m+1}(t), \tag{6.12.88}$$

or, as per equation (6.9.6), we have

$$\log\left(\frac{Z_{\zeta^{(p)}}(T)}{Z_{\zeta^{(p)}}(0)}\right) \;=\; \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log\left(\frac{\mathfrak{D}_p\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{D}_p\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + (1-p)\int_0^T \gamma^*_{\zeta^{(p)}}(t)\,dt - \frac{1}{2}\int_0^T \zeta^{(p)}_{(m)}(t)\,d\mathfrak{L}_{m,m+1}(t). \tag{6.12.89}$$

Now, consider Theorem 6.10.1. Then, by (6.10.22), the *local time component of the drift process* in the more general setting where triple points may exist, is given by

$$d\mathscr{L}_{\zeta^{(p)}}(t) \;=\; -\sum_{k=1}^{m} \zeta_{(k)}^{(p)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) + \sum_{k=1}^{m} \zeta_{(k)}^{(p)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t). \tag{6.12.90}$$

Hence, by (6.10.20) and (6.10.6) of Theorem 6.10.1, the drift process of the $\mathbf{D}_p$-weighted large-stock index portfolio in this more general setting is given by

$$d\Upsilon_{\zeta^{(p)}}(t) \;=\; (1-p)\gamma^*_{\zeta^{(p)}}(t)\,dt - \sum_{k=1}^{m} \zeta_{(k)}^{(p)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) + \sum_{k=1}^{m} \zeta_{(k)}^{(p)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t). \tag{6.12.91}$$

Furthermore, by (6.9.2), the performance of this portfolio $\zeta^{(p)}$, relative to the market portfolio satisfies, for all $t \in [0,T]$, a.s.,

$$\begin{aligned} d\log\Big(Z_{\zeta^{(p)}}(t)/Z_\mu(t)\Big) \;&=\; d\log\mathfrak{D}_p\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + (1-p)\gamma^*_{\zeta^{(p)}}(t)\,dt \\ &\quad - \sum_{k=1}^{m} \zeta_{(k)}^{(p)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) + \sum_{k=1}^{m} \zeta_{(k)}^{(p)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t) \qquad (6.12.92) \\ &=\; d\log\mathfrak{D}_p\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + (1-p)\gamma^*_{\zeta^{(p)}}(t)\,dt \\ &\quad - \sum_{k=1}^{m} \zeta_{(k)}^{(p)}(t)\,\frac{1}{N_t(k)}\Big[\sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) - \sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t)\Big]. \qquad (6.12.93) \end{aligned}$$

6.12.5 A Portfolio with Fixed Weight Ratios

The weight ratios for the portfolio ψ^{bh} are defined by $\psi^{\mathsf{bh}}_{p_t(k)}(t)/\mu_{(k)}(t)$ for $k = 1,2,\ldots,n$. We say that the weight ratios of ψ^{bh} are *increasing* if for $k = 1,2,\ldots,n-1$,

$$\frac{\psi^{\mathsf{bh}}_{p_t(k)}(t)}{\mu_{(k)}(t)} \;\le\; \frac{\psi^{\mathsf{bh}}_{p_t(k+1)}(t)}{\mu_{(k+1)}(t)}, \qquad t \in [0,T], \qquad \text{a.s.}$$

We have seen that the buy-and-hold portfolio in Chapter 5 holds a fixed number of shares of each stock. In a similar fashion, we can fix the weight ratios of a portfolio, at least up to a common multiple [Fernholz (2002)]. Consider constants $w_1,\ldots,w_n \ge 0$ such that $\sum_{i=1}^n w_i > 0$ and consider the following rank-dependent generating function

$$\mathfrak{G}\big(\mathbf{x}_{(\cdot)}\big) \;\equiv\; \mathfrak{G}^{\mathrm{BH}}\big(\mathbf{x}_{(\cdot)}\big) \;=\; \mathfrak{G}^{\mathrm{BH}}\big(x_{(1)},\ldots,x_{(n)}\big) \;\triangleq\; w_1 x_{(1)} + \cdots + w_n x_{(n)} \;=\; \sum_{k=1}^{n} w_k x_{(k)}, \tag{6.12.94}$$

for all $\mathbf{x}$ and $\mathbf{x}_{(\cdot)}$in the open unit $(n-1)-$simplex Δ^{n-1}, $\mathbf{x}, \mathbf{x}_{(\cdot)} \in \Delta^{n-1}$. Then, for $t \in [0,T]$, the corresponding market process is given by

$$\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \;\equiv\; \mathfrak{G}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \;=\; \mathfrak{G}^{\mathrm{BH}}\big(\mu_{(1)}(t),\ldots,\mu_{(n)}(t)\big) \;\triangleq\; w_1\mu_{(1)}(t) + \cdots + w_n\mu_{(n)}(t) \;=\; \sum_{k=1}^{n} w_k\mu_{(k)}(t). \tag{6.12.95}$$

By analogy with (5.4.17) and (5.4.18), Theorem 6.8.1 implies that the rank-dependent generating function $\mathfrak{G}^{\mathrm{BH}}$ generates a portfolio ψ^{bh} with weights

$$\psi^{\mathsf{bh}}_{p_t(k)}(t) \;=\; \frac{w_k\mu_{(k)}(t)}{\mathfrak{G}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \;=\; \frac{w_k\mu_{(k)}(t)}{\sum_{\ell=1}^n w_\ell\mu_{(\ell)}(t)} \;=\; \frac{w_k\mu_{(k)}(t)}{w_1\mu_{(1)}(t) + \cdots + w_n\mu_{(n)}(t)}, \tag{6.12.96}$$

for $k = 1,2,\ldots,n$, and we have the following definition.

Definition 6.12.5 (Portfolio with Fixed Weight Ratios). *Let μ be the market portfolio. The portfolio process $\psi^{\mathsf{bh}} = \big\{\psi^{\mathsf{bh}}(t) = \big(\psi_1^{\mathsf{bh}}(t), \psi_2^{\mathsf{bh}}(t), \ldots, \psi_n^{\mathsf{bh}}(t)\big), t \in [0,T]\big\}$, with weights*

$$\psi_{(k)}^{\mathsf{bh}}(t) \quad = \quad \psi_{p_t(k)}^{\mathsf{bh}}(t) \quad \triangleq \quad \frac{w_k \mu_{(k)}(t)}{\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}, \tag{6.12.97}$$

for all $k = 1, 2, \ldots, n$, and $t \in [0,T]$, is called the ***buy-and-hold portfolio (process)****, i.e., a portfolio with fixed weight ratios.*

Hence, the weight ratios are proportional to the constants w_k. In addition, by analogy with (5.4.21) or (5.4.22), the *smooth component of the drift process* is given by

$$g^{\mathsf{bh}}(t) \quad = \quad g_{\psi^{\mathsf{bh}}}(t) \quad = \quad 0. \tag{6.12.98}$$

By (6.9.14) and (6.12.97), the *local time component of the drift process* is given by

$$\begin{aligned} d\mathscr{L}^{\mathsf{bh}}(t) \quad = \quad d\mathscr{L}_{\psi^{\mathsf{bh}}}(t) \quad &= \quad \frac{1}{2}\sum_{k=1}^{n-1}\Big(\psi_{p_t(k+1)}^{\mathsf{bh}}(t) - \psi_{p_t(k)}^{\mathsf{bh}}(t)\Big)\, d\mathfrak{L}_{k,k+1}(t) \\ &= \quad \frac{1}{2}\sum_{k=1}^{n-1}\left(\frac{w_{k+1}\mu_{(k+1)}(t)}{\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} - \frac{w_k\mu_{(k)}(t)}{\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\right) d\mathfrak{L}_{k,k+1}(t) \\ &= \quad \frac{1}{2\,\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\sum_{k=1}^{n-1}\big(w_{k+1}\mu_{(k+1)}(t) - w_k\mu_{(k)}(t)\big)\, d\mathfrak{L}_{k,k+1}(t). \end{aligned} \tag{6.12.99}$$

Therefore, by (6.12.30), we have

$$d\mathscr{L}_{\psi^{\mathsf{bh}}}(t) \quad = \quad \frac{1}{2\,\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\sum_{k=1}^{n-1}\big(w_{k+1}\mu_{(k+1)}(t) - w_k\mu_{(k)}(t)\big)\mathbb{1}_{\{0\}}\big(\mu_{(k+1)}(t) - \mu_{(k)}(t)\big)\, d\mathfrak{L}_{k,k+1}(t).$$

So, if $\mu_{(k+1)}(t) \neq \mu_{(k)}(t)$ for all $t \in [0,T]$ and for all $k = 1, 2, \ldots, n-1$, then $\mathbb{1}_{\{0\}}\big(\mu_{(k+1)}(t) - \mu_{(k)}(t)\big) = 0$, and we have

$$d\mathscr{L}_{\psi^{\mathsf{bh}}}(t) \quad = \quad 0. \tag{6.12.100}$$

Hence, by (6.9.12) and (6.8.7), the drift process of a portfolio with fixed weight ratios is given by for $t \in [0,T]$, a.s.,

$$d\Upsilon_{\psi^{\mathsf{bh}}}(t) \quad = \quad g_{\psi^{\mathsf{bh}}}(t)\, dt + d\mathscr{L}_{\psi^{\mathsf{bh}}}(t) \quad = \quad 0. \tag{6.12.101}$$

Furthermore, by (6.9.2) or (6.9.15), the performance of such a portfolio ψ^{bh}, relative to the market portfolio satisfies, for all $t \in [0,T]$, a.s.,

$$d\log\Big(Z_{\psi^{\mathsf{bh}}}(t)/Z_\mu(t)\Big) \quad = \quad d\log\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + d\Upsilon_{\psi^{\mathsf{bh}}}(t) \tag{6.12.102}$$

$$= \quad d\log\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big), \tag{6.12.103}$$

or, by (6.9.8), we have the relative performance of a portfolio with fixed weight ratios for $T \in [0,\infty)$,

$$\log\left(\frac{Z_{\psi^{\mathsf{bh}}}(T)}{Z_\mu(T)}\right) \quad = \quad \log\left(\frac{\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \Upsilon_{\psi^{\mathsf{bh}}}(T) \tag{6.12.104}$$

$$= \quad \log\left(\frac{\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right), \tag{6.12.105}$$

if $Z_{\psi^{\mathsf{bh}}}(0) = Z_\mu(0)$, otherwise the integral representation is given by

$$\log\left(\frac{Z_{\psi^{\mathsf{bh}}}(T)}{Z_\mu(T)}\right) \quad = \quad \log\left(\frac{Z_{\psi^{\mathsf{bh}}}(0)}{Z_\mu(0)}\right) + \log\left(\frac{\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right), \tag{6.12.106}$$

or, as per equation (6.9.6), we have

$$\log\left(\frac{Z_{\psi^{\mathsf{bh}}}(T)}{Z_{\psi^{\mathsf{bh}}}(0)}\right) = \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) + \log\left(\frac{\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right). \tag{6.12.107}$$

Clearly, if $\mu_{(k+1)}(t) = \mu_{(k)}(t)$ for all $k = 1, 2, \ldots, n-1$, then we have

$$d\mathscr{L}_{\psi^{\mathsf{bh}}}(t) = \frac{1}{2\,\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \sum_{k=1}^{n-1} (w_{k+1} - w_k)\, \mu_{(k)}(t)\, d\mathfrak{L}_{k,k+1}(t). \tag{6.12.108}$$

Hence, by (6.9.12) and (6.8.7), the drift process of a portfolio with fixed weight ratios is given by for $t \in [0, T]$, a.s.,

$$d\Upsilon_{\psi^{\mathsf{bh}}}(t) = g_{\psi^{\mathsf{bh}}}(t)\, dt + d\mathscr{L}_{\psi^{\mathsf{bh}}}(t) = \frac{1}{2\,\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \sum_{k=1}^{n-1} (w_{k+1} - w_k)\, \mu_{(k)}(t)\, d\mathfrak{L}_{k,k+1}(t). \tag{6.12.109}$$

Furthermore, by (6.9.2) or (6.9.15), the performance of such a portfolio ψ^{bh}, relative to the market portfolio satisfies, for all $t \in [0, T]$, a.s.,

$$d\log\Big(Z_{\psi^{\mathsf{bh}}}(t)/Z_{\mu}(t)\Big) = d\log\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + d\Upsilon_{\psi^{\mathsf{bh}}}(t) \tag{6.12.110}$$

$$= d\log\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + \frac{1}{2\,\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \sum_{k=1}^{n-1} (w_{k+1} - w_k)\, \mu_{(k)}(t)\, d\mathfrak{L}_{k,k+1}(t), \tag{6.12.111}$$

or, by (6.9.8), we have the relative performance of a portfolio with fixed weight ratios for $T \in [0, \infty)$,

$$\log\left(\frac{Z_{\psi^{\mathsf{bh}}}(T)}{Z_{\mu}(T)}\right) = \log\left(\frac{\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \Upsilon_{\psi^{\mathsf{bh}}}(T) \tag{6.12.112}$$

$$= \log\left(\frac{\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \frac{1}{2}\sum_{k=1}^{n-1} (w_{k+1} - w_k) \int_0^T \frac{\mu_{(k)}(t)}{\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\, d\mathfrak{L}_{k,k+1}(t), \tag{6.12.113}$$

if $Z_{\psi^{\mathsf{bh}}}(0) = Z_{\mu}(0)$, otherwise the integral representation is given by

$$\log\left(\frac{Z_{\psi^{\mathsf{bh}}}(T)}{Z_{\mu}(T)}\right) = \log\left(\frac{Z_{\psi^{\mathsf{bh}}}(0)}{Z_{\mu}(0)}\right) + \log\left(\frac{\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \frac{1}{2}\sum_{k=1}^{n-1} (w_{k+1} - w_k) \int_0^T \frac{\mu_{(k)}(t)}{\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\, d\mathfrak{L}_{k,k+1}(t), \tag{6.12.114}$$

or, as per equation (6.9.6), we have

$$\log\left(\frac{Z_{\psi^{\mathsf{bh}}}(T)}{Z_{\psi^{\mathsf{bh}}}(0)}\right) = \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) + \log\left(\frac{\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \frac{1}{2}\sum_{k=1}^{n-1} (w_{k+1} - w_k) \int_0^T \frac{\mu_{(k)}(t)}{\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\, d\mathfrak{L}_{k,k+1}(t). \tag{6.12.115}$$

The weight ratios for a portfolio of this type are given by

$$\frac{\psi^{\mathsf{bh}}_{p_t(k)}(t)}{\mu_{(k)}(t)} = \frac{w_k}{\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}.$$

Thus, if the weight ratios of such a portfolio are increasing, then $w_{k+1} \geq w_k$, and from (6.12.113) this implies that the drift process for the portfolio will be increasing. The weight ratios of a portfolio can be interpreted as the (fractional) number of shares of each corresponding stock held in the portfolio. Since the fractional number of shares of each stock can be calculated for any portfolio, then $d\log\mathfrak{S}^{\mathrm{BH}}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)$ can also be calculated for any portfolio. This term measures the component of relative portfolio return due to changes in the capital distribution curve, namely the *distributional component* of the relative portfolio return. This example was applied in Fernholz (2001b) to determine the effect of company size on portfolio return.

6.13 Applications of Rank-Based Portfolio Generating Functions and Rank-Dependent Functionally Generated Portfolios

There are both theoretical and practical applications in examining the portfolios generated by functions of the ranked market weight processes. In this section, we shall consider the theoretical application of the size effect in equity markets, i.e., the propensity of small stocks to perform well relative to large stocks, and attempt to explain the cause of this effect by simply making an appeal to Theorem 6.8.1. Leakage in diversity-weighted equity indices is the practical application that we shall investigate. Leakage pertains to the effect of crossovers in equity markets, i.e., when the smaller stocks in a large-stock capitalisation-weighted index portfolio cross over to the remaining market portfolio as the market weights are too small to stay in the large-stock index, and these stocks are said to "leak" out of the large-stock index.

6.13.1 The Size Effect

The size effect is defined by Fernholz (2002) as the observed tendency of small stocks to have higher long-term returns than those of the larger stocks. The size effect was first given consideration by Banz (1981) and Reinganum (1981a,b). Banz (1981) and Reinganum (1981a,b) observed that in the U.S. stock market, smaller stocks tend to have higher returns on average than that of larger stocks, even when an adjustment is made to account for risk. This phenomenon has been coined the size effect. Thereafter, there has been a plethora of literature that has attempted to justify and explain the size effect in several different equity markets [see e.g., Roll (1981)]. Employing Theorem 6.8.1, an explanation of the size effect is provided in Fernholz (2002). The higher risk associated with the smaller stocks is sometimes provided as a justification for the size effect, however, Fernholz points out that the explanation comes closer to the alternative hypothesis put forth in Fernholz (1998c).

The explanation of the size effect as proposed by Fernholz (1998c, 2002) is given as follows. We shall compare the performance of the small-stock index relative to the large-stock index. Combining equations (6.12.33) for the large-stock portfolio and (6.12.60) for the small-stock portfolio, for $t \in [0, T]$, a.s., we obtain the relative return of the small-stock index versus the large-stock index as

$$d\log\Big(Z_\eta(t)/Z_\zeta(t)\Big) = d\log\Big(\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)/\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\Big) + \frac{1}{2}\Big(\zeta_{(m)}(t) + \eta_{(m+1)}(t)\Big)\, d\mathfrak{L}_{m,m+1}(t) \tag{6.13.1}$$

$$= d\log\left(\frac{\mu_{(m+1)}(t) + \cdots + \mu_{(n)}(t)}{\mu_{(1)}(t) + \cdots + \mu_{(m)}(t)}\right) + \frac{1}{2}\Big(\zeta_{(m)}(t) + \eta_{(m+1)}(t)\Big)\, d\mathfrak{L}_{m,m+1}(t), \tag{6.13.2}$$

and, furthermore as per Fernholz & Karatzas (2009), in integral form, we have for $T \in [0, \infty)$,

$$\log\left(\frac{Z_\eta(T)}{Z_\zeta(T)}\right) = \log\left(\frac{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \frac{1}{2}\int_0^T \Big(\zeta_{(m)}(t) + \eta_{(m+1)}(t)\Big)\, d\mathfrak{L}_{m,m+1}(t) \tag{6.13.3}$$

$$= \log\left(\frac{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)/\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)/\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \frac{1}{2}\int_0^T \Big(\zeta_{(m)}(t) + \eta_{(m+1)}(t)\Big)\, d\mathfrak{L}_{m,m+1}(t) \tag{6.13.4}$$

$$= \log\left(\frac{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)/\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)/\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \frac{1}{2}\int_0^T \left(\frac{\mu_{(m)}(t)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} + \frac{\mu_{(m+1)}(t)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\right) d\mathfrak{L}_{m,m+1}(t). \tag{6.13.5}$$

If there are no triple points, it follows from equations (6.12.46) and (6.12.73), that

$$\begin{aligned}
d\log\Big(Z_\eta(t)/Z_\zeta(t)\Big) &= d\log\Big(\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)/\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\Big) \\
&\quad - \sum_{k=m+1}^{n} \eta_{(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) + \sum_{k=m+1}^{n} \eta_{(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t) \qquad (6.13.6)\\
&\quad + \sum_{k=1}^{m} \zeta_{(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) - \sum_{k=1}^{m} \zeta_{(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t) \\
&= d\log\Big(\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)/\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\Big) \\
&\quad - \sum_{k=m+1}^{n} \eta_{(k)}(t)\,\frac{1}{N_t(k)}\Big[\sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) - \sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t)\Big] \qquad (6.13.7)\\
&\quad + \sum_{k=1}^{m} \zeta_{(k)}(t)\,\frac{1}{N_t(k)}\Big[\sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) - \sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t)\Big] \\
&= d\log\Big(\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)/\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\Big) \\
&\quad - \sum_{k=m+1}^{n} \frac{\mu_{(k)}(t)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\,\frac{1}{N_t(k)}\Big[\sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) - \sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t)\Big] \qquad (6.13.8)\\
&\quad + \sum_{k=1}^{m} \frac{\mu_{(k)}(t)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\,\frac{1}{N_t(k)}\Big[\sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) - \sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t)\Big] \\
&= d\log\Big(\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)/\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\Big) \\
&\quad - \frac{1}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\sum_{k=m+1}^{n} \mu_{(k)}(t)\,\frac{1}{N_t(k)}\Big[\sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) - \sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t)\Big] \qquad (6.13.9)\\
&\quad + \frac{1}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\sum_{k=1}^{m} \mu_{(k)}(t)\,\frac{1}{N_t(k)}\Big[\sum_{\ell=k+1}^{n} d\mathfrak{L}_{k,\ell}(t) - \sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t)\Big].
\end{aligned}$$

From equation (6.13.1), we obtain a representation of the relative return of a small-stock index versus a large-stock index which decomposes the performance of the small-stock index relative to the large-stock index into two components: the change in the relative capitalisations of the two equity indices and a drift process. The drift component reduces to $\frac{1}{2}\left(\zeta_{(m)}(t) + \eta_{(m+1)}(t)\right) d\mathfrak{L}_{m,m+1}(t)$. Thus, the drift process is monotonically increasing. It can be argued that general market stability imposes bounds on the variation of the relative capitalisations of the small-stock index and the large-stock index. Consequently, if the ratio of the relative capitalisation of the smaller stocks to that of the larger stocks remains stable over time, thereby inducing a form of stability in the equity market, then it does not change much. In this case, over a sufficiently long period of time, the relative return will be dominated by the drift process, i.e., by the local time term. It is, therefore, the increasing nature of the drift process that likely results in the return on the small-stock index to gradually, over time, be higher than that of the large-stock index. Hence, the small stocks generate higher long-term returns than the larger stocks, and in so doing, the small-stock index will outperform the large-stock index over the long term. Note that this argument does not invoke at all any putative assumption about the greater riskiness of the smaller stocks. The size effect can thus be attributed to the long-term effect that the increasing drift term has on the performance of the small-stock index relative to the large-stock index, and the associated risk level of the small-stock index is not taken into consideration in the analysis. This phenomenon is structural, and will occur regardless of whether or not small stocks are more riskier than large stocks. Hence, it would appear that the relative level of small-stock risk is irrelevant, when providing reasoning for the size effect.

6.13.2 Leakage in a $\mathbf{D}_p$-Weighted (Diversity-Weighted) Large-Stock Index Portfolio

Let ζ be the large-stock index generated by $\mathcal{G}_L$ and let $\zeta^{(p)}$ be the portfolio of stocks held in ζ generated by the function $\mathfrak{D}_p$. A calculation of the performance of the $\mathbf{D}_p$-weighted (diversity-weighted) large-stock index (i.e., the diversity-weighted large-capitalisation portfolio) $\zeta^{(p)}$ relative to the large-stock index ζ is what we desire, rather than the performance relative to the entire market. Thus, equation (6.12.33) together with equation (6.12.85), for $t \in [0, T]$, a.s., yields the relative return of the $\mathbf{D}_p$-weighted large-stock index versus the large-stock index as

$$\begin{aligned} d\log\Big(Z_{\zeta^{(p)}}(t)/Z_\zeta(t)\Big) &= d\log\Big(\mathfrak{D}_p\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)/\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\Big) + (1-p)\gamma^*_{\zeta^{(p)}}(t)\,dt \\ &\quad + \frac{1}{2}\Big(\zeta_{(m)}(t) - \zeta^{(p)}_{(m)}(t)\Big)\,d\mathfrak{L}_{m,m+1}(t), \end{aligned} \tag{6.13.10}$$

and, furthermore by equations (6.12.37) and (6.12.87), in integral form, we have for $T \in [0, \infty)$,

$$\begin{aligned} \log\left(\frac{Z_{\zeta^{(p)}}(T)}{Z_\zeta(T)}\right) &= \log\left(\frac{\mathfrak{D}_p\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)\mathfrak{D}_p\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + (1-p)\int_0^T \gamma^*_{\zeta^{(p)}}(t)\,dt \\ &\quad + \frac{1}{2}\int_0^T \Big(\zeta_{(m)}(t) - \zeta^{(p)}_{(m)}(t)\Big)\,d\mathfrak{L}_{m,m+1}(t) \end{aligned} \tag{6.13.11}$$

$$\begin{aligned} &= \log\left(\frac{\mathfrak{D}_p\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)/\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{D}_p\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)/\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + (1-p)\int_0^T \gamma^*_{\zeta^{(p)}}(t)\,dt \\ &\quad + \frac{1}{2}\int_0^T \Big(\zeta_{(m)}(t) - \zeta^{(p)}_{(m)}(t)\Big)\,d\mathfrak{L}_{m,m+1}(t). \end{aligned} \tag{6.13.12}$$

Making use of the scale invariance property of the $\mathbf{D}_p$ diversity function given in equation (4.6.31), implies that for the positive numbers $\mu_{(1)}(t), \ldots, \mu_{(m)}(t)$, that do not necessarily add up to one, we have

$$\frac{\mathbf{D}_p\big(\mu_{(1)}(t), \ldots, \mu_{(m)}(t)\big)}{\mu_{(1)}(t) + \cdots + \mu_{(m)}(t)} = \mathbf{D}_p\left(\frac{\mu_{(1)}(t)}{\mu_{(1)}(t) + \cdots + \mu_{(m)}(t)}, \ldots, \frac{\mu_{(m)}(t)}{\mu_{(1)}(t) + \cdots + \mu_{(m)}(t)}\right). \tag{6.13.13}$$

Therefore, by (6.12.77), (6.12.22) and (6.12.26) of Definition 6.12.2, the above scale invariance property implies the reduction

$$\frac{\mathfrak{D}_p\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} = \mathbf{D}_p\big(\zeta_{(1)}(t), \ldots, \zeta_{(m)}(t)\big) = \mathbf{D}_p\big(\boldsymbol{\zeta}_{(\cdot)}(t)\big), \tag{6.13.14}$$

which could also be written as

$$\frac{\mathfrak{D}_p\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} = \mathfrak{D}_p\big(\zeta_{(1)}(t), \ldots, \zeta_{(m)}(t)\big) = \mathfrak{D}_p\big(\boldsymbol{\zeta}_{(\cdot)}(t)\big). \tag{6.13.15}$$

Hence, employing (6.13.14), we can rewrite (6.13.10) as

$$\begin{aligned} d\log\Big(Z_{\zeta^{(p)}}(t)/Z_\zeta(t)\Big) &= d\log\mathbf{D}_p\big(\zeta_{(1)}(t), \ldots, \zeta_{(m)}(t)\big) + (1-p)\gamma^*_{\zeta^{(p)}}(t)\,dt \\ &\quad + \frac{1}{2}\Big(\zeta_{(m)}(t) - \zeta^{(p)}_{(m)}(t)\Big)\,d\mathfrak{L}_{m,m+1}(t) \end{aligned} \tag{6.13.16}$$

$$\begin{aligned} &= d\log\mathfrak{D}_p\big(\boldsymbol{\zeta}_{(\cdot)}(t)\big) + (1-p)\gamma^*_{\zeta^{(p)}}(t)\,dt \\ &\quad + \frac{1}{2}\Big(\zeta_{(m)}(t) - \zeta^{(p)}_{(m)}(t)\Big)\,d\mathfrak{L}_{m,m+1}(t), \end{aligned} \tag{6.13.17}$$

or, similarly, (6.13.11) has the following form for $T \in [0, \infty)$,

$$\begin{aligned}\log\left(\frac{Z_{\zeta^{(p)}}(T)}{Z_\zeta(T)}\right) &= \log\left(\frac{\mathbf{D}_p\big(\zeta_{(1)}(T), \ldots, \zeta_{(m)}(T)\big)}{\mathbf{D}_p\big(\zeta_{(1)}(0), \ldots, \zeta_{(m)}(0)\big)}\right) + (1-p)\int_0^T \gamma^*_{\zeta^{(p)}}(t)\,dt \\ &\quad + \frac{1}{2}\int_0^T \Big(\zeta_{(m)}(t) - \zeta^{(p)}_{(m)}(t)\Big)\,d\mathfrak{L}_{m,m+1}(t). \end{aligned} \tag{6.13.18}$$

$$\begin{aligned} &= \log\left(\frac{\mathfrak{D}_p\big(\boldsymbol{\zeta}_{(\cdot)}(T)\big)}{\mathfrak{D}_p\big(\boldsymbol{\zeta}_{(\cdot)}(0)\big)}\right) + (1-p)\int_0^T \gamma^*_{\zeta^{(p)}}(t)\,dt \\ &\quad + \frac{1}{2}\int_0^T \Big(\zeta_{(m)}(t) - \zeta^{(p)}_{(m)}(t)\Big)\,d\mathfrak{L}_{m,m+1}(t). \end{aligned} \tag{6.13.19}$$

If there are no triple points, it follows from equations (6.12.46) and (6.12.92), that

$$\begin{aligned} d\log\Big(Z_{\zeta^{(p)}}(t)/Z_\zeta(t)\Big) &= d\log\Big(\mathfrak{D}_p\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)/\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\Big) + (1-p)\gamma^*_{\zeta^{(p)}}(t)\,dt \\ &\quad - \sum_{k=1}^m \zeta^{(p)}_{(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=k+1}^n d\mathfrak{L}_{k,\ell}(t) + \sum_{k=1}^m \zeta^{(p)}_{(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t) \\ &\quad + \sum_{k=1}^m \zeta_{(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=k+1}^n d\mathfrak{L}_{k,\ell}(t) - \sum_{k=1}^m \zeta_{(k)}(t)\,\frac{1}{N_t(k)}\sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t) \end{aligned} \tag{6.13.20}$$

$$\begin{aligned} &= d\log\Big(\mathfrak{D}_p\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)/\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\Big) + (1-p)\gamma^*_{\zeta^{(p)}}(t)\,dt \\ &\quad - \sum_{k=1}^m \zeta^{(p)}_{(k)}(t)\,\frac{1}{N_t(k)}\Big[\sum_{\ell=k+1}^n d\mathfrak{L}_{k,\ell}(t) - \sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t)\Big] \\ &\quad + \sum_{k=1}^m \zeta_{(k)}(t)\,\frac{1}{N_t(k)}\Big[\sum_{\ell=k+1}^n d\mathfrak{L}_{k,\ell}(t) - \sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t)\Big] \end{aligned} \tag{6.13.21}$$

$$\begin{aligned} &= d\log \mathbf{D}_p\big(\zeta_{(1)}(t), \ldots, \zeta_{(m)}(t)\big) + (1-p)\gamma^*_{\zeta^{(p)}}(t)\,dt \\ &\quad - \sum_{k=1}^m \zeta^{(p)}_{(k)}(t)\,\frac{1}{N_t(k)}\Big[\sum_{\ell=k+1}^n d\mathfrak{L}_{k,\ell}(t) - \sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t)\Big] \\ &\quad + \sum_{k=1}^m \zeta_{(k)}(t)\,\frac{1}{N_t(k)}\Big[\sum_{\ell=k+1}^n d\mathfrak{L}_{k,\ell}(t) - \sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t)\Big] \end{aligned} \tag{6.13.22}$$

$$\begin{aligned} &= d\log \mathfrak{D}_p\big(\boldsymbol{\zeta}_{(\cdot)}(t)\big) + (1-p)\gamma^*_{\zeta^{(p)}}(t)\,dt \\ &\quad - \sum_{k=1}^m \zeta^{(p)}_{(k)}(t)\,\frac{1}{N_t(k)}\Big[\sum_{\ell=k+1}^n d\mathfrak{L}_{k,\ell}(t) - \sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t)\Big] \\ &\quad + \sum_{k=1}^m \zeta_{(k)}(t)\,\frac{1}{N_t(k)}\Big[\sum_{\ell=k+1}^n d\mathfrak{L}_{k,\ell}(t) - \sum_{\ell=1}^{k-1} d\mathfrak{L}_{\ell,k}(t)\Big]. \end{aligned} \tag{6.13.23}$$

From equation (6.13.16), we obtain a representation of the relative return of the $\mathbf{D}_p$-weighted large-stock index versus a large-stock index which decomposes the performance of the $\mathbf{D}_p$-weighted large-stock index relative to the large-stock index into three components: the change in the generating function $\log \mathbf{D}_p\big(\boldsymbol{\zeta}_{(\cdot)}(t)\big)$, the smooth component of the drift process and a term involving local times, i.e., the local time component of the drift process. The smooth component of the drift term is given by $(1-p)\gamma^*_{\zeta^{(p)}}(t)\,dt$. Since $0 < p < 1$, the smooth component of the drift term is monotonically increasing. If the equity market exhibits some form of stability over the long time horizon, then that would suggest that $d\log \mathbf{D}_p\big(\boldsymbol{\zeta}_{(\cdot)}(t)\big)$ will be stable and mean-reverting over the long term, and will not change by much over the long term. If this is indeed the case, then the long-term relative return of the $\mathbf{D}_p$-weighted large-stock index will be dominated by the last two terms in the two equations directly above. The last term of the drift process, involving the local times, is called "leakage". In a heuristic fashion, when a capitalisation-weighted portfolio (such as the large-stock index) is contained in a larger market

portfolio, Fernholz (2002) explains that leakage refers to the effect caused by stocks that cross over from the capitalisation-weighted portfolio to the rest of the market portfolio. Thus, leakage measures the effect on the relative return of those stocks whose market value or weighting becomes too small that they are as a result dropped from the large-stock index, i.e., they essentially "leak" out of the large-stock portfolio. Moreover, since $0 < p < 1$, it follows that $\zeta_{(m)}^{(p)}(t) > \zeta_{(m)}(t)$ which implies that leakage is monotonically decreasing. Consequently, the combined effects of the increasing smooth component of drift and the decreasing leakage determines whether overall the drift process is increasing or decreasing. This application of leakage in a diversity-weighted portfolio was discussed in Fernholz, Garvy & Hannon (1998), but a detailed account as is presented here was first carried out by Fernholz (2001c).

6.14 An Extension of Portfolio Generating Functions

Theorem 5.2.2 for portfolio generating functions can be extended in other directions other than the extension employed to cover functions of the ranked market weights. Here we consider an extension to the Gini coefficient (4.6.60) that was introduced in Chapter 4.

6.14.1 The Gini-Coefficient-Weighted Portfolio

Consider the Gini function introduced in (4.6.60),

$$\mathbb{G}(\mathbf{x}) \;=\; \mathbf{S}^{\mathbb{G}}(\mathbf{x}) \;=\; 1 - \frac{1}{2}\sum_{i=1}^{n} \left|x_i - n^{-1}\right|.$$

The corresponding market process is given by

$$\mathbb{G}\big(\boldsymbol{\mu}(t)\big) \;=\; \mathbf{S}^{\mathbb{G}}\big(\boldsymbol{\mu}(t)\big) \;=\; 1 - \frac{1}{2}\sum_{i=1}^{n} \left|\,\mu_i(t) - n^{-1}\right|. \tag{6.14.1}$$

From (6.5.13) a result of (6.3.145) of Definition 6.3.20, we have a.s., for $t \in [0, T]$,

$$d\left|\,\mu_i(t) - n^{-1}\right| \;=\; \operatorname{sgn}\big(\mu_i(t) - n^{-1}\big)\, d\mu_i(t) + 2\, d\Lambda_{\mu_i - n^{-1}}(t),$$

for $i = 1, 2, \ldots, n$. Hence, a.s., for $t \in [0, T]$,

$$\begin{aligned} d\mathbb{G}\big(\boldsymbol{\mu}(t)\big) &= -\frac{1}{2}\sum_{i=1}^{n} d\left|\,\mu_i(t) - n^{-1}\right| \\ &= -\frac{1}{2}\sum_{i=1}^{n} \operatorname{sgn}\big(\mu_i(t) - n^{-1}\big)\, d\mu_i(t) - \sum_{i=1}^{n} d\Lambda_{\mu_i - n^{-1}}(t) \\ &= \frac{1}{2}\sum_{i=1}^{n} \operatorname{sgn}\big(n^{-1} - \mu_i(t)\big)\, d\mu_i(t) - \sum_{i=1}^{n} d\Lambda_{\mu_i - n^{-1}}(t). \end{aligned}$$

Therefore,

$$\begin{aligned} d\left\langle \mathbb{G}(\boldsymbol{\mu})\right\rangle_t &= \frac{1}{4}\sum_{i,j=1}^{n} \operatorname{sgn}\big(n^{-1} - \mu_i(t)\big)\operatorname{sgn}\big(n^{-1} - \mu_j(t)\big)\, d\left\langle \mu_i, \mu_j\right\rangle_t \\ &= \frac{1}{4}\sum_{i,j=1}^{n} \operatorname{sgn}\big(n^{-1} - \mu_i(t)\big)\operatorname{sgn}\big(n^{-1} - \mu_j(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\, dt, \end{aligned}$$

since $d\langle \mu_i, \mu_j \rangle_t = \mu_i(t)\mu_j(t)\,\tau_{ij}(t)\,dt$. Now, by Itô's formula, a.s., for $t \in [0,T]$,

$$\begin{aligned} d\log \mathbb{G}\big(\boldsymbol{\mu}(t)\big) &= \frac{d\mathbb{G}\big(\boldsymbol{\mu}(t)\big)}{\mathbb{G}\big(\boldsymbol{\mu}(t)\big)} - \frac{d\langle \mathbb{G}(\boldsymbol{\mu})\rangle_t}{2\,\big(\mathbb{G}\big(\boldsymbol{\mu}(t)\big)\big)^2} \\ &= \frac{1}{2\,\mathbb{G}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^n \operatorname{sgn}\big(n^{-1} - \mu_i(t)\big)\,d\mu_i(t) - \frac{1}{\mathbb{G}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^n d\Lambda_{\mu_i - n^{-1}}(t) \\ &\quad - \frac{1}{8\,\big(\mathbb{G}\big(\boldsymbol{\mu}(t)\big)\big)^2} \sum_{i,j=1}^n \operatorname{sgn}\big(n^{-1} - \mu_i(t)\big)\operatorname{sgn}\big(n^{-1} - \mu_j(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt. \end{aligned} \tag{6.14.2}$$

From (5.2.17), for the function $\mathbb{G}$, we shall analogously define

$$\phi\big(\boldsymbol{\mu}(t)\big) \triangleq 1 - \sum_{j=1}^n \mu_j(t)\,D_j \log \mathbb{G}\big(\boldsymbol{\mu}(t)\big), \tag{6.14.3}$$

where

$$D_i\mathbb{G}\big(\boldsymbol{\mu}(t)\big) = \frac{\partial \mathbb{G}\big(\boldsymbol{\mu}(t)\big)}{\partial \mu_i(t)} = \frac{1}{2}\operatorname{sgn}\big(n^{-1} - \mu_i(t)\big),$$

so that, for $i = 1, 2, \ldots, n$, we have

$$D_i \log \mathbb{G}\big(\boldsymbol{\mu}(t)\big) = \frac{D_i\mathbb{G}\big(\boldsymbol{\mu}(t)\big)}{\mathbb{G}\big(\boldsymbol{\mu}(t)\big)} = \frac{\operatorname{sgn}\big(n^{-1} - \mu_i(t)\big)}{2\,\mathbb{G}\big(\boldsymbol{\mu}(t)\big)}. \tag{6.14.4}$$

Employing the above result in (6.14.3) yields

$$\phi\big(\boldsymbol{\mu}(t)\big) = 1 - \frac{1}{2\,\mathbb{G}\big(\boldsymbol{\mu}(t)\big)} \sum_{j=1}^n \mu_j(t)\operatorname{sgn}\big(n^{-1} - \mu_j(t)\big), \qquad t \in [0,T]. \tag{6.14.5}$$

Then, analogous to (5.2.18), for $t \in [0,T]$, define the weights of the portfolio $\psi^{\mathfrak{g}}$ by

$$\psi_i^{\mathfrak{g}}(t) = \Big(D_i \log \mathbb{G}\big(\boldsymbol{\mu}(t)\big) + \phi\big(\boldsymbol{\mu}(t)\big)\Big)\,\mu_i(t) = \left(\frac{\operatorname{sgn}\big(n^{-1} - \mu_i(t)\big)}{2\,\mathbb{G}\big(\boldsymbol{\mu}(t)\big)} + \phi\big(\boldsymbol{\mu}(t)\big)\right)\mu_i(t). \tag{6.14.6}$$

The value process $Z_{\psi^{\mathfrak{g}}}$ satisfies, a.s., for $t \in [0,T]$,

$$d\log\Big(Z_{\psi^{\mathfrak{g}}}(t)/Z_\mu(t)\Big) = \sum_{i=1}^n \frac{\psi_i^{\mathfrak{g}}(t)}{\mu_i(t)}\,d\mu_i(t) - \frac{1}{2}\sum_{i,j=1}^n \psi_i^{\mathfrak{g}}(t)\psi_j^{\mathfrak{g}}(t)\tau_{ij}(t)\,dt, \tag{6.14.7}$$

as in (5.2.27). It follows from (6.14.6), that the first term on the right-hand side of the preceding equation is

$$\begin{aligned} \sum_{i=1}^n \frac{\psi_i^{\mathfrak{g}}(t)}{\mu_i(t)}\,d\mu_i(t) &= \sum_{i=1}^n \left(\frac{\operatorname{sgn}\big(n^{-1} - \mu_i(t)\big)}{2\,\mathbb{G}\big(\boldsymbol{\mu}(t)\big)} + \phi\big(\boldsymbol{\mu}(t)\big)\right) d\mu_i(t) \\ &= \frac{1}{2\,\mathbb{G}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^n \operatorname{sgn}\big(n^{-1} - \mu_i(t)\big)\,d\mu_i(t) + \phi\big(\boldsymbol{\mu}(t)\big)\sum_{i=1}^n d\mu_i(t) \\ &= \frac{1}{2\,\mathbb{G}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^n \operatorname{sgn}\big(n^{-1} - \mu_i(t)\big)\,d\mu_i(t) + \phi\big(\boldsymbol{\mu}(t)\big)\,d\left(\sum_{i=1}^n \mu_i(t)\right) \\ &= \frac{1}{2\,\mathbb{G}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^n \operatorname{sgn}\big(n^{-1} - \mu_i(t)\big)\,d\mu_i(t), \end{aligned} \tag{6.14.8}$$

since $\sum_{i=1}^n d\mu_i(t) = d\big(\sum_{i=1}^n \mu_i(t)\big) = 0$. This result also follows from (5.2.29) in conjunction with (6.14.4),

$$\sum_{i=1}^n \frac{\psi_i^{\mathfrak{g}}(t)}{\mu_i(t)}\,d\mu_i(t) = \sum_{i=1}^n D_i \log \mathbb{G}\big(\boldsymbol{\mu}(t)\big)\,d\mu_i(t) = \frac{1}{2\,\mathbb{G}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^n \operatorname{sgn}\big(n^{-1} - \mu_i(t)\big)\,d\mu_i(t).$$

By (6.14.6), the second term on the right-hand side involves

$$\begin{aligned}
\sum_{i,j=1}^n \psi_i^{\mathfrak{g}}(t)\psi_j^{\mathfrak{g}}(t)\tau_{ij}(t) &= \sum_{i,j=1}^n \left[\frac{\operatorname{sgn}\big(n^{-1}-\mu_i(t)\big)}{2\,\mathbb{G}\big(\boldsymbol{\mu}(t)\big)} + \phi\big(\boldsymbol{\mu}(t)\big)\right]\left[\frac{\operatorname{sgn}\big(n^{-1}-\mu_j(t)\big)}{2\,\mathbb{G}\big(\boldsymbol{\mu}(t)\big)} + \phi\big(\boldsymbol{\mu}(t)\big)\right]\mu_i(t)\mu_j(t)\tau_{ij}(t) \\
&= \frac{1}{4\left(\mathbb{G}\big(\boldsymbol{\mu}(t)\big)\right)^2}\sum_{i,j=1}^n \operatorname{sgn}\big(n^{-1}-\mu_i(t)\big)\operatorname{sgn}\big(n^{-1}-\mu_j(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t) \\
&\quad + \frac{\phi\big(\boldsymbol{\mu}(t)\big)}{\mathbb{G}\big(\boldsymbol{\mu}(t)\big)}\sum_{i,j=1}^n \operatorname{sgn}\big(n^{-1}-\mu_i(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t) \\
&\quad + \phi^2\big(\boldsymbol{\mu}(t)\big)\sum_{i,j=1}^n \mu_i(t)\mu_j(t)\tau_{ij}(t) \\
&= \frac{1}{4\left(\mathbb{G}\big(\boldsymbol{\mu}(t)\big)\right)^2}\sum_{i,j=1}^n \operatorname{sgn}\big(n^{-1}-\mu_i(t)\big)\operatorname{sgn}\big(n^{-1}-\mu_j(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t) \\
&\quad + \frac{\phi\big(\boldsymbol{\mu}(t)\big)}{\mathbb{G}\big(\boldsymbol{\mu}(t)\big)}\sum_{i=1}^n \operatorname{sgn}\big(n^{-1}-\mu_i(t)\big)\mu_i(t)\left[\sum_{j=1}^n \mu_j(t)\tau_{ij}(t)\right] \\
&\quad + \phi^2\big(\boldsymbol{\mu}(t)\big)\sum_{i,j=1}^n \mu_i(t)\mu_j(t)\tau_{ij}(t).
\end{aligned}$$

Thus, by (2.12.54) of Lemma 2.12.4 and (2.12.56) (since $\boldsymbol{\mu}(t)$ is in the null space of $\boldsymbol{\tau}(t)$), the last two summations vanish and we have a.s., for $t \in [0,T]$,

$$\sum_{i,j=1}^n \psi_i^{\mathfrak{g}}(t)\psi_j^{\mathfrak{g}}(t)\tau_{ij}(t) = \frac{1}{4\left(\mathbb{G}\big(\boldsymbol{\mu}(t)\big)\right)^2}\sum_{i,j=1}^n \operatorname{sgn}\big(n^{-1}-\mu_i(t)\big)\operatorname{sgn}\big(n^{-1}-\mu_j(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t). \tag{6.14.9}$$

This result also follows from (5.2.31) in conjunction with (6.14.4),

$$\begin{aligned}
\sum_{i,j=1}^n \psi_i^{\mathfrak{g}}(t)\psi_j^{\mathfrak{g}}(t)\tau_{ij}(t) &= \sum_{i,j=1}^n D_i \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\, D_j \log \mathbf{G}\big(\boldsymbol{\mu}(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t) \\
&= \frac{1}{4\left(\mathbb{G}\big(\boldsymbol{\mu}(t)\big)\right)^2}\sum_{i,j=1}^n \operatorname{sgn}\big(n^{-1}-\mu_i(t)\big)\operatorname{sgn}\big(n^{-1}-\mu_j(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t).
\end{aligned}$$

Hence (6.14.8) and (6.14.9) inserted into (6.14.7) gives, a.s., for $t \in [0,T]$,

$$\begin{aligned}
d\log\Big(Z_{\psi^{\mathfrak{g}}}(t)/Z_\mu(t)\Big) &= \frac{1}{2\,\mathbb{G}\big(\boldsymbol{\mu}(t)\big)}\sum_{i=1}^n \operatorname{sgn}\big(n^{-1}-\mu_i(t)\big)\,d\mu_i(t) \\
&\quad - \frac{1}{8\left(\mathbb{G}\big(\boldsymbol{\mu}(t)\big)\right)^2}\sum_{i,j=1}^n \operatorname{sgn}\big(n^{-1}-\mu_i(t)\big)\operatorname{sgn}\big(n^{-1}-\mu_j(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt.
\end{aligned}$$

Comparing this last expression with (6.14.2)

$$\begin{aligned}
d\log \mathbb{G}\big(\boldsymbol{\mu}(t)\big) &= \frac{1}{2\,\mathbb{G}\big(\boldsymbol{\mu}(t)\big)}\sum_{i=1}^n \operatorname{sgn}\big(n^{-1}-\mu_i(t)\big)\,d\mu_i(t) \\
&\quad - \frac{1}{8\left(\mathbb{G}\big(\boldsymbol{\mu}(t)\big)\right)^2}\sum_{i,j=1}^n \operatorname{sgn}\big(n^{-1}-\mu_i(t)\big)\operatorname{sgn}\big(n^{-1}-\mu_j(t)\big)\mu_i(t)\mu_j(t)\tau_{ij}(t)\,dt \\
&\quad - \frac{1}{\mathbb{G}\big(\boldsymbol{\mu}(t)\big)}\sum_{i=1}^n d\Lambda_{\mu_i - n^{-1}}(t),
\end{aligned}$$

yields for $t \in [0,T]$, a.s.,

$$\log\Big(Z_{\psi^{\mathfrak{g}}}(t)/Z_\mu(t)\Big) = d\log \mathbb{G}\big(\boldsymbol{\mu}(t)\big) + \frac{1}{\mathbb{G}\big(\boldsymbol{\mu}(t)\big)}\sum_{i=1}^n d\Lambda_{\mu_i - n^{-1}}(t). \tag{6.14.10}$$

In summary, we have essentially shown that $\mathbb{G}$ generates the portfolio $\psi^{\mathfrak{g}}$ with weights

$$\psi_i^{\mathfrak{g}}(t) = \left(\frac{\operatorname{sgn}\big(n^{-1} - \mu_i(t)\big)}{2\,\mathbb{G}\big(\boldsymbol{\mu}(t)\big)} + 1 - \sum_{j=1}^{n} \frac{\mu_j(t)\operatorname{sgn}\big(n^{-1} - \mu_j(t)\big)}{2\,\mathbb{G}\big(\boldsymbol{\mu}(t)\big)} \right) \mu_i(t), \tag{6.14.11}$$

for $i = 1, 2, \ldots, n$, and a drift process that satisfies

$$d\Theta_{\psi^{\mathfrak{g}}}(t) = \frac{1}{\mathbb{G}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^{n} d\Lambda_{\mu_i - n^{-1}}(t), \qquad t \in [0, T], \qquad \text{a.s.} \tag{6.14.12}$$

Clearly, the drift process Θ is nonnegative as it comprises nonnegative local times. Further notice that it depends on the local times measuring the time $\mu_i(t)$ spends near n^{-1}, for $i = 1, 2, \ldots, n$. Thus the drift captures the behaviour of the market weights when they correspond to an equal-weighted portfolio.

6.15 Estimation of Local Time

Local times can be estimated in practice quite accurately; indeed from (6.12.37) we obtain

$$\frac{1}{2}\int_0^T \zeta_{(m)}(t)\, d\mathfrak{L}_{m,m+1}(t) = \log\left(\frac{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(T)\big) Z_\mu(T)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(0)\big) Z_\zeta(T)} \right).$$

Therefore,

$$\begin{aligned} \zeta_{(m)}(t)\, d\mathfrak{L}_{m,m+1}(t) &= 2\, d\log\left(\frac{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) Z_\mu(t)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(0)\big) Z_\zeta(t)} \right) \\ d\mathfrak{L}_{m,m+1}(t) &= \frac{2}{\zeta_{(m)}(t)}\, d\log\left(\frac{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) Z_\mu(t)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(0)\big) Z_\zeta(t)} \right). \end{aligned}$$

Hence, for $m = 1, 2, \ldots, n-1$, local times can estimated as follows

$$\mathfrak{L}_{m,m+1}(\cdot) = \int_0^{\cdot} \frac{2}{\zeta_{(m)}(t)}\, d\log\left(\frac{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) Z_\mu(t)}{\mathcal{G}_L\big(\boldsymbol{\mu}_{(\cdot)}(0)\big) Z_\zeta(t)} \right). \tag{6.15.1}$$

Alternatively, from (6.12.64), we get

$$\frac{1}{2}\int_0^T \eta_{(m+1)}(t)\, d\mathfrak{L}_{m,m+1}(t) = \log\left(\frac{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(0)\big) Z_\eta(T)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(T)\big) Z_\mu(T)} \right).$$

Therefore,

$$\begin{aligned} \eta_{(m+1)}(t)\, d\mathfrak{L}_{m,m+1}(t) &= 2\, d\log\left(\frac{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(0)\big) Z_\eta(t)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) Z_\mu(t)} \right) \\ d\mathfrak{L}_{m,m+1}(t) &= \frac{2}{\eta_{(m+1)}(t)}\, d\log\left(\frac{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(0)\big) Z_\eta(t)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) Z_\mu(t)} \right). \end{aligned}$$

Hence, for $m = 1, 2, \ldots, n-1$, local times can estimated as follows

$$\mathfrak{L}_{m,m+1}(\cdot) = \int_0^{\cdot} \frac{2}{\eta_{(m+1)}(t)}\, d\log\left(\frac{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(0)\big) Z_\eta(t)}{\mathcal{G}_S\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) Z_\mu(t)} \right). \tag{6.15.2}$$

The quantities on the right-hand side of equations (6.15.1) and (6.15.2) are completely observable, so that local times can be estimated from market observable quantities.

6.16 Summary and Conclusion

Due to the fact that we are more interested in selecting stocks according to their ranking in the equity market rather than by their name, we started this chapter off by providing a formal definition for the rank process (the ranked stock price processes), in terms of the original named processes (the original named stock price processes), as well as in terms of the mathematically amenable maximum and minimum functions. If we let $X_1, X_2, \ldots, X_n$ be processes, then, for $k = 1, 2, \ldots, n$, the kth rank process, $X_{(k)} = \big\{X_{(k)}(t), t \in [0,T]\big\}$ of $\mathcal{M} = \big\{X_1(\cdot), \ldots, X_n(\cdot)\big\}$ is defined by

$$X_{(k)}(t) \quad \triangleq \quad \max_{1 \le i_1 < \cdots < i_k \le n} \min\Big(X_{i_1}(t), \ldots, X_{i_k}(t)\Big), \qquad t \in [0,T], \tag{6.16.1}$$

where $i_1 \ge 1$ and $i_k \le n$. Thus, at any given moment, the values of the rank processes represent the values of the original named processes arranged in decreasing order. Also, if we let $X_1, X_2, \ldots, X_n$ be processes, and for $t \in [0,T]$, let $\boldsymbol{p}_t = \big(p_t(1), p_t(2), \ldots, p_t(n)\big)$ denote the random permutation vector of the set of name elements $\{1, 2, \ldots, n\}$, then for $k = 1, 2, \ldots, n$, the following hold

$$X_{p_t(k)}(t) \quad \triangleq \quad X_{(k)}(t), \tag{6.16.2}$$

and,

$$\text{if} \quad X_{(k)}(t) \quad = \quad X_{(k+1)}(t) \quad \text{then} \quad p_t(k) \quad < \quad p_t(k+1). \tag{6.16.3}$$

Here, $p_t(k)$ represents the kth element of the permutation vector and signifies the name (i.e., index or subscript) of the stock with the kth largest capitalisation at time $t \in [0,T]$. In other words, $p_t(k)$ is the name (i.e., index or subscript) of the stock that occupies the kth rank (the kth ranked stock) in terms of total capitalisation at time $t \in [0,T]$. The permutation vector associates each rank process with one of the original processes that has the same value at time t. In addition, in the event of a tie, we resort to the lowest index for the kth ranked stock. Thus, ties are resolved by assigning the lower index to the higher-ranked participant of the tie. We then went on to introduce the concept of local time. We can interpret local time for a particular process as a measure of the quantity of time that a particular process (be it Brownian motion or any arbitrary semimartingale) spends within a certain locale, i.e. within a local neighbourhood of some level (near or at some level), typically the origin. We first considered the local time for Brownian motion, otherwise known as Brownian local time. The local time for the one-dimensional Brownian motion $W = \{W(t), \mathcal{F}_t, t \in [0,\infty)\}$ at $a \in \mathbb{R}$, denoted $L^a \equiv L(a) = \{L^a(t) \equiv L(a,t), \mathcal{F}_t, t \in [0,\infty)\}$, is defined to be the process given by

$$L^a(t) \quad = \quad \frac{1}{2}\left(\big|W(t) - a\big| - \big|W(0) - a\big| - \int_0^t \operatorname{sgn}\big(W(s) - a\big)\, dW(s)\right). \tag{6.16.4}$$

Equivalently, the local time for the Brownian motion W at the level 0 (i.e., at the origin), denoted $L^0 \equiv L(0) = \{L^0(t) \equiv L(0,t), \mathcal{F}_t, t \in [0,\infty)\}$, is defined to be the process given by

$$L^0(t) \quad = \quad L(t) \quad = \quad \frac{1}{2}\left(\big|W(t)\big| - \big|W(0)\big| - \int_0^t \operatorname{sgn}\big(W(s)\big)\, dW(s)\right). \tag{6.16.5}$$

Next, we considered a generalisation of Brownian local time, i.e. the local time for continuous semimartingales, otherwise known as semimartingale local time, within the context of continuous-time equity markets. The main difference now is that time-integrals, present in Brownian local time, now become integrals with respect to quadratic variation in the case of semimartingale local time. The local time for the continuous semimartingale $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ at $a \in \mathbb{R}$, denoted $\Lambda_X^a \equiv \Lambda_X(a) = \big\{\Lambda_X^a(t) \equiv \Lambda_X(a,t), \mathcal{F}_t, t \in [0,\infty)\big\}$, is defined to be the process given by

$$\Lambda_X^a(t) \quad = \quad \frac{1}{2}\left(\big|X(t) - a\big| - \big|X(0) - a\big| - \int_0^t \operatorname{sgn}\big(X(s) - a\big)\, dX(s)\right). \tag{6.16.6}$$

Equivalently, the local time for the continuous semimartingale X at the level 0 (i.e., at the origin), denoted $\Lambda_X^0 \equiv \Lambda_X(0) = \big\{\Lambda_X^0(t) \equiv \Lambda_X(0,t), \mathcal{F}_t, t \in [0,\infty)\big\}$, is defined to be the process given by

$$\Lambda_X^0(t) \quad = \quad \Lambda_X(t) \quad = \quad \frac{1}{2}\left(\big|X(t)\big| - \big|X(0)\big| - \int_0^t \operatorname{sgn}\big(X(s)\big)\, dX(s)\right). \tag{6.16.7}$$

Thus, the semimartingale local time Λ_X measures the amount of time the process X spends near or at the origin, so $d\Lambda_X(t)$ defines a positive measure on $[0,T]$ that is concentrated on the set of points $\{t \in [0,T] \mid X(t) = 0\}$. Equipped with these definitions of local time, we were then able to establish the associated well-known and well-established Tanaka formulae for Brownian motion and the Tanaka-Meyer formulae for more general continuous semimartingales. We also presented a plethora of fundamental results involving local time, which also demonstrated some of its key characteristics. Thereafter, to enable us to effectively utilise local times in deriving a decomposition of the ranked stock price processes, we invoked certain fundamental nondegeneracy conditions for our cause. This involved an assumption that the original stock price processes considered, exhibit a certain level of nondegeneracy. This led us to the following definition of pathwise mutual nondegeneracy: the processes $X_1, \ldots, X_n$ are said to be pathwise mutually nondegenerate if for all $i, j, k = 1, 2, \ldots, n$, the following conditions hold: for all $i \neq j$,

$$\text{Leb}\left\{t \in [0,\infty) \;\middle|\; X_i(t) = X_j(t)\right\} \;=\; 0, \qquad \text{a.s.;} \tag{6.16.8}$$

and for all $i < j < k$, the set

$$\left\{t \in [0,\infty) \;\middle|\; X_i(t) = X_j(t) = X_k(t)\right\} \;=\; \varnothing, \qquad \text{a.s.} \tag{6.16.9}$$

In particular, the second condition ensures that a triple collision event does not exist, i.e. not more than two stock price processes can ever collide with one another at the same exact point in time. We not only considered this condition of pathwise mutual nondegeneracy as a requisite market condition, but we also considered what is referred to as the absolute continuity property. Then, the continuous semimartingales that exhibit this absolute continuity property are referred to as absolutely continuous semimartingales, and these continuous semimartingales will be of primary focus. Moreover, we showed that certain functions of absolutely continuous semimartingales that also exhibit absolute continuity, are also absolutely continuous semimartingales. We also had a brief look at local times for these absolutely continuous semimartingales. Supplied with these two conditions of pathwise mutual nondegeneracy and absolute continuity, we are now in a position to put forth what are known as the pathwise mutually nondegenerate absolutely continuous semimartingales. These specific semimartingales are given further detailed treatment in what follows next. Thus, the representation and decomposition for the dynamics of the rank processes, i.e. the ranked stock price processes, derived from these pathwise mutually nondegenerate absolutely continuous semimartingales, are given, for $k = 1, 2, \ldots, n$, by

$$dX_{(k)}(t) \;=\; \sum_{i=1}^{n} \mathbb{1}_{\{p_t(k)=i\}}\, dX_i(t) + \frac{1}{2}\, d\Lambda_{X_{(k)}-X_{(k+1)}}(t) - \frac{1}{2}\, d\Lambda_{X_{(k-1)}-X_{(k)}}(t) \tag{6.16.10}$$

$$\;=\; \sum_{i=1}^{n} \mathbb{1}_{\{i\}}\big(p_t(k)\big)\, dX_i(t) + \frac{1}{2}\, d\Lambda_{X_{(k)}-X_{(k+1)}}(t) - \frac{1}{2}\, d\Lambda_{X_{(k-1)}-X_{(k)}}(t), \tag{6.16.11}$$

where $\mathbb{1}_{\{p_t(k)=i\}} = \mathbb{1}_{\{i\}}\big(p_t(k)\big) = \mathbb{1}_{\{0\}}\big(p_t(k)-i\big)$. These representations of the dynamics of the ranked stock price processes demonstrate that the ranked stock price processes, which are derived from these pathwise mutually nondegenerate absolutely continuous semimartingales, can be expressed in terms of the original named stock price processes, adjusted by semimartingale local times. We also introduced and defined the rank market weight process. If we let $\mathcal{M}$ be a market of stocks $X_1, X_2, \ldots, X_n$, and let $\mu_1, \mu_2, \ldots, \mu_n$ be the market weight processes. For $t \in [0,T]$, let $\boldsymbol{p}_t = \big(p_t(1), p_t(2), \ldots, p_t(n)\big)$ denote the random permutation vector of the set of name elements $\{1, 2, \ldots, n\}$, then for $k = 1, 2, \ldots, n$, the following hold

$$\mu_{p_t(k)}(t) \;\triangleq\; \mu_{(k)}(t), \tag{6.16.12}$$

and,

$$\text{if} \quad \mu_{(k)}(t) \;=\; \mu_{(k+1)}(t) \quad \text{then} \quad p_t(k) \;<\; p_t(k+1). \tag{6.16.13}$$

Then, the representation and decomposition for the dynamics of the ranked market weight processes, derived from these pathwise mutually nondegenerate absolutely continuous semimartingales, are given, for $k =$

$1, 2, \ldots, n$, by

$$d\log\mu_{(k)}(t) = \sum_{i=1}^{n} \mathbb{1}_{\{i\}}\big(p_t(k)\big)\, d\log\mu_i(t) + \frac{1}{2}\, d\Lambda_{\log\mu_{(k)}-\log\mu_{(k+1)}}(t) - \frac{1}{2}\, d\Lambda_{\log\mu_{(k-1)}-\log\mu_{(k)}}(t) \tag{6.16.14}$$

$$= \sum_{i=1}^{n} \mathbb{1}_{\{i\}}\big(p_t(k)\big)\, d\log\mu_i(t) + \frac{1}{2}\, d\mathfrak{L}_{k,k+1}(t) - \frac{1}{2}\, d\mathfrak{L}_{k-1,k}(t), \tag{6.16.15}$$

a.s., for $t \in [0, T]$, where $\mathbb{1}_{\{i\}}\big(p_t(k)\big) = \mathbb{1}_{\{p_t(k)=i\}} = \mathbb{1}_{\{0\}}\big(p_t(k) - i\big)$. Here the quantity $\mathfrak{L}_{k,k+1}(t) \equiv \Lambda_{\Xi_k}(t)$ is the semimartingale local time at the origin, accumulated by the nonnegative process

$$\Xi_k(t) \triangleq \log\mu_{(k)}(t) - \log\mu_{(k+1)}(t), \qquad t \in [0, T]. \tag{6.16.16}$$

It measures the cumulative effect of the changes that have occurred during the time interval $[0, t]$ between ranks k and $k+1$. Again, these representations of the dynamics of the ranked market weight processes, demonstrate that the ranked market weight processes, which are derived from these pathwise mutually nondegenerate absolutely continuous semimartingales, can be expressed in terms of the original named market weight processes, adjusted by semimartingale local times. We then went on to consider equity portfolios that are generated by functions of the ranked market weights, referred to as rank-based functionally generated portfolios. In this regard, we presented the following result in a theorem, which is an extended version of the main theorem in the previous chapter which held for generating functions of the named market weight processes, that holds for generating functions of the ranked market weight processes, and contains the main result and crux of this entire chapter. Let $\mathcal{M}$ be a market of stocks $X_1, \ldots, X_n$ that are pathwise mutually nondegenerate, and let $\boldsymbol{p}_t$ be the random permutation vector. Let U be an open neighbourhood in $\mathbb{R}^n$ ($U \subset \mathbb{R}^n$) of the open positive unit $(n-1)-$simplex Δ^{n-1} and let $\mathbf{G} : U \to (0, \infty)$ be a positive twice continuously differentiable function defined on some open neighbourhood U of Δ^{n-1}. Suppose that there exists a positive twice continuously differentiable function $\mathfrak{G}$ defined on some open neighbourhood U of Δ^{n-1} such that for $\boldsymbol{\mu}(t) = \big(\mu_1(t), \ldots, \mu_n(t)\big) \in U$ and $\boldsymbol{\mu}_{(\cdot)}(t) = \big(\mu_{(1)}(t), \ldots, \mu_{(n)}(t)\big) \in U$,

$$\mathbf{G}\big(\mu_1(t), \ldots, \mu_n(t)\big) = \mathfrak{G}\big(\mu_{(1)}(t), \ldots, \mu_{(n)}(t)\big), \tag{6.16.17}$$

and for all $k = 1, 2, \ldots, n$, $\mu_{(k)}(t) D_k \log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)$ is bounded for $\boldsymbol{\mu}(t), \boldsymbol{\mu}_{(\cdot)}(t) \in \Delta^{n-1}$. Then for all $t \in [0, T]$, a.s., and for $k = 1, 2, \ldots, n$, the rank-dependent generating function $\mathfrak{G}$ generates the (rank-based functionally generated) portfolio ψ with weights

$$\psi_{p_t(k)}(t) = \left(D_k \log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + 1 - \sum_{\ell=1}^{n} \mu_{(\ell)}(t)\, D_\ell \log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) \right) \mu_{(k)}(t), \tag{6.16.18}$$

and with drift process Υ, that satisfies for $t \in [0, T]$, a.s.,

$$\begin{aligned} d\Upsilon(t) &= \frac{-1}{2\,\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \sum_{k,\ell=1}^{n} D_{k\ell}\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\, dt \\ &\quad + \frac{1}{2}\sum_{k=1}^{n-1} \Big(\psi_{p_t(k+1)}(t) - \psi_{p_t(k)}(t)\Big)\, d\mathfrak{L}_{k,k+1}(t), \end{aligned} \tag{6.16.19}$$

where, $\tau_{(k\ell)}(t)$, for $k, \ell = 1, 2, \ldots, n$, is defined to be the relative rank covariance processes $\tau_{(k\ell)} = \big\{\tau_{(k\ell)}(t), t \in [0, \infty)\big\}$, i.e., the ranked covariance process relative to the market, and is given by

$$\tau_{(k\ell)}(t) \triangleq \tau_{p_t(k)p_t(\ell)}(t), \qquad t \in [0, T]. \tag{6.16.20}$$

Let μ be the market portfolio and ψ be the rank-based functionally generated portfolio, and let Z_μ and Z_ψ be their portfolio value processes, respectively. Then, a.s., for $t \in [0, T]$, we have

$$\begin{aligned} d\log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) &= d\log\Big(Z_\psi(t)/Z_\mu(t)\Big) + \frac{1}{2\,\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \sum_{k,\ell=1}^{n} D_{k\ell}\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\, dt \\ &\quad - \frac{1}{2}\sum_{k=1}^{n-1} \Big(\psi_{p_t(k+1)}(t) - \psi_{p_t(k)}(t)\Big)\, d\mathfrak{L}_{k,k+1}(t), \end{aligned} \tag{6.16.21}$$

or,

$$\begin{aligned} d\log\Big(Z_\psi(t)/Z_\mu(t)\Big) &= d\log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) - \frac{1}{2\,\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)}\sum_{k,\ell=1}^{n} D_{k\ell}\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\,dt \\ &\quad + \frac{1}{2}\sum_{k=1}^{n-1}\Big(\psi_{p_t(k+1)}(t) - \psi_{p_t(k)}(t)\Big)\,d\mathfrak{L}_{k,k+1}(t). \end{aligned} \tag{6.16.22}$$

We then discussed rank-dependent portfolio generating functions, i.e. rank-dependent generating functions that generate rank-based functionally generated portfolios. First, we provided the stochastic differential equation, associated with rank-based functionally generated portfolios. Let U be an open neighbourhood in $\mathbb{R}^n$ ($U \subset \mathbb{R}^n$) of the open positive unit $(n-1)-$simplex Δ^{n-1}

$$\Delta^{n-1} = \Big\{\boldsymbol{\mu}_{(\cdot)}(t) = \big(\mu_{(1)}(t),\ldots,\mu_{(n)}(t)\big) \in \mathbb{R}^n \,\Big|\, \mu_{(1)}(t)+\cdots+\mu_{(n)}(t) = 1,\ 0 < \mu_{(k)}(t) < 1,\ k = 1,\ldots,n\Big\},$$

and let $\mathfrak{G} : U \to (0,\infty)$ be a positive twice continuously differentiable function defined on some open neighbourhood U of Δ^{n-1}. Then $\mathfrak{G}$ generates a portfolio ψ if there exists a continuous, measurable and adapted process of bounded variation $\Upsilon = \{\Upsilon(t), t \in [0,\infty)\}$, such that

$$\log\Big(Z_\psi(t)/Z_\mu(t)\Big) = \log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + \Upsilon(t), \qquad t \in [0,T], \qquad \text{a.s.}, \tag{6.16.23}$$

or such that we have the equivalent differential form

$$d\log\Big(Z_\psi(t)/Z_\mu(t)\Big) = d\log\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big) + d\Upsilon(t), \qquad t \in [0,T], \qquad \text{a.s.} \tag{6.16.24}$$

The process Υ is called the drift process corresponding to the rank-dependent generating function $\mathfrak{G}$. We say that ψ is the rank-dependent portfolio generated by the function $\mathfrak{G}$. Then $\mathfrak{G}$ is called the rank-dependent generating function of the portfolio ψ, and the portfolio ψ is said to be the rank-dependent functionally generated portfolio corresponding to the rank-dependent portfolio generating function $\mathfrak{G}$. The stochastic differential equation above, associated with the rank-based functionally generated portfolio, decomposes the logarithmic relative return process of the rank-based functionally generated portfolio with respect to the reference benchmark market portfolio, specifically into two separate constituents. The first component is given by the logarithmic change in the value of the rank-dependent portfolio generating function, to wit, the rank-dependent portfolio generating function component. The second component is determined to be the drift process corresponding to the rank-based functionally generated portfolio generated by the rank-dependent portfolio generating function. In turn, this drift process can further be divided up into its own two distinguishable parts: the first part being the so-called smooth component of the drift process, and the second part being the local time component of the drift process. The smooth component of the drift process specified here, is very much like the entire drift component encountered in the preceding chapter. The local time component of the drift process is the other component that incorporates local times to account for changes in rank that arise among the stock price processes. The integral form for this rank-dependent functionally generated portfolio ψ, is given by

$$\log\left(\frac{Z_\psi(T)}{Z_\mu(T)}\right) - \log\left(\frac{Z_\psi(0)}{Z_\mu(0)}\right) = \log\left(\frac{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \int_0^T d\Upsilon(t), \qquad T \in [0,\infty), \tag{6.16.25}$$

or,

$$\log\left(\frac{Z_\psi(T)}{Z_\mu(T)}\right) = \log\left(\frac{Z_\psi(0)}{Z_\mu(0)}\right) + \log\left(\frac{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \int_0^T d\Upsilon(t), \qquad T \in [0,\infty), \tag{6.16.26}$$

which can be equivalently expressed as

$$\log\left(\frac{Z_\psi(T)}{Z_\psi(0)}\right) - \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) = \log\left(\frac{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \int_0^T d\Upsilon(t), \qquad T \in [0,\infty), \tag{6.16.27}$$

or,

$$\log\left(\frac{Z_\psi(T)}{Z_\psi(0)}\right) = \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log\left(\frac{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \int_0^T d\Upsilon(t), \qquad T \in [0,\infty). \tag{6.16.28}$$

Where $Z_\psi(0) = Z_\mu(0)$, the logarithmic relative return process of this rank-dependent functionally generated portfolio ψ, with respect to the market, is given by the so-called *"master formula"* for rank-dependent portfolios, for all $T \in [0,\infty)$,

$$\log\left(\frac{Z_\psi(T)}{Z_\mu(T)}\right) = \log\left(\frac{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \int_0^T d\Upsilon(t), \tag{6.16.29}$$

$$\log\left(\frac{Z_\psi(T)}{Z_\mu(T)}\right) = \log\left(\frac{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) + \Upsilon(T), \tag{6.16.30}$$

alternatively, for $T \in [0,\infty)$, the above can be expressed as

$$\Upsilon(T) = \int_0^T d\Upsilon(t) = \log\left(\frac{Z_\psi(T)}{Z_\mu(T)}\right) - \log\left(\frac{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}{\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}\right) \tag{6.16.31}$$

$$= \log\left(\frac{Z_\psi(T)\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(0)\big)}{Z_\mu(T)\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(T)\big)}\right), \tag{6.16.32}$$

with

$$\begin{aligned}\Upsilon(T) &= \int_0^T \frac{-1}{2\,\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)} \sum_{k,\ell=1}^{n} D_{k\ell}\mathfrak{G}\big(\boldsymbol{\mu}_{(\cdot)}(t)\big)\mu_{(k)}(t)\mu_{(\ell)}(t)\tau_{(k\ell)}(t)\,dt \\ &\quad + \frac{1}{2}\int_0^T \sum_{k=1}^{n-1}\Big(\psi_{p_t(k+1)}(t) - \psi_{p_t(k)}(t)\Big)\,d\mathfrak{L}_{k,k+1}(t).\end{aligned} \tag{6.16.33}$$

We also offered a generalisation of the main theorem of this chapter for rank-based functionally generated portfolios which was extended to include the existence of triple collision events, i.e. three or more stock price processes may collide with one another at the same exact point in time, at any given time. We then presented a few examples of rank-dependent portfolio generating functions together with their affiliated rank-based functionally generated portfolios that they generate. Both the smooth component and the local time component of the drift process accompanying these rank-based functionally generated portfolios were calculated. In addition, we explored and examined the performance of these portfolios relative to the market portfolio. In particular, the examples of rank-based functionally generated portfolios that are each generated by their respective rank-dependent portfolio generating functions, that we considered are: the biggest stock which holds only the largest stock in the equity market, the large-stock index portfolio which consists of the largest stocks in the equity market, the small-stock index portfolio which consists of the smallest stocks in the equity market, the diversity-weighted version of the large-stock index portfolio, and a portfolio with fixed weight ratios. From the topic of examples of rank-based functionally generated portfolios we segued to the topic of the application of the aforementioned examples of rank-based functionally generated portfolios. We first explored the size effect phenomenon in equity markets, by contrasting the performance of the large-stock index portfolio with the performance of the small-stock index portfolio. The size effect is described as the observed propensity of smaller stocks to have higher long-term returns on average than those of their larger counterparts. An explanation of the size effect is offered, a la Fernholz (2002), who attributes the size effect to something other than the supposed higher risk associated with the smaller stocks, which is the usual explanation proffered in the literature. By contrasting the large-stock index portfolio with the small-stock index portfolio, we obtained a representation of the relative return of a small-stock index versus a large-stock index which decomposes the performance of the small-stock index relative to the large-stock index into two components: the change in the relative capitalisations of the two equity indices and a drift process which comprises local times. The derived drift process in terms of local times, is shown to be monotonically increasing. Furthermore, if this ratio of the relative capitalisation of the smaller stocks to that of the larger stocks remains stable over time, thereby inducing a form of

stability in the equity market, then it does not change much. Then the majority of the movement in the relative return will be attributed to the drift process. Thus, over a sufficiently long period of time, the relative return will be dominated by the drift process, i.e., by the local time term. This allows us to conclude that the size effect is actually attributed to the long-term effect that the increasing drift term has on the performance of the small-stock index relative to the large-stock index, and not to the associated risk level of the small-stock index. Thus, the relative level of small-stock risk is irrelevant when providing reasoning for the size effect. Secondly, we introduced the concept of leakage in equity markets, by contrasting the performance of the diversity-weighted large-stock index relative to the performance of the somewhat similar large-stock index portfolio. Leakage refers to the scenario in which the smaller stocks contained within a large-stock index portfolio are dropped from this portfolio and thus move out of this portfolio and cross over to the rest of the market portfolio, since their relative standings are no longer large enough to remain within this large-stock index portfolio. In such a situation, these smaller stocks which are dropped from the large-stock index portfolio are said to "leak" out of the large-stock index portfolio, hence, the terminology leakage. Thus, leakage explains the effect that these crossovers have on the equity market. By contrasting the diversity-weighted large-stock index portfolio with the large-stock index portfolio, we obtained a representation of the relative return of the diversity-weighted large-stock index versus a large-stock index which decomposes the performance of the diversity-weighted large-stock index relative to the large-stock index into three components: the change in the generating function, the smooth component of the drift process and a term involving local times, i.e., the local time component of the drift process. The last local time term of the drift process is what is referred to as "leakage". We, lastly, provided formulas that enable the estimation of local times in practice from market observable quantities, with relative ease.

Chapter 7

Relative Arbitrage Opportunities in Equity Markets and the Consequences

7.1 Introduction

In the field of mathematical finance, and inherently within the stochastic portfolio theory setting, one of the main problems, is the detection and study of riskless opportunities to make a profit, also known as arbitrages. Thus, the concept of arbitrage is of exceptional importance in mathematical finance. In fact, arbitrage is such a wide and well-researched theory that pervades the area of mathematical finance, and lies at its core. There is such a great abundance and array of available literature out there on the topic of arbitrage. As a result, one cannot mention them all, we have considered Björk (2004), Delbaen & Schachermayer (1995a, 2006), Duffie (1992), Evstigneev & Kapoor (2009), Harrison & Kreps (1979), Kardaras (2006) and Platen (2002), to name just a few. Black & Scholes were also most influential in this area. There are many others not mentioned here, and the reader may consult these further texts to gain greater clarity and insight into the area of arbitrage. The absence of arbitrage is also a common hypothesis in current financial mathematics theory [see, e.g., Duffie (1992)]. In particular, the no-arbitrage hypothesis is also central to modern mathematical finance. The no-arbitrage hypothesis is a basic tenet of current mathematical finance [see, e.g., Duffie (1992)]. The no-arbitrage hypothesis states that equity markets do not present opportunities for riskless arbitrage (or "free lunch"). While there are many theoretical examples of equity markets in which arbitrage does indeed exist, these examples appear to be mathematical oddities which do not resemble "real" equity markets at all. In fact, these examples of arbitrage in the literature bear no resemblance whatsoever to actual equity markets, so it seems that arbitrage has been thought to occur only in very unusual circumstances. Such an unusual circumstance, that would imply that arbitrage is indeed possible, would appear to be the assumption such as the impossibility of the entire capital in the equity market to concentrate into a single stock. Of course, this assumption is related to the concept of diversity.

We show that if there is a portfolio that dominates the market portfolio, then arbitrage opportunities exist and the no-arbitrage hypothesis fails. An arbitrage opportunity is a combination of investments in portfolios such that the total initial value of the investments is zero and such that, at some given future time $T > 0$, the total value of the investment in the portfolios will be positive, with probability one [see, e.g., Duffie (1992)]. The no-arbitrage hypothesis states that there exist no arbitrage opportunities composed of investments in admissible portfolios.

In this chapter, we shall concern ourselves for the most part with the allied notion of relative arbitrage as opposed to the former notion of absolute arbitrage, and explore and investigate some of its consequences. The current research carried out on stochastic portfolio theory has grappled with the notion of relative arbitrage opportunities in equity markets and several recent results in stochastic portfolio theory have been concerned with the existence of relative arbitrage in equity markets. Over a given fixed finite time horizon, there exist relative arbitrage

opportunities, if there exists two long-only portfolios φ and η; such that φ is guaranteed not to underperform η, and such that the probability that φ will outperform η is nonzero, i.e., the probability of outperformance is nonzero, with the constraint that the two portfolios start off with the same initial value [Banner & Fernholz (2008)]. Stochastic portfolio theory, unlike classical portfolio theory in the context of classical mathematical finance, is not averse to the existence of relative arbitrage opportunities in equity markets. Stochastic portfolio theory differs in that it instead investigates the conditions on equity market structure and the characteristics that the equity market exhibits that will imply the existence of relative arbitrage opportunities. These and other related ideas are presented in this chapter. Further to this, the aim of this chapter is to define the notion of relative arbitrage by considering the special cases of weak relative arbitrage and strong relative arbitrage, wherein the first portfolio outperforms the second portfolio, with probability one, within the framework of stochastic portfolio theory. The market portfolio is the canonical choice for a reference benchmark portfolio with respect to which relative arbitrage in the equity market is going to be studied.

Before we delve into the constructs of relative arbitrage, we shall first introduce the concept of admissible portfolios, as well as dominating and strictly dominating portfolios. In a nondegenerate equity market in which the stocks pay no dividends, it is possible to construct an admissible portfolio that strictly dominates the market portfolio. When the first component of the relative return of the functionally generated portfolio versus the market portfolio has a lower bound, and if the drift component is increasing and positive, then in the long term, the functionally generated portfolio will outperform the market. We shall use the canonical stochastic differential equation of the previous chapter to establish dominance relationships between certain functionally generated portfolios and the market portfolio. We shall show that if there are appropriate bounds on the portfolio generating function as well as on the corresponding drift process, then such a dominance relationship will indeed hold. In particular, we shall show that functions which are measures of diversity will generate portfolios that dominate the market portfolio if appropriate bounds exist on the portfolio generating function or on the change in diversity. Recall that a measure of diversity will have a nonnegative (positive) drift process, however, this alone is not sufficient to ensure that the portfolio it generates will dominate the market portfolio. In order for a dominance relationship to exist, the drift process must have a positive lower bound. In this chapter, we shall consider an equity market composed of stocks that do not pay dividends, and we shall assume that the relative variance of every stock with respect to the market is bounded away from zero. We show that with this nondegeneracy condition in conjunction with the diversity criterion, it is possible to construct well-behaved portfolios that dominate the market portfolio. The value of the generating function we use has a positive lower bound, and the nondegeneracy condition ensures that the rate of increase of the monotonic drift process is bounded away from zero. This combination of lower bounds implies that the functionally generated portfolio will dominate the market portfolio.

We shall provide numerous examples that demonstrate the principle that if the market is nondegenerate and (weakly) diverse over a fixed finite time horizon $[0, T]$, with $T > 0$ a given real number, then the (weakly) diverse equity market contains strong arbitrage opportunities relative to the market portfolio, at least for sufficiently large real numbers $T > 0$. That is, in a nondegenerate and (weakly) diverse equity market, there exist functionally generated portfolios that represent strong arbitrage opportunities relative to the market portfolio, over the fixed, finite time horizon $[0, T]$, for T sufficiently large, i.e., over sufficiently long time horizons. Hence, it is possible to outperform or underperform such markets over sufficiently long time horizons. It is shown in Fernholz (2002) how to generate fully-invested, all-long portfolios that outperform a (weakly) diverse equity market over sufficiently long time horizons and how to exploit this property for passive equity management. A case in point is the diversity-weighted index portfolio which is shown to represent a strong arbitrage opportunity relative to the market portfolio, for a (weakly) diverse equity market over sufficiently long time horizons. Thus, the diversity-weighted portfolio outperforms significantly any (weakly) diverse market over sufficiently long time horizons, which leads to strong arbitrage opportunities relative to the market portfolio.

In this chapter, we shall also show, that in (weakly) diverse equity markets, that relative arbitrage can actually be constructed and relative arbitrage opportunities exist on any given arbitrary time horizon, i.e., there always exist portfolios that consistently outperform or underperform a (weakly) diverse market. To this end, we shall introduce mirror portfolios and analyse their properties; these are then used to show that, in the context of

a (weakly) diverse market, it is possible to outperform (or underperform) the market portfolio over any given arbitrary time horizon.

Moreover, we shall provide sufficient conditions that will ensure diversity in equity markets. We shall demonstrate that diversity is indeed possible under appropriate, though rather delicate, conditions and can thus be ensured by a strongly negative rate of growth for the largest stock, resulting in a sufficiently strong repelling drift (e.g., a log-pole-type singularity) away from an appropriate boundary, as well as nonnegative, potentially large, growth rates for all the other stocks.

We shall also introduce the volatility-stabilised equity market model, i.e., we shall briefly study and consider an equity market which is volatility-stabilised, in that the return from the market portfolio has constant drift and variance rates. This stabilisation by volatility is achieved by allocating the smallest stocks the highest volatilities and by allocating the largest stocks the smallest volatilities, so that the individual stocks essentially move all "over the place", while the overall market remains stable. The largest stock and the entire market then grow at the same, constant rate, though the smallest individual stocks fluctuate quite widely. The volatility-stabilised equity market model is shown to not be diverse, in that it fails to satisfy the diversity criterion, nevertheless the volatility-stabilised equity market model admits relative arbitrage because the excess growth rate is positive and constant. In fact, it is demonstrated in Fernholz, Karatzas & Kardaras (2005) and Banner & Fernholz (2008), that strong relative arbitrage opportunities exist in non-diverse equity markets with unbounded volatilities over arbitrarily short time horizons within the context of the volatility-stabilised equity market models.

We shall kick this chapter off with Section 7.2, where we shall introduce the concept of admissible portfolios and the conditions under which a portfolio is said to be admissible. These admissibility conditions, within the context of the stochastic portfolio theory literature, are of particular concern as we are interested in arbitrage opportunities primarily composed of admissible portfolios. This section shall also discuss dominating portfolios along with the portfolio dominance relationships established between these portfolios, specifically looking at the dominance relationship between a portfolio and the canonical benchmark market portfolio. We shall establish that, under certain appropriate conditions, there exist dominance relationships between pairs of functionally generated portfolios, i.e. the conditions under which a portfolio dominates another portfolio. In particular, these conditions are determined to be conditions imposed on the portfolio generating function and on the drift process of the corresponding functionally generated portfolio. We shall show that for certain criteria placed on these aforementioned terms that the functionally generated portfolio will either dominate the market portfolio or be dominated by the market portfolio. Since the existence of a portfolio that dominates another portfolio implies the existence of arbitrage, we shall, in Section 7.3, consider the ubiquitous notion of arbitrage which is of considerable importance in the mathematical finance arena. We shall focus on the allied notion of relative arbitrage by providing definitions for weak relative arbitrage, strong relative arbitrage and superior long-term growth opportunities. We shall also show that under these same conditions imposed on the portfolio generating function and the drift process of the corresponding functionally generated portfolio, the portfolio represents a strong arbitrage opportunity relative to the market portfolio, or vice versa. The next two sections, Section 7.4 and Section 7.5, shall extend on all the content in the previous two sections by including dividends into the mix. As a result, in these two sections, we shall work with the total return process of portfolios and the dividend rate. It thus turns out that the second particular condition for strong relative arbitrage which originally involved only the drift process of the functionally generated portfolio is now one that is a combination of that drift process and the dividend rates for both portfolios considered. Having now established relative arbitrage opportunities, in Section 7.6 we shall explore relative arbitrage opportunities over sufficiently long time horizons. We shall do this by looking at the following portfolios: the weighted-average capitalisation generated portfolio, the price-to-book ratio generated portfolio and the single stock with leverage, that were introduced in Chapter 5. We shall demonstrate that these specific portfolios will be dominated by the market portfolio over sufficiently long time horizons, i.e. the market portfolio represents a strong arbitrage opportunity relative to these portfolios over a sufficiently long time horizon. The reasoning behind why the market portfolio will dominate these aforementioned portfolios is that the generating functions that generated each of these portfolios are not measures of diversity and thus, along with the associated drift process of the generated portfolios, do not satisfy the required bounds for a dominating relationship in the opposing direction to exist. This diversity

measure requirement of the generating functions, as we shall see in the next section, is an imperative and essential requirement for the associated functionally generated portfolios to dominate the market portfolio. This leads us to Section 7.7, in which we investigate relative arbitrage opportunities in diverse equity markets. The investigation here is two-fold: we first look at long-term relative arbitrage opportunities and, secondly, short-term relative arbitrage opportunities, in diverse equity markets. We shall show that if the market is nondegenerate and (weakly) diverse then it contains strong arbitrage opportunities relative to the market portfolio over the long term, i.e. there exist certain portfolios that will dominate the market portfolio over the long term. To this end, we shall present several examples of such strong relative arbitrage opportunities, or portfolios, in a nondegenerate and (weakly) diverse equity market, such as the entropy-weighted portfolio, the diversity-weighted index portfolio and the quadratic Gini-coefficient-weighted portfolio, to name just a few. The reasoning behind why the market portfolio will be dominated by these portfolios is that the generating functions that generated these portfolios are all measures of diversity and thus, along with the associated drift process of the generated portfolios, do indeed satisfy the required bounds for this kind of dominating relationship. This diversity measure requirement of the generating functions is a crucial one in order for these generated portfolios to dominate the market portfolio. The latter part of this section, concerning relative arbitrage opportunities over arbitrarily short time horizons, is devoted to mirror portfolios and the seed portfolio. We shall demonstrate that these portfolios, within the same context of a nondegenerate and (weakly) diverse equity market, represent strong relative arbitrage opportunities over arbitrary time horizons, no matter how small. The sufficient conditions for guaranteeing the existence of diversity in equity markets is considered in Section 7.8. Here we shall consider what conditions on certain stock variables will sufficiently ensure that an equity market is diverse. In this section, we shall also briefly discuss one such example of an equity market model that exhibits diversity, to wit, the diverse equity market model. In Section 7.9, we shall introduce another type of equity market model that is of interest, particularly in the stochastic portfolio theory domain, namely, the volatility-stabilised equity market model and its generalisations. Some existing results and conclusions on this volatility-stabilised equity market model shall also be provided. In essence, the volatility-stabilised equity market model assigns the highest volatilities to the smallest stocks in the equity market and the lowest volatilities to the largest stocks in the equity market, but in such a way that the overall behaviour of the entire equity market exhibits a remarkable stability. Hence, the terminology "volatility-stabilised" for this equity market model as it involves and encapsulates the concept of stabilisation by volatility. This brings us to Section 7.10, in which we briefly discuss the possible relative arbitrage opportunities in this volatility-stabilised equity market model. We shall provide the constructs that reveal that weak and strong relative arbitrage opportunities do indeed exist in equity market models of this type over any arbitrary time horizons. Lastly, in Section 7.11 we provide a summary and conclusion to round this chapter off.

7.2 Portfolio Dominance Relationships

7.2.1 Admissible Portfolios

Definition 7.2.1 (Admissible Portfolios). *A portfolio φ is **admissible** if:*

(*i*) *for $i = 1, \ldots, n$, $\varphi_i(t) \geq 0$, $t \in [0, T]$;*

(*ii*) *there exists a constant $c > 0$ such that*

$$\frac{Z_\varphi(t)}{Z_\varphi(0)} \geq c\,\frac{Z_\mu(t)}{Z_\mu(0)}, \qquad t \in [0, T], \qquad a.s.; \tag{7.2.1}$$

(*iii*) *there exists a constant M such that, for $i = 1, \ldots, n$,*

$$\frac{\varphi_i(t)}{\mu_i(t)} \leq M, \qquad t \in [0, T], \qquad a.s. \tag{7.2.2}$$

Admissibility conditions vary in the literature, and a portfolio that satisfies Definition 7.2.1 may not be "admissible" in other settings. Condition (i) is imposed here because we are concerned with portfolios that do

not admit short sales. Condition (ii) implies limited negative performance relative to the market portfolio as numéraire. The market portfolio is a natural numéraire for equity managers whose performance is measured versus the market as the benchmark. Condition (iii) prevents arbitrarily high overweighting of any particular stock relative to the market weighting. Admissible portfolios are of particular concern, as we are interested in arbitrage opportunities composed of admissible portfolios.

7.2.2 Dominating Portfolios and Strictly Dominating Portfolios

Definition 7.2.2 (Dominating Portfolios, Strictly Dominating Portfolios). *Let φ and η be portfolios. Then the portfolio φ **dominates** the portfolio η on $[0,T]$, if there is a number $T>0$, such that for any positive initial capital values $Z_\varphi(0)$ and $Z_\eta(0)$, we have*

$$\frac{Z_\varphi(T)}{Z_\varphi(0)} \geq \frac{Z_\eta(T)}{Z_\eta(0)}, \qquad a.s., \tag{7.2.3}$$

and

$$\mathbb{P}\left(\frac{Z_\varphi(T)}{Z_\varphi(0)} \geq \frac{Z_\eta(T)}{Z_\eta(0)}\right) = 1, \qquad \text{and} \qquad \mathbb{P}\left(\frac{Z_\varphi(T)}{Z_\varphi(0)} > \frac{Z_\eta(T)}{Z_\eta(0)}\right) > 0. \tag{7.2.4}$$

*The portfolio φ **strictly dominates** the portfolio η on $[0,T]$, if there is a number $T>0$, such that for any positive initial capital values $Z_\varphi(0)$ and $Z_\eta(0)$, we have*

$$\frac{Z_\varphi(T)}{Z_\varphi(0)} > \frac{Z_\eta(T)}{Z_\eta(0)}, \qquad a.s., \tag{7.2.5}$$

and

$$\mathbb{P}\left(\frac{Z_\varphi(T)}{Z_\varphi(0)} > \frac{Z_\eta(T)}{Z_\eta(0)}\right) = 1. \tag{7.2.6}$$

It is clear from this definition that if φ strictly dominates η in $[0,T]$, then φ dominates η in $[0,T]$. Suppose that φ and η are admissible portfolios that satisfy Definition 7.2.2, i.e., such that φ dominates η. Proposition 2.2.20 implies that the value of an investment in a portfolio is scalable by setting its initial value. Hence, we can buy one dollar's worth of φ at time $t=0$, and finance this purchase by selling one dollar's worth of η short at the same time. Therefore, the total initial value of our portfolio holdings is zero. At time T, the dollar value of our holdings in φ will be

$$\frac{Z_\varphi(T)}{Z_\varphi(0)}, \tag{7.2.7}$$

and the dollar value we owe on the short sale of η will be

$$\frac{Z_\eta(T)}{Z_\eta(0)}. \tag{7.2.8}$$

Definition 7.2.2 implies that, with probability one, (7.2.7) is not less that (7.2.8), and will be greater than (7.2.8) with positive probability. It follows that the total value of our holdings at time T will be nonnegative, with probability one, and positive with positive probability. Hence, this combination of investments is an arbitrage opportunity. Therefore, proof of the existence of a pair of admissible portfolios, one of which dominates the other, implies an arbitrage opportunity exists and the no-arbitrage hypothesis fails [Fernholz (2002)].

Lemma 7.2.3 ([**Fernholz (1999c)**]). *Let $\mathbf{G}$ be a generating function of the portfolio φ with drift process Θ (or, $\mathfrak{g}$), and suppose that c_1 and c_2 are constants. Then we have the following:*

(i) *If for all $t>0$, $\mathbf{G}\big(\boldsymbol{\mu}(t)\big) > c_1 > 0$, a.s., and $\mathfrak{g}(t) > c_2 > 0$ (i.e., $\mathfrak{g}(t)$ has a **positive lower bound**), a.s., then the portfolio φ **strictly dominates** the market portfolio μ.*

(*ii*) *If for all* $t > 0$, $0 < \mathbf{G}\big(\boldsymbol{\mu}(t)\big) < c_1$, *a.s., and* $\mathfrak{g}(t) < c_2 < 0$ (*i.e.,* $\mathfrak{g}(t)$ *has a* ***negative upper bound***), *a.s., then the market portfolio* μ ***strictly dominates*** *the portfolio* φ.

Proof. Suppose that the first set of conditions with $c_1 > 0$ and $c_2 > 0$ hold, i.e., we have $\mathbf{G}\big(\boldsymbol{\mu}(t)\big) > c_1 > 0$ and $\mathfrak{g}(t) > c_2 > 0$ for all $t > 0$. Then, from equation (5.2.6),

$$\log\left(\frac{Z_\varphi(T)}{Z_\varphi(0)}\right) - \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) \;=\; \log\left(\frac{\mathbf{G}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{G}\big(\boldsymbol{\mu}(0)\big)}\right) + \int_0^T \mathfrak{g}(t)\,dt, \qquad T \in [0,\infty), \tag{7.2.9}$$

it follows that a.s., for $T > 0$,

$$\log\left(\frac{Z_\varphi(T)}{Z_\varphi(0)}\right) - \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) \;>\; \log\left(\frac{c_1}{\mathbf{G}\big(\boldsymbol{\mu}(0)\big)}\right) + c_2\int_0^T dt.$$

Consequently, we have the following inequality

$$\log\left(\frac{Z_\varphi(T)}{Z_\varphi(0)}\right) - \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) \;>\; \log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\,T. \tag{7.2.10}$$

This is equivalently expressed as

$$\log\left(\frac{Z_\varphi(T)}{Z_\varphi(0)}\right) \;>\; \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\,T, \tag{7.2.11}$$

and, we have

$$\frac{Z_\varphi(T)}{Z_\varphi(0)} \;>\; \frac{Z_\mu(T)}{Z_\mu(0)}\,\exp\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\,T\Big). \tag{7.2.12}$$

Therefore, by (7.2.5) and (7.2.6) of Definition 7.2.2, the portfolio φ strictly dominates the market portfolio μ if $\exp\big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\,T\big) > 1$, thus if $\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\,T > 0$, i.e., if (bearing in mind that $c_2 > 0$),

$$T \;>\; \frac{\log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) - \log c_1}{c_2}. \tag{7.2.13}$$

Since, if

$$\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\,T \;>\; 0, \tag{7.2.14}$$

then

$$\log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\,T \;>\; \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right),$$

which results in the following expression

$$\log\left(\frac{Z_\varphi(T)}{Z_\varphi(0)}\right) \;>\; \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\,T \;>\; \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right).$$

Alternatively, if

$$\exp\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\,T\Big) \;>\; 1,$$

then

$$\frac{Z_\mu(T)}{Z_\mu(0)}\,\exp\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\,T\Big) \;>\; \frac{Z_\mu(T)}{Z_\mu(0)},$$

which results in the following expression

$$\frac{Z_\varphi(T)}{Z_\varphi(0)} \quad > \quad \frac{Z_\mu(T)}{Z_\mu(0)} \exp\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\Big) \quad > \quad \frac{Z_\mu(T)}{Z_\mu(0)}.$$

Thus we have

$$\log\left(\frac{Z_\varphi(T)}{Z_\varphi(0)}\right) \quad > \quad \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right), \qquad \text{a.s.} \tag{7.2.15}$$

Hence,

$$\frac{Z_\varphi(T)}{Z_\varphi(0)} \quad > \quad \frac{Z_\mu(T)}{Z_\mu(0)}, \qquad \text{a.s.,} \tag{7.2.16}$$

and, we have

$$\mathbb{P}\left(\frac{Z_\varphi(T)}{Z_\varphi(0)} \quad > \quad \frac{Z_\mu(T)}{Z_\mu(0)}\right) \quad = \quad 1. \tag{7.2.17}$$

The proof for the second set of conditions with $c_1 > 0$ and $c_2 < 0$ is similar. Suppose that the second set of conditions with $c_1 > 0$ and $c_2 < 0$ hold, i.e., we have $0 < \mathbf{G}\big(\boldsymbol{\mu}(t)\big) < c_1$ and $\mathfrak{g}(t) < c_2 < 0$ for all $t > 0$. Then, from equation (5.2.6),

$$\log\left(\frac{Z_\varphi(T)}{Z_\varphi(0)}\right) - \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) \quad = \quad \log\left(\frac{\mathbf{G}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{G}\big(\boldsymbol{\mu}(0)\big)}\right) + \int_0^T \mathfrak{g}(t)\, dt, \qquad T \in [0, \infty), \tag{7.2.18}$$

it follows that a.s., for $T > 0$,

$$\log\left(\frac{Z_\varphi(T)}{Z_\varphi(0)}\right) - \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) \quad < \quad \log\left(\frac{c_1}{\mathbf{G}\big(\boldsymbol{\mu}(0)\big)}\right) + c_2 \int_0^T dt.$$

Consequently, we have the following inequality

$$\log\left(\frac{Z_\varphi(T)}{Z_\varphi(0)}\right) - \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) \quad < \quad \log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T. \tag{7.2.19}$$

This is equivalently expressed as

$$\log\left(\frac{Z_\varphi(T)}{Z_\varphi(0)}\right) \quad < \quad \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T, \tag{7.2.20}$$

and, we have

$$\frac{Z_\varphi(T)}{Z_\varphi(0)} \quad < \quad \frac{Z_\mu(T)}{Z_\mu(0)} \exp\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\Big). \tag{7.2.21}$$

Therefore, by (7.2.5) and (7.2.6) of Definition 7.2.2, the market portfolio μ strictly dominates the portfolio φ if $\exp\big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\big) < 1$, thus if $\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T < 0$, i.e., if (bearing in mind that $c_2 < 0$),

$$T \quad > \quad \frac{\log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) - \log c_1}{c_2}. \tag{7.2.22}$$

Alternatively, if $c_2 := -c_3 < 0$ where $c_3 > 0$, we have $\exp\big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) - c_3\, T\big) < 1$, thus if $\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) - c_3\, T < 0$, i.e., if (bearing in mind that $c_3 > 0$),

$$T \quad > \quad \frac{\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big)}{c_3}. \tag{7.2.23}$$

Since, if

$$\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T \quad < \quad 0, \tag{7.2.24}$$

then

$$\log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T \quad < \quad \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right),$$

which results in the following expression

$$\log\left(\frac{Z_\varphi(T)}{Z_\varphi(0)}\right) \quad < \quad \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T \quad < \quad \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right).$$

Alternatively, if

$$\exp\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\Big) \quad < \quad 1,$$

then

$$\frac{Z_\mu(T)}{Z_\mu(0)} \exp\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\Big) \quad < \quad \frac{Z_\mu(T)}{Z_\mu(0)},$$

which results in the following expression

$$\frac{Z_\varphi(T)}{Z_\varphi(0)} \quad < \quad \frac{Z_\mu(T)}{Z_\mu(0)} \exp\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\Big) \quad < \quad \frac{Z_\mu(T)}{Z_\mu(0)}.$$

Thus we have

$$\log\left(\frac{Z_\varphi(T)}{Z_\varphi(0)}\right) \quad < \quad \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right), \qquad \text{a.s.} \tag{7.2.25}$$

Hence,

$$\frac{Z_\varphi(T)}{Z_\varphi(0)} \quad < \quad \frac{Z_\mu(T)}{Z_\mu(0)}, \qquad \text{a.s.} \tag{7.2.26}$$

Alternatively, we have

$$\log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) \quad > \quad \log\left(\frac{Z_\varphi(T)}{Z_\varphi(0)}\right), \qquad \text{a.s.} \tag{7.2.27}$$

Hence,

$$\frac{Z_\mu(T)}{Z_\mu(0)} \quad > \quad \frac{Z_\varphi(T)}{Z_\varphi(0)}, \qquad \text{a.s.,} \tag{7.2.28}$$

and, we have

$$\mathbb{P}\left(\frac{Z_\mu(T)}{Z_\mu(0)} \quad > \quad \frac{Z_\varphi(T)}{Z_\varphi(0)}\right) \quad = \quad 1. \tag{7.2.29}$$

■

Under appropriate conditions, dominance relationships between pairs of functionally generated portfolios can be similarly established. If we can find appropriate conditions for $\mathbf{G}$ and Θ, we can apply the above lemma and establish dominance relationships between pairs of functionally generated portfolios.

7.3 Relative Arbitrage

The existence of a portfolio which strictly dominates another implies the existence of an arbitrage opportunity because we can buy a dollar's worth of the dominating portfolio and pay for it by selling a dollar's worth of the dominated portfolio short. The notion of arbitrage is of paramount importance in mathematical finance. Here, we present an allied notion, that of *relative arbitrage*. In fact, we shall introduce three concepts of relative arbitrage, and thus provide a formal definition of weak and strong relative arbitrage opportunities in the process.

7.3.1 Weak Relative Arbitrage, Strong Relative Arbitrage and Superior Long-Term Growth Opportunities

Definition 7.3.1 (Relative Arbitrage: Weak Relative Arbitrage, Strong Relative Arbitrage, Superior Long-Term Growth Opportunity). *Given any two portfolios φ and η, with the same initial capital $Z_\varphi(0) = Z_\eta(0) = w > 0$ (sometimes given as, $Z_\varphi(0) = Z_\eta(0) = 1$), we shall say that:*

(*i*) *the portfolio φ represents a* ***weak arbitrage opportunity*** *relative to the portfolio η (i.e., a* ***weak relative arbitrage opportunity****) over the fixed, finite time horizon $[0,T]$, for $T > 0$ a given real number, if there exists a real constant $q \equiv q_{\varphi,\eta,T} > 0$, such that the following condition*

$$\mathbb{P}\left(\frac{Z_\varphi(t)}{Z_\eta(t)} \;\geq\; q, \;\text{ for all } t \in [0,T]\right) \;=\; 1, \tag{7.3.1}$$

holds, or, put slightly differently, such that

$$\mathbb{P}\Big(Z_\varphi(t) \;\geq\; q\, Z_\eta(t), \;\text{ for all } t \in [0,T]\Big) \;=\; 1, \tag{7.3.2}$$

holds, and, if we have

$$\mathbb{P}\Big(Z_\varphi(T) \;\geq\; Z_\eta(T)\Big) \;=\; 1, \qquad \textit{and} \qquad \mathbb{P}\Big(Z_\varphi(T) \;>\; Z_\eta(T)\Big) \;>\; 0, \tag{7.3.3}$$

and we can then say that the market $\mathcal{M}$ admits ***weak relative arbitrage*** *at time T, and we shall say that the portfolio pair $\big(\varphi(t), \eta(t)\big)$ constitutes a weak relative arbitrage opportunity over $[0,T]$, i.e., a weak relative arbitrage opportunity over $[0,T]$ is a pair of portfolios $\big(\varphi(t), \eta(t)\big)$;*

(*ii*) *the portfolio φ represents a* ***strong arbitrage opportunity*** *relative to the portfolio η (i.e., a* ***strong relative arbitrage opportunity****) over the fixed, finite time horizon $[0,T]$, for $T > 0$ a given real number, if instead we have the stronger condition*

$$\mathbb{P}\Big(Z_\varphi(T) \;>\; Z_\eta(T)\Big) \;=\; 1, \tag{7.3.4}$$

and we can then say that the market $\mathcal{M}$ admits ***strong relative arbitrage*** *at time T, and we shall say that the portfolio pair $\big(\varphi(t), \eta(t)\big)$ constitutes a strong relative arbitrage opportunity over $[0,T]$, i.e., a strong relative arbitrage opportunity over $[0,T]$ is a pair of portfolios $\big(\varphi(t), \eta(t)\big)$;*

(*iii*) *the portfolio φ represents relative to the portfolio η, a* ***superior long-term growth opportunity*** *(i.e., the portfolio φ is a* ***superior long-term growth opportunity*** *relative to the portfolio η), if*

$$\mathcal{L}_{\varphi,\eta} \;\triangleq\; \liminf_{T\to\infty} \frac{1}{T} \log\Big(Z_\varphi(T)/Z_\eta(T)\Big) \;>\; 0, \qquad a.s., \tag{7.3.5}$$

holds.

Relative arbitrage can be regarded as a criterion to judge the performance of portfolios. If the portfolio φ is an arbitrage opportunity relative to the portfolio η, then the portfolio φ has a better performance than the portfolio η in the market. Note that no such arbitrage can exist if the market possesses an equivalent martingale measure, but what of other markets? In light of this, we shall further investigate the possibility of arbitrage in diverse equity markets.

Lemma 7.3.2. *Let* $\mathbf{G}$ *be a generating function of the portfolio* φ *with the same initial capital value as the market portfolio* μ*, i.e.,* $Z_\varphi(0) = Z_\mu(0) = w > 0$*, with drift process* Θ *(or,* $\mathfrak{g}$*), and suppose that* c_1 *and* c_2 *are constants. Then we have the following:*

(*i*) *If for all* $t > 0$*,* $\mathbf{G}\big(\boldsymbol{\mu}(t)\big) > c_1 > 0$*, a.s., and* $\mathfrak{g}(t) > c_2 > 0$ *(i.e.,* $\mathfrak{g}(t)$ *has a* ***positive lower bound****), a.s., then the portfolio* φ *represents a* ***strong arbitrage opportunity*** *relative to the market portfolio* μ *(i.e., a* ***strong relative arbitrage opportunity****) over the fixed, finite time horizon* $[0, T]$*, for* $T > 0$ *a given real number.*

(*ii*) *If for all* $t > 0$*,* $0 < \mathbf{G}\big(\boldsymbol{\mu}(t)\big) < c_1$*, a.s., and* $\mathfrak{g}(t) < c_2 < 0$ *(i.e.,* $\mathfrak{g}(t)$ *has a* ***negative upper bound****), a.s., then the market portfolio* μ *represents a* ***strong arbitrage opportunity*** *relative to the portfolio* φ *(i.e., a* ***strong relative arbitrage opportunity****) over the fixed, finite time horizon* $[0, T]$*, for* $T > 0$ *a given real number.*

Proof. Suppose that the first set of conditions with $c_1 > 0$ and $c_2 > 0$ hold, i.e., we have $\mathbf{G}\big(\boldsymbol{\mu}(t)\big) > c_1 > 0$ and $\mathfrak{g}(t) > c_2 > 0$ for all $t > 0$. If the market $\mathcal{M}$ has initial capital equal to that of the portfolio φ, $Z_\varphi(0) = Z_\mu(0) = w > 0$ such that $\log\big(Z_\varphi(0)/Z_\mu(0)\big) = \log(1) = 0$, then from the inequalities (7.2.11) and (7.2.12) of Lemma 7.2.3 and then following the proof of Lemma 7.2.3, the portfolio φ satisfies

$$\begin{aligned} \log\left(\frac{Z_\varphi(T)}{Z_\mu(T)}\right) &> \log\left(\frac{Z_\varphi(0)}{Z_\mu(0)}\right) + \log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T \\ &= \log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T, \end{aligned} \tag{7.3.6}$$

and, then the value process Z_φ of the portfolio φ a.s. satisfies

$$\begin{aligned} \frac{Z_\varphi(T)}{Z_\mu(T)} &> \frac{Z_\varphi(0)}{Z_\mu(0)} \exp\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\Big) \\ &= \exp\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\Big), \end{aligned} \tag{7.3.7}$$

so that

$$Z_\varphi(T) \;>\; Z_\mu(T) \exp\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\Big), \qquad \text{a.s.} \tag{7.3.8}$$

Consequently, in a similar fashion to that shown in the proof of Lemma 7.2.3, if

$$\exp\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\Big) \;>\; 1,$$

then

$$Z_\mu(T) \exp\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\Big) \;>\; Z_\mu(T),$$

and the following results

$$Z_\varphi(T) \;>\; Z_\mu(T) \exp\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\Big) \;>\; Z_\mu(T).$$

Hence,

$$Z_\varphi(T) \;>\; Z_\mu(T), \qquad \text{a.s.} \tag{7.3.9}$$

Therefore, we have in particular

$$\mathbb{P}\Big(Z_\varphi(T) \;>\; Z_\mu(T)\Big) \;=\; 1, \tag{7.3.10}$$

provided that

$$T \;>\; T_* \;\triangleq\; \frac{\log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) - \log c_1}{c_2}. \tag{7.3.11}$$

Consequently, from (7.3.4) of Definition 7.3.1, the portfolio φ is a strong arbitrage opportunity relative to the market portfolio μ, and φ outperforms the market portfolio μ. Notice that the portfolio φ will only outperform the market portfolio over the long term since the lower bound on T

$$T \quad > \quad \frac{\log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) - \log c_1}{c_2},$$

is sufficiently large as it contains the sufficiently small numbers $c_1 > 0$ and $c_2 > 0$, where the latter constant is present in the denominator of the lower bound. Thus, this lemma signifies that, with probability one, the portfolio φ has a return above that of the market portfolio over a sufficiently long time horizon $[0, T]$. The proof for the second set of conditions with $c_1 > 0$ and $c_2 < 0$ is similar. ■

Lemma 7.3.3. *Let* $\mathbf{G}$ *be a generating function of the portfolio* φ *with the same initial capital value as the market portfolio* μ*, i.e.,* $Z_\varphi(0) = Z_\mu(0) = w > 0$*, with drift process* Θ *(or,* $\mathfrak{g}$*), and suppose that* c_1 *and* c_2 *are constants. Then we have the following:*

(*i*) *If for all* $t > 0$*,* $\mathbf{G}\big(\boldsymbol{\mu}(t)\big) > c_1 > 0$*, a.s., and* $\mathfrak{g}(t) > c_2 > 0$ *(i.e.,* $\mathfrak{g}(t)$ *has a* ***positive lower bound****), a.s., then the portfolio* φ *represents relative to the market portfolio* μ*, a* ***superior long-term growth opportunity*** *(i.e., the portfolio* φ *is a* ***superior long-term growth opportunity*** *relative to the market portfolio* μ*).*

(*ii*) *If for all* $t > 0$*,* $0 < \mathbf{G}\big(\boldsymbol{\mu}(t)\big) < c_1$*, a.s., and* $\mathfrak{g}(t) < c_2 < 0$ *(i.e.,* $\mathfrak{g}(t)$ *has a* ***negative upper bound****), a.s., then the market portfolio* μ *represents relative to the portfolio* φ*, a* ***superior long-term growth opportunity*** *(i.e., the market portfolio* μ *is a* ***superior long-term growth opportunity*** *relative to the portfolio* φ*).*

Proof. Suppose that the first set of conditions with $c_1 > 0$ and $c_2 > 0$ hold, i.e., we have $\mathbf{G}\big(\boldsymbol{\mu}(t)\big) > c_1 > 0$ and $\mathfrak{g}(t) > c_2 > 0$ for all $t > 0$. The result from the proof of Lemma 7.3.2 above (7.3.6) gives the a.s. long-term comparison

$$\begin{aligned}
\mathcal{L}_{\varphi,\mu} &= \liminf_{T\to\infty} \frac{1}{T} \log\Big(Z_\varphi(T)/Z_\mu(T)\Big) \\
&> \liminf_{T\to\infty} \frac{1}{T}\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\Big) \\
&= \liminf_{T\to\infty} \left(\frac{\big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big)\big)}{T} + c_2\right) \\
&= \liminf_{T\to\infty} \frac{\big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big)\big)}{T} + \liminf_{T\to\infty} c_2 \\
&= \liminf_{T\to\infty} \frac{\big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big)\big)}{T} + c_2 \\
&= c_2 \quad > \quad 0, \qquad (7.3.12)
\end{aligned}$$

since $c_2 > 0$. Thus, we have

$$\mathcal{L}_{\varphi,\mu} \quad = \quad \liminf_{T\to\infty} \frac{1}{T} \log\Big(Z_\varphi(T)/Z_\mu(T)\Big) \quad > \quad c_2 \quad > \quad 0, \qquad \text{a.s.,} \qquad (7.3.13)$$

and the portfolio φ is a superior long-term growth opportunity relative to the market, i.e., the market portfolio will lag rather significantly behind the portfolio φ over long time horizons. The proof for the second set of conditions with $c_1 > 0$ and $c_2 < 0$ is similar. ■

7.4 Dividends and Portfolio Dominance Relationships

We shall use $\widehat{Z}_\mu$ to represent the total return process of the market portfolio μ, and δ_μ to represent its dividend rate. In this section, we shall extend Definition 7.2.2 so that it applies to stocks and portfolio which include dividends.

7.4.1 Dividends and Admissible Portfolios

Definition 7.4.1 (Admissible Portfolios). *A portfolio φ is **admissible** if:*

(*i*) *for $i = 1, \ldots, n$, $\varphi_i(t) \geq 0$, $t \in [0, T]$;*

(*ii*) *there exists a constant $c > 0$ such that*

$$\frac{\widehat{Z}_\varphi(t)}{\widehat{Z}_\varphi(0)} \geq c\,\frac{\widehat{Z}_\mu(t)}{\widehat{Z}_\mu(0)}, \qquad t \in [0, T], \qquad a.s.; \tag{7.4.1}$$

(*iii*) *there exists a constant M such that, for $i = 1, \ldots, n$,*

$$\frac{\varphi_i(t)}{\mu_i(t)} \leq M, \qquad t \in [0, T], \qquad a.s. \tag{7.4.2}$$

7.4.2 Dividends and Dominating Portfolios and Strictly Dominating Portfolios

Definition 7.4.2 (Dominating Portfolios, Strictly Dominating Portfolios). *Let φ and η be portfolios, with total return processes $\widehat{Z}_\varphi$ and $\widehat{Z}_\eta$, respectively. Then the portfolio φ **dominates** the portfolio η on $[0, T]$, if there is a number $T > 0$, such that for any positive initial capital values $\widehat{Z}_\varphi(0)$ and $\widehat{Z}_\eta(0)$, we have*

$$\frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\varphi(0)} \geq \frac{\widehat{Z}_\eta(T)}{\widehat{Z}_\eta(0)}, \qquad a.s., \tag{7.4.3}$$

and

$$\mathbb{P}\left(\frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\varphi(0)} \geq \frac{\widehat{Z}_\eta(T)}{\widehat{Z}_\eta(0)}\right) = 1, \qquad \text{and} \qquad \mathbb{P}\left(\frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\varphi(0)} > \frac{\widehat{Z}_\eta(T)}{\widehat{Z}_\eta(0)}\right) > 0. \tag{7.4.4}$$

*The portfolio φ **strictly dominates** the portfolio η on $[0, T]$, if there is a number $T > 0$, such that for any positive initial capital values $\widehat{Z}_\varphi(0)$ and $\widehat{Z}_\eta(0)$, we have*

$$\frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\varphi(0)} > \frac{\widehat{Z}_\eta(T)}{\widehat{Z}_\eta(0)}, \qquad a.s., \tag{7.4.5}$$

and

$$\mathbb{P}\left(\frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\varphi(0)} > \frac{\widehat{Z}_\eta(T)}{\widehat{Z}_\eta(0)}\right) = 1. \tag{7.4.6}$$

This definition coincides with Definition 7.2.2 for portfolios of stocks which pay no dividends.

Proposition 7.4.3 ([**Fernholz (1998a)**]). *Suppose $\mathbf{G}$ generates the portfolio φ with drift process Θ (or, $\mathfrak{g}$), and suppose that there exist constants c_1 and c_2. Then we have the following:*

(*i*) *If for all $t > 0$, $\mathbf{G}\big(\boldsymbol{\mu}(t)\big) > c_1 > 0$, a.s., and $\delta_\varphi(t) - \delta_\mu(t) + \mathfrak{g}(t) > c_2 > 0$ (i.e., $\delta_\varphi(t) - \delta_\mu(t) + \mathfrak{g}(t)$ has a **positive lower bound**), a.s., then the portfolio φ **strictly dominates** the market portfolio μ.*

(*ii*) *If for all $t > 0$, $0 < \mathbf{G}\big(\boldsymbol{\mu}(t)\big) < c_1$, a.s., and $\delta_\varphi(t) - \delta_\mu(t) + \mathfrak{g}(t) < c_2 < 0$ (i.e., $\delta_\varphi(t) - \delta_\mu(t) + \mathfrak{g}(t)$ has a **negative upper bound**), a.s., then the market portfolio μ **strictly dominates** the portfolio φ.*

Proof. Suppose that the first set of conditions with $c_1 > 0$ and $c_2 > 0$ hold, i.e., we have $\mathbf{G}\big(\boldsymbol{\mu}(t)\big) > c_1 > 0$ and $\delta_\varphi(t) - \delta_\mu(t) + \mathfrak{g}(t) > c_2 > 0$ for all $t > 0$. Then, from equation (5.8.3), we have

$$\log\left(\frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\varphi(0)}\right) - \log\left(\frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)}\right) = \log\left(\frac{\mathbf{G}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{G}\big(\boldsymbol{\mu}(0)\big)}\right) + \int_0^T \Big(\delta_\varphi(t) - \delta_\mu(t) + \mathfrak{g}(t)\Big)\,dt, \tag{7.4.7}$$

it follows that a.s., for $T > 0$,

$$\log\left(\frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\varphi(0)}\right) - \log\left(\frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)}\right) \quad > \quad \log\left(\frac{c_1}{\mathbf{G}\big(\boldsymbol{\mu}(0)\big)}\right) + c_2 \int_0^T dt.$$

Consequently, we have the following inequality

$$\log\left(\frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\varphi(0)}\right) - \log\left(\frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)}\right) \quad > \quad \log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T. \tag{7.4.8}$$

This is equivalently expressed as

$$\log\left(\frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\varphi(0)}\right) \quad > \quad \log\left(\frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)}\right) + \log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T, \tag{7.4.9}$$

and, we have

$$\frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\varphi(0)} \quad > \quad \frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)} \exp\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\Big). \tag{7.4.10}$$

Therefore, by (7.4.5) and (7.4.6) of Definition 7.4.2, the portfolio φ strictly dominates the market portfolio μ if $\exp\big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\big) > 1$, thus if $\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T > 0$, i.e., if (bearing in mind that $c_2 > 0$),

$$T \quad > \quad \frac{\log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) - \log c_1}{c_2}. \tag{7.4.11}$$

Since, if

$$\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T \quad > \quad 0, \tag{7.4.12}$$

then

$$\log\left(\frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)}\right) + \log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T \quad > \quad \log\left(\frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)}\right),$$

which results in the following expression

$$\log\left(\frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\varphi(0)}\right) \quad > \quad \log\left(\frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)}\right) + \log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T \quad > \quad \log\left(\frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)}\right).$$

Alternatively, if

$$\exp\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\Big) \quad > \quad 1,$$

then

$$\frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)} \exp\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\Big) \quad > \quad \frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)},$$

which results in the following expression

$$\frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\varphi(0)} \quad > \quad \frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)} \exp\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\Big) \quad > \quad \frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)}.$$

Thus we have

$$\log\left(\frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\varphi(0)}\right) \quad > \quad \log\left(\frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)}\right), \qquad \text{a.s.} \tag{7.4.13}$$

Hence,

$$\frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\varphi(0)} \quad > \quad \frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)}, \qquad \text{a.s.,} \tag{7.4.14}$$

and, we have

$$\mathbb{P}\left(\frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\varphi(0)} \quad > \quad \frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)}\right) \quad = \quad 1. \tag{7.4.15}$$

The proof for the second set of conditions with $c_1 > 0$ and $c_2 < 0$ is similar. Suppose that the second set of conditions with $c_1 > 0$ and $c_2 < 0$ hold, i.e., we have $0 < \mathbf{G}\big(\boldsymbol{\mu}(t)\big) < c_1$ and $\delta_\varphi(t) - \delta_\mu(t) + \mathfrak{g}(t) < c_2 < 0$ for all $t > 0$. Then, from equation (5.8.3), we have

$$\log\left(\frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\varphi(0)}\right) - \log\left(\frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)}\right) \quad = \quad \log\left(\frac{\mathbf{G}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{G}\big(\boldsymbol{\mu}(0)\big)}\right) + \int_0^T \Big(\delta_\varphi(t) - \delta_\mu(t) + \mathfrak{g}(t)\Big)\, dt, \tag{7.4.16}$$

it follows that a.s., for $T > 0$,

$$\log\left(\frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\varphi(0)}\right) - \log\left(\frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)}\right) \quad < \quad \log\left(\frac{c_1}{\mathbf{G}\big(\boldsymbol{\mu}(0)\big)}\right) + c_2 \int_0^T dt.$$

Consequently, we have the following inequality

$$\log\left(\frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\varphi(0)}\right) - \log\left(\frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)}\right) \quad < \quad \log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T. \tag{7.4.17}$$

This is equivalently expressed as

$$\log\left(\frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\varphi(0)}\right) \quad < \quad \log\left(\frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)}\right) + \log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T, \tag{7.4.18}$$

and, we have

$$\frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\varphi(0)} \quad < \quad \frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)} \exp\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\Big). \tag{7.4.19}$$

Therefore, by (7.4.5) and (7.4.6) of Definition 7.4.2, the market portfolio μ strictly dominates the portfolio φ if $\exp\big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\big) < 1$, thus if $\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T < 0$, i.e., if (bearing in mind that $c_2 < 0$),

$$T \quad > \quad \frac{\log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) - \log c_1}{c_2}. \tag{7.4.20}$$

Alternatively, if $c_2 := -c_3 < 0$ where $c_3 > 0$, we have $\exp\big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) - c_3\, T\big) < 1$, thus if $\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) - c_3\, T < 0$, i.e., if (bearing in mind that $c_3 > 0$),

$$T \quad > \quad \frac{\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big)}{c_3}. \tag{7.4.21}$$

Since, if

$$\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T \quad < \quad 0, \tag{7.4.22}$$

then

$$\log\left(\frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)}\right) + \log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T \quad < \quad \log\left(\frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)}\right),$$

which results in the following expression

$$\log\left(\frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\varphi(0)}\right) \quad < \quad \log\left(\frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)}\right) + \log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T \quad < \quad \log\left(\frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)}\right).$$

Alternatively, if

$$\exp\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\Big) \quad < \quad 1,$$

then

$$\frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)} \exp\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\Big) \quad < \quad \frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)},$$

which results in the following expression

$$\frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\varphi(0)} \quad < \quad \frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)} \exp\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\Big) \quad < \quad \frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)}.$$

Thus we have

$$\log\left(\frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\varphi(0)}\right) \quad < \quad \log\left(\frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)}\right), \qquad \text{a.s.} \tag{7.4.23}$$

Hence,

$$\frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\varphi(0)} \quad < \quad \frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)}, \qquad \text{a.s.} \tag{7.4.24}$$

Alternatively, we have

$$\log\left(\frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)}\right) \quad > \quad \log\left(\frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\varphi(0)}\right), \qquad \text{a.s.} \tag{7.4.25}$$

Hence,

$$\frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)} \quad > \quad \frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\varphi(0)}, \qquad \text{a.s.,} \tag{7.4.26}$$

and, we have

$$\mathbb{P}\left(\frac{\widehat{Z}_\mu(T)}{\widehat{Z}_\mu(0)} \quad > \quad \frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\varphi(0)}\right) \quad = \quad 1. \tag{7.4.27}$$

■

This proposition shows that the validity of the no-arbitrage hypothesis depends on the behaviour of the processes $\mathbf{G}\big(\boldsymbol{\mu}(t)\big), \delta_\varphi, \delta_\mu$, and Θ. The processes $\mathbf{G}\big(\boldsymbol{\mu}(t)\big), \delta_\varphi$, and δ_μ, are all observable, and Θ can be calculated from observable processes using (5.2.12).

7.5 Dividends and Relative Arbitrage

7.5.1 Dividends and Weak Relative Arbitrage, Strong Relative Arbitrage and Superior Long-Term Growth Opportunities

Definition 7.5.1 (Relative Arbitrage: Weak Relative Arbitrage, Strong Relative Arbitrage, Superior Long-Term Growth Opportunity)**.** *Given any two portfolios φ and η, with the same initial capital $\widehat{Z}_\varphi(0) = \widehat{Z}_\eta(0) = w > 0$ (sometimes given as, $Z_\varphi(0) = Z_\eta(0) = 1$), we shall say that:*

(*i*) *the portfolio φ represents a* ***weak arbitrage opportunity*** *relative to the portfolio η (i.e., a* ***weak relative arbitrage opportunity****) over the fixed, finite time horizon $[0,T]$, for $T>0$ a given real number, if there exists a real constant $q \equiv q_{\varphi,\eta,T} > 0$, such that the following condition*

$$\mathbb{P}\left(\frac{\widehat{Z}_\varphi(t)}{\widehat{Z}_\eta(t)} \;\geq\; q, \;\text{ for all } t \in [0,T]\right) \;=\; 1, \tag{7.5.1}$$

holds, or, put slightly differently, such that

$$\mathbb{P}\left(\widehat{Z}_\varphi(t) \;\geq\; q\,\widehat{Z}_\eta(t), \;\text{ for all } t \in [0,T]\right) \;=\; 1, \tag{7.5.2}$$

holds, and, if we have

$$\mathbb{P}\left(\widehat{Z}_\varphi(T) \;\geq\; \widehat{Z}_\eta(T)\right) \;=\; 1, \qquad \textit{and} \qquad \mathbb{P}\left(\widehat{Z}_\varphi(T) \;>\; \widehat{Z}_\eta(T)\right) \;>\; 0, \tag{7.5.3}$$

and we can then say that the market $\mathcal{M}$ admits ***weak relative arbitrage*** *at time T, and we shall say that the portfolio pair $\big(\varphi(t),\eta(t)\big)$ constitutes a weak relative arbitrage opportunity over $[0,T]$, i.e., a weak relative arbitrage opportunity over $[0,T]$ is a pair of portfolios $\big(\varphi(t),\eta(t)\big)$;*

(*ii*) *the portfolio φ represents a* ***strong arbitrage opportunity*** *relative to the portfolio η (i.e., a* ***strong relative arbitrage opportunity****) over the fixed, finite time horizon $[0,T]$, for $T>0$ a given real number, if instead we have the stronger condition*

$$\mathbb{P}\left(\widehat{Z}_\varphi(T) \;>\; \widehat{Z}_\eta(T)\right) \;=\; 1, \tag{7.5.4}$$

and we can then say that the market $\mathcal{M}$ admits ***strong relative arbitrage*** *at time T, and we shall say that the portfolio pair $\big(\varphi(t),\eta(t)\big)$ constitutes a strong relative arbitrage opportunity over $[0,T]$, i.e., a strong relative arbitrage opportunity over $[0,T]$ is a pair of portfolios $\big(\varphi(t),\eta(t)\big)$;*

(*iii*) *the portfolio φ represents relative to the portfolio η, a* ***superior long-term growth opportunity*** *(i.e., the portfolio φ is a* ***superior long-term growth opportunity*** *relative to the portfolio η), if*

$$\widehat{\mathcal{L}}_{\varphi,\eta} \;\triangleq\; \liminf_{T\to\infty} \frac{1}{T}\,\log\left(\widehat{Z}_\varphi(T)/\widehat{Z}_\eta(T)\right) \;>\; 0, \qquad \textit{a.s.}, \tag{7.5.5}$$

holds.

Lemma 7.5.2. *Let $\mathbf{G}$ be a generating function of the portfolio φ with the same initial capital value as the market portfolio μ, i.e., $\widehat{Z}_\varphi(0) = \widehat{Z}_\mu(0) = w > 0$, with drift process Θ (or, $\mathfrak{g}$), and suppose that c_1 and c_2 are constants. Then we have the following:*

(*i*) *If for all $t>0$, $\mathbf{G}\big(\boldsymbol{\mu}(t)\big) > c_1 > 0$, a.s., and $\delta_\varphi(t) - \delta_\mu(t) + \mathfrak{g}(t) > c_2 > 0$ (i.e., $\delta_\varphi(t) - \delta_\mu(t) + \mathfrak{g}(t)$ has a* ***positive lower bound****), a.s., then the portfolio φ represents a* ***strong arbitrage opportunity*** *relative to the market portfolio μ (i.e., a* ***strong relative arbitrage opportunity****) over the fixed, finite time horizon $[0,T]$, for $T>0$ a given real number.*

(*ii*) *If for all $t>0$, $0 < \mathbf{G}\big(\boldsymbol{\mu}(t)\big) < c_1$, a.s., and $\delta_\varphi(t) - \delta_\mu(t) + \mathfrak{g}(t) < c_2 < 0$ (i.e., $\delta_\varphi(t) - \delta_\mu(t) + \mathfrak{g}(t)$ has a* ***negative upper bound****), a.s., then the market portfolio μ represents a* ***strong arbitrage opportunity*** *relative to the portfolio φ (i.e., a* ***strong relative arbitrage opportunity****) over the fixed, finite time horizon $[0,T]$, for $T>0$ a given real number.*

Proof. Suppose that the first set of conditions with $c_1>0$ and $c_2>0$ hold, i.e., we have $\mathbf{G}\big(\boldsymbol{\mu}(t)\big) > c_1 > 0$ and $\delta_\varphi(t) - \delta_\mu(t) + \mathfrak{g}(t) > c_2 > 0$ for all $t>0$. If the market $\mathcal{M}$ has initial capital equal to that of the portfolio φ, $\widehat{Z}_\varphi(0) = \widehat{Z}_\mu(0) = w > 0$ such that $\log\big(\widehat{Z}_\varphi(0)/\widehat{Z}_\mu(0)\big) = \log(1) = 0$, then from the inequalities (7.4.9) and (7.4.10) of Proposition 7.4.3 and then following the proof of Proposition 7.4.3, the portfolio φ satisfies

$$\begin{aligned}\log\left(\frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\mu(T)}\right) \;&>\; \log\left(\frac{\widehat{Z}_\varphi(0)}{\widehat{Z}_\mu(0)}\right) + \log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\,T \\ &=\; \log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\,T,\end{aligned} \tag{7.5.6}$$

and, then the value process $\widehat{Z}_\varphi$ of the portfolio φ a.s. satisfies

$$\begin{aligned}\frac{\widehat{Z}_\varphi(T)}{\widehat{Z}_\mu(T)} &> \frac{\widehat{Z}_\varphi(0)}{\widehat{Z}_\mu(0)} \exp\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\Big) \\ &= \exp\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\Big), \end{aligned} \tag{7.5.7}$$

so that

$$\widehat{Z}_\varphi(T) \;>\; \widehat{Z}_\mu(T) \exp\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\Big), \qquad \text{a.s.} \tag{7.5.8}$$

Consequently, in a similar fashion to that shown in the proof of Proposition 7.4.3, if

$$\exp\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\Big) \;>\; 1,$$

then

$$\widehat{Z}_\mu(T) \exp\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\Big) \;>\; \widehat{Z}_\mu(T),$$

and the following results

$$\widehat{Z}_\varphi(T) \;>\; \widehat{Z}_\mu(T) \exp\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\Big) \;>\; \widehat{Z}_\mu(T).$$

Hence,

$$\widehat{Z}_\varphi(T) \;>\; \widehat{Z}_\mu(T), \qquad \text{a.s.} \tag{7.5.9}$$

Therefore, we have in particular

$$\mathbb{P}\Big(\widehat{Z}_\varphi(T) \;>\; \widehat{Z}_\mu(T)\Big) \;=\; 1, \tag{7.5.10}$$

provided that

$$T \;>\; T_* \;\triangleq\; \frac{\log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) - \log c_1}{c_2}. \tag{7.5.11}$$

Consequently, from (7.5.4) of Definition 7.5.1, the portfolio φ is a strong arbitrage opportunity relative to the market portfolio μ, and φ outperforms the market portfolio μ. Notice that the portfolio φ will only outperform the market portfolio over the long term since the lower bound on T

$$T \;>\; \frac{\log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) - \log c_1}{c_2},$$

is sufficiently large as it contains the sufficiently small numbers $c_1 > 0$ and $c_2 > 0$, where the latter constant is present in the denominator of the lower bound. Thus, this lemma signifies that, with probability one, the portfolio φ has a return above that of the market portfolio over a sufficiently long time horizon $[0, T]$. The proof for the second set of conditions with $c_1 > 0$ and $c_2 < 0$ is similar. ■

Lemma 7.5.3. *Let* $\mathbf{G}$ *be a generating function of the portfolio* φ *with the same initial capital value as the market portfolio* μ*, i.e.,* $\widehat{Z}_\varphi(0) = \widehat{Z}_\mu(0) = w > 0$*, with drift process* Θ *(or,* $\mathfrak{g}$*), and suppose that* c_1 *and* c_2 *are constants. Then we have the following:*

(*i*) *If for all* $t > 0$*,* $\mathbf{G}\big(\boldsymbol{\mu}(t)\big) > c_1 > 0$*, a.s., and* $\delta_\varphi(t) - \delta_\mu(t) + \mathfrak{g}(t) > c_2 > 0$ *(i.e.,* $\delta_\varphi(t) - \delta_\mu(t) + \mathfrak{g}(t)$ *has a* ***positive lower bound****), a.s., then the portfolio* φ *represents relative to the market portfolio* μ*, a* ***superior long-term growth opportunity*** *(i.e., the portfolio* φ *is a* ***superior long-term growth opportunity*** *relative to the market portfolio* μ*).*

(*ii*) *If for all* $t > 0$, $0 < \mathbf{G}\big(\boldsymbol{\mu}(t)\big) < c_1$, *a.s., and* $\delta_\varphi(t) - \delta_\mu(t) + \mathfrak{g}(t) < c_2 < 0$ (*i.e.,* $\delta_\varphi(t) - \delta_\mu(t) + \mathfrak{g}(t)$ *has a* ***negative upper bound***), *a.s., then the market portfolio* μ *represents relative to the portfolio* φ, *a* ***superior long-term growth opportunity*** (*i.e., the market portfolio* μ *is a* ***superior long-term growth opportunity*** *relative to the portfolio* φ).

Proof. Suppose that the first set of conditions with $c_1 > 0$ and $c_2 > 0$ hold, i.e., we have $\mathbf{G}\big(\boldsymbol{\mu}(t)\big) > c_1 > 0$ and $\delta_\varphi(t) - \delta_\mu(t) + \mathfrak{g}(t) > c_2 > 0$ for all $t > 0$. The result from the proof of Lemma 7.5.2 above (7.5.6) gives the a.s. long-term comparison

$$\begin{aligned}
\widehat{\mathcal{L}}_{\varphi,\mu} &= \liminf_{T\to\infty} \frac{1}{T} \log\Big(\widehat{Z}_\varphi(T)/\widehat{Z}_\mu(T)\Big) \\
&> \liminf_{T\to\infty} \frac{1}{T}\Big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) + c_2\, T\Big) \\
&= \liminf_{T\to\infty} \left(\frac{\big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big)\big)}{T} + c_2\right) \\
&= \liminf_{T\to\infty} \frac{\big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big)\big)}{T} + \liminf_{T\to\infty} c_2 \\
&= \liminf_{T\to\infty} \frac{\big(\log c_1 - \log \mathbf{G}\big(\boldsymbol{\mu}(0)\big)\big)}{T} + c_2 \\
&= c_2 \;>\; 0, \qquad (7.5.12)
\end{aligned}$$

since $c_2 > 0$. Thus, we have

$$\widehat{\mathcal{L}}_{\varphi,\mu} = \liminf_{T\to\infty} \frac{1}{T} \log\Big(\widehat{Z}_\varphi(T)/\widehat{Z}_\mu(T)\Big) > c_2 > 0, \qquad \text{a.s.,} \qquad (7.5.13)$$

and the portfolio φ is a superior long-term growth opportunity relative to the market, i.e., the market portfolio will lag rather significantly behind the portfolio φ over long time horizons. The proof for the second set of conditions with $c_1 > 0$ and $c_2 < 0$ is similar. ■

7.6 Relative Arbitrage Opportunities over Sufficiently Long Time Horizons

7.6.1 The Weighted-Average Capitalisation Generated Portfolio

It can be shown that in an equity market without dividends, the market portfolio μ strictly dominates the weighted-average capitalisation generated portfolio φ^ϖ, for a sufficiently large real number T. This is demonstrated in the next corollary.

Corollary 7.6.1. *Let* μ *be the market portfolio and* φ^ϖ *be the weighted-average capitalisation generated portfolio. Then for a* ***sufficiently large real number*** $T \in [0, \infty)$, *we have*

$$\frac{Z_\mu(T)}{Z_\mu(0)} > \frac{Z_{\varphi^\varpi}(T)}{Z_{\varphi^\varpi}(0)}, \qquad a.s., \qquad (7.6.1)$$

and, consequently

$$\mathbb{P}\left(\frac{Z_\mu(T)}{Z_\mu(0)} > \frac{Z_{\varphi^\varpi}(T)}{Z_{\varphi^\varpi}(0)}\right) = 1. \qquad (7.6.2)$$

That is, the market portfolio μ *represents a* ***strong arbitrage opportunity*** *relative to the weighted-average capitalisation generated portfolio* φ^ϖ *over the fixed, finite time horizon* $[0, T]$, *for* T *sufficiently large.*

Proof. Consider the function $\mathbf{S}^{\mathrm{WC}} : \mathbb{R}^n \to \mathbb{R}^+$ or $\mathbf{S}^{\mathrm{WC}} : \Delta^{n-1} \to \mathbb{R}^+$ defined by (5.4.27). By Theorem 5.2.2, $\mathbf{S}^{\mathrm{WC}}$ generates the weighted-average capitalisation generated portfolio with weights (5.4.32), and drift process

(5.4.39). By equations (5.4.41) and (5.4.44), for the weighted-average capitalisation generated portfolio, we have the following for $T \in [0, \infty)$,

$$\log\left(\frac{Z_{\varphi^{\varpi}}(T)}{Z_{\varphi^{\varpi}}(0)}\right) = \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{WC}}(\boldsymbol{\mu}(T))}{\mathbf{S}^{\mathrm{WC}}(\boldsymbol{\mu}(0))}\right) - \int_0^T \gamma^*_{\varphi^{\varpi}}(t)\,dt. \tag{7.6.3}$$

Furthermore, we have the following

$$\log\left(\frac{\mathbf{S}^{\mathrm{WC}}(\boldsymbol{\mu}(T))}{\mathbf{S}^{\mathrm{WC}}(\boldsymbol{\mu}(0))}\right) = \log \mathbf{S}^{\mathrm{WC}}(\boldsymbol{\mu}(T)) - \log \mathbf{S}^{\mathrm{WC}}(\boldsymbol{\mu}(0)),$$

to yield

$$\log\left(\frac{Z_{\varphi^{\varpi}}(T)}{Z_{\varphi^{\varpi}}(0)}\right) = \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) + \log \mathbf{S}^{\mathrm{WC}}(\boldsymbol{\mu}(T)) - \log \mathbf{S}^{\mathrm{WC}}(\boldsymbol{\mu}(0)) - \int_0^T \gamma^*_{\varphi^{\varpi}}(t)\,dt, \tag{7.6.4}$$

and, from (5.4.28), we get for all $t \in [0, T]$,

$$\log \mathbf{S}^{\mathrm{WC}}(\boldsymbol{\mu}(t)) = \frac{1}{2}\log\left(\sum_{i=1}^{n} \mu_i^2(t)\right).$$

Hence, employing the above equations, we can write (7.6.3) as

$$\log\left(\frac{Z_{\varphi^{\varpi}}(T)}{Z_{\varphi^{\varpi}}(0)}\right) = \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) + \frac{1}{2}\log\left(\sum_{i=1}^{n} \mu_i^2(T)\right) - \frac{1}{2}\log\left(\sum_{i=1}^{n} \mu_i^2(0)\right) - \int_0^T \gamma^*_{\varphi^{\varpi}}(t)\,dt.$$

If the weighted average of the capitalisation weights is the same at the beginning and at the end of a certain time period, such that

$$\frac{1}{2}\log\left(\sum_{i=1}^{n} \mu_i^2(T)\right) = \frac{1}{2}\log\left(\sum_{i=1}^{n} \mu_i^2(0)\right),$$

then the above expression can be reduced to

$$\log\left(\frac{Z_{\varphi^{\varpi}}(T)}{Z_{\varphi^{\varpi}}(0)}\right) = \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) - \int_0^T \gamma^*_{\varphi^{\varpi}}(t)\,dt, \tag{7.6.5}$$

which by Proposition 2.4.8, that states that $\gamma^*_{\varphi^{\varpi}}(t) \geq 0$ for all $t \in [0, T]$, becomes

$$\log\left(\frac{Z_{\varphi^{\varpi}}(T)}{Z_{\varphi^{\varpi}}(0)}\right) < \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right), \qquad \text{a.s.}, \tag{7.6.6}$$

or,

$$\log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) > \log\left(\frac{Z_{\varphi^{\varpi}}(T)}{Z_{\varphi^{\varpi}}(0)}\right), \qquad \text{a.s.} \tag{7.6.7}$$

Hence,

$$\frac{Z_{\mu}(T)}{Z_{\mu}(0)} > \frac{Z_{\varphi^{\varpi}}(T)}{Z_{\varphi^{\varpi}}(0)}, \qquad \text{a.s.}, \tag{7.6.8}$$

and, we have

$$\mathbb{P}\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)} > \frac{Z_{\varphi^{\varpi}}(T)}{Z_{\varphi^{\varpi}}(0)}\right) = 1. \tag{7.6.9}$$

Thus, the weighted-average capitalisation generated portfolio will have a lower return than the market portfolio over a certain time period, if the weighted average of the capitalisation weights is the same at the beginning and end of that period of time. ■

Thus, if the market $\mathcal{M}$ has initial capital equal to that of the weighted-average capitalisation generated portfolio $Z_{\varphi^{\varpi}}(0) = Z_{\mu}(0) = w > 0$ such that $\log\big(Z_{\varphi^{\varpi}}(0)/Z_{\mu}(0)\big) = \log(1) = 0$, the weighted-average capitalisation generated portfolio satisfies

$$\log\left(\frac{Z_{\mu}(T)}{Z_{\varphi^{\varpi}}(T)}\right) \quad > \quad \log\left(\frac{Z_{\mu}(0)}{Z_{\varphi^{\varpi}}(0)}\right) \quad = \quad 0, \tag{7.6.10}$$

and, then the value process $Z_{\varphi^{\varpi}}$ of the weighted-average capitalisation generated portfolio a.s. satisfies

$$\frac{Z_{\mu}(T)}{Z_{\varphi^{\varpi}}(T)} \quad > \quad \frac{Z_{\mu}(0)}{Z_{\varphi^{\varpi}}(0)} \quad = \quad 1. \tag{7.6.11}$$

Hence,

$$Z_{\mu}(T) \quad > \quad Z_{\varphi^{\varpi}}(T), \qquad \text{a.s.} \tag{7.6.12}$$

Therefore, we have in particular

$$\mathbb{P}\Big(Z_{\mu}(T) \quad > \quad Z_{\varphi^{\varpi}}(T)\Big) \quad = \quad 1. \tag{7.6.13}$$

Consequently, from (7.3.4) of Definition 7.3.1, the market portfolio μ is a strong arbitrage opportunity relative to the weighted-average capitalisation generated portfolio φ^{ϖ}, and the market portfolio μ outperforms the weighted-average capitalisation generated portfolio φ^{ϖ}. Thus, this corollary signifies that, with probability one, the market portfolio has a return above that of the weighted-average capitalisation generated portfolio over a sufficiently long time horizon $[0, T]$. Furthermore, the result above (7.6.10) gives the a.s. long-term comparison

$$\mathcal{L}_{\mu,\varphi^{\varpi}} \quad = \quad \liminf_{T\to\infty} \frac{1}{T} \log\Big(Z_{\mu}(T)/Z_{\varphi^{\varpi}}(T)\Big) \quad > \quad 0, \qquad \text{a.s.}, \tag{7.6.14}$$

and the market portfolio is a superior long-term growth opportunity relative to the weighted-average capitalisation generated portfolio φ^{ϖ}, i.e., the weighted-average capitalisation generated portfolio φ^{ϖ} will lag rather significantly behind the market portfolio over long time horizons.

7.6.2 The Price-to-Book Ratio Generated Portfolio

It can be shown that in an equity market without dividends, the market portfolio μ strictly dominates the price-to-book ratio generated portfolio φ^{ς}, for a sufficiently large real number T. This is demonstrated in the next corollary.

Corollary 7.6.2. *Let μ be the market portfolio and φ^{ς} be the price-to-book ratio generated portfolio. Then for a* ***sufficiently large real number*** *$T \in [0, \infty)$, we have*

$$\frac{Z_{\mu}(T)}{Z_{\mu}(0)} \quad > \quad \frac{Z_{\varphi^{\varsigma}}(T)}{Z_{\varphi^{\varsigma}}(0)}, \qquad \textit{a.s.}, \tag{7.6.15}$$

and, consequently

$$\mathbb{P}\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)} \quad > \quad \frac{Z_{\varphi^{\varsigma}}(T)}{Z_{\varphi^{\varsigma}}(0)}\right) \quad = \quad 1. \tag{7.6.16}$$

That is, the market portfolio μ represents a ***strong arbitrage opportunity*** *relative to the price-to-book ratio generated portfolio φ^{ς} over the fixed, finite time horizon $[0, T]$, for T sufficiently large.*

Proof. Consider the function $\mathbf{S}^{\mathrm{PBR}} : \mathbb{R}^n \to \mathbb{R}^+$ or $\mathbf{S}^{\mathrm{PBR}} : \Delta^{n-1} \to \mathbb{R}^+$ defined by (5.4.45). By Theorem 5.2.2, $\mathbf{S}^{\mathrm{PBR}}$ generates the price-to-book ratio generated portfolio with weights (5.4.50), and drift process (5.4.57). By equations (5.4.59) and (5.4.62), for the price-to-book ratio generated portfolio, we have the following for $T \in [0, \infty)$,

$$\log\left(\frac{Z_{\varphi^{\varsigma}}(T)}{Z_{\varphi^{\varsigma}}(0)}\right) \quad = \quad \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(0)\big)}\right) - \int_0^T \gamma^*_{\varphi^{\varsigma}}(t)\, dt. \tag{7.6.17}$$

Furthermore, we have the following

$$\log\left(\frac{\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(0)\big)}\right) = \log \mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(T)\big) - \log \mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(0)\big),$$

to yield

$$\log\left(\frac{Z_{\varphi^{\mathrm{s}}}(T)}{Z_{\varphi^{\mathrm{s}}}(0)}\right) = \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) + \log \mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(T)\big) - \log \mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(0)\big) - \int_0^T \gamma^*_{\varphi^{\mathrm{s}}}(t)\,dt, \tag{7.6.18}$$

and, from (5.4.46), we get for all $t \in [0,T]$,

$$\log \mathbf{S}^{\mathrm{PBR}}\big(\boldsymbol{\mu}(t)\big) = \frac{1}{2}\log\left(\sum_{i=1}^n \frac{\mu_i^2(t)}{b_i}\right).$$

Hence, employing the above equations, we can write (7.6.17) as

$$\log\left(\frac{Z_{\varphi^{\mathrm{s}}}(T)}{Z_{\varphi^{\mathrm{s}}}(0)}\right) = \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) + \frac{1}{2}\log\left(\sum_{i=1}^n \frac{\mu_i^2(T)}{b_i}\right) - \frac{1}{2}\log\left(\sum_{i=1}^n \frac{\mu_i^2(0)}{b_i}\right) - \int_0^T \gamma^*_{\varphi^{\mathrm{s}}}(t)\,dt.$$

If the weighted-average price-to-book ratio of the market is the same at the beginning and at the end of a certain time period, such that

$$\frac{1}{2}\log\left(\sum_{i=1}^n \frac{\mu_i^2(T)}{b_i}\right) = \frac{1}{2}\log\left(\sum_{i=1}^n \frac{\mu_i^2(0)}{b_i}\right),$$

then the above expression can be reduced to

$$\log\left(\frac{Z_{\varphi^{\mathrm{s}}}(T)}{Z_{\varphi^{\mathrm{s}}}(0)}\right) = \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) - \int_0^T \gamma^*_{\varphi^{\mathrm{s}}}(t)\,dt, \tag{7.6.19}$$

which by Proposition 2.4.8, that states that $\gamma^*_{\varphi^{\mathrm{s}}}(t) \geq 0$ for all $t \in [0,T]$, becomes

$$\log\left(\frac{Z_{\varphi^{\mathrm{s}}}(T)}{Z_{\varphi^{\mathrm{s}}}(0)}\right) < \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right), \qquad \text{a.s.}, \tag{7.6.20}$$

or,

$$\log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) > \log\left(\frac{Z_{\varphi^{\mathrm{s}}}(T)}{Z_{\varphi^{\mathrm{s}}}(0)}\right), \qquad \text{a.s.} \tag{7.6.21}$$

Hence,

$$\frac{Z_{\mu}(T)}{Z_{\mu}(0)} > \frac{Z_{\varphi^{\mathrm{s}}}(T)}{Z_{\varphi^{\mathrm{s}}}(0)}, \qquad \text{a.s.}, \tag{7.6.22}$$

and, we have

$$\mathbb{P}\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)} > \frac{Z_{\varphi^{\mathrm{s}}}(T)}{Z_{\varphi^{\mathrm{s}}}(0)}\right) = 1. \tag{7.6.23}$$

Thus, the price-to-book ratio generated portfolio will have a lower return than the market portfolio over a certain time period, if the weighted-average price-to-book ratio of the market remains fixed over that period of time. ■

Thus, if the market $\mathcal{M}$ has initial capital equal to that of the price-to-book ratio generated portfolio $Z_{\varphi^{\mathrm{s}}}(0) = Z_{\mu}(0) = w > 0$ such that $\log\big(Z_{\varphi^{\mathrm{s}}}(0)/Z_{\mu}(0)\big) = \log(1) = 0$, the price-to-book ratio generated portfolio satisfies

$$\log\left(\frac{Z_{\mu}(T)}{Z_{\varphi^{\mathrm{s}}}(T)}\right) > \log\left(\frac{Z_{\mu}(0)}{Z_{\varphi^{\mathrm{s}}}(0)}\right) = 0, \tag{7.6.24}$$

and, then the value process $Z_{\varphi^{\varsigma}}$ of the price-to-book ratio generated portfolio a.s. satisfies

$$\frac{Z_{\mu}(T)}{Z_{\varphi^{\varsigma}}(T)} \quad > \quad \frac{Z_{\mu}(0)}{Z_{\varphi^{\varsigma}}(0)} \quad = \quad 1. \tag{7.6.25}$$

Hence,

$$Z_{\mu}(T) \quad > \quad Z_{\varphi^{\varsigma}}(T), \qquad \text{a.s.} \tag{7.6.26}$$

Therefore, we have in particular

$$\mathbb{P}\Big(Z_{\mu}(T) \quad > \quad Z_{\varphi^{\varsigma}}(T)\Big) \quad = \quad 1. \tag{7.6.27}$$

Consequently, from (7.3.4) of Definition 7.3.1, the market portfolio μ is a strong arbitrage opportunity relative to the price-to-book ratio generated portfolio φ^{ς}, and the market portfolio μ outperforms the price-to-book ratio generated portfolio φ^{ς}. Thus, this corollary signifies that, with probability one, the market portfolio has a return above that of the price-to-book ratio generated portfolio over a sufficiently long time horizon $[0, T]$. Furthermore, the result above (7.6.24) gives the a.s. long-term comparison

$$\mathcal{L}_{\mu,\varphi^{\varsigma}} \quad = \quad \liminf_{T\to\infty} \frac{1}{T} \log\Big(Z_{\mu}(T)/Z_{\varphi^{\varsigma}}(T)\Big) \quad > \quad 0, \qquad \text{a.s.}, \tag{7.6.28}$$

and the market portfolio is a superior long-term growth opportunity relative to the price-to-book ratio generated portfolio φ^{ς}, i.e., the price-to-book ratio generated portfolio φ^{ς} will lag rather significantly behind the market portfolio over long time horizons.

7.6.3 A Single Stock with Leverage

It can be shown that an equity market without dividends, the market portfolio μ strictly dominates a single stock with leverage φ^{χ_1}, for a sufficiently large real number T. This is demonstrated in the next corollary.

Corollary 7.6.3 ([**Fernholz (1999c)**]). *Let μ be the market portfolio and φ^{χ_1} be a single stock with leverage. Then for a* ***sufficiently large real number*** *$T \in [0, \infty)$, we have*

$$\frac{Z_{\mu}(T)}{Z_{\mu}(0)} \quad > \quad \frac{Z_{\varphi^{\chi_1}}(T)}{Z_{\varphi^{\chi_1}}(0)}, \qquad \textit{a.s.}, \tag{7.6.29}$$

and, consequently

$$\mathbb{P}\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)} \quad > \quad \frac{Z_{\varphi^{\chi_1}}(T)}{Z_{\varphi^{\chi_1}}(0)}\right) \quad = \quad 1. \tag{7.6.30}$$

That is, the market portfolio μ represents a ***strong arbitrage opportunity*** *relative to a single stock with leverage φ^{χ_1} over the fixed, finite time horizon $[0, T]$, for T sufficiently large.*

Proof. Consider the function $\mathbf{S}^{\mathrm{SSL}} : \mathbb{R}^n \to \mathbb{R}^+$ or $\mathbf{S}^{\mathrm{SSL}} : \Delta^{n-1} \to \mathbb{R}^+$ defined by (5.4.63). By Theorem 5.2.2, $\mathbf{S}^{\mathrm{SSL}}$ generates a single stock with leverage with weights (5.4.71), and drift process (5.4.76). By equations (5.4.78), (5.4.85) and (5.4.86), for a single stock with leverage, we have the following for $T \in [0, \infty)$,

$$\log\left(\frac{Z_{\varphi^{\chi_1}}(T)}{Z_{\varphi^{\chi_1}}(0)}\right) \quad = \quad \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(0)\big)}\right) - \int_0^T \tau_{11}(t)\, dt \tag{7.6.31}$$

$$= \quad \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) + 2\, \log\left(\frac{\mu_1(T)}{\mu_1(0)}\right) - \int_0^T \tau_{11}(t)\, dt. \tag{7.6.32}$$

Now, since $0 < \mu_i(t) < 1$, for $t \in [0, T]$, for all $i = 1, 2, \ldots, n$, we have $\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big) = \mu_1^2(t) < 1$. Hence, we have, a.s.,

$$\log\left(\frac{\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(0)\big)}\right) \quad = \quad 2\, \log\left(\frac{\mu_1(T)}{\mu_1(0)}\right) \quad = \quad 2\, \log \mu_1(T) - 2\, \log \mu_1(0) \quad < \quad -2\, \log \mu_1(0). \tag{7.6.33}$$

Since for all $i = 1, 2, \ldots, n$, $\tau_{ii}(t) \geq 0$, we have $-\tau_{11}(t) \leq 0$ and the drift process is decreasing (i.e., the integral term in (7.6.31) is negative), so that for $T \in [0, \infty)$, (7.6.31) can be expressed as

$$\log\left(\frac{Z_{\varphi^{\mathrm{x}_1}}(T)}{Z_{\varphi^{\mathrm{x}_1}}(0)}\right) \quad < \quad \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(0)\big)}\right). \tag{7.6.34}$$

This coupled with (7.6.33) provides, in particular, the following upper bound

$$\log\left(\frac{Z_{\varphi^{\mathrm{x}_1}}(T)}{Z_{\varphi^{\mathrm{x}_1}}(0)}\right) \quad < \quad \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) - 2\,\log\mu_1(0), \tag{7.6.35}$$

or, the following lower bound

$$\log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) \quad > \quad \log\left(\frac{Z_{\varphi^{\mathrm{x}_1}}(T)}{Z_{\varphi^{\mathrm{x}_1}}(0)}\right) + 2\,\log\mu_1(0). \tag{7.6.36}$$

Thus, for $T \in [0, \infty)$, we a.s. have

$$\begin{aligned}
\frac{Z_{\mu}(T)}{Z_{\mu}(0)} \quad &> \quad \frac{Z_{\varphi^{\mathrm{x}_1}}(T)}{Z_{\varphi^{\mathrm{x}_1}}(0)}\,\exp\big(2\,\log\mu_1(0)\big) \\
&= \quad \frac{Z_{\varphi^{\mathrm{x}_1}}(T)}{Z_{\varphi^{\mathrm{x}_1}}(0)}\,\exp\big(\log\mu_1^2(0)\big) \\
&= \quad \mu_1^2(0)\,\frac{Z_{\varphi^{\mathrm{x}_1}}(T)}{Z_{\varphi^{\mathrm{x}_1}}(0)}.
\end{aligned} \tag{7.6.37}$$

Now, if the market is weakly diverse over the time horizon $[0, T]$, from the weak diversity condition (4.2.3) of Definition 4.2.2 and (7.7.45), for $0 < \delta < 1$, we have

$$\int_0^T \big(1 - \mu_{(1)}(t)\big)\,dt \quad \geq \quad \delta\,T. \tag{7.6.38}$$

So, by the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\left(\int_0^T 1^2\,dt\right)\left(\int_0^T \big(1 - \mu_{(1)}(t)\big)^2\,dt\right) \quad &\geq \quad \left(\int_0^T \big(1 - \mu_{(1)}(t)\big)\,dt\right)^2, \\
T\int_0^T \big(1 - \mu_{(1)}(t)\big)^2\,dt \quad &\geq \quad \left(\int_0^T \big(1 - \mu_{(1)}(t)\big)\,dt\right)^2, \\
\int_0^T \big(1 - \mu_{(1)}(t)\big)^2\,dt \quad &\geq \quad \frac{1}{T}\left(\int_0^T \big(1 - \mu_{(1)}(t)\big)\,dt\right)^2,
\end{aligned}$$

so that by (7.7.99), the above inequality becomes

$$\begin{aligned}
T\int_0^T \big(1 - \mu_{(1)}(t)\big)^2\,dt \quad &\geq \quad (\delta\,T)^2 \quad = \quad \delta^2\,T^2, \quad \text{and,} \\
\int_0^T \big(1 - \mu_{(1)}(t)\big)^2\,dt \quad &\geq \quad \delta^2\,T.
\end{aligned} \tag{7.6.39}$$

Therefore, we have

$$-\int_0^T \big(1 - \mu_{(1)}(t)\big)^2\,dt \quad \leq \quad -\,\delta^2\,T. \tag{7.6.40}$$

From (2.4.40) of Lemma 2.4.9 for a nondegenerate market, we have $\tau_{11}(t) \geq \varepsilon\big(1 - \mu_1(t)\big)^2$, so that combining this result, with the result (7.6.40), we see that the assumption of weak diversity implies

$$-\int_0^T \tau_{11}(t)\,dt \quad \leq \quad -\,\varepsilon\int_0^T \big(1 - \mu_1(t)\big)^2\,dt \quad \leq \quad -\,\varepsilon\,\delta^2\,T. \tag{7.6.41}$$

It thus follows, from (7.6.33) and (7.6.41), that a.s., for $T > 0$, (7.6.31) amounts to the following

$$\log\left(\frac{Z_{\varphi^{\mathrm{X}_1}}(T)}{Z_{\varphi^{\mathrm{X}_1}}(0)}\right) \quad < \quad \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) - 2\,\log\mu_1(0) - \varepsilon\,\delta^2\,T, \tag{7.6.42}$$

or,

$$\log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) \quad > \quad \log\left(\frac{Z_{\varphi^{\mathrm{X}_1}}(T)}{Z_{\varphi^{\mathrm{X}_1}}(0)}\right) + 2\,\log\mu_1(0) + \varepsilon\,\delta^2\,T, \tag{7.6.43}$$

and, we have

$$\begin{aligned}\frac{Z_{\mu}(T)}{Z_{\mu}(0)} \quad &> \quad \frac{Z_{\varphi^{\mathrm{X}_1}}(T)}{Z_{\varphi^{\mathrm{X}_1}}(0)}\,\exp\big(2\,\log\mu_1(0) + \varepsilon\,\delta^2\,T\big)\\ &= \quad \frac{Z_{\varphi^{\mathrm{X}_1}}(T)}{Z_{\varphi^{\mathrm{X}_1}}(0)}\,\exp\big(\log\mu_1^2(0) + \varepsilon\,\delta^2\,T\big)\\ &= \quad \frac{Z_{\varphi^{\mathrm{X}_1}}(T)}{Z_{\varphi^{\mathrm{X}_1}}(0)}\,\Big(\mu_1^2(0)\,e^{\varepsilon\,\delta^2\,T}\Big).\end{aligned} \tag{7.6.44}$$

Therefore, by (7.2.5) and (7.2.6) of Definition 7.2.2, the market portfolio μ strictly dominates a single stock with leverage φ^{X_1} if $\mu_1^2(0)\,e^{\varepsilon\,\delta^2\,T} > 1$, thus if $2\,\log\mu_1(0) + \varepsilon\,\delta^2\,T > 0$, i.e., if

$$T \quad > \quad \frac{-\,2\,\log\mu_1(0)}{\varepsilon\,\delta^2}. \tag{7.6.45}$$

Since, if

$$2\,\log\mu_1(0) + \varepsilon\,\delta^2\,T \quad > \quad 0, \tag{7.6.46}$$

then

$$\log\left(\frac{Z_{\varphi^{\mathrm{X}_1}}(T)}{Z_{\varphi^{\mathrm{X}_1}}(0)}\right) + 2\,\log\mu_1(0) + \varepsilon\,\delta^2\,T \quad > \quad \log\left(\frac{Z_{\varphi^{\mathrm{X}_1}}(T)}{Z_{\varphi^{\mathrm{X}_1}}(0)}\right),$$

which results in the following expression

$$\log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) \quad > \quad \log\left(\frac{Z_{\varphi^{\mathrm{X}_1}}(T)}{Z_{\varphi^{\mathrm{X}_1}}(0)}\right) + 2\,\log\mu_1(0) + \varepsilon\,\delta^2\,T \quad > \quad \log\left(\frac{Z_{\varphi^{\mathrm{X}_1}}(T)}{Z_{\varphi^{\mathrm{X}_1}}(0)}\right).$$

Alternatively, if

$$\mu_1^2(0)\,e^{\varepsilon\,\delta^2\,T} \quad > \quad 1,$$

then

$$\frac{Z_{\varphi^{\mathrm{X}_1}}(T)}{Z_{\varphi^{\mathrm{X}_1}}(0)}\,\Big(\mu_1^2(0)\,e^{\varepsilon\,\delta^2\,T}\Big) \quad > \quad \frac{Z_{\varphi^{\mathrm{X}_1}}(T)}{Z_{\varphi^{\mathrm{X}_1}}(0)},$$

which results in the following expression

$$\frac{Z_{\mu}(T)}{Z_{\mu}(0)} \quad > \quad \frac{Z_{\varphi^{\mathrm{X}_1}}(T)}{Z_{\varphi^{\mathrm{X}_1}}(0)}\,\Big(\mu_1^2(0)\,e^{\varepsilon\,\delta^2\,T}\Big) \quad > \quad \frac{Z_{\varphi^{\mathrm{X}_1}}(T)}{Z_{\varphi^{\mathrm{X}_1}}(0)}.$$

Thus we have

$$\log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) \quad > \quad \log\left(\frac{Z_{\varphi^{\mathrm{X}_1}}(T)}{Z_{\varphi^{\mathrm{X}_1}}(0)}\right), \qquad \text{a.s.} \tag{7.6.47}$$

Hence,

$$\frac{Z_{\mu}(T)}{Z_{\mu}(0)} \quad > \quad \frac{Z_{\varphi^{\mathrm{X}_1}}(T)}{Z_{\varphi^{\mathrm{X}_1}}(0)}, \qquad \text{a.s.}, \tag{7.6.48}$$

and, we have

$$\mathbb{P}\left(\frac{Z_\mu(T)}{Z_\mu(0)} \;>\; \frac{Z_{\varphi^{\mathsf{x}_1}}(T)}{Z_{\varphi^{\mathsf{x}_1}}(0)}\right) \;=\; 1. \tag{7.6.49}$$

Alternatively, by setting $c_1 := 1$, we can appeal to (ii) of Lemma 7.2.3, since $\mathbf{S}^{\mathrm{SSL}}\big(\boldsymbol{\mu}(t)\big) < 1$, for all $t > 0$, a.s. Moreover, the inequality (2.4.40) of Lemma 2.4.9 implies that, for a nondegenerate equity market, there is a $\varepsilon > 0$ such that $\tau_{ii}(t) \geq \varepsilon\big(1-\mu_i(t)\big)^2$, for $i = 1, 2, \ldots, n$, for all $t \in [0, T]$, a.s., and we have $\tau_{11}(t) \geq \varepsilon\big(1-\mu_1(t)\big)^2$. Hence, since the drift process of a single stock with leverage is given by $\mathfrak{g}_{\varphi^{\mathsf{x}_1}}(t) = -\,\tau_{11}(t)$, the drift process of a single stock with leverage $\mathfrak{g}_{\varphi^{\mathsf{x}_1}}(t)$ has a negative upper bound. Consequently, Lemma 7.2.3 implies that there is a dominance relationship between a single stock with leverage φ^{x_1} and the market portfolio μ, i.e., the market portfolio μ strictly dominates the single stock with leverage φ^{x_1}. ■

Thus, if the market $\mathcal{M}$ is nondegenerate over the finite time horizon $[0, T]$ and with initial capital equal to that of a single stock with leverage $Z_{\varphi^{\mathsf{x}_1}}(0) = Z_\mu(0) = w > 0$ such that $\log\big(Z_{\varphi^{\mathsf{x}_1}}(0)/Z_\mu(0)\big) = \log(1) = 0$, a single stock with leverage satisfies

$$\begin{aligned}\log\left(\frac{Z_\mu(T)}{Z_{\varphi^{\mathsf{x}_1}}(T)}\right) \;&>\; \log\left(\frac{Z_\mu(0)}{Z_{\varphi^{\mathsf{x}_1}}(0)}\right) + 2\,\log\mu_1(0) + \varepsilon\,\delta^2\,T \\ &=\; 2\,\log\mu_1(0) + \varepsilon\,\delta^2\,T,\end{aligned} \tag{7.6.50}$$

and, then the value process $Z_{\varphi^{\mathsf{x}_1}}$ of a single stock with leverage a.s. satisfies

$$\begin{aligned}\frac{Z_\mu(T)}{Z_{\varphi^{\mathsf{x}_1}}(T)} \;&>\; \frac{Z_\mu(0)}{Z_{\varphi^{\mathsf{x}_1}}(0)}\left(\mu_1^2(0)\,e^{\varepsilon\,\delta^2\,T}\right) \\ &=\; \mu_1^2(0)\,e^{\varepsilon\,\delta^2\,T},\end{aligned} \tag{7.6.51}$$

so that

$$Z_\mu(T) \;>\; Z_{\varphi^{\mathsf{x}_1}}(T)\left(\mu_1^2(0)\,e^{\varepsilon\,\delta^2\,T}\right), \qquad \text{a.s.} \tag{7.6.52}$$

Consequently, in a similar fashion to that shown in the foregoing proof, if

$$\mu_1^2(0)\,e^{\varepsilon\,\delta^2\,T} \;>\; 1,$$

then

$$Z_{\varphi^{\mathsf{x}_1}}(T)\left(\mu_1^2(0)\,e^{\varepsilon\,\delta^2\,T}\right) \;>\; Z_{\varphi^{\mathsf{x}_1}}(T),$$

and the following results

$$Z_\mu(T) \;>\; Z_{\varphi^{\mathsf{x}_1}}(T)\left(\mu_1^2(0)\,e^{\varepsilon\,\delta^2\,T}\right) \;>\; Z_{\varphi^{\mathsf{x}_1}}(T).$$

Hence,

$$Z_\mu(T) \;>\; Z_{\varphi^{\mathsf{x}_1}}(T), \qquad \text{a.s.} \tag{7.6.53}$$

Therefore, we have in particular

$$\mathbb{P}\Big(Z_\mu(T) \;>\; Z_{\varphi^{\mathsf{x}_1}}(T)\Big) \;=\; 1, \tag{7.6.54}$$

provided that

$$T \;>\; T_* \;\triangleq\; \frac{-\,2\,\log\mu_1(0)}{\varepsilon\,\delta^2}. \tag{7.6.55}$$

Consequently, from (7.3.4) of Definition 7.3.1, the market portfolio μ is a strong arbitrage opportunity relative to the single stock with leverage φ^{x_1}, and the market portfolio μ outperforms the single stock with leverage

φ^{χ_1}. Notice that the market portfolio will only outperform the single stock with leverage over the long term (i.e., the time required is very large) since the lower bound on T

$$T \quad > \quad \frac{-\,2\,\log \mu_1(0)}{\varepsilon\,\delta^2},$$

is sufficiently large as it contains the sufficiently small numbers $\varepsilon > 0$ and $\delta^2 > 0$, which are both present in the denominator of the lower bound. Thus, this corollary signifies that, with probability one, the market portfolio has a return above that of the single stock with leverage over a sufficiently long time horizon $[0, T]$. This shows that no single company can ever be allowed to dominate the entire market in terms of relative capitalisation. If the market $\mathcal{M}$ is nondegenerate over $[T_*, \infty)$, then (7.6.50) above gives the a.s. long-term comparison

$$\begin{aligned}
\mathcal{L}_{\mu,\varphi^{\chi_1}} &= \liminf_{T\to\infty} \frac{1}{T} \log\Big(Z_\mu(T)/Z_{\varphi^{\chi_1}}(T)\Big) \\
&> \liminf_{T\to\infty} \frac{1}{T}\big(2\,\log \mu_1(0) + \varepsilon\,\delta^2\,T\big) \\
&= \liminf_{T\to\infty} \left(\frac{2\,\log \mu_1(0)}{T} + \varepsilon\,\delta^2\right) \\
&= \liminf_{T\to\infty} \frac{2\,\log \mu_1(0)}{T} + \liminf_{T\to\infty} \varepsilon\,\delta^2 \\
&= \liminf_{T\to\infty} \frac{2\,\log \mu_1(0)}{T} + \varepsilon\,\delta^2 \\
&= \varepsilon\,\delta^2 \quad > \quad 0, \qquad (7.6.56)
\end{aligned}$$

since $\varepsilon > 0$ and $\delta^2 > 0$. Thus, we have

$$\mathcal{L}_{\mu,\varphi^{\chi_1}} \quad = \quad \liminf_{T\to\infty} \frac{1}{T} \log\Big(Z_\mu(T)/Z_{\varphi^{\chi_1}}(T)\Big) \quad > \quad \varepsilon\,\delta^2 \quad > \quad 0, \qquad \text{a.s.,} \qquad (7.6.57)$$

and the market portfolio is a superior long-term growth opportunity relative to the single stock with leverage φ^{χ_1}, i.e., the single stock with leverage φ^{χ_1} will lag rather significantly behind the market portfolio over long time horizons.

7.7 Relative Arbitrage Opportunities in Diverse Equity Markets

7.7.1 Long-Term Relative Arbitrage Opportunities in Diverse Equity Markets

If the market $\mathcal{M}$ is (weakly) diverse over the time horizon $[0, T]$, then it contains arbitrage opportunities relative to the market portfolio. Further to this, if the market $\mathcal{M}$ is both nondegenerate and (weakly) diverse over the time horizon $[0, T]$, then there most certainly exist arbitrage opportunities relative to the market portfolio. Thus, in a nondegenerate, (weakly) diverse market without dividends it is possible to generate portfolios that will a.s. have return higher than that of the market portfolio over a fixed time period (i.e., relative arbitrage opportunities do exist). This violates the no-arbitrage hypothesis of mathematical finance. We provide here in this section some examples of such arbitrage opportunities in a (weakly) diverse equity market. We show that functions which are measures of diversity will generate portfolios that dominate the market portfolio if appropriate bounds exist on the concentration of market capital.

7.7.1.1 The Entropy-Weighted Portfolio

The results (5.6.25) and (5.6.26) of Proposition 5.6.3 can be strengthened in the case of a diverse equity market. It can be shown that in a nondegenerate and diverse equity market, without dividends, the entropy-weighted portfolio φ^{e} strictly dominates the market portfolio μ, for a sufficiently large real number T. Consequently, over the long term, we can expect that the entropy-weighted portfolio will strictly dominate and outperform the market portfolio. This is demonstrated in the next corollary.

Corollary 7.7.1 ([**Fernholz (2002)**]). *Let μ be the market portfolio and φ^{e} be the entropy-weighted portfolio, and suppose that the market $\mathcal{M}$ is* ***nondegenerate*** *and* ***diverse*** *over a fixed, finite time horizon $[0,T]$, with $T>0$ a given real number. Then for a* ***sufficiently large real number*** *$T\in[0,\infty)$, we have*

$$\frac{Z_{\varphi^{\mathrm{e}}}(T)}{Z_{\varphi^{\mathrm{e}}}(0)} \quad > \quad \frac{Z_{\mu}(T)}{Z_{\mu}(0)}, \qquad a.s., \tag{7.7.1}$$

and, consequently

$$\mathbb{P}\left(\frac{Z_{\varphi^{\mathrm{e}}}(T)}{Z_{\varphi^{\mathrm{e}}}(0)} \quad > \quad \frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) \quad = \quad 1. \tag{7.7.2}$$

That is, in a nondegenerate and diverse equity market, the entropy-weighted portfolio φ^{e} represents a ***strong arbitrage opportunity*** *relative to the market portfolio μ over the fixed, finite time horizon $[0,T]$, for T sufficiently large.*

Proof. Consider the entropy function $\mathbf{S}^{\mathrm{E}}:\mathbb{R}^n\to\mathbb{R}^+$ or $\mathbf{S}^{\mathrm{E}}:\Delta^{n-1}\to\mathbb{R}^+$ defined by (4.6.1). By Theorem 5.2.2, $\mathbf{S}^{\mathrm{E}}$ generates the entropy-weighted portfolio with weights (5.6.3), and drift process (5.6.6). By (5.6.8) of Theorem 5.6.2, or by equations (5.6.19) and (5.6.22), for the entropy-weighted portfolio, we have the following for $T\in[0,\infty)$,

$$\log\left(\frac{Z_{\varphi^{\mathrm{e}}}(T)}{Z_{\varphi^{\mathrm{e}}}(0)}\right) \quad = \quad \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right)+\log\left(\frac{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(0)\big)}\right)+\int_0^T\frac{\gamma_{\mu}^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\,dt. \tag{7.7.3}$$

Now, from (4.6.3), for $t\in[0,T]$, we have the bounds $0<\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)\le\log n$, which implies that $\frac{1}{\mathbf{S}^{\mathrm{E}}(\boldsymbol{\mu}(t))}\ge\frac{1}{\log n}$, in particular, $\frac{1}{\mathbf{S}^{\mathrm{E}}(\boldsymbol{\mu}(0))}\ge\frac{1}{\log n}$. The inequality (4.7.1) of Proposition 4.7.1 implies that, for a diverse equity market, there is a $\zeta_1>0$ such that $\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)\ge\zeta_1$, for all $t\in[0,T]$, a.s. In addition, the inequality (4.2.6) of Proposition 4.2.3 implies that, for a nondegenerate and diverse equity market, there is a $\zeta_2>0$ such that $\gamma_{\mu}^*(t)\ge\zeta_2$, for all $t\in[0,T]$, a.s. Hence, it follows that a.s., for $T>0$, we have

$$\log\left(\frac{Z_{\varphi^{\mathrm{e}}}(T)}{Z_{\varphi^{\mathrm{e}}}(0)}\right) \quad > \quad \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right)+\log\left(\frac{\zeta_1}{\log n}\right)+\frac{\zeta_2}{\log n}\int_0^T dt.$$

Consequently, we have the following inequality

$$\log\left(\frac{Z_{\varphi^{\mathrm{e}}}(T)}{Z_{\varphi^{\mathrm{e}}}(0)}\right) \quad > \quad \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right)+\log\zeta_1-\log\log n+\frac{\zeta_2\,T}{\log n}, \tag{7.7.4}$$

and, we have

$$\frac{Z_{\varphi^{\mathrm{e}}}(T)}{Z_{\varphi^{\mathrm{e}}}(0)} \quad > \quad \frac{Z_{\mu}(T)}{Z_{\mu}(0)}\,\exp\left(\log\zeta_1-\log\log n+\frac{\zeta_2\,T}{\log n}\right). \tag{7.7.5}$$

Therefore, by (7.2.5) and (7.2.6) of Definition 7.2.2, the portfolio φ^{e} strictly dominates the market portfolio μ if $\exp\big(\log\zeta_1-\log\log n+\frac{\zeta_2\,T}{\log n}\big)>1$, thus if $\log\zeta_1-\log\log n+\frac{\zeta_2\,T}{\log n}>0$, i.e., if

$$T \quad > \quad \frac{\log n\,\big(\log\log n-\log\zeta_1\big)}{\zeta_2}. \tag{7.7.6}$$

Since, if

$$\log\zeta_1-\log\log n+\frac{\zeta_2\,T}{\log n} \quad > \quad 0, \tag{7.7.7}$$

then

$$\log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right)+\log\zeta_1-\log\log n+\frac{\zeta_2\,T}{\log n} \quad > \quad \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right),$$

which results in the following expression

$$\log\left(\frac{Z_{\varphi^{\mathrm{e}}}(T)}{Z_{\varphi^{\mathrm{e}}}(0)}\right) \quad > \quad \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) + \log\zeta_1 - \log\log n + \frac{\zeta_2\, T}{\log n} \quad > \quad \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right).$$

Alternatively, if

$$\exp\left(\log\zeta_1 - \log\log n + \frac{\zeta_2\, T}{\log n}\right) \quad > \quad 1,$$

then

$$\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\exp\left(\log\zeta_1 - \log\log n + \frac{\zeta_2\, T}{\log n}\right) \quad > \quad \frac{Z_{\mu}(T)}{Z_{\mu}(0)},$$

which results in the following expression

$$\frac{Z_{\varphi^{\mathrm{e}}}(T)}{Z_{\varphi^{\mathrm{e}}}(0)} \quad > \quad \frac{Z_{\mu}(T)}{Z_{\mu}(0)}\exp\left(\log\zeta_1 - \log\log n + \frac{\zeta_2\, T}{\log n}\right) \quad > \quad \frac{Z_{\mu}(T)}{Z_{\mu}(0)}.$$

Thus we have

$$\log\left(\frac{Z_{\varphi^{\mathrm{e}}}(T)}{Z_{\varphi^{\mathrm{e}}}(0)}\right) \quad > \quad \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right), \qquad \text{a.s.} \tag{7.7.8}$$

Hence,

$$\frac{Z_{\varphi^{\mathrm{e}}}(T)}{Z_{\varphi^{\mathrm{e}}}(0)} \quad > \quad \frac{Z_{\mu}(T)}{Z_{\mu}(0)}, \qquad \text{a.s.,} \tag{7.7.9}$$

and, we have

$$\mathbb{P}\left(\frac{Z_{\varphi^{\mathrm{e}}}(T)}{Z_{\varphi^{\mathrm{e}}}(0)} \quad > \quad \frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) \quad = \quad 1. \tag{7.7.10}$$

Alternatively, we can appeal to (i) of Lemma 7.2.3 by setting $c_1 := \zeta_1$ and $c_2 := \frac{\zeta_2}{\log n}$, since $\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$ is positive and finite from (4.7.1) of Proposition 4.7.1, thus for all $t > 0$, $\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) > \zeta_1 > 0$, a.s. Moreover, the inequality (4.2.6) of Proposition 4.2.3 implies that, for a nondegenerate and diverse equity market, there is a $\zeta_2 > 0$ such that $\gamma^*_{\mu}(t) \geq \zeta_2$, for all $t \in [0, T]$, a.s. Thus, $\gamma^*_{\mu}(t)$ has a positive lower bound in a diverse equity market. Also, from (4.6.3), for $t \in [0, T]$, we have the bound $\frac{1}{\mathbf{S}^{\mathrm{E}}(\boldsymbol{\mu}(t))} \geq \frac{1}{\log n}$. Hence, since the drift process of the entropy-weighted portfolio is given by $\mathfrak{g}_{\varphi^{\mathrm{e}}}(t) = \gamma^*_{\mu}(t)/\mathbf{S}^{\mathrm{E}}(\boldsymbol{\mu}(t))$, we have $\mathfrak{g}_{\varphi^{\mathrm{e}}}(t) > \frac{\zeta_2}{\log n} > 0$. Thus, the drift process of the entropy-weighted portfolio $\mathfrak{g}_{\varphi^{\mathrm{e}}}(t)$ has a positive lower bound. Consequently, Lemma 7.2.3 implies that there is a dominance relationship between the entropy-weighted portfolio φ^{e} and the market portfolio μ, i.e., φ^{e} strictly dominates the market portfolio μ. ■

Thus, if the market $\mathcal{M}$ is (weakly) diverse over the finite time horizon $[0, T]$ and with initial capital equal to that of the entropy-weighted portfolio $Z_{\varphi^{\mathrm{e}}}(0) = Z_{\mu}(0) = w > 0$ such that $\log\big(Z_{\varphi^{\mathrm{e}}}(0)/Z_{\mu}(0)\big) = \log(1) = 0$, the entropy-weighted portfolio satisfies

$$\begin{aligned}\log\left(\frac{Z_{\varphi^{\mathrm{e}}}(T)}{Z_{\mu}(T)}\right) \quad &> \quad \log\left(\frac{Z_{\varphi^{\mathrm{e}}}(0)}{Z_{\mu}(0)}\right) + \log\zeta_1 - \log\log n + \frac{\zeta_2\, T}{\log n} \\ &= \quad \log\zeta_1 - \log\log n + \frac{\zeta_2\, T}{\log n},\end{aligned} \tag{7.7.11}$$

and, then the value process $Z_{\varphi^{\mathrm{e}}}$ of the entropy-weighted portfolio a.s. satisfies

$$\begin{aligned}\frac{Z_{\varphi^{\mathrm{e}}}(T)}{Z_{\mu}(T)} \quad &> \quad \frac{Z_{\varphi^{\mathrm{e}}}(0)}{Z_{\mu}(0)}\exp\left(\log\zeta_1 - \log\log n + \frac{\zeta_2\, T}{\log n}\right) \\ &= \quad \exp\left(\log\zeta_1 - \log\log n + \frac{\zeta_2\, T}{\log n}\right),\end{aligned} \tag{7.7.12}$$

so that

$$Z_{\varphi^{\mathsf{e}}}(T) \quad > \quad Z_{\mu}(T)\, \exp\left(\log \zeta_1 - \log\log n + \frac{\zeta_2\, T}{\log n}\right), \qquad \text{a.s.} \tag{7.7.13}$$

Consequently, in a similar fashion to that shown in the foregoing proof, if

$$\exp\left(\log \zeta_1 - \log\log n + \frac{\zeta_2\, T}{\log n}\right) \quad > \quad 1,$$

then

$$Z_{\mu}(T)\, \exp\left(\log \zeta_1 - \log\log n + \frac{\zeta_2\, T}{\log n}\right) \quad > \quad Z_{\mu}(T),$$

and the following results

$$Z_{\varphi^{\mathsf{e}}}(T) \quad > \quad Z_{\mu}(T)\, \exp\left(\log \zeta_1 - \log\log n + \frac{\zeta_2\, T}{\log n}\right) \quad > \quad Z_{\mu}(T).$$

Hence,

$$Z_{\varphi^{\mathsf{e}}}(T) \quad > \quad Z_{\mu}(T), \qquad \text{a.s.} \tag{7.7.14}$$

Therefore, we have in particular

$$\mathbb{P}\left(Z_{\varphi^{\mathsf{e}}}(T) \quad > \quad Z_{\mu}(T)\right) \quad = \quad 1, \tag{7.7.15}$$

provided that

$$T \quad > \quad T_* \quad \triangleq \quad \frac{\log n\, \big(\log\log n - \log \zeta_1\big)}{\zeta_2}. \tag{7.7.16}$$

Consequently, from (7.3.4) of Definition 7.3.1, the entropy-weighted portfolio φ^{e} is a strong arbitrage opportunity relative to the market portfolio μ, and φ^{e} outperforms the market portfolio μ. Notice that the entropy-weighted portfolio will only outperform the market portfolio over the long term since the lower bound on T

$$T \quad > \quad \frac{\log n\, \big(\log\log n - \log \zeta_1\big)}{\zeta_2},$$

is sufficiently large as it contains the sufficiently small numbers $\zeta_1 > 0$ and $\zeta_2 > 0$, where the latter constant is present in the denominator of the lower bound. Thus, this corollary signifies that, with probability one, the entropy-weighted portfolio has a return above that of the market portfolio over a sufficiently long time horizon $[0, T]$, and any condition of this nature must be avoided in normative theories of equilibrium. Dividend payments by the larger stocks could correct this condition. If the market $\mathcal{M}$ is uniformly weakly diverse over $[T_*, \infty)$, then (7.7.11) above gives the a.s. long-term comparison

$$\begin{aligned}
\mathcal{L}_{\varphi^{\mathsf{e}},\mu} \quad &= \quad \liminf_{T\to\infty} \frac{1}{T}\, \log\left(Z_{\varphi^{\mathsf{e}}}(T)/Z_{\mu}(T)\right) \\
&> \quad \liminf_{T\to\infty} \frac{1}{T}\left(\log \zeta_1 - \log\log n + \frac{\zeta_2\, T}{\log n}\right) \\
&= \quad \liminf_{T\to\infty} \left(\frac{\big(\log \zeta_1 - \log\log n\big)}{T} + \frac{\zeta_2}{\log n}\right) \\
&= \quad \liminf_{T\to\infty} \frac{\big(\log \zeta_1 - \log\log n\big)}{T} + \liminf_{T\to\infty} \frac{\zeta_2}{\log n} \\
&= \quad \liminf_{T\to\infty} \frac{\big(\log \zeta_1 - \log\log n\big)}{T} + \frac{\zeta_2}{\log n} \\
&= \quad \frac{\zeta_2}{\log n} \quad > \quad 0,
\end{aligned} \tag{7.7.17}$$

since $\zeta_2 > 0$, and $\log n > 0$ for $n \geq 2$ as $n \in \mathbb{N}$. Thus, we have

$$\mathcal{L}_{\varphi^{\mathsf{e}},\mu} \quad = \quad \liminf_{T\to\infty} \frac{1}{T}\, \log\left(Z_{\varphi^{\mathsf{e}}}(T)/Z_{\mu}(T)\right) \quad > \quad \frac{\zeta_2}{\log n} \quad > \quad 0, \qquad \text{a.s.,} \tag{7.7.18}$$

and the entropy-weighted portfolio φ^{e} is a superior long-term growth opportunity relative to the market, i.e., the market portfolio will lag rather significantly behind the entropy-weighted portfolio over long time horizons.

7.7.1.2 The Modified Entropy-Weighted Portfolio

It can be shown that in a nondegenerate and diverse equity market, without dividends, the modified entropy-weighted portfolio $\varphi^{(\mathsf{e},c)}$ strictly dominates the market portfolio μ, for a sufficiently large real number T. This is demonstrated in the next corollary.

Corollary 7.7.2 ([**Fernholz & Karatzas (2005)**]). *Let μ be the market portfolio and $\varphi^{(\mathsf{e},c)}$ be the modified entropy-weighted portfolio, and suppose that the market $\mathcal{M}$ is **nondegenerate** and **diverse** over a fixed, finite time horizon $[0,T]$, with $T>0$ a given real number. Then for a **sufficiently large real number** $T\in[0,\infty)$, we have*

$$\frac{Z_{\varphi^{(\mathsf{e},c)}}(T)}{Z_{\varphi^{(\mathsf{e},c)}}(0)} \;>\; \frac{Z_\mu(T)}{Z_\mu(0)}, \qquad a.s., \tag{7.7.19}$$

and, consequently

$$\mathbb{P}\left(\frac{Z_{\varphi^{(\mathsf{e},c)}}(T)}{Z_{\varphi^{(\mathsf{e},c)}}(0)} \;>\; \frac{Z_\mu(T)}{Z_\mu(0)}\right) \;=\; 1. \tag{7.7.20}$$

*That is, in a nondegenerate and diverse equity market, the modified entropy-weighted portfolio $\varphi^{(\mathsf{e},c)}$ represents a **strong arbitrage opportunity** relative to the market portfolio μ over the fixed, finite time horizon $[0,T]$, for T sufficiently large.*

Proof. Consider the modified entropy function $\mathbf{S}_c^{\mathrm{E}}:\mathbb{R}^n\to\mathbb{R}^+$ or $\mathbf{S}_c^{\mathrm{E}}:\Delta^{n-1}\to\mathbb{R}^+$ defined by (4.6.7). By Theorem 5.2.2, $\mathbf{S}_c^{\mathrm{E}}$ generates the modified entropy-weighted portfolio with weights (5.6.34), and drift process (5.6.38). By equations (5.6.40) and (5.6.43), for the modified entropy-weighted portfolio, we have the following for $T\in[0,\infty)$,

$$\log\left(\frac{Z_{\varphi^{(\mathsf{e},c)}}(T)}{Z_{\varphi^{(\mathsf{e},c)}}(0)}\right) \;=\; \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right)+\log\left(\frac{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(0)\big)}\right)+\int_0^T\frac{\gamma_\mu^*(t)}{\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\,dt. \tag{7.7.21}$$

Now, from (4.6.11), for $t\in[0,T]$, we have the bounds $c<\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)\le c+\log n$, which implies that $\frac{1}{\mathbf{S}_c^{\mathrm{E}}(\boldsymbol{\mu}(t))}\ge\frac{1}{c+\log n}$, in particular, $\frac{1}{\mathbf{S}_c^{\mathrm{E}}(\boldsymbol{\mu}(0))}\ge\frac{1}{c+\log n}$. In addition, the inequality (4.2.6) of Proposition 4.2.3 implies that, for a nondegenerate and diverse equity market, there is a $\zeta>0$ such that $\gamma_\mu^*(t)\ge\zeta$, for all $t\in[0,T]$, a.s. Hence, it follows that a.s., for $T>0$, we have

$$\log\left(\frac{Z_{\varphi^{(\mathsf{e},c)}}(T)}{Z_{\varphi^{(\mathsf{e},c)}}(0)}\right) \;>\; \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right)+\log\left(\frac{c}{c+\log n}\right)+\frac{\zeta}{c+\log n}\int_0^T dt.$$

Consequently, we have the following inequality

$$\log\left(\frac{Z_{\varphi^{(\mathsf{e},c)}}(T)}{Z_{\varphi^{(\mathsf{e},c)}}(0)}\right) \;>\; \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right)+\log c-\log\big(c+\log n\big)+\frac{\zeta\,T}{c+\log n}, \tag{7.7.22}$$

and, we have

$$\frac{Z_{\varphi^{(\mathsf{e},c)}}(T)}{Z_{\varphi^{(\mathsf{e},c)}}(0)} \;>\; \frac{Z_\mu(T)}{Z_\mu(0)}\exp\left(\log c-\log\big(c+\log n\big)+\frac{\zeta\,T}{c+\log n}\right). \tag{7.7.23}$$

Therefore, by (7.2.5) and (7.2.6) of Definition 7.2.2, the portfolio $\varphi^{(\mathsf{e},c)}$ strictly dominates the market portfolio μ if $\exp\big(\log c-\log\big(c+\log n\big)+\frac{\zeta\,T}{c+\log n}\big)>1$, thus if $\log c-\log\big(c+\log n\big)+\frac{\zeta\,T}{c+\log n}>0$, i.e., if

$$\begin{aligned} T \;&>\; \frac{\big(c+\log n\big)\big(\log\big(c+\log n\big)-\log c\big)}{\zeta} \\ &=\; \frac{\big(c+\log n\big)\,\log\big(\big(c+\log n\big)/c\big)}{\zeta} \\ &=\; \frac{\big(c+\log n\big)\,\log\big(1+\log n/c\big)}{\zeta} \\ &=\; \frac{\big(c+\log n\big)\,\log\big(1+c^{-1}\log n\big)}{\zeta}. \end{aligned} \tag{7.7.24}$$

Since, if

$$\log c - \log\big(c + \log n\big) + \frac{\zeta\, T}{c + \log n} \quad > \quad 0, \tag{7.7.25}$$

then

$$\log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log c - \log\big(c + \log n\big) + \frac{\zeta\, T}{c + \log n} \quad > \quad \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right),$$

which results in the following expression

$$\log\left(\frac{Z_{\varphi^{(\mathsf{e},c)}}(T)}{Z_{\varphi^{(\mathsf{e},c)}}(0)}\right) \quad > \quad \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log c - \log\big(c + \log n\big) + \frac{\zeta\, T}{c + \log n} \quad > \quad \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right).$$

Alternatively, if

$$\exp\left(\log c - \log\big(c + \log n\big) + \frac{\zeta\, T}{c + \log n}\right) \quad > \quad 1,$$

then

$$\frac{Z_\mu(T)}{Z_\mu(0)}\, \exp\left(\log c - \log\big(c + \log n\big) + \frac{\zeta\, T}{c + \log n}\right) \quad > \quad \frac{Z_\mu(T)}{Z_\mu(0)},$$

which results in the following expression

$$\frac{Z_{\varphi^{(\mathsf{e},c)}}(T)}{Z_{\varphi^{(\mathsf{e},c)}}(0)} \quad > \quad \frac{Z_\mu(T)}{Z_\mu(0)}\, \exp\left(\log c - \log\big(c + \log n\big) + \frac{\zeta\, T}{c + \log n}\right) \quad > \quad \frac{Z_\mu(T)}{Z_\mu(0)}.$$

Thus we have

$$\log\left(\frac{Z_{\varphi^{(\mathsf{e},c)}}(T)}{Z_{\varphi^{(\mathsf{e},c)}}(0)}\right) \quad > \quad \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right), \qquad \text{a.s.} \tag{7.7.26}$$

Hence,

$$\frac{Z_{\varphi^{(\mathsf{e},c)}}(T)}{Z_{\varphi^{(\mathsf{e},c)}}(0)} \quad > \quad \frac{Z_\mu(T)}{Z_\mu(0)}, \qquad \text{a.s.}, \tag{7.7.27}$$

and, we have

$$\mathbb{P}\left(\frac{Z_{\varphi^{(\mathsf{e},c)}}(T)}{Z_{\varphi^{(\mathsf{e},c)}}(0)} \quad > \quad \frac{Z_\mu(T)}{Z_\mu(0)}\right) \quad = \quad 1. \tag{7.7.28}$$

Alternatively, we can appeal to (i) of Lemma 7.2.3 by setting $c_1 := c$ and $c_2 := \frac{\zeta}{c+\log n}$, since $\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$ is positive and finite, thus for all $t > 0$, $\mathbf{S}_c^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) > c > 0$, a.s. Moreover, the inequality (4.2.6) of Proposition 4.2.3 implies that, for a nondegenerate and diverse equity market, there is a $\zeta > 0$ such that $\gamma_\mu^*(t) \geq \zeta$, for all $t \in [0, T]$, a.s. Thus, $\gamma_\mu^*(t)$ has a positive lower bound in a diverse equity market. Also, from (4.6.3), for $t \in [0, T]$, we have the bound $\frac{1}{\mathbf{S}_c^{\mathrm{E}}(\boldsymbol{\mu}(t))} \geq \frac{1}{c+\log n}$. Hence, since the drift process of the modified entropy-weighted portfolio is given by $\mathfrak{g}_{\varphi^{(\mathsf{e},c)}}(t) = \gamma_\mu^*(t)/\mathbf{S}_c^{\mathrm{E}}(\boldsymbol{\mu}(t))$, we have $\mathfrak{g}_{\varphi^{(\mathsf{e},c)}}(t) > \frac{\zeta}{c+\log n} > 0$. Thus, the drift process of the modified entropy-weighted portfolio $\mathfrak{g}_{\varphi^{(\mathsf{e},c)}}(t)$ has a positive lower bound. Consequently, Lemma 7.2.3 implies that there is a dominance relationship between the modified entropy-weighted portfolio $\varphi^{(\mathsf{e},c)}$ and the market portfolio μ, i.e., $\varphi^{(\mathsf{e},c)}$ strictly dominates the market portfolio μ. ■

Thus, if the market $\mathcal{M}$ is (weakly) diverse over the finite time horizon $[0, T]$ and with initial capital equal to that of the modified entropy-weighted portfolio $Z_{\varphi^{(\mathsf{e},c)}}(0) = Z_\mu(0) = w > 0$ such that $\log\big(Z_{\varphi^{(\mathsf{e},c)}}(0)/Z_\mu(0)\big) = \log(1) = 0$, the modified entropy-weighted portfolio satisfies

$$\begin{aligned}\log\left(\frac{Z_{\varphi^{(\mathsf{e},c)}}(T)}{Z_\mu(T)}\right) \quad &> \quad \log\left(\frac{Z_{\varphi^{(\mathsf{e},c)}}(0)}{Z_\mu(0)}\right) + \log c - \log\big(c + \log n\big) + \frac{\zeta\, T}{c + \log n} \\ &= \quad \log c - \log\big(c + \log n\big) + \frac{\zeta\, T}{c + \log n},\end{aligned} \tag{7.7.29}$$

and, then the value process $Z_{\varphi^{(\mathrm{e},c)}}$ of the modified entropy-weighted portfolio a.s. satisfies

$$\begin{aligned} \frac{Z_{\varphi^{(\mathrm{e},c)}}(T)}{Z_\mu(T)} &> \frac{Z_{\varphi^{(\mathrm{e},c)}}(0)}{Z_\mu(0)} \exp\left(\log c - \log\big(c+\log n\big) + \frac{\zeta T}{c+\log n}\right) \\ &= \exp\left(\log c - \log\big(c+\log n\big) + \frac{\zeta T}{c+\log n}\right), \end{aligned} \tag{7.7.30}$$

so that

$$Z_{\varphi^{(\mathrm{e},c)}}(T) > Z_\mu(T) \exp\left(\log c - \log\big(c+\log n\big) + \frac{\zeta T}{c+\log n}\right), \qquad \text{a.s.} \tag{7.7.31}$$

Consequently, in a similar fashion to that shown in the foregoing proof, if

$$\exp\left(\log c - \log\big(c+\log n\big) + \frac{\zeta T}{c+\log n}\right) > 1,$$

then

$$Z_\mu(T) \exp\left(\log c - \log\big(c+\log n\big) + \frac{\zeta T}{c+\log n}\right) > Z_\mu(T),$$

and the following results

$$Z_{\varphi^{(\mathrm{e},c)}}(T) > Z_\mu(T) \exp\left(\log c - \log\big(c+\log n\big) + \frac{\zeta T}{c+\log n}\right) > Z_\mu(T).$$

Hence,

$$Z_{\varphi^{(\mathrm{e},c)}}(T) > Z_\mu(T), \qquad \text{a.s.} \tag{7.7.32}$$

Therefore, we have in particular

$$\mathbb{P}\Big(Z_{\varphi^{(\mathrm{e},c)}}(T) > Z_\mu(T)\Big) = 1, \tag{7.7.33}$$

provided that

$$T > T_* \triangleq \frac{\big(c+\log n\big)\,\log\big(1+c^{-1}\log n\big)}{\zeta}. \tag{7.7.34}$$

Consequently, from (7.3.4) of Definition 7.3.1, the modified entropy-weighted portfolio $\varphi^{(\mathrm{e},c)}$ is a strong arbitrage opportunity relative to the market portfolio μ, and $\varphi^{(\mathrm{e},c)}$ outperforms the market portfolio μ. Notice that the modified entropy-weighted portfolio will only outperform the market portfolio over the long term since the lower bound on T

$$T > \frac{\big(c+\log n\big)\,\log\big(1+c^{-1}\log n\big)}{\zeta},$$

is sufficiently large as it contains the sufficiently small number $\zeta > 0$, which is present in the denominator of the lower bound. Thus, this corollary signifies that, with probability one, the modified entropy-weighted portfolio has a return above that of the market portfolio over a sufficiently long time horizon $[0,T]$. If the market $\mathcal{M}$ is uniformly weakly diverse over $[T_*,\infty)$, then (7.7.29) above gives the a.s. long-term comparison

$$\begin{aligned} \mathcal{L}_{\varphi^{(\mathrm{e},c)},\mu} &= \liminf_{T\to\infty} \frac{1}{T} \log\Big(Z_{\varphi^{(\mathrm{e},c)}}(T)/Z_\mu(T)\Big) \\ &> \liminf_{T\to\infty} \frac{1}{T}\left(\log c - \log\big(c+\log n\big) + \frac{\zeta T}{c+\log n}\right) \\ &= \liminf_{T\to\infty} \left(\frac{\big(\log c - \log\big(c+\log n\big)\big)}{T} + \frac{\zeta}{c+\log n}\right) \\ &= \liminf_{T\to\infty} \frac{\big(\log c - \log\big(c+\log n\big)\big)}{T} + \liminf_{T\to\infty} \frac{\zeta}{c+\log n} \\ &= \liminf_{T\to\infty} \frac{\big(\log c - \log\big(c+\log n\big)\big)}{T} + \frac{\zeta}{c+\log n} \\ &= \frac{\zeta}{c+\log n} > 0, \end{aligned} \tag{7.7.35}$$

since $\zeta > 0$, $c > 0$ and $\log n > 0$ for $n \geq 2$, so that $c + \log n > 0$. Thus, we have

$$\mathcal{L}_{\varphi^{(\mathsf{e},c)},\mu} \quad = \quad \liminf_{T\to\infty} \frac{1}{T} \log\Big(Z_{\varphi^{(\mathsf{e},c)}}(T)/Z_\mu(T)\Big) \quad > \quad \frac{\zeta}{c+\log n} \quad > \quad 0, \qquad \text{a.s.,} \tag{7.7.36}$$

and the modified entropy-weighted portfolio $\varphi^{(\mathsf{e},c)}$ is a superior long-term growth opportunity relative to the market, i.e., the market portfolio will lag rather significantly behind the modified entropy-weighted portfolio over long time horizons.

7.7.1.3 The $\mathbf{D}_p$-Weighted (Diversity-Weighted) Index Portfolio

It can be shown that in a nondegenerate and weakly diverse equity market, without dividends, the diversity-weighted index portfolio (or, the $\mathbf{D}_p$-weighted index portfolio) $\varphi^{(p)}$ strictly dominates the market portfolio μ, for a sufficiently large real number T. This is demonstrated in the next corollary.

Corollary 7.7.3 ([**Fernholz (1998a), Fernholz, Karatzas & Kardaras (2005), Fernholz & Karatzas (2009)**]). *Let μ be the market portfolio and $\varphi^{(p)}$ be the diversity-weighted index portfolio (or, the $\mathbf{D}_p$-weighted index portfolio), and suppose that the market $\mathcal{M}$ is* ***nondegenerate*** *and* ***weakly diverse*** *over a fixed, finite time horizon $[0,T]$, with $T > 0$ a given real number. Then for a* ***sufficiently large real number*** *$T \in [0,\infty)$, we have*

$$\frac{Z_{\varphi^{(p)}}(T)}{Z_{\varphi^{(p)}}(0)} \quad > \quad \frac{Z_\mu(T)}{Z_\mu(0)}, \qquad a.s., \tag{7.7.37}$$

and, consequently

$$\mathbb{P}\left(\frac{Z_{\varphi^{(p)}}(T)}{Z_{\varphi^{(p)}}(0)} \quad > \quad \frac{Z_\mu(T)}{Z_\mu(0)}\right) \quad = \quad 1. \tag{7.7.38}$$

That is, in a nondegenerate and (weakly) diverse equity market, the diversity-weighted index portfolio (or, the $\mathbf{D}_p$-weighted index portfolio) $\varphi^{(p)}$ represents a ***strong arbitrage opportunity*** *relative to the market portfolio μ over the fixed, finite time horizon $[0,T]$, for T sufficiently large.*

Proof. Let $0 < p < 1$, and consider the $\mathbf{D}_p$-function $\mathbf{D}_p : \mathbb{R}^n \to \mathbb{R}^+$ or $\mathbf{D}_p : \Delta^{n-1} \to \mathbb{R}^+$ defined by (4.6.15). By Theorem 5.2.2, $\mathbf{D}_p$ generates the diversity-weighted index portfolio (or, the $\mathbf{D}_p$-weighted index portfolio) with weights (5.6.49), and drift process (5.6.59). By equations (5.6.66) and (5.6.69), for the diversity-weighted index portfolio, we have the following for $T \in [0,\infty)$,

$$\log\left(\frac{Z_{\varphi^{(p)}}(T)}{Z_{\varphi^{(p)}}(0)}\right) \quad = \quad \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log\left(\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(T)\big)}{\mathbf{D}_p\big(\boldsymbol{\mu}(0)\big)}\right) + (1-p)\int_0^T \gamma^*_{\varphi^{(p)}}(t)\,dt. \tag{7.7.39}$$

Now, from (4.6.17), for $t \in [0,T]$, we have the bounds $1 < \big(\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big)\big)^p \leq n^{1-p}$ and from (4.6.18), for $t \in [0,T]$, we have the bounds $1 < \mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) \leq n^{(1-p)/p}$, which implies that $\frac{1}{\mathbf{D}_p(\boldsymbol{\mu}(t))} \geq \frac{1}{n^{(1-p)/p}}$, in particular, $\frac{1}{\mathbf{D}_p(\boldsymbol{\mu}(0))} \geq \frac{1}{n^{(1-p)/p}}$. Alternatively, for $t \in [0,T]$, we have the bounds $0 < \log \mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) \leq \frac{(1-p)}{p}\log n$, so that $-\log \mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) \geq -\frac{(1-p)}{p}\log n$, in particular, $-\log \mathbf{D}_p\big(\boldsymbol{\mu}(0)\big) \geq -\frac{(1-p)}{p}\log n$. Hence, we have, a.s.,

$$\log\left(\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(T)\big)}{\mathbf{D}_p\big(\boldsymbol{\mu}(0)\big)}\right) \quad > \quad \log\left(\frac{1}{n^{(1-p)/p}}\right) \quad = \quad -\frac{(1-p)}{p}\log n. \tag{7.7.40}$$

Proposition 2.4.8 implies that $\gamma^*_{\varphi^{(p)}}(t) \geq 0$, so that for $T \in [0,\infty)$, (7.7.39) can be expressed as

$$\log\left(\frac{Z_{\varphi^{(p)}}(T)}{Z_{\varphi^{(p)}}(0)}\right) \quad > \quad \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log\left(\frac{\mathbf{D}_p\big(\boldsymbol{\mu}(T)\big)}{\mathbf{D}_p\big(\boldsymbol{\mu}(0)\big)}\right). \tag{7.7.41}$$

This coupled with (7.7.40) provides, in particular, the following lower bound

$$\log\left(\frac{Z_{\varphi^{(p)}}(T)}{Z_{\varphi^{(p)}}(0)}\right) > \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log\left(\frac{1}{n^{(1-p)/p}}\right) \tag{7.7.42}$$

$$= \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) - \frac{(1-p)}{p}\log n. \tag{7.7.43}$$

Thus, for $T \in [0,\infty)$, we a.s. have

$$\begin{aligned}\frac{Z_{\varphi^{(p)}}(T)}{Z_{\varphi^{(p)}}(0)} &> \frac{Z_\mu(T)}{Z_\mu(0)}\exp\left(-\frac{(1-p)}{p}\log n\right)\\ &= \frac{Z_\mu(T)}{Z_\mu(0)}\left(n^{-(1-p)/p}\right)\\ &= n^{(p-1)/p}\,\frac{Z_\mu(T)}{Z_\mu(0)},\end{aligned} \tag{7.7.44}$$

which shows that $Z_{\varphi^{(p)}}(T)/Z_\mu(T)$ is bounded below by the constant $n^{(p-1)/p}$. Therefore, condition (ii) of Definition 7.2.1 is satisfied with $c := n^{(p-1)/p} > 0$. Moreover, it is obvious from (5.6.49) that $\varphi_i^{(p)}(t) \geq 0$, for all $t \in [0,T]$, for $i = 1,2,\ldots,n$, and since $\frac{1}{\mathbf{D}_p(\boldsymbol{\mu}(t))} < 1$, we have $\varphi_i^{(p)}(t)/\mu_i(t) < 1$, for $i = 1,2,\ldots,n$. So conditions (i) and (iii) of Definition 7.2.1 are satisfied. Hence, this together with (7.7.44) implies that the diversity-weighted index portfolio $\varphi^{(p)}$ is admissible. Also, (7.3.2) is satisfied for $\varphi := \varphi^{(p)}$, $\eta := \mu$ and for $q := n^{(p-1)/p}$. Now, since the market is weakly diverse over the time horizon $[0,T]$, from the weak diversity condition (4.2.3) of Definition 4.2.2, for $0 < \delta < 1$, we have

$$\begin{aligned}\int_0^T \mu_{(1)}(t)\,dt &\leq (1-\delta)\,T, \quad \text{and,}\\ \int_0^T \mu_{(1)}(t)\,dt - T &\leq -\delta\,T,\end{aligned}$$

so that, we have

$$\begin{aligned}T - \int_0^T \mu_{(1)}(t)\,dt &\geq \delta\,T, \quad \text{and,}\\ \int_0^T \big(1-\mu_{(1)}(t)\big)\,dt &\geq \delta\,T.\end{aligned} \tag{7.7.45}$$

Furthermore, from (5.6.53), for all $t \in [0,T]$, we have $\varphi_{(1)}^{(p)}(t) \leq \mu_{(1)}(t)$, so that $1 - \varphi_{(1)}^{(p)}(t) \geq 1 - \mu_{(1)}(t)$, for all $t \in [0,T]$. Hence, combining (2.4.56) of Lemma 2.4.14 for a nondegenerate market, with the results (5.6.53) and (7.7.45), we see that the assumption of weak diversity implies

$$\int_0^T \gamma^*_{\varphi^{(p)}}(t)\,dt \geq \frac{\varepsilon}{2}\int_0^T \left(1-\varphi_{(1)}^{(p)}(t)\right)dt \geq \frac{\varepsilon}{2}\int_0^T \big(1-\mu_{(1)}(t)\big)\,dt \geq \frac{\varepsilon\,\delta\,T}{2}. \tag{7.7.46}$$

It thus follows, from (7.7.40) and (7.7.46), that a.s., for $T > 0$, (7.7.39) amounts to the following

$$\log\left(\frac{Z_{\varphi^{(p)}}(T)}{Z_{\varphi^{(p)}}(0)}\right) > \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log\left(\frac{1}{n^{(1-p)/p}}\right) + (1-p)\,\frac{\varepsilon\,\delta\,T}{2} \tag{7.7.47}$$

$$= \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) - \frac{(1-p)}{p}\log n + (1-p)\,\frac{\varepsilon\,\delta\,T}{2}, \tag{7.7.48}$$

and, we have

$$\begin{aligned}\frac{Z_{\varphi^{(p)}}(T)}{Z_{\varphi^{(p)}}(0)} &> \frac{Z_\mu(T)}{Z_\mu(0)}\exp\left(-\frac{(1-p)}{p}\log n + (1-p)\,\frac{\varepsilon\,\delta\,T}{2}\right)\\ &= \frac{Z_\mu(T)}{Z_\mu(0)}\left(n^{-(1-p)/p}\,e^{((1-p)\,\varepsilon\,\delta\,T)/2}\right)\\ &= \frac{Z_\mu(T)}{Z_\mu(0)}\left(n^{-1/p}\,e^{(\varepsilon\,\delta\,T)/2}\right)^{1-p}.\end{aligned} \tag{7.7.49}$$

Therefore, by (7.2.5) and (7.2.6) of Definition 7.2.2, the portfolio $\varphi^{(p)}$ strictly dominates the market portfolio μ if $\left(n^{-1/p}\, e^{(\varepsilon\,\delta\,T)/2}\right)^{1-p} > 1$, thus if $-\frac{(1-p)}{p}\log n + (1-p)\,\frac{\varepsilon\,\delta\,T}{2} > 0$, i.e., if

$$T \;>\; \frac{2\,\log n}{p\,\varepsilon\,\delta}. \tag{7.7.50}$$

Since, if

$$-\,\frac{(1-p)}{p}\log n + (1-p)\,\frac{\varepsilon\,\delta\,T}{2} \;>\; 0, \tag{7.7.51}$$

then

$$\log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) - \frac{(1-p)}{p}\log n + (1-p)\,\frac{\varepsilon\,\delta\,T}{2} \;>\; \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right),$$

which results in the following expression

$$\log\left(\frac{Z_{\varphi^{(p)}}(T)}{Z_{\varphi^{(p)}}(0)}\right) \;>\; \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) - \frac{(1-p)}{p}\log n + (1-p)\,\frac{\varepsilon\,\delta\,T}{2} \;>\; \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right).$$

Alternatively, if

$$\left(n^{-1/p}\, e^{(\varepsilon\,\delta\,T)/2}\right)^{1-p} \;>\; 1, \tag{7.7.52}$$

then

$$\frac{Z_\mu(T)}{Z_\mu(0)}\left(n^{-1/p}\, e^{(\varepsilon\,\delta\,T)/2}\right)^{1-p} \;>\; \frac{Z_\mu(T)}{Z_\mu(0)},$$

which results in the following expression

$$\frac{Z_{\varphi^{(p)}}(T)}{Z_{\varphi^{(p)}}(0)} \;>\; \frac{Z_\mu(T)}{Z_\mu(0)}\left(n^{-1/p}\, e^{(\varepsilon\,\delta\,T)/2}\right)^{1-p} \;>\; \frac{Z_\mu(T)}{Z_\mu(0)}.$$

Thus we have

$$\log\left(\frac{Z_{\varphi^{(p)}}(T)}{Z_{\varphi^{(p)}}(0)}\right) \;>\; \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right), \qquad \text{a.s.} \tag{7.7.53}$$

Hence,

$$\frac{Z_{\varphi^{(p)}}(T)}{Z_{\varphi^{(p)}}(0)} \;>\; \frac{Z_\mu(T)}{Z_\mu(0)}, \qquad \text{a.s.,} \tag{7.7.54}$$

and, we have

$$\mathbb{P}\left(\frac{Z_{\varphi^{(p)}}(T)}{Z_{\varphi^{(p)}}(0)} \;>\; \frac{Z_\mu(T)}{Z_\mu(0)}\right) \;=\; 1. \tag{7.7.55}$$

Alternatively, we can appeal to (i) of Lemma 7.2.3, since $\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) > 1$, thus for all $t > 0$, $\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) > 1 > 0$, a.s. Moreover, the inequality (2.4.56) of Lemma 2.4.14 implies that, for a nondegenerate equity market, there is an $\varepsilon > 0$ such that $\gamma^*_{\varphi^{(p)}}(t) \geq \frac{\varepsilon}{2}\big(1 - \varphi^{(p)}_{(1)}(t)\big)$, for all $t \in [0, T]$, a.s. Thus, $\gamma^*_{\varphi^{(p)}}(t)$ has a positive lower bound in a nondegenerate equity market. Hence, since the drift process of the diversity-weighted index portfolio is given by $\mathfrak{g}_{\varphi^{(p)}}(t) = (1-p)\gamma^*_{\varphi^{(p)}}(t)$ and $0 < p < 1$, we have $\mathfrak{g}_{\varphi^{(p)}}(t) > 0$. Thus, the drift process of the diversity-weighted index portfolio $\mathfrak{g}_{\varphi^{(p)}}(t)$ has a positive lower bound. Consequently, Lemma 7.2.3 implies that there is a dominance relationship between the diversity-weighted index portfolio $\varphi^{(p)}$ and the market portfolio μ, i.e., $\varphi^{(p)}$ strictly dominates the market portfolio μ. ■

Thus, if the market $\mathcal{M}$ is (weakly) diverse over the finite time horizon $[0,T]$ and with initial capital equal to that of the diversity-weighted index portfolio $Z_{\varphi^{(p)}}(0) = Z_\mu(0) = w > 0$ such that $\log\left(Z_{\varphi^{(p)}}(0)/Z_\mu(0)\right) = \log(1) = 0$, the diversity-weighted index portfolio satisfies

$$\begin{aligned}
\log\left(\frac{Z_{\varphi^{(p)}}(T)}{Z_\mu(T)}\right) &> \log\left(\frac{Z_{\varphi^{(p)}}(0)}{Z_\mu(0)}\right) + \log\left(\frac{1}{n^{(1-p)/p}}\right) + (1-p)\,\frac{\varepsilon\,\delta\,T}{2} \\
&= \log\left(\frac{Z_{\varphi^{(p)}}(0)}{Z_\mu(0)}\right) - \frac{(1-p)}{p}\log n + (1-p)\,\frac{\varepsilon\,\delta\,T}{2} \\
&= -\frac{(1-p)}{p}\log n + (1-p)\,\frac{\varepsilon\,\delta\,T}{2} \qquad (7.7.56) \\
&= (1-p)\left(\frac{\varepsilon\,\delta\,T}{2} - \frac{\log n}{p}\right), \qquad (7.7.57)
\end{aligned}$$

and, then the value process $Z_{\varphi^{(p)}}$ of the diversity-weighted index portfolio a.s. satisfies

$$\begin{aligned}
\frac{Z_{\varphi^{(p)}}(T)}{Z_\mu(T)} &> \frac{Z_{\varphi^{(p)}}(0)}{Z_\mu(0)}\left(n^{-1/p}\,e^{(\varepsilon\,\delta\,T)/2}\right)^{1-p} \\
&= \left(n^{-1/p}\,e^{(\varepsilon\,\delta\,T)/2}\right)^{1-p}, \qquad (7.7.58)
\end{aligned}$$

so that

$$Z_{\varphi^{(p)}}(T) \;>\; Z_\mu(T)\left(n^{-1/p}\,e^{(\varepsilon\,\delta\,T)/2}\right)^{1-p}, \qquad \text{a.s.} \qquad (7.7.59)$$

Consequently, in a similar fashion to that shown in the foregoing proof, if

$$\left(n^{-1/p}\,e^{(\varepsilon\,\delta\,T)/2}\right)^{1-p} \;>\; 1,$$

then

$$Z_\mu(T)\left(n^{-1/p}\,e^{(\varepsilon\,\delta\,T)/2}\right)^{1-p} \;>\; Z_\mu(T),$$

and the following results

$$Z_{\varphi^{(p)}}(T) \;>\; Z_\mu(T)\left(n^{-1/p}\,e^{(\varepsilon\,\delta\,T)/2}\right)^{1-p} \;>\; Z_\mu(T).$$

Hence,

$$Z_{\varphi^{(p)}}(T) \;>\; Z_\mu(T), \qquad \text{a.s.} \qquad (7.7.60)$$

Therefore, we have in particular

$$\mathbb{P}\left(Z_{\varphi^{(p)}}(T) \;>\; Z_\mu(T)\right) \;=\; 1, \qquad (7.7.61)$$

provided that

$$T \;>\; T_* \;\triangleq\; \frac{2\,\log n}{p\,\varepsilon\,\delta}. \qquad (7.7.62)$$

Consequently, from (7.3.4) of Definition 7.3.1, the diversity-weighted index portfolio $\varphi^{(p)}$ is a strong arbitrage opportunity relative to the market portfolio μ, and $\varphi^{(p)}$ outperforms the market portfolio μ. The significance of such a result for practical long-term portfolio management cannot be overstated. Notice that the diversity-weighted index portfolio will only outperform the market portfolio over the long term since the lower bound on T

$$T \;>\; \frac{2\,\log n}{p\,\varepsilon\,\delta},$$

is sufficiently large as it contains the sufficiently small numbers $0 < p < 1$, $0 < \delta < 1$ and $\varepsilon > 0$, which are present in the denominator of the lower bound. Thus, this corollary signifies that, with probability one, the diversity-weighted index portfolio has a return above that of the market portfolio over a sufficiently long time horizon $[0, T]$. If the market $\mathcal{M}$ is uniformly weakly diverse over $[T_*, \infty)$, then (7.7.56) above gives the a.s. long-term comparison

$$\begin{aligned}
\mathcal{L}_{\varphi^{(p)},\mu} &= \liminf_{T\to\infty} \frac{1}{T} \log\Big(Z_{\varphi^{(p)}}(T)/Z_\mu(T)\Big) \\
&> \liminf_{T\to\infty} \frac{1}{T}\bigg(-\frac{(1-p)}{p}\log n + (1-p)\,\frac{\varepsilon\,\delta\,T}{2}\bigg) \\
&= \liminf_{T\to\infty}\bigg(-\frac{(1-p)}{p}\frac{\log n}{T} + (1-p)\,\frac{\varepsilon\,\delta}{2}\bigg) \\
&= \liminf_{T\to\infty} -\frac{(1-p)}{p}\frac{\log n}{T} + \liminf_{T\to\infty}(1-p)\,\frac{\varepsilon\,\delta}{2} \\
&= \liminf_{T\to\infty} -\frac{(1-p)}{p}\frac{\log n}{T} + (1-p)\,\frac{\varepsilon\,\delta}{2} \\
&= (1-p)\,\frac{\varepsilon\,\delta}{2} \quad > \quad 0, \qquad (7.7.63)
\end{aligned}$$

since $\varepsilon > 0$, $0 < \delta < 1$, and $0 < p < 1$ so that $0 < 1 - p < 1$. Thus, we have

$$\mathcal{L}_{\varphi^{(p)},\mu} \;=\; \liminf_{T\to\infty} \frac{1}{T} \log\Big(Z_{\varphi^{(p)}}(T)/Z_\mu(T)\Big) \;>\; (1-p)\,\frac{\varepsilon\,\delta}{2} \;>\; 0, \qquad \text{a.s.,} \qquad (7.7.64)$$

and the diversity-weighted index portfolio $\varphi^{(p)}$ is a superior long-term growth opportunity relative to the market, i.e., the market portfolio will lag rather significantly behind the diversity-weighted index portfolio over long time horizons.

7.7.1.4 The Normalised $\mathbf{D}_p$-Weighted (Diversity-Weighted) Index Portfolio (The $\widetilde{\mathbf{D}}_p$-Weighted Index Portfolio)

It can be shown that in a nondegenerate and weakly diverse equity market, without dividends, the normalised diversity-weighted index portfolio (or, the $\widetilde{\mathbf{D}}_p$-weighted index portfolio) $\widetilde{\varphi}^{(p)}$ strictly dominates the market portfolio μ, for a sufficiently large real number T. This is demonstrated in the next corollary.

Corollary 7.7.4. *Let μ be the market portfolio and $\widetilde{\varphi}^{(p)}$ be the normalised diversity-weighted index portfolio (or, the $\widetilde{\mathbf{D}}_p$-weighted index portfolio), and suppose that the market $\mathcal{M}$ is* ***nondegenerate*** *and* ***weakly diverse*** *over a fixed, finite time horizon $[0, T]$, with $T > 0$ a given real number. Then for a* ***sufficiently large real number*** *$T \in [0, \infty)$, we have*

$$\frac{Z_{\widetilde{\varphi}^{(p)}}(T)}{Z_{\widetilde{\varphi}^{(p)}}(0)} \;>\; \frac{Z_\mu(T)}{Z_\mu(0)}, \qquad a.s., \qquad (7.7.65)$$

and, consequently

$$\mathbb{P}\left(\frac{Z_{\widetilde{\varphi}^{(p)}}(T)}{Z_{\widetilde{\varphi}^{(p)}}(0)} \;>\; \frac{Z_\mu(T)}{Z_\mu(0)}\right) \;=\; 1. \qquad (7.7.66)$$

That is, in a nondegenerate and (weakly) diverse equity market, the normalised diversity-weighted index portfolio (or, the $\widetilde{\mathbf{D}}_p$-weighted index portfolio) $\widetilde{\varphi}^{(p)}$ represents a ***strong arbitrage opportunity*** *relative to the market portfolio μ over the fixed, finite time horizon $[0, T]$, for T sufficiently large.*

Proof. Let $0 < p < 1$, and consider the normalised version of the $\mathbf{D}_p$-function $\widetilde{\mathbf{D}}_p : \mathbb{R}^n \to \mathbb{R}^+$ or $\widetilde{\mathbf{D}}_p : \Delta^{n-1} \to \mathbb{R}^+$ defined by (4.6.32). By Theorem 5.2.2, $\widetilde{\mathbf{D}}_p$ generates the normalised diversity-weighted index portfolio (or, the $\widetilde{\mathbf{D}}_p$-weighted index portfolio) with weights (5.6.76), and drift process (5.6.83). By equations (5.6.85) and

(5.6.88), for the normalised diversity-weighted index portfolio, we have the following for $T \in [0, \infty)$,

$$\log\left(\frac{Z_{\widetilde{\varphi}^{(p)}}(T)}{Z_{\widetilde{\varphi}^{(p)}}(0)}\right) = \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log\left(\frac{\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(T)\big)}{\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(0)\big)}\right) + (1-p)\int_0^T \gamma^*_{\varphi^{(p)}}(t)\,dt. \tag{7.7.67}$$

Now, from (4.6.35), for $t \in [0, T]$, we have the bounds $n^{(p-1)/p} < \widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big) \leq 1$, which implies that $\frac{1}{\widetilde{\mathbf{D}}_p(\boldsymbol{\mu}(t))} \geq 1$, in particular, $\frac{1}{\widetilde{\mathbf{D}}_p(\boldsymbol{\mu}(0))} \geq 1$. Alternatively, for $t \in [0, T]$, we have the bounds $\frac{(p-1)}{p}\log n < \log \widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big) \leq 0$, so that $-\log \widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big) \geq 0$, in particular, $-\log \widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(0)\big) \geq 0$. Hence, we have, a.s.,

$$\log\left(\frac{\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(T)\big)}{\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(0)\big)}\right) > \log\left(\frac{n^{(p-1)/p}}{1}\right) = \frac{(p-1)}{p}\log n. \tag{7.7.68}$$

Proposition 2.4.8 implies that $\gamma^*_{\varphi^{(p)}}(t) \geq 0$, so that for $T \in [0, \infty)$, (7.7.68) can be expressed as

$$\log\left(\frac{Z_{\widetilde{\varphi}^{(p)}}(T)}{Z_{\widetilde{\varphi}^{(p)}}(0)}\right) > \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log\left(\frac{\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(T)\big)}{\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(0)\big)}\right). \tag{7.7.69}$$

This coupled with (7.7.68) provides, in particular, the following lower bound

$$\log\left(\frac{Z_{\widetilde{\varphi}^{(p)}}(T)}{Z_{\widetilde{\varphi}^{(p)}}(0)}\right) > \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log\left(\frac{n^{(p-1)/p}}{1}\right) \tag{7.7.70}$$

$$= \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \frac{(p-1)}{p}\log n. \tag{7.7.71}$$

Thus, for $T \in [0, \infty)$, we a.s. have

$$\begin{aligned}\frac{Z_{\widetilde{\varphi}^{(p)}}(T)}{Z_{\widetilde{\varphi}^{(p)}}(0)} &> \frac{Z_\mu(T)}{Z_\mu(0)}\exp\left(\frac{(p-1)}{p}\log n\right)\\ &= \frac{Z_\mu(T)}{Z_\mu(0)}\left(n^{(p-1)/p}\right)\\ &= n^{(p-1)/p}\,\frac{Z_\mu(T)}{Z_\mu(0)},\end{aligned} \tag{7.7.72}$$

which shows that $Z_{\widetilde{\varphi}^{(p)}}(T)/Z_\mu(T)$ is bounded below by the constant $n^{(p-1)/p}$. Therefore, condition (ii) of Definition 7.2.1 is satisfied with $c := n^{(p-1)/p} > 0$. Moreover, it is obvious from (5.6.76) that $\widetilde{\varphi}^{(p)}_i(t) \geq 0$, for all $t \in [0, T]$, for $i = 1, 2, \ldots, n$, and since $\frac{1}{\widetilde{\mathbf{D}}_p(\boldsymbol{\mu}(t))} < \frac{1}{n^{(p-1)/p}}$, we have $\widetilde{\varphi}^{(p)}_i(t)/\mu_i(t) < 1$, for $i = 1, 2, \ldots, n$. So conditions (i) and (iii) of Definition 7.2.1 are satisfied. Hence, this together with (7.7.72) implies that the normalised diversity-weighted index portfolio $\widetilde{\varphi}^{(p)}$ is admissible. Also, (7.3.2) is satisfied for $\varphi := \widetilde{\varphi}^{(p)}$, $\eta := \mu$ and for $q := n^{(p-1)/p}$. It thus follows, from (7.7.68) and (7.7.46), that a.s., for $T > 0$, (7.7.67) amounts to the following

$$\log\left(\frac{Z_{\widetilde{\varphi}^{(p)}}(T)}{Z_{\widetilde{\varphi}^{(p)}}(0)}\right) > \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \log\left(\frac{n^{(p-1)/p}}{1}\right) + (1-p)\,\frac{\varepsilon\,\delta\,T}{2} \tag{7.7.73}$$

$$= \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \frac{(p-1)}{p}\log n + (1-p)\,\frac{\varepsilon\,\delta\,T}{2}, \tag{7.7.74}$$

and, we have

$$\begin{aligned}\frac{Z_{\widetilde{\varphi}^{(p)}}(T)}{Z_{\widetilde{\varphi}^{(p)}}(0)} &> \frac{Z_\mu(T)}{Z_\mu(0)}\exp\left(\frac{(p-1)}{p}\log n + (1-p)\,\frac{\varepsilon\,\delta\,T}{2}\right)\\ &= \frac{Z_\mu(T)}{Z_\mu(0)}\exp\left(\frac{(p-1)}{p}\log n - (p-1)\,\frac{\varepsilon\,\delta\,T}{2}\right)\\ &= \frac{Z_\mu(T)}{Z_\mu(0)}\left(n^{(p-1)/p}\,e^{-\,((p-1)\,\varepsilon\,\delta\,T)/2}\right)\\ &= \frac{Z_\mu(T)}{Z_\mu(0)}\left(n^{1/p}\,e^{-\,(\varepsilon\,\delta\,T)/2}\right)^{p-1}.\end{aligned} \tag{7.7.75}$$

Therefore, by (7.2.5) and (7.2.6) of Definition 7.2.2, the portfolio $\widetilde{\varphi}^{(p)}$ strictly dominates the market portfolio μ if $\left(n^{1/p}\, e^{-\,(\varepsilon\,\delta\,T)/2}\right)^{p-1} > 1$, thus if $\frac{(p-1)}{p}\log n + (1-p)\,\frac{\varepsilon\,\delta\,T}{2} > 0$, i.e., if

$$T \;>\; \frac{2\,\log n}{p\,\varepsilon\,\delta}, \tag{7.7.76}$$

since $1-p>0$ or $p-1<0$. For, if

$$\frac{(p-1)}{p}\log n + (1-p)\,\frac{\varepsilon\,\delta\,T}{2} \;>\; 0, \tag{7.7.77}$$

then

$$\log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \frac{(p-1)}{p}\log n + (1-p)\,\frac{\varepsilon\,\delta\,T}{2} \;>\; \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right),$$

which results in the following expression

$$\log\left(\frac{Z_{\widetilde{\varphi}^{(p)}}(T)}{Z_{\widetilde{\varphi}^{(p)}}(0)}\right) \;>\; \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) + \frac{(p-1)}{p}\log n + (1-p)\,\frac{\varepsilon\,\delta\,T}{2} \;>\; \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right).$$

Alternatively, if

$$\left(n^{1/p}\, e^{-\,(\varepsilon\,\delta\,T)/2}\right)^{p-1} \;>\; 1, \tag{7.7.78}$$

then

$$\frac{Z_\mu(T)}{Z_\mu(0)}\left(n^{1/p}\, e^{-\,(\varepsilon\,\delta\,T)/2}\right)^{p-1} \;>\; \frac{Z_\mu(T)}{Z_\mu(0)},$$

which results in the following expression

$$\frac{Z_{\widetilde{\varphi}^{(p)}}(T)}{Z_{\widetilde{\varphi}^{(p)}}(0)} \;>\; \frac{Z_\mu(T)}{Z_\mu(0)}\left(n^{1/p}\, e^{-\,(\varepsilon\,\delta\,T)/2}\right)^{p-1} \;>\; \frac{Z_\mu(T)}{Z_\mu(0)}.$$

Thus we have

$$\log\left(\frac{Z_{\widetilde{\varphi}^{(p)}}(T)}{Z_{\widetilde{\varphi}^{(p)}}(0)}\right) \;>\; \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right), \qquad \text{a.s.} \tag{7.7.79}$$

Hence,

$$\frac{Z_{\widetilde{\varphi}^{(p)}}(T)}{Z_{\widetilde{\varphi}^{(p)}}(0)} \;>\; \frac{Z_\mu(T)}{Z_\mu(0)}, \qquad \text{a.s.}, \tag{7.7.80}$$

and, we have

$$\mathbb{P}\left(\frac{Z_{\widetilde{\varphi}^{(p)}}(T)}{Z_{\widetilde{\varphi}^{(p)}}(0)} \;>\; \frac{Z_\mu(T)}{Z_\mu(0)}\right) \;=\; 1. \tag{7.7.81}$$

Alternatively, we can appeal to (i) of Lemma 7.2.3, since $\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big) > n^{(p-1)/p}$, thus for all $t>0$, $\widetilde{\mathbf{D}}_p\big(\boldsymbol{\mu}(t)\big) > n^{(p-1)/p} > 0$, a.s. Moreover, the inequality (2.4.56) of Lemma 2.4.14 implies that, for a nondegenerate equity market, there is an $\varepsilon > 0$ such that $\gamma^*_{\varphi^{(p)}}(t) \geq \frac{\varepsilon}{2}\big(1-\varphi^{(p)}_{(1)}(t)\big)$, for all $t\in[0,T]$, a.s. Thus, $\gamma^*_{\varphi^{(p)}}(t)$ has a positive lower bound in a nondegenerate equity market. Hence, since the drift process of the normalised diversity-weighted index portfolio is given by $\mathfrak{g}_{\widetilde{\varphi}^{(p)}}(t) = (1-p)\gamma^*_{\varphi^{(p)}}(t)$ and $0<p<1$, we have $\mathfrak{g}_{\widetilde{\varphi}^{(p)}}(t) > 0$. Thus, the drift process of the normalised diversity-weighted index portfolio $\mathfrak{g}_{\widetilde{\varphi}^{(p)}}(t)$ has a positive lower bound. Consequently, Lemma 7.2.3 implies that there is a dominance relationship between the normalised diversity-weighted index portfolio $\widetilde{\varphi}^{(p)}$ and the market portfolio μ, i.e., the normalised diversity-weighted index portfolio $\widetilde{\varphi}^{(p)}$ strictly dominates the market portfolio μ. ■

Thus, if the market $\mathcal{M}$ is (weakly) diverse over the finite time horizon $[0, T]$ and with initial capital equal to that of the normalised diversity-weighted index portfolio $Z_{\widetilde{\varphi}^{(p)}}(0) = Z_\mu(0) = w > 0$ such that $\log\left(Z_{\widetilde{\varphi}^{(p)}}(0)/Z_\mu(0)\right) = \log(1) = 0$, the normalised diversity-weighted index portfolio satisfies

$$\begin{aligned}
\log\left(\frac{Z_{\widetilde{\varphi}^{(p)}}(T)}{Z_\mu(T)}\right) &> \log\left(\frac{Z_{\widetilde{\varphi}^{(p)}}(0)}{Z_\mu(0)}\right) + \log\left(\frac{n^{(p-1)/p}}{1}\right) + (1-p)\,\frac{\varepsilon\,\delta\,T}{2} \\
&= \log\left(\frac{Z_{\widetilde{\varphi}^{(p)}}(0)}{Z_\mu(0)}\right) + \frac{(p-1)}{p}\log n + (1-p)\,\frac{\varepsilon\,\delta\,T}{2} \\
&= \frac{(p-1)}{p}\log n + (1-p)\,\frac{\varepsilon\,\delta\,T}{2} \qquad (7.7.82) \\
&= \frac{(p-1)}{p}\log n - (p-1)\,\frac{\varepsilon\,\delta\,T}{2} \qquad (7.7.83) \\
&= (p-1)\left(\frac{\log n}{p} - \frac{\varepsilon\,\delta\,T}{2}\right), \qquad (7.7.84)
\end{aligned}$$

and, then the value process $Z_{\widetilde{\varphi}^{(p)}}$ of the normalised diversity-weighted index portfolio a.s. satisfies

$$\begin{aligned}
\frac{Z_{\widetilde{\varphi}^{(p)}}(T)}{Z_\mu(T)} &> \frac{Z_{\widetilde{\varphi}^{(p)}}(0)}{Z_\mu(0)}\left(n^{1/p}\,e^{-(\varepsilon\,\delta\,T)/2}\right)^{p-1} \\
&= \left(n^{1/p}\,e^{-(\varepsilon\,\delta\,T)/2}\right)^{p-1}, \qquad (7.7.85)
\end{aligned}$$

so that

$$Z_{\widetilde{\varphi}^{(p)}}(T) \;>\; Z_\mu(T)\left(n^{1/p}\,e^{-(\varepsilon\,\delta\,T)/2}\right)^{p-1}, \qquad \text{a.s.} \qquad (7.7.86)$$

Consequently, in a similar fashion to that shown in the foregoing proof, if

$$\left(n^{1/p}\,e^{-(\varepsilon\,\delta\,T)/2}\right)^{p-1} \;>\; 1,$$

then

$$Z_\mu(T)\left(n^{1/p}\,e^{-(\varepsilon\,\delta\,T)/2}\right)^{p-1} \;>\; Z_\mu(T),$$

and the following results

$$Z_{\widetilde{\varphi}^{(p)}}(T) \;>\; Z_\mu(T)\left(n^{1/p}\,e^{-(\varepsilon\,\delta\,T)/2}\right)^{p-1} \;>\; Z_\mu(T).$$

Hence,

$$Z_{\widetilde{\varphi}^{(p)}}(T) \;>\; Z_\mu(T), \qquad \text{a.s.} \qquad (7.7.87)$$

Therefore, we have in particular

$$\mathbb{P}\left(Z_{\widetilde{\varphi}^{(p)}}(T) \;>\; Z_\mu(T)\right) \;=\; 1, \qquad (7.7.88)$$

provided that

$$T \;>\; T_* \;\triangleq\; \frac{2\,\log n}{p\,\varepsilon\,\delta}. \qquad (7.7.89)$$

Consequently, from (7.3.4) of Definition 7.3.1, the normalised diversity-weighted index portfolio $\widetilde{\varphi}^{(p)}$ is a strong arbitrage opportunity relative to the market portfolio μ, and $\widetilde{\varphi}^{(p)}$ outperforms the market portfolio μ. Notice that the normalised diversity-weighted index portfolio will only outperform the market portfolio over the long term since the lower bound on T

$$T \;>\; \frac{2\,\log n}{p\,\varepsilon\,\delta},$$

is sufficiently large as it contains the sufficiently small numbers $0 < p < 1$, $0 < \delta < 1$ and $\varepsilon > 0$, which are present in the denominator of the lower bound. Thus, this corollary signifies that, with probability one, the normalised diversity-weighted index portfolio has a return above that of the market portfolio over a sufficiently long time horizon $[0, T]$. If the market $\mathcal{M}$ is uniformly weakly diverse over $[T_*, \infty)$, then (7.7.82) above gives the a.s. long-term comparison

$$\begin{aligned}
\mathcal{L}_{\widetilde{\varphi}^{(p)},\mu} &= \liminf_{T\to\infty} \frac{1}{T} \log\Big(Z_{\widetilde{\varphi}^{(p)}}(T)/Z_\mu(T)\Big) \\
&> \liminf_{T\to\infty} \frac{1}{T}\bigg(\frac{(p-1)}{p}\log n + (1-p)\,\frac{\varepsilon\,\delta\,T}{2}\bigg) \\
&= \liminf_{T\to\infty} \bigg(\frac{(p-1)}{p}\frac{\log n}{T} + (1-p)\,\frac{\varepsilon\,\delta}{2}\bigg) \\
&= \liminf_{T\to\infty} \frac{(p-1)}{p}\frac{\log n}{T} + \liminf_{T\to\infty}(1-p)\,\frac{\varepsilon\,\delta}{2} \\
&= \liminf_{T\to\infty} \frac{(p-1)}{p}\frac{\log n}{T} + (1-p)\,\frac{\varepsilon\,\delta}{2} \\
&= (1-p)\,\frac{\varepsilon\,\delta}{2} \quad > \quad 0,
\end{aligned} \tag{7.7.90}$$

since $\varepsilon > 0$, $0 < \delta < 1$, and $0 < p < 1$ so that $0 < 1 - p < 1$. Thus, we have

$$\mathcal{L}_{\widetilde{\varphi}^{(p)},\mu} \;=\; \liminf_{T\to\infty} \frac{1}{T} \log\Big(Z_{\widetilde{\varphi}^{(p)}}(T)/Z_\mu(T)\Big) \;>\; (1-p)\,\frac{\varepsilon\,\delta}{2} \;>\; 0, \qquad \text{a.s.,} \tag{7.7.91}$$

and the normalised diversity-weighted index portfolio $\widetilde{\varphi}^{(p)}$ is a superior long-term growth opportunity relative to the market, i.e., the market portfolio will lag rather significantly behind the normalised diversity-weighted index portfolio over long time horizons.

7.7.1.5 The Equal-Weighted Portfolio

Consider the geometric mean function $\mathbf{S}^{\mathrm{GM}} : \mathbb{R}^n \to \mathbb{R}^+$ or $\mathbf{S}^{\mathrm{GM}} : \Delta^{n-1} \to \mathbb{R}^+$ defined by the geometric mean in (4.6.91). By Theorem 5.2.2, $\mathbf{S}^{\mathrm{GM}}$ generates the equal-weighted portfolio with drift process (5.6.102). From (4.6.93), for $t \in [0, T]$, we have the bounds $0 < \mathbf{S}^{\mathrm{GM}}\big(\boldsymbol{\mu}(t)\big) \le \frac{1}{n}$. The value of the geometric mean will be bounded away from zero only if each $\mu_i(t)$ is bounded in the same manner. This condition is quite restrictive and does not necessarily hold even in a diverse equity market. Thus, in this general case, Lemma 7.2.3 cannot be applied and we cannot establish whether there is a dominance relationship between the equal-weighted portfolio φ^{equal} and the market portfolio μ.

7.7.1.6 The Quadratic Gini-Coefficient-Weighted Portfolio

It can be shown that in a nondegenerate and weakly diverse equity market, without dividends, the quadratic Gini-coefficient-weighted portfolio φ^{g} strictly dominates the market portfolio μ, for a sufficiently large real number T. This is demonstrated in the next corollary.

Corollary 7.7.5. *Let μ be the market portfolio and φ^{g} be the quadratic Gini-coefficient-weighted portfolio, and suppose that the market $\mathcal{M}$ is* ***nondegenerate*** *and* ***weakly diverse*** *over a fixed, finite time horizon $[0, T]$, with $T > 0$ a given real number. Then for a* ***sufficiently large real number*** *$T \in [0, \infty)$, we have*

$$\frac{Z_{\varphi^{\mathrm{g}}}(T)}{Z_{\varphi^{\mathrm{g}}}(0)} \;>\; \frac{Z_\mu(T)}{Z_\mu(0)}, \qquad \textit{a.s.,} \tag{7.7.92}$$

and, consequently

$$\mathbb{P}\left(\frac{Z_{\varphi^{\mathrm{g}}}(T)}{Z_{\varphi^{\mathrm{g}}}(0)} \;>\; \frac{Z_\mu(T)}{Z_\mu(0)}\right) \;=\; 1. \tag{7.7.93}$$

That is, in a nondegenerate and (weakly) diverse equity market, the quadratic Gini-coefficient-weighted portfolio φ^{g} represents a ***strong arbitrage opportunity*** *relative to the market portfolio μ over the fixed, finite time horizon $[0,T]$, for T sufficiently large.*

Proof. Consider the quadratic Gini coefficient function $\mathbf{S}^{\mathrm{G}} : \mathbb{R}^n \to \mathbb{R}^+$ or $\mathbf{S}^{\mathrm{G}} : \Delta^{n-1} \to \mathbb{R}^+$ defined by the quadratic Gini coefficient in (4.6.62). By Theorem 5.2.2, $\mathbf{S}^{\mathrm{G}}$ generates the quadratic Gini-coefficient-weighted portfolio with weights (5.6.147), and drift process (5.6.150). Employing (5.6.152) and (5.6.155), for the quadratic Gini-coefficient-weighted portfolio, we have the following for $T \in [0,\infty)$,

$$\log\left(\frac{Z_{\varphi^{\mathrm{g}}}(T)}{Z_{\varphi^{\mathrm{g}}}(0)}\right) = \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(0)\big)}\right) + \frac{1}{2}\int_0^T \frac{1}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^n \mu_i^2(t)\tau_{ii}(t)\,dt. \tag{7.7.94}$$

Now, from (4.6.64), for $t \in [0,T]$, we have the bounds $\frac{1}{2} < \mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) \leq 1$, which implies that $\frac{1}{\mathbf{S}^{\mathrm{G}}(\boldsymbol{\mu}(t))} \geq 1$, in particular, $\frac{1}{\mathbf{S}^{\mathrm{G}}(\boldsymbol{\mu}(0))} \geq 1$. Hence, we have, a.s.,

$$\log\left(\frac{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(0)\big)}\right) > \log\left(\frac{1}{2}\right) = -\log 2. \tag{7.7.95}$$

Since for all $i = 1,2,\ldots,n$, $\tau_{ii}(t) \geq 0$, the drift process is nondecreasing (i.e., the integral in (7.7.94) is nonnegative), so that for $T \in [0,\infty)$, (7.7.94) can be expressed as

$$\log\left(\frac{Z_{\varphi^{\mathrm{g}}}(T)}{Z_{\varphi^{\mathrm{g}}}(0)}\right) > \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(0)\big)}\right). \tag{7.7.96}$$

This coupled with (7.7.95) provides, in particular, the following lower bound

$$\log\left(\frac{Z_{\varphi^{\mathrm{g}}}(T)}{Z_{\varphi^{\mathrm{g}}}(0)}\right) > \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) - \log 2. \tag{7.7.97}$$

Thus, for $T \in [0,\infty)$, we a.s. have

$$\begin{aligned} \frac{Z_{\varphi^{\mathrm{g}}}(T)}{Z_{\varphi^{\mathrm{g}}}(0)} &> \frac{Z_{\mu}(T)}{Z_{\mu}(0)} \exp\big(-\log 2\big) \\ &= \frac{1}{2}\,\frac{Z_{\mu}(T)}{Z_{\mu}(0)}. \end{aligned} \tag{7.7.98}$$

Now, since the market is weakly diverse over the time horizon $[0,T]$, from the weak diversity condition (4.2.3) of Definition 4.2.2 and (7.7.45), for $0 < \delta < 1$, we have

$$\int_0^T \big(1 - \mu_{(1)}(t)\big)\,dt \geq \delta\,T. \tag{7.7.99}$$

So, by the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \left(\int_0^T 1^2\,dt\right)\left(\int_0^T \big(1-\mu_{(1)}(t)\big)^2\,dt\right) &\geq \left(\int_0^T \big(1-\mu_{(1)}(t)\big)\,dt\right)^2, \\ T\int_0^T \big(1-\mu_{(1)}(t)\big)^2\,dt &\geq \left(\int_0^T \big(1-\mu_{(1)}(t)\big)\,dt\right)^2, \\ \int_0^T \big(1-\mu_{(1)}(t)\big)^2\,dt &\geq \frac{1}{T}\left(\int_0^T \big(1-\mu_{(1)}(t)\big)\,dt\right)^2, \end{aligned}$$

so that by (7.7.99), the above inequality becomes

$$\begin{aligned} T\int_0^T \big(1-\mu_{(1)}(t)\big)^2\,dt &\geq (\delta\,T)^2 = \delta^2\,T^2, \quad \text{and,} \\ \int_0^T \big(1-\mu_{(1)}(t)\big)^2\,dt &\geq \delta^2\,T. \end{aligned} \tag{7.7.100}$$

Hence, combining (2.4.54) of Lemma 2.4.11 for a nondegenerate market, with the results (7.7.100) and $\frac{1}{\mathbf{S}^{\mathrm{G}}(\boldsymbol{\mu}(t))} \geq 1$, we see that the assumption of weak diversity implies

$$\begin{aligned}
\frac{1}{2}\int_0^T \frac{1}{\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^n \mu_i^2(t)\tau_{ii}(t)\,dt &\geq \frac{\varepsilon}{2}\int_0^T \sum_{i=1}^n \mu_i^2(t)\big(1-\mu_{(1)}(t)\big)^2\,dt \\
&= \frac{\varepsilon}{2}\int_0^T \big(1-\mu_{(1)}(t)\big)^2 \sum_{i=1}^n \mu_i^2(t)\,dt \\
&\geq \frac{\varepsilon}{2n}\int_0^T \big(1-\mu_{(1)}(t)\big)^2\,dt \\
&\geq \frac{\varepsilon\,\delta^2\,T}{2n}, \qquad (7.7.101)
\end{aligned}$$

since $\sum_{i=1}^n \mu_i^2(t) \geq \frac{1}{n}$, which follows very easily from the Cauchy-Schwarz inequality. It thus follows, from (7.7.95) and (7.7.101), that a.s., for $T > 0$, (7.7.94) amounts to the following

$$\log\left(\frac{Z_{\varphi^{\mathrm{g}}}(T)}{Z_{\varphi^{\mathrm{g}}}(0)}\right) > \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) - \log 2 + \frac{\varepsilon\,\delta^2\,T}{2n}, \qquad (7.7.102)$$

and, we have

$$\begin{aligned}
\frac{Z_{\varphi^{\mathrm{g}}}(T)}{Z_{\varphi^{\mathrm{g}}}(0)} &> \frac{Z_{\mu}(T)}{Z_{\mu}(0)}\exp\left(-\log 2 + \frac{\varepsilon\,\delta^2\,T}{2n}\right) \\
&= \frac{Z_{\mu}(T)}{Z_{\mu}(0)}\left(\frac{1}{2}\,e^{(\varepsilon\,\delta^2\,T)/2n}\right). \qquad (7.7.103)
\end{aligned}$$

Therefore, by (7.2.5) and (7.2.6) of Definition 7.2.2, the portfolio φ^{g} strictly dominates the market portfolio μ if $\frac{1}{2}\,e^{(\varepsilon\,\delta^2\,T)/2n} > 1$, thus if $-\log 2 + \frac{\varepsilon\,\delta^2\,T}{2n} > 0$, i.e., if

$$T > \frac{2n\,\log 2}{\varepsilon\,\delta^2}. \qquad (7.7.104)$$

Since, if

$$-\log 2 + \frac{\varepsilon\,\delta^2\,T}{2n} > 0, \qquad (7.7.105)$$

then

$$\log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) - \log 2 + \frac{\varepsilon\,\delta^2\,T}{2n} > \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right),$$

which results in the following expression

$$\log\left(\frac{Z_{\varphi^{\mathrm{g}}}(T)}{Z_{\varphi^{\mathrm{g}}}(0)}\right) > \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) - \log 2 + \frac{\varepsilon\,\delta^2\,T}{2n} > \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right).$$

Alternatively, if

$$\frac{1}{2}\,e^{(\varepsilon\,\delta^2\,T)/2n} > 1, \qquad (7.7.106)$$

then

$$\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\left(\frac{1}{2}\,e^{(\varepsilon\,\delta^2\,T)/2n}\right) > \frac{Z_{\mu}(T)}{Z_{\mu}(0)},$$

which results in the following expression

$$\frac{Z_{\varphi^{\mathrm{g}}}(T)}{Z_{\varphi^{\mathrm{g}}}(0)} > \frac{Z_{\mu}(T)}{Z_{\mu}(0)}\left(\frac{1}{2}\,e^{(\varepsilon\,\delta^2\,T)/2n}\right) > \frac{Z_{\mu}(T)}{Z_{\mu}(0)}.$$

Thus we have

$$\log\left(\frac{Z_{\varphi^{\mathrm{g}}}(T)}{Z_{\varphi^{\mathrm{g}}}(0)}\right) \quad > \quad \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right), \qquad \text{a.s.} \tag{7.7.107}$$

Hence,

$$\frac{Z_{\varphi^{\mathrm{g}}}(T)}{Z_{\varphi^{\mathrm{g}}}(0)} \quad > \quad \frac{Z_{\mu}(T)}{Z_{\mu}(0)}, \qquad \text{a.s.,} \tag{7.7.108}$$

and, we have

$$\mathbb{P}\left(\frac{Z_{\varphi^{\mathrm{g}}}(T)}{Z_{\varphi^{\mathrm{g}}}(0)} \quad > \quad \frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) \quad = \quad 1. \tag{7.7.109}$$

Alternatively, we can appeal to (i) of Lemma 7.2.3, since $\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) > \frac{1}{2}$, thus for all $t > 0$, $\mathbf{S}^{\mathrm{G}}\big(\boldsymbol{\mu}(t)\big) > \frac{1}{2} > 0$, a.s. Moreover, the inequality (2.4.54) of Lemma 2.4.11 implies that, for a nondegenerate equity market, there is an $\varepsilon > 0$ such that $\tau_{ii}(t) \geq \varepsilon\big(1 - \mu_{(1)}(t)\big)^2$, for all $t \in [0, T]$, a.s. Thus, $\tau_{ii}(t)$ has a positive lower bound in a nondegenerate equity market. Hence, we have $\mathfrak{g}_{\varphi^{\mathrm{g}}}(t) > 0$. Thus, the drift process of the quadratic Gini-coefficient-weighted portfolio $\mathfrak{g}_{\varphi^{\mathrm{g}}}(t)$ has a positive lower bound. Consequently, Lemma 7.2.3 implies that there is a dominance relationship between the quadratic Gini-coefficient-weighted portfolio φ^{g} and the market portfolio μ, i.e., φ^{g} strictly dominates the market portfolio μ. ■

Thus, if the market $\mathcal{M}$ is (weakly) diverse over the finite time horizon $[0, T]$ and with initial capital equal to that of the quadratic Gini-coefficient-weighted portfolio $Z_{\varphi^{\mathrm{g}}}(0) = Z_{\mu}(0) = w > 0$ such that $\log\big(Z_{\varphi^{\mathrm{g}}}(0)/Z_{\mu}(0)\big) = \log(1) = 0$, the quadratic Gini-coefficient-weighted portfolio satisfies

$$\begin{aligned} \log\left(\frac{Z_{\varphi^{\mathrm{g}}}(T)}{Z_{\mu}(T)}\right) \quad &> \quad \log\left(\frac{Z_{\varphi^{\mathrm{g}}}(0)}{Z_{\mu}(0)}\right) - \log 2 + \frac{\varepsilon\,\delta^2\,T}{2n} \\ &= \quad -\log 2 + \frac{\varepsilon\,\delta^2\,T}{2n}, \end{aligned} \tag{7.7.110}$$

and, then the value process $Z_{\varphi^{\mathrm{g}}}$ of the quadratic Gini-coefficient-weighted portfolio a.s. satisfies

$$\begin{aligned} \frac{Z_{\varphi^{\mathrm{g}}}(T)}{Z_{\mu}(T)} \quad &> \quad \frac{Z_{\varphi^{\mathrm{g}}}(0)}{Z_{\mu}(0)}\left(\frac{1}{2}\,e^{(\varepsilon\,\delta^2\,T)/2n}\right) \\ &= \quad \frac{1}{2}\,e^{(\varepsilon\,\delta^2\,T)/2n}, \end{aligned} \tag{7.7.111}$$

so that

$$Z_{\varphi^{\mathrm{g}}}(T) \quad > \quad Z_{\mu}(T)\left(\frac{1}{2}\,e^{(\varepsilon\,\delta^2\,T)/2n}\right), \qquad \text{a.s.} \tag{7.7.112}$$

Consequently, in a similar fashion to that shown in the foregoing proof, if

$$\frac{1}{2}\,e^{(\varepsilon\,\delta^2\,T)/2n} \quad > \quad 1,$$

then

$$Z_{\mu}(T)\left(\frac{1}{2}\,e^{(\varepsilon\,\delta^2\,T)/2n}\right) \quad > \quad Z_{\mu}(T),$$

and the following results

$$Z_{\varphi^{\mathrm{g}}}(T) \quad > \quad Z_{\mu}(T)\left(\frac{1}{2}\,e^{(\varepsilon\,\delta^2\,T)/2n}\right) \quad > \quad Z_{\mu}(T).$$

Hence,

$$Z_{\varphi^{\mathrm{g}}}(T) \quad > \quad Z_{\mu}(T), \qquad \text{a.s.} \tag{7.7.113}$$

Therefore, we have in particular

$$\mathbb{P}\Big(Z_{\varphi^{\mathrm{g}}}(T) \;>\; Z_{\mu}(T)\Big) \;=\; 1, \tag{7.7.114}$$

provided that

$$T \;>\; T_* \;\triangleq\; \frac{2n\,\log 2}{\varepsilon\,\delta^2}. \tag{7.7.115}$$

Consequently, from (7.3.4) of Definition 7.3.1, the quadratic Gini-coefficient-weighted portfolio φ^{g} is a strong arbitrage opportunity relative to the market portfolio μ, and φ^{g} outperforms the market portfolio μ. Notice that the quadratic Gini-coefficient-weighted portfolio will only outperform the market portfolio over the long term since the lower bound on T

$$T \;>\; \frac{2n\,\log 2}{\varepsilon\,\delta^2},$$

is sufficiently large as it contains the sufficiently small numbers $0 < \delta^2 < 1$ and $\varepsilon > 0$, which are present in the denominator of the lower bound. Thus, this corollary signifies that, with probability one, the quadratic Gini-coefficient-weighted portfolio has a return above that of the market portfolio over a sufficiently long time horizon $[0, T]$. If the market $\mathcal{M}$ is uniformly weakly diverse over $[T_*, \infty)$, then (7.7.110) above gives the a.s. long-term comparison

$$\begin{aligned}
\mathcal{L}_{\varphi^{\mathrm{g}},\mu} &= \liminf_{T\to\infty} \frac{1}{T}\,\log\Big(Z_{\varphi^{\mathrm{g}}}(T)/Z_{\mu}(T)\Big) \\
&> \liminf_{T\to\infty} \frac{1}{T}\Big(-\log 2 + \frac{\varepsilon\,\delta^2\,T}{2n}\Big) \\
&= \liminf_{T\to\infty} \Big(-\frac{\log 2}{T} + \frac{\varepsilon\,\delta^2}{2n}\Big) \\
&= \liminf_{T\to\infty} -\frac{\log 2}{T} + \liminf_{T\to\infty} \frac{\varepsilon\,\delta^2}{2n} \\
&= \liminf_{T\to\infty} -\frac{\log 2}{T} + \frac{\varepsilon\,\delta^2}{2n} \\
&= \frac{\varepsilon\,\delta^2}{2n} \;>\; 0,
\end{aligned} \tag{7.7.116}$$

since $\varepsilon > 0$, $0 < \delta^2 < 1$, and $n \geq 2$. Thus, we have

$$\mathcal{L}_{\varphi^{\mathrm{g}},\mu} \;=\; \liminf_{T\to\infty} \frac{1}{T}\,\log\Big(Z_{\varphi^{\mathrm{g}}}(T)/Z_{\mu}(T)\Big) \;>\; \frac{\varepsilon\,\delta^2}{2n} \;>\; 0, \qquad \text{a.s.}, \tag{7.7.117}$$

and the quadratic Gini-coefficient-weighted portfolio φ^{g} is a superior long-term growth opportunity relative to the market, i.e., the market portfolio will lag rather significantly behind the quadratic Gini-coefficient-weighted portfolio over long time horizons.

7.7.1.7 The Gini-Simpson-Weighted Index Portfolio

Consider the Gini-Simpson index function $\mathbf{S}^{\mathrm{GS}} : \mathbb{R}^n \to \mathbb{R}^+$ or $\mathbf{S}^{\mathrm{GS}} : \Delta^{n-1} \to \mathbb{R}^+$ defined by the Gini-Simpson index in (4.6.74). By Theorem 5.2.2, $\mathbf{S}^{\mathrm{GS}}$ generates the Gini-Simpson-weighted index portfolio with drift process (5.6.177). From (4.6.76), for $t \in [0, T]$, we have the bounds $0 < \mathbf{S}^{\mathrm{GS}}\big(\boldsymbol{\mu}(t)\big) < 1$. The value of the Gini-Simpson index will be bounded away from zero only if each $\mu_i(t)$ is bounded in the same manner. This condition is quite restrictive and does not necessarily hold even in a diverse equity market. Thus, in this general case, Lemma 7.2.3 cannot be applied and we cannot establish whether there is a dominance relationship between the Gini-Simpson-weighted index portfolio φ^{gs} and the market portfolio μ.

7.7.1.8 An Admissible Market-Dominating Portfolio

It can be shown that in a nondegenerate and weakly diverse equity market, without dividends, the admissible market-dominating portfolio φ^{a} strictly dominates the market portfolio μ, for a sufficiently large real number T. This is demonstrated in the next corollary.

Corollary 7.7.6 ([**Fernholz (1998b), Fernholz (2002)**]). *Let μ be the market portfolio and φ^{a} be the admissible market-dominating portfolio, and suppose that the market $\mathcal{M}$ is* ***nondegenerate*** *and* ***weakly diverse*** *over a fixed, finite time horizon $[0,T]$, with $T>0$ a given real number. Then for a* ***sufficiently large real number*** *$T \in [0,\infty)$, we have*

$$\frac{Z_{\varphi^{\mathsf{a}}}(T)}{Z_{\varphi^{\mathsf{a}}}(0)} \quad > \quad \frac{Z_{\mu}(T)}{Z_{\mu}(0)}, \qquad a.s., \tag{7.7.118}$$

and, consequently

$$\mathbb{P}\left(\frac{Z_{\varphi^{\mathsf{a}}}(T)}{Z_{\varphi^{\mathsf{a}}}(0)} \quad > \quad \frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) \quad = \quad 1. \tag{7.7.119}$$

That is, in a nondegenerate and (weakly) diverse equity market, the admissible market-dominating portfolio φ^{a} represents a ***strong arbitrage opportunity*** *relative to the market portfolio μ over the fixed, finite time horizon $[0,T]$, for T sufficiently large.*

Proof. Consider the admissible, market-dominating diversity measure function $\mathbf{S}^{\mathrm{A}} : \mathbb{R}^n \to \mathbb{R}^+$ or $\mathbf{S}^{\mathrm{A}} : \Delta^{n-1} \to \mathbb{R}^+$ defined by (4.6.80). By Theorem 5.2.2, $\mathbf{S}^{\mathrm{A}}$ generates the admissible market-dominating portfolio with weights (5.6.185), and drift process (5.6.188). By equations (5.6.190) and (5.6.193), for the admissible market-dominating portfolio, we have the following for $T \in [0,\infty)$,

$$\log\left(\frac{Z_{\varphi^{\mathsf{a}}}(T)}{Z_{\varphi^{\mathsf{a}}}(0)}\right) \quad = \quad \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(0)\big)}\right) + \frac{1}{2}\int_0^T \frac{1}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)} \sum_{i=1}^n \mu_i^2(t)\tau_{ii}(t)\,dt. \tag{7.7.120}$$

Now, from (4.6.82), for $t \in [0,T]$, we have the bounds $\frac{1}{2} < \mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big) < 1$, which implies that $\frac{1}{\mathbf{S}^{\mathrm{A}}(\boldsymbol{\mu}(t))} > 1$, in particular, $\frac{1}{\mathbf{S}^{\mathrm{A}}(\boldsymbol{\mu}(0))} > 1$. Hence, we have, a.s.,

$$\log\left(\frac{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(0)\big)}\right) \quad > \quad \log\left(\frac{1}{2}\right) \quad = \quad -\log 2. \tag{7.7.121}$$

Since for all $i = 1,2,\ldots,n$, $\tau_{ii}(t) \geq 0$, the drift process is nondecreasing (i.e., the integral in (7.7.120) is nonnegative), so that for $T \in [0,\infty)$, (7.7.120) can be expressed as

$$\log\left(\frac{Z_{\varphi^{\mathsf{a}}}(T)}{Z_{\varphi^{\mathsf{a}}}(0)}\right) \quad > \quad \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) + \log\left(\frac{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(T)\big)}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(0)\big)}\right). \tag{7.7.122}$$

This coupled with (7.7.121) provides, in particular, the following lower bound

$$\log\left(\frac{Z_{\varphi^{\mathsf{a}}}(T)}{Z_{\varphi^{\mathsf{a}}}(0)}\right) \quad > \quad \log\left(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) - \log 2. \tag{7.7.123}$$

Thus, for $T \in [0,\infty)$, we a.s. have

$$\begin{aligned} \frac{Z_{\varphi^{\mathsf{a}}}(T)}{Z_{\varphi^{\mathsf{a}}}(0)} \quad &> \quad \frac{Z_{\mu}(T)}{Z_{\mu}(0)} \exp\big(-\log 2\big) \\ &= \quad \frac{1}{2}\,\frac{Z_{\mu}(T)}{Z_{\mu}(0)}. \end{aligned} \tag{7.7.124}$$

Therefore, condition (ii) of Definition 7.2.1 is satisfied with $c := \frac{1}{2} > 0$. Furthermore, since $0 < \mu_i(t) < 1$, for $i = 1,2,\ldots,n$ and $1 < \frac{1}{\mathbf{S}^{\mathrm{A}}(\boldsymbol{\mu}(t))} < 2$, from (5.6.185) we have a.s. for $t \in [0,T]$, and for all $i = 1,2,\ldots,n$,

$$0 \quad < \quad \varphi_i^{\mathsf{a}}(t) \quad < \quad 3\,\mu_i(t), \tag{7.7.125}$$

so that $\varphi_i^{\mathsf{a}}(t) \geq 0$ and the ratio satisfies $\varphi_i^{\mathsf{a}}(t)/\mu_i(t) < 3$, for all $t \in [0,T]$, and for $i = 1,2,\ldots,n$. So conditions (i) and (iii) of Definition 7.2.1 are satisfied. Hence, this together with (7.7.124) implies that the admissible

market-dominating portfolio φ^{a} is admissible. Also, (7.3.2) is satisfied for $\varphi := \varphi^{\mathsf{a}}$, $\eta := \mu$ and for $q := \frac{1}{2}$. Now, since the market is weakly diverse over the time horizon $[0, T]$, from the weak diversity condition (4.2.3) of Definition 4.2.2 and (7.7.45), for $0 < \delta < 1$, we have

$$\begin{aligned} -\frac{1}{T}\int_0^T \mu_{(1)}(t)\,dt &\geq \delta - 1, \quad \text{and,} \\ 1 - \frac{1}{T}\int_0^T \mu_{(1)}(t)\,dt &\geq \delta, \end{aligned}$$

so that, we have

$$\frac{1}{T}\int_0^T \big(1 - \mu_{(1)}(t)\big)\,dt \geq \delta. \tag{7.7.126}$$

So, by the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \Big(\int_0^T \frac{1}{T^2}\,dt\Big)\Big(\int_0^T \big(1 - \mu_{(1)}(t)\big)^2\,dt\Big) &\geq \Big(\int_0^T \frac{1}{T}\big(1 - \mu_{(1)}(t)\big)\,dt\Big)^2, \\ \frac{1}{T}\int_0^T \big(1 - \mu_{(1)}(t)\big)^2\,dt &\geq \Big(\frac{1}{T}\int_0^T \big(1 - \mu_{(1)}(t)\big)\,dt\Big)^2, \end{aligned}$$

so that by (7.7.126), the above inequality becomes

$$\frac{1}{T}\int_0^T \big(1 - \mu_{(1)}(t)\big)^2\,dt \geq \delta^2, \quad \text{and,} \tag{7.7.127}$$

$$\int_0^T \big(1 - \mu_{(1)}(t)\big)^2\,dt \geq \delta^2\,T. \tag{7.7.128}$$

Hence, combining (2.4.54) of Lemma 2.4.11 for a nondegenerate market, with the results (7.7.128) and $\frac{1}{\mathbf{S}^{\mathrm{A}}(\boldsymbol{\mu}(t))} > 1$, we see that the assumption of weak diversity implies

$$\begin{aligned} \frac{1}{2}\int_0^T \frac{1}{\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big)}\sum_{i=1}^n \mu_i^2(t)\tau_{ii}(t)\,dt &\geq \frac{\varepsilon}{2}\int_0^T \sum_{i=1}^n \mu_i^2(t)\big(1 - \mu_{(1)}(t)\big)^2\,dt \\ &= \frac{\varepsilon}{2}\int_0^T \big(1 - \mu_{(1)}(t)\big)^2 \sum_{i=1}^n \mu_i^2(t)\,dt \\ &\geq \frac{\varepsilon}{2n}\int_0^T \big(1 - \mu_{(1)}(t)\big)^2\,dt \\ &\geq \frac{\varepsilon\,\delta^2\,T}{2n}, \end{aligned} \tag{7.7.129}$$

since $\sum_{i=1}^n \mu_i^2(t) \geq \frac{1}{n}$, which follows very easily from the Cauchy-Schwarz inequality. It thus follows, from (7.7.121) and (7.7.129), that a.s., for $T > 0$, (7.7.120) amounts to the following

$$\log\Big(\frac{Z_{\varphi^{\mathsf{a}}}(T)}{Z_{\varphi^{\mathsf{a}}}(0)}\Big) > \log\Big(\frac{Z_{\mu}(T)}{Z_{\mu}(0)}\Big) - \log 2 + \frac{\varepsilon\,\delta^2\,T}{2n}, \tag{7.7.130}$$

and, we have

$$\begin{aligned} \frac{Z_{\varphi^{\mathsf{a}}}(T)}{Z_{\varphi^{\mathsf{a}}}(0)} &> \frac{Z_{\mu}(T)}{Z_{\mu}(0)}\exp\Big(-\log 2 + \frac{\varepsilon\,\delta^2\,T}{2n}\Big) \\ &= \frac{Z_{\mu}(T)}{Z_{\mu}(0)}\Big(\frac{1}{2}\,e^{(\varepsilon\,\delta^2\,T)/2n}\Big). \end{aligned} \tag{7.7.131}$$

Therefore, by (7.2.5) and (7.2.6) of Definition 7.2.2, the portfolio φ^{a} strictly dominates the market portfolio μ if $\frac{1}{2}\,e^{(\varepsilon\,\delta^2\,T)/2n} > 1$, thus if $-\log 2 + \frac{\varepsilon\,\delta^2\,T}{2n} > 0$, i.e., if

$$T > \frac{2n\,\log 2}{\varepsilon\,\delta^2}. \tag{7.7.132}$$

Since, if

$$-\log 2 + \frac{\varepsilon\,\delta^2\,T}{2n} \quad > \quad 0, \tag{7.7.133}$$

then

$$\log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) - \log 2 + \frac{\varepsilon\,\delta^2\,T}{2n} \quad > \quad \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right),$$

which results in the following expression

$$\log\left(\frac{Z_{\varphi^{\mathsf{a}}}(T)}{Z_{\varphi^{\mathsf{a}}}(0)}\right) \quad > \quad \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right) - \log 2 + \frac{\varepsilon\,\delta^2\,T}{2n} \quad > \quad \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right).$$

Alternatively, if

$$\frac{1}{2}\,e^{(\varepsilon\,\delta^2\,T)/2n} \quad > \quad 1, \tag{7.7.134}$$

then

$$\frac{Z_\mu(T)}{Z_\mu(0)}\left(\frac{1}{2}\,e^{(\varepsilon\,\delta^2\,T)/2n}\right) \quad > \quad \frac{Z_\mu(T)}{Z_\mu(0)},$$

which results in the following expression

$$\frac{Z_{\varphi^{\mathsf{a}}}(T)}{Z_{\varphi^{\mathsf{a}}}(0)} \quad > \quad \frac{Z_\mu(T)}{Z_\mu(0)}\left(\frac{1}{2}\,e^{(\varepsilon\,\delta^2\,T)/2n}\right) \quad > \quad \frac{Z_\mu(T)}{Z_\mu(0)}.$$

Thus we have

$$\log\left(\frac{Z_{\varphi^{\mathsf{a}}}(T)}{Z_{\varphi^{\mathsf{a}}}(0)}\right) \quad > \quad \log\left(\frac{Z_\mu(T)}{Z_\mu(0)}\right), \qquad \text{a.s.} \tag{7.7.135}$$

Hence,

$$\frac{Z_{\varphi^{\mathsf{a}}}(T)}{Z_{\varphi^{\mathsf{a}}}(0)} \quad > \quad \frac{Z_\mu(T)}{Z_\mu(0)}, \qquad \text{a.s.}, \tag{7.7.136}$$

and, we have

$$\mathbb{P}\left(\frac{Z_{\varphi^{\mathsf{a}}}(T)}{Z_{\varphi^{\mathsf{a}}}(0)} \quad > \quad \frac{Z_\mu(T)}{Z_\mu(0)}\right) \quad = \quad 1. \tag{7.7.137}$$

Alternatively, we can appeal to (i) of Lemma 7.2.3, since $\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big) > \frac{1}{2}$, thus for all $t > 0$, $\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big) > \frac{1}{2} > 0$, a.s. Moreover, the inequality (2.4.54) of Lemma 2.4.11 implies that, for a nondegenerate equity market, there is an $\varepsilon > 0$ such that $\tau_{ii}(t) \geq \varepsilon\big(1 - \mu_{(1)}(t)\big)^2$, for all $t \in [0, T]$, a.s. Thus, $\tau_{ii}(t)$ has a positive lower bound in a nondegenerate equity market. Hence, we have $\mathfrak{g}_{\varphi^{\mathsf{a}}}(t) > 0$. Thus, the drift process of the admissible market-dominating portfolio $\mathfrak{g}_{\varphi^{\mathsf{a}}}(t)$ has a positive lower bound. Consequently, Lemma 7.2.3 implies that there is a dominance relationship between the admissible market-dominating portfolio φ^{a} and the market portfolio μ, i.e., φ^{a} strictly dominates the market portfolio μ. ■

Thus, if the market $\mathcal{M}$ is (weakly) diverse over the finite time horizon $[0, T]$ and with initial capital equal to that of the admissible market-dominating portfolio $Z_{\varphi^{\mathsf{a}}}(0) = Z_\mu(0) = w > 0$ such that $\log\big(Z_{\varphi^{\mathsf{a}}}(0)/Z_\mu(0)\big) = \log(1) = 0$, the admissible market-dominating portfolio satisfies

$$\begin{aligned}\log\left(\frac{Z_{\varphi^{\mathsf{a}}}(T)}{Z_\mu(T)}\right) \quad &> \quad \log\left(\frac{Z_{\varphi^{\mathsf{a}}}(0)}{Z_\mu(0)}\right) - \log 2 + \frac{\varepsilon\,\delta^2\,T}{2n} \\ &= \quad -\log 2 + \frac{\varepsilon\,\delta^2\,T}{2n},\end{aligned} \tag{7.7.138}$$

and, then the value process $Z_{\varphi^{\mathsf{a}}}$ of the admissible market-dominating portfolio a.s. satisfies

$$\begin{aligned} \frac{Z_{\varphi^{\mathsf{a}}}(T)}{Z_\mu(T)} &> \frac{Z_{\varphi^{\mathsf{a}}}(0)}{Z_\mu(0)} \left(\frac{1}{2}\, e^{(\varepsilon\, \delta^2\, T)/2n} \right) \\ &= \frac{1}{2}\, e^{(\varepsilon\, \delta^2\, T)/2n}, \end{aligned} \tag{7.7.139}$$

so that

$$Z_{\varphi^{\mathsf{a}}}(T) \;>\; Z_\mu(T) \left(\frac{1}{2}\, e^{(\varepsilon\, \delta^2\, T)/2n} \right), \qquad \text{a.s.} \tag{7.7.140}$$

Consequently, in a similar fashion to that shown in the foregoing proof, if

$$\frac{1}{2}\, e^{(\varepsilon\, \delta^2\, T)/2n} \;>\; 1,$$

then

$$Z_\mu(T) \left(\frac{1}{2}\, e^{(\varepsilon\, \delta^2\, T)/2n} \right) \;>\; Z_\mu(T),$$

and the following results

$$Z_{\varphi^{\mathsf{a}}}(T) \;>\; Z_\mu(T) \left(\frac{1}{2}\, e^{(\varepsilon\, \delta^2\, T)/2n} \right) \;>\; Z_\mu(T).$$

Hence,

$$Z_{\varphi^{\mathsf{a}}}(T) \;>\; Z_\mu(T), \qquad \text{a.s.} \tag{7.7.141}$$

Therefore, we have in particular

$$\mathbb{P}\Big(Z_{\varphi^{\mathsf{a}}}(T) \;>\; Z_\mu(T) \Big) \;=\; 1, \tag{7.7.142}$$

provided that

$$T \;>\; T_* \;\triangleq\; \frac{2n\, \log 2}{\varepsilon\, \delta^2}. \tag{7.7.143}$$

Consequently, from (7.3.4) of Definition 7.3.1, the admissible market-dominating portfolio φ^{a} is a strong arbitrage opportunity relative to the market portfolio μ, and φ^{a} outperforms the market portfolio μ. Notice that the admissible, market-dominating portfolio will only outperform the market portfolio over the long term since the lower bound on T

$$T \;>\; \frac{2n\, \log 2}{\varepsilon\, \delta^2},$$

is sufficiently large as it contains the sufficiently small numbers $0 < \delta^2 < 1$ and $\varepsilon > 0$, which are present in the denominator of the lower bound. Thus, this corollary signifies that, with probability one, the admissible market-dominating portfolio has a return above that of the market portfolio over a sufficiently long time horizon $[0, T]$. If the market $\mathcal{M}$ is uniformly weakly diverse over $[T_*, \infty)$, then (7.7.138) above gives the a.s. long-term comparison

$$\begin{aligned} \mathcal{L}_{\varphi^{\mathsf{a}},\mu} &= \liminf_{T\to\infty} \frac{1}{T}\, \log\Big(Z_{\varphi^{\mathsf{a}}}(T)/Z_\mu(T) \Big) \\ &> \liminf_{T\to\infty} \frac{1}{T} \Big(-\log 2 + \frac{\varepsilon\, \delta^2\, T}{2n} \Big) \\ &= \liminf_{T\to\infty} \Big(-\frac{\log 2}{T} + \frac{\varepsilon\, \delta^2}{2n} \Big) \\ &= \liminf_{T\to\infty} -\frac{\log 2}{T} + \liminf_{T\to\infty} \frac{\varepsilon\, \delta^2}{2n} \\ &= \liminf_{T\to\infty} -\frac{\log 2}{T} + \frac{\varepsilon\, \delta^2}{2n} \\ &= \frac{\varepsilon\, \delta^2}{2n} \;>\; 0, \end{aligned} \tag{7.7.144}$$

since $\varepsilon > 0$, $0 < \delta^2 < 1$, and $n \geq 2$. Thus, we have

$$\mathcal{L}_{\varphi^{\mathsf{a}},\mu} \;=\; \liminf_{T\to\infty} \frac{1}{T} \log\Big(Z_{\varphi^{\mathsf{a}}}(T)/Z_{\mu}(T)\Big) \;>\; \frac{\varepsilon\,\delta^2}{2n} \;>\; 0, \qquad \text{a.s.,} \tag{7.7.145}$$

and the admissible market-dominating portfolio φ^{a} is a superior long-term growth opportunity relative to the market, i.e., the market portfolio will lag rather significantly behind the admissible market-dominating portfolio over long time horizons.

7.7.2 Short-Term Relative Arbitrage Opportunities in Diverse Equity Markets

In the previous section, we saw that in weakly diverse equity markets that satisfy the nondegeneracy condition, we can construct explicitly simple long-only portfolios (e.g., the diversity-weighted portfolio) that lead to strong arbitrage opportunities relative to the market over sufficiently long time horizons. Thus, over sufficiently long time horizons, there exist portfolios that represent strong arbitrage opportunities relative to the market portfolio. In this section, we shall demonstrate that under these same market conditions of weak diversity and nondegeneracy, such relative arbitrage opportunities exist indeed over arbitrary time horizons, no matter how small. In a weakly diverse equity market, there always exist portfolios that consistently outperform or underperform the market portfolio over arbitrarily small time horizons.

7.7.2.1 Mirror Portfolios

It is possible to attain arbitrage opportunities relative to the market in a nondegenerate, weakly diverse equity market over $[0,T]$ for any $T > 0$, using *mirror portfolios*.

Definition 7.7.7 ($q-$Mirror Image)**.** *For any given extended (long-short) portfolio π of Definition 2.2.16, and any fixed real number $q \neq 0$, define the* ***$q-$mirror image of π with respect to the market portfolio μ****, $\widetilde{\pi}^{[q]} = \big\{\widetilde{\pi}^{[q]}(t) = \big(\widetilde{\pi}_1^{[q]}(t), \widetilde{\pi}_2^{[q]}(t), \ldots, \widetilde{\pi}_n^{[q]}(t)\big), t \in [0,T]\big\}$, as*

$$\widetilde{\boldsymbol{\pi}}^{[q]}(t) \;\triangleq\; q\,\boldsymbol{\pi}(t) + (1-q)\,\boldsymbol{\mu}(t), \qquad t \in [0,T]. \tag{7.7.146}$$

Therefore, for $i = 1, 2, \ldots, n$, we have

$$\widetilde{\pi}_i^{[q]}(t) \;=\; q\,\pi_i(t) + (1-q)\,\mu_i(t), \qquad t \in [0,T]. \tag{7.7.147}$$

This is clearly an extended (long-short) portfolio; and it is a long-only portfolio (i.e., in the strict "all-long" sense) if π itself is long-only and $0 < q < 1$.

Definition 7.7.8 (Mirror Image)**.** *If $q = -1$, we call*

$$\widetilde{\boldsymbol{\pi}}^{[-1]}(t) \;\triangleq\; 2\,\boldsymbol{\mu}(t) - \boldsymbol{\pi}(t), \qquad t \in [0,T], \tag{7.7.148}$$

the ***mirror image of π with respect to the market portfolio μ****.*

Notice the following

$$\begin{aligned}
\big(\widetilde{\widetilde{\boldsymbol{\pi}}^{[p]}}\big)^{[q]}(t) &= q\,\widetilde{\boldsymbol{\pi}}^{[p]}(t) + (1-q)\,\boldsymbol{\mu}(t) \\
&= q\,\big(p\,\boldsymbol{\pi}(t) + (1-p)\,\boldsymbol{\mu}(t)\big) + (1-q)\,\boldsymbol{\mu}(t) \\
&= pq\,\boldsymbol{\pi}(t) + q\,\boldsymbol{\mu}(t) - pq\,\boldsymbol{\mu}(t) + \boldsymbol{\mu}(t) - q\,\boldsymbol{\mu}(t) \\
&= pq\,\boldsymbol{\pi}(t) + (1-pq)\,\boldsymbol{\mu}(t) \\
&= \widetilde{\boldsymbol{\pi}}^{[pq]}(t),
\end{aligned} \tag{7.7.149}$$

and

$$\begin{aligned}
\big(\widetilde{\widetilde{\boldsymbol{\pi}}^{[q]}}\big)^{[1/q]}(t) &= (1/q)\,\widetilde{\boldsymbol{\pi}}^{[q]}(t) + (1-1/q)\,\boldsymbol{\mu}(t) \\
&= (1/q)\,\big(q\,\boldsymbol{\pi}(t) + (1-q)\,\boldsymbol{\mu}(t)\big) + (1-1/q)\,\boldsymbol{\mu}(t) \\
&= \boldsymbol{\pi}(t) + (1/q)\,\boldsymbol{\mu}(t) - \boldsymbol{\mu}(t) + \boldsymbol{\mu}(t) - (1/q)\,\boldsymbol{\mu}(t) \\
&= \boldsymbol{\pi}(t).
\end{aligned} \tag{7.7.150}$$

By analogy with equations (2.4.17) and (2.4.18) of Definition 2.4.3, and also referring to equations (2.4.15) and (2.4.20), let us define the relative variance of the portfolio π versus the market portfolio μ, as

$$\tau^{\mu}_{\pi\pi}(t) \triangleq \boldsymbol{\pi}(t)\boldsymbol{\tau}^{\boldsymbol{\mu}}(t)\boldsymbol{\pi}^{\mathsf{T}}(t) = \Big(\boldsymbol{\pi}(t)-\boldsymbol{\mu}(t)\Big)\boldsymbol{\sigma}(t)\Big(\boldsymbol{\pi}(t)-\boldsymbol{\mu}(t)\Big)^{\mathsf{T}}, \tag{7.7.151}$$

and the relative variance of the market portfolio μ versus the portfolio π, as

$$\tau^{\pi}_{\mu\mu}(t) \triangleq \boldsymbol{\mu}(t)\boldsymbol{\tau}^{\boldsymbol{\pi}}(t)\boldsymbol{\mu}^{\mathsf{T}}(t) = \Big(\boldsymbol{\mu}(t)-\boldsymbol{\pi}(t)\Big)\boldsymbol{\sigma}(t)\Big(\boldsymbol{\mu}(t)-\boldsymbol{\pi}(t)\Big)^{\mathsf{T}}. \tag{7.7.152}$$

By (2.4.19) of Lemma 2.4.4, we have

$$\tau^{\mu}_{\pi\pi}(t) = \tau^{\pi}_{\mu\mu}(t). \tag{7.7.153}$$

From (7.7.146) of Definition 7.7.7, we have $\widetilde{\boldsymbol{\pi}}^{[q]}(t)-\boldsymbol{\mu}(t) = q\left(\boldsymbol{\pi}(t)-\boldsymbol{\mu}(t)\right)$, which in conjunction with (7.7.151) yields

$$\begin{aligned}
\tau^{\mu}_{\widetilde{\pi}^{[q]}\widetilde{\pi}^{[q]}}(t) &= \Big(\widetilde{\boldsymbol{\pi}}^{[q]}(t)-\boldsymbol{\mu}(t)\Big)\boldsymbol{\sigma}(t)\Big(\widetilde{\boldsymbol{\pi}}^{[q]}(t)-\boldsymbol{\mu}(t)\Big)^{\mathsf{T}} \\
&= \Big(q\left(\boldsymbol{\pi}(t)-\boldsymbol{\mu}(t)\right)\Big)\boldsymbol{\sigma}(t)\Big(q\left(\boldsymbol{\pi}(t)-\boldsymbol{\mu}(t)\right)\Big)^{\mathsf{T}} \\
&= q^2\Big(\boldsymbol{\pi}(t)-\boldsymbol{\mu}(t)\Big)\boldsymbol{\sigma}(t)\Big(\boldsymbol{\pi}(t)-\boldsymbol{\mu}(t)\Big)^{\mathsf{T}} \\
&= q^2\,\tau^{\mu}_{\pi\pi}(t).
\end{aligned} \tag{7.7.154}$$

Alternatively, employing the elementary property (2.12.54) or (2.12.55) of Lemma 2.12.4, $\boldsymbol{\tau}^{\boldsymbol{\mu}}(t)\boldsymbol{\mu}^{\mathsf{T}}(t) = \mathbf{0}^{\mathsf{T}}$, we get

$$\begin{aligned}
\tau^{\mu}_{\widetilde{\pi}^{[q]}\widetilde{\pi}^{[q]}}(t) &= \widetilde{\boldsymbol{\pi}}^{[q]}(t)\boldsymbol{\tau}^{\boldsymbol{\mu}}(t)\big(\widetilde{\boldsymbol{\pi}}^{[q]}(t)\big)^{\mathsf{T}} \\
&= \Big(q\,\boldsymbol{\pi}(t)+(1-q)\,\boldsymbol{\mu}(t)\Big)\boldsymbol{\tau}^{\boldsymbol{\mu}}(t)\Big(q\,\boldsymbol{\pi}(t)+(1-q)\,\boldsymbol{\mu}(t)\Big)^{\mathsf{T}} \\
&= \Big(q\,\boldsymbol{\pi}(t)+(1-q)\,\boldsymbol{\mu}(t)\Big)\boldsymbol{\tau}^{\boldsymbol{\mu}}(t)\Big(q\,\boldsymbol{\pi}^{\mathsf{T}}(t)+(1-q)\,\boldsymbol{\mu}^{\mathsf{T}}(t)\Big) \\
&= q^2\,\boldsymbol{\pi}(t)\boldsymbol{\tau}^{\boldsymbol{\mu}}(t)\boldsymbol{\pi}^{\mathsf{T}}(t)+q\,(1-q)\,\boldsymbol{\pi}(t)\boldsymbol{\tau}^{\boldsymbol{\mu}}(t)\boldsymbol{\mu}^{\mathsf{T}}(t) \\
&\quad + q\,(1-q)\,\boldsymbol{\mu}(t)\boldsymbol{\tau}^{\boldsymbol{\mu}}(t)\boldsymbol{\pi}^{\mathsf{T}}(t)+(1-q)^2\,\boldsymbol{\mu}(t)\boldsymbol{\tau}^{\boldsymbol{\mu}}(t)\boldsymbol{\mu}^{\mathsf{T}}(t) \\
&= q^2\,\boldsymbol{\pi}(t)\boldsymbol{\tau}^{\boldsymbol{\mu}}(t)\boldsymbol{\pi}^{\mathsf{T}}(t) \\
&= q^2\,\tau^{\mu}_{\pi\pi}(t).
\end{aligned} \tag{7.7.155}$$

Lemma 7.7.9 ([**Fernholz, Karatzas & Kardaras (2005), Fernholz & Karatzas (2009)**]). *The wealth of $\widetilde{\pi}^{[q]}$ relative to the market, can be computed as*

$$d\log\Big(Z_{\widetilde{\pi}^{[q]}}(t)/Z_{\mu}(t)\Big) = q\,d\log\Big(Z_{\pi}(t)/Z_{\mu}(t)\Big)+\frac{q\,(1-q)}{2}\,\tau^{\mu}_{\pi\pi}(t)\,dt, \tag{7.7.156}$$

or, in integral form, for $T\in[0,\infty)$, as

$$\log\left(\frac{Z_{\widetilde{\pi}^{[q]}}(T)}{Z_{\mu}(T)}\right) = q\,\log\left(\frac{Z_{\pi}(T)}{Z_{\mu}(T)}\right)+\frac{q\,(1-q)}{2}\int_0^T \tau^{\mu}_{\pi\pi}(t)\,dt \tag{7.7.157}$$

$$= \log\left(\frac{Z_{\pi}(T)}{Z_{\mu}(T)}\right)^{q}+\frac{q\,(1-q)}{2}\int_0^T \tau^{\mu}_{\pi\pi}(t)\,dt. \tag{7.7.158}$$

Proof. From (2.12.64) of Lemma 2.12.9, we have the relative performance of the portfolio π versus the market portfolio μ,

$$d\log\Big(Z_{\pi}(t)/Z_{\mu}(t)\Big) = \big(\gamma^*_{\pi}(t)-\gamma^*_{\mu}(t)\big)\,dt+\sum_{i=1}^{n}\big(\pi_i(t)-\mu_i(t)\big)\,d\log\mu_i(t), \tag{7.7.159}$$

and the relative performance of the portfolio $\widetilde{\pi}^{[q]}$ versus the market portfolio μ,

$$d\log\Big(Z_{\widetilde{\pi}^{[q]}}(t)/Z_\mu(t)\Big) \;=\; \big(\gamma^*_{\widetilde{\pi}^{[q]}}(t)-\gamma^*_\mu(t)\big)\,dt+\sum_{i=1}^n\big(\widetilde{\pi}^{[q]}_i(t)-\mu_i(t)\big)\,d\log\mu_i(t). \tag{7.7.160}$$

Recall, from (7.7.146) of Definition 7.7.7, that $\widetilde{\boldsymbol{\pi}}^{[q]}(t)-\boldsymbol{\mu}(t)=q\,\big(\boldsymbol{\pi}(t)-\boldsymbol{\mu}(t)\big)$, so that for $i=1,2,\ldots,n$, and $t\in[0,T]$, we have

$$\widetilde{\pi}^{[q]}_i(t)-\mu_i(t) \;=\; q\,\big(\pi_i(t)-\mu_i(t)\big). \tag{7.7.161}$$

Employing the above equation, we can write (7.7.160) as

$$d\log\Big(Z_{\widetilde{\pi}^{[q]}}(t)/Z_\mu(t)\Big) \;=\; \big(\gamma^*_{\widetilde{\pi}^{[q]}}(t)-\gamma^*_\mu(t)\big)\,dt+q\sum_{i=1}^n\big(\pi_i(t)-\mu_i(t)\big)\,d\log\mu_i(t),$$

which when compared with (7.7.159), now multiplied by q,

$$q\sum_{i=1}^n\big(\pi_i(t)-\mu_i(t)\big)\,d\log\mu_i(t) \;=\; q\,d\log\Big(Z_\pi(t)/Z_\mu(t)\Big)-q\,\big(\gamma^*_\pi(t)-\gamma^*_\mu(t)\big)\,dt,$$

gives

$$\begin{aligned} d\log\Big(Z_{\widetilde{\pi}^{[q]}}(t)/Z_\mu(t)\Big) &= q\,d\log\Big(Z_\pi(t)/Z_\mu(t)\Big)+\Big(\gamma^*_{\widetilde{\pi}^{[q]}}(t)-\gamma^*_\mu(t)-q\,\gamma^*_\pi(t)+q\,\gamma^*_\mu(t)\Big)\,dt\\ &= q\,d\log\Big(Z_\pi(t)/Z_\mu(t)\Big)+\Big(\gamma^*_{\widetilde{\pi}^{[q]}}(t)-q\,\gamma^*_\pi(t)+(q-1)\,\gamma^*_\mu(t)\Big)\,dt. \end{aligned} \tag{7.7.162}$$

Recall, from (7.7.146) of Definition 7.7.7, that $\widetilde{\boldsymbol{\pi}}^{[q]}(t)-q\,\boldsymbol{\pi}(t)=(1-q)\,\boldsymbol{\mu}(t)$, so that for $i=1,2,\ldots,n$, and $t\in[0,T]$, we have

$$\widetilde{\pi}^{[q]}_i(t)-q\,\pi_i(t) \;=\; (1-q)\,\mu_i(t). \tag{7.7.163}$$

Thus, setting $\eta:=\mu$ in the numéraire invariance property of (2.4.29), and employing equations (7.7.163), (7.7.154) and (2.12.95) (or, setting $\pi:=\mu$ is (2.4.36)), we obtain

$$\begin{aligned} \gamma^*_{\widetilde{\pi}^{[q]}}(t)-q\,\gamma^*_\pi(t) &= \frac{1}{2}\left(\sum_{i=1}^n\widetilde{\pi}^{[q]}_i(t)\tau^\mu_{ii}(t)-\tau^\mu_{\widetilde{\pi}^{[q]}\widetilde{\pi}^{[q]}}(t)\right)-\frac{q}{2}\left(\sum_{i=1}^n\pi_i(t)\tau^\mu_{ii}(t)-\tau^\mu_{\pi\pi}(t)\right)\\ &= \frac{1}{2}\left(\sum_{i=1}^n\Big(\widetilde{\pi}^{[q]}_i(t)-q\,\pi_i(t)\Big)\tau^\mu_{ii}(t)-\tau^\mu_{\widetilde{\pi}^{[q]}\widetilde{\pi}^{[q]}}(t)+q\,\tau^\mu_{\pi\pi}(t)\right)\\ &= \frac{1}{2}\left((1-q)\sum_{i=1}^n\mu_i(t)\tau^\mu_{ii}(t)-q^2\,\tau^\mu_{\pi\pi}(t)+q\,\tau^\mu_{\pi\pi}(t)\right)\\ &= \frac{1}{2}\Big(2\,(1-q)\,\gamma^*_\mu(t)+q\,(1-q)\,\tau^\mu_{\pi\pi}(t)\Big)\\ &= (1-q)\,\gamma^*_\mu(t)+\frac{q\,(1-q)}{2}\,\tau^\mu_{\pi\pi}(t). \end{aligned} \tag{7.7.164}$$

Substituting this back into (7.7.162), yields the desired equality (7.7.156). Therefore, we have

$$\log\left(\frac{Z_{\widetilde{\pi}^{[q]}}(T)}{Z_\mu(T)}\right)-\log\left(\frac{Z_{\widetilde{\pi}^{[q]}}(0)}{Z_\mu(0)}\right) \;=\; q\,\log\left(\frac{Z_\pi(T)}{Z_\mu(T)}\right)-q\,\log\left(\frac{Z_\pi(0)}{Z_\mu(0)}\right)+\frac{q\,(1-q)}{2}\int_0^T\tau^\mu_{\pi\pi}(t)\,dt, \tag{7.7.165}$$

and

$$\log\left(\frac{Z_{\widetilde{\pi}^{[q]}}(T)/Z_\mu(T)}{Z_{\widetilde{\pi}^{[q]}}(0)/Z_\mu(0)}\right) \;=\; q\,\log\left(\frac{Z_\pi(T)/Z_\mu(T)}{Z_\pi(0)/Z_\mu(0)}\right)+\frac{q\,(1-q)}{2}\int_0^T\tau^\mu_{\pi\pi}(t)\,dt, \tag{7.7.166}$$

$$\log\left(\frac{Z_{\widetilde{\pi}^{[q]}}(T)/Z_{\widetilde{\pi}^{[q]}}(0)}{Z_\mu(T)/Z_\mu(0)}\right) \;=\; q\,\log\left(\frac{Z_\pi(T)/Z_\pi(0)}{Z_\mu(T)/Z_\mu(0)}\right)+\frac{q\,(1-q)}{2}\int_0^T\tau^\mu_{\pi\pi}(t)\,dt, \tag{7.7.167}$$

with $Z_{\widetilde{\pi}^{[q]}}(0)=Z_\pi(0)=Z_\mu(0)=w>0$, we get (7.7.157). ■

Lemma 7.7.10 ([**Fernholz, Karatzas & Kardaras (2005), Fernholz & Karatzas (2009)**]). *Suppose that the extended portfolio π satisfies*

$$\mathbb{P}\left(\frac{Z_\pi(T)}{Z_\mu(T)} \geq \beta\right) = 1, \qquad or \qquad \mathbb{P}\left(\frac{Z_\pi(T)}{Z_\mu(T)} \leq \frac{1}{\beta}\right) = 1, \tag{7.7.168}$$

and

$$\mathbb{P}\left(\int_0^T \tau^\mu_{\pi\pi}(t)\,dt \geq \kappa\right) = 1, \tag{7.7.169}$$

for some real numbers $T > 0$, $\kappa > 0$, and $0 < \beta < 1$. Then there exists another extended portfolio $\widehat{\pi}$ with

$$\mathbb{P}\left(Z_{\widehat{\pi}}(T) < Z_\mu(T)\right) = 1. \tag{7.7.170}$$

Proof. Suppose first that we have

$$\mathbb{P}\left(\frac{Z_\pi(T)}{Z_\mu(T)} \leq \frac{1}{\beta}\right) = 1, \tag{7.7.171}$$

then we can just take $\widehat{\pi} \equiv \widetilde{\pi}^{[q]}$ (i.e., $\widehat{\pi}(t) \equiv \widetilde{\pi}^{[q]}(t)$, for all $t \in [0,T]$), with $q > 1 + \frac{2}{\kappa}\log\left(\frac{1}{\beta}\right)$. Since $\kappa > 0$ and $0 < \beta < 1$, we have $\frac{2}{\kappa}\log\left(\frac{1}{\beta}\right) > 0$, so that we have $q > 1$ and $1 - q < 0$. Then, by (7.7.157) of Lemma 7.7.9, (7.7.171), (7.7.169) and the fact that $q > 1$, we arrive at the following a.s.,

$$\begin{aligned}
\log\left(\frac{Z_{\widetilde{\pi}^{[q]}}(T)}{Z_\mu(T)}\right) &= q\log\left(\frac{Z_\pi(T)}{Z_\mu(T)}\right) + \frac{q\,(1-q)}{2}\int_0^T \tau^\mu_{\pi\pi}(t)\,dt \\
&\leq q\log\left(\frac{1}{\beta}\right) + \frac{q\,(1-q)}{2}\,\kappa \\
&= q\left(\log\left(\frac{1}{\beta}\right) + \frac{\kappa}{2}\,(1-q)\right) \\
&= \frac{q\,\kappa}{2}\left(\frac{2}{\kappa}\log\left(\frac{1}{\beta}\right) + 1 - q\right) \\
&= \frac{q\,\kappa}{2}\left(1 + \frac{2}{\kappa}\log\left(\frac{1}{\beta}\right) - q\right) \\
&< 0,
\end{aligned}$$

since $q > 1$, $\kappa > 0$, and $q > 1 + \frac{2}{\kappa}\log\left(\frac{1}{\beta}\right)$, which implies that $1 + \frac{2}{\kappa}\log\left(\frac{1}{\beta}\right) - q < 0$. Therefore, we have

$$\log\left(\frac{Z_{\widetilde{\pi}^{[q]}}(T)}{Z_\mu(T)}\right) < 0, \qquad \text{a.s.}, \tag{7.7.172}$$

so that

$$\frac{Z_{\widetilde{\pi}^{[q]}}(T)}{Z_\mu(T)} < 1, \qquad \text{a.s.}, \tag{7.7.173}$$

and

$$Z_{\widetilde{\pi}^{[q]}}(T) < Z_\mu(T), \qquad \text{a.s.}, \tag{7.7.174}$$

and, we have

$$\mathbb{P}\left(Z_{\widetilde{\pi}^{[q]}}(T) < Z_\mu(T)\right) = 1. \tag{7.7.175}$$

If, on the other hand, we have

$$\mathbb{P}\left(\frac{Z_\pi(T)}{Z_\mu(T)} \geq \beta\right) = 1, \tag{7.7.176}$$

then it suffices to take $\widehat{\pi} \equiv \widetilde{\pi}^{[q]}$ (i.e., $\widehat{\pi}(t) \equiv \widetilde{\pi}^{[q]}(t)$, for all $t \in [0,T]$), with $q < \min\left(0, 1 - \frac{2}{\kappa}\log\left(\frac{1}{\beta}\right)\right)$. Since $\kappa > 0$ and $0 < \beta < 1$, we have $\frac{2}{\kappa}\log\left(\frac{1}{\beta}\right) > 0$ and $1 - \frac{2}{\kappa}\log\left(\frac{1}{\beta}\right) < 1$, so that we have $q < 0$ and $1 - q > 1$. Then, by (7.7.157) of Lemma 7.7.9, (7.7.176), (7.7.169) and the fact that $q < 0$, we arrive at the following a.s.,

$$\begin{aligned}
\log\left(\frac{Z_{\widetilde{\pi}^{[q]}}(T)}{Z_\mu(T)}\right) &= q\,\log\left(\frac{Z_\pi(T)}{Z_\mu(T)}\right) + \frac{q\,(1-q)}{2}\int_0^T \tau^\mu_{\pi\pi}(t)\,dt \\
&\leq q\,\log\beta + \frac{q\,(1-q)}{2}\,\kappa \\
&= q\left(-\log\left(\frac{1}{\beta}\right) + \frac{\kappa}{2}\,(1-q)\right) \\
&= \frac{q\,\kappa}{2}\left(-\frac{2}{\kappa}\log\left(\frac{1}{\beta}\right) + 1 - q\right) \\
&= \frac{q\,\kappa}{2}\left(1 - \frac{2}{\kappa}\log\left(\frac{1}{\beta}\right) - q\right) \\
&< 0,
\end{aligned}$$

since $q < 0$, $\kappa > 0$, and $q < \min\left(0, 1 - \frac{2}{\kappa}\log\left(\frac{1}{\beta}\right)\right)$, thus if $0 < 1 - \frac{2}{\kappa}\log\left(\frac{1}{\beta}\right) < 1$, then $\min\left(0, 1 - \frac{2}{\kappa}\log\left(\frac{1}{\beta}\right)\right) = 0$, which implies that $q < 0$ and $1 - \frac{2}{\kappa}\log\left(\frac{1}{\beta}\right) - q > -q > 0$, otherwise, if $1 - \frac{2}{\kappa}\log\left(\frac{1}{\beta}\right) < 0$, then $\min\left(0, 1 - \frac{2}{\kappa}\log\left(\frac{1}{\beta}\right)\right) = 1 - \frac{2}{\kappa}\log\left(\frac{1}{\beta}\right)$, which implies that $q < 1 - \frac{2}{\kappa}\log\left(\frac{1}{\beta}\right) < 0$ and $1 - \frac{2}{\kappa}\log\left(\frac{1}{\beta}\right) - q > 0$. Therefore, we have

$$\log\left(\frac{Z_{\widetilde{\pi}^{[q]}}(T)}{Z_\mu(T)}\right) < 0, \qquad \text{a.s.}, \tag{7.7.177}$$

so that

$$\frac{Z_{\widetilde{\pi}^{[q]}}(T)}{Z_\mu(T)} < 1, \qquad \text{a.s.}, \tag{7.7.178}$$

and

$$Z_{\widetilde{\pi}^{[q]}}(T) < Z_\mu(T), \qquad \text{a.s.}, \tag{7.7.179}$$

and, we have

$$\mathbb{P}\left(Z_{\widetilde{\pi}^{[q]}}(T) < Z_\mu(T)\right) = 1. \tag{7.7.180}$$

Hence, there exists another extended portfolio $\widehat{\pi}$ such that

$$\mathbb{P}\left(Z_{\widehat{\pi}}(T) < Z_\mu(T)\right) = 1.$$

■

Condition (7.7.168) postulates that the extended portfolio π is not very different from the market portfolio μ. But condition (7.7.169) mandates that π must be sufficiently different from the market portfolio.

7.7.2.2 A Seed Portfolio

Definition 7.7.11 (Seed Portfolio). *Let us consider $\boldsymbol{\pi}(t) = \mathbf{e}_1 = (1, 0, \ldots, 0)$, the portfolio comprising the single stock X_1, and the market portfolio μ. Fix a real number $q > 1$, and define the extended portfolio*

$$\widehat{\boldsymbol{\pi}}(t) \triangleq \widetilde{\boldsymbol{\pi}}^{[q]}(t) = q\,\mathbf{e}_1 + (1-q)\,\boldsymbol{\mu}(t), \qquad t \in [0, T], \tag{7.7.181}$$

*which takes a long position in the first stock X_1 (since $q > 1$), and a short position in the market portfolio (since $1 - q < 0$). This portfolio is referred to as the **"seed" portfolio**.*

In particular, for $t \in [0, T]$, we have

$$\widehat{\pi}_1(t) = q + (1-q)\,\mu_1(t), \tag{7.7.182}$$

and for $i = 2, \ldots, n$, we have

$$\widehat{\pi}_i(t) = (1-q)\,\mu_i(t). \tag{7.7.183}$$

Then, by (7.7.157) of Lemma 7.7.9, or, by using (7.7.165) or (7.7.166), with $\widetilde{\pi}^{[q]} = \widehat{\pi}$ and $\pi = \chi_1 = X_1$, the portfolio comprising the first stock (i.e., $\boldsymbol{\pi}(t) = \mathbf{e}_1$), and with $Z_{\widehat{\pi}}(0) = Z_\mu(0) = w > 0$, we obtain

$$\log\left(\frac{Z_{\widehat{\pi}}(T)}{Z_\mu(T)}\right) = q\,\log\left(\frac{Z_{\chi_1}(T)/Z_\mu(T)}{Z_{\chi_1}(0)/Z_\mu(0)}\right) - \frac{q\,(q-1)}{2}\int_0^T \tau^\mu_{\chi_1\chi_1}(t)\,dt \tag{7.7.184}$$

$$= q\,\log\left(\frac{X_1(T)/Z_\mu(T)}{X_1(0)/Z_\mu(0)}\right) - \frac{q\,(q-1)}{2}\int_0^T \tau^\mu_{11}(t)\,dt. \tag{7.7.185}$$

Thus, employing (2.12.19), we have $X_1(T)/Z_\mu(T) = \mu_1(T)$ and $X_1(0)/Z_\mu(0) = \mu_1(0)$, and we can rewrite the above expression as follows

$$\log\left(\frac{Z_{\widehat{\pi}}(T)}{Z_\mu(T)}\right) = q\,\log\left(\frac{\mu_1(T)}{\mu_1(0)}\right) - \frac{q\,(q-1)}{2}\int_0^T \tau^\mu_{11}(t)\,dt \tag{7.7.186}$$

$$= \log\left(\frac{\mu_1(T)}{\mu_1(0)}\right)^q - \frac{q\,(q-1)}{2}\int_0^T \tau^\mu_{11}(t)\,dt. \tag{7.7.187}$$

Since $\mu_1(t) < 1$ for all $t \in [0, T]$, we have a.s.,

$$\log\left(\frac{\mu_1(T)}{\mu_1(0)}\right) < \log\left(\frac{1}{\mu_1(0)}\right). \tag{7.7.188}$$

Since $q > 1$ and $\tau^\mu_{11}(t) \geq 0$, the drift process is decreasing (i.e., the integral in (7.7.186) is negative), so that for $T \in [0, \infty)$, (7.7.186) can be expressed as

$$\log\left(\frac{Z_{\widehat{\pi}}(T)}{Z_\mu(T)}\right) < q\,\log\left(\frac{\mu_1(T)}{\mu_1(0)}\right) = \log\left(\frac{\mu_1(T)}{\mu_1(0)}\right)^q. \tag{7.7.189}$$

This coupled with (7.7.188) provides, in particular, the following upper bound

$$\log\left(\frac{Z_{\widehat{\pi}}(T)}{Z_\mu(T)}\right) < q\,\log\left(\frac{1}{\mu_1(0)}\right) = \log\left(\frac{1}{\mu_1(0)}\right)^q. \tag{7.7.190}$$

Thus, for $T \in [0, \infty)$, we a.s. have

$$\frac{Z_{\widehat{\pi}}(T)}{Z_\mu(T)} < \left(\frac{\mu_1(T)}{\mu_1(0)}\right)^q \tag{7.7.191}$$

$$< \left(\frac{1}{\mu_1(0)}\right)^q, \tag{7.7.192}$$

which shows that $Z_{\widehat{\pi}}(T)/Z_\mu(T)$ is bounded above by the constant $\left(\frac{1}{\mu_1(0)}\right)^q$. Since the market is weakly diverse on $[0,T]$ and satisfies the strict nondegeneracy condition we obtain from (2.4.54) of Lemma 2.4.11 and (7.7.100)

$$\int_0^T \tau_{11}^\mu(t)\,dt \;\geq\; \varepsilon \int_0^T \left(1-\mu_{(1)}(t)\right)^2 dt \;\geq\; \varepsilon\,\delta^2\,T, \tag{7.7.193}$$

so that

$$-\frac{q\,(q-1)}{2}\int_0^T \tau_{11}^\mu(t)\,dt \;\leq\; -\frac{q\,(q-1)}{2}\,\varepsilon\,\delta^2\,T. \tag{7.7.194}$$

It thus follows, from (7.7.188) and (7.7.194), that a.s., for $T>0$, (7.7.186) amounts to the following

$$\log\left(\frac{Z_{\widehat{\pi}}(T)}{Z_\mu(T)}\right) \;<\; q\,\log\left(\frac{1}{\mu_1(0)}\right) - \frac{q\,(q-1)}{2}\,\varepsilon\,\delta^2\,T \tag{7.7.195}$$

$$= \;\log\left(\frac{1}{\mu_1(0)}\right)^q - \frac{q\,(q-1)}{2}\,\varepsilon\,\delta^2\,T, \tag{7.7.196}$$

and, we have

$$\frac{Z_{\widehat{\pi}}(T)}{Z_\mu(T)} \;<\; \left(\frac{1}{\mu_1(0)}\right)^q e^{-(q\,(q-1)\,\varepsilon\,\delta^2\,T)/2}. \tag{7.7.197}$$

Furthermore, we get

$$\begin{aligned}\log\left(\frac{Z_{\widehat{\pi}}(T)}{Z_\mu(T)}\right) &< q\left(\log\left(\frac{1}{\mu_1(0)}\right) - \frac{\varepsilon\,\delta^2\,T}{2}\,(q-1)\right)\\ &= q\,\frac{\varepsilon\,\delta^2\,T}{2}\left(\frac{2}{\varepsilon\,\delta^2\,T}\log\left(\frac{1}{\mu_1(0)}\right) - q + 1\right)\\ &= q\,\frac{\varepsilon\,\delta^2\,T}{2}\left(1 + \frac{2}{\varepsilon\,\delta^2\,T}\log\left(\frac{1}{\mu_1(0)}\right) - q\right).\end{aligned}$$

Therefore, the market portfolio μ strictly dominates the extended portfolio $\widehat{\pi}$ if $\left(\frac{1}{\mu_1(0)}\right)^q e^{-(q\,(q-1)\,\varepsilon\,\delta^2\,T)/2} < 1$, or since $q>1$, $\varepsilon>0$ and $0<\delta<1$, if $1+\frac{2}{\varepsilon\,\delta^2\,T}\log\left(\frac{1}{\mu_1(0)}\right) - q < 0$, i.e., if

$$q \;>\; q(T) \;\triangleq\; 1 + \frac{2}{\varepsilon\,\delta^2\,T}\log\left(\frac{1}{\mu_1(0)}\right). \tag{7.7.198}$$

Therefore, we have

$$\log\left(\frac{Z_{\widehat{\pi}}(T)}{Z_\mu(T)}\right) \;<\; 0, \qquad \text{a.s.}, \tag{7.7.199}$$

so that

$$\frac{Z_{\widehat{\pi}}(T)}{Z_\mu(T)} \;<\; 1, \qquad \text{a.s.}, \tag{7.7.200}$$

and

$$Z_{\widehat{\pi}}(T) \;<\; Z_\mu(T), \qquad \text{a.s.}, \tag{7.7.201}$$

and, we have

$$\mathbb{P}\left(Z_{\widehat{\pi}}(T) \;<\; Z_\mu(T)\right) \;=\; 1. \tag{7.7.202}$$

Alternatively, we can appeal to Lemma 7.7.10 by taking $\beta := \mu_1(0)$, since $0<\mu_1(0)<1$, we have $0<\beta<1$, and we have

$$\frac{\mu_1(T)}{\mu_1(0)} \;\leq\; \frac{1}{\beta}.$$

Thus, by (7.7.168), the extended portfolio $\pi = \chi_1 = X_1$ satisfies

$$\mathbb{P}\left(\frac{X_1(T)}{Z_\mu(T)} \;\leq\; \frac{1}{\beta}\right) \;=\; \mathbb{P}\left(\frac{\mu_1(T)}{\mu_1(0)} \;\leq\; \frac{1}{\beta}\right) \;=\; 1, \tag{7.7.203}$$

and by taking $\kappa := \varepsilon\,\delta^2\,T$, by (7.7.193) and (7.7.169), the extended portfolio π satisfies

$$\mathbb{P}\left(\int_0^T \tau_{11}^{\mu}(t)\,dt \;\geq\; \kappa := \varepsilon\,\delta^2\,T\right) \;=\; 1. \tag{7.7.204}$$

Then, from Lemma 7.7.10, we see that the market portfolio represents a strong relative arbitrage opportunity with respect to the extended portfolio $\widehat{\pi}$ of (7.7.181), provided that for any given real number $T \in [0, \infty)$, we select

$$q \;>\; q(T) \;\triangleq\; 1 + \frac{2}{\kappa}\log\left(\frac{1}{\beta}\right) \;=\; 1 + \frac{2}{\varepsilon\,\delta^2\,T}\log\left(\frac{1}{\mu_1(0)}\right). \tag{7.7.205}$$

The extended portfolio $\widehat{\pi}$ of (2.12.36), which underperforms the market portfolio, can be used as a "seed" (hence, the term "seed" portfolio) to create long-only portfolios that underperform or outperform the market portfolio over any time horizon $[0, T]$, with given real number $T > 0$. The idea is to immerse $\widehat{\pi}$ in a sea of the market portfolio, swamping the short positions while retaining the essential portfolio characteristics [Fernholz & Karatzas (2009)]. Crucial in these constructions are the a.s. comparisons, taken from (7.7.191), for $t \in [0, \infty)$,

$$Z_{\widehat{\pi}}(t) \;\leq\; \left(\frac{\mu_1(t)}{\mu_1(0)}\right)^q Z_\mu(t), \tag{7.7.206}$$

and

$$Z_\mu(t) \;\geq\; \left(\frac{\mu_1(0)}{\mu_1(t)}\right)^q Z_{\widehat{\pi}}(t). \tag{7.7.207}$$

7.7.2.3 Relative Arbitrage Opportunities in Diverse Equity Markets over Arbitrarily Short Time Horizons

To implement the idea in the previous section, consider a trading strategy $h(t)$ that, at time $t = 0$ invests $\frac{q}{(\mu_1(0))^q}$ dollars in the market portfolio μ, goes one dollar short in the extended portfolio $\widehat{\pi}$ of (7.7.181) (i.e., invests -1 dollars in $\widehat{\pi}$), and makes no change thereafter. The number $q > 1$ is chosen again as in Definition 7.7.11. The value of this trading strategy, with initial capital

$$z \;:=\; \frac{q}{\big(\mu_1(0)\big)^q} - 1 \;>\; 0, \tag{7.7.208}$$

is given, for $t \in [0, \infty)$, by

$$\begin{aligned}
\mathcal{Z}_{z,h}(t) \;&=\; \frac{q}{\big(\mu_1(0)\big)^q}\,Z_\mu(t) - Z_{\widehat{\pi}}(t) \\
&\geq\; \frac{q}{\big(\mu_1(0)\big)^q}\,Z_\mu(t) - \left(\frac{\mu_1(t)}{\mu_1(0)}\right)^q Z_\mu(t) \\
&=\; \frac{Z_\mu(t)}{\big(\mu_1(0)\big)^q}\Big(q - \big(\mu_1(t)\big)^q\Big) \\
&>\; 0,
\end{aligned}$$

which follows from (7.7.206) and $q > 1 > \big(\mu_1(t)\big)^q > 0$, since $q > 1$ and $0 < \mu_1(t) < 1$. This process $\mathcal{Z}_{z,h}(t)$ coincides with the wealth $Z_{z,\eta}(t) = Z_\eta(t)$ that is generated by a portfolio η with weights, for $i = 1, 2, \ldots, n$,

$$\eta_i(t) \;=\; \frac{1}{\mathcal{Z}_{z,h}(t)}\left(\frac{q\,\mu_i(t)}{\big(\mu_1(0)\big)^q}\,Z_\mu(t) - \widehat{\pi}_i(t)\,Z_{\widehat{\pi}}(t)\right) \tag{7.7.209}$$

$$\;=\; \frac{1}{Z_\eta(t)}\left(\frac{q\,\mu_i(t)}{\big(\mu_1(0)\big)^q}\,Z_\mu(t) - \widehat{\pi}_i(t)\,Z_{\widehat{\pi}}(t)\right), \tag{7.7.210}$$

that clearly satisfy $\sum_{i=1}^{n} \eta_i(t) = 1$. Now, from (7.7.183), for $i = 2, \ldots, n$, we have $\widehat{\pi}_i(t) = -(q-1)\,\mu_i(t) < 0$, since $q > 1$. Thus, for $i = 2, \ldots, n$, we obtain

$$\begin{aligned}
\eta_i(t) &= \frac{1}{\mathcal{Z}_{z,h}(t)} \left(\frac{q\,\mu_i(t)}{\big(\mu_1(0)\big)^q}\, Z_\mu(t) + (q-1)\,\mu_i(t)\, Z_{\widehat{\pi}}(t) \right) \\
&\geq \frac{1}{\mathcal{Z}_{z,h}(t)} \left(\frac{q\,\mu_i(t)}{\big(\mu_1(0)\big)^q} \left(\frac{\mu_1(0)}{\mu_1(t)} \right)^q Z_{\widehat{\pi}}(t) + (q-1)\,\mu_i(t)\, Z_{\widehat{\pi}}(t) \right) \\
&= \frac{1}{\mathcal{Z}_{z,h}(t)}\, Z_{\widehat{\pi}}(t)\,\mu_i(t) \left(\frac{q}{\big(\mu_1(t)\big)^q} + q - 1 \right) \\
&> 0,
\end{aligned}$$

which follows from (7.7.207) and $q > 1 > \big(\mu_1(t)\big)^q > 0$. Therefore, the quantities $\eta_2(t), \ldots, \eta_n(t)$ are strictly positive. To check that η is actually a long-only portfolio, it remains to verify that $\eta_1(t) \geq 0$. From (7.7.182), for $i = 1$, we have $\widehat{\pi}_1(t) = q - (q-1)\,\mu_1(t) = q\big(1 - \mu_1(t)\big) + \mu_1(t) > 0$, since $q > 1$ and $0 < \mu_1(t) < 1$. Thus, we obtain

$$\begin{aligned}
\eta_1(t) &= \frac{1}{\mathcal{Z}_{z,h}(t)} \left(\frac{q\,\mu_1(t)}{\big(\mu_1(0)\big)^q}\, Z_\mu(t) - \big(q - (q-1)\,\mu_1(t)\big)\, Z_{\widehat{\pi}}(t) \right) \\
&\geq \frac{1}{\mathcal{Z}_{z,h}(t)} \left(\frac{q\,\mu_1(t)}{\big(\mu_1(0)\big)^q}\, Z_\mu(t) - \big(q - (q-1)\,\mu_1(t)\big) \left(\frac{\mu_1(t)}{\mu_1(0)} \right)^q Z_\mu(t) \right) \\
&= \frac{1}{\mathcal{Z}_{z,h}(t)}\, Z_\mu(t)\, \frac{\mu_1(t)}{\big(\mu_1(0)\big)^q} \left(q - q\,\big(\mu_1(t)\big)^{q-1} + (q-1)\,\big(\mu_1(t)\big)^q \right) \\
&= \frac{1}{\mathcal{Z}_{z,h}(t)}\, Z_\mu(t)\, \frac{\mu_1(t)}{\big(\mu_1(0)\big)^q} \left(q\left(1 - \big(\mu_1(t)\big)^{q-1}\right) + (q-1)\,\big(\mu_1(t)\big)^q \right) \\
&> 0,
\end{aligned}$$

which follows from (7.7.206), $q > 1 > \big(\mu_1(t)\big)^q > 0$ and $q > 1 > \big(\mu_1(t)\big)^{q-1} > 0$, i.e., the dollar amount invested by the portfolio η in the first stock at time t, namely

$$\frac{q\,\mu_1(t)}{\big(\mu_1(0)\big)^q}\, Z_\mu(t) - \big(q - (q-1)\,\mu_1(t)\big)\, Z_{\widehat{\pi}}(t),$$

dominates

$$\frac{q\,\mu_1(t)}{\big(\mu_1(0)\big)^q}\, Z_\mu(t) - \big(q - (q-1)\,\mu_1(t)\big) \left(\frac{\mu_1(t)}{\mu_1(0)} \right)^q Z_\mu(t),$$

or, equivalently,

$$Z_\mu(t)\, \frac{\mu_1(t)}{\big(\mu_1(0)\big)^q} \left(q\left(1 - \big(\mu_1(t)\big)^{q-1}\right) + (q-1)\,\big(\mu_1(t)\big)^q \right) \quad > \quad 0.$$

Hence, η is indeed a long-only portfolio and has a positive value process. On the other hand, η outperforms at $t = T$ a market portfolio that starts with the same initial capital of $z := Z_\eta(0) = \frac{q}{(\mu_1(0))^q} - 1 > 0$ dollars at time $t = 0$, this is because η is long in the market portfolio and short in the extended portfolio $\widehat{\pi}$ which underperforms the market portfolio at time $t = T$. In particular, from (7.7.170) of Lemma 7.7.10, we notice the

following, a.s.,

$$\begin{aligned}
\mathcal{Z}_{z,h}(T) \;=\; Z_\eta(T) &= \frac{q}{\big(\mu_1(0)\big)^q}\, Z_\mu(T) - Z_{\widehat{\pi}}(T) \\
&= \frac{q}{\big(\mu_1(0)\big)^q}\, Z_\mu(T) - Z_\mu(T) + Z_\mu(T) - Z_{\widehat{\pi}}(T) \\
&= \left(\frac{q}{\big(\mu_1(0)\big)^q} - 1\right) Z_\mu(T) + \Big(Z_\mu(T) - Z_{\widehat{\pi}}(T)\Big) \\
&> \left(\frac{q}{\big(\mu_1(0)\big)^q} - 1\right) Z_\mu(T) \\
&= z\, Z_\mu(T) \;=\; Z_{z,\mu}(T),
\end{aligned}$$

since $Z_\mu(T) > Z_{\widehat{\pi}}(T)$, a.s. Thus, we have $Z_\eta(T) > Z_\mu(T)$ for all

$$T \;>\; \frac{-2\,\log \mu_1(0)}{(q-1)\,\varepsilon\,\delta^2},$$

by choosing q large enough as in (7.7.205), we can ensure relative arbitrage on any positive arbitrary time horizon. Note, however, that as $T \to 0$, the initial capital $z(T) := \frac{q(T)}{(\mu_1(0))^{q(T)}} - 1 > 0$ required to do all of this, increases without bound. It may take a huge amount of initial investment to realise the extra basis point's worth of relative arbitrage over a short time horizon.

In addition, consider an investment strategy $h(t)$ that, at time $t = 0$ invests $\frac{(q-1)}{(\mu_1(0))^q}$ dollars in the market portfolio μ, invests one dollar in the extended portfolio $\widehat{\pi}$ of (7.7.181), and makes no change thereafter. The number $q > 1$ is chosen again as in Definition 7.7.11. The value of this investment strategy, with initial capital

$$z \;:=\; 1 + \frac{(q-1)}{\big(\mu_1(0)\big)^q} \;>\; 0, \tag{7.7.211}$$

is given, for $t \in [0, \infty)$, by

$$\mathcal{Z}_{z,h}(t) \;=\; Z_{\widehat{\pi}}(t) + \frac{(q-1)}{\big(\mu_1(0)\big)^q}\, Z_\mu(t) \;>\; 0,$$

which follows from $q > 1 > \big(\mu_1(t)\big)^q > 0$, since $q > 1$ and $0 < \mu_1(t) < 1$. This process $\mathcal{Z}_{z,h}(t)$ coincides with the wealth $Z_{z,\rho}(t) = Z_\rho(t)$ that is generated by a portfolio ρ with weights, for $i = 1, 2, \ldots, n$,

$$\rho_i(t) \;=\; \frac{1}{\mathcal{Z}_{z,h}(t)} \left(\widehat{\pi}_i(t)\, Z_{\widehat{\pi}}(t) + \frac{(q-1)\,\mu_i(t)}{\big(\mu_1(0)\big)^q}\, Z_\mu(t)\right) \tag{7.7.212}$$

$$= \frac{1}{Z_\rho(t)} \left(\widehat{\pi}_i(t)\, Z_{\widehat{\pi}}(t) + \frac{(q-1)\,\mu_i(t)}{\big(\mu_1(0)\big)^q}\, Z_\mu(t)\right), \tag{7.7.213}$$

that clearly satisfy $\sum_{i=1}^n \rho_i(t) = 1$. From (7.7.182), for $i = 1$, we have $\widehat{\pi}_1(t) = q - (q-1)\,\mu_1(t) = q\,\big(1 - \mu_1(t)\big) + \mu_1(t) > 0$, since $q > 1$ and $0 < \mu_1(t) < 1$. Thus, since both $\widehat{\pi}_1(t)$ and $\mu_1(t)$ are positive and $q > 1$, we have $\rho_1(t) > 0$. To check that ρ is actually a long-only portfolio, it remains to verify that $\eta_i(t) \geq 0$, for $i = 2, \ldots, n$. Now, from (7.7.183), for $i = 2, \ldots, n$, we have $\widehat{\pi}_i(t) = -(q-1)\,\mu_i(t) < 0$, since $q > 1$. Thus, for $i = 2, \ldots, n$, we obtain

$$\begin{aligned}
\rho_i(t) &= \frac{1}{\mathcal{Z}_{z,h}(t)} \left(-(q-1)\,\mu_i(t)\, Z_{\widehat{\pi}}(t) + \frac{(q-1)\,\mu_i(t)}{\big(\mu_1(0)\big)^q}\, Z_\mu(t)\right) \\
&\geq \frac{1}{\mathcal{Z}_{z,h}(t)} \left(-(q-1)\,\mu_i(t) \left(\frac{\mu_1(t)}{\mu_1(0)}\right)^q Z_\mu(t) + \frac{(q-1)\,\mu_i(t)}{\big(\mu_1(0)\big)^q}\, Z_\mu(t)\right) \\
&= \frac{1}{\mathcal{Z}_{z,h}(t)}\, Z_\mu(t)\, \frac{(q-1)\,\mu_i(t)}{\big(\mu_1(0)\big)^q} \Big(1 - \big(\mu_1(t)\big)^q\Big) \\
&> 0,
\end{aligned}$$

which follows from (7.7.206), $q > 1 > \big(\mu_1(t)\big)^q > 0$, i.e., observe that the dollar amount invested by the portfolio ρ in any stock at time t, namely

$$-\,(q-1)\,\mu_i(t)\,Z_{\widehat{\pi}}(t) + \frac{(q-1)\,\mu_i(t)}{\big(\mu_1(0)\big)^q}\,Z_\mu(t),$$

dominates

$$Z_\mu(t)\,\frac{(q-1)\,\mu_i(t)}{\big(\mu_1(0)\big)^q}\,\Big(1-\big(\mu_1(t)\big)^q\Big),$$

or, equivalently,

$$Z_\mu(t)\,\frac{(q-1)\,\mu_i(t)}{\big(\mu_1(0)\big)^q}\,\Big(1-\big(\mu_1(t)\big)^q\Big) \quad > \quad 0.$$

Hence, ρ is indeed a long-only portfolio and has a positive value process. On the other hand, ρ underperforms at $t = T$ a market portfolio that starts out with the same initial capital of $z := Z_\rho(0) = 1 + \frac{(q-1)}{(\mu_1(0))^q} > 0$ dollars at time $t = 0$, this is because ρ holds a mix of μ and the extended portfolio $\widehat{\pi}$ which underperforms the market portfolio at time $t = T$. In particular, from (7.7.170) of Lemma 7.7.10, we notice the following, a.s.,

$$\begin{aligned}
\mathcal{Z}_{z,h}(T) \;=\; Z_\rho(T) \;&=\; Z_{\widehat{\pi}}(T) + \frac{(q-1)}{\big(\mu_1(0)\big)^q}\,Z_\mu(T) \\
&=\; Z_{\widehat{\pi}}(T) - Z_\mu(T) + Z_\mu(T) + \frac{(q-1)}{\big(\mu_1(0)\big)^q}\,Z_\mu(T) \\
&=\; \Big(Z_{\widehat{\pi}}(T) - Z_\mu(T)\Big) + \left(1 + \frac{(q-1)}{\big(\mu_1(0)\big)^q}\right) Z_\mu(T) \\
&<\; \left(1 + \frac{(q-1)}{\big(\mu_1(0)\big)^q}\right) Z_\mu(T) \\
&=\; z\,Z_\mu(T) \;=\; Z_{z,\mu}(T),
\end{aligned}$$

since $Z_\mu(T) > Z_{\widehat{\pi}}(T)$, a.s. Thus, we have $Z_\rho(T) < Z_\mu(T)$ for all

$$T \quad > \quad \frac{-2\,\log \mu_1(0)}{(q-1)\,\varepsilon\,\delta^2},$$

by choosing q large enough as in (7.7.205), we can ensure relative arbitrage on any positive arbitrary time horizon.

7.8 Sufficient Conditions for Ensuring Diversity

What conditions on the coefficients $\alpha_i(t)$ for $i = 1, 2, \ldots, n$, and $\xi_{i\nu}(t)$ for $i, \nu = 1, 2, \ldots, n$, of $\mathcal{M}$ are sufficient for ensuring diversity? Suppose that we select a number $\delta \in \big(0, 1 - \mu_{(1)}(0)\big)$ and ask under what conditions we might have

$$\mu_{(1)}(t) \quad \le \quad 1 - \delta, \qquad t \in [0, \infty), \qquad \text{a.s.} \tag{7.8.1}$$

This condition implies the requirement of diversity (4.2.2) on any finite time horizon $[0, T]$, i.e.,

$$\mu_{(1)}(t) \quad \le \quad 1 - \delta, \qquad t \in [0, T], \qquad \text{a.s.} \tag{7.8.2}$$

To simplify the analysis we shall assume $\frac{1}{2} \le \mu_{(1)}(0) < 1 - \delta$, and consider

$$R \;:=\; \inf\left\{t \in [0,\infty) \;\middle|\; \mu_{(1)}(t) \le \frac{1}{2}\right\}, \qquad S \;:=\; \inf\left\{t \in [0,\infty) \;\middle|\; \mu_{(1)}(t) \ge 1 - \delta\right\}, \tag{7.8.3}$$

as well as the stopping times

$$S_k \;:=\; \inf\left\{t \in [0,\infty) \;\middle|\; \mu_{(1)}(t) \geq 1-\delta_k\right\} \;=\; \inf\left\{t \in [0,\infty) \;\middle|\; \mu_{(1)}(t) \geq 1-\delta-\frac{1}{k}\right\}, \tag{7.8.4}$$

where $\delta_k := \delta + \frac{1}{k}$, for all $k \in \mathbb{N}$ sufficiently large. For diversity it will be enough to guarantee

$$\limsup_{k\to\infty} \mathbb{P}\left(S_k \;<\; R\right) \;=\; \overline{\lim_{k\to\infty}}\, \mathbb{P}\left(S_k \;<\; R\right) \;=\; 0, \tag{7.8.5}$$

because then

$$\mathbb{P}\left(S \;<\; R\right) \;\leq\; \limsup_{k\to\infty} \mathbb{P}\left(S_k \;<\; R\right) \;=\; \overline{\lim_{k\to\infty}}\, \mathbb{P}\left(S_k \;<\; R\right) \;=\; 0, \tag{7.8.6}$$

and this leads to (7.8.1).

Theorem 7.8.1 ([**Fernholz, Karatzas & Kardaras (2005)**]). *Suppose that on the event* $\left\{\frac{1}{2} \leq \mu_{(1)}(t) < 1-\delta\right\}$, *i.e.* $\mu_{(1)}(t) \in [0.5, 1-\delta)$, *we have for all* $k = 2, \ldots, n$,

$$\gamma_{(k)}(t) \;\geq\; 0 \;\geq\; \gamma_{(1)}(t), \tag{7.8.7}$$

and,

$$\min_{2\leq k\leq n} \gamma_{(k)}(t) - \gamma_{(1)}(t) + \frac{\varepsilon}{2} \;\geq\; \frac{K}{\delta Q(t)}, \tag{7.8.8}$$

where

$$Q(t) \;\triangleq\; \log\left(\frac{1-\delta}{\mu_{(1)}(t)}\right). \tag{7.8.9}$$

Then (7.8.1) *and* (7.8.5) *are satisfied. On any given finite time horizon* $[0,T]$, *the equity market is diverse and we have*

$$\int_0^T \left(Q(t)\right)^{-2} \;=\; \int_0^T \frac{1}{\left(Q(t)\right)^2} \;<\; \infty, \qquad a.s., \tag{7.8.10}$$

holds.

Proof. Refer to the proof provided in Fernholz, Karatzas & Kardaras (2005, Proof of Theorem 6.1). ■

The condition (7.8.8) holds, in particular, if all the stocks but the largest stock have nonnegative growth rates, whereas the growth rate of the largest stock is negative and exhibits a log-pole-type singularity as the relative capitalisation of the largest stock approaches $1-\delta$, namely

$$\gamma_{(1)}(t) \;\leq\; \frac{-K}{\delta Q(t)}, \tag{7.8.11}$$

on the event $\left\{\frac{1}{2} \leq \mu_{(1)}(t) < 1-\delta\right\}$. Thus, diversity is ensured by a strongly negative rate of growth for the largest stock, resulting in a sufficiently strong repelling drift (e.g., a log-pole-type singularity) away from an appropriate right boundary, and by nonnegative growth rates for all the other stocks. Slightly more generally, in order to guarantee diversity it is enough to require what follows in the theorem below, and thus the inequalities of (7.8.7) and (7.8.8) can be replaced below by the following inequalities.

Theorem 7.8.2 ([**Fernholz, Karatzas & Kardaras (2005), Fernholz & Karatzas (2009)**]). *Suppose that on the event* $\left\{\frac{1}{2} \leq \mu_{(1)}(t) < 1-\delta\right\}$, *i.e.* $\mu_{(1)}(t) \in [0.5, 1-\delta)$, *we have for all* $k = 2, \ldots, n$,

$$\gamma_{(k)}(t) \;\geq\; \min_{2\leq k\leq n} \gamma_{(k)}(t) \;\geq\; 0 \;\geq\; \gamma_{(1)}(t), \tag{7.8.12}$$

and,

$$\min_{2\le k\le n} \gamma_{(k)}(t) - \gamma_{(1)}(t) + \frac{\varepsilon}{2} \quad \ge \quad \frac{KF\big(Q(t)\big)}{\delta}, \tag{7.8.13}$$

where

$$Q(t) \quad \triangleq \quad \log\left(\frac{1-\delta}{\mu_{(1)}(t)}\right), \tag{7.8.14}$$

and here the function $F : (0,\infty) \to (0,\infty)$ is taken to be a continuous function with the property that the associated scale function $U(x)$, given by

$$U(x) \quad \triangleq \quad \int_1^x \exp\left(-\int_1^y F(z)\,dz\right) dy, \qquad x \in (0,\infty), \tag{7.8.15}$$

satisfies $U(0+) = -\infty$. For instance, we have $U(x) := \log x$ when $F(x) := \frac{1}{x}$ as in the preceding theorem. Then (7.8.1) *and* (7.8.5) *are satisfied. On any given finite time horizon $[0,T]$, the equity market is diverse and under these conditions, it can then be shown that the process $Q(t)$ satisfies*

$$\int_0^T \big(Q(t)\big)^{-2}\,dt \quad = \quad \int_0^T \frac{1}{\big(Q(t)\big)^2}\,dt \quad < \quad \infty, \qquad a.s., \tag{7.8.16}$$

holds.

Proof. Refer to the proof provided in Fernholz, Karatzas & Kardaras (2005, Proof of Theorem 6.1) as well as the comments provided further on. ■

7.8.1 A Diverse Equity Market Model

Do there exist equity market models that are indeed diverse, or at least weakly diverse? Now, we shall mention rather briefly an example of such an equity market model $\mathcal{M}$ which is diverse over any given time horizon $[0,T]$ with $T > 0$ a given real number. For the details of this construction and on the diverse equity market model, refer to Fernholz, Karatzas & Kardaras (2005) and Fernholz & Karatzas (2009). Let us consider an equity market comprising an equal number of stocks and driving Brownian motions (that is, $d = n$), a constant volatility matrix and nonnegative numbers $g_1, \ldots, g_n$. The growth rates of the stocks in this equity market are specified as,

$$\gamma_i(t) \quad = \quad g_i \quad \ge \quad 0, \tag{7.8.17}$$

if the ith stock does not have the largest capitalisation, and

$$\gamma_i(t) \quad = \quad \frac{-K}{\delta \log\big((1-\delta)/\mu_i(t)\big)}, \tag{7.8.18}$$

if the ith stock does have the largest capitalisation. With this particular specification, all stocks but the largest behave like geometric Brownian motions, with growth rates $g_i \ge 0$ as long as $i \ne p_t(1)$, and variance $\sigma_{ii}(t) = \sum_{\nu=1}^n \xi_{i\nu}^2(t)$, whereas the log capitalisation or log price of the largest stock, i.e., the stock with the largest capitalisation, is subjected to a log-pole-type singularity in its drift, away from an appropriate right boundary. For additional examples, and for an interesting probabilistic construction of diverse equity markets that leads to arbitrage, see Osterrieder & Rheinländer (2006).

7.9 The Volatility-Stabilised Equity Market Model

In this section, we introduce the representation of the stock prices that characterise the *volatility-stabilised equity market model*. We will give a brief overview of the existing results on volatility-stabilised equity markets from

Fernholz & Karatzas (2005). Let us consider the following equity market model $\mathcal{M}$, for $i = 1, 2, \ldots, n$, with

$$d \log X_i(t) = \frac{1}{\sqrt{\mu_i(t)}} \, dW_i(t), \qquad t \in [0, \infty), \tag{7.9.1}$$

where $W(t) = \big(W_1(t), \ldots, W_n(t)\big)$ is a standard $n-$dimensional Brownian motion. The above processes are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and are adapted to a given filtration which satisfies the "usual" conditions of right continuity and augmentation by $\mathbb{P}-$negligible sets. Equivalently, (7.9.1), can be written in the following form, for $i = 1, 2, \ldots, n$,

$$dX_i(t) = \frac{1}{2}\Big(X_1(t) + \cdots + X_n(t)\Big) \, dt + \sqrt{X_i(t)\Big(X_1(t) + \cdots + X_n(t)\Big)} \, dW_i(t), \qquad t \in [0, \infty). \tag{7.9.2}$$

Therefore, for $1 \leq i, \nu \leq n$, we have

$$\xi_{i\nu}(t) \equiv \frac{\delta_{i\nu}}{\sqrt{\mu_i(t)}}, \tag{7.9.3}$$

and for $i = 1, 2, \ldots, n$, we have

$$\gamma_i(t) \equiv 0. \tag{7.9.4}$$

The equity market model (7.9.1) is referred to as the *volatility-stabilised equity market model*, which was first described by Fernholz & Karatzas (2005). Therefore, the volatility-stabilised equity market model of (7.9.1) assigns to all stocks log-drifts $\gamma_i(t) \equiv 0$, for $i = 1, 2, \ldots, n$, and volatilities $\xi_{i\nu}(t) = \delta_{i\nu}/\sqrt{\mu_i(t)}$. In other words, the quantities $\gamma_i(t)$ are zero for all individual stocks and we are thus selecting equal growth rates $\gamma_i(t) \equiv 0$ for all the individual stocks and selecting volatilities $\xi_{i\nu}(t) = \delta_{i\nu}/\sqrt{\mu_i(t)}$, which are very high for the smallest stocks and very low for the largest stocks. Consider the above model given by (7.9.1), then elementary computations yield the following quantities in the form, namely the variances $\sigma_{ii}(t)$ for $i = 1, 2, \ldots, n$,

$$\begin{aligned} \sigma_{ii}(t) = \sum_{\nu=1}^{n} \xi_{i\nu}^2(t) &= \sum_{\nu=1}^{n} \left(\frac{\delta_{i\nu}}{\sqrt{\mu_i(t)}}\right)^2 \\ &= \sum_{\nu=1}^{n} \frac{\delta_{i\nu}^2}{\mu_i(t)} \\ &= \frac{1}{\mu_i(t)} \sum_{\nu=1}^{n} \delta_{i\nu}^2 \\ &= \frac{1}{\mu_i(t)}, \end{aligned} \tag{7.9.5}$$

and the covariances $\sigma_{ij}(t)$ for $i \neq j$, $i, j = 1, 2, \ldots, n$,

$$\begin{aligned} \sigma_{ij}(t) = \sum_{\nu=1}^{n} \xi_{i\nu}(t)\xi_{j\nu}(t) &= \sum_{\nu=1}^{n} \left(\frac{\delta_{i\nu}}{\sqrt{\mu_i(t)}}\right)\left(\frac{\delta_{j\nu}}{\sqrt{\mu_j(t)}}\right) \\ &= \sum_{\nu=1}^{n} \frac{\delta_{i\nu}\delta_{j\nu}}{\sqrt{\mu_i(t)\mu_j(t)}} \\ &= \frac{1}{\sqrt{\mu_i(t)\mu_j(t)}} \sum_{\nu=1}^{n} \delta_{i\nu}\delta_{j\nu} \\ &= 0, \end{aligned} \tag{7.9.6}$$

and the variance of the market portfolio $\sigma_{\mu\mu}(t)$,

$$\begin{aligned}\sigma_{\mu\mu}(t) &= \sum_{i,j=1}^{n} \mu_i(t)\sigma_{ij}(t)\mu_j(t) &&= \sum_{i=1}^{n} \mu_i^2(t)\sigma_{ii}(t) \\ &&&= \sum_{i=1}^{n} \mu_i^2(t)\left[\frac{1}{\mu_i(t)}\right] \\ &&&= \sum_{i=1}^{n} \mu_i(t) \\ &&&= 1, \end{aligned} \tag{7.9.7}$$

and the excess growth rate of the market portfolio $\gamma_\mu^*(t)$,

$$\begin{aligned}\gamma_\mu^*(t) &= \frac{1}{2}\left(\sum_{i=1}^{n} \mu_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^{n} \mu_i(t)\sigma_{ij}(t)\mu_j(t)\right) \\ &= \frac{1}{2}\left(\sum_{i=1}^{n} \mu_i(t)\sigma_{ii}(t) - \sigma_{\mu\mu}(t)\right) \\ &= \frac{1}{2}\left(\sum_{i=1}^{n} \mu_i(t)\left[\frac{1}{\mu_i(t)}\right] - 1\right) \\ &= \frac{1}{2}(n-1) \;=\; \frac{n-1}{2} \;=:\; \gamma^*. \end{aligned} \tag{7.9.8}$$

A key feature of the volatility-stabilised equity market model is that its cumulative volatility, as measured by the excess growth rate of the corresponding market portfolio, is constant through time, i.e., $\gamma_\mu^*(t) := \gamma^*$ for all $t \in [0, \infty)$. For the growth rate of the market portfolio, we obtain

$$\gamma_\mu(t) \;=\; \sum_{i=1}^{n} \mu_i(t)\gamma_i(t) + \gamma_\mu^*(t) \;\equiv\; \gamma_\mu^*(t) \;=\; \frac{n-1}{2} \;=:\; \gamma^* \;>\; 0. \tag{7.9.9}$$

Now, we shall briefly consider the following generalisation of the equity market model $\mathcal{M}$, (7.9.1), for $i = 1, 2, \ldots, n$, in the following form

$$d\log X_i(t) \;=\; \frac{\alpha}{2\,\mu_i(t)}\,dt + \frac{1}{\sqrt{\mu_i(t)}}\,dW_i(t), \qquad t \in [0, \infty), \tag{7.9.10}$$

or, equivalently for $i = 1, 2, \ldots, n$,

$$dX_i(t) \;=\; \frac{1+\alpha}{2}\Big(X_1(t) + \cdots + X_n(t)\Big)\,dt + \sqrt{X_i(t)\Big(X_1(t) + \cdots + X_n(t)\Big)}\,dW_i(t), \qquad t \in [0, \infty), \tag{7.9.11}$$

as in (7.9.2), where the parameter α is a given real constant and is nonnegative, thus we shall take $\alpha \geq 0$ and set $m := 2(1+\alpha)$. The model considered in (7.9.1) corresponds to the choice $\alpha = 0$ and $m = 2$, so we shall concentrate here on $\alpha > 0$ and $m > 2$. However, in Fernholz & Karatzas (2005), the core of this exposition is devoted to the first case where $\alpha = 0$. Therefore, we have for $1 \leq i, \nu \leq n$,

$$\xi_{i\nu}(t) \;\equiv\; \frac{\delta_{i\nu}}{\sqrt{\mu_i(t)}}, \tag{7.9.12}$$

and, for $i = 1, 2, \ldots, n$, we have

$$\gamma_i(t) \;\equiv\; \frac{\alpha}{2\,\mu_i(t)}. \tag{7.9.13}$$

Therefore, the generalised volatility-stabilised equity market model of (7.9.10) assigns to all stocks log-drifts $\gamma_i(t) \equiv \alpha/\big(2\,\mu_i(t)\big) > 0$, i.e., $\gamma_i(t) \neq 0$, for all $i = 1, 2, \ldots, n$, and volatilities $\xi_{i\nu}(t) = \delta_{i\nu}/\sqrt{\mu_i(t)}$, that are

largest for the smallest stocks and smallest for the largest stocks. Consider the above model given by (7.9.10), straightforward computations give constant variance and growth rates for the resulting market, namely

$$\begin{aligned}
\gamma_\mu(t) &= \sum_{i=1}^{n} \mu_i(t)\gamma_i(t) + \gamma_\mu^*(t) &= \sum_{i=1}^{n} \mu_i(t)\left[\frac{\alpha}{2\,\mu_i(t)}\right] + \gamma_\mu^*(t) \\
&&= \sum_{i=1}^{n} \frac{\alpha}{2} + \gamma_\mu^*(t) \\
&&= \frac{\alpha n}{2} + \frac{n-1}{2} \\
&&= \frac{(1+\alpha)n - 1}{2} \\
&&= \frac{mn}{4} - \frac{1}{2} \quad =: \quad \gamma \quad > \quad 0. \qquad (7.9.14)
\end{aligned}$$

It should be noted that the volatility-stabilised equity market model of (7.9.10) with $\alpha > 0$, assigns both big variances and big growth rates to the smallest stocks, but in such a manner that makes the overall market performance remarkably stable, as witnessed by the constant variance and growth rates. Volatility-stabilised equity market models are remarkable because in these models the entire market (as a whole) itself behaves in a rather sedate fashion, i.e., as an exponential Brownian motion with drift, while the individual stocks are going all "over the place" (in a rigorously defined manner, of course). These volatility-stabilised equity market models reflect and encapsulate the fact that in real markets, the smaller stocks tend to have greater volatility than the larger stocks. Thus, the smaller stocks enjoy these extreme volatilities. The volatilities are the largest for the smallest stocks and smallest for the largest stocks. Thus, not surprisingly then, individual stocks fluctuate widely in a market of this type. Yet, despite these fluctuations and the erratic widely fluctuating behaviour of the individual stocks, the overall market has a quite stable behaviour and the overall market performance is remarkably stable and relatively well-behaved. Hence the term volatility stabilisation. We call this phenomenon stabilisation by volatility, in the case where $\alpha = 0$ in (7.9.1) and (7.9.10), and stabilisation by both volatility and drift, in the case where $\alpha > 0$ in (7.9.10). The volatility-stabilised equity market model prescribes big volatility swings for the smallest stocks and smaller volatility swings for the largest stocks in a way that ends up stabilising the overall market by producing constant, positive overall growth and variance rates. In fact, diversity fails on every $[0, T]$, and the volatility-stabilised equity market does not satisfy the diversity condition and is thus not diverse.

7.10 Relative Arbitrage Opportunities in Volatility-Stabilised Equity Markets

The extreme volatilities enjoyed by the small-cap stocks in the volatility-stabilised equity market model lead to strong relative arbitrage opportunities on time horizons greater than a fixed constant which usually depends on the number of stocks in the market. In this section, we shall explain how the volatility-stabilised equity market model exhibits relative arbitrage.

7.10.1 Long-Term Relative Arbitrage Opportunities in Volatility-Stabilised Equity Markets

7.10.1.1 Relative Arbitrage Opportunities in Volatility-Stabilised Equity Markets over Sufficiently Long Time Horizons

Since $\gamma_\mu^*(t)$ is a positive constant, there should be enough relative volatility to achieve relative arbitrage. In Fernholz & Karatzas (2005), it is shown that a strong relative arbitrage opportunity exists in the volatility-stabilised equity market model of (7.9.10) over the time horizon $[0, T]$ for any T strictly greater than T_*, i.e.,

$T > T_*$, for

$$T \;>\; T_* \;:=\; \frac{2\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(0)\big)}{n-1} \;\leq\; \frac{2\log n}{n-1}. \tag{7.10.1}$$

A pair of portfolios which provides this arbitrage opportunity is given by $\big(\varphi^{(\mathsf{e},c)}(t), \mu(t)\big)$, where μ is the market portfolio and $\varphi^{(\mathsf{e},c)}$ is the modified entropy-weighted portfolio, for some sufficiently large constant c depending on T. Note that as the number of stocks in the equity market $n \to \infty$, $T_* \to 0$, i.e., the upper estimate represented above goes to zero as the number of stocks in the market tends to infinity or as the number of stocks in the market increases. Therefore, the model of (7.9.10) admits long-term strong relative arbitrage opportunities relative to the market portfolio, at least on time horizons $[0, T]$, with $T > T_*$. Thus, relative arbitrage can exist in a non-diverse market with unbounded volatilities and we can guarantee the existence of relative arbitrage opportunities even when the volatilities are unbounded and diversity fails.

7.10.2 Short-Term Relative Arbitrage Opportunities in Volatility-Stabilised Equity Markets

Do there exist relative arbitrage opportunities over arbitrarily short time horizons in the context of certain volatility-stabilised equity market models, i.e. in this equity market model of (7.9.10)? Indeed, does there exist a weak and strong relative arbitrage opportunity over arbitrarily short time horizons in the context of certain volatility-stabilised equity market models, i.e. in this equity market model of (7.9.10)? Banner & Fernholz (2008) answer this question, which was posed as an open question in Fernholz & Karatzas (2005), in the affirmative. In fact, the same result holds for any market whose smallest-cap stock possesses sufficient relative volatility.

7.10.2.1 Weak Relative Arbitrage Opportunities in Volatility-Stabilised Equity Markets over Arbitrarily Short Time Horizons

We shall first take a look at the simpler argument which establishes the existence of strictly weak relative arbitrage opportunities in the volatility-stabilised equity market model of (7.9.10) over arbitrarily short time horizons. Banner & Fernholz (2008) demonstrate that weak relative arbitrage opportunities do indeed exist in the volatility-stabilised equity market model of (7.9.10) over arbitrarily short time horizons. The premise is provided in the following proposition.

Proposition 7.10.1 ([**Banner & Fernholz (2008)**]). *For any $T > 0$ a given arbitrarily small real number, a **weak relative arbitrage opportunity** exists in the **volatility-stabilised equity market model** of* (7.9.10) *over the fixed, finite arbitrary time horizon $[0, T]$, for some $T \in [0, \infty)$ arbitrarily small, i.e., the volatility-stabilised equity market $\mathcal{M}$ of* (7.9.10) *admits **weak relative arbitrage** at time T, for T arbitrarily small.*

Proof. See Banner & Fernholz (2008). ■

7.10.2.2 Strong Relative Arbitrage Opportunities in Volatility-Stabilised Equity Markets over Arbitrarily Short Time Horizons

The goal is to show that a strong relative arbitrage opportunity exists over $[0, T]$ for *any* $T > 0$. Here, we state the main result of Banner & Fernholz (2008) that strong relative arbitrage opportunities also exist in the volatility-stabilised equity market model of (7.9.10) over arbitrarily short time horizons. Banner & Fernholz (2008) show that, in fact, a strong relative arbitrage opportunity exists in the volatility-stabilised equity market model of (7.9.10) over the fixed, finite arbitrary time horizon $[0, T]$, for some arbitrary $T > 0$ arbitrarily small. For any given fixed, finite arbitrary time horizon $[0, T]$, it is possible to construct a portfolio which is guaranteed to beat the market portfolio μ over that time horizon.

Proposition 7.10.2 ([**Banner & Fernholz (2008)**]). *For any $T > 0$ a given arbitrarily small real number, a **strong relative arbitrage opportunity** exists in the **volatility-stabilised equity market model** of* (7.9.10)

over the fixed, finite arbitrary time horizon $[0,T]$*, for some* $T \in [0,\infty)$ *arbitrarily small, i.e., the volatility-stabilised equity market* $\mathcal{M}$ *of* (7.9.10) *admits* ***strong relative arbitrage*** *at time* T*, for* T *arbitrarily small.*

Proof. See Banner & Fernholz (2008, Proof of Proposition 1) for the proof. In this proof Banner & Fernholz (2008) make use of (E.2.1) of Lemma E.2.1, (E.2.5) of Lemma E.2.2, (E.2.8) of Lemma E.2.3 and (E.2.11) of Lemma E.2.4. ■

Fernholz & Banner (2008), after providing the proof above, note that the precise form of the growth rate term $\frac{\alpha}{2\mu_i(t)}\,dt$ of the volatility-stabilised equity market model of (7.9.10), does not appear in their proof. In other words, the relative arbitrage is driven purely by volatility considerations. In principle, the growth rate term could essentially be replaced by another suitable growth rate $\gamma_i(t)\,dt$, although the resulting market may lack the long-term stability of the volatility-stabilised equity market of (7.9.10).

7.11 Summary and Conclusion

In this chapter, we introduced the concept of admissible portfolios, as well as the admissibility conditions under which a portfolio is deemed to be admissible. Essentially, these conditions amount to: considering long-only portfolios and preventing arbitrarily high overweighting of any particular stock relative to the market weighting. Recall, that the reasoning behind considering these admissibility conditions is that we are mainly intrigued with the idea of arbitrage linked to the admissible portfolios. We then provided definitions for dominating portfolios and strictly dominating portfolios: the portfolio φ dominates the portfolio η on $[0,T]$, if there exists a number $T > 0$, such that for any positive initial capital values $Z_\varphi(0)$ and $Z_\eta(0)$, we have

$$\frac{Z_\varphi(T)}{Z_\varphi(0)} \geq \frac{Z_\eta(T)}{Z_\eta(0)}, \qquad \text{a.s.,} \tag{7.11.1}$$

and

$$\mathbb{P}\left(\frac{Z_\varphi(T)}{Z_\varphi(0)} \geq \frac{Z_\eta(T)}{Z_\eta(0)}\right) = 1, \qquad \text{and} \qquad \mathbb{P}\left(\frac{Z_\varphi(T)}{Z_\varphi(0)} > \frac{Z_\eta(T)}{Z_\eta(0)}\right) > 0. \tag{7.11.2}$$

The portfolio φ strictly dominates the portfolio η on $[0,T]$, if there exists a number $T > 0$, such that for any positive initial capital values $Z_\varphi(0)$ and $Z_\eta(0)$, we have

$$\frac{Z_\varphi(T)}{Z_\varphi(0)} > \frac{Z_\eta(T)}{Z_\eta(0)}, \qquad \text{a.s.,} \tag{7.11.3}$$

and

$$\mathbb{P}\left(\frac{Z_\varphi(T)}{Z_\varphi(0)} > \frac{Z_\eta(T)}{Z_\eta(0)}\right) = 1. \tag{7.11.4}$$

Next, we showed that in order for a dominance relationship to occur between a particular functionally generated portfolio and the market portfolio, certain conditions need to be imposed on the corresponding portfolio generating function of the functionally generated portfolio, $\mathbf{G}\big(\boldsymbol{\mu}(t)\big)$, and on the drift process of the functionally generated portfolio, $\mathfrak{g}(t)$. Thus, we established that for certain criteria placed on these aforementioned terms that the functionally generated portfolio will either dominate the market portfolio or be dominated by the market portfolio. In particular, in order for a functionally generated portfolio to strictly dominate the market portfolio, we require that $\mathbf{G}\big(\boldsymbol{\mu}(t)\big) > c_1 > 0$, and $\mathfrak{g}(t) > c_2 > 0$, where c_1 and c_2 are constants. Thus, both the portfolio generating function and the drift process must be positive and have a positive lower bound, i.e., be bounded below by a positive constant. On the other hand, in order for a functionally generated portfolio to be strictly dominated by the market portfolio, we require that $0 < \mathbf{G}\big(\boldsymbol{\mu}(t)\big) < c_1$, and $\mathfrak{g}(t) < c_2 < 0$. Thus, the portfolio generating function must be positive and have a positive upper bound, i.e., be bounded above by a positive constant, and the drift process must be negative and have a negative upper bound, i.e., be bounded

above by a negative constant. Note that, in each case, this will only occur if the following lower bound on T holds

$$T \;>\; T_* \;\triangleq\; \frac{\log \mathbf{G}\big(\boldsymbol{\mu}(0)\big) - \log c_1}{c_2}. \tag{7.11.5}$$

This suggests that both of the dominance relationships will only occur over the long term, i.e. over a sufficiently long time horizon $[0,T]$, since the lower bound on T above is sufficiently large as it contains the sufficiently small numbers c_1 and c_2, where the latter constant is present in the denominator of the lower bound. The natural next step was to introduce and define relative arbitrage, since the notion of dominating portfolios naturally leads us to the notion of relative arbitrage. In this regard, we provided definitions for weak relative arbitrage and strong relative arbitrage as follows: the portfolio φ represents a weak arbitrage opportunity relative to the portfolio η (i.e., a weak relative arbitrage opportunity) over the fixed, finite time horizon $[0,T]$, for $T > 0$ a given real number, if we have

$$\mathbb{P}\Big(Z_\varphi(T) \;\geq\; Z_\eta(T)\Big) \;=\; 1, \qquad \text{and} \qquad \mathbb{P}\Big(Z_\varphi(T) \;>\; Z_\eta(T)\Big) \;>\; 0, \tag{7.11.6}$$

and we can then say that the market $\mathcal{M}$ admits weak relative arbitrage at time T; the portfolio φ represents a strong arbitrage opportunity relative to the portfolio η (i.e., a strong relative arbitrage opportunity) over the fixed, finite time horizon $[0,T]$, for $T > 0$ a given real number, if instead we have the stronger condition

$$\mathbb{P}\Big(Z_\varphi(T) \;>\; Z_\eta(T)\Big) \;=\; 1, \tag{7.11.7}$$

and we can then say that the market $\mathcal{M}$ admits strong relative arbitrage at time T. We also showed that under these same conditions imposed on the portfolio generating function and the drift process of the corresponding functionally generated portfolio, the portfolio represents a strong arbitrage opportunity relative to the market portfolio, or vice versa. Armed with the preceding knowledge, we then moved on to relative arbitrage opportunities over sufficiently long time horizons. With this in mind, we looked at the following portfolios: the weighted-average capitalisation generated portfolio, the price-to-book ratio generated portfolio and the single stock with leverage. We demonstrated that these specific portfolios will be strictly dominated by the market portfolio over sufficiently long time horizons, i.e. the market portfolio represents a strong arbitrage opportunity relative to these portfolios over a sufficiently long time horizon. The reasoning behind why the market portfolio will dominate these aforementioned portfolios is that the generating functions that generated each of these portfolios are not measures of diversity and thus, along with the associated drift process of the generated portfolios, do not satisfy the required bounds for a dominating relationship in the opposing direction to exist. We followed this with an investigation of long-term relative arbitrage opportunities in diverse equity markets. We showed that if the market is nondegenerate and (weakly) diverse then it contains strong arbitrage opportunities relative to the market portfolio over the long term, i.e. there exist certain portfolios that will dominate the market portfolio over the long term. To this end, we presented several examples of such strong relative arbitrage opportunities, or portfolios, in a nondegenerate and (weakly) diverse equity market, such as: the entropy-weighted portfolio, the diversity-weighted index portfolio, the quadratic Gini-coefficient-weighted portfolio and the admissible market-dominating portfolio, to name just a few. Thus, we considered the entropy function $\mathbf{S}^{\mathrm{E}}$ which generates the entropy-weighted portfolio, φ^{e}. We showed that if the market $\mathcal{M}$ is nondegenerate and (weakly) diverse over a fixed, finite time horizon $[0,T]$, with $T > 0$ a given real number. Then for a sufficiently large real number $T \in [0,\infty)$, we have

$$\frac{Z_{\varphi^{\mathrm{e}}}(T)}{Z_{\varphi^{\mathrm{e}}}(0)} \;>\; \frac{Z_\mu(T)}{Z_\mu(0)}, \qquad \text{a.s.}, \tag{7.11.8}$$

and, consequently

$$\mathbb{P}\left(\frac{Z_{\varphi^{\mathrm{e}}}(T)}{Z_{\varphi^{\mathrm{e}}}(0)} \;>\; \frac{Z_\mu(T)}{Z_\mu(0)}\right) \;=\; 1. \tag{7.11.9}$$

Moreover, we have

$$\mathbb{P}\Big(Z_{\varphi^{\mathrm{e}}}(T) \;>\; Z_\mu(T)\Big) \;=\; 1. \tag{7.11.10}$$

That is, in a nondegenerate and (weakly) diverse equity market, the entropy-weighted portfolio φ^{e} represents a strong arbitrage opportunity relative to the market portfolio μ over the fixed, finite time horizon $[0,T]$, for T sufficiently large. This was essentially proved by appealing to the requisite bound conditions on the entropy function and the drift process of the entropy-weighted portfolio, that were mentioned earlier. We set $c_1 := \zeta_1$ and $c_2 := \frac{\zeta_2}{\log n}$, and since $\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$ is positive and finite from (4.7.1) of Proposition 4.7.1, thus for all $t > 0$, $\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) > \zeta_1 > 0$, a.s. Moreover, the inequality (4.2.6) of Proposition 4.2.3 implies that, for a nondegenerate and diverse equity market, there is a $\zeta_2 > 0$ such that $\gamma^*_\mu(t) \geq \zeta_2$, i.e., positive, for all $t \in [0,T]$, a.s. Thus, $\gamma^*_\mu(t)$ has a positive lower bound in a diverse equity market. Also, from (4.6.3), for $t \in [0,T]$, we have the bound $\frac{1}{\mathbf{S}^{\mathrm{E}}(\boldsymbol{\mu}(t))} \geq \frac{1}{\log n}$. Hence, since the drift process of the entropy-weighted portfolio is given by $\mathfrak{g}_{\varphi^{\mathrm{e}}}(t) = \gamma^*_\mu(t)/\mathbf{S}^{\mathrm{E}}(\boldsymbol{\mu}(t))$, we have $\mathfrak{g}_{\varphi^{\mathrm{e}}}(t) > \frac{\zeta_2}{\log n} > 0$. Thus, the drift process of the entropy-weighted portfolio $\mathfrak{g}_{\varphi^{\mathrm{e}}}(t)$ has a positive lower bound. Consequently, this implies that there is a dominance relationship between the entropy-weighted portfolio φ^{e} and the market portfolio μ, i.e., φ^{e} strictly dominates the market portfolio μ, provided that the following lower bound on T holds

$$T \;>\; T_* \;\triangleq\; \frac{\log n\,\big(\log\log n - \log\zeta_1\big)}{\zeta_2}. \tag{7.11.11}$$

This suggests that that the entropy-weighted portfolio will only outperform the market portfolio over the long term, i.e. over a sufficiently long time horizon $[0,T]$, since the lower bound on T above is sufficiently large as it contains the sufficiently small numbers $\zeta_1 > 0$ and $\zeta_2 > 0$, where the latter constant is present in the denominator of the lower bound. Thus, we also considered the $\mathbf{D}_p$-function $\mathbf{D}_p$, for $0 < p < 1$, which generates the diversity-weighted index portfolio (or, the $\mathbf{D}_p$-weighted index portfolio), $\varphi^{(p)}$. We showed that if the market $\mathcal{M}$ is nondegenerate and (weakly) diverse over a fixed, finite time horizon $[0,T]$, with $T > 0$ a given real number. Then for a sufficiently large real number $T \in [0,\infty)$, we have

$$\frac{Z_{\varphi^{(p)}}(T)}{Z_{\varphi^{(p)}}(0)} \;>\; \frac{Z_\mu(T)}{Z_\mu(0)}, \qquad \text{a.s.}, \tag{7.11.12}$$

and, consequently

$$\mathbb{P}\left(\frac{Z_{\varphi^{(p)}}(T)}{Z_{\varphi^{(p)}}(0)} \;>\; \frac{Z_\mu(T)}{Z_\mu(0)}\right) \;=\; 1. \tag{7.11.13}$$

Moreover, we have

$$\mathbb{P}\Big(Z_{\varphi^{(p)}}(T) \;>\; Z_\mu(T)\Big) \;=\; 1. \tag{7.11.14}$$

That is, in a nondegenerate and (weakly) diverse equity market, the diversity-weighted index portfolio (or, the $\mathbf{D}_p$-weighted index portfolio) $\varphi^{(p)}$ represents a strong arbitrage opportunity relative to the market portfolio μ over the fixed, finite time horizon $[0,T]$, for T sufficiently large. This was essentially proved by appealing to the requisite bound conditions on the $\mathbf{D}_p$-function and the drift process of the diversity-weighted index portfolio, that were mentioned earlier. Since $\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) > 1$, thus for all $t > 0$, $\mathbf{D}_p\big(\boldsymbol{\mu}(t)\big) > 1 > 0$, a.s. Moreover, the inequality (2.4.56) of Lemma 2.4.14 implies that, for a nondegenerate equity market, there is an $\varepsilon > 0$ such that $\gamma^*_{\varphi^{(p)}}(t) \geq \frac{\varepsilon}{2}\big(1 - \varphi^{(p)}_{(1)}(t)\big)$, for all $t \in [0,T]$, a.s. Thus, $\gamma^*_{\varphi^{(p)}}(t)$ is positive and has a positive lower bound in a nondegenerate equity market. Hence, since the drift process of the diversity-weighted index portfolio is given by $\mathfrak{g}_{\varphi^{(p)}}(t) = (1-p)\gamma^*_{\varphi^{(p)}}(t)$ and $0 < p < 1$, we have $\mathfrak{g}_{\varphi^{(p)}}(t) > 0$. Thus, the drift process of the diversity-weighted index portfolio $\mathfrak{g}_{\varphi^{(p)}}(t)$ is positive and has a positive lower bound. Consequently, this implies that there is a dominance relationship between the diversity-weighted index portfolio $\varphi^{(p)}$ and the market portfolio μ, i.e., $\varphi^{(p)}$ strictly dominates the market portfolio μ, provided that the following lower bound on T holds

$$T \;>\; T_* \;\triangleq\; \frac{2\,\log n}{p\,\varepsilon\,\delta}. \tag{7.11.15}$$

This suggests that the diversity-weighted index portfolio will only outperform the market portfolio over the long term, i.e. over a sufficiently long time horizon $[0,T]$, since the lower bound on T above is sufficiently large as it contains the sufficiently small numbers $0 < p < 1$, $0 < \delta < 1$ and $\varepsilon > 0$, which are present in the

denominator of the lower bound. Thus, we also considered the admissible, market-dominating diversity measure function $\mathbf{S}^{\mathrm{A}}$, which generates the admissible market-dominating portfolio, φ^{a}. We showed that if the market $\mathcal{M}$ is nondegenerate and (weakly) diverse over a fixed, finite time horizon $[0,T]$, with $T>0$ a given real number. Then for a sufficiently large real number $T \in [0,\infty)$, we have

$$\frac{Z_{\varphi^{\mathsf{a}}}(T)}{Z_{\varphi^{\mathsf{a}}}(0)} \quad > \quad \frac{Z_{\mu}(T)}{Z_{\mu}(0)}, \qquad \text{a.s.,} \tag{7.11.16}$$

and, consequently

$$\mathbb{P}\left(\frac{Z_{\varphi^{\mathsf{a}}}(T)}{Z_{\varphi^{\mathsf{a}}}(0)} \quad > \quad \frac{Z_{\mu}(T)}{Z_{\mu}(0)}\right) \quad = \quad 1. \tag{7.11.17}$$

Moreover, we have

$$\mathbb{P}\Big(Z_{\varphi^{\mathsf{a}}}(T) \quad > \quad Z_{\mu}(T)\Big) \quad = \quad 1. \tag{7.11.18}$$

That is, in a nondegenerate and (weakly) diverse equity market, the admissible market-dominating portfolio φ^{a} represents a strong arbitrage opportunity relative to the market portfolio μ over the fixed, finite time horizon $[0,T]$, for T sufficiently large. This was essentially proved by appealing to the requisite bound conditions on the admissible, market-dominating diversity measure function and the drift process of the admissible market-dominating portfolio, that were mentioned earlier. Since $\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big) > \frac{1}{2}$, thus for all $t>0$, $\mathbf{S}^{\mathrm{A}}\big(\boldsymbol{\mu}(t)\big) > \frac{1}{2} > 0$, a.s. Moreover, the inequality (2.4.54) of Lemma 2.4.11 implies that, for a nondegenerate equity market, there is an $\varepsilon > 0$ such that $\tau_{ii}(t) \geq \varepsilon\big(1-\mu_{(1)}(t)\big)^2$, for all $t \in [0,T]$, a.s. Thus, $\tau_{ii}(t)$ has a positive lower bound in a nondegenerate equity market. Hence, we have $\mathfrak{g}_{\varphi^{\mathsf{a}}}(t) > 0$. Thus, the drift process of the admissible market-dominating portfolio $\mathfrak{g}_{\varphi^{\mathsf{a}}}(t)$ is positive and has has a positive lower bound. Consequently, this implies that there is a dominance relationship between the admissible market-dominating portfolio φ^{a} and the market portfolio μ, i.e., φ^{a} strictly dominates the market portfolio μ, provided that the following lower bound on T holds

$$T \quad > \quad T_* \quad \triangleq \quad \frac{2n\,\log 2}{\varepsilon\,\delta^2}. \tag{7.11.19}$$

This suggests that the admissible market-dominating portfolio will only outperform the market portfolio over the long term, i.e. over a sufficiently long time horizon $[0,T]$, since the lower bound on T above is sufficiently large as it contains the sufficiently small numbers $0<\delta^2<1$ and $\varepsilon>0$, which are present in the denominator of the lower bound. Basically, the reasoning behind why the market portfolio will be dominated by these portfolios is that the generating functions that generated these portfolios are all measures of diversity and thus, along with the associated drift process of the generated portfolios, do indeed satisfy the required bounds for this kind of dominating relationship. That is, measures of diversity are portfolio generating functions that are positive and thus have positive lower bounds, and the drift process of a functionally generated portfolio generated by a measure of diversity, is also strictly positive and thus has a positive lower bound. This diversity measure requirement of the generating functions is an imperative one in order for these associated functionally generated portfolios to dominate the market portfolio. We then slightly digressed from the discussion of relative arbitrage, for awhile, to exploit a discussion of an equity market model that exhibits diversity, to wit, the diverse equity market model. We also mentioned the conditions on certain stock variables that will sufficiently guarantee the existence of a diverse equity market. It turns out that the condition that must hold is that all the stocks but the largest stock must have nonnegative growth rates, whereas the growth rate of the largest stock must be strongly negative resulting in a sufficiently strong repelling drift (e.g., a log-pole-type singularity) away from an appropriate right boundary. We then went on to introduce the volatility-stabilised equity market model along with its generalisation:

$$d\log X_i(t) \quad = \quad \frac{1}{\sqrt{\mu_i(t)}}\,dW_i(t), \qquad t \in [0,\infty), \tag{7.11.20}$$

and

$$d\log X_i(t) \quad = \quad \frac{\alpha}{2\,\mu_i(t)}\,dt + \frac{1}{\sqrt{\mu_i(t)}}\,dW_i(t), \qquad t \in [0,\infty). \tag{7.11.21}$$

In essence, the volatility-stabilised equity market model assigns the highest volatilities to the smallest stocks in the equity market and the lowest volatilities to the largest stocks in the equity market, but in such a way that the overall behaviour of the entire equity market exhibits a remarkable stability. Hence, the terminology "volatility-stabilised" for this equity market model as it involves and encapsulates the concept of stabilisation by volatility. Weak and strong relative arbitrage opportunities in this volatility-stabilised equity market model were considered next, over any arbitrary time horizons.

Appendix A

Stochastic Calculus

A.1 Local Martingales

Stopping times are frequently used to generalise certain properties of stochastic processes to situations in which the required property is satisfied in only a local sense. First, if X is a process and τ is a stopping time, then X_τ is used to denote the process X stopped at time τ. A process X is a *local martingale* if it is càdlàg[1] and there exists a sequence of stopping times τ_n increasing to infinity, such that the process $\mathbb{1}_{\{\tau_n>0\}}X^{\tau_n}$ is a martingale for each n. Thus, satisfying the localised version of the martingale property.

A.2 Semimartingales

A real-valued process X is called a *semimartingale* if it can be decomposed as the sum of a local martingale and an adapted finite-variation process. Semimartingales are "good integrators", forming the largest class of processes with respect to which the Itô integral can be defined. The class of semimartingales is quite large (including, for example, all continuously differentiable processes, Brownian motion and Poisson processes). Submartingales and supermartingales together represent a subset of the semimartingales.

A real-valued process X defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathcal{F}_t, t \in [0,\infty)\}$, is called a semimartingale if it can be decomposed as

$$X(t) \quad = \quad X(0) + M(t) + V(t), \qquad t \in [0,\infty), \qquad \text{a.s.,} \tag{A.2.1}$$

where $M = \{M(t), \mathcal{F}_t, t \in [0,\infty)\}$ is a local martingale and $V = \{V(t), \mathcal{F}_t, t \in [0,\infty)\}$ is a càdlàg adapted process of locally bounded variation.[2]

A.3 Continuous Semimartingales

By definition, every semimartingale is a sum of a local martingale and a finite variation process. However, this decomposition is not unique.

A *continuous semimartingale* $X = \{X(t), \mathcal{F}_t, t \in [0,\infty)\}$ is a measurable, adapted process, defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathcal{F}_t, t \in [0,\infty)\}$, that has the following decomposition

$$X(t) \quad = \quad X(0) + M_X(t) + V_X(t), \qquad t \in [0,\infty), \qquad \text{a.s.,} \tag{A.3.1}$$

[1] A càdlàg function (from the French "continue à droite, limite gauche") is a function defined on the real numbers (or a subset thereof) that is everywhere right-continuous and has left limits everywhere.

[2] A function of *bounded variation* refers to a real-valued function whose total variation is bounded (finite): the graph of a function having this property is well behaved in a precise sense.

where $X(0)$ is a $\mathcal{F}_0$-measurable random variable, $M_X = \{M_X(t), \mathcal{F}_t, t \in [0,\infty)\}$ is a continuous, square-integrable[3] local martingale and $V_X = \{V_X(t), \mathcal{F}_t, t \in [0,\infty)\}$ is a continuous càdlàg adapted process of locally bounded variation. It can be shown that the decomposition is a.s. unique [see Karatzas & Shreve (1991)].

A.4 Quadratic Variation and Cross-Variation

For continuous, square-integrable martingales $M = \{M(t), \mathcal{F}_t, t \in [0,\infty)\}$ and $N = \{N(t), \mathcal{F}_t, t \in [0,\infty)\}$, we can define the *cross-variation process* $\langle M, N\rangle$. The cross-variation process is adapted, continuous and of bounded variation, and the operation $\langle\,\cdot\,,\cdot\,\rangle$ is bilinear on the real vector space of continuous, square-integrable martingales. If $M = N$, we shall use the notation $\langle M\rangle = \langle M, M\rangle$, $\langle M\rangle$ is called the *quadratic variation process* of M, and has continuous, nondecreasing sample paths. The Brownian motion process is a continuous, square-integrable martingale, and it is characterised by its cross-variation processes

$$\begin{aligned} \langle W_\nu, W_\upsilon\rangle_t &= \rho_{\nu\upsilon} t, \qquad t \in [0,\infty), \\ d\,\langle W_\nu, W_\upsilon\rangle_t &= \rho_{\nu\upsilon}\, dt, \qquad t \in [0,\infty), \end{aligned}$$

where $\rho_{\nu\upsilon} = 1$ if $\nu = \upsilon$ and 0 otherwise, i.e.,

$$\rho_{\nu\upsilon} \triangleq \begin{cases} 1 & \text{if} \quad \nu = \upsilon, \\ 0 & \text{if} \quad \nu \neq \upsilon, \end{cases} \tag{A.4.1}$$

which represents the well-known *Kronecker delta function*, $\delta_{\nu\upsilon}$. Therefore, we have

$$d\,\langle W_\nu, W_\upsilon\rangle_t = \begin{cases} dt & \text{if} \quad \nu = \upsilon, \\ 0 & \text{if} \quad \nu \neq \upsilon. \end{cases}$$

We can also define the cross-variation process for continuous semimartingales X and Y by

$$\langle X, Y\rangle = \langle M_X, M_Y\rangle,$$

where M_X and M_Y are the martingale parts of X and Y, respectively. The quadratic variation process for X is similarly defined by $\langle X\rangle = \langle M_X\rangle$.

[3] A real-valued function of a real variable is *square-integrable* on an interval if the integral of the square of its absolute value, over that interval, is finite. The set of all measurable functions that are square-integrable forms a Hilbert space, the so-called L^2 space.

Appendix B

Itô Calculus

In Itô calculus we can express the Itô stochastic differential equation (SDE) in the following form

$$dX(t) \quad = \quad \mu\big(t, X(t)\big)\, dt + \sigma\big(t, X(t)\big)\, dW(t),$$

or equivalently in integral form as

$$X(t) \quad = \quad X(0) + \int_0^t \mu\big(s, X(s)\big)\, ds + \int_0^t \sigma\big(s, X(s)\big)\, dW(s).$$

Definition B.0.1 (The Itô Integral)**.** *Suppose that $W(t)$ is a Brownian motion process and that $X(t)$ is a stochastic process. Consider a partition of $[0, T]$, $0 = t_0 < t_1 < \cdots < t_n = T$, then the Itô integral of X w.r.t. W is the random variable*

$$\int_0^T X(t)\, dW(t) \quad \triangleq \quad \lim_{n\to\infty} \sum_{j=0}^{n-1} X(t_j)\big(W(t_{j+1}) - W(t_j)\big).$$

Notice, in the summation, that the function X is defined at the left-hand point, i.e., the value of X at the beginning of each timestep is used, this is of crucial importance.

Theorem B.0.2 (Itô's Formula)**.** *Let $X(t)$ be a generalised Brownian motion process or an Itô process. That is, let $X(t)$ have the following dynamics*

$$dX(t) \quad = \quad a\big(t, X(t)\big)\, dt + b\big(t, X(t)\big)\, dW(t),$$

where $W(t)$ is a standard Brownian motion process.

Let $F(t, X(t))$ be a function with continuous second derivatives, where F and X have a functional dependence. Then $F(t, X(t))$ is also an Itô process and has the following dynamics

$$\begin{aligned} dF\big(t, X(t)\big) \quad &= \quad \frac{\partial F}{\partial t}(t,x)\, dt + \frac{\partial F}{\partial x}(t,x)\, dX(t) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t,x)\, \big(dX(t)\big)^2 \\ &= \quad \left(\frac{\partial F}{\partial t}(t,x) + a(t,x)\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}b^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x)\right) dt + b(t,x)\frac{\partial F}{\partial x}(t,x)\, dW(t). \end{aligned}$$

Hence, F is also an Itô process, but with adjusted drift rate given by $\frac{\partial F}{\partial t}(t,x) + a(t,x)\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}b^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x)$ and a scaled variance, $b^2(t,x)\frac{\partial F}{\partial x}(t,x)$.

Appendix C

Auxiliary Proofs of Selected Results

C.1 An Alternative Proof of Proposition 2.2.20

In this proof we set out to show that (2.2.65) holds given the knowledge of (2.2.64).

Proof. Let us apply Itô's formula to $\log Z_\pi(t)$. So, by letting $Z_\pi(t) = Y(t)$, the form of the function to be used in Itô's formula is given by $F\big(t, Y(t)\big) = \log\big(Y(t)\big)$, and the following are easily obtained

$$\begin{aligned}
\frac{\partial F}{\partial t}(t,y) &= 0, \\
\frac{\partial F}{\partial y}(t,y) &= \frac{1}{Y(t)}, \\
\frac{\partial^2 F}{\partial y^2}(t,y) &= -\frac{1}{Y^2(t)}.
\end{aligned}$$

We then arrive at the following for $t \in [0,\infty)$, a.s.,

$$dF\big(t, Y(t)\big) = \frac{1}{Y(t)}\, dY(t) - \frac{1}{2Y^2(t)}\, d\langle Y\rangle_t.$$

Since, $Y(t) = Z_\pi(t)$, the following formula for $d\log Z_\pi(t)$ is obtained

$$\begin{aligned}
d\log Z_\pi(t) &= \frac{dZ_\pi(t)}{Z_\pi(t)} - \frac{1}{2\big(Z_\pi(t)\big)^2}\, d\big\langle Z_\pi\big\rangle_t \\
&= \sum_{i=1}^n \pi_i(t)\, \frac{dX_i(t)}{X_i(t)} - \frac{1}{2\big(Z_\pi(t)\big)^2}\, d\big\langle Z_\pi\big\rangle_t, \qquad \text{(C.1.1)}
\end{aligned}$$

where (C.1.1) follows from (2.2.64). Recall, that another application of Itô's formula subsequently yields

$$\begin{aligned}
dZ_\pi(t) &= Z_\pi(t) \sum_{i=1}^n \pi_i(t)\, \frac{dX_i(t)}{X_i(t)} \\
&= Z_\pi(t) \sum_{i=1}^n \pi_i(t) \left[\alpha_i(t)\, dt + \sum_{\nu=1}^n \xi_{i\nu}(t)\, dW_\nu(t)\right] \\
&= Z_\pi(t) \sum_{i=1}^n \pi_i(t) \left[\left(\gamma_i(t) + \frac{1}{2}\,\sigma_{ii}(t)\right) dt + \sum_{\nu=1}^n \xi_{i\nu}(t)\, dW_\nu(t)\right] \\
&= Z_\pi(t) \sum_{i=1}^n \pi_i(t) \left(\gamma_i(t) + \frac{1}{2}\,\sigma_{ii}(t)\right) dt + Z_\pi(t) \sum_{i,\nu=1}^n \pi_i(t)\xi_{i\nu}(t)\, dW_\nu(t).
\end{aligned}$$

Thus, the quadratic variation of the process, Z_π, is obtained as follows

$$\begin{aligned}
\langle Z_\pi \rangle_t &= \left\langle \int_0^\cdot Z_{\pi,s} \sum_{i,\nu=1}^n \pi_{i,s}\xi_{i\nu,s}\, dW_{\nu,s} \right\rangle_t \\
&= \left\langle \sum_{\nu=1}^n \int_0^\cdot Z_{\pi,s} \sum_{i=1}^n \pi_{i,s}\xi_{i\nu,s}\, dW_{\nu,s} \right\rangle_t \\
&= \sum_{\nu=1}^n \left\langle \int_0^\cdot Z_{\pi,s} \sum_{i=1}^n \pi_{i,s}\xi_{i\nu,s}\, dW_{\nu,s} \right\rangle_t \\
&= \sum_{\nu=1}^n \int_0^t \big(Z_\pi(s)\big)^2 \left(\sum_{i=1}^n \pi_i(s)\xi_{i\nu}(s) \right)^2 d\langle W_\nu \rangle_s \\
&= \sum_{\nu=1}^n \int_0^t \big(Z_\pi(s)\big)^2 \sum_{i,j=1}^n \pi_i(s)\pi_j(s)\xi_{i\nu}(s)\xi_{j\nu}(s)\, ds \\
&= \int_0^t \big(Z_\pi(s)\big)^2 \sum_{i,j=1}^n \pi_i(s)\pi_j(s) \sum_{\nu=1}^n \xi_{i\nu}(s)\xi_{j\nu}(s)\, ds \\
&= \int_0^t \big(Z_\pi(s)\big)^2 \sum_{i,j=1}^n \pi_i(s)\sigma_{ij}(s)\pi_j(s)\, ds. \qquad \text{(C.1.2)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
d\langle Z_\pi \rangle_t &= d\left\langle \int_0^\cdot Z_{\pi,s} \sum_{i,\nu=1}^n \pi_{i,s}\xi_{i\nu,s}\, dW_{\nu,s} \right\rangle_t \\
&= d\left(\int_0^t \big(Z_\pi(s)\big)^2 \sum_{i,j=1}^n \pi_i(s)\sigma_{ij}(s)\pi_j(s)\, ds \right) \\
&= \big(Z_\pi(t)\big)^2 \sum_{i,j=1}^n \pi_i(t)\sigma_{ij}(t)\pi_j(t)\, dt. \qquad \text{(C.1.3)}
\end{aligned}$$

So, by substituting (C.1.3) into (C.1.1), we arrive at the following

$$d\log Z_\pi(t) = \sum_{i=1}^n \pi_i(t)\, \frac{dX_i(t)}{X_i(t)} - \frac{1}{2} \sum_{i,j=1}^n \pi_i(t)\sigma_{ij}(t)\pi_j(t)\, dt.$$

Recalling (2.2.91) allows us to rewrite the above expression in the following form

$$\begin{aligned}
d\log Z_\pi(t) &= \sum_{i=1}^n \pi_i(t) \left[\alpha_i(t)\, dt + \sum_{\nu=1}^n \xi_{i\nu}(t)\, dW_\nu(t) \right] - \frac{1}{2} \sum_{i,j=1}^n \pi_i(t)\sigma_{ij}(t)\pi_j(t)\, dt \\
&= \sum_{i=1}^n \pi_i(t)\alpha_i(t)\, dt - \frac{1}{2} \sum_{i,j=1}^n \pi_i(t)\sigma_{ij}(t)\pi_j(t)\, dt + \sum_{i,\nu=1}^n \pi_i(t)\xi_{i\nu}(t)\, dW_\nu(t),
\end{aligned}$$

which by (2.2.88) can be rewritten as follows

$$\begin{aligned} d\log Z_\pi(t) &= \sum_{i=1}^n \pi_i(t)\Big(\gamma_i(t)+\frac{1}{2}\,\sigma_{ii}(t)\Big)\,dt - \frac{1}{2}\sum_{i,j=1}^n \pi_i(t)\sigma_{ij}(t)\pi_j(t)\,dt + \sum_{i,\nu=1}^n \pi_i(t)\xi_{i\nu}(t)\,dW_\nu(t) \\ &= \sum_{i=1}^n \pi_i(t)\gamma_i(t)\,dt + \frac{1}{2}\sum_{i=1}^n \pi_i(t)\sigma_{ii}(t)\,dt - \frac{1}{2}\sum_{i,j=1}^n \pi_i(t)\sigma_{ij}(t)\pi_j(t)\,dt \\ &\quad + \sum_{i,\nu=1}^n \pi_i(t)\xi_{i\nu}(t)\,dW_\nu(t) \end{aligned} \tag{C.1.4}$$

$$\begin{aligned} &= \sum_{i=1}^n \pi_i(t)\gamma_i(t)\,dt + \frac{1}{2}\left(\sum_{i=1}^n \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^n \pi_i(t)\sigma_{ij}(t)\pi_j(t)\right)dt \\ &\quad + \sum_{i,\nu=1}^n \pi_i(t)\xi_{i\nu}(t)\,dW_\nu(t). \end{aligned} \tag{C.1.5}$$

This, in conjunction with the notation offered in (2.2.66), yields the desired result

$$d\log Z_\pi(t) = \gamma_\pi(t)\,dt + \sum_{i,\nu=1}^n \pi_i(t)\xi_{i\nu}(t)\,dW_\nu(t). \tag{C.1.6}$$

Remark C.1.1. Note, from (C.1.3) and (2.2.75) that

$$\frac{d\left\langle Z_\pi\right\rangle_t}{\big(Z_\pi(t)\big)^2} = d\left\langle \log Z_\pi\right\rangle_t = \sum_{i,j=1}^n \pi_i(t)\sigma_{ij}(t)\pi_j(t)\,dt = \sigma_{\pi\pi}(t)\,dt, \tag{C.1.7}$$

this parallels the result obtained for the market portfolio, (2.12.50),

$$\frac{d\left\langle \mu_i\right\rangle_t}{\mu_i^2(t)} = d\left\langle \log \mu_i\right\rangle_t. \tag{C.1.8}$$

■

C.2 An Alternative Formulation and Proof of Lemma 2.4.14

The following lemma is taken and adapted from Fernholz (1999a, Lemma 2.4).

Lemma C.2.1 ([**Fernholz (1999a, Lemma 2.4)**]). *Let π be a portfolio with nonnegative weights (i.e., $0 \le \pi_i(t) < 1$ for all $i = 1,\dots,n$) in a nondegenerate market. Then there exists an $\varepsilon > 0$ such that for all $i = 1,\dots,n$,*

$$\gamma_\pi^*(t) \ge \frac{\varepsilon}{2}\big(\pi_i(t)-\pi_i^2(t)\big) = \frac{\varepsilon}{2}\pi_i(t)\big(1-\pi_i(t)\big), \qquad t\in[0,\infty), \qquad a.s. \tag{C.2.1}$$

We provide a concise version of the proof given in Fernholz (1999a), which is analogous to the proof of Lemma 2.4.15.

Proof. Firstly, since we assume that the weights of the portfolio π are nonnegative (i.e., we exclusively consider long-only portfolios), we have for all $i = 1,\dots,n$ and $t \in [0,\infty)$, $0 \le \pi_i(t) < 1$. Furthermore, the market is assumed to exhibit strong nondegeneracy and consequently satisfies the expression (2.2.55) provided in Definition 2.2.13. Thus, we can choose $\varepsilon > 0$ such that

$$\mathbf{x}\boldsymbol{\sigma}(t)\mathbf{x}^\mathsf{T} \ge \varepsilon\,\|\mathbf{x}\|^2, \qquad \mathbf{x}\in\mathbb{R}^n, \qquad t\in[0,\infty), \qquad \text{a.s.} \tag{C.2.2}$$

For any integer k, $k = 1,\dots,n$, the long-only condition on the portfolio suggests that $\pi_k(t) < 1$. Let η be defined as in (2.4.60). Since $\pi_k(t) < 1$ by our assumption, this implies that $1-\pi_k(t) > 0$. Recall that from Lemma

2.4.15, $\eta = \big\{\boldsymbol{\eta}(t) = \big(\eta_1(t), \ldots, \eta_n(t)\big), t \in [0, \infty)\big\}$ defines a portfolio, in conformity with Definition 2.2.16, with nonnegative weights. Recall (2.4.29) we thus obtain a.s., for $t \in [0, \infty)$

$$\sum_{i=1}^{n} \eta_i(t)\sigma_{ii}(t) - \sigma_{\eta\eta}(t) \quad = \quad 2\gamma_\eta^*(t) \quad \geq \quad 0, \tag{C.2.3}$$

by Proposition 2.4.8. Now, let $\mathbf{e}_k$ be the kth unit vector in $\mathbb{R}^n$, i.e., $\mathbf{e}_k = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 is in the kth position. Consider, for $1 \leq k \leq n$, and $t \in [0, \infty)$, the vector $\mathbf{x}(t) \triangleq \boldsymbol{\eta}(t) - \mathbf{e}_k$, i.e.,

$$\begin{aligned}
\mathbf{x}(t) \quad &\triangleq \quad \Big(\eta_1(t), \ldots, \eta_{k-1}(t), \eta_k(t) - 1, \eta_{k+1}(t), \ldots, \eta_n(t)\Big), \\
&= \quad \Big(\eta_1(t), \ldots, \eta_{k-1}(t), -1, \eta_{k+1}(t), \ldots, \eta_n(t)\Big),
\end{aligned}$$

i.e., with the -1 in the kth position. Then

$$\tau_{kk}^\eta(t) \quad = \quad \sigma_{kk}(t) - 2\sigma_{k\eta}(t) + \sigma_{\eta\eta}(t). \tag{C.2.4}$$

Recall the strong nondegeneracy condition in Definition 2.2.13, given here in this proof as (C.2.2). Thus, we have for $t \in [0, \infty)$, a.s.,

$$\tau_{kk}^\eta(t) \quad = \quad \Big(\boldsymbol{\eta}(t) - \mathbf{e}_k\Big)\boldsymbol{\sigma}(t)\Big(\boldsymbol{\eta}(t) - \mathbf{e}_k\Big)^\mathsf{T} \quad = \quad \mathbf{x}(t)\boldsymbol{\sigma}(t)\mathbf{x}^\mathsf{T}(t) \quad \geq \quad \varepsilon \left\|\mathbf{x}(t)\right\|^2.$$

Since $\mathbf{x}(t) = \boldsymbol{\eta}(t) - \mathbf{e}_k$,

$$\begin{aligned}
\left\|\mathbf{x}(t)\right\|^2 \quad = \quad \left\|\boldsymbol{\eta}(t) - \mathbf{e}_k\right\|^2 \quad &= \quad \eta_1^2(t) + \cdots + \eta_{k-1}^2(t) + \big(\eta_k(t) - 1\big)^2 + \eta_{k+1}^2(t) + \cdots + \eta_n^2(t) \\
&= \quad \sum_{i \neq k}^{n} \eta_i^2(t) + \big(\eta_k(t) - 1\big)^2 \\
&= \quad \sum_{i \neq k}^{n} \eta_i^2(t) + (-1)^2 \\
&= \quad \sum_{i \neq k}^{n} \eta_i^2(t) + 1.
\end{aligned}$$

Therefore,

$$\left\|\mathbf{x}(t)\right\|^2 \quad = \quad \left\|\boldsymbol{\eta}(t) - \mathbf{e}_k\right\|^2 \quad \geq \quad 1.$$

So, for $k = 1, \ldots, n$, $t \in [0, \infty)$, a.s.,

$$\tau_{kk}^\eta(t) \quad \geq \quad \varepsilon \left\|\mathbf{x}(t)\right\|^2 \quad = \quad \varepsilon \left\|\boldsymbol{\eta}(t) - \mathbf{e}_k\right\|^2 \quad \geq \quad \varepsilon. \tag{C.2.5}$$

From (2.4.71), a.s., for $t \in [0, \infty)$, we have for $k = 1, \ldots, n$,

$$\begin{aligned}
2\gamma_\pi^*(t) \quad &= \quad \pi_k(t)\big(1 - \pi_k(t)\big)\Big(\sigma_{kk}(t) - 2\sigma_{k\eta}(t) + \sigma_{\eta\eta}(t)\Big) + \big(1 - \pi_k(t)\big)\left(\sum_{i=1}^{n} \eta_i(t)\sigma_{ii}(t) - \sigma_{\eta\eta}(t)\right) \\
&= \quad \pi_k(t)\big(1 - \pi_k(t)\big)\big(\tau_{kk}^\eta(t)\big) + \big(1 - \pi_k(t)\big)\big(2\gamma_\eta^*(t)\big) \\
&\geq \quad \varepsilon\pi_k(t)\big(1 - \pi_k(t)\big) \qquad \text{(C.2.6)} \\
&= \quad \varepsilon\big(\pi_k(t) - \pi_k^2(t)\big), \qquad \text{(C.2.7)}
\end{aligned}$$

where (C.2.6) follows from (C.2.3) and the fact that $1 - \pi_k(t) > 0$, and (C.2.5). Since k, $k = 1, \ldots, n$, was arbitrary, we obtain the required result (C.2.1) for all $i = 1, \ldots, n$,

$$\gamma_\pi^*(t) \quad \geq \quad \frac{\varepsilon}{2}\big(\pi_i(t) - \pi_i^2(t)\big).$$

■

C.3 An Alternative Proof of Lemma 2.12.9

Proof. From (2.12.35), we have

$$d\mu_i(t) = \mu_i(t)\, d\log\mu_i(t) + \frac{1}{2}\,\mu_i(t)\,\tau_{ii}(t)\,dt,$$

which when rearranged and altered with the summation gives the following

$$\begin{aligned}\sum_{i=1}^{n}\mu_i(t)\, d\log\mu_i(t) &= \sum_{i=1}^{n} d\mu_i(t) - \frac{1}{2}\sum_{i=1}^{n}\mu_i(t)\tau_{ii}(t)\,dt \\ &= d\left(\sum_{i=1}^{n}\mu_i(t)\right) - \frac{1}{2}\sum_{i=1}^{n}\mu_i(t)\tau_{ii}(t)\,dt \\ &= -\frac{1}{2}\sum_{i=1}^{n}\mu_i(t)\tau_{ii}(t)\,dt. \end{aligned} \tag{C.3.1}$$

Recall (2.4.32) of Lemma 2.4.7, by setting $\pi := \mu$, we obtain

$$\gamma_\mu^*(t) = \frac{1}{2}\sum_{i=1}^{n}\mu_i(t)\tau_{ii}^{\mu}(t) = \frac{1}{2}\sum_{i=1}^{n}\mu_i(t)\tau_{ii}(t),$$

which when substituted into (C.3.1) gives

$$\sum_{i=1}^{n}\mu_i(t)\, d\log\mu_i(t) = -\gamma_\mu^*(t)\,dt.$$

Following the remainder of the proof of Lemma 2.12.9 yields the required result. ■

C.4 An Alternative Proof of Theorem 5.6.2

The following proof of Theorem 5.6.2 is taken and adapted from Fernholz (1999a, Proof of Theorem 4.1).

Proof. An application of Itô's formula to $\log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)$ yields

$$d\log \mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) = \frac{d\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} - \frac{1}{2\left(\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)\right)^2}\, d\left\langle \mathbf{S}^{\mathrm{E}}(\boldsymbol{\mu})\right\rangle_t. \tag{C.4.1}$$

By (4.6.2) of Definition 4.6.1, we have

$$\begin{aligned} d\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) = d\left(-\sum_{i=1}^{n}\mu_i(t)\log\mu_i(t)\right) &= -\sum_{i=1}^{n} d\Big(\mu_i(t)\log\mu_i(t)\Big) \\ &= -\sum_{i=1}^{n}\Big(\mu_i(t)\, d\log\mu_i(t) + \log\mu_i(t)\, d\mu_i(t) + d\left\langle \mu_i, \log\mu_i\right\rangle_t\Big), \end{aligned}$$

which follows from Itô's formula. Now, from (2.12.35), we have

$$d\mu_i(t) = \mu_i(t)\, d\log\mu_i(t) + \frac{1}{2}\,\mu_i(t)\,\tau_{ii}(t)\,dt, \tag{C.4.2}$$

so that

$$\mu_i(t)\, d\log\mu_i(t) = d\mu_i(t) - \frac{1}{2}\,\mu_i(t)\,\tau_{ii}(t)\,dt. \tag{C.4.3}$$

Moreover, from (2.12.35) we have

$$\begin{aligned}\left\langle \mu_i, \log\mu_i\right\rangle_t &= \left\langle \int_0^{\cdot}\mu_{i,s}\, d\log\mu_{i,s}, \int_0^{\cdot} d\log\mu_{i,s}\right\rangle_t \\ &= \int_0^t \mu_i(s)\, d\left\langle \log\mu_i\right\rangle_s \\ &= \int_0^t \mu_i(s)\tau_{ii}(s)\,ds, \end{aligned} \tag{C.4.4}$$

where (C.4.4) follows from (2.12.27). Thus, we equivalently have

$$d\left\langle \mu_i, \log \mu_i \right\rangle_t \quad = \quad \mu_i(t)\tau_{ii}(t)\,dt. \tag{C.4.5}$$

Now, employing (C.4.3) and (C.4.5), gives

$$\begin{aligned}
d\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) &= -\sum_{i=1}^{n} \mu_i(t)\,d\log\mu_i(t) - \sum_{i=1}^{n} \log\mu_i(t)\,d\mu_i(t) - \sum_{i=1}^{n} d\left\langle \mu_i, \log\mu_i \right\rangle_t \\
&= -\sum_{i=1}^{n} \left[d\mu_i(t) - \frac{1}{2}\,\mu_i(t)\,\tau_{ii}(t)\,dt \right] - \sum_{i=1}^{n} \log\mu_i(t)\,d\mu_i(t) - \sum_{i=1}^{n} \mu_i(t)\tau_{ii}(t)\,dt \\
&= -\sum_{i=1}^{n} d\mu_i(t) - \sum_{i=1}^{n} \log\mu_i(t)\,d\mu_i(t) + \frac{1}{2}\sum_{i=1}^{n} \mu_i(t)\tau_{ii}(t)\,dt - \sum_{i=1}^{n} \mu_i(t)\tau_{ii}(t)\,dt \\
&= -\sum_{i=1}^{n} d\mu_i(t) - \sum_{i=1}^{n} \log\mu_i(t)\,d\mu_i(t) - \frac{1}{2}\sum_{i=1}^{n} \mu_i(t)\tau_{ii}(t)\,dt \\
&= -d\left(\sum_{i=1}^{n} \mu_i(t) \right) - \sum_{i=1}^{n} \log\mu_i(t)\,d\mu_i(t) - \frac{1}{2}\sum_{i=1}^{n} \mu_i(t)\tau_{ii}(t)\,dt.
\end{aligned}$$

Thus, by (2.4.32) of Lemma 2.4.7, or by equations (2.4.36) and (2.12.95), we have

$$d\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) \quad = \quad -\sum_{i=1}^{n} \log\mu_i(t)\,d\mu_i(t) - \gamma_\mu^*(t)\,dt,$$

since $\sum_{i=1}^{n} d\mu_i(t) = d\left(\sum_{i=1}^{n} \mu_i(t)\right) = 0$. Now, appealing to (C.4.2) gives

$$\begin{aligned}
d\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) &= -\sum_{i=1}^{n} \log\mu_i(t) \left[\mu_i(t)\,d\log\mu_i(t) + \frac{1}{2}\,\mu_i(t)\,\tau_{ii}(t)\,dt \right] - \gamma_\mu^*(t)\,dt \\
&= -\sum_{i=1}^{n} \mu_i(t)\log\mu_i(t)\,d\log\mu_i(t) - \frac{1}{2}\sum_{i=1}^{n} \mu_i(t)\log\mu_i(t)\tau_{ii}(t)\,dt - \gamma_\mu^*(t)\,dt.
\end{aligned} \tag{C.4.6}$$

Therefore,

$$\frac{d\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \quad = \quad \sum_{i=1}^{n} \left[\frac{-\mu_i(t)\log\mu_i(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \right] d\log\mu_i(t) + \frac{1}{2}\sum_{i=1}^{n} \left[\frac{-\mu_i(t)\log\mu_i(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \right] \tau_{ii}(t)\,dt - \frac{\gamma_\mu^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\,dt,$$

which by (5.6.3) of Definition 5.6.1 simplifies to

$$\frac{d\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)} \quad = \quad \sum_{i=1}^{n} \varphi_i^{\mathsf{e}}(t)\,d\log\mu_i(t) + \frac{1}{2}\sum_{i=1}^{n} \varphi_i^{\mathsf{e}}(t)\tau_{ii}(t)\,dt - \frac{\gamma_\mu^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\,dt. \tag{C.4.7}$$

Furthermore, from (C.4.6), we obtain

$$\begin{aligned}
\left\langle \mathbf{S}^{\mathrm{E}}(\boldsymbol{\mu}) \right\rangle_t &= \left\langle -\int_0^{\cdot} \sum_{i=1}^{n} \mu_{i,s}\log\mu_{i,s}\,d\log\mu_{i,s}, -\int_0^{\cdot} \sum_{j=1}^{n} \mu_{j,s}\log\mu_{j,s}\,d\log\mu_{j,s} \right\rangle_t \\
&= \left\langle \sum_{i=1}^{n} \int_0^{\cdot} \mu_{i,s}\log\mu_{i,s}\,d\log\mu_{i,s}, \sum_{j=1}^{n} \int_0^{\cdot} \mu_{j,s}\log\mu_{j,s}\,d\log\mu_{j,s} \right\rangle_t \\
&= \sum_{i,j=1}^{n} \left\langle \int_0^{\cdot} \mu_{i,s}\log\mu_{i,s}\,d\log\mu_{i,s}, \int_0^{\cdot} \mu_{j,s}\log\mu_{j,s}\,d\log\mu_{j,s} \right\rangle_t \\
&= \sum_{i,j=1}^{n} \int_0^{t} \mu_i(s)\mu_j(s)\log\mu_i(s)\log\mu_j(s)\,d\left\langle \log\mu_i, \log\mu_j \right\rangle_s \\
&= \int_0^{t} \sum_{i,j=1}^{n} \mu_i(s)\mu_j(s)\log\mu_i(s)\log\mu_j(s)\tau_{ij}(s)\,ds,
\end{aligned} \tag{C.4.8}$$

where (C.4.8) follows from (2.12.25), so that

$$\begin{aligned} d\left\langle \mathbf{S}^{\mathrm{E}}(\boldsymbol{\mu})\right\rangle_t &= d\left\langle -\int_0^{\cdot}\sum_{i=1}^n \mu_{i,s}\log\mu_{i,s}\,d\log\mu_{i,s}, -\int_0^{\cdot}\sum_{j=1}^n \mu_{j,s}\log\mu_{j,s}\,d\log\mu_{j,s}\right\rangle_t \\ &= d\left(\int_0^t \sum_{i,j=1}^n \mu_i(s)\mu_j(s)\log\mu_i(s)\log\mu_j(s)\tau_{ij}(s)\,ds\right) \\ &= \sum_{i,j=1}^n \mu_i(t)\mu_j(t)\log\mu_i(t)\log\mu_j(t)\tau_{ij}(t)\,dt. \end{aligned} \tag{C.4.9}$$

Now, substituting (C.4.7) and (C.4.9) into (C.4.1), we arrive at the following

$$\begin{aligned} d\log\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) &= \sum_{i=1}^n \varphi_i^{\mathrm{e}}(t)\,d\log\mu_i(t) + \frac{1}{2}\sum_{i=1}^n \varphi_i^{\mathrm{e}}(t)\tau_{ii}(t)\,dt - \frac{\gamma_\mu^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\,dt \\ &\quad - \frac{1}{2\left(\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)\right)^2}\sum_{i,j=1}^n \mu_i(t)\mu_j(t)\log\mu_i(t)\log\mu_j(t)\tau_{ij}(t)\,dt \\ &= \sum_{i=1}^n \varphi_i^{\mathrm{e}}(t)\,d\log\mu_i(t) + \frac{1}{2}\sum_{i=1}^n \varphi_i^{\mathrm{e}}(t)\tau_{ii}(t)\,dt - \frac{\gamma_\mu^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\,dt \\ &\quad - \frac{1}{2}\sum_{i,j=1}^n \left[\frac{-\mu_i(t)\log\mu_i(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\right]\left[\frac{-\mu_j(t)\log\mu_j(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\right]\tau_{ij}(t)\,dt \\ &= \sum_{i=1}^n \varphi_i^{\mathrm{e}}(t)\,d\log\mu_i(t) + \frac{1}{2}\sum_{i=1}^n \varphi_i^{\mathrm{e}}(t)\tau_{ii}(t)\,dt - \frac{\gamma_\mu^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\,dt \\ &\quad - \frac{1}{2}\sum_{i,j=1}^n \varphi_i^{\mathrm{e}}(t)\varphi_j^{\mathrm{e}}(t)\tau_{ij}(t)\,dt, \end{aligned}$$

where the last expression follows from (5.6.3) of Definition 5.6.1. Rearranging the last expression gives

$$\begin{aligned} d\log\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) &= \sum_{i=1}^n \varphi_i^{\mathrm{e}}(t)\,d\log\mu_i(t) + \frac{1}{2}\sum_{i=1}^n \varphi_i^{\mathrm{e}}(t)\tau_{ii}(t)\,dt - \frac{1}{2}\sum_{i,j=1}^n \varphi_i^{\mathrm{e}}(t)\varphi_j^{\mathrm{e}}(t)\tau_{ij}(t)\,dt - \frac{\gamma_\mu^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\,dt \\ &= \sum_{i=1}^n \varphi_i^{\mathrm{e}}(t)\,d\log\mu_i(t) + \frac{1}{2}\left(\sum_{i=1}^n \varphi_i^{\mathrm{e}}(t)\tau_{ii}(t) - \sum_{i,j=1}^n \varphi_i^{\mathrm{e}}(t)\tau_{ij}(t)\varphi_j^{\mathrm{e}}(t)\right)dt - \frac{\gamma_\mu^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\,dt \\ &= \sum_{i=1}^n \varphi_i^{\mathrm{e}}(t)\,d\log\mu_i(t) + \gamma_{\varphi^{\mathrm{e}}}^*(t)\,dt - \frac{\gamma_\mu^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\,dt, \end{aligned}$$

where the last expression follows from (2.4.26) of Lemma 2.4.5. Now, from the first term in the last expression, we obtain

$$\begin{aligned} \sum_{i=1}^n \varphi_i^{\mathrm{e}}(t)\,d\log\mu_i(t) &= \sum_{i=1}^n \varphi_i^{\mathrm{e}}(t)\,d\log\Big(X_i(t)/Z_\mu(t)\Big) \\ &= \sum_{i=1}^n \varphi_i^{\mathrm{e}}(t)\Big[d\log X_i(t) - d\log Z_\mu(t)\Big] \\ &= \sum_{i=1}^n \varphi_i^{\mathrm{e}}(t)\,d\log X_i(t) - d\log Z_\mu(t)\left(\sum_{i=1}^n \varphi_i^{\mathrm{e}}(t)\right) \\ &= d\log Z_{\varphi^{\mathrm{e}}}(t) - \gamma_{\varphi^{\mathrm{e}}}^*(t)\,dt - d\log Z_\mu(t), \end{aligned}$$

which follows from (2.2.112) of Corollary 2.2.25. Hence, we have a.s. for $t \in [0,T]$,

$$\begin{aligned} d\log\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big) &= d\log Z_{\varphi^{\mathrm{e}}}(t) - d\log Z_\mu(t) - \gamma_{\varphi^{\mathrm{e}}}^*(t)\,dt + \gamma_{\varphi^{\mathrm{e}}}^*(t)\,dt - \frac{\gamma_\mu^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\,dt \\ &= d\log\Big(Z_{\varphi^{\mathrm{e}}}(t)/Z_\mu(t)\Big) - \frac{\gamma_\mu^*(t)}{\mathbf{S}^{\mathrm{E}}\big(\boldsymbol{\mu}(t)\big)}\,dt. \end{aligned}$$

■

Appendix D

Higher Order Derivatives

Here we shall provide a brief discussion of higher order derivatives. Let $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$ be a differentiable real-valued function defined on an open set U in $\mathbb{R}^n$. Its derivative $f'(\mathbf{x})$ depends on a particular point $\mathbf{x}$ for which the limit is taken, so that it is, in general, a function of $\mathbf{x}$. Given the function $g(\mathbf{x}) \equiv f'(\mathbf{x})$, we may define its derivative $g'(\mathbf{x})$ (when it exists). We denote this by $f''(\mathbf{x})$, and we call $f''(\mathbf{x})$ the *second-order derivative* of f at $\mathbf{x}$. Again, in general, $f''(\mathbf{x})$ depends on $\mathbf{x}$ and thus it is a function of $\mathbf{x}$. Repeating this process we may analogously define the *kth order derivative* of f at $\mathbf{x}$ (when it exists), and we denote it by $f^{(k)}(\mathbf{x})$. If $f''(\mathbf{x})$ exists, f is said to be *twice differentiable* at $\mathbf{x}$. Similarly, if $f^{(k)}(\mathbf{x})$ exists, f is *k-times differentiable* at $\mathbf{x}$. $f'(\mathbf{x})$ and $f''(\mathbf{x})$ are the conventional notations for $f^{(1)}(\mathbf{x})$ and $f^{(2)}(\mathbf{x})$, respectively. If $f^{(k)}(\mathbf{x})$ is differentiable and its derivative $f^{(k+1)}(\mathbf{x})$ is continuous for each $\mathbf{x}$ in U (i.e., if $f^{(k)}$ is continuously differentiable), then f is said to be a function of class C^{k+1} (on the domain U). Thus, if $f \in C^1$, f is continuously differentiable. If $f \in C^2$, f is said to be a *twice continuously differentiable function*. Recall that $f'(\mathbf{x}) = \nabla f(\mathbf{x}) = \left[\frac{\partial f(\mathbf{x})}{\partial x_1}, \ldots, \frac{\partial f(\mathbf{x})}{\partial x_n}\right]$, when $f : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$. Note that $g_i(\mathbf{x}) \equiv \frac{\partial f(\mathbf{x})}{\partial x_i}$ for $i = 1, \ldots, n$, is also a real-valued function. Then define its partial derivatives in a similar fashion (if they exist), and denote these partials by $\frac{\partial g_i(\mathbf{x})}{\partial x_j} = \frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j}$, etc. They are called *second-order partial derivatives* of f at $\mathbf{x}$. The partial derivatives of higher orders, $k = 3, 4, \ldots$, can be defined analogously.

Definition D.0.1 (Continuously Differentiable Function)**.** *A real-valued function of n real variables, $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$, defined on an open subset U of $\mathbb{R}^n$, i.e., $f : \mathbb{R}^n \to \mathbb{R}$ or $f : U \to \mathbb{R}$ for $U \subset \mathbb{R}^n$, is said to be* ***continuously differentiable*** *at $\mathbf{x} \in \mathbb{R}^n$, if the first-order partial derivatives with respect to the ith variable, i.e., $\frac{\partial f(\mathbf{x})}{\partial x_i}$, for $i = 1, \ldots, n$, exist and are continuous. Moreover, a function f is said to be continuously differentiable in an open region $U \subset \mathbb{R}^n$, denoted by $f \in C^1(U)$, if it is continuously differentiable at every point $\mathbf{x}$ in U. Thus, a real-valued function defined on an open subset U of $\mathbb{R}^n$ is of class C^1, $f \in C^1$, if it is continuously differentiable in all n variables.*

Appendix E

Concave and Convex Functions

E.1 General Concave and Convex Functions

The inequality (4.5.3) admits the following generalisation, which can be established by means of induction.

Lemma E.1.1 (General Concave (or Strictly Concave) Function)**.** *The function $f : \mathbb{R}^n \to \mathbb{R}$ or $f : U \to \mathbb{R}$, where $U \subset \mathbb{R}^n$ and U is convex in $\mathbb{R}^n$, is* ***concave*** *(resp.,* ***strictly concave****) if and only if for every integer $k \geq 1$,*

$$f\Big(\theta_1\mathbf{x}_1 + \theta_2\mathbf{x}_2 + \cdots + \theta_k\mathbf{x}_k\Big) \quad \geq \quad \theta_1 f(\mathbf{x}_1) + \theta_2 f(\mathbf{x}_2) + \cdots + \theta_k f(\mathbf{x}_k), \tag{E.1.1}$$

$$\Bigg(\textit{resp.,} \quad f\Big(\theta_1\mathbf{x}_1 + \theta_2\mathbf{x}_2 + \cdots + \theta_k\mathbf{x}_k\Big) \quad > \quad \theta_1 f(\mathbf{x}_1) + \theta_2 f(\mathbf{x}_2) + \cdots + \theta_k f(\mathbf{x}_k)\Bigg), \tag{E.1.2}$$

for all $\mathbf{x}_i \in U \subset \mathbb{R}^n$, $\theta_i \in \mathbb{R}$, $\theta_i \geq 0$, $i = 1, 2, \ldots, k$, with $\theta_1 + \theta_2 + \cdots + \theta_k = 1$, i.e., θ lies in the unit $(k-1)-$simplex, $\overline{\Delta}^{k-1}$, defined as in Definition 2.2.16 by

$$\overline{\Delta}^{k-1} \quad \triangleq \quad \Big\{\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k) \in \mathbb{R}^k \;\Big|\; \theta_1 \geq 0, \ldots, \theta_k \geq 0,\ \theta_1 + \cdots + \theta_k = 1\Big\}.$$

(resp., for all $\mathbf{x}_i \neq \mathbf{x}_j$, $i, j = 1, 2, \ldots, k$ and $\theta_i > 0$).

Proof. Let us suppose that $\theta_1 \triangleq \theta$ and $\theta_2 \triangleq 1 - \theta$, are two arbitrary positive real-valued numbers such that $\theta_1 + \theta_2 = 1$, and further suppose that the function $f : \mathbb{R}^n \to \mathbb{R}$ or $f : U \to \mathbb{R}$, where $U \subset \mathbb{R}^n$ and U is convex in $\mathbb{R}^n$ is concave. Then the concavity of f implies that for all $\mathbf{x}_1, \mathbf{x}_2 \in U \subset \mathbb{R}^n$,

$$f\big(\theta_1\mathbf{x}_1 + \theta_2\mathbf{x}_2\big) \quad \geq \quad \theta_1 f(\mathbf{x}_1) + \theta_2 f(\mathbf{x}_2). \tag{E.1.3}$$

Consequently, in congruence with the concavity premise, equation (E.1.1) is true for $k = 2$. Now, let us surmise that the statement specified in (E.1.1) is true for some $k \geq 3$, we need to prove the statement true for $k + 1$. Let $\theta_i \in \mathbb{R}$, $\theta_i \geq 0$, $i = 1, 2, \ldots, k + 1$, with $\theta_1 + \theta_2 + \cdots + \theta_{k+1} = 1$. At least one of the θ_i is strictly positive, say θ_1, therefore by (4.5.3) in Definition 4.5.4, we ascertain the following

$$f\Big(\theta_1\mathbf{x}_1 + \theta_2\mathbf{x}_2 + \cdots + \theta_k\mathbf{x}_k + \theta_{k+1}\mathbf{x}_{k+1}\Big) \quad = \quad f\left(\theta_1\mathbf{x}_1 + \sum_{i=2}^{k+1}\theta_i\mathbf{x}_i\right) \tag{E.1.4}$$

$$= \quad f\left(\theta_1\mathbf{x}_1 + (1-\theta_1)\sum_{i=2}^{k+1}\frac{\theta_i}{1-\theta_1}\mathbf{x}_i\right) \tag{E.1.5}$$

$$\geq \quad \theta_1 f(\mathbf{x}_1) + (1-\theta_1) f\left(\sum_{i=2}^{k+1}\frac{\theta_i}{1-\theta_1}\mathbf{x}_i\right). \tag{E.1.6}$$

Therefore, we have

$$f\left(\sum_{i=1}^{k+1}\theta_i\mathbf{x}_i\right) \quad \geq \quad \theta_1 f(\mathbf{x}_1)+(1-\theta_1)f\left(\sum_{i=2}^{k+1}\frac{\theta_i}{1-\theta_1}\mathbf{x}_i\right). \tag{E.1.7}$$

We can apply the induction hypotheses to the last term in the previous formula (since we assumed the expression in (E.1.1) to be true for some k) to obtain the desired result. This is obtained by noticing the following $\frac{\theta_i}{1-\theta_1}\in\mathbb{R}$, $\frac{\theta_i}{1-\theta_1}\geq 0$ for $i=2,\ldots,k+1$ and $\sum_{i=2}^{k+1}\frac{\theta_i}{1-\theta_1}=1$, which when combined with the assumption, yields

$$f\left(\sum_{i=2}^{k+1}\frac{\theta_i}{1-\theta_1}\mathbf{x}_i\right) \quad \geq \quad \frac{\theta_2}{1-\theta_1}f(\mathbf{x}_2)+\cdots+\frac{\theta_{k+1}}{1-\theta_1}f(\mathbf{x}_{k+1}) \tag{E.1.8}$$

$$= \quad \frac{1}{1-\theta_1}\Big(\theta_2 f(\mathbf{x}_2)+\cdots+\theta_{k+1}f(\mathbf{x}_{k+1})\Big). \tag{E.1.9}$$

Thus, the proof is completed by substituting (E.1.9) into (E.1.7). ■

We shall also offer the allied notion of convexity and that of a convex function.

Definition E.1.2 (Convex Function, Strictly Convex Function). *Let $f(\mathbf{x})=f(x_1,x_2,\ldots,x_n)$ be a real-valued function of n real variables defined on a convex (not necessarily open) subset U of $\mathbb{R}^n$, $U\subset\mathbb{R}^n$, i.e., $f:\mathbb{R}^n\to\mathbb{R}$ or $f:U\to\mathbb{R}$. The function f is called a* ***convex function*** *if for all $0\leq\theta\leq 1$ and $\mathbf{x},\mathbf{y}\in U\subset\mathbb{R}^n$,*[1]

$$f\big(\theta\mathbf{x}+(1-\theta)\mathbf{y}\big) \quad \leq \quad \theta f(\mathbf{x})+(1-\theta)f(\mathbf{y}). \tag{E.1.10}$$

If the inequalities are strict for all $\mathbf{x},\mathbf{y}\in U$, with $\mathbf{x}\neq\mathbf{y}$, and $0<\theta<1$, i.e.,

$$f\big(\theta\mathbf{x}+(1-\theta)\mathbf{y}\big) \quad < \quad \theta f(\mathbf{x})+(1-\theta)f(\mathbf{y}), \tag{E.1.11}$$

then f is called a ***strictly convex function****.*

An alternative definition relating to the differentiability of the function f of a ***single variable*** *is as follows: Let $f(x)$, $f:\mathbb{R}\to\mathbb{R}$ or $f:U\to\mathbb{R}$ be a real-valued continuously differentiable function of a single variable $x\in U\subset\mathbb{R}$, defined on an open convex subset U in $\mathbb{R}$. Then the function f is called a* ***convex function*** *on U if the function f' is an* ***increasing function*** *on U.*

On the other hand, f is called a ***convex*** *(or* ***strictly convex****) function if $-f$ is* ***concave*** *(or* ***strictly concave****).*

The proof of the latter claim made in the preceding definition is trivial and is left as an exercise for the reader. Intuitively, the geometrical interpretation is that f is a convex function if the chord joining any two points on the function lies on or above the function, or if the differentiable function f lies above all of its tangent lines. Clearly, if a function is strictly convex, it is automatically convex.

Lemma E.1.3 (General Convex (or Strictly Convex) Function). *The function $f:\mathbb{R}^n\to\mathbb{R}$ or $f:U\to\mathbb{R}$, where $U\subset\mathbb{R}^n$ and U is convex in $\mathbb{R}^n$, is* ***convex*** *(resp.,* ***strictly convex****) if and only if for every integer $k\geq 1$,*

$$f\Big(\theta_1\mathbf{x}_1+\theta_2\mathbf{x}_2+\cdots+\theta_k\mathbf{x}_k\Big) \quad \leq \quad \theta_1 f(\mathbf{x}_1)+\theta_2 f(\mathbf{x}_2)+\cdots+\theta_k f(\mathbf{x}_k), \tag{E.1.12}$$

$$\left(\textit{resp.}, \quad f\Big(\theta_1\mathbf{x}_1+\theta_2\mathbf{x}_2+\cdots+\theta_k\mathbf{x}_k\Big) \quad < \quad \theta_1 f(\mathbf{x}_1)+\theta_2 f(\mathbf{x}_2)+\cdots+\theta_k f(\mathbf{x}_k)\right), \tag{E.1.13}$$

for all $\mathbf{x}_i\in U\subset\mathbb{R}^n$, $\theta_i\in\mathbb{R}$, $\theta_i\geq 0$, $i=1,2,\ldots,k$, with $\theta_1+\theta_2+\cdots+\theta_k=1$, i.e., θ lies in the unit $(k-1)-$simplex, $\overline{\Delta}^{k-1}$ (resp., for all $\mathbf{x}_i\neq\mathbf{x}_j$, $i,j=1,2,\ldots,k$ and $\theta_i>0$).

[1] More commonly identified as Jensen's inequality: if $f:\mathbb{R}^n\to\mathbb{R}$ is a convex function and $\sum_{i=1}^k\theta_i=1$, $\theta_i\geq 0$ for $i=1,2,\ldots,k$ then $f\left(\sum_{i=1}^k\theta_i\mathbf{x}_i\right)\leq\sum_{i=1}^k\theta_i f(\mathbf{x}_i)$, for all $\mathbf{x}_i\in\mathbb{R}^n$. This is the representation of Jensen's inequality in its discrete finite form as is adopted herein. The more familiar continuous version is as follows: for $\theta(x)\geq 0$ and $\int\theta(x)\,dx=1$ then $f\left(\int\theta(x)x\,dx\right)\leq\int\theta(x)f(x)\,dx$. This form relates the value of a convex function of an integral to the integral of the convex function. In its most general form, Jensen's inequality can be expressed as: if $f:\mathbb{R}^n\to\mathbb{R}$ is a convex function such that the random variables X and $f(X)$ have finite expectation, then $f\big(\mathbb{E}[X]\big)\leq\mathbb{E}\big[f(X)\big]$.

E.2 Some Useful Properties of Concave and Convex Functions

At this stage, we shall briefly digress to impart some of the useful properties of concave functions which will be required to proceed at certain junctures, in particular in Chapter 7. The following selected results are taken from Banner & Fernholz (2008), which should be consulted for a detailed account on this subject.

Lemma E.2.1 ([**Banner & Fernholz (2008)**]). *If f is concave on $[A, B]$, i.e., $f : [A, B] \to \mathbb{R}$, and $a_1, \ldots, a_m \in [A, B]$, then*

$$\sum_{i=1}^{m} f(a_i) \quad \leq \quad m\, f\Big(\frac{1}{m}\sum_{i=1}^{m} a_i\Big). \tag{E.2.1}$$

Proof. From the concavity property (4.5.3), we have for $0 \leq \theta \leq 1$ and a_1, a_2 in the domain of the concave function f,

$$(1-\theta)f(a_1) + \theta f(a_2) \quad \leq \quad f\big((1-\theta)a_1 + \theta a_2\big). \tag{E.2.2}$$

Thus, it follows easily by induction that the concave function on $[A, B]$ satisfies the following inequality (as per Lemma E.1.1, (E.1.1))

$$\sum_{i=1}^{m} w_i f(a_i) \quad \leq \quad f\Big(\sum_{i=1}^{m} w_i a_i\Big), \tag{E.2.3}$$

whenever $a_1, \ldots, a_m \in [A, B]$ and $\mathbf{w} \triangleq (w_1, \ldots, w_m) \in \Delta^{m-1}$. Where, Δ^{m-1} is the open unit $(m-1)-$simplex, defined as in Definition 2.2.16 by

$$\Delta^{m-1} \quad \triangleq \quad \Big\{\mathbf{w} = (w_1, \ldots, w_m) \in \mathbb{R}^m \;\Big|\; w_1 > 0, \ldots, w_m > 0,\; w_1 + \cdots + w_m = 1\Big\}.$$

Thus, by setting $w_i := \frac{1}{m}$ for all $i = 1, \ldots, m$, we obtain the required result

$$\begin{aligned}
\sum_{i=1}^{m} \frac{1}{m} f(a_i) \quad &\leq \quad f\Big(\sum_{i=1}^{m} \frac{1}{m} a_i\Big) \\
\frac{1}{m}\sum_{i=1}^{m} f(a_i) \quad &\leq \quad f\Big(\frac{1}{m}\sum_{i=1}^{m} a_i\Big) \\
\sum_{i=1}^{m} f(a_i) \quad &\leq \quad m f\Big(\frac{1}{m}\sum_{i=1}^{m} a_i\Big).
\end{aligned} \tag{E.2.4}$$

■

Lemma E.2.2 ([**Banner & Fernholz (2008)**]). *Suppose that f is concave on $[A, B]$, $f : [A, B] \to \mathbb{R}$, and $a_1, \ldots, a_m \in [A, B]$ are chosen such that $a - (m-1)A \leq B$, where $a \triangleq \sum_{i=1}^{m} a_i$. Then*

$$\sum_{i=1}^{m} f(a_i) \quad \geq \quad (m-1)f(A) + f\big(a - (m-1)A\big). \tag{E.2.5}$$

Proof. Working off the proof offered in Banner & Fernholz (2008), we shall consider and set $\theta_i := \frac{a_i - A}{a - mA}$. It is evident that $a_i - A \geq 0$ for all $i = 1, \ldots, m$, and $mA \leq a \leq mB$, which implies that $a - mA \geq 0$. Moreover, it is clear that $a_i - A \leq a - mA = (a_1 - A) + \cdots + (a_i - A) + \cdots + (a_m - A)$. Consequently, $\frac{a_i - A}{a - mA} = \theta_i \leq 1$ for all $i = 1, \ldots, m$, this establishes that $0 \leq \theta_i \leq 1$ for all $i = 1, \ldots, m$. In addition, we obtain

$$(1-\theta_i)A + \theta_i\big(a - (m-1)A\big) \quad = \quad a_i.$$

Thus, it follows from the concavity property (E.2.2), with θ replaced by θ_i, that

$$f\Big((1-\theta_i)A + \theta_i\big(a - (m-1)A\big)\Big) \quad \geq \quad (1-\theta_i)f(A) + \theta_i f\big(a - (m-1)A\big) \tag{E.2.6}$$

$$f(a_i) \quad \geq \quad (1-\theta_i)f(A) + \theta_i f\big(a - (m-1)A\big), \tag{E.2.7}$$

for any $i = 1, \ldots, m$. Furthermore, we observe the following

$$\sum_{i=1}^{m} \theta_i \quad = \quad \sum_{i=1}^{m} \frac{a_i - A}{a - mA} \quad = \quad \frac{1}{a - mA} \sum_{i=1}^{m} \big(a_i - A\big) \quad = \quad \frac{1}{a - mA}\big(a - mA\big) \quad = \quad 1.$$

Employing this fact and by adding all m inequalities, we arrive at the following

$$\begin{aligned} \sum_{i=1}^{m} f(a_i) \quad &\geq \quad f(A) \sum_{i=1}^{m} \big(1 - \theta_i\big) + f\big(a - (m-1)A\big) \sum_{i=1}^{m} \theta_i \\ &= \quad f(A)(m-1) + f\big(a - (m-1)A\big), \end{aligned}$$

which establishes (E.2.5). ■

Now, let us consider the case where $\mathbf{x} = (x_1, \ldots, x_n)$, where $\mathbf{x}$ lies in Δ^{n-1},

$$\Delta^{n-1} \quad \triangleq \quad \Big\{ \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \;\Big|\; x_1 > 0, \ldots, x_n > 0,\; x_1 + \cdots + x_n = 1 \Big\}.$$

Moreover, let us adopt the reverse-order-statistics notation and set the following

$$x_{(n)} \quad \triangleq \quad \min_{1 \leq i \leq n} x_i.$$

Note that $0 < x_{(n)} < \frac{1}{n}$. There is no loss of generality in assuming that $x_n = x_{(n)}$. We then have the following lemma adapted from the results obtained in Banner & Fernholz (2008).

Lemma E.2.3. *Suppose that f is concave on $[0,1]$, $f : [0,1] \to \mathbb{R}$, and $x_1, \ldots, x_n \in [0,1]$, then*

$$\sum_{i=1}^{n} f(x_i) \quad \leq \quad f\big(x_{(n)}\big) + (n-1) f\left(\frac{1 - x_{(n)}}{n-1}\right) \quad \leq \quad n f\big(1/n\big). \tag{E.2.8}$$

Proof. By Lemma E.2.1 with $A = 0$, $B = 1$, $m = n - 1$ and $a_i = x_i$ for each $i = 1, \ldots, n-1$ or $i = 1, \ldots, m$ as stipulated above, we have

$$\sum_{i=1}^{n} f(x_i) \quad = \quad f\big(x_{(n)}\big) + \sum_{i=1}^{n-1} f(x_i) \quad \leq \quad f\big(x_{(n)}\big) + (n-1) f\left(\frac{1 - x_{(n)}}{n-1}\right). \tag{E.2.9}$$

Reapplying Lemma E.2.1, this time with $m = n$, $a_1 = \cdots = a_{n-1} = \frac{1 - x_{(n)}}{n-1}$ and $a_n = x_{(n)}$, with this new characterisation of x_i, we have

$$\sum_{i=1}^{n} x_i \quad = \quad (n-1) \frac{1 - x_{(n)}}{n-1} + x_{(n)} \quad = \quad 1,$$

which is consistent with the supposition that $\mathbf{x}$ lies in the open unit $(n-1)-$simplex. We thus obtain

$$f\big(x_{(n)}\big) + (n-1) f\left(\frac{1 - x_{(n)}}{n-1}\right) \quad = \quad \sum_{i=1}^{n} f(x_i) \quad \leq \quad n f\left(\frac{1}{n} \sum_{i=1}^{n} x_i\right) \quad = \quad n f\big(1/n\big). \tag{E.2.10}$$

Taken together, (E.2.9) and (E.2.10), imply (E.2.8). ■

Lemma E.2.4. *Suppose that f is concave on $[0,1]$, $f : [0,1] \to \mathbb{R}$, and $x_1, \ldots, x_n \in [0,1]$, then*

$$\sum_{i=1}^{n} f(x_i) \quad \geq \quad (n-1) f\big(x_{(n)}\big) + f\big(1 - (n-1) x_{(n)}\big) \quad \geq \quad (n-1) f(0) + f(1). \tag{E.2.11}$$

Proof. By applying Lemma E.2.2 with $m = n$, $A = x_{(n)}$, $B = 1$ and $a_i = x_i$ for all $i = 1, \ldots, n$, and noting that $\mathbf{x} \in \Delta^{n-1}$, and thus $a = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} x_i = 1$, gives

$$\sum_{i=1}^{n} f(x_i) \quad \geq \quad (n-1)f\big(x_{(n)}\big) + f\big(1-(n-1)x_{(n)}\big). \tag{E.2.12}$$

Once again, reapplying Lemma E.2.2, this time with $m = n$, $A = 0$, $a_1 = \cdots = a_{n-1} = x_{(n)}$ and $a_n = 1-(n-1)x_{(n)}$, we obtain

$$(n-1)f\big(x_{(n)}\big) + f\big(1-(n-1)x_{(n)}\big) \quad = \quad \sum_{i=1}^{n} f(x_i) \quad \geq \quad (n-1)f(0) + f(1), \tag{E.2.13}$$

this in conjunction with (E.2.12), yields the required result. ■

We shall also require another useful result, regarding both differentiable and concave (or convex) functions, that provides a complete characterisation of concave (or convex) and strictly concave (or strictly convex) functions. This result adapted from Takayama (1994), concerning a multivariable function $f : \mathbb{R}^n \to \mathbb{R}$ or $f : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$, will be stated without proof.

Lemma E.2.5 ([**Takayama (1994)**]). *Let $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$, $f : \mathbb{R}^n \to \mathbb{R}$ or $f : U \to \mathbb{R}$ be a* ***continuously differentiable*** *real-valued function of n real variables, i.e., f is of class C^2, defined on an open convex subset U in $\mathbb{R}^n$, and let $\nabla f(\mathbf{x}) = Df(\mathbf{x}) = \big(D_i f(\mathbf{x})\big)_{1\leq i\leq n} \triangleq \big(\frac{\partial f(\mathbf{x})}{\partial x_i}\big)_{1\leq i\leq n}$, be the* ***gradient vector*** *of f at $\mathbf{x}$.*[2] *Then:*

(*i*) *The function f is* ***concave*** *on U if and only if*

$$f(\mathbf{x}) \quad \leq \quad f(\mathbf{x_0}) + Df(\mathbf{x_0})\big(\mathbf{x}-\mathbf{x_0}\big)^{\mathsf{T}}, \tag{E.2.14}$$

for all $\mathbf{x}, \mathbf{x_0} \in U$, where $Df(\mathbf{x_0})$ is the gradient vector evaluated at $\mathbf{x_0}$;

(*ii*) *The function f is* ***strictly concave*** *on U if and only if the above inequality is strict, i.e.*

$$f(\mathbf{x}) \quad < \quad f(\mathbf{x_0}) + Df(\mathbf{x_0})\big(\mathbf{x}-\mathbf{x_0}\big)^{\mathsf{T}}, \tag{E.2.15}$$

for all $\mathbf{x}, \mathbf{x_0} \in U$, with $\mathbf{x} \neq \mathbf{x_0}$;

(*iii*) *The function f is* ***convex*** *on U if and only if*

$$f(\mathbf{x}) \quad \geq \quad f(\mathbf{x_0}) + Df(\mathbf{x_0})\big(\mathbf{x}-\mathbf{x_0}\big)^{\mathsf{T}}, \tag{E.2.16}$$

for all $\mathbf{x}, \mathbf{x_0} \in U$;

(*iv*) *The function f is* ***strictly convex*** *on U if and only if the above inequality is strict, i.e.*

$$f(\mathbf{x}) \quad > \quad f(\mathbf{x_0}) + Df(\mathbf{x_0})\big(\mathbf{x}-\mathbf{x_0}\big)^{\mathsf{T}}, \tag{E.2.17}$$

for all $\mathbf{x}, \mathbf{x_0} \in U$, with $\mathbf{x} \neq \mathbf{x_0}$.

Furthermore, we have the following lemma for concave and convex functions of a single variable, which we shall state without proof.

[2]The *gradient vector* is the $1 \times n$ vector of first-order partial derivatives of the real-valued function f at $\mathbf{x} \in \mathbb{R}^n$:

$$\nabla f(\mathbf{x}) \quad = \quad Df(\mathbf{x}) \quad = \quad \Big[\tfrac{\partial f(\mathbf{x})}{\partial x_1} \quad \tfrac{\partial f(\mathbf{x})}{\partial x_2} \quad \cdots \quad \tfrac{\partial f(\mathbf{x})}{\partial x_n}\Big].$$

Given the real-valued function $f(x_1, \ldots, x_n)$, if all first-order partial derivatives of f exist, then the gradient vector of f evaluated at $\mathbf{x}$ is the vector $\nabla f(\mathbf{x}) = Df(\mathbf{x}) = \big(D_i f(\mathbf{x})\big)_{1\leq i\leq n} \triangleq \big(\frac{\partial f(\mathbf{x})}{\partial x_i}\big)_{1\leq i\leq n}$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and we use the notation D_i for the first partial derivative with respect to the ith variable. D_i is also referred to as the *differentiation operator* with respect to the ith variable.

Lemma E.2.6. *Let $f(x)$, $f : \mathbb{R} \to \mathbb{R}$ or $f : U \to \mathbb{R}$ be a* ***continuously differentiable*** *real-valued function of a single variable $x \in U \subset \mathbb{R}$ defined on an open convex subset U in $\mathbb{R}$. Then:*

(*i*) *The function f is* ***concave*** *on U if and only if*

$$f(x) \quad \leq \quad f(x_0) + f'(x_0)\big(x - x_0\big), \tag{E.2.18}$$

for all $x, x_0 \in U$;

(*ii*) *The function f is* ***strictly concave*** *on U if and only if the above inequality is strict, i.e.*

$$f(x) \quad < \quad f(x_0) + f'(x_0)\big(x - x_0\big), \tag{E.2.19}$$

for all $x, x_0 \in U$, with $x \neq x_0$;

(*iii*) *The function f is* ***convex*** *on U if and only if*

$$f(x) \quad \geq \quad f(x_0) + f'(x_0)\big(x - x_0\big), \tag{E.2.20}$$

for all $x, x_0 \in U$;

(*iv*) *The function f is* ***strictly convex*** *on U if and only if the above inequality is strict, i.e.*

$$f(x) \quad > \quad f(x_0) + f'(x_0)\big(x - x_0\big), \tag{E.2.21}$$

for all $x, x_0 \in U$, with $x \neq x_0$.

We shall also require another useful result, regarding both twice differentiable and concave (or convex) functions, that provides an important characterisation of concave (or convex) and strictly concave (or strictly convex) functions in terms of Hessians. This result taken from Takayama (1994), concerning a multivariable function $f : \mathbb{R}^n \to \mathbb{R}$ or $f : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$, will be stated without proof.

Lemma E.2.7 ([**Takayama (1994)**]). *Let $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$, $f : \mathbb{R}^n \to \mathbb{R}$ or $f : U \to \mathbb{R}$ be a* ***twice continuously differentiable*** *real-valued function of n real variables, i.e., f is of class C^2, defined on an open convex subset U in $\mathbb{R}^n$, and let $\mathbf{H} = \nabla^2 f(\mathbf{x}) = D^2 f(\mathbf{x}) = \big(D_i D_j f(\mathbf{x})\big)_{1 \leq i,j \leq n} = \big(D_{ij} f(\mathbf{x})\big)_{1 \leq i,j \leq n} \triangleq \big(\frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j}\big)_{1 \leq i,j \leq n}$, be its Hessian* [3] *(matrix). Then:*

(*i*) *The function f is* ***concave*** *on U if and only if $\mathbf{H} = D^2 f(\mathbf{x})$ is* ***negative semidefinite***[4] *for all $\mathbf{x} \in U$;*

(*ii*) *The function f is* ***strictly concave*** *on U if $\mathbf{H} = D^2 f(\mathbf{x})$ is* ***negative definite***[5] *for all $\mathbf{x} \in U$;*

[3]The *Hessian* (*matrix*) is the $n \times n$ square symmetric matrix of second-order partial derivatives of the real-valued function f at $\mathbf{x} \in \mathbb{R}^n$:

$$\mathbf{H}f(\mathbf{x}) \quad = \quad \begin{bmatrix} \frac{\partial^2 f(\mathbf{x})}{\partial x_1^2} & \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_2 \partial x_1} & \frac{\partial^2 f(\mathbf{x})}{\partial x_2^2} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_n \partial x_1} & \frac{\partial^2 f(\mathbf{x})}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(\mathbf{x})}{\partial x_n^2} \end{bmatrix}.$$

Given the real-valued function $f(x_1, \ldots, x_n)$, if all second-order partial derivatives of f exist, then the Hessian matrix of f is the matrix $\mathbf{H}f(\mathbf{x}) = \nabla^2 f(\mathbf{x}) = D^2 f(\mathbf{x}) = \big(D_i D_j f(\mathbf{x})\big)_{1 \leq i,j \leq n} = \big(D_{ij} f(\mathbf{x})\big)_{1 \leq i,j \leq n} \triangleq \big(\frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j}\big)_{1 \leq i,j \leq n}$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and we use the notation D_i for the first partial derivative with respect to the ith variable, D_{ij} for the second partial derivative with respect to the ith and jth variables and D_{ii} for the second partial derivative with respect to the ith variable.

[4]An $n \times n$ symmetric matrix $\mathbf{A}$ is said to be negative semidefinite if $\mathbf{x}\mathbf{A}\mathbf{x}^\mathsf{T} \leq 0$ for all $\mathbf{x} \in \mathbb{R}^n$, sometimes expressed as $\mathbf{A} \preceq 0$. An $n \times n$ symmetric matrix $\mathbf{A}$ is negative semidefinite if and only if all the eigenvalues of $\mathbf{A}$ are nonpositive. Another criterion is that a symmetric matrix $\mathbf{A}$ is negative semidefinite if and only if the determinant of every odd-numbered principal submatrix is nonpositive and the determinant of every even-numbered principal submatrix is positive. The proof is omitted here, consult Anton & Rorres (2000) for details on principal submatrices.

[5]An $n \times n$ symmetric matrix $\mathbf{A}$ is said to be negative definite if $\mathbf{x}\mathbf{A}\mathbf{x}^\mathsf{T} < 0$ for all $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \neq \mathbf{0}$, sometimes expressed as $\mathbf{A} \prec 0$. An $n \times n$ symmetric matrix $\mathbf{A}$ is negative definite if and only if all the eigenvalues of $\mathbf{A}$ are strictly negative. Another criterion is that a symmetric matrix $\mathbf{A}$ is negative definite if and only if the determinant of every odd-numbered principal submatrix is negative and the determinant of every even-numbered principal submatrix is positive.

(*iii*) *The function f is* ***convex*** *on U if and only if $\mathbf{H} = D^2 f(\mathbf{x})$ is* ***positive semidefinite***[6] *for all $\mathbf{x} \in U$;*

(*iv*) *The function f is* ***strictly convex*** *on U if $\mathbf{H} = D^2 f(\mathbf{x})$ is* ***positive definite***[7] *for all $\mathbf{x} \in U$.*

Furthermore, we have the following lemma for concave and convex functions of a single variable, which we shall state without proof. This is a well-known result that can be found in any standard Calculus textbook, it essentially constitutes the second-order derivative test for concavity of a function $f : \mathbb{R} \to \mathbb{R}$ or $f : U \to \mathbb{R}$, on the open set $U \subset \mathbb{R}$. The subsequent lemma does not stem from only one source but has rather been adapted so as to comply with the notation adopted throughout this chapter.

Lemma E.2.8. *Let $f(x)$, $f : \mathbb{R} \to \mathbb{R}$ or $f : U \to \mathbb{R}$ be a* ***twice continuously differentiable*** *real-valued function of a single variable $x \in U \subset \mathbb{R}$, i.e., f is of class C^2, defined on an open convex subset U in $\mathbb{R}$. Then:*

(*i*) *The function f is* ***concave*** *if and only if the second-order derivative of f at x is nonpositive for all $x \in U$, i.e., $f''(x) \leq 0$, and if $f''(x) < 0$, for all $x \in U$, then the function f is* ***strictly concave****;*

(*ii*) *The function f is* ***convex*** *if and only if the second-order derivative of f at x is nonnegative for all $x \in U$, i.e., $f''(x) \geq 0$, and if $f''(x) > 0$, for all $x \in U$, then the function f is* ***strictly convex****.*

A proof of this lemma follows directly from the alternative definition of concavity and convexity given in Definitions 4.5.4 and E.1.2, as well as from the nature of increasing and decreasing functions. To clarify this for the concave case, we have in accordance with Definition 4.5.4, for the concave function f, the function f' is decreasing and we know that for a decreasing function g, the function $g' \leq 0$. Consequently, for the decreasing function f', we have $f'' \leq 0$. The next property establishes the concavity (or convexity) of a linear combination of concave (or convex) functions, this property will be of vital importance in corroborating the concavity of the functions of interest in this section, namely the measures of diversity.

Lemma E.2.9 (Linear Combinations of Concave (or Convex) Functions)**.** *Suppose that $f_1 : U \to \mathbb{R}, f_2 : U \to \mathbb{R}, \ldots, f_k : U \to \mathbb{R}$ are all real-valued* ***concave (or convex) functions*** *of n real variables $f_i(\mathbf{x}) = f_i(x_1, x_2, \ldots, x_n)$, for $i = 1, \ldots, k$, $\mathbf{x} \in U \subset \mathbb{R}^n$, where U is a convex set in $\mathbb{R}^n$. Then for any $\alpha_1, \alpha_2, \ldots, \alpha_k$, for which each $\alpha_i \geq 0$, $i = 1, \ldots, k$, the* ***nonnegative linear combination*** *$f(\mathbf{x}) = \sum_{i=1}^k \alpha_i f_i(\mathbf{x}) = \alpha_1 f_1(\mathbf{x}) + \alpha_2 f_2(\mathbf{x}) + \cdots + \alpha_k f_k(\mathbf{x})$ is also a* ***concave (or convex) function****.*

Proof. We shall substantiate the claim made in the aforementioned lemma by considering the concave case, the convex case immediately follows by employing Definition E.1.2. Let f_i, $i = 1, \ldots, k$, be real-valued functions defined on a convex (not necessarily open) subset U of $\mathbb{R}^n$, $U \subset \mathbb{R}^n$, i.e., $f_i : \mathbb{R}^n \to \mathbb{R}$ or $f_i : U \to \mathbb{R}$. Consider any $\mathbf{x}, \mathbf{y} \in U \subset \mathbb{R}^n$ and $0 \leq \theta \leq 1$. If for all $i = 1, \ldots, k$, f_i is a concave function, then they exhibit the concavity property as put forth in Definition 4.5.4, we have for all $i = 1, \ldots, k$,

$$f_i\big(\theta\mathbf{x} + (1-\theta)\mathbf{y}\big) \quad \geq \quad \theta f_i(\mathbf{x}) + (1-\theta) f_i(\mathbf{y}). \tag{E.2.22}$$

Therefore,

$$\begin{aligned} f\big(\theta\mathbf{x} + (1-\theta)\mathbf{y}\big) &= \sum_{i=1}^k \alpha_i f_i\big(\theta\mathbf{x} + (1-\theta)\mathbf{y}\big) && \text{(E.2.23)} \\ &\geq \sum_{i=1}^k \alpha_i \Big[\theta f_i(\mathbf{x}) + (1-\theta) f_i(\mathbf{y})\Big] && \text{(E.2.24)} \\ &= \theta \sum_{i=1}^k \alpha_i f_i(\mathbf{x}) + (1-\theta) \sum_{i=1}^n \alpha_i f_i(\mathbf{y}) \\ &= \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y}). \end{aligned}$$

[6] An $n \times n$ symmetric matrix $\mathbf{A}$ is said to be positive semidefinite if $\mathbf{x}\mathbf{A}\mathbf{x}^\mathsf{T} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$, sometimes expressed as $\mathbf{A} \succeq 0$. An $n \times n$ symmetric matrix $\mathbf{A}$ is positive semidefinite if and only if all the eigenvalues of $\mathbf{A}$ are nonnegative. Another criterion is that a symmetric matrix $\mathbf{A}$ is positive semidefinite if and only if the determinant of every principal submatrix is nonnegative.

[7] An $n \times n$ symmetric matrix $\mathbf{A}$ is said to be positive definite if $\mathbf{x}\mathbf{A}\mathbf{x}^\mathsf{T} > 0$ for all $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \neq \mathbf{0}$, sometimes expressed as $\mathbf{A} \succ 0$. An $n \times n$ symmetric matrix $\mathbf{A}$ is positive definite if and only if all the eigenvalues of $\mathbf{A}$ are strictly positive. Another criterion is that a symmetric matrix $\mathbf{A}$ is positive definite if and only if the determinant of every principal submatrix is positive.

Therefore,

$$f\big(\theta\mathbf{x}+(1-\theta)\mathbf{y}\big) \quad \geq \quad \theta f(\mathbf{x})+(1-\theta)f(\mathbf{y}). \tag{E.2.25}$$

The analogous convexity result follows by replacing (E.2.22) with the convex version:

$$f_i\big(\theta\mathbf{x}+(1-\theta)\mathbf{y}\big) \quad \leq \quad \theta f_i(\mathbf{x})+(1-\theta)f_i(\mathbf{y}). \tag{E.2.26}$$

■

Lemma E.2.9 establishes that any nonnegative linear combination of concave functions is itself a concave function and that any nonnegative linear combination of convex functions is itself a convex function. In addition to this property, is a property that considers the nondecreasing concave transformation of a concave (or convex) function. This is yet another property that will be useful in demonstrating the concavity of the measures of diversity. We shall offer this result in the following lemma.

Lemma E.2.10 (Composite Concave Functions)**.** *Let $h(\mathbf{x})=h(x_1,x_2,\ldots,x_n)$, $h:\mathbb{R}^n\to\mathbb{R}$ or $h:U\to\mathbb{R}$ be a real-valued function of n variables, $\mathbf{x}\in U$, where $U\subset\mathbb{R}^n$ and U is convex in $\mathbb{R}^n$ and let $g(x)$, $g:\mathbb{R}\to\mathbb{R}$ be a real-valued function of a single variable, $x\in\mathbb{R}$. Consider the* ***composite function*** *$f(\mathbf{x})=f(x_1,x_2,\ldots,x_n)$, $f:\mathbb{R}^n\to\mathbb{R}$ or $f:U\to\mathbb{R}$ defined by $f(\mathbf{x})=(g\circ h)(\mathbf{x})=g\big(h(\mathbf{x})\big)$. The composite function f is a* ***concave function*** *if:*

(*i*) *both h and g are* ***concave*** *functions, and also if g is* ***nondecreasing****;*

(*ii*) *g is a* ***nonincreasing concave*** *function and h is a* ***convex*** *function.*

Proof. We thus require to show that

$$f\big(\theta\mathbf{x}+(1-\theta)\mathbf{y}\big) \quad \geq \quad \theta f(\mathbf{x})+(1-\theta)f(\mathbf{y}), \tag{E.2.27}$$

where $0\leq\theta\leq 1$ and $\mathbf{x},\mathbf{y}\in U\subset\mathbb{R}^n$. By the definition of f, we have

$$f\big(\theta\mathbf{x}+(1-\theta)\mathbf{y}\big) \quad = \quad g\Big(h\big(\theta\mathbf{x}+(1-\theta)\mathbf{y}\big)\Big). \tag{E.2.28}$$

Let us initially consider the first case (i), where h is a concave function and g is a nondecreasing concave function. Thus, making use of the fact that h is a concave function, we have by Definition 4.5.4,

$$h\big(\theta\mathbf{x}+(1-\theta)\mathbf{y}\big) \quad \geq \quad \theta h(\mathbf{x})+(1-\theta)h(\mathbf{y}). \tag{E.2.29}$$

Furthermore, since g is nondecreasing, i.e., $x'\geq y'$, where $x',y'\in\mathbb{R}$ implies that $g(x')\geq g(y')$, we obtain the following using the concavity result in (E.2.29),

$$g\Big(h\big(\theta\mathbf{x}+(1-\theta)\mathbf{y}\big)\Big) \quad \geq \quad g\Big(\theta h(\mathbf{x})+(1-\theta)h(\mathbf{y})\Big). \tag{E.2.30}$$

Now, by the concavity of g, where $h(\mathbf{x})\in\mathbb{R}$, we have

$$g\Big(\theta h(\mathbf{x})+(1-\theta)h(\mathbf{y})\Big) \quad \geq \quad \theta g\big(h(\mathbf{x})\big)+(1-\theta)g\big(h(\mathbf{y})\big) \tag{E.2.31}$$

$$= \quad \theta f(\mathbf{x})+(1-\theta)f(\mathbf{y}). \tag{E.2.32}$$

Thus, from (E.2.28), (E.2.30) and (E.2.32), we achieve the desired result

$$f\big(\theta\mathbf{x}+(1-\theta)\mathbf{y}\big) \quad \equiv \quad g\Big(h\big(\theta\mathbf{x}+(1-\theta)\mathbf{y}\big)\Big) \tag{E.2.33}$$

$$\geq \quad g\Big(\theta h(\mathbf{x})+(1-\theta)h(\mathbf{y})\Big). \tag{E.2.34}$$

Therefore,

$$f\big(\theta\mathbf{x}+(1-\theta)\mathbf{y}\big) \quad \geq \quad \theta f(\mathbf{x})+(1-\theta)f(\mathbf{y}). \tag{E.2.35}$$

This completes the proof for the first part. The proof of the second case (ii) is synonymous to the proof offered for the first case. The only departures therein are those of equation (E.2.29), which is altered to

$$h\big(\theta\mathbf{x} + (1-\theta)\mathbf{y}\big) \quad \leq \quad \theta h(\mathbf{x}) + (1-\theta)h(\mathbf{y}), \tag{E.2.36}$$

since h is now a convex function, and thus (E.2.36) follows from Definition E.1.2. Now, by employing the other departure, that being the nonincreasing nature of g, we have for $x', y' \in \mathbb{R}$ and $x' \leq y'$, that $g(x') \geq g(y')$. This result together with (E.2.36), renders the following equivalent result to (E.2.30)

$$g\Big(h\big(\theta\mathbf{x} + (1-\theta)\mathbf{y}\big)\Big) \quad \geq \quad g\Big(\theta h(\mathbf{x}) + (1-\theta)h(\mathbf{y})\Big). \tag{E.2.37}$$

Once again, by appealing to the concavity property of g, the remainder of the proof follows in an analogous fashion. ■

Lemma E.2.11 (Composite Convex Functions). *Let $h(\mathbf{x}) = h(x_1, x_2, \ldots, x_n)$, $h : \mathbb{R}^n \to \mathbb{R}$ or $h : U \to \mathbb{R}$ be a real-valued function of n variables, $\mathbf{x} \in U$, where $U \subset \mathbb{R}^n$ and U is convex in $\mathbb{R}^n$ and let $g(\mathbf{x})$, $g : \mathbb{R} \to \mathbb{R}$ be a real-valued function of a single variable, $x \in \mathbb{R}$. Consider the* ***composite function*** *$f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$, $f : \mathbb{R}^n \to \mathbb{R}$ or $f : U \to \mathbb{R}$ defined by $f(\mathbf{x}) = (g \circ h)(\mathbf{x}) = g\big(h(\mathbf{x})\big)$. The composite function f is a* ***convex function*** *if:*

(i) both h and g are ***convex*** *functions, and also if g is* ***nondecreasing****;*

(ii) g is a ***nonincreasing convex*** *function and h is a* ***concave*** *function.*

Proof. The proof of the statements for the composite convex case is similar to the proof obtained for the composite concave case. Refer to the proof of Lemma E.2.10, for further details. ■

The result above can also be established for twice continuously differentiable real-valued functions. However, even though the above result is true for all functions h and g, we include the undermentioned result for the sake of completeness and to demonstrate how a version of Lemma E.2.10 for twice continuously differentiable functions may be established by adopting the characterisation of concavity outlined in Lemma E.2.8. Thus, the undermentioned result brings to the fore, this characterisation of concavity which will be required throughout the remainder of this chapter.

Lemma E.2.12 (Twice Continuously Differentiable Composite Concave Functions). *Let $h(x)$, $h : \mathbb{R} \to \mathbb{R}$ or $h : U \to \mathbb{R}$, $x \in U$ and $g(x)$, $g : \mathbb{R} \to \mathbb{R}$, where $x \in \mathbb{R}$, $U \subset \mathbb{R}$ and U is convex in $\mathbb{R}$ be* ***twice continuously differentiable real-valued functions*** *of a single variable. Consider the* ***composite function*** *$f(x)$ of a single variable, $f : \mathbb{R} \to \mathbb{R}$ or $f : U \to \mathbb{R}$ defined by $f(x) = (g \circ h)(x) = g\big(h(x)\big)$. The composite function f is a* ***concave function*** *if:*

(i) both h and g are ***concave*** *functions, and also if g is* ***nondecreasing****;*

(ii) g is a ***nonincreasing concave*** *function and h is a* ***convex*** *function.*

Proof. We shall prove the result provided in the first statement, the proof of the second statement is similar. By appealing to Lemma E.2.8 regarding the condition for concavity for twice continuously differentiable functions, we can deduce that for the twice continuously differentiable concave functions, g and h, that $g''(x) \leq 0$, for all $x \in \mathbb{R}$, and $h''(x) \leq 0$, for all $x \in U$. In addition, since the function g is nondecreasing, we have $g'(x) \geq 0$, for all $x \in \mathbb{R}$. Once again, we shall invoke Lemma E.2.8, to show that the composite function f is indeed a concave function, by examining the second-order derivative of f at x:

$$f'(x) \quad = \quad g'\big(h(x)\big)h'(x) \tag{E.2.38}$$

$$f''(x) \quad = \quad g''\big(h(x)\big)h'(x)h'(x) + g'\big(h(x)\big)h''(x) \tag{E.2.39}$$

$$= \quad g''\big(h(x)\big)\big(h'(x)\big)^2 + g'\big(h(x)\big)h''(x) \quad \leq \quad 0. \tag{E.2.40}$$

From E.2.40, it is evident that $f''(x) \leq 0$, since $g''\big(h(x)\big) \leq 0$, $g'\big(h(x)\big) \geq 0$ and $h''(x) \leq 0$. Consequently, in accordance with Lemma E.2.8, the function f is concave for all $x \in U \subset \mathbb{R}$. Notice that we do not require any knowledge of the monotonicity of the concave function h, yet the requirement that the concave function g be nondecreasing is essential. ■

Lemma E.2.13 (Twice Continuously Differentiable Composite Convex Functions)**.** *Let $h(x)$, $h : \mathbb{R} \to \mathbb{R}$ or $h : U \to \mathbb{R}$, $x \in U$ and $g(x)$, $g : \mathbb{R} \to \mathbb{R}$, where $x \in \mathbb{R}$, $U \subset \mathbb{R}$ and U is convex in $\mathbb{R}$ be* ***twice continuously differentiable real-valued functions*** *of a single variable. Consider the* ***composite function*** *$f(x)$ of a single variable, $f : \mathbb{R} \to \mathbb{R}$ or $f : U \to \mathbb{R}$ defined by $f(x) = (g \circ h)(x) = g\big(h(x)\big)$. The composite function f is a* ***convex function*** *if:*

(*i*) *both h and g are* ***convex*** *functions, and also if g is* ***nondecreasing****;*

(*ii*) *g is a* ***nonincreasing convex*** *function and h is a* ***concave*** *function.*

Proof. We shall prove the result provided in the second statement, the proof of the first statement is similar. By appealing to Lemma E.2.8 regarding the condition for concavity for twice continuously differentiable functions, we can deduce that for the twice continuously differentiable concave function h, that $h''(x) \leq 0$, for all $x \in U$. Moreover, exploiting the conjugate concavity condition, namely the convexity condition for twice continuously differentiable functions, we can deduce that for the twice continuously differentiable convex function g, that $g''(x) \geq 0$, for all $x \in U$. In addition, since the function g is nonincreasing, we have $g'(x) \leq 0$, for all $x \in \mathbb{R}$. Once again, we shall invoke Lemma E.2.8, to show that the composite function f is indeed a convex function, by examining the second-order derivative of f at x:

$$f'(x) = g'\big(h(x)\big)h'(x) \tag{E.2.41}$$

$$f''(x) = g''\big(h(x)\big)h'(x)h'(x) + g'\big(h(x)\big)h''(x) \tag{E.2.42}$$

$$= g''\big(h(x)\big)\big(h'(x)\big)^2 + g'\big(h(x)\big)h''(x) \geq 0. \tag{E.2.43}$$

From E.2.43, it is evident that $f''(x) \geq 0$, since $g''\big(h(x)\big) \geq 0$, $g'\big(h(x)\big) \leq 0$ and $h''(x) \leq 0$. Consequently, in accordance with Lemma E.2.8, the function f is convex for all $x \in U \subset \mathbb{R}$. This completes the proof. ■

Appendix F

Dr. E. Robert Fernholz

Dr. E. Robert Fernholz is founder and Co-Chief Investment Officer of INTECH, an institutional equity manager and subsidiary of the Janus Capital Group. The innovative and groundbreaking concept of stochastic portfolio theory is indeed the brainchild of Dr. E. Robert Fernholz, and credit in this dissertation is duly given. In April 2002, he published the research monograph titled *Stochastic Portfolio Theory* [Fernholz (2002)], which provides a general mathematical framework for equity investment and details the applications of stochastic calculus to portfolio theory and management. He speaks extensively around the globe regarding his work in the field of mathematical finance and his research continues to advance new and innovative ideas.

Figure F.1: Dr. E. Robert Fernholz

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