A Study of Monoidal t-norm based Logic

by

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A dissertation submitted in fulfillment of the

academic requirements for the degree of

Master of Science,

in the

School of Mathematics,

University of the Witwatersrand,

Johannesburg, 2013

To my late mother 'Matsekeli, my late father Ratsuonyana, my husband Motlalepule and my son Thuto

Abstract

The logical system MTL (for Monoidal t-norm Logic) is a formalism of the logic of left-continuous t-norms, which are operations that arise in the study of fuzzy sets and fuzzy logic. The objective is to investigate the important results on MTL and collect them together in a coherent form. The main results considered will be the completeness results for the logic with respect to MTL-algebras, MTL-chains (linearly ordered MTL-algebras) and standard MTL-algebras (left-continuous t-norm algebras). Completeness of MTL with respect to standard MTL-algebras means that MTL is indeed the logic of left-continuous t-norms. The logical system BL (for Basic Logic) is an axiomatic extension of MTL; we will consider the same completeness results for BL; that is we will show that BL is complete with respect to BL-algebras, BL-chains and standard BL-algebras (continuous t-norm algebras). Completeness of BL with respect to standard BL-algebras means that BL is the logic of continuous t-norms.

Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

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Ellen Mohau Toloane

This ——— day of ——————— 2012, at Johannesburg, South Africa.

Acknowledgements

I would like to express my sincere appreciation and gratitute to my supervisor, Prof. Clint van Alten, for the patient guidance and mentorship he provided to me. I would also like to thank him for arranging some funding for me.

I am truly indebted and thankful to my entire family for their support, love and encouragement.

I gratefully acknowledge the partial funding made towards this study by the school of Computer Science, school of Mathematics and the National University of Lesotho.

Contents

CHAPTER 1

Introduction

T-norms were introduced by Menger in [23]. The name t-norm originates from 'triangular norm' which is a generalized type of metric. Since their introduction t-norms have been applied in various other mathematical disciplines including game theory, the theory of non-additive measures and integrals, the theory of measure-free conditioning, fuzzy set theory, preference modeling, decision analysis and artificial intelligence [20]. More details on t-norms can be found in [21].

Recently t-norms have been widely used in the formal study of fuzzy logic. Whereas in classical logic there are only two truth values $\{0,1\}$ in fuzzy logic the set of truth-values is the whole unit interval $[0,1]$, which allows one to speak of degrees of truth. In this context, a t-norm is a binary operation on $[0,1]$ representing a form of conjunction which is different to the classical one and serves as a model of the 'AND' connective in a typical fuzzy if-then rule. Typically t-norms are assumed to be continuous or, more generally, left-continuous operations. A left-continuous t-norm $\circ$ has an associated operation $\rightarrow$, called its residuum, which is a form of implication and models the logical consequence of an if-then rule.

Monoidal t-norm Logic, or MTL for short, was introduced in [5] as a formalization of the logic of left-continuous t-norms. This logic has as its class of algebraic semantics the class of MTL-algebras. These algebras are bounded lattice-ordered algebras with additional operations $\circ$ and $\rightarrow$ which model the t-norm and its residuum, respectively. Fuzzy logic and fuzzy sets have been used successfully as tools for mathematical applications since their introduction in the 1960's. Since then the connection between fuzzy logic and formal mathematical logic has been explored. The connection with the existing theory of many-valued logic was established as well as with certain classes of ordered algebraic structures. The logic MTL extends this general area of study. While the continuous t-norms on [0, 1] have been completely classified, the study of left-continuous t-norms is relatively new and there is good scope for future investigations.

The objective is to gain a complete understanding of the methodology and techniques used in the study of MTL as well as in the study of algebraic systems related to this logic. This thesis intends to bring together in a single framework a coherent collection of important results related to MTL and MTL-algebras. The main mathematical results that will be considered are the completeness of the logic MTL. The Completeness Theorem states that a formula φ is provable in a logic $\mathcal{L}$ if and only if the identity $\varphi = 1$ holds in some class of algebraic models. The completeness results that will be investigated are with respect to the classes of MTL-algebras, MTL-chains and standard MTL-algebras.

Basic Logic (BL for short) was introduced by H*á*jek in [14] as a formalization of the logic of continuous t-norms. Since BL is closely related to MTL, we are also going to investigate the same completeness results for BL. Completeness is the key requirement of any logic. The significance of completeness was first realized by Hilbert and Ackermann, who posed it as an open question in their book [16]; the question asks whether there are axioms of a formal system sufficient to derive every statement that is true in all models of the system. An English translation of the second edition of this book can be found in [17]. The first proof of a the Completeness Theorem for classical first order logic was given by G*ö*del in his doctoral thesis, and it was published in [11]. An English translation of it can be found in [12].

The work in this thesis is organised as follows. Chapter 3 deals with t-norms. In this chapter we give examples of left and right continuous t-norms and their properties. We use these properties to prove a Decomposition Theorem of continuous t-norms, which says any continuous t-norm is isomorphic to an ordinal sum of Lukasiewicz, Product and Gödel t-norms. Most of the material in this chapter comes from Hájek's book [14].

Chapter 4 is about the logic MTL. We list some of the formulas provable in MTL and give proofs for some formulas. Again we discuss the corresponding algebraic structures, namely MTL-algebras. In particular, we prove some important properties of MTL-algebras. We also prove that the class of all MTL-algebras forms a variety of algebras. The material in this chapter is taken from the work done by Esteva and Godo in [5].

In chapter 5, we prove completeness of MTL with respect to MTL-algebras and MTL-chains (linearly ordered MTL-algebras), that is we prove that a formula φ is a theorem of MTL if and only if the identity $\varphi = 1$ holds in all MTL-algebras. We show that every MTL-algebra is a subdirect product of MTL-chains and use this result to show that MTL is complete with respect to MTL-chains. The results in this chapter are from the work of Esteva and Godo [5] and Hájek [14].

In chapter 6, we present a proof of completenesss of MTL with respect to standard MTL-algebras (left-continuous t-norm algebras), which shows that a formula φ is a theorem of MTL if and only if the identity $\varphi = 1$ holds in all standard MTL-algebras. To achieve this we use the completeness result from chapter 5 and prove that every finitely generated MTL-chain can be embedded into a standard MTL-algebra. The material in this chapter comes from the work of Jenei and Montagna [20].

Chapter 7 is devoted to a standard completeness proof of BL. We only prove standard completeness of BL with respect to standard BL-algebras since the proof of completeness of BL with respect to BL-algebras and BL-chains is similar to that of MTL. The standard completeness of BL was first proved by Gignoli, Esteva, Godo and Torrens in [4]. It was later also proved by Angliano, Ferreirim and Montagna in [1]. We prove this completeness using the approach in [1]. We also discuss Wajsberg hoops and the characterization theorem of subdirectly irreducible BL-algebras which play a significant role in proving the results leading to the completeness theorem. We also give an example to show that the proof of completeness for MTL with respect to standard MTL-algebras does not extend to BL. Lastly, we present our conclusions in chapter 8.

CHAPTER 2

Preliminaries

We assume familiarity with basic concepts from mathematical analysis and universal algebra. In this section we give the definitions of the main concepts we shall use. For more background on universal algebra we refer the reader to [3].

We use $\mathbf{N}$ to denote the set of natural numbers, $\mathbf{Q}$ to denote the set of rational numbers and $\mathbf{R}$ to denote the set of real numbers.

Definition 2.0.1. Let $f : \mathbf{R} \to \mathbf{R}$ be a function with domain $D \subseteq \mathbf{R}$ and $a \in D$. We say f is *continuous* at a if for every $\epsilon > 0$, there is a $\delta > 0$ such that
$|f(x) - f(a)| < \epsilon$ whenever $x \in D$ and $|x - a| < \delta$.
f is said to be *left-continuous* at a if for every $\epsilon > 0$, there is a $\delta > 0$ such that
$|f(x) - f(a)| < \epsilon$ whenever $x \in D$ and $a - \delta < x < a$.

We can also define continuity and left-continuity equivalently in terms of sequences and it is this definition we are mainly going to use. The definition is as follows:

Definition 2.0.2. Let $f : \mathbf{R} \to \mathbf{R}$ be a function with domain $D \subseteq \mathbf{R}$ and $a \in D$.

Then f is continuous at a iff whenever $\{x_n\}$ is a sequence in D which converges $\mathbf{R}$to a, then the sequence $\{f(x_n)\}$ converges to $f(a)$. f is left-continuous at a iff for all increasing sequences $\{x_n\}$ converging to a, $\{f(x_n)\}$ converges to $f(a)$.

Definition 2.0.3. For a non-empty set A and a non-negative integer n, we define $A^0 = \emptyset$, and for $n > 0$, A^n is the set of n-tuples of elements from A.

Definition 2.0.4. An *n-ary operation* is any function f from A^n to A.

Definition 2.0.5. A *type* (or *language*) of algebras is a set $\mathcal{F}$ of function symbols such that a non-negative integer n is assigned to each member f of $\mathcal{F}$. This integer is called the *arity* of f.

Definition 2.0.6. An *algebra* $\mathbf{A}$ of type $\mathcal{F}$ is an ordered pair (A, F), where A is a non-empty set and F is a family of operations on A indexed by the type $\mathcal{F}$ such that corresponding to each function symbol f, of arity n, in $\mathcal{F}$ there is an n-ary operation $f^{\mathbf{A}}$ on A. The set A is called the *universe* of $\mathbf{A}$.

Definition 2.0.7. An *embedding* of an algebra $\mathbf{A}$ into an algebra $\mathbf{B}$ (of the same type) is a 1-1 map $e : A \to B$ that preserves all existing operations; i.e., for each (k-ary) operation symbol f and $b_1, \ldots, b_k \in A$ we have that $e(f^{\mathbf{A}}(b_1, \ldots, b_k)) = f^{\mathbf{B}}(e(b_1), \ldots, e(b_k))$.

Definition 2.0.8. Let $\mathcal{K}$ be a class of algebras of the same type. Then
$I(\mathcal{K})$ is the class of all algebras which are isomorphic to some member of $\mathcal{K}$.
$H(\mathcal{K})$ is the class of homomorphic images of algebras in $\mathcal{K}$.
$S(\mathcal{K})$ is the class of subalgebras of algebras in $\mathcal{K}$.
$P(\mathcal{K})$ is the class of direct products of non-empty families of algebras in $\mathcal{K}$.
The class $\mathcal{K}$ is said to be *closed* under a class operator O if $O(\mathcal{K})$ is contained in $\mathcal{K}$.

Definition 2.0.9. A non-empty class $\mathcal{K}$ of algebras of the same type is called a *variety* if it is closed under subalgebras, homomorphic images and direct products.

Definition 2.0.10. For a class $\mathcal{K}$ of algebras of the same type, let $V(\mathcal{K})$ denote the smallest variety containing $\mathcal{K}$. We say that $V(\mathcal{K})$ is the *variety generated by* $\mathcal{K}$. A variety V is *finitely generated* if $V = V(\mathcal{K})$ for some finite set $\mathcal{K}$ of finite algebras.

Theorem 2.0.1 (Tarski). *For any class $\mathcal{K}$ of algebras of the same type, $V(\mathcal{K}) = HSP(\mathcal{K})$.*

Definition 2.0.11. An algebra $\mathbf{A}$ is a *subdirect product* of algebras $\mathbf{B}_i$, $i \in I$, if $\mathbf{A}$ is embeddable into a direct product of $\mathbf{B}_i$'s in such a way that the projection of the image of B to each factor algebra $\mathbf{A}_i$ is onto.

Definition 2.0.12. Let $\mathbf{A}$ be an algebra of type $\mathcal{F}$, p, q terms of type $\mathcal{F}$ with variables $x_1, x_2, ..., x_n$ and $p \approx q$ an identity of type $\mathcal{F}$. Then $\mathbf{A}$ *satisfies* $p \approx q$, if for all $a_1, a_2, ..., a_n \in A$,

$$p^{\mathbf{A}}(a_1, a_2, ..., a_n) = q^{\mathbf{A}}(a_1, a_2, ..., a_n)$$

where $p^{\mathbf{A}}(a_1, a_2, ..., a_n)$ is the evaluation of the term p in $\mathbf{A}$ under the assignment $x_i \mapsto a_i$. Let $\mathcal{K}$ be a class of algebras of type $\mathcal{F}$. Then $\mathcal{K}$ *satisfies* $p \approx q$, if for all $\mathbf{A} \in \mathcal{K}$, $\mathbf{A}$ satisfies $p \approx q$.

Definition 2.0.13. Let X be a set of identities of type $\mathcal{F}$ and define $M(X)$ to be the class of all algebras of type $\mathcal{F}$ that satisfy every identity in X. A class $\mathcal{K}$ of algebras is an *equational class* if there is a set of identities X such that $\mathcal{K} = M(X)$. In this case we say $\mathcal{K}$ is *axiomatized* by X.

Theorem 2.0.2 (Birkoff). *Let $\mathcal{K}$ be a class of algebras of the same type. Then $\mathcal{K}$ is an equational class iff $\mathcal{K}$ is a variety.*

Definition 2.0.14. A binary relation $\sim$ on a set A is said to be an *equivalence relation* if for all $a, b, c \in A$:

(1) $a \sim a$ (reflexivity)

(2) if $a \sim b$ then $b \sim a$ (symmetry)

(3) if $a \sim b$ and $b \sim c$ then $a \sim c$. (transitivity)

The *equivalence class* of a under $\sim$, denoted $[a]_\sim$ or $a/\sim$, is defined as $a/\sim = \{b \in A : a \sim b\}$. The set $\{a/\sim : a \in A\}$ is denoted by $A/\sim$.

Definition 2.0.15. If $\mathbf{A}$ is an algebra and $\sim$ is a binary relation on A, then $\sim$ is a *congruence* if it is an equivalence relation and is compatible with the operations of $\mathbf{A}$ in the sense that: if $f^{\mathbf{A}}$ is an n-ary operation of $\mathbf{A}$ and $a_1 \sim b_1, a_2 \sim b_2, ..., a_n \sim b_n$, then $f^{\mathbf{A}}(a_1, a_2, ..., a_n) \sim f^{\mathbf{A}}(b_1, b_2, ..., b_n)$. The sets $A \times A$ and $\{(a,a) : a \in A\}$ are congruences on A; $\{(a,a) : a \in A\}$ is called the *trivial congruence*. A *proper congruence* is one not equal to $A \times A$.

Definition 2.0.16. Let $\sim$ be a congruence on an algebra $\mathbf{A}$. Then the *quotient algebra of* $\boldsymbol{A}$ by $\sim$, denoted by $\mathbf{A}/\sim$ is the algebra whose universe is $A/\sim$ and whose fundamental operations are defined by:

$$f^{\mathbf{A}/\sim}(a_1/\sim, a_2/\sim, ..., a_n/\sim) = f^{\mathbf{A}}(a_1, a_2, ..., a_n)/\sim$$

where $a_1, a_2, ..., a_n \in A$ and f is an n-ary function symbol. Note that the quotient algebras of $\mathbf{A}$ are of the same type as $\mathbf{A}$.

Definition 2.0.17. An algebra $\mathbf{A}$ is said to be *simple* if it has no proper non-trivial congruences.

Definition 2.0.18. A *monoid* is an algebra $(A, *, e)$, where $*$ is a binary operation and e is a constant such that $*$ is associative and e is an identity element for $*$. $(A, *, e)$ is called a *commutative monoid* if, in addition, $*$ is commutative.

Definition 2.0.19. Given a set X, an *ultrafilter* on X is a set U consisting of subsets of X such that

(i) $\emptyset \notin U$.

(ii) If $A, B \subseteq X, A \subseteq B$ and $A \in U$, then $B \in U$.

(iii) If $A, B \in U$, then $A \cap B \in U$.

(iv) If $A \subseteq X$, then either $A \in U$ or $X \setminus A \in U$.

Definition 2.0.20. Let $\{\mathbf{A}_{i\in I}\}$ be a family of algebras of the given type and let U be an ultrafilter on I. Define θ_U on $\prod_{i\in I} A_i$ as follows:

$$(a,b) \in \theta_U \text{ iff } \{i \in I : a_i = b_i\} \in U.$$

Then an *ultraproduct* denoted by $\prod_{i\in I} A_i/U$ is defined to be $\prod_{i\in I} A_i/\theta_U$.
We denote by $P_u(\mathcal{K})$ the class of ultraproducts of non-empty families of algebras in $\mathcal{K}$.

Theorem 2.0.3. *Any algebra $\boldsymbol{L}$ is a subalgebra of an ultraproduct of finitely generated subalgebras of $\boldsymbol{L}$.*

Definition 2.0.21. A *quasivariety* is a class of algebras closed under I, S, P and P_u and contains a trivial algebra (one-element algebra).

Definition 2.0.22. A *partially ordered set* (or *poset*) is a set S together with a binary relation $\preceq$ on S called a *partial order* such that the following axioms are satisfied:

(1) $x \preceq x$ (reflexivity)

(2) $x \preceq y$ and $y \preceq x$ implies $x = y$ (antisymmetry)

(3) $x \preceq y$ and $y \preceq z$ implies $x \preceq z$. (transitivity)

In a poset we use the expression $x \prec y$ to mean $x \preceq y$ but $x \neq y$.

Definition 2.0.23. If $\preceq$ is a partial order on a set S such that $x \preceq y$ or $y \preceq x$ for all $x, y \in S$, then $(S, \preceq)$ is a *totally ordered set* or a *linearly odered set* or simply a *chain*.

Definition 2.0.24. A linearly ordered set $(S, \preceq)$ is *densely ordered* if for all $x, y \in S$ for which $x \prec y$, there is a z in S such that $x \prec z \prec y$.

Definition 2.0.25. Let $(S, \preceq)$ be a poset and $A \subseteq S$. An element u in S is said to be an *upper bound* for A if $x \preceq u$ for each $x \in A$. The element u is the *least upper bound (l.u.b)* for A or *supremum* for A if it is an upper bound for A and $u \preceq v$ for each upper bound v for A. An element u in S is said to be a *lower bound* for A if $u \preceq x$ for each $x \in A$. The element u is the *greatest lower bound (g.l.b)* for A or *infimum* for A if it is a lower bound for A and $v \preceq u$ for each lower bound v for A.

We can define a lattice in two different ways. Lattices can be characterized as algebras and also as partially ordered sets.
Lattices as partially ordered sets:

Definition 2.0.26. A *lattice* is a poset in which each pair of elements has a least upper bound and a greatest lower bound. The l.u.b of elements a and b in a lattice will be denoted by $a \vee b$. The g.l.b of elements a and b will be denoted by $a \wedge b$. The operations $\vee$ and $\wedge$ are called *join* and *meet* respectively. Note that every chain is a lattice.

Lattices as algebras:

Definition 2.0.27. An algebra $(L, \vee, \wedge)$ with binary operations $\vee$ and $\wedge$ (read *join* and *meet* respectively) is called a *lattice* if it satisfies the following identities:

L1: (a) $x \vee y = y \vee x$
(b) $x \wedge y = y \wedge x$ (commutative laws)

L2: (a) $x \vee (y \vee z) = (x \vee y) \vee z$
(b) $x \wedge (y \wedge z) = (x \wedge y) \vee z$ (associative laws)

L3: (a) $x \vee x = x$
(b) $x \wedge x = x$ (idempotent laws)

L4: (a) $x = x \vee (x \wedge y)$
(b) $x = x \wedge (x \vee y)$ (absorption laws)

Connection between the two definitions:
If $(L, \vee, \wedge)$ is a lattice as an algebra, we can define $\preceq$ on L by $a \preceq b$ iff $a = a \wedge b$ or $a \preceq b$ iff $b = a \vee b$. First we show that the conditions $a \vee b = b$ and $a \wedge b = a$ are equivalent.
Assume $a \vee b = b$. Then

$$\begin{aligned} a \wedge b &= a \wedge (a \vee b) \quad (\text{since } a \vee b = b) \\ &= a \quad (\text{by an absorption law}) \end{aligned}$$

The proof of the other implication is similar.
We now show that $(L, \preceq)$ is a poset.
$\preceq$ is reflexive:
Let $a \in L$. Then $a \wedge a = a$ by an idempotent law. Thus $a \preceq a$.

$\preceq$ is antisymmetric:

Suppose $a \preceq b$ and $b \preceq a$. Then $a = a \vee b$ and $b = b \vee a$. Hence $a = b$ by a commutative law.

$\preceq$ is transitive:

Suppose $a \preceq b$ and $b \preceq c$. Then $a \wedge b = a$ and $b \wedge c = b$. Hence

$$\begin{aligned} a \wedge c &= (a \wedge b) \wedge c \quad \text{(since } a \wedge b = a) \\ &= a \wedge (b \wedge c) \quad \text{(by an associative law)} \\ &= a \wedge b \quad \text{(since } b \wedge c = b) \\ &= a. \end{aligned}$$

Lastly we show that for each pair of elements $\{a, b\}$, the l.u.b is $a \vee b$ and the g.l.b. is $a \wedge b$.

$a \preceq a \vee b$ since $a \wedge (a \vee b) = a$. Similarly, $b \preceq a \vee b$. Hence $a \vee b$ is an upper bound of a and b.

Now assume that $a \preceq u$ and $b \preceq u$. Then $a \vee u = u$ and $b \vee u = u$. Thus

$$\begin{aligned} (a \vee b) \vee u &= a \vee (b \vee u) \quad \text{(by associative law)} \\ &= a \vee u \quad \text{(since } b \vee u = u) \\ &= u. \end{aligned}$$

Thus $a \vee b \preceq u$. Hence $a \vee b$ is the l.u.b of $\{a, b\}$. Similarly, $a \wedge b$ is the g.l.b of $\{a, b\}$.

If $(L, \preceq)$ is a lattice as a poset, then it can be easily verified that the operations $\vee$ and $\wedge$ satisfy $L1$ to $L4$.

CHAPTER 3

T-norms and their residua

In this chapter we introduce the notion of a t-norm which is the fundamental object of study in this thesis. We also define the subclasses of t-norms that we shall mainly be concerned with, namely, left-continuous and continuous t-norms. We give some examples of such t-norms and derive a number of basic properties. The main result of this chapter is the Decomposition Theorem of continuous t-norms, which shows that every continuous t-norm can be decomposed as an ordinal sum of the three basic t-norms: Lukasiewicz, Product and Gödel. Most of the material in this chapter comes from Hájek's book [14].

3.1 Left-continuous t-norms

Definition 3.1.1. A *t-norm* is a binary operation $\circ$ on the unit interval $[0, 1]$ satisfying the following conditions.

(1) $\circ$ is commutative and associative.

(2) $\circ$ is order preserving in both arguments (i.e., $x \leq y$ implies $x \circ z \leq y \circ z$ and $z \circ x \leq z \circ y$).

(3) $1 \circ a = a$ and $0 \circ a = 0$, for all $a \in [0, 1]$.

Definition 3.1.2. A t-norm is *left-continuous* if it is left-continuous as a function from $[0,1]^2$ to $[0, 1]$. Equivalently, for each $a \in [0, 1]$, the function $f_a(x) := a \circ x$ is left-continuous on $[0, 1]$, i.e., for every increasing sequence $\{y_i\}$ in $[0, 1]$,

$$a \circ \sup \{y_i : i \in \mathbf{N}\} = \sup \{a \circ y_i : i \in \mathbf{N}\}.$$

(Recall that the limit of an increasing sequence in $[0, 1]$ is its supremum.)
It follows that $\circ$ is left-continuous if, for every set $Y \subseteq [0, 1]$ and $a \in [0, 1]$,

$$a \circ \sup Y = \sup \{a \circ y : y \in Y\}.$$

Example 1. A nilpotent minimum is a standard example of a t-norm which is left-continuous but not continuuous. It was introduced by Fodor in [9], where it is claimed as the first example of a left-continuous t-norm which is not continuous. It is defined as:

$$x \circ y = \begin{cases} \min \{x, y\} & \text{if } x + y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Every t-norm $\circ$ has a corresponding binary operation $\rightarrow$ on $[0, 1]$ called its *residuum*, defined as follows:

$$x \rightarrow y = \sup \{z : x \circ z \leq y\}.$$

The residuum of a nilpotent minimum is:

$$x \rightarrow y = \begin{cases} \max \{1 - x, y\} & \text{if } x > y \\ 1 & \text{otherwise.} \end{cases}$$

Definition 3.1.3. The *residuation property* for a t-norm $\circ$ is as follows: For all $x, y, z \in [0, 1]$,

$$x \circ y \leq z \text{ iff } x \leq y \rightarrow z.$$

Lemma 3.1.1. *A t-norm $\circ$ is left-continuous if and only if the residuation property holds.*

Proof. Let $x, y, z \in [0,1]$. Assume $\circ$ is left-continuous. This means that for every set $Y \subseteq [0,1]$,

$$x \circ \sup Y = \sup \{x \circ t : t \in Y\}.$$

Assume $x \leq y \to z$, i.e., $x \leq \sup \{t : y \circ t \leq z\}$. Then $x \circ y = y \circ x \leq y \circ \sup \{t : y \circ t \leq z\} = \sup \{y \circ t : y \circ t \leq z\} \leq z$. Conversely, suppose $x \circ y \leq z$. From $y \to z = \sup \{t : y \circ t \leq z\}$, we get $x \leq y \to z$ since $x \in \{t : y \circ t \leq z\}$.

We now show that $\circ$ is left-continuous whenever the residuation property is satisfied. Let $\{y_i : i \in \mathbf{N}\}$ be an increasing sequence in $[0,1]$. Also let $S = \{x \circ y_i : i \in \mathbf{N}\}$ and $z \in S$. Then $z = x \circ y_k$ for some $k \in \mathbf{N}$. But $y_k \leq \sup \{y_i : i \in \mathbf{N}\}$, so $x \circ y_k \leq x \circ \sup \{y_i : i \in \mathbf{N}\}$. Suppose there exists w such that $z \leq w$ for every $z \in S$. Then $x \circ y_i \leq w$ for all $i \in \mathbf{N}$. Thus $y_i \leq x \to w$ for all $i \in \mathbf{N}$, by residuation. Hence $x \to w$ is an upper bound of $\{y_i : i \in \mathbf{N}\}$. Thus $\sup \{y_i : i \in \mathbf{N}\} \leq x \to w$. Hence $x \circ \sup \{y_i : i \in \mathbf{N}\} \leq w$, by residuation. Therefore $x \circ \sup \{y_i : i \in \mathbf{N}\} = \sup \{x \circ y_i : i \in \mathbf{N}\}$. □

Note the following: If $\circ$ is left-continuous, then for $x, y \in [0,1]$,

$$x \circ (x \to y) = x \circ \sup \{z : x \circ z \leq y\} = \sup \{x \circ z : x \circ z \leq y\} \leq y.$$

Thus the supremum of $\{z : x \circ z \leq y\}$, i.e., $x \to y$, belongs to the set, so it is a maximum. Hence $x \to y = \max \{z : x \circ z \leq y\}$.

Lemma 3.1.2. *The following hold for each left-continuous t-norm $\circ$ and its residuum $\to$:*

(1) $x \leq y$ *if and only if* $x \to y = 1$

(2) $x \circ (x \to y) \leq y$

(3) $x \circ y \leq y$

(4) $1 \to x = x$.

Proof.

(1)

$$\begin{aligned} x \leq y &\Leftrightarrow 1 \circ x \leq y \\ &\Leftrightarrow 1 \leq x \to y \text{ (by residuation)} \\ &\Leftrightarrow 1 = x \to y. \end{aligned}$$

(2) From $x \to y \leq x \to y$ we get $x \circ (x \to y) \leq y$ by residuation.

(3) From $x \leq 1$ we get $x \circ y \leq 1 \circ y = y$, since $\circ$ is order preserving.

(4) From (2), if we let x be 1 and y be x, then $1 \circ (1 \to x) \leq x$. Thus $1 \to x \leq x$. Also $1 \circ x \leq x$. Hence $x \leq 1 \to x$ by residuation. Therefore $1 \to x = x$.

□

In the lemma that will follow, we shall consider a *left-continuous t-norm algebra* $\mathbf{L} = ([0,1], \circ, \to, \wedge, \vee, 0, 1)$, where $\circ$ is a fixed left-continuous t-norm and $\to$ its residuum, $\wedge$ and $\vee$ denote min and max respectively, with respect to the standard ordering $\leq$ on $[0,1]$.

Lemma 3.1.3. *The following are true in* $\boldsymbol{L} = ([0,1], \circ, \to, \wedge, \vee, 0, 1)$*:*

(1) $x \circ (x \to y) \leq x \wedge y$

(2) $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$

Proof.

(1) By Lemma 3.1.2(2), $x \circ (x \to y) \leq y$. Also $x \circ (x \to y) \leq x$ by Lemma 3.1.2(3), hence $x \circ (x \to y) \leq x \wedge y$.

(2) For any $x, y \in [0,1]$, either $x \leq y$ or $y \leq x$. Suppose $x \leq y$. Then $x \to y = 1$ by Lemma 3.1.2(1). Hence $(x \to y) \to y = 1 \to y = y$ by Lemma 3.1.2(4). Also $y \circ (y \to x) \leq x$ by Lemma 3.1.2(2). Thus $y \leq (y \to x) \to x$ by residuation. Therefore $((x \to y) \to y) \wedge ((y \to x) \to x) = y = x \vee y$. The proof of the case $y \leq x$ is similar.

□

3.2 Continuous t-norms

Definition 3.2.1. A t-norm is *continuous* if it is continuous as a function on $[0,1]^2$. Equivalently, for each $a \in [0,1]$, the function $f_a(x) := a \circ x$ is continuous on $[0,1]$, i.e., for every convergent sequence $\{y_i\}$ in $[0,1]$,

$$a \circ \lim \{y_i : i \in \mathbf{N}\} = \lim \{a \circ y_i : i \in \mathbf{N}\}.$$

The definition of the *residuum* of a continuous t-norm is the same as that of a left-continuous t-norm: $x \to y = \sup\{z : x \circ z \leq y\} = \max\{z : x \circ z \leq y\}$.

Example 2. The following are the main examples of continuous t-norms:

(1) Lukasiewicz t-norm: $x \circ_L y = \max\{0, x + y - 1\} = 0 \vee (x + y - 1)$

(2) Gödel t-norm: $x \circ_G y = \min\{x, y\} = x \wedge y$

(3) Product t-norm: $x \circ_\Pi y = x \cdot y$.

They are the most prominent examples of continuous t-norms because it is possible to describe all continuous t-norms in terms of these three by using the notion of ordinal sum (see Theorem 3.3.1 where we prove the result). Their residua are, respectively, the following: For $x \leq y$, $x \to y = 1$ and for $x > y$,

(1) $x \to_L y = 1 - x + y$

(2) $x \to_G y = y$

(3) $x \to_\Pi y = y/x$.

All properties that are satisfied by left-continuous t-norms are also satisfied by continuous t-norms.

Definition 3.2.2. An element a of $[0,1]$ is an *idempotent* of a t-norm $\circ$ if $a \circ a = a$.

Lemma 3.2.1. *The following are properties of continuous t-norms:*

(1) If $x \leq y$, then $x = y \circ (y \to x)$.

(2) If $x \leq u \leq y$ and u is an idempotent, then $x \circ y = x$.

Proof.

(1) Let $y \in [0,1]$ and let f be the function on $[0,1]$ defined by: $f(z) = z \circ y$. Then f is continuous on $[0,1]$. Also $f(0) = 0$ and $f(1) = y$. Thus for some z with $0 \leq z \leq 1$, $f(z) = x$ by the Intermediate Value Theorem. Hence for the maximum z satisfying $f(z) = x$, i.e., $z \circ y = x$, we have $z = y \to x$ by the definition of $\to$.

(2) Assume $x \leq u$ in $[0,1]$ and u is an idempotent of $\circ$. Then $x = u \circ (u \to x)$ by (1). Thus $x \circ u = u \circ (u \to x) \circ u = u \circ u \circ (u \to x) = u \circ (u \to x) = x$. Let $u \leq y$ in $[0,1]$. Then $x \circ y \geq x \circ u = x$ and also $x \circ y \leq x$ by Lemma 3.1.2(3). Therefore $x \circ y = x$.

□

Lemma 3.2.2. *Let $\circ$ be a continuous t-norm and $\boldsymbol{L} = ([0,1], \circ, \to, \wedge, \vee, 0, 1)$. Then the following is true in $\boldsymbol{L}$:*

$$x \wedge y = x \circ (x \to y)\,.$$

Proof. For any $x, y \in [0,1]$, either $x \leq y$ or $y \leq x$. If $x \leq y$, then $x \to y = 1$ by Lemma 3.1.2(1). Hence $x \circ (x \to y) = x = x \wedge y$. If $x > y$, then $x \circ (x \to y) = y = x \wedge y$ by Lemma 3.2.1(1). □

We can see from Lemma 3.2.2, that if $\circ$ is continuous, then $\wedge$ is definable in terms of $\circ$ and $\to$ and this is not the case for left-continuous t-norms.

3.3 Decomposition of continuous t-norms in terms of Product, Lukasiewicz and Gödel t-norms

Definition 3.3.1.

(1) A t-norm is called *Archimedean* if it is continuous and has no idempotents except 0 and 1. (Recall that $a \in [0,1]$ is an idempotent of $\circ$ if $a \circ a = a$.)

(2) An element $a \in [0,1]$ is called a *nilpotent* of a t-norm $\circ$ if there is a natural number n such that $a \circ a \circ ... \circ a$ (n times) $= 0$. We shall write a^n for $a \circ a \circ ... \circ a$ with n factors, and a^0 for 1.

(3) An Archimedean t-norm is called *strict* if it has no nilpotent elements except 0; otherwise it is called *nilpotent*.

Lemma 3.3.1. *If $\circ$ is an Archimedean t-norm, then for each $x \in [0,1)$:*

(1) $\lim_{n\to\infty} x^n = 0$.

(2) If $\circ$ is nilpotent, then x is nilpotent.

(3) If $0 < x < 1$, $n < m$ and $x^n > 0$, then $x^m < x^n$.

Proof.

(1) The sequence $\{x^n\}$ is non-increasing and bounded by zero from below, so $\lim_{n\to\infty} x^n$ exists and we denote it by b. We note that:

$$\begin{aligned} b \circ b &= \lim_{n\to\infty} x^n \circ \lim_{n\to\infty} x^n \\ &= \lim_{n\to\infty} x^n \circ x^n \quad \text{(by continuity of } \circ) \\ &= \lim_{n\to\infty} x^{n+n} \\ &= b. \end{aligned}$$

Hence b is an idempotent. Thus we must have $b = 0$ since $\circ$ is Archimedean.

(2) Suppose $y > 0$ is nilpotent and $0 < x < y$. Then for some natural number n, $y^n = 0$. Hence $x^n \leq y^n = 0$. Thus x^n must equal zero. If $0 < y < x < 1$ such that y is nilpotent, then for some m, $x^m < y$ since $\lim\limits_{m\to\infty} x^m = 0$. Thus $(x^m)^n \leq y^n = 0$. Hence $(x^m)^n$ must equal zero. Therefore every $x < 1$ is nilpotent.

(3) Suppose $n < m$ and $x^n = x^m$. Then $x^{n+1} = x^{m+1}$. But then $x^{n+1} = x^m$, since $n < n+1 \leq m$ and $x^{m+1} \leq x^m \leq x^{n+1} \leq x^n$. Thus $x^{m+1} = x^m$. Hence $x^k = x^m$ for all $k \geq m$, so $\lim\limits_{k\to\infty} x^k = 0 = x^m$, whence $x^n = x^m$, a contradiction.

□

Lemma 3.3.2. *If $\circ$ is an Archimedean t-norm, then for each $x \in (0,1]$ and each $n \in \boldsymbol{N}$ with $n \geq 1$, there is a unique $y \in [0,1]$ such that $y^n = x$.*

Proof. Let $n \in \mathbf{N}$ such that $n \geq 1$. If $x = 1$ then we may take $y = 1$. If $x < 1$ then, by the Intermediate Value Theorem, there exists $y \in [0,1]$ such that $y^n = x$ since $f(y) = y^n$ is continuous and we also have that $f(0) = 0$ and $f(1) = 1$. We now show that y is unique.

If $y^n = x$, then $0 < x < y < 1$. Let $x < z < y$ and $z^n = y^n$. Then $z = y \circ t$ for some $t \in (0,1)$ by Lemma 3.2.1(1). Hence $y^n = z^n = y^n \circ t^n = y^n \circ t^{(kn)}$ for every $k > 0$. But $\lim_k t^{(kn)} = 0$ by Lemma 3.3.1(1) and $x = y^n = y^n \circ 0 = 0$ by continuity and hence we have a contradiction. □

Definition 3.3.2. For an Archimedean t-norm $\circ$, $x \in (0,1]$ and $n \in \mathbf{N}$ with $n \geq 1$, let $x^{\frac{1}{n}}$ denote the unique $y \in [0,1]$ with $y^n = x$. For any rational number $r = \frac{m}{n}$, let $x^r = \left(x^{\frac{1}{n}}\right)^m$.

Lemma 3.3.3. *Let $\circ$ be an Archimedean t-norm.*

(1) If $\frac{m}{n}$ and $\frac{m'}{n'}$ are positive rational numbers such that $\frac{m}{n} = \frac{m'}{n'}$, then $x^{\frac{m}{n}} = x^{\frac{m'}{n'}}$.

(2) $x^r \circ x^s = x^{r+s}$ for all $x \in [0,1]$ and positive rational numbers r and s.

(3) If $x > 0$, then $\lim_{n\to\infty} x^{\frac{1}{n}} = 1$.

Proof.

(1) Suppose $\frac{m}{n} = \frac{m'}{n'}$. Then $m' = km, n' = kn$ for some $k \in \mathbf{N}$.
Hence $x^{\frac{m'}{n'}} = \left(x^{\frac{1}{kn}}\right)^{km} = \left(\left(x^{\frac{1}{kn}}\right)^{k}\right)^{m} = \left(x^{\frac{1}{n}}\right)^{m} = x^{\frac{m}{n}}$.

(2) Let $r = \frac{m}{n}, s = \frac{k}{n}$, where m, n and k are positive integers. Then
$x^r \circ x^s = \left(x^{\frac{1}{n}}\right)^{m} \circ \left(x^{\frac{1}{n}}\right)^{k} = \left(x^{\frac{1}{n}}\right)^{m+k} = x^{r+s}$.

(3) If $x > 0$, then the sequence $\left\{x^{\frac{1}{n}}\right\}$ is increasing and bounded from above by 1, so its limit exists. Let the limit of this sequence be a. We have that:

$$\begin{aligned} a \circ a &= \lim_{n\to\infty} x^{\frac{1}{n}} \circ \lim_{n\to\infty} x^{\frac{1}{n}} \\ &= \lim_{n\to\infty} x^{\frac{1}{n}} \circ x^{\frac{1}{n}} \quad (\text{ by continuity of } \circ) \\ &= \lim_{n\to\infty} x^{\frac{1}{n}+\frac{1}{n}} \quad (\text{ by (2) }) \\ &= a. \end{aligned}$$

Thus a is an idempotent. Therefore $\lim_{n\to\infty} x^n x^{\frac{1}{n}} = 1$, since $\circ$ is Archimedean.

□

Lemma 3.3.4. *Let $\circ$ be an Archimedean t-norm.*

(1) If $\circ$ is strict, then $([0,1], \circ)$ is isomorphic to $([0,1], \circ_{\Pi})$, where $x \circ_{\Pi} y = x \cdot y$ (product).

(2) If $\circ$ is nilpotent, then $([0,1], \circ)$ is isomorphic to $\left(\left[\frac{1}{4}, 1\right], \circ_{CP}\right)$, where $x \circ_{CP} y = max\left\{\frac{1}{4}, x \cdot y\right\}$.

Proof.

(1) Let $C = \left\{c_r = \frac{1}{2^r} : r \in \mathbf{Q}, 0 \le r < \infty\right\}$ and $D = \left\{d_r = \left(\frac{1}{2}\right)^r : r \in \mathbf{Q}, 0 \le r < \infty\right\}$, where $\left(\frac{1}{2}\right)^r$ denotes $\frac{1}{2} \circ \frac{1}{2} \circ \cdots \circ \frac{1}{2}$ (r times) and define $f : C \to D$ by $f(c_r) = d_r$.
We take the following steps:

(i) We show that C and D are dense in $[0,1]$ and f is an isomorphism from $(C, \circ_\Pi)$ to $(D, \circ)$.

(ii) We show that there exists an isomorphism $g : [0,1] \to [0,1]$ defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \in C \\ \bigvee_i f(c_i) & \text{if } x \in [0,1] \setminus C \end{cases}$$

where $x = \bigvee_i c_i$ and $\{c_i\}$ is an increasing sequence in C.

We first prove (i). We start by showing that C and D are dense subsets of $[0,1]$.

Density of C in $[0,1]$:
Let $x \in (0,1]$ and $y = \log_2 \frac{1}{x}$. Then $y \in \mathbf{R}$ and $y \geq 0$. By the density of $\mathbf{Q}$ in $\mathbf{R}$, there exists a sequence $\{q_n\}$ in $\mathbf{Q}$, $q_n \geq 0$, such that $\lim_{n\to\infty} q_n = y$. It follows that:

$$\lim_{n\to\infty} \frac{1}{2^{q_n}} = \frac{1}{2^y} = \frac{1}{2^{\log_2 \frac{1}{x}}} = x.$$

But $\frac{1}{2^{q_n}} \in C$, so $\left\{\frac{1}{2^{q_n}}\right\}$ is a sequence in C converging to $x \in (0,1]$. Also $\left\{\frac{1}{2^n}\right\}$ is a sequence in C converging to 0. Therefore C is dense in $[0,1]$.

Density of D in $[0,1]$:
Let $x \in (0,1)$. We shall approximate x from above by the elements $d_r = \left(\frac{1}{2}\right)^r$, where r has the form $\frac{m}{2^n}$. If $m_2 \geq m_1$, then $\frac{m_2}{2^n} \geq \frac{m_1}{2^n}$ and

$$\left(\tfrac{1}{2}\right)^{\frac{m_2}{2^n}} = \left(\left(\tfrac{1}{2}\right)^{\frac{1}{2^n}}\right)^{m_2} \leq \left(\left(\tfrac{1}{2}\right)^{\frac{1}{2^n}}\right)^{m_1} = \left(\tfrac{1}{2}\right)^{\frac{m_1}{2^n}} \text{ (by Lemma 3.3.1(3))}.$$

Hence, for a fixed n, $\left\{\left(\frac{1}{2}\right)^{\frac{m}{2^n}} : m \in \mathbf{N}\right\}$ is a decreasing sequence and its limit is 0.

Since $\lim_{n\to\infty} x^n \left(\dfrac{1}{2}\right)^{\frac{1}{2^n}} = 1$ by Lemma 3.3.3(3) there exists n_0 such that $\left(\frac{1}{2}\right)^{\frac{1}{2^{n_0}}} \geq x$. For $n \geq n_0$, let $m(n)$ be the largest m such that $\left(\frac{1}{2}\right)^{\frac{m}{2^n}} \geq x$. Since $\lim_{m\to\infty} x^n \left(\dfrac{1}{2}\right)^{\frac{m}{2^n}} = 0$ the largest such m exists. We have that

$$\left(\tfrac{1}{2}\right)^{\frac{m(n)+1}{2^n}} = \left(\tfrac{1}{2}\right)^{\frac{m(n)}{2^n}} \circ \left(\tfrac{1}{2}\right)^{\frac{1}{2^n}} \text{ (by Lemma 3.3.3(2))}.$$

Thus

$$\begin{aligned}
\lim_{n\to\infty}\left(\tfrac{1}{2}\right)^{\frac{m(n)+1}{2^n}} &= \lim_{n\to\infty}\left(\tfrac{1}{2}\right)^{\frac{m(n)}{2^n}} \circ \left(\tfrac{1}{2}\right)^{\frac{1}{2^n}} \\
&= \lim_{n\to\infty}\left(\tfrac{1}{2}\right)^{\frac{m(n)}{2^n}} \circ \lim_{n\to\infty}\left(\tfrac{1}{2}\right)^{\frac{1}{2^n}} \text{ (since } \circ \text{ is continuous)} \\
&= \lim_{n\to\infty}\left(\tfrac{1}{2}\right)^{\frac{m(n)}{2^n}} \circ 1 \text{ (by Lemma 3.3.3(3))} \\
&= \lim_{n\to\infty}\left(\tfrac{1}{2}\right)^{\frac{m(n)}{2^n}}.
\end{aligned}$$

We have that for any $n \geq n_0$, $\left(\frac{1}{2}\right)^{\left(\frac{m(n)}{2^n}\right)} \geq x$, so $\lim_{n\to\infty} x^n \left(\dfrac{1}{2}\right)^{\left(\frac{m(n)}{2^n}\right)} \geq x$. Also for any $n \geq n_0$, $\left(\frac{1}{2}\right)^{\left(\frac{m(n)+1}{2^n}\right)} \leq x$, so $\lim_{n\to\infty} x^n \left(\dfrac{1}{2}\right)^{\left(\frac{m(n)+1}{2^n}\right)} \leq x$. But $\lim_{n\to\infty} x^n \left(\dfrac{1}{2}\right)^{\left(\frac{m(n)+1}{2^n}\right)} = \lim_{n\to\infty} x^n \left(\dfrac{1}{2}\right)^{\left(\frac{m(n)}{2^n}\right)}$, therefore the limits of these two sequences must equal x. Thus, D is dense in $[0,1]$. Also, $(\frac{1}{2})^0 = 1 \in D$ and $\{(\frac{1}{2})^n\}$ converges to 0.

<u>f is 1-1 and strict order-preserving:</u>
Assume $c_s > c_r$. Then $s < r$. Let m_1, m_2, n be such that $r = \frac{m_1}{n}$ and $s = \frac{m_2}{n}$. Since $s < r$ this means that $m_2 < m_1$.
Now, $d_r = \left(\frac{1}{2}\right)^r = \left(\frac{1}{2}\right)^{\frac{m_1}{n}} = \left(\frac{1}{2}^{\frac{1}{n}}\right)^{m_1}$, $d_s = \left(\frac{1}{2}\right)^s = \left(\frac{1}{2}\right)^{\frac{m_2}{n}} = \left(\frac{1}{2}^{\frac{1}{n}}\right)^{m_2}$.
Letting $\left(\frac{1}{2}\right)^{\frac{1}{n}} = x$, we have $d_r = x^{m_1}$ and $d_s = x^{m_2}$. But $0 < x < 1$, $m_2 < m_1$ and $x^{m_2} > 0$, so $x^{m_1} < x^{m_2}$ by Lemma 3.3.1(3). It follows that f is strictly order preserving.

<u>f is onto:</u>
Given $d_r \in D$, there exists $c_r \in C$ such that $f(c_r) = d_r$ since both elements of D and C depend on r, where $0 \leq r < \infty$.

f is operation preserving:

$$\begin{aligned} f\left(c_r \circ_\Pi c_s\right) &= f\left(c_r \cdot c_s\right) \\ &= f\left(\left(\tfrac{1}{2^r}\right) \cdot \left(\tfrac{1}{2^s}\right)\right) \\ &= f\left(\tfrac{1}{2^{r+s}}\right) \\ &= f\left(c_{r+s}\right) \\ &= d_{r+s}, \end{aligned}$$

$$\begin{aligned} f\left(c_r\right) \circ f\left(c_s\right) &= d_r \circ d_s \\ &= \left(\tfrac{1}{2}\right)^r \circ \left(\tfrac{1}{2}\right)^s \\ &= \left(\tfrac{1}{2}\right)^{(r+s)} \\ &= d_{r+s}. \end{aligned}$$

Therefore $f\left(c_r \circ_\Pi c_s\right) = f\left(c_r\right) \circ f\left(c_r\right)$.

We now prove (ii). We first show that g is well-defined; that is, we show that if $\{c_i\}$ and $\{d_j\}$ are increasing sequences in C converging to x, then $\bigvee_i f\left(c_i\right) = \bigvee_j f\left(d_j\right)$.

Suppose $\{c_i\}$ and $\{d_j\}$ are increasing sequences converging to x. Then for any c_i there exists d_j such that $c_i \leq d_j$. Thus $f\left(c_i\right) \leq f\left(d_j\right)$. Similarly, for any d_j there exists c_i such that $d_j \leq c_i$. Hence $f\left(d_j\right) \leq f\left(c_i\right)$. Thus $\{f\left(c_i\right)\}$ and $\{f\left(d_j\right)\}$ have the same upper bounds. Therefore $\bigvee_i f\left(c_i\right) = \bigvee_j f\left(d_j\right)$.

g is 1-1 and strict order preserving:

Assume $x_1, x_2 \in [0,1]$ such that $x_1 < x_2$. Then $x_1 = \bigvee_i c_i$ and $x_2 = \bigvee_j b_j$, for increasing sequences $\{b_j\}$, $\{c_i\}$ in C. Thus there exists $b_j > x_1$. Hence $b_j > c_i$ for all i. Thus:

$$\begin{aligned} & f\left(b_j\right) > f\left(c_i\right) \text{ for all } i \quad \text{(since } f \text{ is strict order-preserving)} \\ \Rightarrow\ & \bigvee_i f\left(c_i\right) \leq f\left(b_j\right) \\ \Rightarrow\ & \bigvee_i f\left(c_i\right) \leq f\left(b_j\right) < f\left(b_{j+1}\right) \quad \text{(since } b_j < b_{j+1}, \text{ for } b_{j+1} \in C\text{)} \\ \Rightarrow\ & \bigvee_i f\left(c_i\right) < \bigvee_j f\left(b_j\right) \\ \Rightarrow\ & g\left(x_1\right) < g\left(x_2\right). \end{aligned}$$

g is onto:

Let $y \in [0,1]$. Since D is dense in $[0,1]$ and f is onto, $y = \bigvee_i f(c_i)$ for some $\{c_i\}$ in C. Therefore there exists a corresponding $x \in [0,1]$ such that $g(x) = y$, where $x = \bigvee_i c_i$.

g is operation preserving:

Let $x, y \in [0,1]$. Then $x = \bigvee_i c_i$ and $y = \bigvee_j b_j$ for increasing sequences $\{b_j\}$, $\{c_i\}$ in C. Thus

$$\begin{aligned} g(x \circ_\Pi y) &= g(x \cdot y) \\ &= g\left(\bigvee_i c_i \cdot \bigvee_j b_j\right) \\ &= g\left(\bigvee_{i,j} (c_i \cdot b_j)\right) \quad \text{(since } \cdot \text{ is continuous)} \\ &= \bigvee_{i,j} f(c_i \cdot b_j) \\ &= \bigvee_{i,j} (f(c_i) \circ f(b_j)) \quad \text{(since } f \text{ is operation preserving)} \\ &= \bigvee_i f(c_i) \circ \bigvee_j f(b_j) \quad \text{(since } \circ \text{ is continuous)} \\ &= g(x) \circ g(y). \end{aligned}$$

Therefore g is an isomorphism, which completes the proof of (ii) and also (1).

(2) Let $d = \max\{x : x \circ x = 0\}$; the maximum exists since $\circ$ is continuous. Let $C = \left\{c_r = \frac{1}{2^r} : r \in \mathbf{Q}, 0 \le r \le 2\right\}$ and $D = \{d_r = d^r : r \in \mathbf{Q}, 0 \le r \le 2, d^r > 0\}$. Also define $f : C \to D$ by $f(c_r) = d_r$. We follow the same steps as in (1). The proof of density of C and D is similar to that of (1). Note that in this case C is dense in $\left[\frac{1}{4}, 1\right]$.

We want to show that f is an isomorphism from $(C, \circ_{CP})$ to $(D, \circ)$.

f is 1-1 and strict order-preserving:

Assume $c_s > c_r$, where $0 \le] r, s \le 2$. Then $s < r$. Let m_1, m_2, n be such that $r = \frac{m_1}{n}$ and $s = \frac{m_2}{n}$. Since $s < r$ this means that $m_2 < m_1$. Now, $d_r = d^r = d^{\frac{m_1}{n}} = \left(d^{\frac{1}{n}}\right)^{m_1}$, $d_s = d^s = d^{\frac{m_2}{n}} = \left(d^{\frac{1}{n}}\right)^{m_2}$. Letting $d^{\frac{1}{n}} = x$, we have $d_r = x^{m_1}$ and $d_s = x^{m_2}$. But $0 < x < 1$ and $m_2 < m_1$, so $x^{m_1} < x^{m_2}$ by

Lemma 3.3.1(3).

f is onto:
For any $d_r \in D$ such that $0 \leq r \leq 2$, there exists $c_r \in C$ such that $f(c_r) = d_r$.

f is operation preserving:

$$\begin{aligned} f(c_r \circ_{CP} c_s) &= f\left(\max\left\{\tfrac{1}{4}, c_r \cdot c_s\right\}\right) \\ ] \qquad &= f\left(\max\left\{\tfrac{1}{4}, \tfrac{1}{2^r} \cdot \tfrac{1}{2^s}\right\}\right) \\ &= f\left(\max\left\{\tfrac{1}{4}, \tfrac{1}{2^{r+s}}\right\}\right). \end{aligned}$$

If $r + s \leq 2$, then $f(c_r \circ_{CP} c_s) = \frac{1}{2^{r+s}} = d_{r+s}$. In this case,

$$\begin{aligned} f(c_r) \circ f(c_r) &= d_r \circ d_s \\ &= d^r \circ d^s \\ &= d^{(r+s)} \\ &= d_{r+s}. \end{aligned}$$

If $r + s > 2$, then $f(c_r \circ_{CP} c_s) = \frac{1}{4}$. In this case, $d^{(r+s)} \leq d^2 = d \circ d = 0$. Therefore $f(c_r \circ_{CP} c_s) = f(c_r) \circ f(c_r)$.

The extension of f to an isomorphism g from $([\frac{1}{4}, 1], \circ_{CP})$ to $([0, 1], \circ)$ is as in (1). □

Lemma 3.3.5. *$\left(\left[\frac{1}{4}, 1\right], \circ_{CP}\right)$, where $x \circ_{CP} y = max\left\{\frac{1}{4}, x \cdot y\right\}$, is isomorphic to $([0, 1], \circ_L)$, where $\circ_L$ is the Lukasiewicz t-norm defined by $x \circ_L y = max\{0, x + y - 1\}$.*

Proof. Let $f : [0, 1] \to \left[\frac{1}{4}, 1\right]$ be defined by $f(x) = 2^{2(x-1)}$. We shall show that f is an isomorphism from $([0, 1], \circ_L)$ to $\left(\left[\frac{1}{4}, 1\right], \circ_{CP}\right)$.

f is 1-1:
Suppose $f(x_1) = f(x_2)$. Then:

$$\begin{aligned} & 2^{2(x_1-1)} = 2^{2(x_2-1)} \\ \Rightarrow\ & 2(x_1 - 1) = 2(x_2 - 1) \\ \Rightarrow\ & x_1 = x_2. \end{aligned}$$

Thus f is 1-1.

f is onto:

Let $y \in \left[\frac{1}{4}, 1\right]$ and $x = \frac{1}{2}\log_2 y + 1$. Then $x \in [0,1]$ and

$$\begin{aligned} f(x) &= 2^{2\left(\frac{1}{2}\log_2 y + 1 - 1\right)} \\ &= 2^{\log_2 y} \\ &= y. \end{aligned}$$

Hence f is onto.

f is operation preserving:

$$\begin{aligned} f(x \circ_L y) &= f(\max\{x+y-1, 0\}) \\ &= 2^{2(\max\{x+y-1,0\}-1)} \end{aligned} \tag{3.1}$$

$$\begin{aligned} f(x) \circ_{CP} f(y) &= \max\left\{\tfrac{1}{4}, f(x) \cdot f(y)\right\} \\ &= \max\left\{\tfrac{1}{4}, 2^{2(x-1)} \cdot 2^{2(y-1)}\right\} \\ &= \max\left\{\tfrac{1}{4}, 2^{2(x+y-2)}\right\}. \end{aligned} \tag{3.2}$$

If $\max\{x+y-1, 0\} = 0$, then (3.1) becomes $f(x \circ_L y) = 2^{-2} = \frac{1}{4}$ and (3.2) becomes $f(x) \circ_{CP} f(y) = \frac{1}{4}$ since $x + y - 2 < -1$. If $\max\{x+y-1, 0\} = x+y-1$, then (3.1) becomes $f(x \circ_L y) = 2^{2(x+y-2)}$ and (3.2) becomes $f(x) \circ_{CP} f(y) = 2^{2(x+y-2)}$ since $x + y - 2 > -1$. Therefore $f(x \circ_L y) = f(x) \circ_{CP} f(y)$. □

Lemma 3.3.6. *Let $\circ$ be a continuous t-norm and let $E = \{a \in [0,1] : a \circ a = a\}$. Then E is a closed subset of $[0,1]$ (in the usual topology).*

Proof. Let b be a limit point of E. Then there exists a sequence $\{a_n\}$ in E such that $\lim_{n\to\infty} a_n = b$. Hence,

$$\begin{aligned} b \circ b &= \lim_{n\to\infty} a_n \circ \lim_{n\to\infty} a_n \\ &= \lim_{n\to\infty} a_n \circ a_n \quad \text{(by continuity of } \circ\text{)} \\ &= \lim_{n\to\infty} a_n \quad \text{(since } a_n \circ a_n = a_n\text{)} \\ &= b. \end{aligned}$$

Thus $b \in E$, so E is closed. □

Note that since $E = \{a \in [0,1] : a \circ a = a\}$ is a closed subset of the set of $[0,1]$, it is a countable union of singletons and closed intervals. Thus its complement is a countable union of non-overlapping open intervals. Also 1 and 0 are in E.

Theorem 3.3.1 (Decomposition)**.** *Let $\circ$ be a continuous t-norm and E its set of idempotents, $E = \{a \in [0,1] : a \circ a = a\}$, and denote the set of open intervals in its complement by $\mathcal{I}_{open}(E)$. Also let $[a,b] \in \mathcal{I}(E)$ iff $(a,b) \in \mathcal{I}_{open}(E)$. For a closed interval $I = [a,b] \subseteq [0,1]$, let $(\circ|I)$ be the restriction of $\circ$ to $[a,b]^2$. Then:*

(1) For each $I \in \mathcal{I}(E)$, $(I, (\circ|I))$ is isomorphic either to $([0,1], \circ_\Pi)$ (Product t-norm) or $([0,1], \circ_L)$ (Lukasiewicz t-norm).

(2) If $I = [a,b]$ is a closed interval in E with $a < b$, then $(I, (\circ|I))$ is isomorphic to $([0,1], \circ_G)$ (Gödel t-norm).

(3) If $x, y \in [0,1]$ are such that there is no $I \in \mathcal{I}(E)$ with $x, y \in I$, then $x \circ y = x \wedge y$.

Proof.

(1) Let $I = [a,b] \in \mathcal{I}(E)$. There are no idempotents in I except a, b. If $x, y \in [a,b]$, then $a \le x \le b$ and $a \le y \le b$, so $a \circ a \le x \circ y \le b \circ b$. Thus $a \le x \circ y \le b$. Hence $x(\circ|I)y = x \circ y \in [a,b]$. Again, for $a \le x \le b$, we have $a \circ x = a$ and $b \circ x = x$ by Lemma 3.2.1(2). Thus a is a zero element on $[a,b]$ and b is an identity on $[a,b]$. Let $f : [a,b] \to [0,1]$ be defined by $f(x) = \frac{x-a}{b-a}$. Define an operation $*$ on $[0,1]$ by $x * y = f\left(f^{-1}(x)(\circ|I) f^{-1}(y)\right)$. Since f is an order-preserving isomorphism, it follows that $*$ is a continuous t-norm on $[0,1]$ whose only idempotents are 0 and 1. Therefore $(I, (\circ|I))$ is isomorphic to an Archimedean t-norm. An Archimedian t-norm is either nilpotent or strict, so since we have shown in Lemma 3.3.4 and Lemma 3.3.5 that each strict Archimedean t-norm is isomorphic to the product t-norm and each nilpotent Archimedean t-norm is isomorphic to the Lukasiewicz t-norm, the result follows.

(2) Let $I = [a,b]$ be a closed interval in the set E. Hence I is a closed interval of idempotents. If $x, y \in I$, then $x \circ y = x \wedge y$ by Lemma 3.2.1(2). Thus $(I, (I|\circ))$ is isomorphic to the Gödel t-norm on $[0,1]$.

(3) Suppose $x, y \in [0,1]$ such that x and y are not from the same interval $I \in \mathcal{I}(E)$. If $x < y$, then there exists an idempotent e such that $x \leq e \leq y$. Thus $x \circ y = x = x \wedge y$ by Lemma 3.2.1(2).

□

Definition 3.3.3. Let $\{[a_i, b_i] : i \in I\}$ be a countable family of closed subintervals of $[0,1]$ such that their interiors are pairwise disjoint and their union is $[0,1]$. For every $i \in I$, let $\circ_i$ be a t-norm defined on $[a_i, b_i]^2$. Then the *ordinal sum* of this family of t-norms is defined as:

$$x \circ y = \begin{cases} x \circ_k y & \text{if } \exists k \in I \text{ such that } x, y \in [a_k, b_k] \\ x \wedge y & \text{otherwise.} \end{cases}$$

Starting with a continuous t-norm $\circ$ on $[0,1]$, we have shown that its set of idempotents $E = \{a \in [0,1] : a \circ a = a\}$ is a closed subset of $[0,1]$ and hence is a disjoint union of singletons and closed intervals. Therefore its complement is a countable union of non-overlapping open intervals. We decomposed the interval $[0,1]$ into closed intervals $[a,b]$ containing no idempotents other than a, b and the closed intervals consisting entirely of idempotents. In the first case, we have seen that the restriction of $\circ$ to $[a,b]^2$ is isomorphic to an Archimedean t-norm. An Archimedean t-norm is either nilpotent or strict and we have shown that each strict Archimedean t-norm is isomorphic to the product t-norm and each nilpotent Archimedean t-norm is isomorphic to the Lukasiewicz t-norm. Hence any interval $[a,b]$ containing no idempotents except a, b together with $(\circ|I)$ is isomorphic to either the product or the Lukasiewicz t-norm. We have further shown that any interval of idempotents together with $(\circ|I)$ is isomorphic to the G*ödel* t-norm and that if x, y are not from the same interval, then $x \circ y = x \wedge y$. It follows from the definition of an ordinal sum that we can equivalently state the Decomposition Theorem as follows: Any continuous t-norm is isomorphic to an ordinal sum of Lukasiewicz, Product and Gödel t-norms.

CHAPTER 4

The logic MTL and MTL-algebras

In this chapter we define, in section 4.1, the logic MTL by stating its language, derivation rule and axioms. We list some of the formulas provable in MTL and give proofs for some of the listed formulas. In section 4.2, we give the definition of an MTL-algebra. We prove some properties of MTL-algebras. We also prove that the class of all MTL-algebras forms a variety of algebras. The results in this chapter will play a significant role in some chapters that will follow. The material in this chapter is taken from work done by Esteva and Godo in [5].

4.1 The logic MTL

Monoidal t-norm based logic or MTL for short is the logic of left-continuous t-norms. A left-continuous t-norm is used to represent a conjunction in MTL and its residuum operation represents an implication.

A Hilbert-style deductive system for MTL has been introduced in [5]. The language of the propositional calculus is defined from a countable set of propositional variables $p_1, p_2, p_3, \ldots$, three binary connectives $\circ, \rightarrow, \wedge$ and the truth constant $\bar{0}$. *Formulas* are defined inductively as follows: all propositional variables and the truth constant $\bar{0}$ are formulas; if φ and ψ are formulas, then so are $\varphi \circ \psi$, $\varphi \rightarrow \psi$ and $\varphi \wedge \psi$. Other connectives are defined in terms of the primitive connectives as follows:

(1) $\neg\varphi := \varphi \rightarrow \bar{0}$

(2) $\varphi \vee \psi := ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$

(3) $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \circ (\psi \rightarrow \varphi)$

(4) $\bar{1} := \neg\bar{0}$.

The Hilbert-style derivation rule for MTL is modus ponens: $\varphi, \varphi \rightarrow \psi \vdash \psi$.

The following are the axioms of MTL:
$(A1)$ $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
$(A2)$ $(\varphi \circ \psi) \rightarrow \varphi$
$(A3)$ $(\varphi \circ \psi) \rightarrow (\psi \circ \varphi)$
$(A4)$ $(\varphi \wedge \psi) \rightarrow \varphi$
$(A5)$ $(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$
$(A6)$ $(\varphi \circ (\varphi \rightarrow \psi)) \rightarrow (\varphi \wedge \psi)$
$(A7a)$ $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \circ \psi) \rightarrow \chi)$
$(A7b)$ $((\varphi \circ \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
$(A8)$ $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$
$(A9)$ $\bar{0} \rightarrow \varphi$.

Definition 4.1.1. A *proof* of φ_n in MTL is a sequence $\varphi_1, \varphi_2, \ldots, \varphi_n$ such that for each i $(1 \leq i \leq n)$ either φ_i is an axiom of MTL or φ_i follows from two previous members of the sequence, say φ_j and φ_k $(j < i, k < i)$ as a direct consequence of using the rule of modus ponens. In this situation, we say that φ_n is a *theorem* of MTL, or that φ_n is *provable* in MTL, and denote this by $\vdash_{MTL} \varphi$. More generally, if Γ is a set of formulas in MTL and a proof sequence $\varphi_1, \varphi_2, \ldots, \varphi_n$ as above exists but with the additional option for each i that $\varphi_i \in \Gamma$, then we say that φ_n is *provable* from Γ in MTL, denoted by $\Gamma \vdash_{MTL} \varphi_n$.

In the following lemma we give proofs of a number of theorems of MTL that will be used later in the study as well as some that we feel are of special interest.

Lemma 4.1.1. *The following formulas are provable in MTL:*

(1) $\varphi \to (\psi \to \varphi)$

(2) $(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$

(3) $\varphi \to \varphi$

(4) $(\varphi \circ (\varphi \to \psi)) \to \psi$

(5) $\varphi \to (\psi \to (\varphi \circ \psi))$

(6) $(\varphi \to \psi) \to ((\varphi \circ \chi) \to (\psi \circ \chi))$

(7) $((\varphi_1 \to \psi_1) \circ (\varphi_2 \to \psi_2)) \to ((\varphi_1 \circ \varphi_2) \to (\psi_1 \circ \psi_2))$

(8) $((\varphi \circ \psi) \circ \chi) \to (\varphi \circ (\psi \circ \chi)), (\varphi \circ (\psi \circ \chi)) \to ((\varphi \circ \psi) \circ \chi)$

(9) $(\varphi \circ \psi) \to (\varphi \wedge \psi)$

(10) $(\varphi \to \psi) \to (\varphi \to (\varphi \wedge \psi))$

(11) $(\varphi \to (\varphi \wedge \chi)) \to (((\psi \to \varphi) \wedge (\psi \to \chi)) \to (\psi \to (\varphi \wedge \chi)))$

(12) $((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi))$

(13) $\varphi \to (\varphi \wedge \varphi)$

(14) $\varphi \to (\varphi \vee \psi), \psi \to (\varphi \vee \psi), (\varphi \vee \psi) \to (\psi \vee \varphi)$

(15) $(\varphi \to \psi) \to ((\varphi \vee \psi) \to \psi)$

(16) $(\varphi \to \psi) \vee (\psi \to \varphi)$

(17) $(\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))$

(18) $((\varphi \vee \psi) \to \psi) \to (((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi))$

(19) $((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi)$

(20) $(\varphi \vee \varphi) \to \varphi$

(21) $((\varphi \to \psi) \circ (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi))$

(22) $((\varphi \to \chi) \circ (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi)$

(23) $\varphi \to (\neg\varphi \to \psi)\,, \varphi \to \neg\neg\varphi, (\varphi \circ \neg\varphi) \to \bar{0}$

(24) $(\varphi \to (\psi \circ \neg\varphi)) \to \neg\varphi$

(25) $(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$

(26) $(\varphi \to \neg\psi) \to (\psi \to \neg\varphi)$

(27) $\bar{1}$

(28) $\varphi \to (\bar{1} \circ \varphi)$

(29) $(\bar{1} \to \varphi) \to \varphi$

(30) $(\varphi \wedge (\psi \wedge \chi)) \to ((\varphi \wedge \psi) \wedge \chi)\,, ((\varphi \wedge \psi) \wedge \chi) \to (\varphi \wedge (\psi \wedge \chi))$

(31) $(\varphi \vee (\psi \vee \chi)) \to ((\varphi \vee \psi) \vee \chi)\,, ((\varphi \vee \psi) \vee \chi) \to (\varphi \vee (\psi \vee \chi))$

(32) $\varphi \to (\varphi \wedge (\varphi \vee \psi))\,, (\varphi \vee (\varphi \wedge \psi)) \to \varphi$

(33) $\varphi \leftrightarrow \varphi, (\varphi \leftrightarrow \psi) \to (\psi \leftrightarrow \varphi)\,, ((\varphi \leftrightarrow \psi) \circ (\varphi \leftrightarrow \chi)) \to (\psi \leftrightarrow \chi)$

(34) $(\varphi \leftrightarrow \psi) \to (\varphi \to \psi)\,, (\varphi \leftrightarrow \psi) \to (\psi \to \varphi)$

(35) $(\varphi \leftrightarrow \psi) \to ((\varphi \circ \chi) \leftrightarrow (\psi \circ \chi))$

(36) $(\varphi \leftrightarrow \psi) \to ((\varphi \to \chi) \leftrightarrow (\psi \to \chi))$

(37) $(\varphi \leftrightarrow \psi) \to ((\chi \to \varphi) \leftrightarrow (\chi \to \psi))$

(38) $(\varphi \leftrightarrow \psi) \to ((\varphi \to \psi) \wedge (\psi \to \varphi))$

(39) $(\varphi \circ (\psi \vee \chi)) \leftrightarrow ((\varphi \circ \psi) \vee (\varphi \circ \chi))\,, (\varphi \circ (\psi \wedge \chi)) \leftrightarrow ((\varphi \circ \psi) \wedge (\varphi \circ \chi))$

(40) $(\varphi \wedge (\psi \vee \chi)) \leftrightarrow ((\varphi \wedge \psi) \vee (\varphi \wedge \chi)), (\varphi \vee (\psi \wedge \chi)) \leftrightarrow ((\varphi \vee \psi) \wedge (\varphi \vee \chi))$

(41) $((\varphi \vee \psi) \circ (\varphi \vee \psi)) \to ((\varphi \circ \varphi) \vee (\psi \circ \psi)), ((\varphi \wedge \psi) \circ (\varphi \wedge \psi)) \to ((\varphi \circ \varphi) \wedge (\psi \circ \psi))$

(42) $(\varphi \to \psi)^n \vee (\psi \to \varphi)^n$, *for each* $n \in \boldsymbol{N}$ *where* $(\varphi \to \psi)^n$ *means* $(\varphi \to \psi) \circ (\varphi \to \psi) \circ \ldots \circ (\varphi \to \psi)$ *(n times)*

(43) $(\neg\varphi \wedge \neg\psi) \leftrightarrow \neg(\varphi \vee \psi)$

(44) $(\neg\varphi \vee \neg\psi) \leftrightarrow \neg(\varphi \wedge \psi)$

Proof. We shall prove the formulas we are going to use:

(1) $\underline{\vdash_{MTL} \varphi \to (\psi \to \varphi)}$:

(a) $\vdash_{MTL} (\varphi \circ \psi) \to \varphi$ by (A2)

(b) $\vdash_{MTL} ((\varphi \circ \psi) \to \varphi) \to (\varphi \to (\psi \to \varphi))$ by (A7b)

(c) $\vdash_{MTL} \varphi \to (\psi \to \varphi)$ by (a),(b) and modus ponens

(2) $\underline{\vdash_{MTL} (\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))}$:

(a) $\vdash_{MTL} (\varphi \to (\psi \to \chi)) \to ((\varphi \circ \psi) \to \chi)$ by (A7a)

(b) $\vdash_{MTL} \psi \circ \varphi \to \varphi \circ \psi$ by (A3)

(c) $\vdash_{MTL} ((\psi \circ \varphi) \to (\varphi \circ \psi)) \to (((\varphi \circ \psi) \to \chi) \to ((\psi \circ \varphi) \to \chi))$ by (A1)

(d) $\vdash_{MTL} ((\varphi \circ \psi) \to \chi) \to ((\psi \circ \varphi) \to \chi)$ by (b),(c) and modus ponens

(e) $\vdash_{MTL} ((\psi \circ \varphi) \to \chi) \to (\psi \to (\varphi \to \chi))$ by (A7b)

(f) $\vdash_{MTL} ((\varphi \circ \psi) \to \chi) \to (\psi \to (\varphi \to \chi))$ by (d),(e) and (A1)

(g) $\vdash_{MTL} ((\varphi \to (\psi \to \chi)) \to ((\varphi \circ \psi) \to \chi)) \to$
$((((\varphi \circ \psi) \to \chi) \to (\psi \to (\varphi \to \chi))) \to ((\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))))$
by (A1)

(h) $\vdash_{MTL} (((\varphi \circ \psi) \to \chi) \to (\psi \to (\varphi \to \chi))) \to ((\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi)))$
by (a),(g) and modus ponens

(i) $\vdash_{MTL} ((\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi)))$ by (f),(h) and modus ponens

(3) $\underline{\vdash_{MTL} \varphi \to \varphi}$:

(a) $\vdash_{MTL} (\varphi \to (\psi \to \varphi)) \to (\psi \to (\varphi \to \varphi))$ by (2)

(b) $\vdash_{MTL} \psi \to (\varphi \to \varphi)$ by (1),(a) and modus ponens

(c) $\vdash_{MTL} \psi$ for any axiom ψ

(d) $\vdash_{MTL} \varphi \to \varphi$ by (b),(c) and modus ponens

(4) $\underline{\vdash_{MTL} (\varphi \circ (\varphi \to \psi)) \to \psi}$:

(a) $\vdash_{MTL} (\varphi \to \psi) \to (\varphi \to \psi)$ by (3)

(b) $\vdash_{MTL} ((\varphi \to \psi) \to (\varphi \to \psi)) \to (\varphi \to ((\varphi \to \psi) \to \psi))$ by (2)

(c) $\vdash_{MTL} \varphi \to ((\varphi \to \psi) \to \psi)$ by (a),(b) and modus ponens

(d) $\vdash_{MTL} (\varphi \to ((\varphi \to \psi) \to \psi)) \to ((\varphi \circ (\varphi \to \psi)) \to \psi)$ by (A7b)

(e) $\vdash_{MTL} ((\varphi \circ (\varphi \to \psi)) \to \psi)$ by (c),(d) and modus ponens

(5) $\underline{\vdash_{MTL} \varphi \to (\psi \to (\varphi \circ \psi))}$:

(a) $\vdash_{MTL} (\varphi \circ \psi) \to (\varphi \circ \psi)$ by (3)

(b) $\vdash_{MTL} ((\varphi \circ \psi) \to (\varphi \circ \psi)) \to (\varphi \to (\psi \to (\varphi \circ \psi)))$ by (A7b)

(c) $\vdash_{MTL} \varphi \to (\psi \to (\varphi \circ \psi))$ by (a),(b) and modus ponens

(6) $\underline{\vdash_{MTL} (\varphi \to \psi) \to ((\varphi \circ \chi) \to (\psi \circ \chi))}$:

(a) $\vdash_{MTL} (\varphi \circ (\varphi \to \psi)) \to \psi$ by (4)

(b) $\vdash_{MTL} \psi \to (\chi \to (\psi \circ \chi))$ by (5)

(c) $\vdash_{MTL} (\varphi \circ (\varphi \to \psi)) \to (\chi \to (\psi \circ \chi))$ by (a),(b) and (A1)

(d) $\vdash_{MTL} ((\varphi \circ (\varphi \to \psi)) \to (\chi \to (\psi \circ \chi))) \to (\varphi \to ((\varphi \to \psi) \to (\chi \to (\psi \circ \chi))))$
by (A7b)

(e) $\vdash_{MTL} \varphi \to ((\varphi \to \psi) \to (\chi \to (\psi \circ \chi)))$ by (c),(d) and modus ponens

(f) $\vdash_{MTL} ((\varphi \to \psi) \to (\chi \to (\psi \circ \chi))) \to (\chi \to ((\varphi \to \psi) \to (\psi \circ \chi)))$ by (2)

(g) $\vdash_{MTL} \varphi \to (\chi \to ((\varphi \to \psi) \to (\psi \circ \chi)))$ by (e),(f) and (A1)

(h) $\vdash_{MTL} (\varphi \to (\chi \to ((\varphi \to \psi) \to (\psi \circ \chi)))) \to ((\varphi \circ \chi) \to ((\varphi \to \psi) \to (\psi \circ \chi)))$ by (A7a)

(i) $\vdash_{MTL} (\varphi \circ \chi) \to ((\varphi \to \psi) \to (\psi \circ \chi))$ by (g),(h) and modus ponens

(j) $\vdash_{MTL} ((\varphi \circ \chi) \to ((\varphi \to \psi) \to (\psi \circ \chi))) \to ((\varphi \to \psi) \to ((\varphi \circ \chi) \to (\psi \circ \chi)))$ by (2)

(k) $\vdash_{MTL} (\varphi \to \psi) \to ((\varphi \circ \chi) \to (\psi \circ \chi))$ by (i),(j) and modus ponens

(8) $\vdash_{MTL}$ (i) $(\varphi \circ (\psi \circ \chi)) \to ((\varphi \circ \psi) \circ \chi)$, (ii) $((\varphi \circ \psi) \circ \chi) \to (\varphi \circ (\psi \circ \chi))$:

(a) $\vdash_{MTL} (((\varphi \circ \psi) \circ \chi) \to \delta) \to ((\varphi \circ \psi) \to (\chi \to \delta))$ by (A7b)

(b) $\vdash_{MTL} ((\varphi \circ \psi) \to (\chi \to \delta)) \to (\varphi \to (\psi \to (\chi \to \delta)))$ by (A7b)

(c) $\vdash_{MTL} (((\varphi \circ \psi) \circ \chi) \to \delta) \to (\varphi \to (\psi \to (\chi \to \delta)))$ by (a),(b) and (A1)

(d) $\vdash_{MTL} (\varphi \to (\psi \to (\chi \to \delta))) \to (\varphi \to ((\psi \circ \chi) \to \delta))$ by (A7a)

(e) $\vdash_{MTL} (((\varphi \circ \psi) \circ \chi) \to \delta) \to (\varphi \to ((\psi \circ \chi) \to \delta))$ by (c),(d) and (A1)

(f) $\vdash_{MTL} (\varphi \to ((\psi \circ \chi) \to \delta)) \to ((\varphi \circ (\psi \circ \chi) \to \delta))$ by (A7a)

(g) $\vdash_{MTL} (((\varphi \circ \psi) \circ \chi) \to \delta) \to ((\varphi \circ (\psi \circ \chi) \to \delta))$ by (e),(f) and (A1)

(h) $\vdash_{MTL} (\varphi \circ (\psi \circ \chi)) \to ((\varphi \circ \psi) \circ \chi)$ (if we set $\delta = (\varphi \circ \psi) \circ \chi$)

Similarly, $\vdash_{MTL} ((\varphi \circ \psi) \circ \chi) \to (\varphi \circ (\psi \circ \chi))$

(9) $\vdash_{MTL} (\varphi \circ \psi) \to (\varphi \wedge \psi)$:

(a) $\vdash_{MTL} \psi \to (\varphi \to \psi)$ by (1)

(b) $\vdash_{MTL} (\psi \to (\varphi \to \psi)) \to ((\varphi \circ \psi) \to (\varphi \circ (\varphi \to \psi)))$ by (6)

(c) $\vdash_{MTL} (\varphi \circ \psi) \to (\varphi \circ (\varphi \to \psi))$ by (a),(b) and modus ponens

(d) $\vdash_{MTL} (\varphi \circ (\varphi \to \psi)) \to (\varphi \wedge \psi)$ by (A6)

(e) $\vdash_{MTL} (\varphi \circ \psi) \to (\varphi \wedge \psi)$ by (c),(d) and (A1)

(10) $\vdash_{MTL} (\varphi \to \psi) \to (\varphi \to (\varphi \wedge \psi))$:

(a) $\vdash_{MTL} ((\varphi \to \psi) \circ \varphi) \to (\varphi \circ (\varphi \to \psi))$ by (A3)

(b) $\vdash_{MTL} (\varphi \circ (\varphi \to \psi)) \to (\varphi \wedge \psi)$ by (A6)

(c) $\vdash_{MTL} ((\varphi \to \psi) \circ \varphi) \to (\varphi \wedge \psi)$ by (A1)

(d) $\vdash_{MTL} (((\varphi \to \psi) \circ \varphi) \to (\varphi \wedge \psi)) \to ((\varphi \to \psi) \to (\varphi \to (\varphi \wedge \psi)))$ by (A7b)

(d) $\vdash_{MTL} (\varphi \to \psi) \to (\varphi \to (\varphi \wedge \psi))$ by (c),(d) and modus ponens

(11) $\underline{\vdash_{MTL} (\varphi \to (\varphi \wedge \chi)) \to (((\psi \to \varphi) \wedge (\psi \to \chi)) \to (\psi \to (\varphi \wedge \chi)))}$:

(a) $\vdash_{MTL} ((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\psi \to \varphi)$ by (A4)

(b) $\vdash_{MTL} (\psi \to \varphi) \to ((\varphi \to (\varphi \wedge \chi)) \to (\psi \to (\varphi \wedge \chi)))$ by (A1)

(c) $\vdash_{MTL} ((\varphi \to \psi) \wedge (\varphi \to \chi)) \to ((\varphi \to (\varphi \wedge \chi)) \to (\psi \to (\varphi \wedge \chi)))$ by (a),(b) and (A1)

(d) $\vdash_{MTL} (((\varphi \to \psi) \wedge (\varphi \to \chi)) \to ((\varphi \to (\varphi \wedge \chi)) \to (\psi \to (\varphi \wedge \chi)))) \to$ $((\varphi \to (\varphi \wedge \chi)) \to (((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\psi \to (\varphi \wedge \chi))))$ by (2)

(e) $\vdash_{MTL} (\varphi \to (\varphi \wedge \chi)) \to (((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\psi \to (\varphi \wedge \chi)))$ by (c),(d) and modus ponens

(12) $\underline{\vdash_{MTL} ((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi))}$:

(a) $\vdash_{MTL} (\psi \to \chi) \to (\psi \to (\psi \wedge \chi))$ by (10)

(b) $\vdash_{MTL} (\psi \to (\psi \wedge \chi)) \to (((\varphi \to \psi) \wedge (\psi \to \varphi)) \to (\varphi \to (\psi \wedge \chi)))$ by (11)

(c) $\vdash_{MTL} (\psi \to \chi) \to (((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi)))$ by (a),(b) and (A1);
Similarly

(d) $\vdash_{MTL} (\chi \to \psi) \to (((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi)))$

(e) $\vdash_{MTL} ((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi))$ by (c),(d) and (A8)

(13) $\underline{\vdash_{MTL} \varphi \to (\varphi \wedge \varphi)}$:

(a) $\vdash_{MTL} \varphi \to \varphi$ by (3)

(b) $\vdash_{MTL} (\varphi \to \varphi) \to (\varphi \to (\varphi \wedge \varphi))$ by (10)

(c) $\vdash_{MTL} \varphi \to (\varphi \wedge \varphi)$ by (a),(b) and modus ponens

(14) $\underline{\vdash_{MTL} \text{(i) } \varphi \to (\varphi \vee \psi), \text{(ii) } \psi \to (\varphi \vee \psi), \text{(iii) } (\varphi \vee \psi) \to (\psi \vee \varphi)}$:

(i) :

(a) $\vdash_{MTL} (\varphi \circ (\varphi \to \psi)) \to \psi$ by (4)

(b) $\vdash_{MTL} ((\varphi \circ (\varphi \to \psi)) \to \psi) \to (\varphi \to ((\varphi \to \psi) \to \psi))$ by (A7b)

(c) $\vdash_{MTL} \varphi \to ((\varphi \to \psi) \to \psi)$ by (a),(b) and modus ponens

(d) $\vdash_{MTL} \varphi \to ((\psi \to \varphi) \to \varphi)$ by (1)

(e) $\vdash_{MTL} \varphi \to (((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi))$ by (c),(d) and (12)

(f) $\vdash_{MTL} \varphi \to (\varphi \vee \psi)$ by (e) and the definition of $\vee$

(ii) :

$\vdash_{MTL} (\varphi \vee \psi) \to (\psi \vee \varphi)$ by the definition of $\vee$ and (A5)

(iii) :

(a) $\vdash_{MTL} \psi \to (\psi \vee \varphi)$ by (i)

(b) $\vdash_{MTL} (\psi \vee \varphi) \to (\varphi \vee \psi)$ by (ii)

(c) $\vdash_{MTL} \psi \to (\varphi \vee \psi)$ by (a),(b) and (A1)

(15) $\underline{\vdash_{MTL} (\varphi \to \psi) \to ((\varphi \vee \psi) \to \psi)}$:

(a) $\vdash_{MTL} (\varphi \vee \psi) \to (((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi))$ by the definition of $\vee$

(b) $\vdash_{MTL} (((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)) \to ((\varphi \to \psi) \to \psi)$ by (A4)

(c) $\vdash_{MTL} (\varphi \vee \psi) \to ((\varphi \to \psi) \to \psi)$ by (a),(b) and (A1)

(d) $\vdash_{MTL} ((\varphi \vee \psi) \to ((\varphi \to \psi) \to \psi)) \to ((\varphi \to \psi) \to ((\varphi \vee \psi) \to \psi))$ by (2)

(e) $\vdash_{MTL} (\varphi \to \psi) \to ((\varphi \vee \psi) \to \psi)$ by (c),(d) and modus ponens

(16) $\underline{\vdash_{MTL} (\varphi \to \psi) \vee (\psi \to \varphi)}$:

(a) $\vdash_{MTL} (\varphi \to \psi) \to ((\varphi \to \psi) \vee (\psi \to \varphi))$ by (14)

(b) $\vdash_{MTL} (\psi \to \varphi) \to ((\varphi \to \psi) \vee (\psi \to \varphi))$ by (14)

(c) $\vdash_{MTL} (\varphi \to \psi) \vee (\psi \to \varphi)$ by (a),(b) and (A8)

(17) $\underline{\vdash_{MTL} (\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))}$:

(a) $\vdash_{MTL} (\chi \to \varphi) \to ((\varphi \to \psi) \to (\chi \to \psi))$ by (A1)

(b) $\vdash_{MTL} ((\chi \to \varphi) \to ((\varphi \to \psi) \to (\chi \to \psi))) \to ((\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi)))$ by (2)

(c) $\vdash_{MTL} (\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))$ by (a),(b) and modus ponens.

(18) $\underline{\vdash_{MTL} ((\varphi \vee \psi) \to \psi) \to (((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi))}$:

(a) $\vdash_{MTL} ((\varphi \to \chi) \wedge (\psi \to \chi)) \to (\psi \to \chi)$ by (A4)

(b) $\vdash_{MTL} (\psi \to \chi) \to (((\varphi \vee \psi) \to \psi) \to ((\varphi \vee \psi) \to \chi))$ by (17)

(c) $\vdash_{MTL} ((\varphi \to \chi) \wedge (\psi \to \chi)) \to (((\varphi \vee \psi) \to \psi) \to ((\varphi \vee \psi) \to \chi))$ by (a),(b) and (A1)

(d) $\vdash_{MTL} (((\varphi \to \chi) \wedge (\psi \to \chi)) \to (((\varphi \vee \psi) \to \psi) \to ((\varphi \vee \psi) \to \chi))) \to (((\varphi \vee \psi) \to \psi) \to (((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi)))$ by (2)

(e) $\vdash_{MTL} ((\varphi \vee \psi) \to \psi) \to (((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi))$ by (c),(d) and modus ponens

(19) $\underline{\vdash_{MTL} ((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi)}$:

(a) $\vdash_{MTL} (\varphi \to \psi) \to ((\varphi \vee \psi) \to \psi)$ by (15)

(b) $\vdash_{MTL} ((\varphi \vee \psi) \to \psi) \to (((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi))$ by (18)

(c) $\vdash_{MTL} (\varphi \to \psi) \to (((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi))$ by (a),(b) and (A1); Similarly

(d) $\vdash_{MTL} (\psi \to \varphi) \to (((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi))$

(e) $\vdash_{MTL} ((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi)$ by (c),(d) and (A8)

(20) $\underline{\vdash_{MTL} (\varphi \vee \varphi) \to \varphi}$:

(a) $\vdash_{MTL} \varphi \to \varphi$ by (3)

(b) $\vdash_{MTL} (\varphi \to \varphi) \to ((\varphi \vee \varphi) \to \varphi)$ by (15)

(c) $\vdash_{MTL} (\varphi \vee \varphi) \to \varphi$ by (a),(b) and modus ponens

(21) $\underline{\vdash_{MTL} ((\varphi \to \psi) \circ (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi))}$:

(a) $\vdash_{MTL} ((\varphi \to \psi) \circ (\varphi \to \chi)) \to ((\varphi \to \psi) \wedge (\varphi \to \chi))$ by (9)

(b) $\vdash_{MTL} ((\varphi \to \psi) \wedge (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi))$ by (12)

(c) $\vdash_{MTL} ((\varphi \to \psi) \circ (\varphi \to \chi)) \to (\varphi \to (\psi \wedge \chi))$ by (a),(b) and (A1)

(22) $\underline{\vdash_{MTL} ((\varphi \to \chi) \circ (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi)}$:

(a) $\vdash_{MTL} ((\varphi \to \psi) \circ (\varphi \to \chi)) \to ((\varphi \to \psi) \wedge (\varphi \to \chi))$ by (9)

(b) $\vdash_{MTL} ((\varphi \to \chi) \wedge (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi)$ by (19)

(c) $\vdash_{MTL} ((\varphi \to \chi) \circ (\psi \to \chi)) \to ((\varphi \vee \psi) \to \chi)$ by (a),(b) and modus ponens

(27) $\underline{\vdash_{MTL} \bar{1}}$:

(a) $\vdash_{MTL} \bar{0} \to \bar{0}$ by (3)

(b) $\vdash_{MTL} \bar{1}$ by (a) and the definition of $\bar{1}$

(28) $\underline{\vdash_{MTL} \varphi \to (\bar{1} \circ \varphi)}$:

(a) $\vdash_{MTL} (\bar{1} \circ \varphi) \to (\bar{1} \circ \varphi)$ by (3)

(b) $\vdash_{MTL} ((\bar{1} \circ \varphi) \to (\bar{1} \circ \varphi)) \to (\bar{1} \to (\varphi \to (\bar{1} \circ \varphi)))$ by (A7b)

(c) $\vdash_{MTL} \varphi \to (\bar{1} \circ \varphi)$ by (a),(b) and modus ponens

(30) $\underline{(\varphi \wedge (\psi \wedge \chi)) \to ((\varphi \wedge \psi) \wedge \chi)\,, ((\varphi \wedge \psi) \wedge \chi) \to (\varphi \wedge (\psi \wedge \chi))}$:

(a) $\vdash_{MTL} (\varphi \wedge (\psi \wedge \chi)) \to \delta$ for δ being $\varphi, \psi \wedge \chi, \psi, \chi, \varphi \wedge \psi, ((\varphi \wedge \psi) \wedge \chi)$ by (A4),(A1) and (21); Similarly
$\vdash_{MTL} ((\varphi \wedge \psi) \wedge \chi) \to (\varphi \wedge (\psi \wedge \chi))$

(31) $\underline{\vdash_{MTL} (\varphi \vee (\psi \vee \chi)) \to ((\varphi \vee \psi) \vee \chi)\,, ((\varphi \vee \psi) \vee \chi) \to (\varphi \vee (\psi \vee \chi))}$:

(a) $\vdash_{MTL} \delta \to (\varphi \vee (\psi \vee \chi))$ for δ being $\varphi, \psi \vee \chi, \psi, \chi, \varphi \vee \psi, ((\varphi \vee \psi) \vee \chi)$ by (14),(A1) and (22); Similarly
$\vdash_{MTL} ((\varphi \vee \psi) \vee \chi) \to (\varphi \vee (\psi \vee \chi))$

(32) $\underline{\vdash_{MTL} \text{(i) } \varphi \to (\varphi \wedge (\varphi \vee \psi))\,, \text{(ii) } (\varphi \vee (\varphi \wedge \psi)) \to \varphi}$:

(i) :

(a) $\vdash_{MTL} \varphi \to (\varphi \vee \psi)$ by (14)

(b) $\vdash_{MTL} \varphi \to \varphi$ by (3)

(c) $\vdash_{MTL} (\varphi \to \varphi) \to ((\varphi \to (\varphi \vee \psi)) \to ((\varphi \to \varphi) \circ (\varphi \to (\varphi \vee \psi))))$ by (5)

(d) $\vdash_{MTL} (\varphi \to (\varphi \vee \psi)) \to ((\varphi \to \varphi) \circ (\varphi \to (\varphi \vee \psi)))$ by (b),(c) and modus ponens

(e) $\vdash_{MTL} (\varphi \to \varphi) \circ (\varphi \to (\varphi \vee \psi))$ by (a),(d) and modus ponens

(f) $\vdash_{MTL} ((\varphi \to \varphi) \circ (\varphi \to (\varphi \vee \psi))) \to (\varphi \to (\varphi \wedge (\varphi \vee \psi)))$ by (21)

(g) $\vdash_{MTL} \varphi \to (\varphi \wedge (\varphi \vee \psi))$ by (e),(f) and modus ponens

(ii) :

(a) $\vdash_{MTL} \varphi \to \varphi$ by (3)

(b) $\vdash_{MTL} (\varphi \wedge \psi) \to \varphi$ by (A4)

(c) $\vdash_{MTL} (\varphi \to \varphi) \to (((\varphi \wedge \psi) \to \varphi) \to ((\varphi \to \varphi) \circ ((\varphi \wedge \psi) \to \varphi)))$ by (5)

(d) $\vdash_{MTL} ((\varphi \wedge \psi) \to \varphi) \to ((\varphi \to \varphi) \circ ((\varphi \wedge \psi) \to \varphi))$ by (a),(c) and modus ponens

(e) $\vdash_{MTL} (\varphi \to \varphi) \circ ((\varphi \wedge \psi) \to \varphi)$ by (b),(d) and modus ponens

(f) $\vdash_{MTL} ((\varphi \to \varphi) \circ ((\varphi \wedge \psi) \to \varphi)) \to ((\varphi \vee (\varphi \wedge \psi)) \to \varphi)$ by (22)

(g) $\vdash_{MTL} (\varphi \vee (\varphi \wedge \psi)) \to \varphi$ by (e),(f) and modus ponens.

□

4.2 MTL-algebras

Definition 4.2.1. A *residuated lattice* is an algebra $(L, \circ, \to, \wedge, \vee, 0, 1)$ satisfying the following conditions.

(1) $(L, \wedge, \vee, 0, 1)$ is a lattice with largest element 1 and least element 0 (w.r.t. the lattice ordering $\leq$).

(2) $(L, \circ, 1)$ is a commutative monoid with identity element 1.

(3) $\to$ is the residuum of $\circ$, that is, the residuation property holds: for all $x, y, z \in L$, $x \circ z \leq y$ iff $z \leq x \to y$.

Definition 4.2.2. An *MTL-algebra* is a residuated lattice $(L, \circ, \to, \wedge, \vee, 0, 1)$ such that the following *prelinearity property* holds for all $x, y \in L$:

$$(x \to y) \vee (y \to x) = 1.$$

Observe that any algebra of the form $([0,1], \circ, \to, \wedge, \vee, 0, 1)$ in which $\circ$ is a left-continuous t-norm, $\to$ its residuum and $\wedge$ and $\vee$ are min and max, respectively, is an MTL-algebra. The prelinearity property is easy to see in this case since the order is linear: either $x \leq y$ and then $x \to y = 1$, or $y \leq x$ and then $y \to x = 1$.

Lemma 4.2.1. *In each residuated lattice, the following hold:*

(1) $x \circ (x \to y) \leq y$ *and* $x \leq y \to (x \circ y)$

(2) $x \leq y$ *implies* $x \circ z \leq y \circ z$, $z \to x \leq z \to y$ *and* $y \to z \leq x \to z$

(3) $x \leq y$ *iff* $x \to y = 1$

(4) $(x \vee y) \circ z = (x \circ z) \vee (y \circ z)$

(5) $x \circ y \leq y$

(6) $(x \to y) \leq (z \to x) \to (z \to y)$ *and* $(x \to y) \leq (y \to z) \to (x \to z)$

(7) $(x \to y) \leq (x \circ z) \to (y \circ z)$

(8) $x \circ y \leq x \wedge y$

(9) $x \to (x \wedge y) = x \to y$

(10) $(x \to y) \circ (x \vee z) \leq y \vee z$

(11) $(x \wedge z) \circ (x \to y) \leq y \wedge z$

(12) $x \to 1 = 1$

(13) $1 \to x = x$

(14) $x \to (y \to z) = (x \circ y) \to z$

(15) $x \to (y \to z) = y \to (x \to z)$

(16) $x \leq (x \to y) \to y$

(17) $x \leq y \to x$

(18) $(x \to y)^n \leq x^n \to y^n$, *for all* $n \in \boldsymbol{N}$

Proof.

(1) The proof of the firt part is similar to that of Lemma 3.1.2(2). For the second part, we have that $(x \circ y) \leq (x \circ y)$. Thus $x \leq y \to (x \circ y)$ by residuation.

(2) Suppose $x \leq y$. Then by (1), $y \leq (z \to (y \circ z))$. Thus $x \leq (z \to (y \circ z))$. Hence $x \circ z \leq y \circ z$ by residuation. Again, letting $x \leq y$, we have $z \circ (z \to x) \leq x \leq y$ by (1). Hence $(z \to x) \leq (z \to y)$ by residuation. Also $x \circ (y \to z) \leq y \circ (y \to z) \leq z$. Therefore $(y \to z) \leq (x \to z)$.

(3) The proof is similar to that of Lemma 3.1.2(1).

(4) Since $x \leq x \vee y$, $x \circ z \leq (x \vee y) \circ z$ by (2). Similarly, $y \circ z \leq (x \vee y) \circ z$. Thus $(x \circ z) \vee (y \circ z) \leq (x \vee y) \circ z$. Also $x \circ z \leq (x \circ z) \vee (y \circ z)$. Hence $x \leq z \to ((x \circ z) \vee (y \circ z))$ by residuation. Similarly, $y \leq z \to ((x \circ z) \vee (y \circ z))$. Thus $(x \vee y) \leq z \to ((x \circ z) \vee (y \circ z))$. Therefore $(x \vee y) \circ z \leq ((x \circ z) \vee (y \circ z))$.

(5) The proof is similar to that of Lemma 3.1.2(3).

(6) $z \circ (z \to x) \leq x$. Hence $z \circ (z \to x) \circ (x \to y) \leq x \circ (x \to y) \leq y$ by (2). Thus $(z \to x) \circ (x \to y) \leq (z \to y)$ by residuation. Therefore $(x \to y) \leq (z \to x) \to (z \to y)$ by residuation.
The proof of the second part is similar.

(7) By (1) and (2), $x \circ (x \to y) \circ z \leq y \circ z$. Thus $(x \to y) \circ (x \circ z) \leq y \circ z$ by commutativity and associativity of $\circ$. Therefore $(x \to y) \leq (x \circ z) \to (y \circ z)$ by residuation.

(8) By (5) and commutativity, $x \circ y \leq y$ and $x \circ y \leq x$. Therefore $x \circ y \leq x \wedge y$.

(9) By (2), $x \to (x \wedge y) \leq (x \to y)$. Again we have that $x \circ (x \to y) \leq y$ and $x \circ (x \to y) \leq x$ by (1) and (5). Hence $x \circ (x \to y) \leq (x \wedge y)$. Therefore $(x \to y) \leq (x \to (x \wedge y))$ by residuation.

(10) $(x \to y) \circ (x \vee z) = (x \circ (x \to y)) \vee ((x \to y) \circ z) \leq y \vee z$ by (4),(1) and (5).

(11) Since $x \wedge z \leq x$ and $x \wedge z \leq z$, $(x \wedge z) \circ (x \to y) \leq (x \circ (x \to y)) \leq y$ and $(x \wedge z) \circ (x \to y) \leq (z \circ (x \to y)) \leq z$ by (2), (1) and (5). Therefore $(x \wedge z) \circ (x \to y) \leq y \wedge z$.

(12) By (3), $x \to 1 = 1$ since $x \leq 1$, for all x.

(13) The proof is similar to that of Lemma 3.1.2(4).

(14)

$$\begin{aligned} & x \circ (x \to (y \to z)) \leq (y \to z) \\ \Leftrightarrow \quad & y \circ x \circ (x \to (y \to z)) \leq z \quad \text{(by residuation)} \\ \Leftrightarrow \quad & (x \to (y \to z)) \leq ((x \circ y) \to z) \quad \text{(by residuation).} \end{aligned} \tag{4.1}$$

Also,

$$\begin{aligned} & (x \circ y) \circ ((x \circ y) \to z) \leq z \\ \Leftrightarrow \quad & x \circ ((x \circ y) \to z) \leq y \to z \quad \text{(by residuation)} \\ \Leftrightarrow \quad & ((x \circ y) \to z) \leq (x \to (y \to z)) \quad \text{(by residuation).} \end{aligned} \tag{4.2}$$

Therefore, $x \to (y \to z) = (x \circ y) \to z$ by (4.1) and (4.2).

(15)

$$\begin{aligned} x \to (y \to z) &= (x \circ y) \to z \quad \text{(by (14))} \\ &= (y \circ x) \to z \\ &= y \to (x \to z)\,. \end{aligned}$$

(16) By (1), $x \circ (x \to y) \leq y$. Therefore $x \leq (x \to y) \to y$ by residuation.

(17) By (5), $x \circ y \leq x$. Therefore $x \leq y \to x$ by residuation.

(18) By associativity and commutativity of $\circ$, $x^n \circ (x \to y)^n = (x \circ (x \to y))^n \leq y^n$. Therefore $(x \to y)^n \leq x^n \to y^n$ by residuation.

□

Lemma 4.2.2. *In each MTL-algebra, the following hold:*

(1) $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$

(2) $(x \to y)^n \vee (y \to x)^n = 1$, *for all* $n \in \boldsymbol{N}$.

Proof.

(1) $((x \to y) \to y) \wedge ((y \to x) \to x)$
$= [((x \to y) \to y) \wedge ((y \to x) \to x)] \circ ((x \to y) \vee (y \to x))$
(since $(x \to y) \vee (y \to x) = 1$)
$= ([((x \to y) \to y) \wedge ((y \to x) \to x)] \circ (x \to y))$
$\vee ([((x \to y) \to y) \wedge ((y \to x) \to x)] \circ (y \to x))$ (by Lemma 4.2.1(4)).
$\leq (((x \to y) \to y) \circ (x \to y)) \vee (((y \to x) \to x) \circ (y \to x))$
(since $a \wedge b \leq a$ and by Lemma 4.2.1(2))
$\leq y \vee x = x \vee y$ (by Lemma 4.2.1(1)).
Also $(x \to y) \circ (x \vee y) = (x \circ (x \to y)) \vee (y \circ (x \to y)) \leq y \vee y = y$ by Lemma 4.2.1(4),(1) and (5). Hence $x \vee y \leq (x \to y) \to y$ by residuation. Similarly, $x \vee y \leq (y \to x) \to x$. Therefore $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$.

(2) For $n = 1$, the identity $(x \to y) \vee (y \to x) = 1$ holds, by the definition of an MTL-algebra. Assume, inductively, that MTL-algebras satisfy the identity

$$(x \to y)^m \vee (y \to x)^m = 1. \tag{4.3}$$

We need to prove that $(x \to y)^{m+1} \vee (y \to x)^{m+1} = 1$. We first show that

$$x^k \circ y^{m+1-k} \leq x^{m+1} \vee y^{m+1} \text{ for } 0 \leq k \leq m+1. \tag{4.4}$$

We know that $x^k \circ y^{m+1-k} \circ \left(y^{m+1-k} \to x^{m+1-k}\right) \leq x^k \circ x^{m+1-k} = x^{m+1}$ and $x^k \circ y^{m+1-k} \circ \left(x^k \to y^k\right) \leq y^k \circ y^{m+1-k} = y^{m+1}$. Hence

$$x^k \circ y^{m+1-k} \circ \left(\left(y^{m+1-k} \to x^{m+1-k}\right) \vee \left(x^k \to y^k\right)\right) \leq x^{m+1} \vee y^{m+1}.$$

For (4.4) to hold, we must have

$$\left(y^{m+1-k} \to x^{m+1-k}\right) \vee \left(x^k \to y^k\right) = 1. \tag{4.5}$$

To prove (4.5), we prove that $(y \to x)^{m+1-k} \vee (x \to y)^k = 1$ and then use Lemma 4.2.1(18).
Now, for any k such that $1 \leq k \leq m$,

$$\begin{aligned} & (y \to x)^{m+1-k} \vee (x \to y)^k \\ \geq\ & (y \to x)^m \vee (x \to y)^m \text{ (since } k \leq m \text{ and } m+1-k \leq m) \\ =\ & 1 \text{ by (4.3).} \end{aligned}$$

Thus $\left(y^{m+1-k} \to x^{m+1-k}\right) \vee \left(x^k \to y^k\right) = 1$ by Lemma 4.2.1(18). Thus

$$\begin{aligned} x^k \circ y^{m+1-k} &= x^k \circ y^{m+1-k} \circ \left(\left(y^{m+1-k} \to x^{m+1-k}\right) \vee \left(x^k \to y^k\right)\right) \\ &\leq x^{m+1} \vee y^{m+1}. \end{aligned} \tag{4.6}$$

Now,

$$\begin{aligned} (x \vee y)^{m+1} &= \bigvee_{k=0}^{m+1} x^k \circ y^{m+1-k} \\ &= x^{m+1} \vee y^{m+1} \vee \bigvee_{k=1}^{m} x^k \circ y^{m+1-k}. \end{aligned}$$

By (4.6), for $1 \leq k \leq m$, each of $x^k \circ y^{m+1-k}$ is less or equal to $x^{m+1} \vee y^{m+1}$. Hence the identity $(x \vee y)^{m+1} = x^{m+1} \vee y^{m+1}$ holds, so

$$\begin{aligned} & (x \to y)^{m+1} \vee (y \to x)^{m+1} \\ =\ & ((x \to y) \vee (y \to x))^{m+1} \\ =\ & 1^{m+1} \\ =\ & 1. \end{aligned}$$

Therefore MTL-algebras satisfy $(x \to y)^n \vee (y \to x)^n = 1$, for all $n \in \mathbf{N}$.

□

Theorem 4.2.1. *The class of all residuated lattices is a variety, hence the class of all MTL-algebras is a variety.*

Proof. The class of all lattices forms a variety from the definition of a lattice in the preliminary section. The conditions on 0 and 1 can be expressed by identities $x \vee 1 = 1$, $x \wedge 0 = 0$. Also commutativity and associativity of $\circ$, $1 \circ x = x$ and $(x \to y) \vee (y \to x) = 1$ are identities. We verify that the residuation property is expressed by the following identities:

(1) $((x \to y) \circ x) \vee y = y$,

(2) $x \to (x \vee y) = 1$,

(3) $x \to (y \to z) = (x \circ y) \to z$.

We first show that the above identities hold in each residuated lattice. By Lemma 4.2.1(1), $(x \to y) \circ x \leq y$, and $(x \to y) \circ x \leq y$ iff $((x \to y) \circ x) \vee y = y$, so (1) holds. By Lemma 4.2.1(3), $x \to (x \vee y) = 1$, so (2) holds. By Lemma 4.2.1(14), (3) holds. We now prove that the residuation property can be derived from (1), (2), (3) and the other identities of MTL-algebras. We first prove the following property:

$$x \leq y \text{ iff } x \to y = 1.$$

If $x \leq y$, then $x \vee y = y$. Thus $x \to y = 1$ by (2). If $x \to y = 1$, then $((x \to y) \circ x) \vee y = x \vee y = y$ by (1). Hence $x \leq y$.
We now prove the residuation property ($x \circ y \leq z$ iff $x \leq y \to z$).
Suppose $x \circ y \leq z$. Then $(x \circ y) \to z = 1$. But $(x \circ y) \to z = x \to (y \to z)$ by (3), so $x \to (y \to z) = 1$. Thus $x \leq y \to z$. Conversely, if $x \leq y \to z$, then $x \to (y \to z) = 1$. Thus $(x \circ y) \to z = 1$. Therefore $x \circ y \leq z$. □

In this chapter, we started by proving some of the formulas provable in MTL with an intension of making use of them. We have proved some of the properties of MTL-algebras which we will use in the subsequent chapters. Most importantly, we have shown that a class of MTL-algebras forms a variety of algebras. We will use this result in proving Lemma 5.1.2, which in turn will be used in proving completeness of MTL with respect to MTL-algebras and MTL-chains.

CHAPTER 5

Completeness of MTL with respect to MTL-algebras and MTL-chains

In this chapter we clarify the connection between the logic MTL, MTL-algebras and MTL-chains. As we shall show, MTL is complete with respect to the variety of MTL-algebras, in the sense that a formula φ is a theorem of MTL if and only if the identity $\varphi = 1$ holds in all MTL-algebras. In addition we show that every MTL-algebra is a subdirect product of MTL-chains, that is linearly ordered MTL-algebras. Consequently, the variety of MTL-algebras is generated by the class of MTL-chains, and hence MTL is also complete with respect to the class of MTL-chains. The material in this chapter is taken from the work done by Esteva and Godo [5] and Hájek [14].

5.1 MTL completeness with respect to MTL-algebras and MTL-chains

Definition 5.1.1. Let Fm be the set of all MTL formulas and let $\sim$ be a relation on Fm defined by $\varphi \sim \psi$ iff $\vdash_{MTL} \varphi \leftrightarrow \psi$. Then $\sim$ is an equivalence relation by Lemma 4.1.1(33). Let $Fm/\sim$ be the set of equivalence classes $[\varphi]_\sim$ of $\sim$. We define on $Fm/\sim$ the following operations:

(1) $0 = [\bar{0}]_\sim$

(2) $1 = [\bar{1}]_\sim$

(3) $[\varphi]_\sim \wedge [\psi]_\sim = [\varphi \wedge \psi]_\sim$

(4) $[\varphi]_\sim \to [\psi]_\sim = [\varphi \to \psi]_\sim$

(5) $[\varphi]_\sim \circ [\psi]_\sim = [\varphi \circ \psi]_\sim$

(6) $[\varphi]_\sim \vee [\psi]_\sim = [\varphi \vee \psi]_\sim$.

These operations are well-defined by Lemma 4.1.1(35-37).

Lemma 5.1.1. *With the above definitions* $\boldsymbol{Fm}/\sim = (Fm/\sim, \circ, \to, \wedge, \vee, 0, 1)$ *is an MTL-algebra.*

Proof.
$(Fm/\sim, \wedge, \vee)$ is a lattice:

(1) $[\varphi]_\sim \wedge [\varphi]_\sim = [\varphi \wedge \varphi]_\sim = [\varphi]_\sim$, $\quad [\varphi]_\sim \vee [\varphi]_\sim = [\varphi \vee \varphi]_\sim = [\varphi]_\sim$ by Lemma 4.1.1(13),(A4), Lemma 4.1.1(14) and (20).

(2) $[\varphi]_\sim \wedge [\psi]_\sim = [\varphi \wedge \psi]_\sim = [\psi \wedge \varphi]_\sim = [\psi]_\sim \wedge [\varphi]_\sim$, $\quad [\varphi]_\sim \vee [\psi]_\sim = [\varphi \vee \psi]_\sim = [\psi \vee \varphi]_\sim = [\psi]_\sim \vee [\varphi]_\sim$ by (A5) and Lemma 4.1.1(14).

(3) $[\varphi]_\sim \wedge ([\psi]_\sim \wedge [\chi]_\sim) = [\varphi \wedge (\psi \wedge \chi)]_\sim = [(\psi \wedge \varphi) \wedge \chi]_\sim = ([\psi]_\sim \wedge [\varphi]_\sim) \wedge [\chi]_\sim$, $[\varphi]_\sim \vee ([\psi]_\sim \vee [\chi]_\sim) = [\varphi \vee (\psi \vee \chi)]_\sim = [(\psi \vee \varphi) \vee \chi]_\sim = ([\psi]_\sim \vee [\varphi]_\sim) \vee [\chi]_\sim$ by (A5) Lemma 4.1.1(30),(31).

(4) $[\varphi]_\sim \wedge ([\varphi]_\sim \vee [\psi]_\sim) = [\varphi \wedge (\varphi \vee \psi)]_\sim = [\varphi]_\sim$, $\quad [\varphi]_\sim \vee ([\varphi]_\sim \wedge [\psi]_\sim) = [\varphi \vee (\varphi \wedge \psi)]_\sim = [\varphi]_\sim$ by (A5) Lemma 4.1.1(32),(A4) and Lemma 4.1.1(14).

$(Fm/\sim, \circ, 1)$ is a commutative monoid:

(1) $[\varphi]_\sim \circ ([\psi]_\sim \circ [\chi]_\sim) = [\varphi \circ (\psi \circ \chi)]_\sim = [(\psi \circ \varphi) \circ \chi]_\sim = ([\psi]_\sim \circ [\varphi]_\sim) \circ [\chi]_\sim$ by Lemma 4.1.1(8).

(2) $[\varphi]_\sim \circ [\psi]_\sim = [\varphi \circ \psi]_\sim = [\psi \circ \varphi]_\sim = [\psi]_\sim \circ [\varphi]_\sim$ by (A3).

(3) $[\varphi]_\sim \circ [\bar{1}]_\sim = [\varphi \circ \bar{1}]_\sim = [\varphi]_\sim$ by (A4) and Lemma 4.1.1(28).

$[\chi]_\sim \leq [\varphi]_\sim \to [\psi]_\sim$ iff $[\chi]_\sim \circ [\varphi]_\sim \leq [\psi]_\sim$:

We first show that $[\varphi]_\sim \leq [\psi]_\sim$ iff $\vdash_{MTL} \varphi \to \psi$.

$$\begin{aligned}
[\varphi]_\sim \leq [\psi]_\sim \quad &\Leftrightarrow \quad [\varphi]_\sim \wedge [\psi]_\sim = [\varphi]_\sim \\
&\Leftrightarrow \quad [\varphi \wedge \psi]_\sim = [\varphi]_\sim \\
&\Leftrightarrow \quad \vdash_{MTL} (\varphi \wedge \psi) \leftrightarrow \varphi \\
&\Leftrightarrow \quad \vdash_{MTL} ((\varphi \wedge \psi) \to \varphi) \circ (\varphi \to (\varphi \wedge \psi)) \\
&\Leftrightarrow \quad \vdash_{MTL} (\varphi \to (\varphi \wedge \psi)) \qquad \text{(by (A4))} \\
&\Leftrightarrow \quad \vdash_{MTL} \varphi \to \psi \qquad \text{(by (A4),(A1), and Lemma 4.1.1(10)).}
\end{aligned} \tag{5.1}$$

We now prove the residuation property.

$$\begin{aligned}
[\chi]_\sim \leq [\varphi]_\sim \to [\psi]_\sim \quad &\Leftrightarrow \quad \vdash_{MTL} \ \chi \to (\varphi \to \psi) \quad \text{(by (5.1))} \\
&\Leftrightarrow \quad \vdash_{MTL} (\chi \circ \varphi) \to \psi \quad \text{(by (A7a))} \\
&\Leftrightarrow \quad [\chi \circ \varphi]_\sim \leq [\psi]_\sim \quad \text{(by (5.1))} \\
&\Leftrightarrow \quad [\chi]_\sim \circ [\varphi]_\sim \leq [\psi]_\sim .
\end{aligned}$$

The prelinearity condition follows from Lemma 4.1.1(16).
Therefore $\mathbf{Fm}/\sim$ is an MTL-algebra. □

Throughout the remainder of this section, let $\mathbf{L} = (L, \circ, \to, \wedge, \vee, 0, 1)$ be an MTL-algebra.

Definition 5.1.2. A *filter* F of $\mathbf{L}$ is a non-empty subset of L satisfying, for all $a, b \in L$,

(1) $a \in F$ and $b \in F$ implies $a \circ b \in F$,

(2) $a \in F$ and $a \leq b$ implies $b \in F$.

A filter F of $\mathbf{L}$ is called a *prime filter* iff for all $a, b \in L$, $a \to b \in F$ or $b \to a \in F$.

Lemma 5.1.2. *Let F be a filter of $\boldsymbol{L}$ and define a binary relation $\sim_F$ on L by:*

$$x \sim_F y \;\; \textit{iff} \;\; x \to y \in F \textit{ and } y \to x \in F.$$

Then

(1) $\sim_F$ is a congruence of $\boldsymbol{L}$ and the quotient algebra $\boldsymbol{L}/\sim_F$ is an MTL-algebra.

(2) $\boldsymbol{L}/\sim_F$ is linearly ordered iff F is a prime filter.

Proof. We first prove (1).
Reflexivity:
We have $1 \in F$ since F is nonempty. But $x \to x = 1$ so $x \to x \in F$. Hence $\sim_F$ is reflexive.
Symmetry:
Suppose $x \sim_F y$. Then $x \to y \in F$ and $y \to x \in F$. Hence $y \to x \in F$ and $x \to y \in F$, i.e., $y \sim_F x$, hence $\sim_F$ is symmetric.
Transitivity:
Assume $x \sim_F y$ and $y \sim_F z$. Then $x \to y \in F$, $y \to x \in F$, $y \to z \in F$ and $z \to y \in F$. By Lemma 4.2.1(6) and residuation, $(x \to y) \circ (y \to z) \leq x \to z$ and $(z \to y) \circ (y \to x) \leq z \to x$. Since F is a filter and $x \to y$, $y \to x$, $y \to z$ and $z \to y$ are elements of F, $(x \to y) \circ (y \to z) \in F$ and $(z \to y) \circ (y \to x) \in F$. Hence $x \to z \in F$ and $z \to x \in F$. Thus $x \sim_F z$, hence $\sim_F$ is transitive.
Therefore $\sim_F$ is an equivalence relation.
$\sim_F$ is operation preserving:
Assume $x \sim_F y$. Then $x \to y \in F$ and $y \to x \in F$. By Lemma 4.2.1(7), $x \to y \leq$

$(x \circ z) \to (y \circ\ z)$ and $y \to x \leq (y \circ z) \to (x \circ\ z)$. This implies $(x \circ z) \to (y \circ\ z) \in F$ and $(y \circ z) \to (x \circ\ z) \in F$ since $x \to y \in F$ and $y \to x \in F$.
Thus

$$x \circ z \sim_F y \circ z. \tag{5.2}$$

As for (5.2), if $z \sim_F w$, then $z \circ y \sim_F w \circ y$. Hence by commutativity of $\circ$ we have

$$y \circ z \sim_F y \circ w. \tag{5.3}$$

By (5.2) and (5.3) and transitivity of $\sim_F$ we have that

$$x \circ z \sim_F y \circ w.$$

Therefore $\sim_F$ preserves $\circ$.
We now show that $\sim_F$ preserves $\to$.
Suppose $x \sim_F y$. Then $x \to y \in F$ and $y \to x \in F$. By Lemma 4.2.1(6), $x \to y \leq (z \to x) \to (z \to y)$ and $y \to x \leq (z \to y) \to (z \to x)$, hence $(z \to x) \to (z \to y) \in F$ and $(z \to y) \to (z \to x) \in F$. Thus

$$z \to x \sim_F z \to y. \tag{5.4}$$

Suppose $w \sim_F z$. Then $w \to z \in F$ and $z \to w \in F$. By Lemma 4.2.1(6), $w \to z \leq (z \to x) \to (w \to x)$ and $z \to w \leq (w \to x) \to (z \to x)$. Hence $(z \to x) \to (w \to x) \in F$ and $(w \to x) \to (z \to x) \in F$. Thus, by (5.4), symmetry and transitivity of $\sim_F$, we get

$$w \to x \sim_F z \to y.$$

Therefore $\sim_F$ preserves $\to$.
We show that $\sim_F$ preserves $\vee$.
Assume $x \sim_F y$. Then $x \to y \in F$ and $y \to x \in F$. By Lemma 4.2.1(10) and residuation, $x \to y \leq (x \vee z) \to (y \vee z)$ and $y \to x \leq (y \vee z) \to (x \vee z)$. This implies $(x \vee z) \to (y \vee z) \in F$ and $(y \vee z) \to (x \vee z) \in F$ since $x \to y \in F$ and $y \to x \in F$.
Thus

$$x \vee z \sim_F y \vee z. \tag{5.5}$$

Assume $z \sim_F w$. As above, $z \vee y \sim_F w \vee y$. Thus, by commutativity of $\vee$, we have

$$y \vee z \sim_F y \vee w. \tag{5.6}$$

By (5.5) and (5.6) and transitivity of $\sim_F$ we have

$$x \vee z \sim_F y \vee w.$$

Therefore $\sim_F$ preserves $\vee$.

Lastly, we show that $\sim_F$ preserves $\wedge$.

Assume $x \sim_F y$. Then $x \to y \in F$ and $y \to x \in F$. By Lemma 4.2.1(11) and residuation, $x \to y \leq (x \wedge z) \to (y \wedge z)$ and $y \to x \leq (y \wedge z) \to (x \wedge z)$. This implies $(x \wedge z) \to (y \wedge z) \in F$ and $(y \wedge z) \to (x \wedge z) \in F$ since $x \to y \in F$ and $y \to x \in F$. Thus

$$x \wedge z \sim_F y \wedge z. \tag{5.7}$$

Assume $z \sim_F w$. As above, $z \wedge y \sim_F w \wedge y$, hence by commutativity of $\wedge$, we get

$$y \wedge z \sim_F y \wedge w. \tag{5.8}$$

By (5.7) and (5.8) and transitivity of $\sim_F$ we have

$$x \wedge z \sim_F y \wedge w.$$

Hence $\sim_F$ is a congruence of $\mathbf{L}$. Also MTL-algebras form a variety of algebras by Theorem 4.2.1. Therefore $\mathbf{L}/\sim_F$ is an MTL-algebra.

We now prove (2).

Assume F is a prime filter and $x, y \in L$. Then $x \to y \in F$ or $y \to x \in F$. If $x \to y \in F$, then $x \to (x \wedge y) \in F$, by Lemma 4.2.1(9). Thus $x \to (x \wedge y) \in F$ and $(x \wedge y) \to x \in F$ since $x \wedge y \leq x$, hence $x \wedge y \sim_F x$. Thus,

$$\begin{aligned} & [x \wedge y]_{\sim_F} = [x]_{\sim_F} \\ \Rightarrow \quad & [x]_{\sim_F} \wedge [y]_{\sim_F} = [x]_{\sim_F} \\ \Rightarrow \quad & [x]_{\sim_F} \leq [y]_{\sim_F}. \end{aligned}$$

Similarly, if $y \to x \in F$, then $[y]_{\sim_F} \leq [x]_{\sim_F}$. Hence $\mathbf{L}/\sim_F$ is linearly ordered.
Conversely, let F be a filter such that $\mathbf{L}/\sim_F$ is linearly ordered. Then $[x]_{\sim_F} \leq [y]_{\sim_F}$ or $[y]_{\sim_F} \leq [x]_{\sim_F}$. In the first case,

$$\begin{aligned} & [x]_{\sim_F} \wedge [y]_{\sim_F} = [x]_{\sim_F} \\ \Rightarrow\ & [x \wedge y]_{\sim_F} = [x]_{\sim_F} \\ \Rightarrow\ & x \wedge y \sim_F x \\ \Rightarrow\ & x \to (x \wedge y) \in F \text{ and } (x \wedge y) \to x \in F \\ \Rightarrow\ & x \to y \in F \text{ by Lemma 4.2.1(9)}. \end{aligned}$$

Similarly if $[y]_{\sim_F} \leq [x]_{\sim_F}$, then $y \to x \in F$. Therefore F is a prime filter. □

Lemma 5.1.3. *If θ is a congruence of $\boldsymbol{L}$, then $[1]_\theta$ is a filter.*

Proof. Since θ is reflexive, $1 \in [1]_\theta$. Let $a, b \in [1]_\theta$, i.e., $a\theta 1$ and $b\theta 1$. Hence $(a \circ b)\,\theta\,(1 \circ 1)$. Thus $(a \circ b)\,\theta 1$. Hence $a \circ b \in [1]_\theta$.
Let $a \in [1]_\theta$ and $a \leq b$. Then $a\theta 1$. Thus $(a \wedge b)\,\theta\,(1 \wedge b)$. This implies $a\theta b$. By symmetry and transitivity of θ, we get $b\theta 1$. Hence $b \in [1]_\theta$. □

Lemma 5.1.4.

(1) For any congruence θ, $\sim_{[1]_\theta} = \theta$.

(2) For any filter F, $[1]_{\sim_F} = F$.

Proof.

(1) Suppose $a \sim_{[1]_\theta} b$. Then $a \to b, b \to a \in [1]_\theta$. Since θ is a congruence, the factor algebra $\mathbf{L}/\theta$ is an MTL-algebra by Lemma 5.1.2(1) and in this algebra $[a]_\theta$ and $[b]_\theta$ are two elements such that $[a]_\theta \to [b]_\theta = [a \to b]_\theta = [1]_\theta$. Thus $[a]_\theta \leq [b]_\theta$ in $\mathbf{L}/\theta$. Similarly, $[b]_\theta \leq [a]_\theta$. Thus $[a]_\theta = [b]_\theta$. Hence $a\theta b$.
Conversely, assume $a\theta b$. Then $[a]_\theta = [b]_\theta$. Thus $[1]_\theta = [a]_\theta \to [b]_\theta = [a \to b]_\theta$. Hence $a \to b \in [1]_\theta$. Similarly, $b \to a \in [1]_\theta$. Therefore $a \sim_{[1]_\theta} b$.

(2) $[1]_{\sim_F} = \{x \in F : x \to 1 \in F \text{ and } 1 \to x \in F\}$. But $x \to 1 = 1 \in F$ and $1 \to x = x \in F$ for all $x \in F$, so $[1]_{\sim_F} = \{x : x \in F\} = F$.

□

Lemma 5.1.5. *The congruence lattice of* $\boldsymbol{L}$ *is isomorphic to the filter lattice of* $\boldsymbol{L}$.

Proof. It follows from Lemma 5.1.4 that the map $\theta \mapsto [1]_\theta$ is an isomorphism between the congruence lattice of $\mathbf{L}$ and the filter lattice of $\mathbf{L}$. □

Lemma 5.1.6. *Let* $\{F_i : i \in I\}$ *be a collection of filters of* $\boldsymbol{L}$. *Then* $\bigcap_{i\in I} F_i$ *is a filter of* $\boldsymbol{L}$.

Proof. For every $i \in I$, $1 \in F_i$ hence $1 \in \bigcap_{i\in I} F_i$. Let $a, b \in \bigcap_{i\in I} F_i$. Then $a, b \in F_i$ for every $i \in I$. Hence $a \circ b \in F_i$ for every $i \in I$. Thus $a \circ b \in \bigcap_{i\in I} F_i$. Also let $a \in \bigcap_{i\in I} F_i$ and $a \leq b$. Then $a \in F_i$ for every $i \in I$. Hence $b \in F_i$ for every $i \in I$. Thus $b \in \bigcap_{i\in I} F_i$. Therefore $\bigcap_{i\in I} F_i$ is a filter. □

Definition 5.1.3. Let $X \subseteq L$. Then $\langle X\rangle = \bigcap\{F : F \text{ is a filter of } \mathbf{L} \text{ and } X \subseteq F\}$ is the *filter generated by* X. That is, $\langle X\rangle$ is the smallest filter of $\mathbf{L}$ containing X.

Lemma 5.1.7. $\langle X\rangle = \{a \in L : (\exists n \in \boldsymbol{N})\,(\exists b_1, b_2, ..., b_n \in X)\, b_1 \circ b_2 \circ ... \circ b_n \leq a\}$.

Proof. Suppose $a \in \{a \in L : (\exists n \in \mathbf{N})\,(\exists b_1, b_2, ..., b_n \in X)\, b_1 \circ b_2 \circ ... \circ b_n \leq a\}$. Then for some $b_1, b_2, ..., b_n \in X$, $b_1 \circ b_2 \circ ... \circ b_n \leq a$. Since $X \subseteq \langle X\rangle$ and $\langle X\rangle$ is a filter, $b_1 \circ b_2 \circ ... \circ b_n \in \langle X\rangle$, so $a \in \langle X\rangle$.
Thus $\{a \in L : (\exists n \in \mathbf{N})\,(\exists b_1, b_2, ..., b_n \in X)\, b_1 \circ b_2 \circ ... \circ b_n \leq a\} \subseteq \langle X\rangle$.
Next we show that $\langle X\rangle \subseteq \{a \in L : (\exists n \in \mathbf{N})\,(\exists b_1, b_2, ..., b_n \in X)\, b_1 \circ b_2 \circ ... \circ b_n \leq a\}$.
Let $a \in X$. Then $a \in \{a \in L : (\exists n \in \mathbf{N})\,(\exists b_1, b_2, ..., b_n \in X)\, b_1 \circ b_2 \circ ... \circ b_n \leq a\}$.
Hence $X \subseteq \{a \in L : (\exists n \in \mathbf{N})\,(\exists b_1, b_2, ..., b_n \in X)\, b_1 \circ b_2 \circ ... \circ b_n \leq a\}$.
Suppose $a, c \in \{a \in L : (\exists n \in \mathbf{N})\,(\exists b_1, b_2, ..., b_n \in X)\, b_1 \circ b_2 \circ ... \circ b_n \leq a\}$. Then for some $b_1, b_2, ..., b_n \in X$, $b_1 \circ b_2 \circ ... \circ b_n \leq a$ and for some $d_1, d_2, ..., d_m \in X$, $d_1 \circ d_2 \circ ... \circ d_m \leq c$. Thus $(b_1 \circ b_2 \circ ... \circ b_n) \circ (d_1 \circ d_2 \circ ... \circ d_m) \leq a \circ c$. Thus $a \circ c \in \{a \in L : (\exists n \in \mathbf{N})\,(\exists b_1, b_2, ..., b_n \in X)\, b_1 \circ b_2 \circ ... \circ b_n \leq a\}$.
Assume $a \in \{a \in L : (\exists n \in \mathbf{N})\,(\exists b_1, b_2, ..., b_n \in X)\, b_1 \circ b_2 \circ ... \circ b_n \leq a\}$ and $a \leq c$. Then for some $b_1, b_2, ..., b_n \in X$, $b_1 \circ b_2 \circ ... \circ b_n \leq a$. By transitivity of $\leq$, $b_1 \circ b_2 \circ ... \circ b_n \leq c$, so $c \in \{a \in L : (\exists n \in \mathbf{N})\,(\exists b_1, b_2, ..., b_n \in X)\, b_1 \circ b_2 \circ ... \circ b_n \leq a\}$. Thus,

$\{a \in L : (\exists n \in \mathbf{N})\, (\exists b_1, b_2, ..., b_n \in X)\, b_1 \circ b_2 \circ ... \circ b_n \leq a\}$ is a filter containing X and hence $\langle X \rangle \subseteq \{a \in L : (\exists n \in \mathbf{N})\, (\exists b_1, b_2, ..., b_n \in X)\, b_1 \circ b_2 \circ ... \circ b_n \leq a\}$.
Therefore $\langle X \rangle = \{a \in L : (\exists n \in \mathbf{N})\, (\exists b_1, b_2, ..., b_n \in X)\, b_1 \circ b_2 \circ ... \circ b_n \leq a\}$. □

Lemma 5.1.8. *Let $a \in L$ and $a \neq 1$. Then there is a prime filter F of $\boldsymbol{L}$, not containing a.*

Proof. Note that $\{1\}$ is a filter of $\mathbf{L}$ not containing a. Assume $\langle F_i \rangle_{i \in I}$ is a chain of filters not containing a. Then $\bigcup F_i$ does not contain a since each of the F_is does not contain a. We want to show that $\bigcup F_i$ is a filter.
If $b \in \bigcup F_i$ and $b \leq c$, then $c \in \bigcup F_i$ since for some F_i, $b \in F_i$ and $b \leq c$ implies $c \in F_i$. Let $b, c \in \bigcup F_i$. Then $b \in F_k$ and $c \in F_j$ for some k and j. We have that either $F_k \subseteq F_j$ or $F_j \subseteq F_k$. If $F_k \subseteq F_j$, then $b \in F_j$. Hence $b \circ c \in F_j$ since F_j is a filter. Thus $b \circ c \in \bigcup F_i$. Similarly, if $F_j \subseteq F_k$, we have that $b \circ c \in F_k$. Hence $b \circ c \in \bigcup F_i$. Therefore $\bigcup F_i$ is a filter not containing a.
We have shown that whenever $\langle F_i \rangle_{i \in I}$ is a chain of filters not containing a, then $\bigcup F_i$ is also a filter not containing a. Hence by Zorn's lemma, there exists a maximal filter not containing a, say F. Suppose F is not prime. Then there exist $x, y \in L$ such that $x \to y \notin F$ and $y \to x \notin F$. Let F_j and F_k be filters generated by $F \cup \{y \to x\}$ and $F \cup \{x \to y\}$ respectively. By Lemma 5.1.7,
$\langle F \cup \{x \to y\} \rangle = F_j = \{u \in L : (\exists v \in F)\, (\exists n \in \mathbf{N})\; v \circ (x \to y)^n \leq u\}$ and
$\langle F \cup \{y \to x\} \rangle = F_k = \{u \in L : (\exists v \in F)\, (\exists n \in \mathbf{N})\; v \circ (y \to x)^n \leq u\}$.
Now $F \subsetneq F_j$, so $a \in F_j$. Similarly $a \in F_k$. This means that $(\exists v_1 \in F)\, (\exists n_1 \in \mathbf{N})$ such that $v_1 \circ (x \to y)^{n_1} \leq a$ and $(\exists v_2 \in F)\, (\exists n_2 \in \mathbf{N})$ such that $v_2 \circ (y \to x)^{n_2} \leq a$.
Let $v = v_1 \circ v_2$. Then $v \in F$ since $v_1, v_2 \in F$. Also let $n = max\,\{n_1, n_2\}$.
Then $v \circ (x \to y)^n \leq v_1 \circ (x \to y)^{n_1} \leq a$ and $v \circ (y \to x)^n \leq v_2 \circ (y \to x)^{n_2} \leq a$. This implies that $(v \circ (x \to y)^n) \vee (v \circ (y \to x)^n) \leq a$. It follows that
$v \circ ((x \to y)^n \vee (y \to x)^n) \leq a$. Thus $v \circ 1 \leq a$ by Lemma 4.2.2(2). But $v \in F$ hence $a \in F$, a contradiction. Therefore F is prime. □

Let U be the set of all prime filters of $\mathbf{L}$ and let $f : L \to \prod_{F \in U} L_F$ be defined by $f(x) = \left\{[x]_{\sim_F} : F \in U\right\}$. Then f is a homomorphism. For $F \in U$, $\mathbf{L}/\sim_F$ is an MTL-algebra. Also $\mathbf{L}/\sim_F$ is linearly ordered since F is prime. For $F \in U$, let $\mathbf{L}_F = \mathbf{L}/\sim_F$ and $\mathbf{L}^* = \prod_{F \in U} \mathbf{L}_F$. Then $\mathbf{L}^*$ is an MTL-algebra since each $\mathbf{L}_F$ is an MTL-algebra.

Theorem 5.1.1. *Each MTL-algebra is a subdirect product of linearly ordered MTL-algebras.*

Proof. Following the above discussion, we need only show that $\mathbf{L}$ is isomorphic to a subalgebra of $\mathbf{L}^*$. In particular, we want to show that f is $1-1$, which is equivalent to showing that $\bigcap U = \{1\}$. If $a \in U$ and $a \neq 1$, then by Lemma 5.1.8, there is a prime filter $F \in U$ such that $a \notin F$, hence $a \notin \bigcap U$. Thus, $\bigcap U = \{1\}$ and so f is an embedding of $\mathbf{L}$ into $\mathbf{L}^*$. □

Definition 5.1.4. Let $\mathbf{L}$ be an MTL-algebra. An ***L**-evaluation* of propositional variables is any mapping e assigning to each propositional variable p an element $e(p)$ of L. An evaluation e extends to arbitrary formulas of MTL as follows:

(1) $e(\varphi \wedge \psi) = e(\varphi) \wedge e(\psi)$

(2) $e(\varphi \rightarrow \psi) = e(\varphi) \rightarrow e(\psi)$

(3) $e(\varphi \circ \psi) = e(\varphi) \circ e(\psi)$

(4) $e(\bar{0}) = 0$.

Definition 5.1.5. Let $\mathbf{L}$ be an MTL-algebra. An (MTL) formula φ is an ***L**-tautology* if $e(\varphi) = 1$ for each $\mathbf{L}$-evaluation e. That is, φ is an ***L**-tautology* iff $\mathbf{L}$ satisfies the identity $\varphi = 1$.

Theorem 5.1.2 (Completeness)**.** *MTL is complete with respect to MTL-algebras and MTL-chains, that is, for each formula φ the following are equivalent.*

(1) φ is provable in MTL.

*(2) For each linearly ordered MTL-algebra **L**, φ is an **L**-tautology.*

*(3) For each MTL-algebra **L**, φ is an **L**-tautology.*

Proof.
(1)⇒(2):
Suppose that φ is provable in MTL. We use induction on the number of steps in the

proof of φ. For the base step, assume that the proof of φ has only one step in it. Then φ must be an axiom of MTL. We have to prove that all axioms of MTL are **L**-tautologies. $\underline{(A1)\,(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)):}$

It suffices to show that: $1 \leq (x \to y) \to ((y \to z) \to (x \to z))$.

Using the residuation three times on the above, we get

$(x \to y) \circ (y \to z) \circ x \leq z$.

This is true in **L** since $x \circ (x \to y) \leq y$ and similarly $y \circ (y \to z) \leq z$.

$\underline{(A2)\,(\varphi \circ \psi) \to \varphi:}$

Since $x \circ y \leq x$, $(x \circ y) \to x = 1$.

$\underline{(A3)\,(\varphi \circ \psi) \to (\psi \circ \varphi):}$

Since $x \circ y = y \circ x$, $(x \circ y) \to (y \circ x) = 1$.

The proof of $(A4)$ is similar to that of $(A2)$.

The proof of $(A5)$ is similar to that of $(A3)$.

$\underline{(A6)\,(\varphi \circ (\varphi \to \psi)) \to (\varphi \wedge \psi):}$

We have $x \circ (x \to y) \leq y$ and $x \circ (x \to y) \leq x$, so $x \circ (x \to y) \leq x \wedge y$.

$\underline{(A7a) \text{ and } (A7b)}$:

The proof follows from Lemma 4.2.1(14).

$\underline{(A8)\,((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi):}$

$((x \to y) \to z) \circ ((y \to x) \to z)$

$= [((x \to y) \to z) \circ ((y \to x) \to z)] \circ ((x \to y) \vee (y \to x))$

(since $(x \to y) \vee (y \to x) = 1$)

$= ([((x \to y) \to z) \circ ((y \to x) \to z)] \circ (x \to y))$

$\vee ([((x \to y) \to z) \circ ((y \to x) \to z)] \circ (y \to x))$ (by Lemma 4.2.1(4))

$\leq (((x \to y) \to z) \circ (x \to y)) \vee (((y \to x) \to z) \circ (y \to x))$ (by Lemma 4.2.1(5))

$\leq z \vee z = z$ (by Lemma 4.2.1(1)).

$\underline{(A9)\,\bar{0} \to \varphi:}$

This follows from $0 \leq x$.

Now suppose that the proof of φ contains n steps, where $n > 1$ and suppose as induction hypothesis that all theorems of MTL which have proofs less than n steps are **L**-tautologies. Either φ is an axiom of MTL in which case φ is an **L**-tautology, or φ follows from previous formulas by modus ponens. These formulas must have the forms ψ and $\psi \to \varphi$. But ψ and $\psi \to \varphi$ are theorems of **L** with proof sequences containing less than n steps. Hence $e(\psi) = 1$ and $e(\psi \to \varphi) = e(\psi) \to e(\varphi) = 1$, so

$e(\varphi) = 1$. Therefore, by the Principle of Mathematical induction, every theorem of MTL is an **L**-tautology.

(2)⇒(3):

By Theorem 5.1.1, every MTL-algebra **L** is isomorphic to a subdirect product of linearly ordered MTL-algebras. If $\varphi = 1$ holds in every linearly ordered MTL-algebra, then it also holds in **L**.

(3)⇒(1):

Suppose that for every MTL-algebra **L**, $\varphi = 1$ holds. Consider the MTL-algebra **Fm**/∼ in Definition 5.1.1. In this MTL-algebra we have that $[\varphi] = [\bar{1}]$, hence

$$\begin{aligned} & \vdash_{MTL} \varphi \leftrightarrow \bar{1} \\ \Rightarrow\ & \vdash_{MTL} (\varphi \rightarrow \bar{1}) \circ (\bar{1} \rightarrow \varphi) \\ \Rightarrow\ & \vdash_{MTL} (\bar{1} \rightarrow \varphi) \\ \Rightarrow\ & \vdash_{MTL} \varphi. \end{aligned}$$

□

We have established the connection between the logic MTL and the variety of MTL-algebras: that MTL is complete with respect to the variety of MTL-algebras. In addition, we have shown that MTL is complete with respect to the class of MTL-chains. In particular, we have that the variety of MTL-algebras is generated by the class of MTL-chains. The fact that MTL is complete with respect to MTL-chains will contribute towards proving completeness of MTL with respect to standard MTL-algebras.

CHAPTER 6

Compeleteness of MTL with respect to standard MTL-algebras

We present in this chapter a proof of completeness of MTL with respect to 'standard MTL-algebras', where a standard MTL-algebra is a left-continuous t-norm algebra. The proof of the Completeness Theorem of MTL with respect to standard MTL-algebras shows that a formula φ is a theorem of MTL if and only if the identity $\varphi = 1$ holds in all standard MTL-algebras. To prove this Completeness Theorem we use the fact that MTL is complete with respect to linearly ordered MTL-algebras and show that every finitely generated linearly ordered MTL-algebra can be embedded into a standard MTL-algebra. It will be enough to use finitely generated MTL-algebras since any algebra is a subalgebra of an ultraproduct of its finitely generated subalgebras, by [3, Theorem 2.14]. The results discussed in this chapter come from the work of Jenei and Montagna [20].

6.1 Standard completeness of MTL

Theorem 6.1.1. *If* $\boldsymbol{L} = (L, \circ_L, \rightarrow_L, \wedge_L, \vee_L, 0_L, 1_L)$ *is a countable linearly ordered MTL-algebra, then there exists a countable densely linearly ordered MTL-algebra* $\boldsymbol{X} = (X, *, \rightarrow_*, \wedge_X, \vee_X, 0_X, 1_X)$ *and an embedding* ϕ *from* $\boldsymbol{L}$ *into* $\boldsymbol{X}$.

Proof.
We first construct $\mathbf{X} = (X, *, \rightarrow_*, \wedge_X, \vee_X, 0_X, 1_X)$ that is a densely and linearly ordered MTL-algebra.
Let $X = \{(s,q) : s \in L, s \neq 0_L, q \in \mathbf{Q} \cap (0,1]\} \cup \{(0_L, 1)\}$. Then X is countable since L and $\mathbf{Q}$ are. For $(s,q), (t,r) \in X$, we define:
$(s,q) \preceq (t,r)$ iff either $s <_L t$, or $s = t$ and $q \leq r$.
(Note that this is a lexicographical ordering.)
$\preceq$ is a partial order:
Suppose $(s,q) \in X$. Then $(s,q) \preceq (s,q)$ since $s = s$ and $q = q$ so that $q \leq q$.
Hence $\preceq$ is reflexive.
Suppose $(s,q) \preceq (t,r)$ and $(t,r) \preceq (s,q)$. Then we have the following cases.
case (i): $s <_L t$ and $t <_L s$. This case is impossible.
case (ii): $s <_L t$ and ($s = t$ and $r \leq q$). The case is impossible.
case (iii): ($s = t$ and $q \leq r$) and $t <_L s$. The case is impossible.
case (iv): ($s = t$ and $q \leq r$) and ($s = t$ and $r \leq q$). Then $s = t$ and $q = r$ since $\leq$ is antisymmetric. Hence $(s,q) = (t,r)$. Thus $\preceq$ is antisymmetric.
Suppose $(s,q) \preceq (t,r)$ and $(t,r) \preceq (u,p)$. Then we have the following cases.
case (i): $s <_L t$ and $t <_L u$. Then $s <_L u$ since $\leq_L$ is transitive. Hence $(s,q) \preceq (u,p)$.
case (ii): $s <_L t$ and ($t = u$ and $r \leq p$). Then $s <_L u$. Hence $(s,q) \preceq (u,p)$.
case (iii): ($s = t$ and $q \leq r$) and $t <_L u$. Then $s <_L u$. Hence $(s,q) \preceq (u,p)$.
case (iv): ($s = t$ and $q \leq r$) and ($t = u$ and $r \leq p$). Then $s = u$ and $q \leq p$ since $=$ and $\leq$ are transitive. Hence $(s,q) \preceq (u,p)$. Therefore $\preceq$ is transitive.
$\preceq$ is a linear order:
This follows from the fact that $\mathbf{L}$ and $\mathbf{Q}$ are linearly ordered.
$(X, \preceq)$ is densely ordered:
Assume $(s,q) \prec (t,r)$. Then either $s <_L t$ or $s = t$ and $q < r$. If $s <_L t$, then

$(s,q) \prec \left(t, \frac{r}{2}\right) \prec (t,r)$. If $s = t$ and $q < r$, then $(s,q) \prec \left(s, \frac{q+r}{2}\right) \prec (t,r)$.
$(1_L, 1)$ is the maximum of $(X, \preceq)$:
Let $(s,q) \in X$. Then $s \leq_L 1_L$, since 1_L is the maximum of L. If $s <_L 1_L$, then $(s,q) \preceq (1_L, 1)$. If $s = 1_L$, then $(s,q) \preceq (1_L, 1)$ as $q \leq 1$.
$(0_L, 1)$ is the minimum of $(X, \preceq)$:
Let $(s,q) \in X$. Then $0_L \leq_L s$, since 0_L is the minimum of L. If $0_L <_L s$, then $(0_L, 1) \preceq (s,q)$. If $0_L = s$, the only element in X with 0_L as the first co-ordinate is $(0_L, 1)$, so $(0_L, 1) \preceq (s,q)$.

For $(s,q), (t,r) \in X$, we define $*$ as follows:

$$(s,q) * (t,r) = \begin{cases} (s,q) \wedge_X (t,r) & \text{if } \ s \circ_L t = s \wedge_L t \\ (s \circ_L t, 1) & \text{otherwise} \end{cases}$$

where $(s,q) \wedge_X (t,r)$ is the minimum of (s,q) and (t,r) with respect to $\preceq$ and $s \wedge_L t$ is the minimum of s and t with respect to $\leq_L$.
$*$ is commutative:

$$\begin{aligned} (s,q) * (t,r) \ &= \begin{cases} (s,q) \wedge_X (t,r) & \text{if } \ s \circ_L t = s \wedge_L t \\ (s \circ_L t, 1) & \text{otherwise} \end{cases} \\ &= \begin{cases} (t,r) \wedge_X (s,q) & \text{if } \ t \circ_L s = t \wedge_L s \\ (t \circ_L s, 1) & \text{otherwise} \end{cases} \\ &= (t,r) * (s,q) \end{aligned}$$

$*$ is associative:

$$(s,q)*((t,r)*(u,p)) = \begin{cases} (s,q)*((t,r)\wedge_X (u,p)) & \text{if } t\circ_L u = t\wedge_L u \\ (s,q)*(t\circ_L u, 1) & \text{otherwise} \end{cases}$$

$$= \begin{cases} \begin{cases} (s,q)\wedge_X ((t,r)\wedge_X (u,p)) & \text{if } s\circ_L (t\circ_L u) = s\wedge_L (t\circ_L u) \\ & \text{and } t\circ_L u = t\wedge_L u \\ (s\circ_L (t\circ_L u), 1) & \text{otherwise} \end{cases} \\ \\ \begin{cases} (s,q)\wedge_X (t\circ_L u, 1) & \text{if } s\circ_L (t\circ_L u) = s\wedge_L (t\circ_L u) \\ & \text{and } t\circ_L u < t\wedge_L u \\ (s\circ_L (t\circ_L u), 1) & \text{otherwise} \end{cases} \end{cases}$$

$$= \begin{cases} (s,q)\wedge_X ((t,r)\wedge_X (u,p)) & \text{if } s\circ_L (t\circ_L u) = s\wedge_L (t\circ_L u) \\ & \text{and } t\circ_L u = t\wedge_L u \\ (s\circ_L (t\circ_L u), 1) & \text{if } s\circ_L (t\circ_L u) <_L s\wedge_L (t\circ_L u) \\ & \text{and } t\circ_L u = t\wedge_L u \\ (s,q)\wedge_X (t\circ_L u, 1) & \text{if } s\circ_L (t\circ_L u) = s\wedge_L (t\circ_L u) \\ & \text{and } t\circ_L u <_L t\wedge_L u \\ (s\circ_L (t\circ_L u), 1) & \text{if } s\circ_L (t\circ_L u) <_L s\wedge_L (t\circ_L u) \\ & \text{and } t\circ_L u <_L t\wedge_L u \end{cases} \quad (6.1)$$

Note that $(t,r)\wedge_X (u,p)$ is either (t,r) or (u,p) and the first co-ordinate is $t\wedge_L u = t\circ_L u$.

We also have that:

$$((s,q)*(t,r))*(u,p) = \begin{cases} ((s,q)\wedge_X (t,r))*(u,p) & \text{if } s\circ_L t = s\wedge_L t \\ (s\circ_L t,1)*(u,p) & \text{otherwise} \end{cases}$$

$$= \begin{cases} \begin{cases} ((s,q)\wedge_X (t,r))\wedge_X (u,p) & \text{if } (s\circ_L t)\circ_L u = (s\circ_L t)\wedge_L u \\ & \text{and } s\circ_L t = s\wedge_L t \\ ((s\circ_L t)\circ_L u, 1) & \text{otherwise} \end{cases} \\ \\ \begin{cases} (s\circ_L t,1)\wedge_X (u,p) & \text{if } (s\circ_L t)\circ_L u = (s\circ_L t)\wedge_L u \\ & \text{and } s\circ_L t < s\wedge_L t \\ ((s\circ_L t)\circ_L u, 1) & \text{otherwise} \end{cases} \end{cases}$$

$$= \begin{cases} ((s,q)\wedge_X (t,r))\wedge_X (u,p) & \text{if } (s\circ_L t)\circ_L u = (s\circ_L t)\wedge_L u \\ & \text{and } s\circ_L t = s\wedge_L t \\ ((s\circ_L t)\circ_L u, 1) & \text{if } (s\circ_L t)\circ_L u <_L (s\circ_L t)\wedge_L u \\ & \text{and } s\circ_L t = s\wedge_L t \\ (s\circ_L t,1)\wedge_X (u,p) & \text{if } (s\circ_L t)\circ_L u = (s\circ_L t)\wedge_L u \\ & \text{and } s\circ_L t <_L s\wedge_L t \\ ((s\circ_L t)\circ_L u, 1) & \text{if } (s\circ_L t)\circ_L u <_L (s\circ_L t)\wedge_L u \\ & \text{and } s\circ_L t <_L s\wedge_L t \end{cases} \tag{6.2}$$

We are now comparing (6.1) and (6.2):

Case (i): If $s\circ_L (t\circ_L u) = s\wedge_L (t\circ_L u)$ and $t\circ_L u = t\wedge_L u$, then $s\circ_L (t\circ_L u) = s\wedge_L t\wedge_L u$. Thus

$$(s,q)*((t,r)*(u,p)) = (s,q)\wedge_X (t,r)\wedge_X (u,p)$$

$$((s,q)*(t,r))*(u,p) = (s,q)\wedge_X (t,r)\wedge_X (u,p)\,.$$

Therefore $(s,q)*((t,r)*(u,p)) = ((s,q)*(t,r))*(u,p)$.

Case (ii): If $s\circ_L (t\circ_L u) <_L s\wedge_L (t\circ_L u)$ and $t\circ_L u = t\wedge_L u$, then we have:

$$(s,q)*((t,r)*(u,p)) = (s\circ_L (t\circ_L u)\,, 1)$$

$$((s,q)*(t,r))*(u,p) = ((s\circ_L t)\circ_L u, 1)$$

But $(s \circ_L (t \circ_L u), 1) = ((s \circ_L t) \circ_L u, 1)$ since $\circ_L$ is associative.
Therefore $(s,q) * ((t,r) * (u,p)) = ((s,q) * (t,r)) * (u,p)$.
<u>Case (iii):</u> $s \circ_L (t \circ_L u) = s \wedge_L (t \circ_L u)$ and $t \circ_L u <_L t \wedge_L u$.
If $(t \circ_L u) <_L s$, then $s \circ_L (t \circ_L u) = t \circ_L u$.
Thus

$$(s,q) * ((t,r) * (u,p)) = (t \circ_L u, 1).$$

If $(s \circ_L t) <_L u$, then $(s \circ_L t) \circ_L u = s \circ_L t$.
Hence

$$\begin{aligned} ((s,q) * (t,r)) * (u,p) &= (s \circ_L t, 1) \\ &= (t \circ_L u, 1) \text{ since } s \circ_L (t \circ_L u) = t \circ_L u = s \circ_L t. \end{aligned}$$

If $s <_L t \circ_L u$, then $s \wedge_L (t \circ_L u) = s$. Thus $s \circ_L (t \circ_L u) = s$. But $s \circ_L (t \circ_L u) \leq_L t$ and $s \circ_L (t \circ_L u) \leq_L u$, so $s = s \wedge_L t \wedge_L u$. Hence $s \circ_L (t \circ_L u) = s \wedge_L t \wedge_L u$ and we fall into case (i).
Therefore $(s,q) * ((t,r) * (u,p)) = ((s,q) * (t,r)) * (u,p)$.
<u>Case (iv):</u> $s \circ_L (t \circ_L u) = s \wedge_L (t \circ_L u)$, $t \circ_L u <_L t \wedge_L u$ and $s \circ_L t <_L u$. The proof of this case is similar to that of case (iii).
<u>Case (v):</u> If $s \circ_L (t \circ_L u) <_L s \wedge_L (t \circ_L u)$ and $t \circ_L u <_L t \wedge_L u$, then:

$$(s,q) * ((t,r) * (u,p)) = (s \circ_L (t \circ_L u), 1)$$

$$((s,q) * (t,r)) * (u,p) = ((s \circ_L t) \circ_L u, 1).$$

But $(s \circ_L (t \circ_L u), 1) = ((s \circ_L t) \circ_L u, 1)$ since $\circ_L$ is associative.
Therefore $(s,q) * ((t,r) * (u,p)) = ((s,q) * (t,r)) * (u,p)$.
<u>$*$ is order-preserving:</u>
Since $*$ is commutative, it is enough to show that if $(s,q) \preceq (t,r)$, then for all $(u,p) \in X$, $(u,p) * (s,q) \preceq (u,p) * (t,r)$.

$$(u,p) * (s,q) = \begin{cases} (u,p) \wedge_X (s,q) & \text{if } u \circ_L s = u \wedge_L s \\ (u \circ_L s, 1) & \text{if } u \circ_L s < t \wedge_L s \end{cases} \quad (6.3)$$

$$(u,p) * (t,r) = \begin{cases} (u,p) \wedge_X (t,r) & \text{if } u \circ_L t = u \wedge_L t \\ (u \circ_L t, 1) & \text{if } u \circ_L t < u \wedge_X s. \end{cases} \quad (6.4)$$

We compare (6.3) and (6.4):
If $u \circ_L s = u \wedge_L s$ and $u \circ_L t = u \wedge_L t$, then

$$(u,p) * (s,q) = (u,p) \wedge_X (s,q)$$

$$(u,p) * (t,r) = (u,p) \wedge_X (t,r)\,.$$

Therefore $(u,p) * (s,q) \preceq (u,p) * (t,r)$ since $(s,q) \preceq (t,r)$. If $u \circ_L s = u \wedge_L s$ and $u \circ_L t <_L u \wedge_L t$, then

$$(u,p) * (s,q) = (u,p) \wedge_X (s,q)$$

$$(u,p) * (t,r) = (u \circ_L t, 1)\,.$$

But $u \circ_L s \leq_L u \circ_L t$ and the first component of $(u,p) \wedge_X (s,q)$ is $u \circ_L s$, so $(u,p) * (s,q) \preceq (u,p) * (t,r)$.
If $u \circ_L s <_L u \wedge_L s$ and $u \circ_L t <_L u \wedge_L t$, then

$$(u,p) * (s,q) = (u \circ_L s, 1)$$

$$(u,p) * (t,r) = (u \circ_L t, 1)\,.$$

But $u \circ_L s \leq_L u \circ_L t$, so $(u,p) * (s,q) \preceq (u,p) * (t,r)$.
If $u \circ_L s <_L u \wedge_L s$ and $u \circ_L t = u \wedge_L t$, then

$$(u,p) * (s,q) = (u \circ_L s, 1)$$

$$(u,p) * (t,r) = (u,p) \wedge_X (t,r)\,.$$

But $u \circ_L s \leq_L u \circ_L t = u \wedge_L t$ and the first component of $(u,p) \wedge_X (t,r)$ is $u \circ_L t$, so $(u,p) * (s,q) \preceq (u,p) * (t,r)$.
<u>$*$ is left-continuous:</u>
$*$ is order-preserving and commutative so it will be enough to show that if $\{(s_i, q_i) : i \in \mathbf{N}\}$ is any increasing sequence of elements of X such that $\sup\{(s_i, q_i) : i \in \mathbf{N}\} = (s,q)$, then for all $(t,r) \in X$, $\sup\{(s_i, q_i) * (t,r) : i \in \mathbf{N}\} = (s,q) * (t,r)$. We note that for almost every i we must have $s_i = s$ since if $s_i \prec s$ for every i then $(s_i, q_i) \prec \left(s, \frac{q}{2}\right) \prec (s,q)$ for every i. This contradicts the fact that (s,q) is the supremum of the sequence. After deleting a finite number of elements of the sequence, we can suppose, without

loss of generality that for every i, $s_i = s$ and $\sup\{q_i : i \in \mathbf{N}\} = q$. Thus we must show that $\sup\{(s, q_i) * (t, r) : i \in \mathbf{N}\} = (s, q) * (t, r)$.

$$(s, q) * (t, r) = \begin{cases} (s, q) \wedge_X (t, r) & \text{if } s \circ_L t = s \wedge_L t \\ (s \circ_L t, 1) & \text{if } s \circ_L t <_L s \wedge_L t. \end{cases}$$

If $s \circ_L t = s \wedge_L t$, then

$$(s, q) * (t, r) = (s, q) \wedge_X (t, r)$$

and

$$(s, q_i) * (t, r) = (s, q_i) \wedge_X (t, r)\,.$$

We now show that $\sup\{(s, q_i) \wedge_X (t, r) : i \in \mathbf{N}\} = (s, q) \wedge_X (t, r)$.
If $(s, q) \preceq (t, r)$, then

$$\sup\{(s, q_i) \wedge_X (t, r) : i \in \mathbf{N}\} = \sup\{(s, q_i) : i \in \mathbf{N}\} = (s, q) = (s, q) \wedge_X (t, r)\,.$$

If $(t, r) \prec (s, q)$, then there exists j such that $(t, r) \preceq (s, q_j)$, so

$$\sup\{(s, q_i) \wedge_X (t, r) : i \in \mathbf{N}\} = (t, r) = (s, q) \wedge_X (t, r) \quad (\forall i \geq j)\,.$$

If $s \circ_L t <_L s \wedge_L t$, then

$$(s, q) * (t, r) = (s \circ_L t, 1)$$

and

$$(s, q_i) * (t, r) = (s \circ_L t, 1)\,.$$

Therefore $\sup\{(s, q_i) \wedge_X (t, r) : i \in I\} = (s, q) * (t, r)$.
<u>ϕ is an embedding of the structure $(L, \circ_L, \wedge_L, \vee_L, 0_L, 1_L)$ into $(X, *, \wedge_X, \vee_X, 0_X, 1_X)$:</u>
Define, for every $s \in S$, $\phi(s) = (s, 1)$. Hence if $s <_L t$, then $(s, 1) \prec (t, 1)$. Thus ϕ is increasing and therefore one-to-one. Also $\phi(1_L) = (1_L, 1)$ is the greatest element of $(X, \preceq)$ and the neutral element with respect to $*$. Again $\phi(0_L) = (0_L, 1)$ is the least element of $(X, \preceq)$.
We now show that $\phi(s) * \phi(t) = (s, 1) * (t, 1) = (s \circ_L t, 1) = \phi(s \circ_L t)$.
If $s \circ_L t = s \wedge_L t$, then $(s, 1) * (t, 1) = (s, 1) \wedge_X (t, 1)$. If $s \leq_L t$, then $s \circ_L t = s$ and $(s, 1) \wedge_X (t, 1) = (s, 1)$. But $(s, 1) = (s \circ_L t, 1)$, so $(s, 1) * (t, 1) = (s, 1) = (s \circ_L t, 1)$.

If $t \leq_L s$ the proof is similar. If $s \circ_L t <_L s \wedge_L t$, then $(s,1) * (t,1) = (s \circ_L t, 1)$. Therefore $\phi(s) * \phi(t) = \phi(s \circ_L t)$.

For all $s, t \in L$, the residuum $\phi(s) \rightarrow_* \phi(t)$ of $\phi(s)$ and $\phi(t)$ exists in X, and $\phi(s \rightarrow_L t) = \phi(s) \rightarrow_* \phi(t)$, where

$$\begin{aligned} \phi(s) \rightarrow_* \phi(t) &= \max\{(u,p) : \phi(s) * (u,p) \preceq \phi(t)\} \\ &= \max\{(u,p) : (s,1) * (u,p) \preceq (t,1)\}. \end{aligned}$$

We first show that $\phi(s \rightarrow_L t) = (s \rightarrow_L t, 1) \in \{(u,p) : (s,1) * (u,p) \preceq (t,1)\}$.
$(s,1) * (s \rightarrow_L t, 1) = (s \circ_L (s \rightarrow_L t), 1) \preceq (t,1)$ since $s \circ_L (s \rightarrow_L t) \leq_L t$.
Thus $(s \rightarrow_L t, 1) \in \{(u,p) : (s,1) * (u,p) \preceq (t,1)\}$.
Lastly we show that $(s \rightarrow_L t, 1)$ is the maximum element of the set $\{(u,p) : (s,1) * (u,p) \preceq (t,1)\}$. Suppose $(s \rightarrow_L t, 1)$ is not the maximum element of the set $\{(u,p) : (s,1) * (u,p) \preceq (t,1)\}$. This means there exists $(u,p) > (s \rightarrow_L t, 1)$ such that $(s,1) * (u,p) \preceq (t,1)$. But $p \leq_L 1$, so we must have $s \rightarrow_L t <_L u$. This implies $s \circ_L u \not\leq_L t$. Hence $t < s \circ_L u$. Thus $(t,1) \prec (s,1) * (u,p)$, since the first component of $(s,1) * (u,p)$ is $s \circ_L u$ in each case. This contradicts the fact that $(s,1) * (u,p) \preceq (t,1)$. $\square$

Theorem 6.1.2. *Every countable linearly ordered MTL-algebra can be embedded into a standard MTL-algebra.*

Proof. Let $\mathbf{L}$ and $\mathbf{X}$ be as in Theorem 6.1.1. Then $(X, \preceq)$ is a countable, dense, linearly-ordered set with maximum and minimum elements, hence it is order isomorphic to $(\mathbf{Q} \cap [0,1], \leq)$. Let ψ be an isomorphism from $(X, \preceq)$ to $(\mathbf{Q} \cap [0,1], \leq)$. Also let $\alpha, \beta \in \mathbf{Q} \cap [0,1]$ and define $\alpha *' \beta = \psi(\psi^{-1}(\alpha) * \psi^{-1}(\beta))$. Then $(\mathbf{Q} \cap [0,1], *', \wedge, \vee, 0, 1)$ is isomorphic to $\mathbf{X}$ by ψ. Also define, for all $s \in L, \phi'(s) = \psi(\phi(s))$. Thus $\mathbf{L}$ is embeddable into $(\mathbf{Q} \cap [0,1], *', \wedge, \vee, 0, 1)$ by ϕ' and $\phi'(s \rightarrow_* t) = \phi'(s) \rightarrow_{*'} \phi'(t)$ for all $s, t \in X$, where $\rightarrow_{*'}$ is the residuum in $(\mathbf{Q} \cap [0,1], *', \wedge, \vee, 0, 1)$.
Now define for $\alpha, \beta \in [0,1]$,

$$\alpha \hat{*} \beta = \sup\{x *' y : x, y \in \mathbf{Q} \cap [0,1], x \leq \alpha, y \leq \beta\}.$$

We need to show that $([0,1], \hat{*}, \rightarrow_{\hat{*}}, \wedge, \vee, 0, 1)$ is an MTL-algebra, where $\rightarrow_{\hat{*}}$ is the residuum of $\hat{*}$ in $[0,1]$.

$\hat{*}$ is a t-norm:

$\hat{*}$ is commutative since it is defined in terms of $*'$ which is commutative. Also $\hat{*}$ is associative from associativity of $*'$. The fact that $\hat{*}$ is order preserving and has 1 as a neutral element is also a consequence of the definition.

$\hat{*}$ is left continuous:

Let $\{\alpha_n : n \in \mathbf{N}\}$ and $\{\beta_n : n \in \mathbf{N}\}$ be increasing sequences of reals in $[0,1]$ such that $\sup\{\alpha_n : n \in \mathbf{N}\} = \alpha$ and $\sup\{\beta_n : n \in \mathbf{N}\} = \beta$. Now $\alpha_n \leq \alpha$ and $\beta_n \leq \beta$ for every n, so $\alpha_n \hat{*} \beta_n \leq \alpha \hat{*} \beta$. Hence $\sup\{\alpha_n \hat{*} \beta_n : n \in \mathbf{N}\} \leq \alpha \hat{*} \beta$.

Since the restriction of $\hat{*}$ to $\mathbf{Q} \cap [0,1]$ is left continuous, we have:

$$\begin{aligned} \alpha \hat{*} \beta &= \sup\left\{x *' y : x, y \in \mathbf{Q} \cap [0,1], x \leq \alpha, y \leq \beta\right\} \\ &= \sup\left\{x *' y : x, y \in \mathbf{Q} \cap [0,1], x < \alpha, y < \beta\right\}. \end{aligned}$$

We have that for every $q < \alpha$ and for every $r < \beta$ there is n such that $q < \alpha_n$ and $r < \beta_n$.

Therefore $\alpha \hat{*} \beta = \sup\left\{x *' y : x, y \in \mathbf{Q} \cap [0,1], x < \alpha, y < \beta\right\} \leq \sup\left\{\alpha_n \hat{*} \beta_n : n \in \mathbf{N}\right\}$.

The prelinearity property is satisfied since $([0,1], \leq)$ is linearly ordered.

Hence $([0,1], \hat{*}, \rightarrow_{\hat{*}}, \wedge, \vee, 0, 1)$ is an MTL-algebra.

$\hat{*}$ is an extension of $*'$ to $[0,1]$:

For $x, y \in \mathbf{Q} \cap [0,1]$, we have

$$x \hat{*} y = \sup\left\{p *' q : p, q \in \mathbf{Q}, p \leq x, q \leq y\right\}.$$

Since $x \leq x$ and $y \leq y$, $x *' y \in \{p *' q : p, q \in \mathbf{Q}, p \leq x, q \leq y\}$.

If $p \leq x$ and $q \leq y$ then $p *' q \leq x *' y$.

Thus $x \hat{*} y = \sup\left\{p *' q : p, q \in \mathbf{Q}, p \leq x, q \leq y\right\} = x *' y$.

φ defined by $\varphi(x) = x$ is an embedding of $(\mathbf{Q} \cap [0,1], *', \wedge, \vee, 0, 1)$ into $([0,1], \hat{*}, \wedge, \vee, 0, 1)$:

φ is clearly 1-1. Also φ is operation preserving since $\hat{*}$ is an extension of $*'$.

We also show that if $s \rightarrow_{*'} t$ exists in $\mathbf{Q} \cap [0,1]$, then $\varphi(s \rightarrow_{*'} t) = \varphi(s) \rightarrow_{\hat{*}} \varphi(t)$ for $s, t \in \mathbf{Q} \cap [0,1]$. This is equivalent to showing that

$$s \rightarrow_{*'} t = s \rightarrow_{\hat{*}} t = \max\left\{r \in [0,1] : r \hat{*} s \leq t\right\}.$$

Suppose there exists $u \in \mathbf{R}$ such that $u \hat{*} s \leq t$ and $u > s \rightarrow_{*'} t$. Then there exists $q \in \mathbf{Q}$ such that $s \rightarrow_{*'} t < q \leq u$, since $\mathbf{Q} \cap [0,1]$ is dense in $[0,1]$. Thus, $q \hat{*} s = q *' s \nleq t$. Hence $q *' s > t$. Hence $u \hat{*} s \geq q \hat{*} s > t$. This contradicts the fact that $u \hat{*} s \leq t$.

Let $\rho : L \rightarrow [0,1]$ be the composition of the maps φ and ϕ'. Since each of the maps is an embedding, ρ is an embedding from $(L, \circ_L, \rightarrow_L, \wedge_L, \vee_L, 0_L, 1_L)$ to $([0,1], \hat{*}, \rightarrow_{\hat{*}}, \wedge, \vee, 0, 1)$. □

Theorem 6.1.3. *The variety of all MTL-algebras is generated by the class of all standard MTL-algebras.*

Proof. We know the following:

(1) The class of all MTL-algebras forms a variety by Theorem 4.2.1.

(2) Any MTL-algebra is a subdirect product of linearly ordered MTL-algebras by Theorem 5.1.1.

(3) Any algebra $\mathbf{L}$ is a subalgebra of an ultraproduct of finitely generated subalgebras of $\mathbf{L}$ by Theorem 2.0.3.

Combining (1) and (2) we have that the variety of MTL-algebras is the variety generated by a class of linearly ordered MTL-algebras. It follows from (3) that the variety of MTL-algebras is generated by a class of finitely generated subalgebras of linearly ordered MTL-algebras. Since a finitely generated subalgebra of linearly ordered MTL-algebra is countable and linearly ordered, we have that the variety of MTL-algebras is generated by a class of countable, linearly ordered MTL-algebras. From Theorem 6.1.2, we get that the variety of all MTL-algebras is generated by the class of all standard MTL-algebras.

□

Theorem 6.1.4 (completeness)**.** *MTL is complete with respect to the class of all standard MTL-algebras.*

Proof. The result follows immediately from Theorem 6.1.3. □

In this chapter, we showed that MTL is complete with respect to the class of all standard MTL-algebras. This establishes one of the main goals of this study which was to investigate the connection between MTL and left-continuous t-norms. Similar completeness results will be investigated for BL and continuous t-norms; these will be carried out in the next chapter. This chapter marks the end of discussions of completeness results pertaining to MTL.

CHAPTER 7

The logic BL and its compeleteness

Basic logic or BL for short has been introduced by Hájek in [14] as a formalization of the logic of continuous t-norms. BL is MTL with axiom (A6) replaced by $(\varphi \circ (\varphi \to \psi)) \leftrightarrow (\varphi \wedge \psi)$ (see page 30). Therefore BL is an axiomatic extension of MTL. In this chapter we give a proof of completeness of BL with respect to BL-algebras, BL-chains (linearly ordered BL-algebras) and standard BL-algebras (continuous t-norm algebras), that is we show that a formula φ is a theorem of BL if and only if $\varphi = 1$ holds in any of these classes of BL-algebras. In section 7.1 we define BL-algebras as a subclass of MTL-algebras. We further prove some extra identities that hold in BL-algebras. In section 7.2 we prove a Characterization Theorem of subdirectly irreducible BL-algebras. This Characterization Theorem plays a major role in proving the results leading to the Completeness Theorem of BL with respect to standard BL-algebras. In section 7.3 we prove completeness of BL with respect to BL-algebras and BL-chains. To prove standard completeness of BL we first show that each finite subdirectly irreducible BL-algebra is isomorphic to an ordinal sum of finite Wajsberg algebras. Next we show that the class of linearly ordered BL-algebras has the finite embeddability property. Combining these facts and [8, Theorem 4], we get that the variety of BL-algebras is generated as a quasivariety by its finite members. Since each finite BL-algebra is a subdirect product of subdirectly irreducible finite BL-algebras that are homomorphic images of itself by [3, Theorem 8.6] and each subdirectly irreducible finite BL-algebra is an ordinal sum of Wajsberg algebras, to get the completeness result it will be enough to show that an ordinal sum of finite Wajsberg algebras is isomorphic to a subalgebra of a standard BL-algebra. The results we present in this chapter come from the journal papers [2] and [1].

7.1 Basic logic(BL), BL-algebras and Hoops

Definition 7.1.1. *Basic Logic*, or BL for short, is the axiomatic extension of MTL obtained by adding the axiom: $(\varphi \wedge \psi) \to (\varphi \circ (\varphi \to \psi))$.
Equivalently, BL is obtained by replacing axiom (A6): $(\varphi \circ (\varphi \to \psi)) \to (\varphi \wedge \psi)$ in the definition of MTL by $(\varphi \circ (\varphi \to \psi)) \leftrightarrow (\varphi \wedge \psi)$.

All formulas provable in MTL are also provable in BL. Hence BL proves all formulas listed in Lemma 4.1.1.

Definition 7.1.2. A *BL-algebra* is an MTL-algebra satisfying the identity:

$$x \wedge y = x \circ (x \to y)\,.$$

Since BL-algebras are an axiomatic extension of MTL-algebras, and the class of MTL-algebras is a variety, the class of BL-algebras is also a variety.

Recall that any algebra $\mathbf{L} = ([0,1], \circ, \to, \wedge, \vee, 0, 1)$, in which $\circ$ is a left-continuous t-norm and $\to$ its residuum, is an MTL-algebra. Any such algebra in which $\circ$ is a continuous t-norm is a BL-algebra, by Lemma 3.2.2. In particular, if $\circ$ is the Lukasiewicz, Product or Gödel t-norm, then $\mathbf{L}$ is a BL-algebra.

Lemma 7.1.1. *All identities of MTL-algebras also hold in BL-algebras. The following* *identity holds in BL-algebras:*

$$(x \to y) \circ x = (y \to x) \circ y.$$

Proof.

$$(x \to y) \circ x = x \wedge y = y \wedge x = (y \to x) \circ y.$$

□

Lemma 7.1.2. *The following identity holds in BL-algebras and also in MTL-algebras.*

$$(x \to y) \to (y \to x) = (y \to x)\,.$$

Proof. By Lemma 4.2.1(15),

$$(x \to y) \to (y \to x) = y \to ((x \to y) \to x)\,. \tag{7.1}$$

Letting $(x \to y) \to x = z$, (7.1) becomes

$$(x \to y) \to (y \to x) = y \to z. \tag{7.2}$$

By Lemma 4.2.1(17), $x \leq z$. This implies $z \to y \leq x \to y$ by Lemma 4.2.1(2). Also $x \to y \leq z \to x$ by Lemma 4.2.1(16). Hence $z \to y \leq z \to x$ by transitivity. Thus $(z \to y) \circ z \leq (z \to x) \circ z \leq x$. By Lemma 7.1.1, $(y \to z) \circ y = (z \to y) \circ z \leq x$. Hence

$$y \to z \leq y \to x. \tag{7.3}$$

Thus $(x \to y) \to (y \to x) \leq y \to x$ by (7.2) and (7.3). Also $y \to x \leq (x \to y) \to (y \to x)$ by Lemma 4.2.1(17). Therefore $(x \to y) \to (y \to x) = y \to x$. □

Definition 7.1.3. A *hoop* is an algebra $\mathbf{L} = (L, \circ, \to, 1)$ such that the following hold.

(1) $(L, \circ, 1)$ is a commutative monoid.

(2) $x \to x = 1$

(3) $x \to (y \to z) = (x \circ y) \to z$

(4) $x \circ (x \to y) = y \circ (y \to x)$.

Definition 7.1.4. A *bounded hoop* is an algebra $\mathbf{L} = (L, \circ, \to, 0, 1)$ such that $(L, \circ, \to, 1)$ is a hoop and $\mathbf{L}$ satisfies $0 \to x = 1$.

Definition 7.1.5. A *Wajsberg hoop* is a hoop satisfying the following identity:

$$(x \to y) \to y = (y \to x) \to x.$$

We shall refer to a bounded Wajsberg hoop as a *Wajsberg algebra*. This term is used for a slightly different type of algebra in [10], however these algebras are termwise equivalent to bounded Wajsberg hoops.

Note that every hoop is partially ordered by the relation:

$$x \leq y \iff x \to y = 1.$$

Definition 7.1.6. For each $n \in \mathbf{N}$, let $\mathbf{C}_n = (C_n, \circ, \to, 1)$ denote the finite Wajsberg hoop with universe $C_n = \{1 = a^0, a, a^2, ..., a^n\}$, operations $a^k \circ a^m = a^{\min\{k+m,n\}}$ and $a^k \to a^m = a^{\max\{m-k,0\}}$ for $0 \leq k, m \leq n$. Let $\mathbf{Wa}_n$ denote the finite Wajsberg algebra $(C_n, \circ, \to, a^n, 1)$. Note that $\mathbf{C}_n$ and $\mathbf{Wa}_n$ are linearly ordered by the relation $x \leq y \Leftrightarrow x \to y = 1$. Thus $\mathbf{C}_n$ and $\mathbf{Wa}_n$ are lattice-ordered.

Lemma 7.1.3. *Let $\boldsymbol{L} = ([0,1], \circ_L, \to_L, 0, 1)$, where $\circ_L$ is the Lukasiewicz t-norm and $\to_L$ its residuum. Then:*

(1) $\boldsymbol{L}$ is a Wajsberg algebra.

(2) If $a, b \in \boldsymbol{R}$ and $a < b$, then we can define $\circ$ and $\to$ on $[a, b]$ in such a way that $([a, b], \circ, \to, a, b)$ is isomorphic to $\boldsymbol{L}$.

(3) Each $\boldsymbol{Wa}_n$ can be embedded into $\boldsymbol{L}$.

Proof.

(1) Since $([0,1], \circ_L, \to_L, \wedge_L, \vee_L, 0, 1)$ is a BL-algebra conditions (2)-(4) in the definition of a hoop are satisfied, by Lemma 4.2.1(3),(14) and Lemma 7.1.1. Hence

$([0,1], \circ_L, \to_L, 0, 1)$ is a bounded hoop. We now show that $([0,1], \circ_L, \to_L, 0, 1)$ is a Wajsberg hoop. Let $x, y \in [0,1]$.
If $x \leq y$, then

$$(x \to_L y) \to_L y = 1 \to_L y = y$$

$$(y \to_L x) \to_L x = x - x + y - 1 + 1 = y.$$

If $y < x$, then

$$(x \to_L y) \to_L y = y - y + x - 1 + 1 = x$$

$$(y \to_L x) \to_L x = 1 \to_L x = x.$$

Thus $(x \to_L y) \to_L y = (y \to_L x) \to_L x$, so $([0,1], \circ_L, \to_L, 0, 1)$ is a Wajsberg algebra.

(2) Define for all $u, v \in [a,b]$, $u \circ v = \max\{u + v - b, a\}$, $u \to v = \min\{b - u + v, b\}$. Also define the functions f and g by $f(u) = a + (b-a)u$ for all $u \in [0,1]$, $g(u) = \dfrac{u-a}{b-a}$ for all $u \in [a,b]$. Clearly, f and g are order-preserving maps. We show that g and f are mutually inverse isomorphisms between $[0,1]$ and $[a,b]$. $g(a) = \dfrac{a-a}{b-a} = 0$, $g(b) = \dfrac{b-a}{b-a} = 1$, $f(0) = a$ and $f(1) = a + (b-a) = b$.
<u>g is operation preserving:</u>
Let $u, v \in [a,b]$. Then

$$\begin{aligned} g(u \circ v) &= g(\max\{u + v - a - b\}) \\ &= \frac{\max\{u + v - b, a\} - a}{b-a}, \end{aligned} \tag{7.4}$$

$$\begin{aligned}
g(u) \circ_L g(v) &= \frac{u-a}{b-a} \circ_L \frac{v-a}{b-a} \\
&= \max\left\{\frac{u-a}{b-a} + \frac{v-a}{b-a} - 1, 0\right\} \\
&= \max\left\{\frac{u+v-2a-(b-a)}{b-a}, 0\right\} \\
&= \max\left\{\frac{u+v-a-b}{b-a}, 0\right\}.
\end{aligned} \tag{7.5}$$

If $u+v-a-b<0$, then $u+v-b<a$. Hence $\max\{u+v-a-b, a\} = a$. Thus (7.4) becomes $g(u \circ v) = \dfrac{a-a}{b-a} = 0$ and (7.5) becomes $g(u) \circ_L g(v) = 0$.
If $u+v-a-b \geq 0$, then $u+v-b \geq a$. Thus $\max\{u+v-a-b, a\} = u+v-b$. Hence (7.4) and (7.5) become, respectively:

$$g(u \circ v) = \frac{u+v-b-a}{b-a} \text{ and } g(u) \circ_L g(v) = \frac{u+v-b-a}{b-a}.$$

Therefore in both cases $g(u \circ v) = g(u) \circ_L g(v)$.

$$\begin{aligned}
g(u \to v) &= g(\min\{b-u+v, b\}) \\
&= \frac{\min\{b-u+v, b\} - a}{b-a} \\
&= \min\left\{\frac{b-u+v-a}{b-a}, \frac{b-a}{b-a}\right\} \\
&= \min\left\{\frac{b-u+v-a}{b-a}, 1\right\},
\end{aligned}$$

$$\begin{aligned}
g(u) \to_L g(v) &= \frac{u-a}{b-a} \to_L \frac{v-a}{b-a} \\
&= \min\left\{\frac{v-a}{b-a} - \frac{u-a}{b-a} + 1, 1\right\} \\
&= \min\left\{\frac{v-u+b-a}{b-a}, 1\right\}.
\end{aligned}$$

Therefore $g(u \to v) = g(u) \to_L g(v)$.

<u>f is operation preserving:</u>

Let $u, v \in [0,1]$. Then

$$\begin{aligned}
f(u \circ_L v) &= f(\max\{u+v-1, 0\}) \\
&= \max\{a + (b-a)(u+v-1), a\} \\
&= \max\{a + bu + bv - b - au - av + a, a\} \\
&= \max\{2a + bu + bv - b - au - av, a\},
\end{aligned}$$

$$\begin{aligned}
f(u) \circ f(v) &= (a + (b-a)u) \circ (a + (b-a)v) \\
&= \max\{a + (b-a)u + a + (b-a)v - b, a\} \\
&= \max\{a + bu - au + a + bv - av - b, a\} \\
&= \max\{2a + bu + bv - b - au - av, a\}.
\end{aligned}$$

Therefore $f(u \circ_L v) = f(u) \circ f(v)$.

$$\begin{aligned}
f(u \to_L v) &= f(\min\{v-u+1, 1\}) \\
&= a + (b-a)(\min\{v-u+1, 1\}) \\
&= \min\{a + (b-a)(v-u+1), a + (b-a)\} \\
&= \min\{a + bv - bu + b - av + au - a, b\} \\
&= \min\{b + bv - bu - av + au, b\},
\end{aligned}$$

$$\begin{aligned}
f(u) \to f(v) &= a + (b-a)u \to a + (b-a)v \\
&= \min\{b - (a + (b-a)u) + a + (b-a)v, b\} \\
&= \min\{b - a - bu + au + a + bv - av, b\} \\
&= \min\{b + bv - bu - av + au, b\}.
\end{aligned}$$

Therefore $f(u \to_L v) = f(u) \to f(v)$.

$\underline{f \text{ is the inverse of } g:}$

$$\begin{aligned} f(g(u)) &= f\left(\frac{u-a}{b-a}\right) \\ &= a + (b-a)\left(\frac{u-a}{b-a}\right) \\ &= a + u - a \\ &= u. \end{aligned}$$

Also,

$$\begin{aligned} g(f(u)) &= g(a + (b-a)u) \\ &= \frac{a + (b-a)u - a}{b-a} \\ &= \frac{a + (b-a)u - a}{b-a} \\ &= \frac{(b-a)u}{b-a} \\ &= u. \end{aligned}$$

Therefore f and g are inverses of each other.

$\underline{f \text{ and } g \text{ are 1-1 and onto:}}$

Since f and g are inverses to each other, it follows that they are 1-1 and onto.

(3) Let $f : C_n \to [0,1]$ be defined by $f\left(a^k\right) = \dfrac{n-k}{n}$, for $k \le n$.
We will show that f is an embedding of $\mathbf{C}_n$ into $([0,1], \circ_L, \to_L, 0, 1)$.
$f(a^n) = \dfrac{n-n}{n} = 0$, $f(a^0) = \dfrac{n}{n} = 1$.
$\underline{f \text{ is 1-1:}}$
Let $a^k, a^m \in C_n$ such that $f\left(a^k\right) = f(a^m)$. Then $\dfrac{n-k}{n} = \dfrac{n-m}{n}$. Thus $k = m$.

f is operation preserving:

$$
\begin{aligned}
f\left(a^k \circ a^m\right) &= f\left(a^{\min\{k+m,n\}}\right) \\
&= \max\left\{\frac{n-(k+m)}{n}, 0\right\} \\
&= \max\left\{\frac{n-k-m}{n}, 0\right\},
\end{aligned}
$$

$$
\begin{aligned}
f\left(a^k\right) \circ_L f\left(a^m\right) &= \max\left\{\frac{n-k}{n} + \frac{n-m}{n} - 1, 0\right\} \\
&= \max\left\{\frac{n-k-m}{n}, 0\right\}.
\end{aligned}
$$

Therefore $f\left(a^k \circ a^m\right) = f\left(a^k\right) \circ_L f\left(a^m\right)$.

$$
\begin{aligned}
f\left(a^k \rightarrow a^m\right) &= f\left(a^{\max\{m-k,0\}}\right) \\
&= \min\left\{\frac{n-(m-k)}{n}, 1\right\} \\
&= \min\left\{\frac{n-m+k}{n}, 1\right\},
\end{aligned}
$$

$$
\begin{aligned}
f\left(a^k\right) \rightarrow_L f\left(a^m\right) &= \min\left\{\frac{n-m}{n} - \left(\frac{n-k}{n}\right) + 1, 1\right\} \\
&= \min\left\{\frac{n-m+k}{n}, 1\right\}.
\end{aligned}
$$

Therefore $f\left(a^k \rightarrow a^m\right) = f\left(a^k\right) \rightarrow_L f\left(a^m\right)$. □

7.2 Subdirectly irreducible BL-algebras

Definition 7.2.1. An algebra **L** is *subdirectly irreducible* if **L** has a non-trivial congruence that is contained in every non-trivial congruence of **L**. If **L** is a BL-algebra then, equivalently, **L** is subdirectly irreducible if it has a non-trivial filter that is contained in every non-trivial filter of **L** by Lemma 5.1.5.

We are going to describe the structure of subdirectly irreducible BL-algebras.

Definition 7.2.2. An algebra **A** has the *congruence extension property* if for any subalgebra **B** of **A**, and any congruence relation θ on **B**, there exists a congruence relation σ on **A** such that $\sigma \cap (B \times B) = \theta$.

Lemma 7.2.1. *Every BL-algebra has the congruence extension property.*

Proof. Let **A** be a BL-algebra and **B** a subalgebra of **A** . By Lemma 5.1.5 it is enough to show that for every filter F of **B** there exists a filter F' of **A** such that $F' \cap B = F$. Let F the filter of **B** and F' be a filter of **A** generated by F. Since F is contained in F', $F \subseteq F' \cap B$. To show that $F' \cap B \subseteq F$, let $a \in F' \cap B$. Then there exist $b_1, b_2, ..., b_n \in F$ such that $b_1 \circ b_2 \circ ... \circ b_n \leq a$. But $a \in B$ and F is a filter of B, Hence $b_1 \circ b_2 \circ ... \circ b_n \in F$ and consequently $a \in F$. □

Lemma 7.2.2. *If* ***L*** *is a BL-algebra, define* $x^0 = 1$, $x \overset{0}{\to} y = y$ *and* $x \overset{n+1}{\to} y = x \to (x \overset{n}{\to} y)$ *and* $x^{n+1} = x \circ x^n$ *for all* $n \in \boldsymbol{N}$. *Then* $x \overset{n}{\to} y = x^n \to y$ *for all* $n \in \boldsymbol{N}$.

Proof. If $n = 1$, then $x \overset{1}{\to} y = x \to (x \overset{0}{\to} y) = x \to y$. Assume $x \overset{k}{\to} y = x^k \to y$. Then $x \overset{k+1}{\to} y = x \to (x \overset{k}{\to} y) = x \to (x^k \to y) = (x \circ x^k) \to y = x^{k+1} \to y$ by Lemma 4.2.1(14). The result follows from Induction. □

Definition 7.2.3. An algebra is said to be *simple* if it has no proper non-trivial congruences. Thus, a BL-algebra is simple if it has only two filters, the trivial filter $\{1\}$ and the universe of the algebra.

Lemma 7.2.3. *Let* ***L*** *be a BL-algebra. Then*

(1) ***L*** *is simple iff for all* $a, b \in L$, $a \neq 1$, *there exists* $n \in \boldsymbol{N}$ *such that* $a \overset{n}{\to} b = 1$.

(2) If $\mathbf{L}$ *is simple and* $a, b \in L$*, then* $b \to a = a$ *implies* $a = 1$ *or* $b = 1$*.*

Proof.

(1) $\underline{\Rightarrow:}$ Suppose $\mathbf{L}$ is simple and $a \in L$ such that $a \neq 1$. Then $\langle a \rangle = L$, where $\langle a \rangle$ is the filter generated by a (filters are covered in chapter 5). But $\langle a \rangle = \{b \in L : a^n \leq b, \text{ for some } n \in \mathbf{N}\} = \left\{b \in L : a \overset{n}{\to} b = 1, \text{ for some } n \in \mathbf{N}\right\}$.
$\underline{\Leftarrow:}$ Suppose for all $a, b \in L$, $a \neq 1$, there exists $n \in \mathbf{N}$ such that $a \overset{n}{\to} b = 1$. This implies that $b \in \langle a \rangle$ for all $b \in L$. Thus $\mathbf{L}$ is simple.

(2) Suppose $\mathbf{L}$ is simple and $a, b \in L$ such that $b \to a = a$. Then for all $n \in \mathbf{N}$, $b \overset{n}{\to} a = a$. By (1), if $b \neq 1$ then there exists m such that $b \overset{m}{\to} a = 1$. Thus $a = 1$.

□

Lemma 7.2.4. *Let* $\mathbf{L}$ *be a BL-algebra such that for all* $a, b \in L$*,* $b \to a = a$ *implies* $a = 1$ *or* $b = 1$*. Then* $\mathbf{L}$ *is linearly ordered and satisfies*

$$(x \to y) \to y = (y \to x) \to x. \tag{7.6}$$

Proof. We first show that $\mathbf{L}$ is linearly ordered. We have that $(a \to b) \to (b \to a) = b \to a$ for all $a, b \in L$, by Lemma 7.1.2. Hence $a \to b = 1$ or $b \to a = 1$. Thus $a \leq b$ or $b \leq a$.
We now prove (7.6).
Let $a, b \in L$ and assume $a < b$. Then $(a \to b) \to b = 1 \to b = b$. Hence it is enough to show that $b = (b \to a) \to a$. Now,

$$
\begin{aligned}
(((b \to a) \to a) \to b) \to (b \to a) &= (((b \to a) \to a) \to b) \to (((b \to a) \to a) \to a) \\
&\quad \text{(since } b \to a = ((b \to a) \to a) \to a \\
&\qquad \text{by Lemma 4.2.1(2) and (16))} \\
&= ((((b \to a) \to a) \to b) \circ (((b \to a) \to a))) \to a \\
&\quad \text{(by Lemma 4.2.1(14))} \\
&= ((b \to ((b \to a) \to a)) \circ b) \to a \\
&\quad \text{(by Lemma 7.1.1)} \\
&= (b \to ((b \to a) \to a)) \to (b \to a) \\
&\quad \text{(by Lemma 4.2.1(14))} \\
&= 1 \to (b \to a) \ \text{(by Lemma 4.2.1(16))} \\
&= b \to a.
\end{aligned}
$$

Thus, $((b \to a) \to a) \to b = 1$ or $b \to a = 1$. But $b \to a \neq 1$ since $a < b$. Hence we must have $((b \to a) \to a) \to b = 1$. It follows that $(b \to a) \to a \leq b$. Also, $b \leq (b \to a) \to a$ by Lemma 4.2.1(16), so $(b \to a) \to a = b$. If $b < a$ the proof is similar. If $a = b$ the proof is immediate. □

Lemma 7.2.5. *Every simple BL-algebra is linearly ordered and satisfies* (7.6).

Proof. The result follows immediately from Lemmas 7.2.3(2) and 7.2.4. □

Definition 7.2.4. Let **L** be a subdirectly irreducible BL-algebra with least non-trivial filter U. An element $a \in L$ is said to be *fixed* if for all $u \in U$, $u \to a = a$. The set of fixed elements of **L** is denoted by F. The set $S = (L \setminus F) \cup \{1\}$ is called the *support of* U.

Definition 7.2.5. A *0-free* BL-algebra is any BL-algebra without 0 in its language. Hence it does not satisfy the property $0 \leq x$ and consequently does not necessarily have a smallest element.

Lemma 7.2.6. *Let* $\boldsymbol{L}$ *be a subdirectly irreducible BL-algebra. Then the least non-trivial filter* U *is a subuniverse of* $\boldsymbol{L}$ *on the language* $\{\circ, \to, \wedge, \vee, 1\}$ *and* $(U, \circ, \to, \wedge, \vee, 1)$ *is a linearly ordered, 0-free BL-algebra satisfying* (7.6).

Proof. U is a subuniverse of $\mathbf{L}$ on the language $\{\circ, \to, \wedge, \vee, 1\}$ since it is a filter and every filter of $\mathbf{L}$ is closed under $\circ$, contains 1 and is also closed under $\to$ by Lemma 4.2.1(17). Every filter is also closed under $\vee$ and $\wedge$ since $x \circ y \leq x \wedge y \leq x \vee y$. U is generated as a filter by any $a \in U \setminus \{1\}$ since U is the least filter of $\mathbf{L}$ different from $\{1\}$. Thus $U = \left\{ b \in L : a \overset{n}{\to} b = 1 \text{ for some } n \in \mathbf{N} \right\}$. Hence U is simple by Lemma 7.2.3(1). Therefore U is linearly ordered and satisfies (7.6), by Lemma 7.2.5. □

Lemma 7.2.7. *Let $\mathbf{L}$ be a subdirectly irreducible BL-algebra with least non-trivial filter U and set of fixed elements F. Then*

(1) For all $a \in L$, $a \neq 1$, there exists $u \in U$, $u \neq 1$ such that $a \leq u$.

(2) $U \cap F = \{1\}$.

(3) An element $a \in L$ is fixed if and only if for some $b \in U \setminus \{1\}$, $b \to a = a$.

Proof.

(1) Let $a \in L$, $a \neq 1$. Then $\langle a \rangle = \left\{ b \in L : a \overset{n}{\to} b = 1 \text{ for some } n \in \mathbf{N} \right\}$ and $\langle a \rangle \neq \{1\}$, and $U \subseteq \langle a \rangle$ since U is contained in every non-trivial filter of $\mathbf{L}$. Let $b \in U \setminus \{1\}$. Then $a \overset{n}{\to} b = 1$ for some $n \in \mathbf{N}$. Let's choose the smallest n such that $a \overset{n}{\to} b = 1$. Let $u = a \overset{n-1}{\to} b$. Then $u \neq 1$ and $b \leq a^{n-1} \to b = u$ by Lemma 4.2.1(17). Thus $u \in U \setminus \{1\}$ since $b \in U$ and U is a filter. Also $a \to u = a \to \left(a \overset{n-1}{\to} b \right) = a \overset{n}{\to} b = 1$. Therefore $a \leq u$.

(2) Let $a \in U \cap F$ and $u \in U \setminus \{1\}$. Then $u \to a = a$, since $a \in F$. By Lemma 7.2.6, U is the universe of a simple 0-free BL-algebra, so $a = 1$ by Lemma 7.2.3(2).

(3) The implication from left to right is clear since $U \neq \{1\}$. For the converse, assume $a \neq 1$ and $b \to a = a$ for some $b \in U \neq \{1\}$. To show that a is fixed we have to verify that $u \to a = a$ for every $u \in U$. We have that $u \leq (u \to a) \to a$, by Lemma 4.2.1(16). Hence $(u \to a) \to a \in U$ since U is a filter. Since U is the universe of a simple 0-free BL-algebra and $b \neq 1$, there exists $m \in \mathbf{N}$ such that

$b \overset{m}{\to} ((u \to a) \to a) = 1$. But

$$\begin{aligned} b \overset{m}{\to} ((u \to a) \to a) &= (u \to a) \to (b \overset{m}{\to} a) \text{ (by Lemma 4.2.1(15))} \\ &= (u \to a) \to a \text{ (since } b \overset{n}{\to} a = a \text{ for every } n \in \mathbf{N}). \end{aligned}$$

Thus $(u \to a) \to a = 1$. This implies $u \to a \leq a$. Also, $a \leq u \to a$ by Lemma 4.2.1(17). Therefore $u \to a = a$.

□

Lemma 7.2.8. *Let* $\boldsymbol{L}$ *be a subdirectly irreducible BL-algebra with least non-trivial filter* U*,* F *its set of fixed elements and* S *the support of* U*. Let* $a \in F$*,* $a \neq 1$*. Then*

(1) for all $u \in U$*,* $a \leq u$*,*

(2) for all $u \in U$*,* $u \circ a = a$*,*

(3) for all $b \in L$*, if* $b \leq a$ *then* $b \in F$*,*

(4) for all $b \in L$*,* $a \to b, b \to a \in F$*,*

(5) for all $b \in S$*,* $a \leq b$*,* $b \to a = a$ *and* $a \circ b = a$*.*

Proof.

(1) $U \subseteq \langle a \rangle$ since $a \neq 1$ and U is the least filter different from $\{1\}$. Hence for every $u \in U$, there exists $m \in \mathbf{N}$ such that $a \overset{m}{\to} u = 1$. We have that

$$\begin{aligned} a \to u &= (u \to a) \to (a \to u) \text{ (by Lemma 7.1.2)} \\ &= a \to (a \to u) \text{ (since } a \text{ is fixed)} \\ &= a \overset{2}{\to} u. \end{aligned}$$

By induction, one gets $a \overset{n}{\to} u = a \to u$, for every $n \in \mathbf{N}$. If $n = m$, then $1 = a \overset{m}{\to} u = a \to u$. Hence $a \leq u$.

(2) Let $u \in U$. Then $u \circ a = u \circ (u \to a)$ since a is fixed. But $u \circ (u \to a) = u \wedge a = a$ by (1), so $u \circ a = a$.

(3) Let $b \in L$ such that $b \leq a$. Also let $u \in U$. We want to show that $u \to b = b$. Now,

$$\begin{aligned}(u \to b) \to b &= 1 \to (u \to b) \to b \text{ (by Lemma 4.2.1(13))} \\ &= ((u \to b) \to a) \to ((u \to b) \to b) \\ &(b \leq a \text{ implies } u \to b \leq u \to a = a) \text{ (by Lemma 4.2.1(2))} \\ &= (((u \to b) \to a) \circ (u \to b)) \to b \text{ (by Lemma 4.2.1(14))} \\ &= ((a \to (u \to b)) \circ a) \to b \text{ (by Lemma 7.1.1)} \\ &= ((a \circ u \to b) \circ a \circ u) \to b \text{ (by (2) and Lemma 4.2.1(14))} \\ &= 1 \text{ (since } (a \circ u \to b) \circ a \circ u \leq b) \,.\end{aligned}$$

Thus $u \to b \leq b \leq u \to b$. Hence $u \to b = b$. Therefore $b \in F$.

(4) Let $b \in L$ and $u \in U$. Then

$$\begin{aligned}u \to (a \to b) &= u \circ a \to b \text{ (by Lemma 4.2.1(14))} \\ &= a \to b \text{ (by (2))} \,,\end{aligned}$$

Hence $a \to b \in F$.

We now show that $b \to a \in F$.

$$\begin{aligned}u \to (b \to a) &= b \to (u \to a) \text{ (by Lemma 4.2.1(15))} \\ &= b \to a \text{ (since } a \text{ is fixed)} \,.\end{aligned}$$

Thus $b \to a \in F$.

(5) We first show that $a \leq b$.

Let $b \in S$. Then $b \in (L \setminus F) \cup \{1\}$. If $b = 1$, then all statements are true. If $b \neq 1$, then $a \to b \in F$ by (4). But $b \leq a \to b$ (Lemma 4.2.1(17) and $b \notin F$, so $a \to b = 1$, by (3). Hence $a \leq b$.

We now show that $b \to a = a$.

Let $u \in U$, $u \neq 1$. Then

$$\begin{aligned}u \to ((b \to a) \to a) &= (b \to a) \to (u \to a) \text{ (by Lemma 4.2.1(15))} \\ &= (b \to a) \to a \text{ (since } a \text{ is fixed)} \,.\end{aligned}$$

Thus $(b \to a) \to a$ is fixed. But $b \leq (b \to a) \to a$ (Lemma 4.2.1(16)) and $b \notin F$, so $(b \to a) \to a = 1$, by (3). Hence $b \to a \leq a \leq b \to a$. Hence $b \to a = a$.

Lastly we show that $a \circ b = a$:

$$\begin{aligned} a \circ b &= (b \to a) \circ b \ \text{(since } b \to a = a\text{)} \\ &= a \wedge b \\ &= a \ \text{(since } a \leq b\text{)}. \end{aligned}$$

□

Definition 7.2.6. Let $\mathbf{L}_i = (L_i, \wedge_i, \vee_i, \circ_i, \to_i, 0_i, 1_i)$, $i = 1, 2$, be BL-algebras and assume $1_1 = 0_2$ and $(L_1 \setminus \{1_1\}) \cap (L_2 \setminus \{0_2\}) = \emptyset$, for the sake of simplicity. The *ordinal sum* $\mathbf{L}_1 \oplus \mathbf{L}_2 = (L_1 \cup L_2, \wedge, \vee, \circ, \to, 0_1, 1_2)$ is a new BL-algebra whose operations $\wedge, \vee, \circ$ coincide with those of L_i when applied to pairs of elements from the same L_i. For $x \in L_1$ and $y \in L_2$:

(1) $x \wedge y = y \wedge x = x$

(2) $x \vee y = y \vee x = y$

(3) $x \circ y = y \circ x = x$.

$\to$ is defined by:

$$x \to y = \begin{cases} 1_2 & \text{if } x \leq y \\ x \to_i y & \text{if } x > y \text{ and } x, y \in L_i \\ y & \text{if } x > y \text{ and } x \in L_2, y \in L_1. \end{cases}$$

Note that every element of L_1 is below every element of L_2.

Theorem 7.2.1. *Let* $\boldsymbol{L}$ *be a subdirectly irreducible BL-algebra,* U *its least non-trivial filter,* F *its set of fixed elements and* S *the support of* U*. Then*

(1) F *is a subuniverse of* $\boldsymbol{L}$ *and* S *is a subuniverse of* $\boldsymbol{L}$ *on the language* $(\circ, \to, \wedge, \vee, 1)$*. Also* S *is a filter of* $\boldsymbol{L}$*.*

(2) $U \subseteq S$ *and* $\boldsymbol{S} = (S, \circ, \rightarrow, \wedge, \vee, 1)$ *is a subdirectly irreducible 0-free BL-algebra satisfying* (7.6) *and it is linearly ordered. In particular,* $(S, \circ, \rightarrow, 1)$ *is a subdirectly irreducible Wajsberg hoop.*

(3) $\boldsymbol{L} = \boldsymbol{F} \oplus \boldsymbol{S}$.

Proof.

(1) We first show that F is a subuniverse of $\mathbf{L}$.
<u>F is closed under $\circ$:</u>
$1 \in F$ since $u \rightarrow 1 = 1$, for every $u \in U$. Let $a, b \in F$. If $a = 1$, then $a \circ b = 1 \circ b = b \in F$. If $a \neq 1$, then $a \circ b \leq b$. Hence $a \circ b \in F$, by Lemma 7.2.8(3).
<u>F is closed under $\rightarrow$:</u>
Let $a, b \in F$. If $a = 1$, then $a \rightarrow b = 1 \rightarrow b = b \in F$. If $a \neq 1$, then $a \rightarrow b, b \rightarrow a \in F$, by Lemma 7.2.8(4).
<u>0 is in F:</u>
Let $a \in F, a \neq 1$. Then $0 \leq a$. Thus $0 \in F$ by Lemma 7.2.8(3).
Hence F is a subuniverse of $\mathbf{L}$.
Next we show that S is a subuniverse of $\mathbf{L}$ on the language $(\circ, \rightarrow, \wedge, \vee, 1)$.
<u>S is closed under $\circ$:</u>
Let $a, b \in S$ such that $a, b \neq 1$ and assume $a \circ b \notin S$. Then $a \circ b \neq 1$, $a \circ b \in F$, $a \notin F$ and $b \notin F$. By residuation, $a \circ b \leq a \circ b$ implies $a \leq b \rightarrow (a \circ b)$. Hence $a \leq b \rightarrow (a \circ b)$ and $a \notin F$ implies $b \rightarrow (a \circ b) = 1$, by Lemma 7.2.8(3). Thus $b \leq a \circ b \leq b$. Hence $b = a \circ b$. But $a \circ b \in S$ since $b \in S$, so this contradicts the fact that $a \circ b \notin S$.
<u>S is closed under $\rightarrow$:</u>
Let $a, b \in S$ such that $a, b \neq 1$. Since $b \leq a \rightarrow b$ (Lemma 4.2.1(17)) and $b \notin F$, $a \rightarrow b \notin F$, by Lemma 7.2.8(3). Hence $a \rightarrow b \in S$.
Since any filter is closed under $\wedge$ and $\vee$, S is closed under $\wedge$ and $\vee$.
<u>1 is in S:</u>
$1 \in S$ from the definition of S.
Hence S is a subuniverse of $\mathbf{L}$ on the language $(\circ, \rightarrow, \wedge, \vee, 1)$.

S is a filter of $\mathbf{L}$:

$1 \in S$ and we have already shown that S is closed under $\circ$. It remains for us to show that S is upward closed. Let $a \in S$ such that $a \leq b$. Then $a \notin F$. Hence $a \notin F$ and $a \leq b$ implies $b \notin F$, by (3). Thus $b \in S$.

(2) S is subdirectly irreducible:

$U \subseteq S$, by Lemma 7.2.7(2) and the definition of S. Since U is the least filter of $\mathbf{L}$ distinct from $\{1\}$, U is also the least filter of $\mathbf{S}$ distinct from $\{1\}$, by the congruence extension property (Lemma 7.2.1). Therefore $\mathbf{S}$ is subdirectly irreducible.

S is linearly ordered:

To prove that S is linearly ordered, it is enough to show that for all $a, b \in S$, $b \to a = a$ implies $a = 1$ or $b = 1$, by Lemma 7.2.4. Let $a, b \in S$ such that $b \neq 1$ and $b \to a = a$. Then there exists $u \in U$ such that $u \neq 1$ and $b \leq u$, by Lemma 7.2.7(1). Thus $u \to a \leq b \to a = a$, by Lemma 4.2.1(2). Also $a \leq u \to a$ by Lemma 4.2.1(17), so $u \to a = a$. Hence $a \in F$ by Lemma 7.2.7(3). Therefore $a \in F \cap S$ and it follows from Lemma 7.2.7(2) that $a = 1$.

(3) The result follows immediately from (1) and Lemma 7.2.8(5).

□

7.3 Completeness Theorems for BL

Theorem 7.3.1. *Every BL-algebra is a subdirect product of linearly ordered BL-algebras.*

Proof. Every BL-algebra $\mathbf{L}$ is an MTL-algebra and hence can be embedded into a product of linearly ordered MTL-algebras of the form $\mathbf{L}/\sim_F$, where F is a prime filter of $\mathbf{L}$ by Theorem 5.1.1. But since $\sim_F$ is a congruence and BL is a variety, each $\mathbf{L}/\sim_F$ is a BL-algebra. Thus the direct product of BL-algebras of the form $\mathbf{L}/\sim_F$ is a BL-algebra. Therefore every BL-algebra is a subdirect product of linearly ordered BL-algebras. □

Theorem 7.3.2 (completeness)**.** *BL is complete with respect to BL-algebras and BL-chains, that is for each formula φ the following are equivalent:*

(1) φ is provable in BL;

(2) for each linearly ordered BL-algebra $\boldsymbol{L}$, φ is an $\boldsymbol{L}$-tautology;

(3) for each BL-algebra $\boldsymbol{L}$, φ is an $\boldsymbol{L}$-tautology.

Proof. The proof of the completeness theorem of BL with respect to BL-algebras and BL-chains is entirely similar to that of MTL-algebras and MTL-chains (see Theorem 5.1.2). □

Next we shall deal with completeness of BL with respect to standard BL-algebras.

Lemma 7.3.1. *Let $\boldsymbol{L}$ be an MTL-algebra and $a, b \in L$. Then $\langle a \vee b\rangle = \langle a\rangle \cap \langle b\rangle$, where $\langle c\rangle$ denotes the filter generated by c.*

Proof. We first show that MTL-algebras satisfy: $(x \vee y)^n = x^n \vee y^n$. In a linearly ordered MTL-algebra, either $x \leq y$ or $y \leq x$. If $x \leq y$, then $(x \vee y)^n = y^n$ and $x^n \leq y^n$ so $x^n \vee y^n = y^n$. If $y \leq x$, the proof is similar. Since every MTL-algebra is a subdirect product of linearly ordered MTL-algebras (Theorem 5.1.1), the identity must hold in all MTL-algebras.

Recall that $\langle a\rangle = \left\{c \in L : a \overset{n}{\to} c = 1 \text{ for some } n \in \mathbf{N}\right\}$, so $c \in \langle a\rangle$ iff $a^n \leq c$ for some $n \in \mathbf{N}$. Let $c \in \langle a \vee b\rangle$, so $(a \vee b)^n \leq c$ for some $n \in \mathbf{N}$, hence $a^n \vee b^n \leq c$. Therefore $a^n \leq c$ and $b^n \leq c$, so $c \in \langle a\rangle \cap \langle b\rangle$. Also, if $c \in \langle a\rangle \cap \langle b\rangle$, then $a^n \leq c$ for some $n \in \mathbf{N}$ and $b^m \leq c$ for some $m \in \mathbf{N}$. Then $(a \vee b)^{max\{n,m\}} \leq c$, so $c \in \langle a \vee b\rangle$. □

Lemma 7.3.2. *A finite BL-algebra is subdirectly irreducible if and only if it is linearly ordered.*

Proof. Let $\mathbf{L}$ be a finite subdirectly irreducible BL-algebra.

$\Rightarrow$:

Then $(a \to b) \vee (b \to a) = 1$. Now, $\langle 1\rangle = \{1\}$, but $\langle (a \to b) \vee (b \to a)\rangle = \langle a \to b\rangle \cap$

$\langle b \to a \rangle$ by Lemma 7.3.1. Thus $\langle a \to b \rangle = \{1\}$ or $\langle b \to a \rangle = \{1\}$ since $\mathbf{L}$ is subdirectly irreducible. Hence $a \to b = 1$ or $b \to a = 1$. Therefore, $a \leq b$ or $b \leq a$.

$\underline{\Leftarrow}$:

Since $\mathbf{L}$ is finite and linearly ordered the set $\{x \in \mathbf{L}\}$ has a largest element, say b. Thus $\langle b \rangle$ is the least nontrivial filter. $\square$

Recall the algebras $\mathbf{Wa}_n$ from Definition 7.1.6.

Theorem 7.3.3. *$\boldsymbol{L}$ is a finite subdirectly irreducible BL-algebra if and only if there are $k, n_1, n_2, ..., n_k \in \boldsymbol{N}$ such that*

$$\boldsymbol{L} \cong \boldsymbol{Wa}_{n_1} \oplus \boldsymbol{Wa}_{n_2} \oplus \cdots \oplus \boldsymbol{Wa}_{n_k}.$$

Proof. Suppose $\mathbf{L}$ is a finite subdirectly irreducible BL-algebra. The proof is by induction on $|L|$, the cardinality of L. If $|L| = 1$, then we are done. Assume $|L| = r$ and that $\mathbf{L}' \cong \mathbf{Wa}_{n_1} \oplus \mathbf{Wa}_{n_2} \oplus \cdots \oplus \mathbf{Wa}_{n_{k-1}}$ for some $k, n_1, n_2, ..., n_{k-1} \in \mathbf{N}$ for all $\mathbf{L}'$ with $|L'| < r$. Then by Theorem 7.2.1(3), there is a BL-algebra $\mathbf{F}$ and a linearly ordered Wajsberg hoop $\mathbf{S}$ such that $\mathbf{L} \cong \mathbf{F} \oplus \mathbf{S}$ and $|S| > 1$. Note that $\mathbf{F}$ is linearly ordered since $\mathbf{L}$ is linearly ordered by Lemma 7.3.2. Hence by Lemma 7.3.2, $\mathbf{F}$ is finite and subdirectly irreducible. We have that $|F| < r$, so by the induction hypothesis $\mathbf{F} \cong \mathbf{Wa}_{n_1} \oplus \mathbf{Wa}_{n_2} \oplus \cdots \oplus \mathbf{Wa}_{n_{k-1}}$ for some $k, n_1, n_2, ..., n_{k-1} \in \mathbf{N}$. Also $\mathbf{S}$ is a finite linearly ordered Wajsberg hoop, so it is isomorphic to $\mathbf{Wa}_{n_k}$ for some $n_k \in \mathbf{N}$, by [22, Theorem 3.13]. Therefore $\mathbf{L} \cong \mathbf{Wa}_{n_1} \oplus \mathbf{Wa}_{n_2} \oplus \cdots \oplus \mathbf{Wa}_{n_k}$. For the converse, note that $\mathbf{Wa}_{n_1} \oplus \mathbf{Wa}_{n_2} \oplus \cdots \oplus \mathbf{Wa}_{n_{k-1}}$ is a finite and linearly ordered, hence subdirectly irreducible, BL-algebra. $\square$

Definition 7.3.1. Given an algebra $\mathbf{A}$ (of any type) and any $A' \subseteq A$, the *partial subalgebra* $\mathbf{A}'$ of $\mathbf{A}$ is the partial algebra with universe A' and for each operation f of $\mathbf{A}$ (say k-ary) and $b_1, \ldots, b_k \in A'$,

$$f^{\mathbf{A}'}(b_1, \ldots, b_k) = \begin{cases} f^{\mathbf{A}}(b_1, \ldots, b_k) & \text{if} \quad f^{\mathbf{A}}(b_1, \ldots, b_k) \in A', \\ \textit{undefined} & \text{if} \quad f^{\mathbf{A}}(b_1, \ldots, b_k) \notin A'. \end{cases}$$

A class $\mathcal{K}$ of algebras has the *finite embeddability property* (FEP, for short) if every finite partial subalgebra of any given member of $\mathcal{K}$ can be embedded into some finite member of $\mathcal{K}$.

By [2] (and also [8]) if a variety $\mathcal{K}$ of algebras has the FEP, then it is generated as a quasivariety by its finite members. That is,

$$\mathcal{K} = ISPP_u(\{\text{finite members in } \mathcal{K}\}).$$

We use the following result from [2, Theorem 3.9] without proof:

Theorem 7.3.4. *The class of subdirectly irreducible Wajsberg hoops has the FEP.*

Theorem 7.3.5. *The class of linearly ordered BL-algebras has the finite embeddability property.*

Proof. Let $\mathbf{L}$ be a linearly ordered BL-algebra and $\mathbf{L}'$ a finite partial subalgebra of $\mathbf{L}$ such that $L' = \{l_1, l_2, ..., l_n\}$, where $l_1 > l_2 > ... > l_n$. We prove the theorem by induction on n, the cardinality of L'. If $n = 1$, then $\mathbf{L}'$ embeds into the trivial BL-algebra. If $n > 1$, then the set $\{l_i \to l_j : 1 \leq i < j \leq n\}$ has a largest element say $l_k \to l_m$. We want to show that any maximal congruence θ of $\mathbf{L}$ not containing (l_k, l_m) as an element cannot contain (l_i, l_j) with $i \neq j$. Suppose on the contrary that $(l_i, l_j) \in \theta$. Then $(l_i \to l_j, l_j \to l_j) \in \theta$. But $l_j \to l_j = 1$, so $(l_i \to l_j, 1) \in \theta$. Since $F = 1/\theta$ is a filter and $l_i \to l_j \leq l_k \to l_m$, we have $l_k \to l_m \in F$. Also $l_m \to l_k = 1 \in F$, so $(l_k, l_m) \in \theta$, contradicting our assumption. Hence $(l_i, l_j) \notin \theta$ for all $i \neq j$ and so $\mathbf{L}'$ embeds into $\mathbf{L}/\theta$ by the map $l_i \mapsto l_i/\theta$.

Now, $\mathbf{L}/\theta$ is a linearly ordered BL-algebra (since $\mathbf{L}$ is). Also, $\mathbf{L}/\theta$ is subdirectly irreducible since the congruence generated by $([l_k]_\theta, [l_m]_\theta)$ is the least nontrivial congruence: recall that θ is the maximal congruence not containing (l_k, l_m) so any nontrivial congruence of $\mathbf{L}/\theta$ contains $([l_k]_\theta, [l_m]_\theta)$ by the correspondence theorem (see [3, Theorem 6.20]). Thus, $\langle [l_k \to l_m]_\theta \rangle$ is the least nontrivial filter of $\mathbf{L}/\theta$. By Theorem 7.2.1(3), $\mathbf{L}/\theta = \mathbf{F} \oplus \mathbf{S}$, where $\mathbf{F}$ is a linearly ordered subalgebra of $\mathbf{L}/\theta$ and $\mathbf{S}$ a linearly ordered subalgebra of $\mathbf{L}/\theta$ that is also a filter and hence $[l_k \to l_m]_\theta \in S$. Hence we cannot have both $[l_k]_\theta, [l_m]_\theta \in F$ otherwise $[l_k \to l_m]_\theta \in F$. Thus $(L'/\theta) \cap F$ has fewer elements than L'. Hence by the induction hypothesis, there exists a finite linearly ordered BL-algebra $\mathbf{F}_1$ such that the partial algebra $\mathbf{L}'/\theta \cap \mathbf{F}$ embeds into $\mathbf{F}_1$. Also $\mathbf{L}'/\theta \cap \mathbf{S}$ is a partial subalgebra of a subdirectly irreducible Wajsberg hoop, so it can be embedded into a finite subdirectly irreducible Wajsberg hoop $\mathbf{S}_1$ by Theorem 7.3.4. Consequently, $\mathbf{L}'/\theta$ embeds into $\mathbf{F}_1 \oplus \mathbf{S}_1$ which is a linearly ordered BL-algebra. Thus, $\mathbf{L}'$ embeds into a linearly ordered BL-algebra, hence the class of linearly ordered BL-algebras has the FEP. □

Theorem 7.3.6. *The variety $\mathcal{BL}$ of BL-algebras is generated as a quasivariety by its finite members. In fact*

$$\mathcal{BL} = \boldsymbol{SPP}_u\left(\boldsymbol{Wa}_{n_1} \oplus \boldsymbol{Wa}_{n_2} \oplus \cdots \oplus \boldsymbol{Wa}_{n_k} : k, n_1, n_2, ..., n_k \in \boldsymbol{N}\right)$$

Proof. By Theorem 7.3.1, every BL-algebra is a subdirect product of linearly ordered BL-algebras. By the FEP for linearly ordered BL-algebras (Theorem 7.3.5), the class of linearly ordered BL-algebras is generated as a quasivariety by the finite linearly ordered BL-algebras. By Theorem 7.3.3, every finite linearly ordered BL-algebra is of the form $\mathbf{Wa}_{n_1} \oplus \mathbf{Wa}_{n_2} \oplus \cdots \oplus \mathbf{Wa}_{n_k}$ and so the result follows. □

Theorem 7.3.7. *The variety of BL-algebras is generated as a quasivariety by all standard BL-algebras, that is, by all algebras of the form $([0,1], \circ, \rightarrow, \wedge, \vee, 1)$, where $\circ$ is a continuous t-norm and $\rightarrow$ its residual.*

Proof. By Theorem 7.3.6 it suffices to show that every algebra of type $\mathbf{Wa}_{n_1} \oplus \mathbf{Wa}_{n_2} \oplus \cdots \oplus \mathbf{Wa}_{n_k}$ for some $k, n_1, n_2, ..., n_k \in \mathbf{N}$ can be embedded into a standard BL-algebra. Each $\mathbf{Wa}_{n_i}$ is embeddable in $([0,1], \circ_L, \rightarrow_L, \wedge_L, \vee_L, 1)$, by Lemma 7.1.3(3). Let $\mathbf{B}_i$ be a copy of $([0,1], \circ_L, \rightarrow_L, \wedge_L, \vee_L, 1)$ in which $\mathbf{Wa}_{n_i}$ is embedded. Then by Lemma 7.1.3(1), $\mathbf{B}_i$ is isomorphic to $\mathbf{D}_i = ([a_i, b_i], \cdot_i, \rightarrow_i, 1)$, where $a_i = \frac{i-1}{k}$, $b_i = \frac{i}{k}$ and the operations are defined accordingly. Thus $\mathbf{Wa}_{n_1} \oplus \mathbf{Wa}_{n_2} \oplus \cdots \oplus \mathbf{Wa}_{n_k}$ can be embedded into $\mathbf{D} = \mathbf{D}_1 \oplus \mathbf{D}_2 \oplus \cdots \oplus \mathbf{D}_k$ and the universe of $\mathbf{D}$ is $[0,1]$. The binary operation $\circ$ defined on $\mathbf{D}$ is obviously a continuous t-norm. Thus the algebra $\mathbf{Wa}_{n_1} \oplus \mathbf{Wa}_{n_2} \oplus \cdots \oplus \mathbf{Wa}_{n_k}$ is isomorphic to a subalgebra of a standard BL-algebra, so the class of all BL-algebras is generated by the standard BL-algebras. □

Theorem 7.3.8. *BL is complete with respect to the class of standard BL-algebras.*

Proof. The result follows immediately from Theorem 7.3.7. □

Below is a justification of why the proof of standard completeness for MTL does not work for BL and hence why a different proof for BL was needed.

Consider the algebra $\mathbf{L} = (\mathbf{Q} \cap [0,1], \circ, \rightarrow, \wedge, \vee, 0, 1)$, where $\circ$ is the Lukasiewicz t-norm: $x \circ y = (x + y - 1) \vee 0$, and its residuum: $x \rightarrow y = (1 - x + y) \wedge 1$. Note that $\mathbf{L}$ is a subalgebra of $([0,1], \circ, \rightarrow, \wedge, \vee, 0, 1)$ which is a BL-algebra, so $\mathbf{L}$ is a BL-algebra.
Let $\mathbf{X} = (X, *, \rightarrow_*, \wedge, \vee, 0, 1)$ be as in Theorem 6.1.1, so
$X = \{(s,t) : s \in \mathbf{Q} \cap [0,1], s \neq 0, t \in \mathbf{Q} \cap (0,1]\} \cup \{(0,1)\}$.
We show that there exist $a, b \in X$ such that $a \preceq b$ but $b * (b \rightarrow_* a) \prec a$. This implies that the standard MTL-algebra constructed from X as in Chapter 6 cannot be a BL-algebra as the identity $x \circ (x \rightarrow y) = x \wedge y$ fails.
Let $a = \left(\frac{1}{2}, q_1\right) \in X$ and $b = \left(\frac{1}{2}, q_2\right) \in X$ such that $q_1 < q_2 < 1$. Then $a \preceq b$.
We claim that:

$$\begin{aligned} & \{(s,t) : b * (s,t) \preceq a,\ s,t \in \mathbf{Q} \cap (0,1] \text{ or } (s,t) = (0,1)\} \\ = & \left\{(s,t) : \left(\tfrac{1}{2}, q_2\right) * (s,t) \preceq \left(\tfrac{1}{2}, q_1\right),\ s,t \in \mathbf{Q} \cap (0,1] \text{ or } (s,t) = (0,1)\right\} \\ = & \{(s,t) : s,t \in \mathbf{Q} \cap (0,1] \text{ and } s < 1\}. \end{aligned}$$

To see the above, note that: If $s = 1$ and $t \in (0,1]$, then

$$\left(\tfrac{1}{2}, q_2\right) * (1,t) = \left(\tfrac{1}{2}, q_2\right) \npreceq \left(\tfrac{1}{2}, q_1\right) \quad \text{since } q_1 < q_2.$$

If $0 < s < 1$ and $t \in (0,1]$, then

$$\begin{aligned} s \circ_L \tfrac{1}{2} &= \left(\tfrac{1}{2} + s - 1\right) \vee 0 \\ &= \left(s - \tfrac{1}{2}\right) \vee 0 \\ &= \begin{cases} 0 & \text{if } 0 < s \leq \frac{1}{2} \\ s - \frac{1}{2} & \text{if } \frac{1}{2} < s < 1 \end{cases} \\ &< s \wedge \tfrac{1}{2}. \end{aligned}$$

Therefore in this case, $\left(\frac{1}{2}, q_2\right) * (s,t) = \left(s \circ_L \frac{1}{2}, 1\right) \prec \left(\frac{1}{2}, q_1\right)$. Thus the above claim holds. The supremum of $\{(s,t) : s,t \in \mathbf{Q} \cap (0,1] \text{ and } s < 1\}$ does not exist in $\mathbf{X}$ since the upper bounds of this set are $\{(1,t) : t \in (0,1]\}$ which has no infimum in $\mathbf{X}$. In the completion of $\mathbf{X}$, the set $\{(s,t) : s,t \in \mathbf{Q} \cap (0,1] \text{ and } s < 1\}$ has a supremum, which is $b \rightarrow_* a$.

$$\begin{aligned} b * (b \rightarrow_* a) &= b * \sup\{(s,t) : s,t \in \mathbf{Q} \cap (0,1] \text{ and } s < 1\} \\ &= \sup\{b * (s,t) : s,t \in \mathbf{Q} \cap (0,1] \text{ and } s < 1\} \text{ (by left-continuity)} \\ &= \sup\left\{\left(\tfrac{1}{2}, q_2\right) * (s,t) : s,t \in \mathbf{Q} \cap (0,1] \text{ and } s < 1\right\} \\ &= \sup\left\{\left(\tfrac{1}{2} \circ_L s, 1\right) : s,t \in \mathbf{Q} \cap (0,1] \text{ and } s < 1\right\} \\ &\prec \left(\tfrac{1}{2}, q_1\right) = a. \end{aligned}$$

Our key objective of this chapter was to show that BL is complete with respect to standard BL-algebras. The following lemmas contributed towards proving this Completeness Theorem:

(1) Every BL-algebra is a subdirect product of linearly ordered BL-algebras.

(2) A finite BL-algebra is subdirectly irreducible if and only if it is linearly ordered.

(3) $\mathbf{L}$ is a finite subdirectly irreducible BL-algebra if and only if there are $k, n_1, n_2, ..., n_k \in \mathbf{N}$ such that

$$\mathbf{L} \cong \mathbf{Wa}_{n_1} \oplus \mathbf{Wa}_{n_2} \oplus \cdots \oplus \mathbf{Wa}_{n_k}.$$

(4) The class of linearly ordered BL-algebras has the finite embeddability property.

(5) $\mathcal{BL} = \mathbf{SPP}_u\left(\mathbf{Wa}_{n_1} \oplus \mathbf{Wa}_{n_2} \oplus \cdots \oplus \mathbf{Wa}_{n_k} : k, n_1, n_2, ..., n_k \in \mathbf{N}\right)$.

(6) The variety of BL-algebras is generated as a quasivariety by the class of standard BL-algebras.

Below we explain how these lemmas were used to give the Completeness Theorem. Lemmas (1) and (4) imply that the variety of BL-algebras is generated as a quasivariety by its finite linearly ordered members. From (2) each finite linearly ordered BL-algebra is subdirectly irreducible, so it is an ordinal sum of $\mathbf{W}_n$-algebras, by (3). To show that (6) holds, it was sufficient to show that each ordinal sum of $\mathbf{W}_n$-algebras can be embedded into a standard BL-algebra since by (5), the variety of BL-algebras is generated as a quasivariety by the class of ordinal sums of $\mathbf{W}_n$-algebras. As a consequence of (6) and the fact that BL is complete with respect to BL-algebras, BL is complete with respect to standard BL-algebras. We have also provided an example to show that the proof technique used to establish standard completeness of MTL does not apply to BL.

CHAPTER 8

Conclusion

We started by considering t-norms on the unit interval $[0,1]$, which are associative, commutative, order-preserving binary operations with identity 1. In particular, we considered t-norms that are continuous or left-continuous in the sense of classical analysis. The left-continuity property means that there exists an associated binary operation called the residuum of the t-norm and acts like an implication operation. In addition, the meet and join operations on $[0,1]$ act like logical conjunction and disjunction and 1, 0 as logical constants. Thus, the classes of continuous and left-continuous t-norms have a natural connection with logic. What we did in this thesis was to clarify this connection by collecting a number of related results together in one place. The logic MTL was defined and shown to be the logic of left-continuous t-norms. What this means is that MTL is complete with respect to left-continuous t-norms. The way we showed this was by proving that MTL is complete with respect to MTL-algebras and, in fact, with respect to linearly ordered MTL-algebras. Every finitely generated linearly ordered MTL-algebra was shown to be embeddable into a standard MTL-algebra, which is a left-continuous t-norm algebra. It was sufficient to use finitely generated algebras since any algebra is a subalgebra of an ultraproduct of its finitely generated subalgebras. Another way of expressing this is to say that the variety of MTL-algebras is generated by the standard MTL-algebras.

We also showed that the logic BL is the logic of continuous t-norms, that is, we showed that BL is complete with respect to standard BL-algebras, which are continuous t-norm algebras. To achieve this we first showed that, as for MTL-algebras, every BL-algebra is a subdirect product of linearly ordered BL-algebras. Then we showed that the class of linearly ordered BL-algebras has the finite embeddability property and consequently the variety of BL-algebras is generated as a quasivariety by its finite linearly ordered members. Since each finite linearly ordered BL-algebra is a subdirectly irreducible Wajsberg algebra, it is an ordinal sum of $\mathbf{W}_n$-algebras, and we showed that each such ordinal sum can be embedded into a standard BL-algebra. Thus, the variety of BL-algebras is generated as a quasivariety by the standard BL-algebras. Hence BL is complete with respect to the class of standard BL-algebras.

Since the variety of BL-algebras is a subvariety of the variety of MTL-algebras, it may seem that the proof of completeness for MTL with respect to standard MTL-algebras can be extended to a proof of completeness for BL with respect to standard BL-algebras. We gave an example which showed that this approach would not work.

We further proved the theorem that characterizes all continuous t-norms, namely that each continuous t-norm can be decomposed as an ordinal sum of Lukasiewicz, Product and Gödel t-norms. Unlike the situation pertaining to continuous t-norms, knowledge of left-continuous t-norms is limited, so there is no structural theorem that would allow us to classify them in a similar way [6].

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