A dissertation submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, in fulfilment of the requirements for the degree of Master of Science.

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Giant Graviton Oscillators

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Declaration

This dissertation is being submitted for the degree of Master of Science in the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

In part 1 of this dissertation I have given an introduction and background which puts the project in context. Part 2 is based on the publication

> "Giant Graviton Oscillators," R. de Mello Koch, M. Dessein, D. Giataganas and C. Mathwin, JHEP **10** (2011) 009 [arXiv:1108.2761v1 [hep-th]]

and I have given more detail which I hope will make the work more accessible and clear. Any results from elsewhere that have been used are referenced accordingly.

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Abstract

We study the action of the dilatation operator on restricted Schur polynomials labeled by Young diagrams with p long columns or p long rows. A new version of Schur-Weyl duality provides a powerful approach to the computation and manipulation of the symmetric group operators appearing in the restricted Schur polynomials. Using this new technology, we are able to evaluate the action of the one loop dilatation operator. The result has a direct and natural connection to the Gauss Law constraint for branes with a compact world volume. We find considerable evidence that the dilatation operator reduces to a decoupled set of harmonic oscillators. This strongly suggests that integrability in $\mathcal{N} = 4$ super Yang-Mills theory is not just a feature of the planar limit, but extends to other large N but non-planar limits.

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I would like to thank my supervisor Robert de Mello Koch for teaching me so much, and always being supportive and understanding. His inspiring guidance through this journey in Theoretical Physics has just been wonderful.

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Part I

Introduction & Background

Chapter 1

Introduction

An astounding discovery from string theory is the AdS/CFT correspondence[1] which asserts dualities between certain superconformal field theories and supergravity on the product of anti-de Sitter spacetimes with compact manifolds. By duality we mean that two dual theories are physically completely equivalent. The most understood duality is that of $\mathcal{N} = 4$ super Yang-Mills theory on four dimensional Minkowski spacetime with type IIB string theory on $AdS_5 \times S^5$. Since $\mathcal{N} = 4$ super Yang-Mills is maximally supersymmetric it is the most tractable gauge theory in the class of dualities conjectured in [1]. A formal proof of the conjecture has yet to be revealed, and so the correspondence might be contemplated with much skepticism; how can it be that a theory without gravity on a four dimensional flat space contains all the physics of a theory on a ten dimensional curved space that has gravity as an integral part? Considering that the conjecture has not conflicted with a single one of the 10535 (to date) publications that cite it, faith in its validity does not require much of a leap.

String theory in curved spacetimes has proven to be highly inaccessible, especially compared to $\mathcal{N} = 4$ super Yang-Mills which has been shown to be integrable in certain limits. A dictionary for translating between these dual theories would allow us to understand string theory by studying the dual conformal field theory. [2, 3] have provided a powerful piece of this dictionary that applies to gravitons; the recipe maps states in the gravity theory to operators in the dual conformal field theory.

1.1 Integrability in $\mathcal{N} = 4$ Super Yang-Mills

For a system to be integrable a conservation law is needed for each degree of freedom. Since super Yang-Mills is a quantum field theory, its operators are *fields*, thus integrability requires an infinite number of conservation laws. Methods of finding equations that lead to an infinite number of conservation laws and showing that these laws are independent are referred to as methods of integrability[1].

Super Yang-Mills is a cousin of Quantum Chromodynamics in the sense that it is based on the same types of fundamental particles and interactions, but also enjoys supersymmetry and conformal symmetry. The latter implies that there are no dimensionful parameters in the theory (see appendix A). In normal quantum field theory the basic observable is the S-matrix. It provides predictions for scattering experiments. A conformal field theory is scale invariant so the scattering framework cannot be used; there is no notion of the distant past and future required for non-interacting initial and final states. What is of primary interest in a conformal field theory is the scaling dimension Δ of operators

$$\langle \mathcal{O}_\alpha(x)\mathcal{O}_\beta(y)\rangle \propto \frac{\delta_{\alpha\beta}}{|x-y|^{2\Delta}} \ ,$$
$$\Delta = \Delta_0 + g_{\rm YM}^2\Delta_2 + g_{\rm YM}^2\Delta_4 \cdots \ .$$

$\mathcal{O}$ is a local operator meaning it is composed from the fundamental fields, all residing at a common point in spacetime. Δ_0 is the classical or bare dimension of $\mathcal{O}$, given by the sum of the constituent dimensions. $\Delta_2, \Delta_4, \ldots$ are the anomalous dimensions of $\mathcal{O}$ which are quantum corrections to the classical dimension due to interactions between the constituents. Δ_2 is a one loop level Feynman diagram contribution, Δ_4 a two loop level contribution and so on.

In 't Hooft's planar limit[50] of $\mathcal{N} = 4$ super Yang-Mills, integrability has proven a powerful tool in that it has made it possible to express the scaling dimension of some local operator $\mathcal{O}$ as a function of the coupling constant[2] λ

$$\Delta = f(\lambda) \ .$$

In general this function is given as the solution of a set of integral equations which follow from the thermodynamic Bethe ansatz[25] or related techniques. The equations simplify to a set of algebraic equations in a certain limit (the asymptotic Bethe equations) which have been solved numerically for a wide range of λ's in particular cases. This has given hope that the elusive dream

[1]For a review of integrability in the context of the AdS/CFT correspondence as of the year 2010 see [24].

[2]Recall that in 't Hooft's formalism, the rank of the gauge group N is taken to infinity while keeping the coupling $\lambda = g_{\rm YM}^2 N$ fixed. This corresponds to an infinite number of colours and infinitesimal interaction strength.

of understanding QCD at strong coupling is reachable, especially since any four dimensional gauge theory can be viewed as $\mathcal{N}=4$ Super Yang-Mills with some particles and interactions added or removed[24]. The spectrum that the solutions provide are not simple formulae like that of the harmonic oscillator

$$E_{HO} = \omega(n + \frac{1}{2}) \; .$$

Until recently, it seemed that such simplicity was too much to hope for. In [26], which forms the basis of this dissertation, we extend the results of [9, 14, 15] solidifying evidence that the spectrum of one loop anomalous dimensions of a specific class of operators is precisely a set of harmonic oscillators! What is most powerful about our findings is that this elegance emerges from beyond the planar limit, where integrability has supposedly been proven not to exist[8]. The reason why the planar limit is not sufficient for the operators we consider is that they have a bare dimension of order N, and so huge combinatoric factors (arising from the number of ways one can form the Feynman diagrams out of so many fields) enhance the non-planar contributions and completely overpower the usual $\frac{1}{N^2}$ suppression of non-planar diagrams[6].

1.2 The Conformal Field Theory Duals of Giant Gravitons

The objects we wish to study in IIB string theory on $AdS_5 \times S^5$ are giant gravitons[18]. These are $D3$-branes with a spherical world volume, stable due to their orbital angular momentum and the five form flux[17]. From the state-operator map of [2, 3] we know that the mass of states in the string theory map to the dimension of operators in the CFT. Thus operators dual to giant gravitons must have a large bare dimension and so calculating correlation functions of these operators involves summing a lot more than just the planar diagrams. In an inspired article, [4] showed that this daunting task can be achieved by moving from the trace basis to the basis of Schur polynomials

$$\prod_{n_i} \mathrm{Tr}(Z^{n_i}) = \sum_j \alpha_j \chi_{R_j}(Z) \; ,$$

where the operators Z are complex combinations of two of the six scalar fields in $\mathcal{N}=4$ SYM, which are in the adjoint representation of $U(N)$; $Z = \phi_1 + i\phi_2$ is an $N \times N$ matrix whose elements are gluons. The Schur polynomial is defined by

$$\chi_R(Z) = \frac{1}{n!}\sum_{\sigma\in S_n}\chi_R(\sigma)\mathrm{Tr}\left(\sigma Z^{\otimes n}\right) ,$$

where $\mathrm{Tr}\left(\sigma Z^{\otimes n}\right) = Z^{i_1}_{i_{\sigma(1)}}Z^{i_2}_{i_{\sigma(2)}}\cdots Z^{i_n}_{i_{\sigma(n)}}$. R is a Young diagram[3] with n boxes and hence labels an irreducible representation of the symmetric group of degree n (S_n). $\chi_R(\sigma)$ is the character of group element σ in representation R, given by $\chi_R(\sigma) = \mathrm{Tr}\left(\Gamma_R(\sigma)\right)$, where $\Gamma_R(\sigma)$ is the representation matrix of σ in the vector space carrying R.

[4] showed that in the $\frac{1}{2}$-BPS sector, the two point functions are diagonal in the Schur polynomial labels

$$\left\langle \chi_R(Z)\chi_S(Z)^\dagger\right\rangle = \delta_{RS}f_R \,. \tag{1.1}$$

f_R is the product of the factors in the Young diagram R; the factor of a box in row i and column j is $N+i-j$. The fact that the two point correlators are exact in N means that all Feynman diagrams (not just the planar diagrams) are summed.

Soon after this initial work, an elegant explanation of the results of [4] were given in terms of projection operators[27]. One of the basic observations made in [27] is the fact that two point functions of operators of the form

$$\hat{A}_n \equiv A^{i_1\, i_2\cdots i_n}_{j_1\, j_2\cdots j_n}Z^{j_1}_{i_1}Z^{j_2}_{i_2}\cdots Z^{j_n}_{i_n} = \mathrm{Tr}(AZ^{\otimes n})$$

are given by

$$\left\langle \hat{A}_n\hat{B}^\dagger_n\right\rangle = \sum_{\sigma\in S_n}\mathrm{Tr}(\sigma A\sigma^{-1}B^\dagger) \,.$$

By choosing A and B to be projection operators projecting onto irreducible representations of the symmetric group, they clearly commute with σ (rendering the above sum trivial) and are orthogonal. With this choice for A, $\hat{A}_n$ is nothing but a Schur polynomial, so that we obtain a rather simple understanding of how and why the Schur polynomials diagonalize the two point function.

Certain Schur polynomials with order N Z's were quickly identified[6, 4, 29, 28] as the operators of $\mathcal{N}=4$ SYM dual to the giant gravitons of IIB string theory on $AdS_5\times S^5$, while Schur polynomials with order N^2 fields were identified with $\frac{1}{2}$-BPS geometries[30, 31].

[3]Recall that a Young diagram is an arrangement of boxes in rows and columns in a single, contiguous cluster such that the left borders of all rows are aligned and each row is not longer than the one above. The empty Young diagram consisting of no boxes is a valid Young diagram.

Excited giant gravitons can be described in terms of open strings which end on the D-brane. Operators dual to excited giant gravitons were proposed in [5]. Since giant gravitons have a compact world volume, Gauss's Law requires that the total charge on the worldvolume must vanish[32]. A highly non-trivial test of the proposal of [5] is that the number of operators that can be defined matches the number of states obeying this Gauss Law constraint. The operators of [5] are defined in terms of symmetric group operators that project from the carrier space of some irreducible representation of the symmetric group to a subspace defined using the carrier space of an irreducible representation of a subgroup. Although the construction of the operators proposed in [5] is a highly non-trivial problem in the representation theory of the symmetric group, the two point functions of these operators, the *restricted Schur polynomials*, were computed exactly, in the free field theory limit, in [13], by exploiting the technology developed in [10, 11, 12]. It was also shown that the restricted Schur polynomials provide a basis for the gauge invariant local operators built using only scalar (adjoint Higgs) fields[7]. Further, it is a convenient description. Indeed, the restricted Schur basis diagonalizes the two point function in the free field theory limit and it mixes weakly at one loop level[11, 12].

1.2.1 The Dilatation Operator

The quest for the gauge theory/gravity dictionary is guided by symmetries. Super Yang-Mills in (3+1) dimensions has the symmetry group $SO(2,4) \times SO(6)$; conformal invariance is given by $SO(2,4)$, while $SO(6)$ is the symmetry of rotating the theory's six scalar fields into each other. AdS_5 has the isometry group[4] $SO(2,4)$, while S_5 has the rotational symmetry of $SO(6)$. So the two theories enjoy the same symmetries. It is thus natural to associate generators of symmetries that are shared by the theories. The eigenvalues of these generators label states so we get a direct correspondence between states of the two theories. These eigenvalues are of course conserved charges, as every global symmetry of the action is associated with a conserved charge (Noether's theorem). The generator of time translations in string theory, the Hamiltonian, is identified as the dual operator to the generator of scale invariance in conformal field theory, the dilatation operator D. The Hamiltonian's eigenvalues (energies) should thus agree with those of the dilatation operator, which are the anomalous scaling dimensions of the operator it acts

[4] Recall that an isometry group is the group of all smooth, one-to-one maps of the space into itself that leave all distances invariant.

on

$$D\mathcal{O} = \Delta\mathcal{O} \ .$$

Solving this eigenvalue problem will yield the dimensions of $\mathcal{O}$ and its corresponding eigenstates, which will *also* be the eigenenergies and eigenstates of the dual string theory system. This is all that is needed to describe time evolution, and hence the dynamics, of the string theory system[5].

Numerical studies of the dilatation operator when acting on restricted Schur polynomials dual to a two sphere giant system showed that the spectrum of the one loop anomalous dimensions is that of a set of decoupled harmonic oscillators[9, 14]. Using insights gained from these numerical studies, an analytic study of the dilatation operator in the sector of the theory with either two sphere giants or two AdS giants has been carried out in [15]. The crucial new ingredient in [15] is the realization that the problem of computing the symmetric group operators needed to define the restricted Schur polynomial can be performed using an auxiliary spin chain. This is essentially an application of Schur-Weyl duality. The suggestion that Schur-Weyl duality may play an important role in the study of gauge theory/gravity duality was first made in [16].

In this dissertation we will recover the two giant graviton results of [15] by clarifying the role of Schur-Weyl duality. An auxiliary spin chain will not be used. The advantage of the new approach is that it will allow us to study the p giant graviton sector of the theory. This generalization is highly non-trivial as we now explain. The two giant graviton problem is too simple to see the full complexity of the problem. Indeed, the symmetric group operators needed to define the restricted Schur polynomials in this case are simple because the subspaces they project to appear without multiplicity. For $p > 2$ giant gravitons, this multiplicity problem must be solved. Our present approach, based on Schur Weyl duality, allows us to

- Construct the restricted Schur polynomials for the p giant graviton problem using the representation theory of $U(p)$. For the case of p giant gravitons we obtain an example of Schur-Weyl duality that is, as far as we know, novel.

- Organize the multiplicity of $S_n \times S_m$ irreducible representations subduced from a given S_{n+m} irreducible representation by mapping it into the *inner multiplicity* appearing in $U(p)$ representation theory. As far as we know, this connection has not been pointed out in the maths

[5] Any state can be expanded in terms of its eigenstates, and the evolution of an energy eigenstate is simply $|\psi(t)\rangle = e^{-iEt}\,|\psi(0)\rangle$.

literature, although it follows as a rather simple consequence of the Schur-Weyl duality we have found.

- Evaluate the action of the dilatation operator in terms of known Clebsch-Gordan coefficients of $U(p)$.

Thus, we achieve a complete generalization of the results of [15] together with a much clearer understanding of the general problem. One noteworthy feature of our results is that the action of the one loop dilatation operator has a direct and natural connection to the Gauss Law constraint we discussed above.

Although we have focused on the restricted Schur polynomials in this dissertation, they are not the only basis for local gauge invariant operators of a matrix model. Another interesting basis to consider is the Brauer basis[33, 34]. This basis is built using elements of the Brauer algebra. The structure constants of the Brauer algebra are N dependent. There is an elegant construction of a class of BPS operators[35] in which the natural N dependence appearing in the definition of the operator[36] is reproduced by the Brauer algebra projectors[35]. Alternatively, another natural approach to the problem is to adopt a basis that has sharp quantum numbers for the global symmetries of the theory[37, 38]. The action of the anomalous dimension operator in this sharp quantum number basis is very similar to the action in the restricted Schur basis: again operators which mix can differ at most by moving one box around on the Young diagram labeling the operator[39]. For further related interesting work see [40, 41]. Finally, for a rather general approach which correctly counts and constructs the weak coupling BPS operators see [42]. The results obtained in [42] can be translated into any of the bases we have considered.

This dissertation is organized as follows:

- The following chapter is dedicated to the AdS/CFT correspondence,
- In chapter 3 we give a detailed review of the discovery of giant gravitons of [18]. This concludes part I.
- In the first chapter of part II we construct the CFT duals of excited giant gravitons by using our new version of Schur-Weyl duality,
- We then evaluate the action of the one-loop dilatation operator on these restricted Schur polynomials in chapter 5,
- The dilatation operator is diagonalized in chapter 6.
- We conclude with an analysis of our results in the final chapter.

In an attempt to keep this disseration self contained, we have included the following appendices

- Appendix A is on conformal symmetry,
- The background representation theory needed to develop our construction is reviewed in Appendices B and C,
- In Appendix D we study a continuum limit of the dilatation operator.

Chapter 2

The AdS/CFT Correspondence

This chapter will illuminate the origins of Maldacena's powerful conjecture. We start with a brief review of anti-de Sitter space and its relation to D-branes.

2.1 Anti-de Sitter Space and D-branes

Anti-de Sitter space is the maximally symmetric solution of Einstein's equations with an attractive cosmological constant included[1]. n dimensional anti-de Sitter space can be represented as a hyperboloid of 'radius' R

$$\left(X^0\right)^2 + \left(X^{-1}\right)^2 - \left(X^1\right)^2 - \left(X^2\right)^2 - \cdots - \left(X^{n-1}\right)^2 = R^2 \tag{2.1}$$

embedded in a flat $n+1$ dimensional space with metric

$$ds^2 = -\left(dX^0\right)^2 - \left(dX^{-1}\right)^2 + \left(dX^1\right)^2 + \left(dX^2\right)^2 + \cdots + \left(dX^{n-1}\right)^2 \ .$$

The second 'time-like' coordinate X_{-1} can be absorbed by introducing the light cone coordinates[49]

$$u = \frac{X^{-1} - X^{n-1}}{R^2} \ , \qquad v = \frac{X^{-1} + X^{n-1}}{R^2} \ , \tag{2.2}$$

and redefining the other coordinates as

$$t = \frac{X^0}{Ru} \ , \qquad x^i = \frac{X^i}{Ru} \ , \tag{2.3}$$

[1]de Sitter space has a repulsive cosmological constant, making it a positively curved Lorentzian manifold, such as the one we live in.

where i runs from 1 to $n-2$. In these coordinates the hyperboloid equation (2.1) takes the form

$$R^4 uv + R^2 u^2 (t^2 - \bar{x}^2) = R^2 \ ,$$

where $\bar{x}^2 \equiv (x^1)^2 + \cdots + (x^{n-2})^2$. From this equation we can express v in terms of u, t, and x^i, and find from (2.2) and (2.3)

$$\begin{aligned} X^{-1} &= \frac{1}{2u}\left(1 + u^2(R^2 + \bar{x}^2 - t^2)\right) \ , \\ X^{n-1} &= \frac{1}{2u}\left(1 + u^2(-R^2 + \bar{x}^2 - t^2)\right) \ , \\ X^i &= Rux^i \\ X^0 &= Rut \ . \end{aligned}$$

It is useful to change to the coordinate $z = \frac{1}{u}$ as we then obtain[2] the Poincaré AdS metric which takes the nice form

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 + (d\bar{x})^2 - dt^2\right) \ . \tag{2.4}$$

The coordinate z divides AdS space into two regions ($z > 0$ and $z < 0$). In each region z behaves as a radial coordinate; at each value of z we have a copy of $n-1$ dimensional Minkowski space scaled by $\frac{R^2}{z^2}$. Anti-de Sitter space has many unusual properties, one of which is that it has a boundary 'at' spatial infinity. This can be seen by considering $z = 0$ which corresponds to spatial infinity in global coordinates (for details see [49]). When $z = 0$ we have the largest possible copy of M_{n-1}, which can thus be identified as the Poincaré patch[3] of the AdS_n boundary.

Another unusual property of anti-de Sitter space is the nature of its time-like geodesics. Any time-like geodesic that departs from the origin will return to it. For this reason anti-de Sitter space can be thought of as 'putting gravity in a box'.

Maldacena considered a system of D-branes in Type IIB string theory to arrive at the conjecture. These Dp-branes are extended objects with p spatial dimensions. Open strings in the presence of a D-brane have their

[2]Recall that the induced metric on a manifold embedded in a space with metric G is obtained from $ds^2 = G_{ij}\frac{\partial X^i}{\partial y^a}\frac{\partial X^j}{\partial y^b}dy^a dy^b$, with X^i embedding coordinates and y^a coordinates parameterizing the manifold, $g_{ab} \equiv G_{ij}\frac{\partial X^i}{\partial y^a}\frac{\partial X^j}{\partial y^b}$ is the induced metric.

[3]The AdS boundary in global coordinates also includes 'points at infinity' of the Poincaré coordinates x^i and t for which $z \neq 0$.

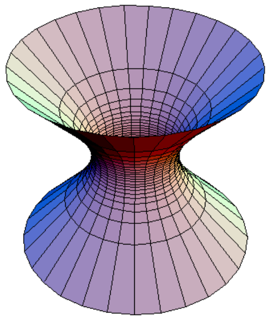


Figure 2.1: Two dimensional de Sitter space embedded in a three dimensional flat space. In this special case the flat space is Minkowski and we can interpret the X^1 coordinate as time and the X^0, X^{-1} coordinates as spacial coordinates. In the figure the X^1 coordinate lies along the axis of rotational symmetry. The circles are lines of constant time and the lines perpendicular to them are time-like geodesics.

endpoints constrained to lie on the brane. The string coordinates normal to the D-brane must thus satisfy Dirichlet boundary conditions

$$Y^a(\tau,\sigma)\Big|_{\sigma=0} = Y^a(\tau,\sigma)\Big|_{\sigma=\pi} = \bar{y}^a \ , \qquad a = p+1, \ldots, d \ ,$$

where the $\bar{y}^a$ are the $(d-p)$ constants which specify the D-brane. The open string endpoints are free to move along the directions tangential to the D-brane, and thus satisfy Neumann boundary conditions

$$\frac{\partial X^m}{\partial Y^a}(\tau,\sigma)\Big|_{\sigma=0} = \frac{\partial X^m}{\partial Y^a}(\tau,\sigma)\Big|_{\sigma=\pi} = 0 \ , \qquad m = 0, \ldots, p \ .$$

D-branes have mass (they have tension), and so curve the spacetime in which they reside. The metric induced by a $D3$-brane in a flat ten dimensional space (as an example) is

$$\begin{aligned} ds^2 = {} & \frac{1}{\sqrt{1+\frac{R^4}{r^4}}}\left(-dt^2 + \left(dX^1\right)^2 + \left(dX^2\right)^2 + \left(dX^3\right)^2\right) \\ & + \sqrt{1+\frac{R^4}{r^4}}\left(\left(dY^1\right)^2 + \left(dY^2\right)^2 + \cdots + \left(dY^6\right)^2\right) \ . \end{aligned} \quad (2.5)$$

Near to the brane, $r \sim 0$, so we can drop the '1' in the square root of the coefficients and the metric becomes (after transforming the Y coordinates to spherical coordinates)

$$ds^2 = \frac{r^2}{R^2}\left(-dt^2 + \left(dX^1\right)^2 + \left(dX^2\right)^2 + \left(dX^3\right)^2\right) + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2 \ , \quad (2.6)$$

which is the metric of a five sphere times[4] AdS_5. Notice that the S^5 and AdS_5 share a radius of curvature R which is given by $R = \ell_s(4\pi g_s)^{\frac{1}{4}}$[1], where ℓ_s is the string length scale and g_s is the string coupling constant.

2.2 Origins of The Correspondence

Imagine N parallel $D3$-branes in Type IIB string theory separated by distances r. We take a low energy limit by taking $\ell_s^2 \to 0$, and also bring the branes together ($r \to 0$) so that the mass of the strings stretching between the branes is kept small. The open strings on the brane do not have enough energy to have oscillatory excitations and so behave like point particles. The resulting theory on the brane is $\mathcal{N} = 4$ $U(N)$ super Yang-Mills.

In the bulk (9+1)-dimensional spacetime we have a theory of closed strings, that is, a theory of quantum gravity. In the low energy limit the gravity theory decouples from the Yang-Mills theory on the brane as the wavelength of the gravitons in the bulk is far too large to allow for interactions with the branes.

We can also consider the system of $D3$-branes from the viewpoint of the curved spacetime they induce. The $D3$-brane solution to Einstein's equations (2.5) includes a horizon at the end of an infinite throat. The closed strings near the horizon will have low energy due to being so far down the throat and so their oscillatory excitation modes will have a negligible contribution to their energy. These excitation modes will thus never have sufficient energy to influence the system far from the horizon. The physics near the branes will decouple from the physics far from the branes, which is that of closed strings in asymptotically flat space.

What is the geometry down the throat? We have seen that near to a $D3$-brane M_{10} is curved to $AdS_5 \times S^5$. The same holds true for a system of N $D3$-branes that have been brought together as all that changes in (2.5) is that the curvature constant R now includes the number of D-branes: $R = \ell_s(4\pi g_s N)^{\frac{1}{4}}$.

We have investigated the configuration of N $D3$-branes from two viewpoints; the dynamics of open strings attached to the branes, and as a gravitational configuration. In the low energy limit each has led to two decoupled subsystems, with that of closed strings in flat spacetime being common to the two viewpoints. The other two subsystems, $\mathcal{N} = 4$ super Yang-Mills on $d = 3+1$ Minkowski spacetime and IIB superstring theory on $AdS_5 \times S^5$, also enjoy the same symmetry group $SO(2,4) \times SO(6)$ (section 1.2.1), which of

[4]The AdS component of (2.6) is related to the Poincaré AdS metric (2.4) by the coordinate transformation $r^2 = \frac{R^4}{z^2}$.

course is not dependent on energy. We are thus led to Maldacena's conjecture that these two theories are in fact *equivalent* for all energies!

One might expect the supergravity solution to break down in the large N limit as bringing such a large number of branes together would induce a huge gravitational field. This is not the case because the radius of curvature of the $AdS \times S$ space increases with N, and so general relativity remains intact.

2.3 State-Operator Map

We have already seen that the Poincaré patch of the AdS_5 boundary is 4 dimensional Minkowski space, and that this is the spacetime on which the CFT on the one side of the correspondence lives. In fact the boundary of AdS_5 in global coordinates is isomorphic to the M_4 of the CFT. To see this clearly we firstly perform a Wick rotation $t \to it$ on M_4 to obtain 4d Euclidean space, and then transform to spherical coordinates

$$\begin{aligned} ds^2 &= dt^2 + dx^i dx^i \\ &= dr^2 + r^2 d\Omega_3^2 \,, \end{aligned} \tag{2.7}$$

making the transformation $r = e^t$ we get

$$ds^2 = e^{2t}\left(dt^2 + d\Omega_3^2\right) \,.$$

e^{2t} is an overall scale factor of the metric (see Appendix A) and so can be eliminated by acting with the conformal group $SO(2,4)$, yielding

$$ds^2 = dt^2 + d\Omega_3^2 \,, \tag{2.8}$$

which, after another Wick rotation, is the boundary of AdS_5 in global coordinates [49]. We can thus identify the conformal field theory of the correspondence as living on the boundary of the ant-de Sitter space in which the dual gravity theory lives. The correspondence thus serves as a concrete example of the holographic hypothesis of [51, 52], which states that a quantum theory of gravity in a d dimensional space can be fully described by a theory on the $d-1$ dimensional boundary of that space.

The map discovered by [2, 3] follows from the isomorphism between the boundary of AdS space and Minkowski space, as we now explain. To describe interacting systems on the boundary of AdS_5 we need well separated initial and final states which live on the S^3 of (2.8) in the infinitely far past and future. From the transformation $r = e^t$ we see that $t \to -\infty$ on the boundary of AdS_5 corresponds to a point in the M_4 of the CFT (2.7), which is where operators, not wavefunctions, are defined. It is thus natural to identify states in the string theory as mapping to operators in the gauge theory and we find a matching of observables of the two theories

$$\langle\psi_i|\psi_f\rangle = \langle\hat{\mathcal{O}}_{\psi_i}\hat{\mathcal{O}}^\dagger_{\psi_f}\rangle\ .$$

Chapter 3

Giant Gravitons

Giant gravitons are spherical $D3$-branes orbiting on a disc in the spherical component of $AdS \times S$. Their invasion from anti-de Sitter space was first publicised by McGreevy, Susskind and Toumbas in [18]. Giant gravitons expand with increasing angular momentum and since the giants of [18] expand into the spherical component of $AdS \times S$, their size is limited as the spherical manifold is compact. This puts an upper bound on a giant's angular momentum and was advertised as an explanation of the origin of the stringy exclusion principle[19] which puts a limit on the number of single particle BPS states in supergravity. The discovery by [20] and [21] of dual giant gravitons expanded in the AdS component of $AdS \times S$ posed a problem to giant gravitons taking responsibility for stringy exclusion as AdS space is not compact.

3.1 A Dipole Analogy

The simple dipole system we describe here has strong similarities to a graviton coupled to the n-form field strength and so helps to conceptualize the giant graviton system. Imagine a dipole constrained to move on the surface of a sphere of radius R. The sphere has magnetic flux N due to a magnetic monopole at the centre of the sphere. Quantization of flux requires

$$2\pi N = \Omega_2 B R^2 \ .$$

When the dipole has a tiny momentum it moves along the equator of the sphere. Increasing its momentum will cause the charges to split and so they will move along parallel circles equally separated from the equator. Once the momentum of the dipole is about $2BR$ the charges will be at the poles, so its size is limited to the size of the sphere. At this point the angular momentum takes its maximum value

$$L = PR \sim BR^2 \ ,$$

which is of order of the total magnetic flux N.

For the Lagrangian of the dipole we parametrize the sphere by two angles ϕ, θ, where the azimuthal angle ϕ goes from 0 to 2π and θ is the angle from the equator, taking values $\pm\frac{\pi}{2}$ at the poles. The positions of the positive and negative charges are given by (ϕ, θ) and $(\phi, -\theta)$. If we consider a slow-moving dipole whose mass is so small that its kinetic term in the Lagrangian can be ignored compared to the other terms, then the Lagrangian of the dipole takes the form

$$\begin{aligned} \mathcal{L} &= -\frac{k}{2} R^2 \sin^2\theta - N \sin\theta \dot{\phi} \\ &= \mathcal{L}_S + \mathcal{L}_B \end{aligned}$$

where $\mathcal{L}_S$ is the Coulombic spring coupling term and $\mathcal{L}_B$ is the term coupling the dipole to the magnetic field. From $L = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$ we find that the angular momentum is given by

$$L = -N \sin\theta$$

which will reach its maximum value when $\theta = \frac{\pi}{2}$ at which point

$$|L_{\max}| = N \ ,$$

so the maximum angular momentum of the dipole is exactly N.

3.2 S^5 Giants of $AdS_5 \times S^5$

We will firstly study gravitons that expand into the S^5 component of $AdS_5 \times S^5$. The five-sphere considered has a radius of curvature R much larger than the 10 dimensional Planck length ℓ_p. The 5-form field strength is analogous to the magnetic field featuring in the dipole system, and so we denote the flux density B. Quantization of flux requires

$$\Omega_5 B R^5 = 2\pi N \ .$$

The radius of curvature R follows from the supergravity equations of motion[1]

$$R = \ell_p \left(\pi N\right)^{\frac{1}{3}} \ .$$

The full bosonic Lagrangian of the giant graviton consists of two terms: the kinetic energy term $\mathcal{L}_K$ (or Dirac-Born-Infeld term), and the term coupling the giant to the 5-form field strength $\mathcal{L}_B$ (or Chern-Simons term)

$$\mathcal{L} = \mathcal{L}_K + \mathcal{L}_B \ .$$

3.2.1 The Kinetic Energy Term $\mathcal{L}_K$

We parametrize S^5 by five angles $\theta^1, \cdots, \theta^5$ which are related to the cartesian coordinates X^i of the six dimensional flat space in which the S^5 is embedded by

$$\begin{aligned}
X^1 &= R\cos\theta^1 \\
X^2 &= R\sin\theta^1\cos\theta^2 \\
X^3 &= R\sin\theta^1\sin\theta^2\cos\theta^3 \\
X^4 &= R\sin\theta^1\sin\theta^2\sin\theta^3\cos\theta^4 \\
X^5 &= R\sin\theta^1\sin\theta^2\sin\theta^3\sin\theta^4\cos\theta^5 \\
X^6 &= R\sin\theta^1\sin\theta^2\sin\theta^3\sin\theta^4\sin\theta^5 \ .
\end{aligned}$$

The angles $\theta^1, \cdots, \theta^4$ go from 0 to π, while the azimuthal angle θ^5 goes from 0 to 2π. These coordinate transformations satisfy the equation of a five sphere

$$\left(X^1\right)^2 + \left(X^2\right)^2 + \left(X^3\right)^2 + \left(X^4\right)^2 + \left(X^5\right)^2 + \left(X^6\right)^2 = R^2 \ .$$

We wrap the $D3$-brane around an S^3 embedded in the S^5 and so parametrize its surface by the angles $\theta^3, \theta^4, \theta^5$. The brane is allowed to move in the X^1, X^2 plane and its size is determined by its position in the plane according to

$$r = R\sin\theta^1\sin\theta^2 \ .$$

We see that when the brane is at its maximum size ($r = R$) it is at the origin $X^1 = X^2 = 0$, analogous to the dipole constrained to a sphere having maximum size when the charges are at the poles. The brane moves around a circle in the X^1, X^2 plane according to

$$\left(X^1\right)^2 + \left(X^2\right)^2 = R^2 - r^2 \ ,$$

so the graviton is a point particle when circling at the edge of the X^1, X^2 disc of radius R, and blows up as it moves in towards the centre of the disc. The radius $\sqrt{R^2 - r^2}$ of the giant's orbit decreases as its angular momentum increases. Introducing the angle ϕ measuring the position of the giant on the plane, we have

$$\begin{aligned}
X^1 &= \sqrt{R^2 - r^2}\cos\phi \\
X^2 &= \sqrt{R^2 - r^2}\sin\phi \ .
\end{aligned}$$

In terms of the coordinates $r, \phi, \theta^3, \theta^4, \theta^5$ the metric on the 5-sphere becomes

$$ds^2_{sph} = \frac{R^2}{\left(R^2 - r^2\right)} dr^2 + \left(R^2 - r^2\right) d\phi^2 + r^2 d\Omega_3^2 \ , \tag{3.1}$$

where $d\Omega_3^2$ is the metric of a unit 3-sphere parametrized by $\theta^3, \theta^4, \theta^5$.

The kinetic energy term of the full $D3$-brane action is determined from

$$S_K = -T_{D3} \int d^4y \sqrt{-\text{det}G} \ , \tag{3.2}$$

where T_{D3} is the membrane tension, given by

$$T_{D3} = \frac{1}{(2\pi)^3 \ell_s^4 g_s} \ .$$

The y coordinates are the coordinates parameterizing the $D3$-brane world-volume

$$\begin{aligned} y^0 &= t \\ y^1 &= \theta^3 \\ y^2 &= \theta^4 \\ y^3 &= \theta^5 \ . \end{aligned}$$

G is the induced metric on the $D3$-brane, given by

$$G_{\alpha\beta} = \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta} g_{\mu\nu} \ ,$$

where x^μ are the coordinates of $AdS_5 \times S^5$: $x^1, \cdots, x^5$ are the S^5 coordinates

$$\begin{aligned} x^1 &= r = R \sin\theta^1 \sin\theta^2 \\ x^2 &= \phi = \phi(t) \\ x^3 &= \theta^3 \\ x^4 &= \theta^4 \\ x^5 &= \theta^5 \ ; \end{aligned}$$

and $x^0 = t$, $x^6, \cdots, x^9$ are the coordinates of AdS_5. In terms of these coordinates the induced metric takes the form

$$G_{\alpha\beta} = \begin{bmatrix} -1 + (R^2 - r^2)\,\dot{\phi}^2 & & & \mathbf{0} \\ & r^2 & & \\ & & r^2 \sin^2\theta^3 & \\ \mathbf{0} & & & r^2 \sin^2\theta^3 \sin^2\theta^4 \end{bmatrix} .$$

With this metric the kinetic energy term of the giant graviton action becomes (after integrating over the angular coordinates)

$$\begin{aligned} S_K &= -T_{D3} 2\pi^2 r^3 \int \sqrt{1 - (R^2 - r^2)\dot{\phi}^2}\, dt \\ &= -M_{D3} \int \sqrt{1 - v^2}\, dt \ , \end{aligned}$$

where M_{D3} is the mass of the giant and v is its velocity. So the kinetic energy term of the Lagrangian is

$$\mathcal{L}_K = -T_{D3} 2\pi^2 r^3 \sqrt{1 - (R^2 - r^2)\dot{\phi}^2} \ .$$

3.2.2 The Coupling Term $\mathcal{L}_B$

The contribution of the five-form field strength to the action of the brane per orbit around the S^5 is

$$S_B = \int_{wv} P\left[A^{(4)}\right] = \int_{\Sigma} dA^{(4)} = \int_{\Sigma} F^{(5)} \ . \tag{3.3}$$

The first integral is over the worldvolume of the brane. $P\left[A^{(4)}\right]$ is the pull-back of the 4-form gauge potential onto the brane's worldvolume, given by $P\left[A^{(4)}\right]_{\alpha_1\alpha_2\alpha_3\alpha_4} = \frac{\partial x^{\mu_1}}{\partial y^{\alpha_1}}\frac{\partial x^{\mu_2}}{\partial y^{\alpha_2}}\frac{\partial x^{\mu_3}}{\partial y^{\alpha_3}}\frac{\partial x^{\mu_4}}{\partial y^{\alpha_4}} A_{\mu_1\mu_2\mu_3\mu_4}$, where the y^{α}'s are brane worldvolume coordinates and the x^{μ}'s are coordinates of $AdS_5 \times S^5$. $F^{(5)}$ is the five-form field strength. Σ is the five-manifold in S^5 whose boundary is the 4-dimensional surface swept out by the $D3$-brane in one orbit. $\Sigma = S^3 \times D_2$ where D_2 is a disc on the X^1, X^2 plane whose boundary is the orbit of the brane. The background flux is $F^{(5)} = B d\mathrm{vol}$ where B is the constant flux density and $d\mathrm{vol}$ is the volume form on S^5, that is

$$\begin{aligned} F^{(5)} &= B\sqrt{\det g}\, dr \wedge d\phi \wedge d\Omega_3 \\ &= BRr^3 dr \wedge d\phi \wedge d\Omega_3 \ . \end{aligned}$$

Thus the coupling action is

$$\begin{aligned} S_B &= B\mathrm{vol}\,(\Sigma) \\ &= B \times R \int_{D_2} d\Omega_3 \int_0^{2\pi} d\phi \int_0^{r} r'^3 dr' \\ &= BR\Omega_3 2\pi \frac{r^4}{4} \\ &= BR\Omega_5 r^4 \ , \end{aligned}$$

where the last equality made use of the relation $\Omega_n = \frac{2\pi}{n-1}\Omega_{n-2}$. With the flux quantization condition $BR^5\Omega_5 = 2\pi N$, the coupling action becomes

$$S_B = 2\pi N \frac{r^4}{R^4} \ .$$

The coupling term of the $D3$-brane Lagrangian is thus

$$\mathcal{L}_B = \frac{S_B}{T} = S_B \frac{\dot{\phi}}{2\pi} = \dot{\phi} N \frac{r^4}{R^4} \ .$$

Oppositely Charged Antipodes

Every point on the $D3$-brane's world volume is oppositely charged (and equal in magnitude) to its antipodal point, so it really is a dipole! This is what causes the brane to expand into a three-sphere when moving through the background field. To see how these oppositely charged antipodes manifest, consider the coupling action (3.3)

$$S_B = \int_{wv} P\left[A^{(4)}\right] \sqrt{-\det G}\, dy^{\alpha_1} \wedge dy^{\alpha_2} \wedge dy^{\alpha_3} \wedge dy^{\alpha_4} \, .$$

To get to the antipode of a point on the brane's worldvolume, we would reverse the direction of dy^{α_1}, replacing dy^{α_1} with $-dy^{\alpha_1}$. The action will have its sign reversed and so the antipodal point will be oppositely charged, and be of the same magnitude.

3.2.3 Angular Momentum, Energy and Stability Analysis

The giant's full Lagrangian is

$$\mathcal{L} = \mathcal{L}_K + \mathcal{L}_B = -T_{D3} 2\pi^2 r^3 \sqrt{1-(R^2-r^2)\dot{\phi}^2} + \dot{\phi} N \frac{r^4}{R^4} \, .$$

From this Lagrangian we obtain an expression for the giant's angular momentum via $L = \frac{d\mathcal{L}}{d\dot{\phi}}$

$$L = M_{D3} \frac{(R^2-r^2)\,\dot{\phi}}{\sqrt{1-(R^2-r^2)\dot{\phi}^2}} + N\frac{r^4}{R^4} \, ,$$

where $M_{D3} = T_{D3} 2\pi^2 r^3$. The giant attains its maximum angular momentum when it has reached the size of the sphere, $r = R$. Thus the compact S^5 space imposes a cut off on the giant's angular momentum

$$L_{max} = N \, .$$

This is precisely the cut off predicted by the dual conformal field theory[19] which provides strong evidence that giant gravitons are responsible for the stringy exclusion principle.

The energy of the giant is obtained from $E = \dot{\phi} L - \mathcal{L}$, yielding

$$\begin{aligned} E &= \sqrt{M_{D3}^2 + \frac{\left(L - N\frac{r^4}{R^4}\right)^2}{R^2-r^2}} \\ &= \sqrt{T_{D3} 2\pi^2 r^3 + \frac{\left(L - N\frac{r^4}{R^4}\right)^2}{R^2-r^2}} \, . \end{aligned} \tag{3.4}$$

There are two forces competing to change the $D3$-brane's size: the five-form field strength expands the brane, while the membrane's tension contracts the brane. If the brane's energy as a function of its size r has a stable minimum for fixed L, then the competing tension and coupling forces will be in equilibrium at this r, and so the brane will take this size (at this value of L). The existence of a stable minimum will indicate that the giant is a classically stable state.

Differentiating the energy with respect to r and equating to zero, we get 8 zeros, but the only positive real ones that also restrict r to be less than R are

$$r = 0 \quad \text{and} \quad r = \sqrt{\frac{L}{N}} R \, .$$

The first minimum corresponds to a point-like graviton, while the second is the radius of a stable expanded $D3$-brane which again yields N as the maximum value for L. The point-like graviton solution will be singular from the perspective of the gravitational field equations since for angular momenta of order N, it represents a huge energy concentrated at a point. Thus it is subject to uncontrolled quantum corrections and backreactions on the spacetime will no longer be negligible. There are quantum corrections proportional to powers of the momentum times the flux density, which are large at angular momenta of order N. The expanded brane on the other hand effectively 'smooths out' these quantum corrections due to its energy being spread over its macroscopic size ($r = \sqrt{\frac{L}{N}} R$).

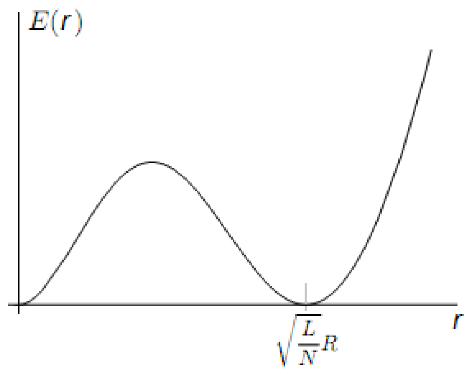


Figure 3.1: Energy of a giant graviton as a function of its radius for a specific angular momentum L.

3.3 AdS Giants

We now consider $D3$-branes that orbit[1] on the S^5 but expand into the AdS component of $AdS_5 \times S^5$. The full metric of $AdS_5 \times S^5$ in global coordinates can be written as

$$\begin{aligned} ds^2 &= \left(ds^2_{AdS}\right) + \left(ds^2_{sph}\right) \\ &= \left(-\left(1+\frac{\rho^2}{R^2}\right)dt^2 + \frac{d\rho^2}{1+\frac{\rho^2}{R^2}} + \rho^2 d\Omega_3^2\right) + \left(\frac{R^2}{(R^2-r^2)}dr^2 + \left(R^2-r^2\right)d\phi^2 + r^2 d\Omega_3^2\right) \ , \end{aligned}$$

where R is the curvature parameter of AdS_5 and S^5. We wrap the $D3$-brane around the S^3 featuring in the AdS metric, and so it is parametrized by the three angles of the Ω_3

$$d\Omega_3^2 = \left(d\psi^1\right)^2 + \sin^2\psi^1\left(d\psi^2\right)^2 + \sin^2\psi^1\sin^2\psi^2\left(d\psi^3\right)^2 \ . \tag{3.5}$$

The radius of the brane is given by ρ.

The action of the giant is again the sum of the kinetic energy term and the term coupling the giant to the five form field strength (of AdS_5)

$$S = S_K + S_B = -T_{D3}\int d^4\sigma\sqrt{-\det\tilde{G}} + T_{D3}\int_{wv} P\left[\tilde{A}^{(4)}\right] \ . \tag{3.6}$$

The AdS_5 four form potential $\tilde{A}^{(4)}$ has opposite sign to that of the S^5 (and takes a different form), and so effectively reverses the sign of the coupling term S_B in the brane's action. Thus this action describes a $D3$-brane of opposite charge to the S^5 giant, and so the AdS_5 giant is an *anti-brane*. While the brane on S^5 couples magnetically to the background field and should be thought of as a dimagnetic brane, the brane in AdS_5 couples electrically and should be thought of as a dielectric brane.

Calculating the pullbacks of the metric and the four form potential, substituting the trial solution into (3.6) and integrating over the angular coordinates yields the Lagrangian

$$\mathcal{L} = 2\pi^2 T_{D3}\left(-\rho^3\sqrt{1+\frac{\rho^2}{R^2} - R^2\dot{\phi}^2} + \frac{\rho^4}{R}\right) \ .$$

From this Lagrangian we obtain an expression for the giant's angular momentum via $L = \dfrac{d\mathcal{L}}{d\dot{\phi}}$

$$L = N\frac{\rho^3}{R^2}\frac{\dot{\phi}}{\sqrt{1+\frac{\rho^2}{R^2} - R^2\dot{\phi}^2}} \ ,$$

[1] The radius of the AdS giant's orbit on the X_1, X_2 plane of S^5 is always R; it does not move in towards the centre of the disc as its angular momentum increases.

where we have used the relationship between the brane's tension and the total flux N

$$T_{D3} = \frac{N}{2\pi^2 R^4} \, .$$

There is no longer a limit on the giant's angular momentum as its radius ρ is no longer restricted by R since AdS space is not compact. This poses a problem for giant gravitons explaining the stringy exclusion principle.

The energy of the giant is computed from $E = \dot{\phi} L - \mathcal{L}$

$$E = \frac{N}{L}\left(\sqrt{\left(1+\frac{\rho^2}{R^2}\right)\left(\frac{L^2}{N^2}+\frac{\rho^6}{R^6}\right)} - \frac{\rho^4}{R^4}\right) \, .$$

Investigating $\dfrac{\partial E}{\partial \rho} = 0$, the energy minima are found at

$$\rho = 0 \quad \text{and} \quad \rho = \sqrt{\frac{L}{N}} R \, ,$$

the point-like solution is again unphysical due to concentrating so much energy at a point. The brane expands to its equilibrium size $\rho = \sqrt{\frac{L}{N}} R$ where the competing forces of its tension (contraction) and background field coupling (expansion) are balanced.

3.4 The Giant Graviton Invasion is Not a Fiction

The giant graviton solution makes the drastic simplifying assumption that the brane is always a perfect three-sphere and orbits on the X^1, X^2 plane. In general the $D3$-brane can deform in complicated ways and so its radius will vary depending on the position on the brane. The general solution must thus have the giant's radius as a function of the angles parameterizing the brane and time $\rho = \rho(t, \theta^3, \theta^4, \theta^5)$, and its trajectory cannot be constrained to lie on the X^1, X^2 plane.

How do we know then that the giant graviton solution can be trusted? Firstly, both the S^5 giants and the AdS_5 giants have exactly the same quantum numbers (angular momentum and energy) as the point-like graviton. Secondly, since antipodal patches of the spherical $D3$-brane can be regarded as parallel portions of branes and anti-branes, preserving any supersymmetry is highly nontrivial. Thus if it can be shown that the giants are BPS preserving precisely the same supersymmetries as the point-like graviton, then the giant graviton solution should be trusted. [20] and [21] performed a detailed analysis of the residual supersymmetries of the giants and showed that they preserve 16 of the 32 supersymmetries, identical to the point-like graviton.

An interesting question is whether there is quantum mixing between the three graviton states (sphere, AdS and point-like). [20] found instanton solutions describing tunneling between the expanded branes and the point-like graviton, and [22] then performed numerical simulations which indicated that there is no instanton solution for tunneling directly between the sphere and AdS giants.

Although it's not clear how giant gravitons could still be responsible for the stringy exclusion principle since AdS space does not impose a limit on the angular momentum of AdS giants, there is a limit on the number of AdS giants. The maximum flux a giant can have is N as $AdS_5 \times S^5$ originated from stacking N $D3$-branes. A second AdS giant can thus have a maximum flux of $N-1$. Clearly the $(N+1)^{\text{th}}$ graviton will have a flux of zero making it a point-like graviton and so the number of giant gravitons AdS space can contain is restricted to N.

Part II

Dancing Giants

Chapter 4

Constructing Restricted Schur Polynomials

We wish to study a system of p giant gravitons interacting via strings stretching between the giants. In order to determine the energy spectrum and eigenstates we work in the dual conformal field theory, and our goal then becomes diagonalizing the dilatation operator acting on the CFT duals of excited giant gravitons. These are the restricted Schur polynomials labeled by Young diagrams of p $O(N)$ long rows (for AdS giants) or p long columns (for sphere giants). To achieve our goal a key new idea is needed: Schur-Weyl duality is used to construct the restricted Schur polynomials. In this chapter we will explain how Schur-Weyl duality arises and how it is exploited.

4.1 Why it is difficult to build a Restricted Schur Polynomial

There are six scalar fields ${\phi^i}_{ab}$ taking values in the adjoint of $u(N)$ in $\mathcal{N}=4$ super Yang-Mills theory. Assemble these scalars into the three complex combinations

$$Z=\phi_1+i\phi_2, \qquad Y=\phi_3+i\phi_4, \qquad X=\phi_5+i\phi_6\ .$$

We will study restricted Schur polynomials built using $n\sim O(N)$ Z and $m\sim O(N)$ Y fields and will often refer to the Y fields as 'impurities'. These operators have a large $\mathcal{R}$-charge and belong to the $SU(2)$ sector of the theory. The definition of the restricted Schur polynomial is

$$\chi_{R,(r,s)jk}(Z,Y)=\frac{1}{n!m!}\sum_{\sigma\in S_{n+m}}\chi_{R,(r,s)jk}(\sigma)Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_n}_{i_{\sigma(n)}}Y^{i_{n+1}}_{i_{\sigma(n+1)}}\cdots Y^{i_{n+m}}_{i_{\sigma(n+m)}}\ .$$

In this definition R is a Young diagram with $n+m$ boxes and hence labels an irreducible representation of S_{n+m}, r is a Young diagram with n boxes and labels an irreducible representation of S_n and s is a Young diagram with m boxes and labels an irreducible representation of S_m. The group S_{n+m} has an $S_n \times S_m$ subgroup, which will shuffle the n Z indices amongst each other and the m Y indices, but will not mix Z indices with Y indices. Taken together r and s label an irreducible representation of this subgroup. A single irreducible representation R will in general subduce many possible representations of the $S_n \times S_m$ subgroup; obtained by removing m boxes from R to get r, and assembling the m boxes into a Young diagram s. This assembly has some restrictions; s can have no more than p rows, and further, boxes coming from the same row in R cannot be stacked directly on top of each other (thus preserving their symmetry). For the $p > 2$ giant case, a particular irreducible representation (r, s) of the subgroup may be subduced more than once. For example, consider removing three 'disconnected' boxes from R to obtain r. The possible S_3 irreducible representations are obtained via

$$\begin{aligned} \ydiagram{1} \otimes \ydiagram{1} \otimes \ydiagram{1} &= \ydiagram{1} \otimes \left(\ydiagram{1,1} \oplus \ydiagram{2} \right) \\ &= \ydiagram{1,1,1} \oplus \ydiagram{2,1}^{(1)} \oplus \ydiagram{3} \oplus \ydiagram{2,1}^{(2)} . \end{aligned} \tag{4.1}$$

The representation $(r, \ydiagram{2,1})$ is subduced twice, with the representations being orthogonal. We thus need to introduce a multiplicity label to keep track of the different copies subduced. The indices j and k appearing in the restricted Schur polynomial above are these multiplicity labels. The object $\chi_{R,(r,s)jk}(\sigma)$ is called a restricted character[10], different to the character mentioned in section 1.2: $\chi_R(\sigma) = \mathrm{Tr}\,(\Gamma_R(\sigma))$, where $\Gamma_R(\sigma)$ is the representation matrix of σ in the vector space carrying R. To compute the restricted character $\chi_{R,(r,s),jk}(\sigma)$, trace the row index of $\Gamma_R(\sigma)$ only over the subspace associated to the j^{th} copy of (r, s) and the column index over the subspace associated to the k^{th} copy of (r, s). It is now clear why two multiplicity labels appear: when performing the 'trace' over the carrier space of (r, s) the row and column indices can come from different copies of (r, s) so that if $i \neq j$ we are not in fact summing diagonal elements of $\Gamma_R(\sigma)$. Operators constructed by summing these 'off diagonal' elements are needed to obtain a complete basis of local operators[7]. In terms of the symmetric group operator $P_{R\to(r,s)jk}$ which obeys

$$\Gamma_{(r,s)j}(\sigma) P_{R\to(r,s)jk} = P_{R\to(r,s)jk} \Gamma_{(r,s)k}(\sigma) \qquad \sigma \in S_n \times S_m$$

$$\Gamma_{(r,s)l}(\sigma)P_{R\to(r,s)jk} = 0 = P_{R\to(r,s)jk}\Gamma_{(r,s)q}(\sigma) \quad \sigma\in S_n\times S_m \quad l\neq j, \quad k\neq q\ ,$$

we can write the restricted character as

$$\chi_{R,(r,s)jk}(\sigma) = \mathrm{Tr}\left(P_{R\to(r,s)jk}\Gamma_R(\sigma)\right)\ .$$

When there are no multiplicities, $P_{R\to(r,s)jk} = P_{R\to(r,s)}$ is a projection operator which projects from the carrier space of R to the (r,s) subspace. When there are multiplicities, $P_{R\to(r,s)jk}$ is an intertwiner[43]; a map between the two isomorphic spaces $(r,s)j$ and $(r,s)k$. However, it is constructed in essentially the same way as a projector and satisfies very similar identities. For these reasons we will sometimes be guilty of an abuse of language and refer to $P_{R\to(r,s)jk}$ simply as a projector even when there are multiplicities.

Key Idea: *It is not easy to construct the operator $P_{R\to(r,s)jk}$ explicitly. This is the most serious obstacle in working with restricted Schur polynomials. An important result of our work is the use of a new version of Schur-Weyl duality to provide an efficient, transparent construction of this operator.*

Our construction is not quite completely general, but it does capture many interesting situations and proves a useful tool to explore semi-classical physics dual to the restricted Schur polynomials.

The restricted Schur polynomials are a very convenient basis for gauge invariant operators in the theory built using only the adjoint scalars. This follows because

- The restricted Schur polynomials are complete in the sense that any multitrace operator or linear combination of multitrace operators can be written as a linear combination of restricted Schur polynomials[7].

- The free theory two point function of the restricted Schur polynomial has been computed exactly[13]

$$\langle\chi_{R,(r,s)jk}(Z,Y)\chi_{T,(t,u)lm}(Z,Y)^\dagger\rangle = \delta_{R,(r,s)\,T,(t,u)}\delta_{kl}\delta_{jm}f_R\frac{\mathrm{hooks}_R}{\mathrm{hooks}_r\,\mathrm{hooks}_s}\ . \tag{4.2}$$

  In this expression hooks_R is the product of the hook lengths of Young diagram R and f_R is the product of the factors in Young diagram R[1]. Just as for the two point function of the Schur polynomials (1.1), the fact that the restricted Schur correlator is known exactly as a function

[1] See section B.7 for a definition of factors and hook lengths of a Young diagram.

of N implies that all Feynman diagrams (not just the planar diagrams) have been summed and this is what allows one to go beyond the planar limit.

- Restricted Schur polynomials have highly constrained mixing at the quantum level[11, 12].

In order to construct the operator $P_{R\to(r,s)jk}$ we will need to build a basis from the carrier space of an S_{n+m} irreducible representation R for the carrier space of an $S_n \times S_m$ irreducible representation $(r,s)j$. This is accomplished in two steps: first we project from S_{n+m} to $S_n \times (S_1)^m$ (this is easy) and second, we assemble the $S_n \times (S_1)^m$ representations into $S_n \times S_m$ representations (this is the trying step). It is this second step that is accomplished using Schur-Weyl duality. As a consequence we learn that the multiplicity index can be organized using $U(p)$ representations, with p the number of rows or columns in R. The background material from representation theory needed to understand our construction is collected in Appendices B and C.

4.2 From S_{n+m} to $S_n \times (S_1)^m$

Start from the carrier space for an irreducible representation R of S_{n+m}. In the Young-Yamonouchi basis, each vector of this carrier space is labeled by a Young tableau with shape R in which the boxes are numbered in one of the possible ways that if the boxes are dropped in that order a valid Young diagram would remain at each step. For example, two of the basis vectors in the carrier space of $\begin{array}{|c|c|c|}\hline \; & \; & \; \\ \hline \; & \; \\ \cline{1-2}\end{array}$ would be

$$\left|\begin{array}{|c|c|c|}\hline 5 & 4 & 3 \\ \hline 2 & 1 \\ \cline{1-2}\end{array}\right\rangle , \qquad \left|\begin{array}{|c|c|c|}\hline 5 & 3 & 1 \\ \hline 4 & 2 \\ \cline{1-2}\end{array}\right\rangle ,$$

while there would be no vectors labeled by

$$\left|\begin{array}{|c|c|c|}\hline 5 & 2 & 1 \\ \hline 4 & 3 \\ \cline{1-2}\end{array}\right\rangle , \qquad \left|\begin{array}{|c|c|c|}\hline 4 & 3 & 2 \\ \hline 5 & 1 \\ \cline{1-2}\end{array}\right\rangle .$$

When restricting to the $S_n \times (S_1)^m$ subgroup, we need not include a label for S_1 as it only has a single irreducible representation. Consequently, to specify an irreducible representation of the $S_n \times (S_1)^m$ subgroup, we only need to specify an irreducible representation of S_n, that is, a Young diagram r with n boxes. The only representations r that are subduced by R are those with Young diagrams that can be obtained by removing m boxes from R. Pulling off the same set of m boxes in different orders leads to different subspaces which all carry the same irreducible representation r. To resolve this multiplicity, we only need to specify the order in which the boxes are removed. By numerically labelling just the m boxes in the manner above, and leaving the other n boxes blank, we obtain a partially labeled Young diagram with shape R. This partially labeled Young diagram represents a *collection* of states in which each state has all $n+m$ boxes labeled as the ones above. In this way, the partially labeled Young diagram represents a subspace carrying an irreducible representation of the $S_n \times (S_1)^m$ subgroup. See Appendix C.3 for a more detailed discussion.

To build an operator which projects from the carrier space of the S_{n+m} irreducible representation R to the carrier space of an $S_n \times S_m$ irreducible representation $(r, s)j$, we now need to assemble the partially labeled Young diagrams (which already carry a representation r of S_n) in such a way that the resulting linear combinations carry an irreducible representation of $S_n \times S_m$. We turn to this task in the next section.

4.3 Basic Idea for Young Diagrams with p Rows

We will consider Young diagrams built using $n + m \sim O(N)$ boxes and with p rows. Thus, for the generic diagram, each row has $O(N)$ boxes. We set $m = \alpha N$ with $\alpha \ll 1$. After labeling the m boxes, two labeled boxes in different rows with labels i and j will have associated factors c_i and c_j respectively, with $c_i - c_j \sim O(N)$. We will refer to this property as the 'displaced corners approximation' (see Figure 4.1).

Consider the S_m subgroup of S_{n+m} which acts on the labeled boxes. As discussed in Appendix C.4, the fact that $c_i - c_j \sim O(N)$ for boxes in different rows implies a significant simplification in the action of the S_m subgroup on these partially labeled Yound diagrams. When adjacent permutations $(i, i+1)$ act on labeled boxes that belong to the same row, the Young diagram is unchanged and when acting on labeled boxes that belong to the different rows, the labeled boxes are swapped.

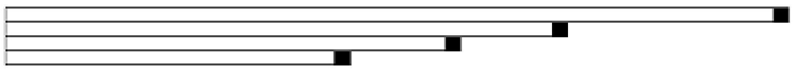


Figure 4.1: An example of a Young diagram with $p = 4$ rows. The rows are shown; the columns are not shown. There are $O(N)$ boxes in each row. The m numbered boxes have been colored black. The difference in factors associated to any two numbered boxes that are in different rows is $O(N)$. This is easily seen by recalling that the difference in the factors counts the number of boxes one needs to step through to move between the two boxes. The difference in the number of boxes in any two rows is generically $O(N)$ so that to move from one of the black tips to another one, generically, one needs to step through $O(N)$ boxes.

If we have a Young diagram with p rows and we label m boxes in all possible ways consistent with the rule of the previous section, we find a total of p^m possible partially labeled Young diagrams. We associate a particular p-dimensional vector to each box that is labeled. This gives a total of m vectors $\vec{v}(i)$ with $i = 1, 2, \cdots, m$. We will denote the components of these vectors as $\vec{v}(i)_n$ where $n = 1, ..., p$. If box i is pulled from the j^{th} row we have

$$\vec{v}(i)_n = \delta_{nj} \, .$$

For each index i (equivalently, for each labeled box) we have a vector space V_p. Taking the tensor product of these spaces we obtain a set of p^m dimensional vectors, of the form

$$\vec{v}(1) \otimes \vec{v}(2) \otimes \vec{v}(3) \otimes \cdots \otimes \vec{v}(m-1) \otimes \vec{v}(m) \, .$$

Call the vector space spanned by these vectors $V_p^{\otimes m}$. When we talk about vectors of the above form we will say that "vector $\vec{v}(i)$ occupies the i^{th} slot". The matrix action of S_m on the partially labeled Young diagrams described above implies the following action on $V_p^{\otimes m}$

$$\sigma \cdot (\vec{v}(1) \otimes \vec{v}(2) \otimes \cdots \otimes \vec{v}(m)) = \vec{v}\,(\sigma(1)) \otimes \vec{v}\,(\sigma(2)) \otimes \cdots \otimes \vec{v}\,(\sigma(m)) \, .$$

Thus, $\sigma \in S_m$ will move the vector in the i^{th} slot to the $\sigma(i)^{\text{th}}$ slot, but does not change its entries. We can also define an action of $U(p)$ on $V_p^{\otimes m}$

$$U \cdot (\vec{v}(1) \otimes \vec{v}(2) \otimes \cdots \otimes \vec{v}(m)) = D(U)\vec{v}\,(1) \otimes D(U)\vec{v}\,(2) \otimes \cdots \otimes D(U)\vec{v}\,(m) \, ,$$

where $D(U)$ is the $p \times p$ unitary matrix representing group element $U \in U(p)$ in the fundamental representation. Thus, $U \in U(p)$ will change the entries of the vector in the i^{th} slot but it will not move it to a different slot. It acts in exactly the same way on each slot. It is quite clear that these are commuting actions of $U(p)$ and S_m on $V_p^{\otimes m}$

$$\begin{aligned} U\cdot(\sigma\cdot(\vec{v}(1)\otimes\cdots\otimes\vec{v}(m))) &= U\cdot(\vec{v}\,(\sigma(1))\otimes\cdots\otimes\vec{v}\,(\sigma(m))) \\ &= D(U)\vec{v}\,(\sigma(1))\otimes\cdots\otimes D(U)\vec{v}\,(\sigma(m)) \\ &= \sigma\cdot(D(U)\vec{v}\,(1)\otimes\cdots\otimes D(U)\vec{v}\,(m)) \\ &= \sigma\cdot(U\cdot(\vec{v}(1)\otimes\cdots\otimes\vec{v}(m))) \end{aligned}$$

and consequently by Schur-Weyl duality the space can be organized as[2][44]

$$V_p^{\otimes m}=\oplus_s V_s^{U(p)}\otimes V_s^{S_m}\ , \tag{4.3}$$

where the sum runs over all Young diagrams s built from m boxes and each has at most p rows. One consequence of this formula is that

$$p^m=\sum_s \mathrm{Dim}(s)\,d_s\ ,$$

where $\mathrm{Dim}(s)$ is the dimension of s as an irreducible representation of $U(p)$ and d_s is the dimension of s as an irreducible representation of S_m. Thus, by identifying states with good $U(p)$ labels we have identified states with good S_m labels. Therefore an important consequence of (4.3) is that it provides an efficient method to construct the projectors which are used to define the restricted Schur polynomials[3].

Key Idea: *Using Schur-Weyl duality it follows that the symmetric group operators $P_{R\to(r,s)jk}$ carry good $U(p)$ labels (where p is the number of rows in R) and, consequently, can be constructed using nothing more than $U(p)$ group theory.*

A necessary step towards building the projectors entails constructing a dictionary between the original labels $R,(r,s)jk$ of the restricted Schur polynomial $\chi_{R,(r,s)jk}(Z,Y)$ and the new $U(p)$ labels. Exactly the same Young diagram s that originally specifies an S_m irreducible representation, specifies a $U(p)$ irreducible representation. The Young diagram r is included among

[2]Part of what is behind Shur-Weyl duality is simple and familiar: any two operators that commute can be simultaneously diagonalized.

[3]The reader may be familiar with the usual use of Schur-Weyl duality, to construct projectors onto good $U(p)$ irreducible representations using the Young symmetrizers i.e. by symmetrizing and antisymmetrizing indices on a tensor. We are turning this argument on its head by using the irreducible representations of the unitary group to build symmetric group projectors. Bear in mind that the details of our Schur-Weyl duality are different to the usual construction.

the new labels and it still specifies an irreducible representation of S_n. The final label is the choice of a state from the carrier space of $U(p)$ representation s, labeled by its Gelfand-Tsetlin pattern. The Δ weight of this state[4] tells us how boxes were removed from R to obtain r. This point deserves some explanation. The state chosen from the carrier space s can be put into one-to-one correspondence with a semi-standard Young tableau and this correspondence plays a central role. Consider for example the $U(3)$ state with Gelfand-Tsetlin pattern

$$\begin{bmatrix} 4 & & 3 & & 1 \\ & 3 & & 2 & \\ & & 2 & & \end{bmatrix} .$$

The uppermost row of the pattern gives the shape of the Young diagram. Each row (starting from the bottom row) tells us how to distribute 1s, then 2s and so on till the semi standard Young tableau is obtained. This connection is reviewed in detail in Appendix B.4. For the Gelfand-Tsetlin pattern shown above the semi-standard Young tableau is

$$\begin{bmatrix} \ast & & \ast & & \ast \\ & \ast & & \ast & \\ & & 2 & & \end{bmatrix} \leftrightarrow \begin{ytableau} 1 & 1 & \ast & \ast \\ \ast & \ast & \ast \\ \ast \end{ytableau} \qquad \begin{bmatrix} \ast & & \ast & & \ast \\ & 3 & & 2 & \\ & & 2 & & \end{bmatrix} \leftrightarrow \begin{ytableau} 1 & 1 & 2 & \ast \\ 2 & 2 & \ast \\ \ast \end{ytableau} \qquad \begin{bmatrix} 4 & & 3 & & 1 \\ & 3 & & 2 & \\ & & 2 & & \end{bmatrix} \leftrightarrow \begin{ytableau} 1 & 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 \end{ytableau} .$$

Each row in the pattern corresponds to a particular number in the semi standard tableau. From the definition of the Gelfand-Tsetlin pattern, we also know that each row in the pattern corresponds to a particular subgroup in the chain of subgroups $U(1) \subset U(2) \subset \cdots \subset U(p-1) \subset U(p)$. So, from the point of view of the semi-standard Young tableau or of the Gelfand-Tsetlin pattern, going to the $U(p-1)$ subgroup implies that we consider a subgroup that does not act on one of the numbers appearing in the semi-standard tableau. What does it mean to consider a $U(p-1)$ subgroup of our action of $U(p)$ on the boxes that have been removed from R? Recall from Appendix B.6 that the particular state that is assigned to each removed box depends on the row it was removed from:

$$\begin{ytableau} {} & {} & {} & {} & {} & {} & {} & {} & {} & {} & {} & 1 \\ {} & {} & {} & {} & {} & {} \end{ytableau} \quad \leftrightarrow \quad \begin{bmatrix} 1 & & 0 \\ & 1 & \end{bmatrix}$$

$$\begin{ytableau} {} & {} & {} & {} & {} & {} & {} & {} & {} & {} & {} & {} \\ {} & {} & {} & {} & {} & 1 \end{ytableau} \quad \leftrightarrow \quad \begin{bmatrix} 1 & & 0 \\ & 0 & \end{bmatrix} \qquad \text{(for } p = 2 \text{ } AdS \text{ giants)}$$

[4]Briefly, the Δ weight of a Gelfand-Tsetlin pattern is simply a sequence of p integers, with each integer given by the sum of the entries in one row of the pattern minus the sum of entries of the row below it. So the Gelfand-Tsetlin pattern $M = \begin{bmatrix} 4 & & 2 & & 0 \\ & 2 & & 1 & \\ & & 1 & & \end{bmatrix}$ has $\Delta(M) = (3, 2, 1)$. See Appendix B.3 for a formal definition.

Thus going to a $U(p-1)$ subgroup corresponds to considering a subgroup that does not act on the boxes belonging to a particular row. Clearly then, the numbers in the semi-standard tableau can be identified with the row from which the corresponding box has been removed from R. Since the Δ weight $\Delta(M) = (\delta_n(M), \delta_{n-1}(M), \cdots \delta_1(M))$ gives the differences between the sum of the entries in a row of a Gelfand-Tsetlin M and the sum of the entries of the row below it, we can conclude that the number of boxes labeled i in the semi-standard Young tableau which is the number of boxes removed from row i of R to produce r, is given by $\delta_i(M)$. Thus, given r and the delta weight we can reconstruct R.

From Schur-Weyl duality we know we can construct the symmetric group operators $P_{R\to(r,s)jk}$ using $U(p)$ group theory. Why is this useful? The multiplicity problem, due to the fact that a particular irreducible representaion (r,s) of the $S_n \times S_m$ subgroup can be subduced more than once from the carrier space of R, is intrinsically handled by the Gelfand-Tsetlin basis[5]. The number of states that carry the same $U(p)$ representation s and have the same Δ weight is called the inner multiplicity of the state $I(\Delta M)$. One interpretation of the inner multiplicity is that it simply counts the number of ways of distributing the relevant fixed set of entries (two 1s, three 2s and three 3s for the example above) in accordance with the rules of a semi-standard Young tableau: the entries of the rows are weakly increasing, while entries down the columns are strictly increasing. Notice now that these rules match those for assembling the m boxes pulled off R into possible irreducible representations s of S_m subduced from R! Thus we can map the multiplicity of the different copies of s subduced from R to the inner multiplicity $I(\Delta M)$ of the corresponding $U(p)$ state. Finally note that each $U(p)$ representation s will also appear with a particular multiplicity. However, thanks to Schur-Weyl duality, we know that this multiplicity is organized by the S_m representation s.

Key Idea: *The Gelfand-Tsetlin patterns of $U(p)$ provide a non-degenerate set of multiplicity labels jk for the symmetric group operators $P_{R\to(r,s)jk}$.*

[5] An alternative approach to resolving these multiplicities has been outlined in [45]. The idea is to consider elements in the group algebra CS_{n+m} which are invariant under conjugation by $CS_n \times CS_m$. The Cartan subalgebra of these elements are the natural generalization of the Jucys-Murphy elements which define a Cartan subalgebra for S_n[46]. The multiplicities will be labeled by the eigenvalues of this Cartan subalgebra[45].

In summary then we trade the labels

R	an irreducible representation of S_{n+m}
r	an irreducible representation of S_n
s	an irreducible representation of S_m
j	multiplicity label resolving copies of (r,s)

for the new labels

r	an irreducible representation of S_n
s	an irreducible representation of $U(p)$
M^i	a state in the carrier space of s where
	i runs over the inner multiplicity.

At this point we have identified an orthonormal set of states spanning any particular carrier space $(r,s)j$ of the $S_n \times S_m$ subgroup. Writing down the corresponding projector is now straight forward.

4.4 From $S_n \times (S_1)^m$ to $S_n \times S_m$

We can now write the symmetric group operator used to define the restricted Schur polynomial as

$$P_{R\to(r,s)jk} = \sum_{\alpha=1}^{d_s} |s, M^j, \alpha\rangle\langle s, M^k, \alpha| \otimes \mathbf{I}_r \ ,$$

where, by Schur-Weyl duality, the multiplicity label α for the $U(p)$ states is organized by the irreducible representation s of the symmetric group S_m. The indices j and k pick out states M that have a particular Δ weight and hence range over $1, 2, ..., I(\Delta(M))$. The components δ_i of the particular Δ that must be used are equal to the number of boxes removed from row i of R to produce r. $\mathbf{I}_r$ is simply the identity matrix in the carrier space of the S_n irreducible representation labeled by r.

As an example of the translation from the labels $R, (r,s)j$ to the new labels, consider the labels

$$R = \yng(10,5)\ , \quad r = \yng(8,3)\ , \quad s = \yng(2,2)\ .$$

These become

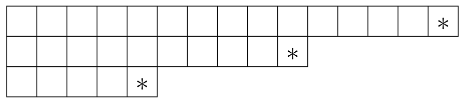
$$r = \yng(8,3)\ , \quad s = \yng(2,2)\ , \quad M^1 = \begin{bmatrix} 2 & & 2 \\ & 2 & \end{bmatrix} \ .$$

For this example $\Delta = (2,2)$ corresponding to 2 boxes being removed from the first row and two from the second row of R to produce r. The first row of M is read off s and the second row is chosen to obtain the correct Δ. The inner multiplicity for this case is 1, so that there is a single possible projection operator.

We now explicitly construct a projector for the simplest possible case where multiplicities appear.

4.4.1 A Three Row Projector using $U(3)$

Consider the following three row Young diagram



.

The starred boxes are to be removed. There are six possible ways to distribute the labels $1,2,3$ between these boxes, each giving an irreducible representation of $S_n \times (S_1)^m$ (here $m = 3$). One possible representation of $S_n \times S_m$ that can be suduced has r as given above but with the starred boxes removed and $s = \yng(2,1)$. To build the projector $P_{R\to(r,s)jk}$ we need to build the projector onto the $U(3)$ irreducible representation labeled by $s = \yng(2,1)$. Further, since one box is pulled off each row, the relevant $U(3)$ states have a Δ weight of $(1,1,1)$. This representation is 8 dimensional and the corresponding Gelfand-Tsetlin patterns are

$$\begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{bmatrix} \quad \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 1 & & \end{bmatrix} \quad \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 2 & & \end{bmatrix} \quad \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix}$$

$$\begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 0 & & \end{bmatrix} \quad \begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix} \quad \begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{bmatrix} \quad \begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{bmatrix}$$

The fourth and sixth states in the above list have the correct Δ weight, so that for weight $\Delta = (1,1,1)$ we have inner multiplicity $I(\Delta) = 2$. The fact that there are two states with the correct Δ weight implies that this particular (r,s) is subduced twice from the carrier space of R, which agrees with (4.1). This in turn implies that there are four possible projection operators and hence four possible restricted Schur polynomials that can be defined.

To build the projector we need to take linear combinations of the above subspaces of $S_n \times (S_1)^m$ in such a way that the resulting combination is an invariant subspace of $S_n \times S_m$ and further that this invariant subspace carries the correct irreducible representation of $S_n \times S_m$ i.e. $\left(r, \ydiagram{2,1}\right)$. To streamline our notation for the six subspaces we work with, we will set

$$|a,b,c\rangle = \begin{ytableau} \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & \ & a \\ \ & \ & \ & \ & \ & \ & \ & \ & \ & b \\ \ & \ & \ & \ & c \end{ytableau} .$$

The $U(3)$ action is defined on the labeled boxes. The box labeled 1 is always in the first slot of the tensor product of $V_p^{\otimes m}$; its position inside the ket tells you what row (and hence what $U(3)$ state) it is in. Notice that all reference to the carrier space of r_n is omitted. This is perfectly consistent because this subspace is common to all the subspaces we consider and it plays no role in the problem of finding good S_m invariant subspaces. Thus, for example,

$$|1,2,3\rangle = \begin{bmatrix} 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{bmatrix} \otimes \begin{bmatrix} 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{bmatrix} \otimes \begin{bmatrix} 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{bmatrix}$$

and

$$|2,1,3\rangle = \begin{bmatrix} 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{bmatrix} \otimes \begin{bmatrix} 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{bmatrix} \otimes \begin{bmatrix} 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{bmatrix} .$$

Using the Clebsch-Gordan coefficients given in Appendix B.5 we easily find that the subspaces considered above break up into subspaces labeled by states from $U(3)$ representations.

$$\begin{aligned} |1,2,3\rangle = {} & \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix} - \frac{1}{\sqrt{12}} \begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix}^{(1)} + \frac{1}{2} \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix}^{(2)} \\ & + \frac{1}{2} \begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix}^{(2)} + \frac{1}{\sqrt{12}} \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix}^{(1)} + \frac{1}{\sqrt{6}} \begin{bmatrix} 3 & & 0 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix} , \end{aligned}$$

$$\begin{aligned} |2,1,3\rangle = {} & -\frac{1}{\sqrt{6}} \begin{bmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix} + \frac{1}{\sqrt{12}} \begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix}^{(1)} + \frac{1}{2} \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix}^{(2)} \\ & - \frac{1}{2} \begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix}^{(2)} + \frac{1}{\sqrt{12}} \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix}^{(1)} + \frac{1}{\sqrt{6}} \begin{bmatrix} 3 & & 0 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix} , \end{aligned}$$

$$|3,1,2\rangle = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix} - \frac{1}{\sqrt{12}} \begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix}^{(1)} - \frac{1}{2} \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix}^{(2)}$$

$$-\frac{1}{2}\begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix}^{(2)} + \frac{1}{\sqrt{12}}\begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix}^{(1)} + \frac{1}{\sqrt{6}}\begin{bmatrix} 3 & & 0 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix},$$

$$|1,3,2\rangle = -\frac{1}{\sqrt{6}}\begin{bmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix} + \frac{1}{\sqrt{12}}\begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix}^{(1)} - \frac{1}{2}\begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix}^{(2)}$$
$$+\frac{1}{2}\begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix}^{(2)} + \frac{1}{\sqrt{12}}\begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix}^{(1)} + \frac{1}{\sqrt{6}}\begin{bmatrix} 3 & & 0 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix},$$

$$|2,3,1\rangle = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix} + \frac{1}{\sqrt{3}}\begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix}^{(1)} - \frac{1}{\sqrt{3}}\begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix}^{(1)} + \frac{1}{\sqrt{6}}\begin{bmatrix} 3 & & 0 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix},$$

$$|3,2,1\rangle = -\frac{1}{\sqrt{6}}\begin{bmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix} - \frac{1}{\sqrt{3}}\begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix}^{(1)} - \frac{1}{\sqrt{3}}\begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix}^{(1)} + \frac{1}{\sqrt{6}}\begin{bmatrix} 3 & & 0 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix}.$$

Given these results it is straight forward to write down the two possible sets of states that carry the S_m irreducible representation $\yng(2,1)$

$$\left|\yng(2,1),1\right\rangle^{(1)} = \begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix}^{(1)} = \frac{1}{\sqrt{12}}\Big(-|1,2,3\rangle + |2,1,3\rangle - |3,1,2\rangle$$
$$+|1,3,2\rangle + 2|2,3,1\rangle - 2|3,2,1\rangle\Big)$$

$$\left|\yng(2,1),2\right\rangle^{(1)} = \begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix}^{(2)} = \frac{1}{2}\Big(|1,2,3\rangle - |2,1,3\rangle - |3,1,2\rangle + |1,3,2\rangle\Big)$$

and

$$\left|\yng(2,1),1\right\rangle^{(2)} = \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix}^{(2)} = \frac{1}{2}\Big(|1,2,3\rangle + |2,1,3\rangle - |3,1,2\rangle - |1,3,2\rangle\Big)$$

$$
\left|\,\yng(2,1)\,,2\right\rangle^{(2)}=\begin{bmatrix}2\ \ 1\ \ 0\\ 2\ \ 0\\ 1\end{bmatrix}^{(1)}=\frac{1}{\sqrt{12}}\Big(|1,2,3\rangle+|2,1,3\rangle+|3,1,2\rangle
$$

$$
+|1,3,2\rangle-2|2,3,1\rangle-2|3,2,1\rangle\Big)
$$

The superscripts on the kets on the left hand sides of these equations are multiplicity labels and the integer inside each ket indexes states in the carrier space. The four possible projectors that can be defined are now given by

$$
P_{R\to(r,\yng(2,1)),jk}=\sum_{\alpha=1}^{2}\left|\,\yng(2,1)\,,\alpha\right\rangle^{(j)}\ {}^{(k)}\!\left\langle\,\yng(2,1)\,,\alpha\right|.
$$

The formulas above have all been obtained using the Clebsch-Gordan coefficients of $U(3)$ - we have not used any symmetric group theory. However, as a consequence of Schur-Weyl duality, we claim that the above states fill out representations of S_3. This is easily verified and thus provides a concrete validation of our construction of symmetric group projectors using $U(p)$ states.

4.5 Young Diagrams with p Columns

We will consider Young diagrams with a total of p columns. In this case, boxes that are in different columns, will again have associated factors with $c_i - c_j \sim O(N)$. As discussed in Appendix C.4, the fact that $c_i - c_j \sim O(N)$ for boxes in different rows again implies a significant simplification in the representations of S_m. When adjacent permutations $(i, i+1)$ act on labeled boxes that belong to the same column, the Young diagram changes sign and when acting on labeled boxes that belong to the different columns, the labeled boxes are swapped. This change in sign for the case that boxes belong to the same column is the only difference to what was considered in section 4.3.

The number of states that can be obtained when m boxes are labeled is again p^m and we again associate a p-dimensional vector to each box that is labeled. This again allows us to put partially labeled Young diagrams into one-to-one correspondence with vectors in $V_p^{\otimes m}$. In this case however, we will include some additional phases when we identify vectors in $V_p^{\otimes m}$ with partially labeled Young diagrams. These extra phases occur precisely because adjacent permutations $(i, i+1)$ acting on labeled boxes that belong to the same column flip the sign of the Young diagram. Choose any specific state with a particular set of labels. This state plays the role of a reference state. Any other state with the same boxes labeled but with a different assignment

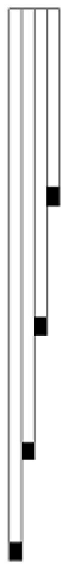


Figure 4.2: An example of a Young diagram with $p = 4$ columns. The columns are shown; the rows are not shown. There are $O(N)$ boxes in each column. The m numbered boxes have been colored black. The difference in factors associated to any two boxes that are in different columns is $O(N)$.

of the labels can be obtained by acting on the reference state with adjacent permutations $(i, i+1)$. Further, the only adjacent permutation $(i, i+1)$ that we are allowed to apply to the reference state to reach any other given state have boxes labeled i and $i+1$ in different columns when $(i, i+1)$ acts. If we act with q adjacent permutations of this type to get from the reference state to another distinct state, it is assigned a phase $(-1)^q$. With this choice for the phases, it is easy to see that the action of S_m on the partially labeled Young diagrams induces the following action on $V_p^{\otimes m}$

$$\sigma \cdot (\vec{v}(1) \otimes \vec{v}(2) \otimes \cdots \otimes \vec{v}(m)) = \mathrm{sgn}(\sigma)\vec{v}\,(\sigma(1)) \otimes \vec{v}\,(\sigma(2)) \otimes \cdots \otimes \vec{v}\,(\sigma(m)) \ ,$$

where $\mathrm{sgn}(\sigma)$ denotes the signature of permutation σ: it is $+1$ for even permutations and -1 for odd permutations[6]. Thus, $\sigma \in S_m$ will move the vector in the i^{th} slot to the $\sigma(i)^{\text{th}}$ slot and may change the overall phase. We can also define an action of $U(p)$ on $V_p^{\otimes m}$

$$U \cdot (\vec{v}(1) \otimes \vec{v}(2) \otimes \cdots \otimes \vec{v}(m)) = D(U)\vec{v}\,(1) \otimes D(U)\vec{v}\,(2) \otimes \cdots \otimes D(U)\vec{v}\,(m) \ ,$$

where $D(U)$ is the $p \times p$ unitary matrix representing group element $U \in U(p)$. Thus, $U \in U(p)$ will change the entries of the vector in the i^{th} slot but it will

[6] Recall that a permutation is even (odd) if it can be written as a product of an even (odd) number of two cycles.

not move it to a different slot. It acts in exactly the same way on each slot. It is quite clear that again these are commuting actions of $U(p)$ and S_m on $V_p^{\otimes m}$

$$\begin{aligned} U\cdot(\sigma\cdot(\vec{v}(1)\otimes\cdots\otimes\vec{v}(m))) &= U\cdot\,\mathrm{sgn}(\sigma)\,(\vec{v}\,(\sigma(1))\otimes\cdots\otimes\vec{v}\,(\sigma(m))) \\ & \quad \mathrm{sgn}(\sigma)D(U)\vec{v}\,(\sigma(1))\otimes\cdots\otimes D(U)\vec{v}\,(\sigma(m)) \\ & \quad \sigma\cdot(D(U)\vec{v}\,(1)\otimes\cdots\otimes D(U)\vec{v}\,(m)) \\ & \quad \sigma\cdot(U\cdot(\vec{v}(1)\otimes\cdots\otimes\vec{v}(m))) \end{aligned}$$

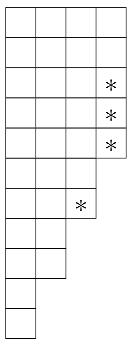
and consequently by Schur-Weyl duality we can again use $U(p)$ to organize the multiplicity label of the S_m irreducible representations. In this case, the space can be organized as

$$V_p^{\otimes m}=\oplus_s V_{s^T}^{U(p)}\otimes V_s^{S_m}\ , \tag{4.4}$$

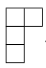
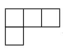
where s^T is obtained by exchanging row and columns in s. The discussion from here on is identical to the case of p rows.

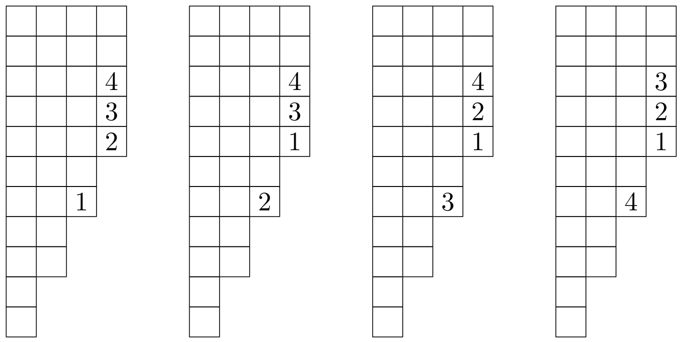
4.5.1 A Four Column Projector using $U(4)$

Consider the following four column Young diagram



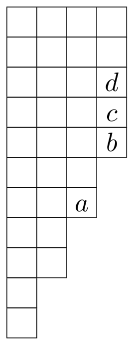
.

The starred boxes are to be removed. There are four possible ways to distribute the labels $1,2,3,4$ between these boxes. One possible $S_n\times S_m$ irreducible representation that can be subduced has r as given above but with the starred boxes removed and $s=$ . To build the corresponding projector we need to build the projector onto the $U(4)$ irreducible representation labeled by $s^T=$ . Since we pull three boxes off the right most column and one box off the neighboring column, the states we are interested in will have a Δ weight of $(0,0,1,3)$. For this example, we will need to assign nontrivial phases between the states in $V_p^{\otimes m}$ and the Young diagrams. The four possible ways to distribute the labels are



Take the first state shown as the reference state. To get the second state from the first we need to act with (12), so that the second state has a phase of -1. The get the third state from the first we need to act with (12) and then with (23), so that it has a phase of 1. Finally, to get the fourth state from the first we need to act with (12) and then (23) and then (34) giving a phase of -1. Writing our states as

$$|a,b,c,d\rangle =$$



we have

$$\begin{aligned}
|1,2,3,4\rangle &= \begin{bmatrix} 1 & & 0 & & 0 & & 0 \\ & 1 & & 0 & & 0 & \\ & & 1 & & 0 & & \\ & & & 0 & & & \end{bmatrix} \otimes \begin{bmatrix} 1 & & 0 & & 0 & & 0 \\ & 1 & & 0 & & 0 & \\ & & 1 & & 0 & & \\ & & & 1 & & & \end{bmatrix} \otimes \begin{bmatrix} 1 & & 0 & & 0 & & 0 \\ & 1 & & 0 & & 0 & \\ & & 1 & & 0 & & \\ & & & 1 & & & \end{bmatrix} \otimes \begin{bmatrix} 1 & & 0 & & 0 & & 0 \\ & 1 & & 0 & & 0 & \\ & & 1 & & 0 & & \\ & & & 1 & & & \end{bmatrix} \\
&= -\frac{\sqrt{3}}{2} \begin{bmatrix} 3 & & 1 & & 0 & & 0 \\ & 3 & & 1 & & 0 & \\ & & 3 & & 1 & & \\ & & & 3 & & & \end{bmatrix}^{(1)} + \frac{1}{2} \begin{bmatrix} 4 & & 0 & & 0 & & 0 \\ & 4 & & 0 & & 0 & \\ & & 4 & & 0 & & \\ & & & 3 & & & \end{bmatrix},
\end{aligned}$$

$$
|2,1,3,4\rangle = -\begin{bmatrix} 1&&0&&0&&0\\ &1&&0&&0& \\ &&1&&0&& \\ &&&1&&& \end{bmatrix} \otimes \begin{bmatrix} 1&&0&&0&&0\\ &1&&0&&0& \\ &&1&&0&& \\ &&&0&&& \end{bmatrix} \otimes \begin{bmatrix} 1&&0&&0&&0\\ &1&&0&&0& \\ &&1&&0&& \\ &&&1&&& \end{bmatrix} \otimes \begin{bmatrix} 1&&0&&0&&0\\ &1&&0&&0& \\ &&1&&0&& \\ &&&1&&& \end{bmatrix}
$$

$$
= \sqrt{\frac{2}{3}} \begin{bmatrix} 3&&1&&0&&0\\ &3&&1&&0& \\ &&3&&1&& \\ &&&3&&& \end{bmatrix}^{(2)} - \frac{1}{\sqrt{12}} \begin{bmatrix} 3&&1&&0&&0\\ &3&&1&&0& \\ &&3&&1&& \\ &&&3&&& \end{bmatrix}^{(1)} + \frac{1}{2} \begin{bmatrix} 4&&0&&0&&0\\ &4&&0&&0& \\ &&4&&0&& \\ &&&3&&& \end{bmatrix} ,
$$

$$
|3,1,2,4\rangle = \begin{bmatrix} 1&&0&&0&&0\\ &1&&0&&0& \\ &&1&&0&& \\ &&&1&&& \end{bmatrix} \otimes \begin{bmatrix} 1&&0&&0&&0\\ &1&&0&&0& \\ &&1&&0&& \\ &&&1&&& \end{bmatrix} \otimes \begin{bmatrix} 1&&0&&0&&0\\ &1&&0&&0& \\ &&1&&0&& \\ &&&0&&& \end{bmatrix} \otimes \begin{bmatrix} 1&&0&&0&&0\\ &1&&0&&0& \\ &&1&&0&& \\ &&&1&&& \end{bmatrix}
$$

$$
= -\frac{1}{\sqrt{2}} \begin{bmatrix} 3&&1&&0&&0\\ &3&&1&&0& \\ &&3&&1&& \\ &&&3&&& \end{bmatrix}^{(3)} + \frac{1}{\sqrt{6}} \begin{bmatrix} 3&&1&&0&&0\\ &3&&1&&0& \\ &&3&&1&& \\ &&&3&&& \end{bmatrix}^{(2)} + \frac{1}{\sqrt{12}} \begin{bmatrix} 3&&1&&0&&0\\ &3&&1&&0& \\ &&3&&1&& \\ &&&3&&& \end{bmatrix}^{(1)} + \frac{1}{2} \begin{bmatrix} 4&&0&&0&&0\\ &4&&0&&0& \\ &&4&&0&& \\ &&&3&&& \end{bmatrix} ,
$$

$$
|4,1,2,3\rangle = -\begin{bmatrix} 1&&0&&0&&0\\ &1&&0&&0& \\ &&1&&0&& \\ &&&1&&& \end{bmatrix} \otimes \begin{bmatrix} 1&&0&&0&&0\\ &1&&0&&0& \\ &&1&&0&& \\ &&&1&&& \end{bmatrix} \otimes \begin{bmatrix} 1&&0&&0&&0\\ &1&&0&&0& \\ &&1&&0&& \\ &&&1&&& \end{bmatrix} \otimes \begin{bmatrix} 1&&0&&0&&0\\ &1&&0&&0& \\ &&1&&0&& \\ &&&0&&& \end{bmatrix}
$$

$$
= -\frac{1}{\sqrt{2}} \begin{bmatrix} 3&&1&&0&&0\\ &3&&1&&0& \\ &&3&&1&& \\ &&&3&&& \end{bmatrix}^{(3)} - \frac{1}{\sqrt{6}} \begin{bmatrix} 3&&1&&0&&0\\ &3&&1&&0& \\ &&3&&1&& \\ &&&3&&& \end{bmatrix}^{(2)} - \frac{1}{\sqrt{12}} \begin{bmatrix} 3&&1&&0&&0\\ &3&&1&&0& \\ &&3&&1&& \\ &&&3&&& \end{bmatrix}^{(1)} - \frac{1}{2} \begin{bmatrix} 4&&0&&0&&0\\ &4&&0&&0& \\ &&4&&0&& \\ &&&3&&& \end{bmatrix} .
$$

Given these results, it is a simple matter to write down the states that carry the S_m irreducible representation $\begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1} \ \\ \cline{1-1}\end{array}$

$$
\left| \begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1} \ \\ \cline{1-1}\end{array}, 1 \right\rangle = \begin{bmatrix} 3&&1&&0&&0\\ &3&&1&&0& \\ &&3&&1&& \\ &&&3&&& \end{bmatrix}^{(1)} = \frac{1}{\sqrt{12}} \Big(-3|1,2,3,4\rangle - |2,1,3,4\rangle + |3,1,2,4\rangle - |4,1,2,3\rangle \Big) ,
$$

$$\left| \ydiagram{2,1,1}, 2 \right\rangle = \begin{bmatrix} 3 & & 1 & & 0 & & 0 \\ & 3 & & 1 & & 0 & \\ & & 3 & & 1 & & \\ & & & 3 & & & \end{bmatrix}^{(2)} = \frac{1}{\sqrt{6}} \Big(2|2,1,3,4\rangle + |3,1,2,4\rangle - |4,1,2,3\rangle \Big) \ ,$$

$$\left| \ydiagram{2,1,1}, 3 \right\rangle = \begin{bmatrix} 3 & & 1 & & 0 & & 0 \\ & 3 & & 1 & & 0 & \\ & & 3 & & 1 & & \\ & & & 3 & & & \end{bmatrix}^{(3)} = -\frac{1}{\sqrt{2}} \Big(|3,1,2,4\rangle + |4,1,2,3\rangle \Big) \ .$$

These formulas use only the Clebsch-Gordan coefficients of $U(4)$. It is again easy to verify that the above states fill out the representation $\ydiagram{2,1,1}$ of S_4. The projector is now given by

$$P_{R\to(r,\ydiagram{2,1,1})} = \sum_{\alpha=1}^{3} \left| \ydiagram{2,1,1}, \alpha \right\rangle \ \left\langle \ydiagram{2,1,1}, \alpha \right| \ .$$

Chapter 5

Action of The Dilatation Operator

The action of the one loop dilatation operator of $\mathcal{N}=4$ super Yang-Mills on restricted Schur polynomials has been studied in [9, 14, 15]. We will start this chapter by reviewing the the derivation of this action given in [14] emphasizing those features important for our discussion, and then move onto evaluating it. We will show that the action of the dilatation operator on restricted Schur polynomials labeled by Young diagrams with $O(1)$ long columns is easily obtained from the action on restricted Schur polynomials with $O(1)$ long rows, and so from section 5.1.1 onwards we will focus on the AdS giant dual operators.

5.1 Evaluation of The Dilatation Operator

The one loop dilatation operator in the $SU(2)$ sector[8] of $\mathcal{N}=4$ super Yang-Mills is

$$D=-g_{\rm YM}^2 \mathrm{Tr}\left[Y,Z\right]\left[\partial_Y,\partial_Z\right] .$$

Acting on a restricted Schur polynomial

$$\chi_{R,(r,s)jk}(Z,Y)=\frac{1}{n!m!}\sum_{\sigma\in S_{n+m}}\mathrm{Tr}_{(r,s)jk}\left(\Gamma_R(\sigma)\right)Y^{i_1}_{i_{\sigma(1)}}\cdots Y^{i_m}_{i_{\sigma(m)}}Z^{i_{m+1}}_{i_{\sigma(m+1)}}\cdots Z^{i_{m+n}}_{i_{\sigma(m+n)}}$$

we obtain[1]

$$\begin{aligned} D\chi_{R,(r,s)jk} \;&=\; \frac{g_{\rm YM}^2}{(n-1)!(m-1)!}\sum_{\psi\in S_{n+m}}\mathrm{Tr}_{(r,s)jk}\Big(\Gamma_R\big((1,m+1)\psi-\psi(1,m+1)\big)\Big)\times \\ &\times\delta^{i_1}_{i_{\psi(1)}}Y^{i_2}_{i_{\psi(2)}}\cdots Y^{i_m}_{i_{\psi(m)}}(YZ-ZY)^{i_{m+1}}_{i_{\psi(m+1)}}Z^{i_{m+2}}_{i_{\psi(m+2)}}\cdots Z^{i_{m+n}}_{i_{\psi(m+n)}} \end{aligned} \tag{5.1}$$

[1] Our index conventions are $(YZ)^i_k=Y^i_jZ^j_k$.

As a consequence of the $\delta^{i_1}_{i_{\psi(1)}}$ appearing in the summand, the sum over ψ runs only over permutations for which $\psi(1) = 1$. To perform the sum over ψ, write the sum over S_{n+m} as a sum over cosets of the S_{n+m-1} subgroup obtained by keeping those permutations that satisfy $\psi(1) = 1$. The result follows immediately from the reduction rule for Schur polynomials (see [47] and Appendix C of [10])

$$D\chi_{R,(r,s)jk} = \frac{g^2_{\rm YM}}{(n-1)!(m-1)!}\sum_{\psi\in S_{n+m-1}}\sum_{R'} c_{RR'}\mathrm{Tr}_{(r,s)jk}\Big(\Gamma_R\left((1,m+1)\right)\Gamma_{R'}(\psi)$$

$$-\Gamma_{R'}(\psi)\Gamma_R\left((1,m+1)\right)\Big)Y^{i_2}_{i_{\psi(2)}}\cdots Y^{i_m}_{i_{\psi(m)}}(YZ-ZY)^{i_{m+1}}_{i_{\psi(m+1)}}Z^{i_{m+2}}_{i_{\psi(m+2)}}\cdots Z^{i_{m+n}}_{i_{\psi(m+n)}}\ .$$

The sum over R' runs over all Young diagrams that can be obtained from R by dropping a single box; $c_{RR'}$ is the factor of the box that must be removed from R to obtain R'. The appearance of $\Gamma_R\big((1,m+1)\big)$ is very natural. $\Gamma_R\big((1,m+1)\big)$ is not an element of the $S_n\times S_m$ subgroup - it mixes indices belonging to Zs and indices belonging to Ys. The dilatation operator has derivatives with respect to Z and Y in the same trace and so does indeed naturally mix Zs and Ys. We will make use of the following notation

$$\mathrm{Tr}(\sigma Z^{\otimes n}Y^{\otimes m}) = Z^{i_1}_{i_{\sigma(1)}}\cdots Z^{i_n}_{i_{\sigma(n)}}Y^{i_{n+1}}_{i_{\sigma(n+1)}}\cdots Y^{i_{n+m}}_{i_{\sigma(n+m)}}\ .$$

Now, use the identities (bear in mind that $\psi(1)=1$)

$$Y^{i_2}_{i_{\psi(2)}}\cdots Y^{i_m}_{i_{\psi(m)}}(YZ-ZY)^{i_{m+1}}_{i_{\psi(m+1)}}Z^{i_{m+2}}_{i_{\psi(m+2)}}\cdots Z^{i_{n+m}}_{i_{\psi(n+m)}} = \mathrm{Tr}\Big(\big((1,m{+}1)\,\psi-\psi\,(1,m{+}1)\big)Z^{\otimes n}Y^{\otimes m}\Big)$$

and (this identity is proved in [7])

$$\mathrm{Tr}\left(\sigma Z^{\otimes n}Y^{\otimes m}\right) = \sum_{T,(t,u)lm}\frac{d_T n!m!}{d_t d_u (n+m)!}\mathrm{Tr}_{(t,u)lm}\left(\Gamma_T(\sigma^{-1})\right)\chi_{T,(t,u)ml}(Z,Y)$$

to obtain

$$D\chi_{R,(r,s)jk}(Z,Y) = \sum_{T,(t,u)lm} M_{R,(r,s)jk;T,(t,u)lm}\chi_{T,(t,u)ml}(Z,Y)\ ,$$

$$M_{R,(r,s)jk;T,(t,u)lm} = g^2_{\rm YM}\sum_{\psi\in S_{n+m-1}}\sum_{R'}\frac{c_{RR'}d_T nm}{d_t d_u (n+m)!}\mathrm{Tr}_{(r,s)jk}\Big(\Gamma_R\big((1,m{+}1)\big)\Gamma_{R'}(\psi)$$

$$-\Gamma_{R'}(\psi)\Gamma_R\big((1,m{+}1)\big)\Big)\mathrm{Tr}_{(t,u)lm}\Big(\Gamma_{T'}(\psi^{-1})\Gamma_T\big((1,m{+}1)\big)-\Gamma_T\big((1,m{+}1)\big)\Gamma_{T'}(\psi^{-1})\Big) \tag{5.2}$$

The sum over ψ can be evaluated using the fundamental orthogonality relation

$$\sum_{\sigma} \left[\Gamma_R(\sigma)\right]_{ij} \left[\Gamma_S(\sigma^{-1})\right]_{kl} = \frac{(n+m)!}{d_R} \delta_{RS}\delta_{il}\delta_{jk}$$

to obtain

$$\begin{aligned} M_{R,(r,s)jk;T,(t,u)lm} &= -g_{\rm YM}^2 \sum_{R'} \frac{c_{RR'} d_T nm}{d_{R'} d_t d_u (n+m)} \mathrm{Tr}\Big(\Big[\Gamma_R\big((1,m+1)\big), P_{R\to(r,s)jk}\Big] I_{R'\,T'} \times \\ &\qquad \times \Big[\Gamma_T\big((1,m+1)\big), P_{T\to(t,u)lm}\Big] I_{T'\,R'}\Big) \ . \end{aligned}$$

Sums of this type and the intertwiners $I_{R'\,T'}$ which arise are discussed in detail in the next section . This expression for the one loop dilatation operator is exact in N.

To obtain the spectrum of anomalous dimensions, we need to consider the action of the dilatation operator on normalized operators. The two point function for the restricted Schur polynomials (4.2) is not unity. Normalized operators which do have unit two point function can be obtained from

$$\chi_{R,(r,s)jk}(Z,Y) = \sqrt{\frac{f_R \, \mathrm{hooks}_R}{\mathrm{hooks}_r \, \mathrm{hooks}_s}} O_{R,(r,s)jk}(Z,Y) \ .$$

In terms of these normalized operators

$$DO_{R,(r,s)jk}(Z,Y) = \sum_{T,(t,u)lm} N_{R,(r,s)jk;T,(t,u)ml} O_{T,(t,u)ml}(Z,Y) \tag{5.3}$$

$$N_{R,(r,s)jk;T,(t,u)ml} = -g_{\rm YM}^2 \sum_{R'} \frac{c_{RR'} d_T nm}{d_{R'} d_t d_u (n+m)} \sqrt{\frac{f_T \, \mathrm{hooks}_T \, \mathrm{hooks}_r \, \mathrm{hooks}_s}{f_R \, \mathrm{hooks}_R \, \mathrm{hooks}_t \, \mathrm{hooks}_u}} \times$$

$$\times \mathrm{Tr}\Big(\Big[\Gamma_R\big((1,m+1)\big), P_{R\to(r,s)jk}\Big] I_{R'\,T'} \Big[\Gamma_T\big((1,m+1)\big), P_{T\to(t,u)lm}\Big] I_{T'\,R'}\Big) \ .$$

It is this last expression that we evaluate explicitely. The bulk of the work entails evaluating the trace. There are three objects which appear: the symmetric group operators $P_{R\to(r,s)jk}$, the intertwiners $I_{T'\,R'}$ and the symmetric group element $\Gamma_R((1,m+1))$. We have already discussed the operators $P_{R\to(r,s)jk}$. The next two subsections are used to discuss $I_{T'\,R'}$ and $\Gamma_R((1,m+1))$.

5.1.1 Intertwiners

In this section we will consider the sum over S_{n+m-1} which was performed to obtain (5.2). This will give a very explicit understanding of the intertwiners appearing in the expression for the dilatation operator. When S^n acts on $V^{\otimes n}$ $n > 1$ it furnishes a reducible representation. Imagine that this includes the irreducible representations R and T. Representing the action of σ as a matrix $\Gamma(\sigma)$, in a suitable basis we can write

$$\Gamma(\sigma) = \begin{bmatrix} \Gamma_R(\sigma) & 0 & \cdots \\ 0 & \Gamma_T(\sigma) & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} .$$

If we restrict ourselves to an S_{n-1} subgroup of S_n, then in general, both R and T will subduce a number of representations. Assume for the sake of this discussion that R subduces R'_1 and R'_2 and that T subduces T'_1 and T'_2. This is precisely the situation that arises in the sum performed to obtain (5.2). Then, for $\sigma \in S_{n-1}$ we have

$$\Gamma(\sigma) = \begin{bmatrix} \Gamma_{R'_1}(\sigma) & 0 & 0 & 0 & \cdots \\ 0 & \Gamma_{R'_2}(\sigma) & 0 & 0 & \cdots \\ 0 & 0 & \Gamma_{T'_1}(\sigma) & 0 & \cdots \\ 0 & 0 & 0 & \Gamma_{T'_2}(\sigma) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} .$$

Imagine that as Young diagrams $T'_1 = R'_1$, that is, one of the irreducible representations subduced by R is isomorphic to one of the representations subduced by T. Then, a simple application of the fundamental orthogonality relation gives

$$\sum_{\sigma \in S_{n-1}} \begin{bmatrix} \Gamma_{R'_1}(\sigma) & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{ij} \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \Gamma_{T'_1}(\sigma) & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{ab}$$

$$= \frac{(n-1)!}{d_{R'_1}} \delta_{R'_1 T'_1} \begin{bmatrix} 0 & 0 & \mathbf{1} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{ib} \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \mathbf{1} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}_{aj}$$

$$\equiv \frac{(n-1)!}{d_{R'_1}} \delta_{R'_1 T'_1} \left(I_{R'_1 T'_1}\right)_{ib} \left(I_{T'_1 R'_1}\right)_{aj}$$

where the form of the intertwiners has been spelled out. Intertwiners are maps between two isomorphic spaces. For $\sigma \in S_{n+m-1}$

$$I_{R'T'}\Gamma_{T'}(\sigma) = \Gamma_{R'}(\sigma)I_{R'T'}$$

The box removed to obtain R' and T' can be removed from any corner of the Young diagram.

It is useful to make a few comments on how the intertwiners are realized in our calculation. Since the first box is removed from R or T the intertwiner acts on the first slot of $V_p^{\otimes m}$. Now, look back at formula (5.1). The delta function which appears freezes the 1 index and hence the S_{n+m-1} subgroup of S_{n+m} is obtained by keeping all elements of S_{n+m} that leave index 1 inert. Consequently, with our choice that the intertwiner acts on the first slot of $V_p^{\otimes m}$, we see that the first slot corresponds to index i_1. Recall that the particular vector a box corresponds to is determined by the row/column the box belongs to. Thus, the explicit form of the intertwiner is determined once the location of the box removed from T and the box removed from R are specified. As an example, for the Young diagrams shown below we have

$$I_{R'T'} = E_{1,5} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}\,, \qquad I_{T'R'} = E_{5,1} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}\,.$$

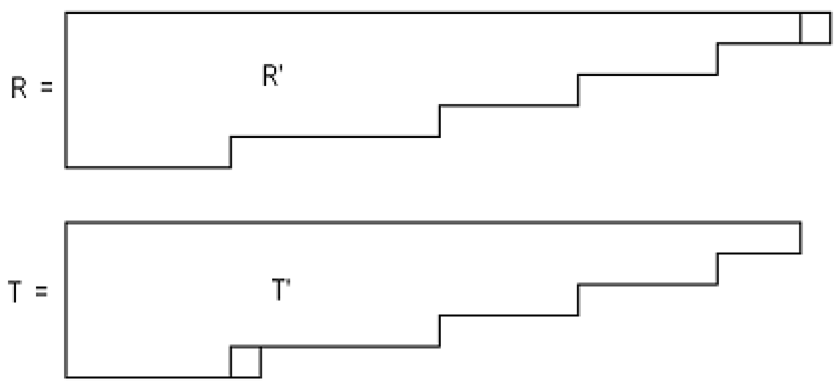


Figure 5.1: A figure showing R and the box that must be removed to obtain R' and T and the box that must be removed to obtain T'. As Young diagrams, $T' = R'$. T and R both have 5 rows.

It is straight forward to extract the general rule from this example. Consider first the case that $R \neq T$. To obtain R' from R we remove a box from row i and to obtain T' from T we remove a box from row j. In this situation we have

$$I_{R'T'} = E_{ij} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}\,, \qquad I_{T'R'} = E_{ji} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}\,.$$

In the case that $R = T$, the box that must be removed can be removed from any row and we get a contribution to the dilatation operator from each possible removal. Each possible removal must be represented by a different intertwiner and one needs to sum over all possible intertwiners. In this situation, the possible intertwiners are

$$I_{R'T'} = E_{kk} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} = I_{T'R'}, \qquad k = 1, 2, \cdots, p\,.$$

5.1.2 $\Gamma_R(1, m+1)$

This group element acts on one slot from the Ys and one slot from the Zs. The box removed from R to get R' is the box acted on by the intertwiner and it is a Y box. This is one of the boxes that $\Gamma_R(1, m+1)$ acts on. The second box that $\Gamma_R(1, m+1)$ acts on can be any box associated to the Zs. Up to now we have discussed the projectors and intertwiners. These only have an action on the boxes corresponding to Ys and as a result, our discussion has always taken place in the vector space $V_p^{\otimes m}$. However, because $\Gamma_R(1, m+1)$ acts on a Z box we must include one more slot and work in $V_p^{\otimes m+1}$. The intertwiners and projectors have a trivial action on the $(m+1)^{\text{th}}$ slot and hence the $(m+1)^{\text{th}}$ slot is simply occupied with the identity. For the rest of this subsection we work in $V_p^{\otimes m+1}$ and not in $V_p^{\otimes m}$. Acting in $V_p^{\otimes m+1}$, $\Gamma_R(1, m+1)$ has a very simple action: it simply swaps the 1^{st} and the $(m+1)^{\text{th}}$ slots. The projectors when acting on $V_p^{\otimes m+1}$ are given by

$$\mathcal{P}_{R\to(r,s)ij} = p_{R,(r,s)ij} \otimes \mathbf{1}$$

where the $p \times p$ unit matrix $\mathbf{1}$ acts on the $(m+1)^{\text{th}}$ slot. $p_{R,(r,s)ij}$ acts only in $V_p^{\otimes m}$. For comparison, the projectors appearing in the defintion of the restricted Schur polynomial are

$$P_{R\to(r,s)ij} = p_{R,(r,s)ij} \otimes \mathbf{I}_r$$

where $\mathbf{I}_r$ is the identity matrix acting on the carrier space of the S_n irreducible representation r. Below we will make use of the obvious formula

$$\mathbf{1} = \sum_{k=1}^{p} E_{kk}\,.$$

In evaluating the dilatation operator, we will need to take products of the intertwiners and $\Gamma(1, m+1)$. These products are easily evaluated

$$\begin{aligned}
\Gamma_R(1, m+1)\, E_{ij} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} &= \Gamma_R(1, m+1) \sum_{k=1}^{p} E_{ij} \otimes \mathbf{1} \otimes \cdots \otimes E_{kk} \\
&= \sum_{k=1}^{p} E_{kj} \otimes \mathbf{1} \otimes \cdots \otimes E_{ik}
\end{aligned}$$

$$
\begin{aligned}
E_{ij} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}\, \Gamma_R(1, m+1) \qquad &= \sum_{k=1}^{p} E_{ij} \otimes \mathbf{1} \otimes \cdots \otimes E_{kk}\, \Gamma_R(1, m+1) \\
&= \sum_{k=1}^{p} E_{ik} \otimes \mathbf{1} \otimes \cdots \otimes E_{kj}
\end{aligned}
$$

$$
\Gamma_R(1, m+1)\, E_{ij} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}\, \Gamma_R(1, m+1) = \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes E_{ij}\ .
$$

From now on we will write the E_{ij} with a superscript, indicating which slot E_{ij} acts on. In this notation we have

$$
E_{ik} \otimes \mathbf{1} \otimes \cdots \otimes E_{kj} = E_{ik}^{(1)} E_{kj}^{(m+1)}\ .
$$

5.1.3 Dilatation Operator Coefficient

In this secton we explain how to evaluate the value of the coefficient

$$
g_{\rm YM}^2 \frac{c_{RR'} d_T n m}{d_{R'} d_t d_u (n+m)} \sqrt{\frac{f_T\, \mathrm{hooks}_T\, \mathrm{hooks}_r\, \mathrm{hooks}_s}{f_R\, \mathrm{hooks}_R\, \mathrm{hooks}_t\, \mathrm{hooks}_u}}
$$

in the large N limit. The Young diagrams R, T, r, t, s and u each have p-rows. We use the symbols R_i, T_i, r_i, t_i, s_i and u_i $i = 1, 2, ..., p$ to denote the number of boxes in each row respectively. We assume p is fixed to be $O(1)$. The top row (which is also the longest row) is the value $i = 1$ and the bottom row (shortest row) has $i = p$. It is straight forward to argue that the product of hook lengths, in r for example, is

$$
\mathrm{hooks}_R = \frac{\prod_{i=1}^{p} (r_i + p - i)!}{\prod_{j<k} (r_j - r_k + k - j)}\ .
$$

For the diagrams R and T, the row lengths R_i are of order N. Further, R and T differ by at most the placement of a single box. This implies that $R_i = T_i$ for all except two values of i, say $i = a, b$. For these values of i we have

$$
R_b = T_b + 1, \qquad R_a = T_a - 1\ .
$$

This implies that

$$
\frac{\mathrm{hooks}_R}{\mathrm{hooks}_T} = \frac{(T_a - 1 + p - a)!(T_b + 1 + p - b)!}{((T_a + p - a)!(T_b + p - b)!} \prod_{\substack{k \neq a \\ k \neq b}} \frac{|T_a - T_k| + |k - a|}{|T_a - 1 - T_k| + |k - a|} \times
$$

$$\times \prod_{\substack{k\neq a\\ k\neq b}} \frac{|T_b-T_k|+|k-b|}{|T_b+1-T_k|+|k-b|}\frac{|T_b-T_a|+|a-b|}{|T_a-T_b-2|+|a-b|} = \frac{R_b}{R_a}\left(1+O(N^{-1})\right)\ .$$

Use R_+ to denote the row length of the row in R that is longer than the corresponding row in T and let R_- denote the row length of the row in R that is shorter than the corresponding row in T. With this notation

$$\frac{\text{hooks}_R}{\text{hooks}_T} = \frac{R_+}{R_-}\left(1+O(N^{-1})\right)\ .$$

This argument has an obvious generalization to the other hook factors $\frac{\text{hooks}_r}{\text{hooks}_t}$ and $\frac{\text{hooks}_s}{\text{hooks}_u}$. Now consider a Young diagram R' that is obtained by removing a single box from Young diagram R. Assuming this box is removed from row a, we have the following relation between the lengths of the rows in R and the lengths of the rows in R'

$$R_i = R'_i \quad i\neq a, \qquad R_a = R'_a + 1\ .$$

Thus, we find

$$\frac{\text{hooks}_R}{\text{hooks}_{R'}} = \frac{(R_a+p-a)!}{(R_a+p-1-a)!}\prod_{j\neq a}\frac{|R_j-R_a-1|+|a-j|}{|R_j-R_a|+|a-j|} = R_a\left(1+O(N^{-1})\right).$$

The coefficient quoted at the start of this subsection is multiplied by the trace over an (r,s) subspace. This trace produces a number of order 1 multiplied by $d_{r'}d_s$. The product of the coefficient and the trace now reduces to quantities that we have studied. Thus, we now have all the ingredients needed to estimate the large N values of the combinations of symmetric group dimensions and hook factors that appear in the dilatation operator. Notice that both the product of the hook lengths and the dimensions of symmetric group irreducible representations are invariant under the flip of the Young diagram which exchanges columns and rows. Thus, these conclusions can immediately be recycled when studying the case of p long columns.

Next, recalling that f_R is the product of factors in Young diagram R and $R'=T'$ we learn that

$$c_{RR'}\sqrt{\frac{f_T}{f_R}} = \sqrt{c_{RR'}c_{TT'}}$$

where $c_{RR'}$ is the factor associated to the box that must be removed from R to obtain R' and $c_{TT'}$ is the factor associated to the box that must be removed from T to obtain T'.

5.1.4 Evaluating Traces

We now turn to the task of evaluating the trace which appears in (5.3)

$$\mathcal{T}=\mathrm{Tr}\Big(\Big[\Gamma_R((1,m+1)),P_{R\to(r,s)jk}\Big]I_{R'\,T'}\Big[\Gamma_T((1,m+1)),P_{T\to(t,u)lm}\Big]I_{T'\,R'}\Big)\,. \tag{5.4}$$

We start by writing this trace as a sum of traces over $m+1$ slots (all the Y slots plus one Z slot) times a trace over $n-1$ slots (the remaining Z slots). The trace over the $n-1$ slots is over the carrier space R^{m+1} which is described by a Young diagram that can be obtained by removing $m+1$ boxes from R, or equivalently by removing one box from r or equivalently by removing one box from t - these all give the same Young diagram describing R^{m+1}. R^{m+1} has different shapes depending on where the $(m+1)^{\text{th}}$ box is removed. The results from the last subsection clearly imply that the dimension of symmetric group representation R^{m+1}, denoted $d_{R^{m+1}}$, depends on the details of this shape. If the $(m+1)^{\text{th}}$ box is removed from row i, denote this dimension by $d^i_{R^{m+1}}$. Our general strategy is then to trace over the last Z slot (the $(m+1)^{\text{th}}$ slot) which then leaves a trace over $V_p^{\otimes m}$. This trace is then evaluated using elementary $U(p)$ representation theory.

The box removed from R to obtain R' is removed from the b^{th} row of R and the box removed from T to obtain T' is removed from the a^{th} row of T. If we multiply out the expression for $\mathcal{T}$ given above we get four terms. We will treat these terms separately. Consider first

$$\begin{aligned}\mathcal{T}_1 &= -\mathrm{Tr}_{V_p^{\otimes(n+m)}}\left(P_{R\to(r,s)jk}\Gamma_R\left((1,m+1)\right)I_{R'T'}\Gamma_T\left((1,m+1)\right)P_{T\to(t,u)lm}I_{T'R'}\right)\\ &= -d^i_{R^{m+1}}\mathrm{Tr}_{V_p^{\otimes(1+m)}}\left(\Gamma_R\left((1,m+1)\right)E^{(1)}_{ba}\Gamma_T\left((1,m+1)\right)P_{T\to(t,u)lm}E^{(1)}_{ab}P_{R\to(r,s)jk}\right)\\ &= -d^i_{R^{m+1}}\mathrm{Tr}_{V_p^{\otimes(1+m)}}\left(E^{(m+1)}_{ba}P_{T\to(t,u)lm}E^{(1)}_{ab}P_{R\to(r,s)jk}\right)\end{aligned}$$

tracing over the $(m+1)^{\text{th}}$ slot, we find that only when $a=b$ do we have a non-zero trace,

$$\begin{aligned}\mathcal{T}_1 &= -\delta_{ab}d^b_{R^{m+1}}\mathrm{Tr}_{V_p^{\otimes m}}\left(P_{R\to(r,s)jk}P_{T\to(t,u)lm}E^{(1)}_{bb}\right)\\ &= -\delta_{ab}\delta_{RT}\delta_{(r,s)(t,u)}\delta_{kl}d^b_{R^{m+1}}\mathrm{Tr}_{V_p^{\otimes m}}\left(P_{R\to(r,s)jm}E^{(1)}_{bb}\right)\,.\end{aligned}$$

The next term follows similarly

$$\begin{aligned}\mathcal{T}_2 &= -\mathrm{Tr}_{V_p^{\otimes(n+m)}}\left(\Gamma_R\left((1,m+1)\right)P_{R\to(r,s)jk}I_{R'T'}P_{T\to(t,u)lm}\Gamma_T\left((1,m+1)\right)I_{T'R'}\right)\\ &= -d^i_{R^{m+1}}\mathrm{Tr}_{V_p^{\otimes(1+m)}}\left(P_{R\to(r,s)jk}E^{(1)}_{ba}P_{T\to(t,u)lm}E^{(m+1)}_{ab}\right)\\ &= -\delta_{ab}\delta_{RT}\delta_{(r,s)(t,u)}\delta_{mj}d^b_{R^{m+1}}\mathrm{Tr}_{V_p^{\otimes m}}\left(P_{R\to(r,s)lk}E^{(1)}_{bb}\right)\,.\end{aligned}$$

Now consider

$$\begin{aligned}\mathcal{T}_3 &= \mathrm{Tr}_{V_p^{\otimes(n+m)}}\left(P_{R\to(r,s)jk}\Gamma_R\left((1,m+1)\right)I_{R'T'}P_{T\to(t,u)lm}\Gamma_T\left((1,m+1)\right)I_{T'R'}\right)\\ &= d^i_{R^{m+1}}\mathrm{Tr}_{V_p^{\otimes(1+m)}}\left(P_{R\to(r,s)jk}E^{(1)}_{qa}E^{(m+1)}_{bq}P_{T\to(t,u)lm}E^{(1)}_{fb}E^{(m+1)}_{af}\right)\ ,\end{aligned}$$

where the repeated indices q and f are summed. Taking the trace over the $(m+1)^{\text{th}}$ slot, we find that only when $q=a$ and $f=b$ do we have a non-zero trace,

$$\mathcal{T}_3 = d^a_{R^{m+1}}\mathrm{Tr}_{V_p^{\otimes m}}\left(P_{R\to(r,s)jk}E^{(1)}_{aa}P_{T\to(t,u)lm}E^{(1)}_{bb}\right)\ .$$

The final term follows similarly

$$\begin{aligned}\mathcal{T}_4 &= \mathrm{Tr}_{V_p^{\otimes(n+m)}}\left(\Gamma_R\left((1,m+1)\right)P_{R\to(r,s)jk}I_{R'T'}\Gamma_T\left((1,m+1)\right)P_{T\to(t,u)lm}I_{T'R'}\right)\\ &= d^i_{R^{m+1}}\mathrm{Tr}_{V_p^{\otimes(1+m)}}\left(E^{(1)}_{ag}E^{(m+1)}_{gb}P_{R\to(r,s)jk}E^{(1)}_{bh}E^{(m+1)}_{ha}P_{T\to(t,u)lm}\right)\\ &= d^b_{R^{m+1}}\mathrm{Tr}_{V_p^{\otimes m}}\left(P_{R\to(r,s)jk}E^{(1)}_{bb}P_{T\to(t,u)lm}E^{(1)}_{aa}\right)\ .\end{aligned}$$

Thus the full trace (5.4) becomes

$$\begin{aligned}\mathcal{T} &= -\delta_{ab}\delta_{RT}\delta_{(r,s)(t,u)}d^b_{R^{m+1}}\left[\delta_{kl}\mathrm{Tr}_{V_p^{\otimes m}}\left(P_{R\to(r,s)jm}E^{(1)}_{bb}\right)+\delta_{mj}\mathrm{Tr}_{V_p^{\otimes m}}\left(P_{R\to(r,s)lk}E^{(1)}_{bb}\right)\right]\\ &+d^a_{R^{m+1}}\mathrm{Tr}_{V_p^{\otimes m}}\left(P_{R\to(r,s)jk}E^{(1)}_{aa}P_{T\to(t,u)lm}E^{(1)}_{bb}\right)+d^b_{R^{m+1}}\mathrm{Tr}_{V_p^{\otimes m}}\left(P_{R\to(r,s)jk}E^{(1)}_{bb}P_{T\to(t,u)lm}E^{(1)}_{aa}\right)\ .\end{aligned} \tag{5.5}$$

We now need to evaluate the traces over $V_p^{\otimes m}$. Towards this end, write the projector as

$$p_{R\to(r,s)ij} = \sum_{\alpha=1}^{d_s}\left|M^i_s,\alpha\right\rangle\left\langle M^j_s,\alpha\right|\ .$$

M^i_s and M^j_s label states from $U(p)$ irreducible representation s which have the same Δ weight. The indices i,j range from $1,...,I(\Delta(M))$. Index α is a multiplicity index that, as a consequence of Schur-Weyl duality, is organized by representation s of the symmetric group S_m. To evaluate the traces over $V_p^{\otimes m}$ we need to allow $E^{(1)}_{kk}$ to act on the state $|M^i_s,\alpha\rangle$. The state $|M^i_s,\alpha\rangle$ was obtained by taking a tensor product of m copies (one for each slot) of the fundamental representation of $U(p)$. It is possible and useful to rewrite this state as a linear combination of states which are each the tensor product of the fundamental representation for the first slot with a state obtained by taking the tensor product of states of the remaining $m-1$ slots. This is a useful thing to do because then $E^{(1)}_{kk}$ has a particularly simple action on each state in the linear combination. Towards this end we can write (in the following $\mathbf{0}$ stands for a string of $p-1$ 0s)

$$\left|M^i_s,\alpha\right\rangle = \sum_{s'}\sum_{M_{\mathbf{10}},M_{s'}} C^{M^i_s}_{M_{s'},M_{\mathbf{10}}}\left|M_{\mathbf{10}}\right\rangle \otimes \left|M_{s'},\beta\right\rangle$$

where $M_{\mathbf{10}}$ indexes states in the carrier space of the fundamental representation and $C^{M^i_s}_{M_{s'},M_{\mathbf{10}}}$ are the Clebsch Gordan coefficients (discussed in detail in Appendix B.5)

$$C^{M^i_s}_{M_{s'},M_{\mathbf{10}}} = \left(\left\langle M_{\mathbf{10}}\right| \otimes \left\langle M_{s'},\beta\right|\right)\left|M^i_s,\alpha\right\rangle \ .$$

s' is obtained by removing a single box from s, and each $M_{\mathbf{10}}$ state corresponds to this box coming from a particular row in R. Since the box removed from s can come from different rows of s, we sum over s'. Note that for a particular s' and a particular $M_{\mathbf{10}}$ there may be multiple $M_{s'}$ with the correct Δ weight, which is why we also sum over $M_{s'}$. By appealing to the Schur-Weyl duality which organizes the space $V_p^{\otimes m-1}$, we know that the multiplicity index β of the state $\left|M_{s'},\beta\right\rangle$ is organized by the irreducible representation s' of S_{m-1}. This allows us to easily evaluate the action of $E^{(1)}_{kk}$: it simply projects onto the state corresponding to box 1 sitting in the k^{th} row. Evaluating the traces over $V_p^{\otimes m}$ is now straight forward. For example, consider

$$\mathrm{Tr}_{V_p^{\otimes m}}\left(p_{R\to(r,s)jm}E^{(1)}_{bb}\right)$$

where the repeated index b is not summed. Inserting the projector

$$\begin{aligned}
&\mathrm{Tr}_{V_p^{\otimes m}}\left(\sum_{\alpha=1}^{d_s}\left|M^j_s,\alpha\right\rangle\left\langle M^m_s,\alpha\right| E^{(1)}_{bb}\right)\\
&\quad= \quad \mathrm{Tr}_{V_p^{\otimes m}}\left(\sum_{\alpha=1}^{d_s}\left|M^j_s,\alpha\right\rangle\sum_{s'}\sum_{M_{\mathbf{10}},M_{s'}} C^{M^m_s}_{M_{s'},M_{\mathbf{10}}}\left\langle M_{\mathbf{10}}\right|\otimes\left\langle M_{s'},\beta\right| E^{(1)}_{bb}\right)
\end{aligned}$$

$E^{(1)}_{bb}$ picks out the state where the box pulled off s to give s' comes from the b^{th} row of R, so we have

$$\begin{aligned}
&\mathrm{Tr}_{V_p^{\otimes m}}\left(\sum_{\alpha=1}^{d_s}\left|M^j_s,\alpha\right\rangle\sum_{s'}\sum_{M_{s'}} C^{M^m_s}_{M_{s'},M^b_{\mathbf{10}}}\left\langle M^b_{\mathbf{10}}\right|\otimes\left\langle M_{s'},\beta\right|\right)\\
&= \mathrm{Tr}_{V_p^{\otimes m}}\left(\sum_{\alpha=1}^{d_s}\sum_{s'}\sum_{M_{\mathbf{10}},M_{s'}} C^{M^j_s}_{M_{s'},M_{\mathbf{10}}}\left|M_{\mathbf{10}}\right\rangle\otimes\left|M_{s'},\gamma\right\rangle\sum_{M_{s'}} C^{M^m_s}_{M_{s'},M^b_{\mathbf{10}}}\left\langle M^b_{\mathbf{10}}\right|\otimes\left\langle M_{s'},\beta\right|\right)
\end{aligned}$$

taking the trace, we find that the $\left\langle M^b_{\mathbf{10}}\right|$ will pick out the $\left|M^b_{\mathbf{10}}\right\rangle$ states since the Gelfand-Tsetlin basis is orthogonal. Further, only when $|M_{s'}\rangle$ and $\langle M_{s'}|$ have the same Gelfand-Tsetlin pattern do we have a non-zero contribution to the trace, yielding

$$\mathrm{Tr}_{V_p^{\otimes m}}\left(p_{R\to(r,s)jm}E^{(1)}_{bb}\right) = \sum_{s'} d_{s'} \sum_{M_{s'}} C^{M^j_s}_{M_{s'},M^b_{\mathbf{10}}} C^{M^m_s}_{M_{s'},M^b_{\mathbf{10}}} \, .$$

5.1.5 Long Columns

Our computation of the action of the dilatation operator for restricted Schur polynomials labeled by Young diagrams that have a total of p long rows has made extensive use of the fact that we can organize the space of partially labeled Young diagrams into $S_n \times S_m$ irreducible representations (r,s) by appealing to Schur-Weyl duality. We have already argued that it is also possible to perform this organization when considering restricted Schur polynomials labeled by Young diagrams that have a total of p long columns - all that is required is that we fine tune a few phases in our map between partially labeled Young diagrams and vectors in $V_p^{\otimes m}$. The same irreducible representations of $U(p)$ are used for both of these organizations, and further since $d_s = d_{s^T}$, each $U(p)$ representation s appears with the same multiplicity in these two cases[2]. Consequently, the traces computed in the last subsection for labels with p long rows are equal to the values for labels with p long columns. To obtain the action of the dilatation operator all that remains is the computation of the coefficient discussed in (5.1.3). The only quantity appearing in (5.1.3) which is not invariant under exchanging rows and columns is

$$c_{RR'}\sqrt{\frac{f_T}{f_R}} = \sqrt{c_{RR'}c_{TT'}}$$

This factor is the only difference between the case of p long rows and p long columns. Consequently, the action of the dilatation operator on restricted Schur polynomials with p long columns is obtained from its action on restricted Schur polynomials with p long rows by making substitutions of the form $N + b \to N - b$. For concrete examples of this substitution see the end of sections 5.2.1 and 5.2.2.1. This generalizes the two row/column relation observed in [15] to an arbitrary number of rows and columns.

This completes the evaluation of the dilatation operator.

[2] Recall that s^T is obtained by exchanging rows and columns in s.

5.2 Explicit Action of the Dilatation Operator

We can now explicitly evaluate the matrix elements $N_{R,(r,s)jk;T,(t,u)ml}$ of the dilatation operator (5.3). We will do so for the case that the Young diagram labels have either two or three rows or columns.

5.2.1 Young Diagrams with Two Rows or Columns

In this case, we will be using $U(2)$ representation theory. The Gelfand-Tsetlin patterns are extremely useful for understanding the structure of the carrier space of a particular $U(2)$ representation. However, the betweenness conditions make it awkward to work directly with the labels m_{ij} which appear in the pattern. For this reason we will employ a new notation: trade the m_{ij} for j, j^3 specified by

$$\begin{bmatrix} m_{12} & & m_{22} \\ & m_{11} & \end{bmatrix} = \begin{bmatrix} m_{22}+2j & & m_{22} \\ & m_{22}+j^3+j & \end{bmatrix} .$$

The new labels are just the familiar angular momenta we usually use for $SU(2)$. It looks as if this trade in labels is not well defined because we have traded three labels m_{12}, m_{22}, m_{11} for two labels j, j^3. There is no need for concern: recall that m is fixed, and further,

$$m = 2(m_{22}+j)$$

so that knowing j, j^3 and m we can indeed reconstruct m_{12}, m_{22}, m_{11}. The benefit of the new labels is that the betweenness conditions are replaced by

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots \qquad -j \le j^3 \le j$$

which are significantly easier to handle. Write our states as kets $|j, j^3\rangle$. The Clebsch-Gordan coefficients we need are (it's simple to compute these using Appendix B.5)

$$\left\langle j-\frac{1}{2}, j^3-\frac{1}{2}; \frac{1}{2}, \frac{1}{2} | j, j^3 \right\rangle = \sqrt{\frac{j+j^3}{2j}}, \quad \left\langle j+\frac{1}{2}, j^3-\frac{1}{2}; \frac{1}{2}, \frac{1}{2} | j, j^3 \right\rangle = -\sqrt{\frac{j-j^3+1}{2(j+1)}},$$

$$\left\langle j-\frac{1}{2}, j^3+\frac{1}{2}; \frac{1}{2}, -\frac{1}{2} | j, j^3 \right\rangle = \sqrt{\frac{j-j^3}{2j}}, \quad \left\langle j+\frac{1}{2}, j^3+\frac{1}{2}; \frac{1}{2}, -\frac{1}{2} | j, j^3 \right\rangle = \sqrt{\frac{j+j^3+1}{2(j+1)}},$$

which are the same for both cases of R having two long rows or columns. In terms of two long rows, the top two Clebsch-Gordon coefficients $\langle *, *; \frac{1}{2}, \frac{1}{2} | j, j^3 \rangle$

correspond to the box pulled off s to obtain s' coming from the first row of R, with the $j-\frac{1}{2}, j^3-\frac{1}{2}$ term corresponding to this box being pulled off the first row of s, and the $j+\frac{1}{2}, j^3-\frac{1}{2}$ term corresponding to the box being pulled off the second row of s. The lower two terms $\langle *, *; \frac{1}{2}, -\frac{1}{2}|j, j^3\rangle$ correspond to the box pulled off s coming from the second row of R, with the $j-\frac{1}{2}, j^3+\frac{1}{2}$ term corresponding to this box being pulled off the first row of s, and the $j+\frac{1}{2}, j^3+\frac{1}{2}$ term corresponding to the box being pulled off the second row of s.

Consider first the case of two rows. To specify r we will specify the number of columns with 2 boxes ($= b_0$) and the number of columns with a single box ($= b_1$). Thus, our operators are labeled as $O(b_0, b_1, j, j^3)$.

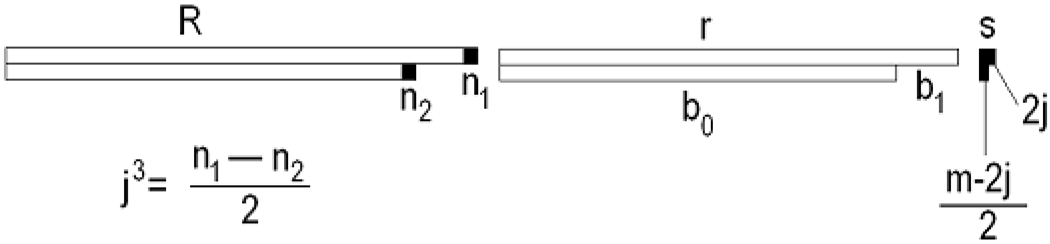


Figure 5.2: This figure summarizes how to translate between the original Young diagram labeling $O_{R,(r,s)}$ and the new $O(b_0, b_1, j, j^3)$ labeling. The boxes that must be removed from R to obtain r have been colored black. The number of boxes to be removed from the i^{th} row of R to obtain r is denoted n_i. The label $j^3 = \frac{n_1-n_2}{2}$. In addition, $m = n_1 + n_2$. The number of columns in r with 2 boxes is b_0 and the number of columns with 1 box is b_1. The number of columns in s with 2 boxes is given by $\frac{m-2j}{2}$ and the number of columns with one box is $2j$.

We will evaluate the diagonal terms where $s = u$ (that is, the terms that don't change the value of j) in detail and simply quote the complete result.

The first contribution to the diagonal terms is when $R = T$, in which case we need to evaluate

$$-\frac{2g_{\text{YM}}^2 c_{RR'} r_k m}{R_k d_s} \sum_{s'} d_{s'} \left[(C^{M_s}_{M_{s'}, M^k_{10}})^2 - (C^{M_s}_{M_{s'}, M^k_{10}})^4 \right] . \tag{5.6}$$

For the case of two rows, there are no multiplicity labels and further for each s' only a single state contributes, so that there is no sum over $M_{s'}$. Consider the contribution obtained when R' is related to R by removing a box from the first row of R ($k = 1$). In this case

$$c_{RR'} = (N + b_0 + b_1)\left(1 + O\left(\frac{n_1}{N + b_0 + b_1}\right)\right), \qquad \frac{r_1}{R_1} = 1 + O\left(\frac{n_1}{b_0 + b_1}\right)$$

and

$$M_{10}^1 \leftrightarrow \left|\frac{1}{2},\frac{1}{2}\right\rangle, \qquad M_s \leftrightarrow \left|j,j^3\right\rangle .$$

When we pull a box from the first row of s to obtain s' we have

$$m\frac{d_{s'}}{d_s}=\frac{\text{hooks}_s}{\text{hooks}_{s'}}=\frac{2j}{2j+1}\frac{m+2j+2}{2}, \qquad M_{s'}=\left|j-\frac{1}{2},j^3-\frac{1}{2}\right\rangle .$$

When we pull a box from the second row of s to obtain s' we have

$$m\frac{d_{s'}}{d_s}=\frac{\text{hooks}_s}{\text{hooks}_{s'}}=\frac{2j+2}{2j+1}\frac{m-2j}{2}, \qquad M_{s'}=\left|j+\frac{1}{2},j^3-\frac{1}{2}\right\rangle .$$

It is now a simple matter to show that (5.6) evaluates to

$$-\frac{g_{\rm YM}^2}{2}\left(m-\frac{(m+2)(j^3)^2}{j(j+1)}\right) . \tag{5.7}$$

The second contribution to the diagonal terms is obtained when $R\neq T$, in which case we need to evaluate

$$\frac{2g_{\rm YM}^2\sqrt{c_{RR'}c_{TT'}}\sqrt{r_w t_x}m}{\sqrt{R_w T_x}d_u}\sum_{s'} d_{s'}(C^{M_s}_{\tilde{M}_{s'},M^2_{\mathbf{10}}})^2(C^{M_s}_{M_{s'},M^1_{10}})^2 . \tag{5.8}$$

When s' is obtained by removing a box from the first row of s we computed $m\frac{d_{s'}}{d_s}$ above and we have

$$(C^{M_s}_{\tilde{M}_{s'},M^2_{\mathbf{10}}})^2(C^{M_s}_{M_{s'},M^1_{\mathbf{10}}})^2=\left\langle j-\frac{1}{2},j^3-\frac{1}{2};\frac{1}{2},\frac{1}{2}|j,j^3\right\rangle^2\left\langle j-\frac{1}{2},j^3+\frac{1}{2};\frac{1}{2},-\frac{1}{2}|j,j^3\right\rangle^2 .$$

When s' is obtained by removing a box from the second row of s we computed $m\frac{d_{s'}}{d_s}$ above and we have

$$(C^{M_s}_{\tilde{M}_{s'},M^2_{\mathbf{10}}})^2(C^{M_s}_{M_{s'},M^1_{\mathbf{10}}})^2=\left\langle j+\frac{1}{2},j^3-\frac{1}{2};\frac{1}{2},\frac{1}{2}|j,j^3\right\rangle^2\left\langle j+\frac{1}{2},j^3+\frac{1}{2};\frac{1}{2},-\frac{1}{2}|j,j^3\right\rangle^2 .$$

It is now easy to show that (5.8) evaluates to

$$\frac{g_{\rm YM}^2}{2}\left(m-\frac{(m+2)(j^3)^2}{j(j+1)}\right) . \tag{5.9}$$

Notice that although they were computed in completely different ways (5.7) and (5.9) are identical up to a sign. Evaluating the off diagonal terms where $s\neq u$, we find that the terms where the first row of u is longer than that of

s are also identical up to a sign, as are the terms where the second row of u is longer than that of s. Consequently, the 9 terms of the dilatation operator can be grouped into three collections of three terms each. Indeed in terms of

$$\begin{aligned}\Delta O(b_0, b_1, j, j^3) &= \sqrt{(N+b_0)(N+b_0+b_1)}(O(b_0+1, b_1-2, j, j^3) + O(b_0-1, b_1+2, j, j^3)) \\ &\quad -(2N+2b_0+b_1)O(b_0, b_1, j, j^3) \ ,\end{aligned}$$

the dilatation operator is

$$\begin{aligned} DO(b_0, b_1, j, j^3) = g_{\text{YM}}^2 \Bigg[&-\frac{1}{2}\left(m - \frac{(m+2)(j^3)^2}{j(j+1)}\right)\Delta O(b_0, b_1, j, j^3) \\ &+ \sqrt{\frac{(m+2j+4)(m-2j)}{(2j+1)(2j+3)}}\frac{(j+j^3+1)(j-j^3+1)}{2(j+1)}\Delta O(b_0, b_1, j+1, j^3) \\ &+ \sqrt{\frac{(m+2j+2)(m-2j+2)}{(2j+1)(2j-1)}}\frac{(j+j^3)(j-j^3)}{2j}\Delta O(b_0, b_1, j-1, j^3)\Bigg] \ . \end{aligned} \tag{5.10}$$

This reproduces the result of [15] and is a nice check of our method. Notice that the dilatation operator does not change the j^3 label of the operator it acts on, which means that the Δ weight of the operator is preserved. This is a consequence of the fact that the $\Gamma(1, m+1)$ factor in D ensures that the box removed comes from the same row of R and r to produce T and t (in the term $\chi_{T,(tu)}$ produced by the action of D on $\chi_{R,(rs)}$). This conclusion only follows in the simplification of Young's representation obtained by considering Young diagrams with row/column separations of $O(N)$ (section C.4 of Appendix C).

Using the results of section 5.1.5 we can immediately obtain the action of the dilatation operator on restricted Schur polynomials with p long columns. Transpose the Young diagram labels. In this case, for example, the number of rows in r with 2 boxes is b_0 and the number of rows with 1 box is b_1, while the number of rows in s with 2 boxes is given by $\frac{m-2j}{2}$ and the number of rows with one box is $2j$. Denote the corresponding normalized operators by

$Q(b_0, b_1, j, j^3)$. The action of the dilatation operator in this case is given by

$$DQ(b_0, b_1, j, j^3) = g_{\rm YM}^2 \Bigg[-\frac{1}{2}\left(m - \frac{(m+2)(j^3)^2}{j(j+1)}\right)\Delta Q(b_0, b_1, j, j^3)$$
$$+ \sqrt{\frac{(m+2j+4)(m-2j)}{(2j+1)(2j+3)}\frac{(j+j^3+1)(j-j^3+1)}{2(j+1)}}\Delta Q(b_0, b_1, j+1, j^3)$$
$$+ \sqrt{\frac{(m+2j+2)(m-2j+2)}{(2j+1)(2j-1)}\frac{(j+j^3)(j-j^3)}{2j}}\Delta Q(b_0, b_1, j-1, j^3)\Bigg] ,$$

where

$$\begin{aligned}\Delta Q(b_0, b_1, j, j^3) &= \sqrt{(N-b_0)(N-b_0-b_1)}(Q(b_0+1, b_1-2, j, j^3) + Q(b_0-1, b_1+2, j, j^3)) \\ &\quad -(2N-2b_0-b_1)Q(b_0, b_1, j, j^3) .\end{aligned}$$

So the sphere giant and AdS giant cases are related by replacing expressions like $N + b_0$ with $N - b_0$.

5.2.2 Young Diagrams with Three Rows or Columns

In this case, we will be using $U(3)$ representation theory. It is again useful to trade the m_{ij} appearing in the Gelfand-Tsetlin patterns for a new set of labels j, k, j^3, k^3, l^3 specified by

$$\begin{bmatrix} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{22} & \\ & & m_{11} & & \end{bmatrix} = \begin{bmatrix} j+k+m_{33} & & k+m_{33} & & m_{33} \\ & j^3+k+m_{33} & & k^3+m_{33} & \\ & & l^3+k^3+m_{33} & & \end{bmatrix} .$$

It again looks like we are trading 5 variables for 6. However, we can again recover the value of m_{33} from the value of m using

$$m = 3m_{33} + 2k + j .$$

The variables satisfy

$$j \geq 0, \quad k \geq 0, \quad j \geq j^3 \geq 0, \quad k \geq k^3 \geq 0, \quad k + j^3 - k^3 \geq l^3 \geq 0,$$

which are again much easier to handle than the betweenness conditions. We will write our states as kets $|j, k, j^3, k^3, l^3\rangle$. The Clebsch-Gordan coefficients we will need are (its simple to compute these using Appendix B.5)

$$\langle j-1, k, j^3, k^3, l^3; m_1 | j, k, j^3, k^3, l^3\rangle = \sqrt{\frac{(j-j^3)(j+k-k^3+1)}{j(j+k+1)}} \equiv f_{\langle m_1 s_1'\rangle}(j, k, j^3, k^3, l^3) ,$$

$$\left\langle j+1,k-1,j^3+1,k^3,l^3;m_1|j,k,j^3,k^3,l^3\right\rangle = \sqrt{\frac{(j^3+1)(k-k^3)}{k(j+2)}} \equiv f_{\langle m_1 s_2'\rangle}(j,k,j^3,k^3,l^3) \ ,$$

$$\left\langle j,k+1,j^3,k^3+1,l^3;m_1|j,k,j^3,k^3,l^3\right\rangle = \sqrt{\frac{(k^3+1)(k+j^3+2)}{(j+k+3)(k+2)}} \equiv f_{\langle m_1 s_3'\rangle}(j,k,j^3,k^3,l^3) \ ,$$

$$\left\langle j-1,k,j^3-1,k^3,l^3;m_2|j,k,j^3,k^3,l^3\right\rangle =$$
$$\sqrt{\frac{(j+k-k^3+1)j^3(k+j^3+1)(j^3-k^3-l^3+k)}{j(j+k+1)(k+j^3-k^3+1)(j^3+k-k^3)}} \equiv f_{\langle m_2 s_1', m_{12}-1\rangle}(j,k,j^3,k^3,l^3) \ ,$$

$$\left\langle j-1,k,j^3,k^3-1,l^3+1;m_2|j,k,j^3,k^3,l^3\right\rangle =$$
$$\sqrt{\frac{(j-j^3)(k-k^3+1)k^3(k+j^3-k^3-l^3+1)}{j(j+k+1)(k+j^3-k^3+1)(k+j^3-k^3+2)}} \equiv f_{\langle m_2 s_1', m_{22}-1\rangle}(j,k,j^3,k^3,l^3) \ ,$$

$$\left\langle j+1,k-1,j^3,k^3,l^3;m_2|j,k,j^3,k^3,l^3\right\rangle =$$
$$-\sqrt{\frac{(k-k^3)(j-j^3+1)(k+j^3+1)(k+j^3-k^3-l^3)}{(j+2)k(j^3+k-k^3+1)(k+j^3-k^3)}} \equiv f_{\langle m_2 s_2', m_{12}-1\rangle}(j,k,j^3,k^3,l^3) \ ,$$

$$\left\langle j+1,k-1,j^3+1,k^3-1,l^3+1;m_2|j,k,j^3,k^3,l^3\right\rangle =$$
$$\sqrt{\frac{(j^3+1)(j+k-k^3+2)k^3(k+j^3-k^3-l^3+1)}{(j+2)k(k+j^3-k^3+1)(k+j^3-k^3+2)}} \equiv f_{\langle m_2 s_2', m_{22}-1\rangle}(j,k,j^3,k^3,l^3) \ ,$$

$$\left\langle j,k+1,j^3-1,k^3+1,l^3;m_2|j,k,j^3,k^3,l^3\right\rangle =$$
$$-\sqrt{\frac{(k^3+1)(j-j^3+1)j^3(k+j^3-k^3-l^3)}{(j+k+3)(k+2)(k+j^3-k^3+1)(k+j^3-k^3)}} \equiv f_{\langle m_2 s_3', m_{12}-1\rangle}(j,k,j^3,k^3,l^3) \ ,$$

$$\left\langle j,k+1,j^3,k^3,l^3+1;m_2|j,k,j^3,k^3,l^3\right\rangle =$$

$$-\sqrt{\frac{(k+j^3+2)(j+k-k^3+2)(k-k^3+1)(k+j^3-k^3-l^3+1)}{(j+k+3)(k+2)(k+j^3-k^3+1)(k+j^3-k^3+2)}} \equiv f_{\langle m_2 s_3', m_{22}-1\rangle}(j,k,j^3,k^3,l^3) \ ,$$

$$\left\langle j-1,k,j^3-1,k^3,l^3-1;m_3|j,k,j^3,k^3,l^3\right\rangle =$$

$$\sqrt{\frac{(j+k-k^3+1)j^3(k+j^3+1)l^3}{j(j+k+1)(k+j^3-k^3+1)(j^3+k-k^3)}} \equiv f_{\langle m_3 s_1', m_{12}-1\rangle}(j,k,j^3,k^3,l^3) \ ,$$

$$\left\langle j-1,k,j^3,k^3-1,l^3;m_3|j,k,j^3,k^3,l^3\right\rangle =$$

$$-\sqrt{\frac{(j-j^3)(k-k^3+1)k^3(l^3+1)}{j(j+k+1)(k+j^3-k^3+1)(k+j^3-k^3+2)}} \equiv f_{\langle m_3 s_1', m_{22}-1\rangle}(j,k,j^3,k^3,l^3) \ ,$$

$$\left\langle j+1,k-1,j^3,k^3,l^3-1;m_3|j,k,j^3,k^3,l^3\right\rangle =$$

$$-\sqrt{\frac{(k-k^3)(j-j^3+1)(k+j^3+1)l^3}{(j+2)k(j^3+k-k^3+1)(k+j^3-k^3)}} \equiv f_{\langle m_3 s_2', m_{12}-1\rangle}(j,k,j^3,k^3,l^3) \ ,$$

$$\left\langle j+1,k-1,j^3+1,k^3-1,l^3;m_3|j,k,j^3,k^3,l^3\right\rangle =$$

$$-\sqrt{\frac{(j^3+1)(j+k-k^3+2)k^3(l^3+1)}{(j+2)k(k+j^3-k^3+1)(k+j^3-k^3+2)}} \equiv f_{\langle m_3 s_2', m_{22}-1\rangle}(j,k,j^3,k^3,l^3) \ ,$$

$$\left\langle j,k+1,j^3-1,k^3+1,l^3-1;m_3|j,k,j^3,k^3,l^3\right\rangle =$$

$$-\sqrt{\frac{(k^3+1)(j-j^3+1)j^3l^3}{(j+k+3)(k+2)(k+j^3-k^3+1)(k+j^3-k^3)}} \equiv f_{\langle m_3 s_3', m_{12}-1\rangle}(j,k,j^3,k^3,l^3) \ ,$$

$$\left\langle j,k+1,j^3,k^3,l^3;m_3|j,k,j^3,k^3,l^3\right\rangle =$$

$$\sqrt{\frac{(k+j^3+2)(j+k-k^3+2)(k-k^3+1)(l^3+1)}{(j+k+3)(k+2)(k+j^3-k^3+1)(k+j^3-k^3+2)}} \equiv f_{\langle m_3 s_3', m_{22}-1\rangle}(j,k,j^3,k^3,l^3)\ ,$$

where

$$m_1 = 1,0,0,0,0, \quad m_2 = 1,0,1,0,0, \quad m_3 = 1,0,1,0,1\ .$$

The first three Clebsch-Gordon coefficients correspond to the box removed from s to obtain s' coming from the first row of R, and either the first, second or third row of s respectively. For the m_2 and m_3 Clebsch-Gordon coefficients, there are two possibilities for each row of s the box is pulled off; either m_{12} or m_{22} can decrease.

Consider first the case of three rows. To specify r we specify the number of columns with three boxes (b_0), the number of columns with two boxes (b_1) and the number of columns with a single box (b_2). Thus, our operators $O(b_1,b_2,j,k,j^3,k^3,l^3)$ carry seven labels. To simplify the notation a little we do not explicitly display b_0 since it is fixed once b_1 and b_2 are chosen by $b_0=(n-b_2-2b_1)/3$. To obtain r from R we remove n_i boxes from each row where

$$n_1 = \frac{m+2j+k-3k^3-3j^3}{3}, \qquad n_2 = \frac{m+k-j+3j^3-3l^3}{3},$$

$$n_3 = \frac{m-j-2k+3l^3+3k^3}{3}\ .$$

We can read j, k and m directly from the Young diagram label s. One might have thought that by employing the above expressions for the n_i one could obtain a formula for j^3,k^3,l^3 in terms of the n_i. This is not possible. Indeed, this conclusion follows immediately upon noting that

$$n_1+n_2+n_3=m\ .$$

The reason why it is not possible to express j^3,k^3,l^3 in terms of the n_i is simply that in all situations where the inner multiplicity is greater than 1, there is no unique j^3,k^3,l^3 given the n_i. When acting on restricted Schur polynomials labeled by Young diagrams with two rows, the dilatation operator preserved the j^3 label of the operator which corresponded to the Δ weight of the operator being preserved. This is true for any number of giant gravitons: The one loop dilatation operator in the displaced corners approximation always preserves the Δ weight of the operator it acts on. Further, the reason why the Δ weight is preserved can again be traced back to the factors of $\Gamma(1,m+1)$ appearing in the dilatation operator and again this conclusion only follows in the displaced corners approximation outlined in section C.4 of Appendix C. For the case of three rows it is simple to give this inner multiplicity a nice characterization: States that belong to the same inner multiplicity multiplet

- Have the same first row in their Gelfand-Tsetlin pattern because they belong to the same $U(3)$ irreducible representation.
- Have the same last row because the Δ weight is conserved.
- Have the same sum of numbers in the second row of the Gelfand-Tsetlin pattern again because the Δ weight is conserved.

This implies that states in the same inner multiplet can be written as

$$\begin{bmatrix} m_{13} & & m_{23} & & m_{33} \\ & m_{12}-i & & m_{22}+i & \\ & & m_{11} & & \end{bmatrix}$$

with different values of i giving the different states, and that the number of states in the inner multiplet is

$$N = \max(m_{12}-m_{11}, m_{12}-m_{23}, \frac{m_{12}-m_{22}}{2}) + \min(m_{13}-m_{12}, m_{22}-m_{33}) + 1,$$

where $\max(a,b,c)$ means take the largest of a,b,c and $\min(a,b)$ means take the smallest of a,b.

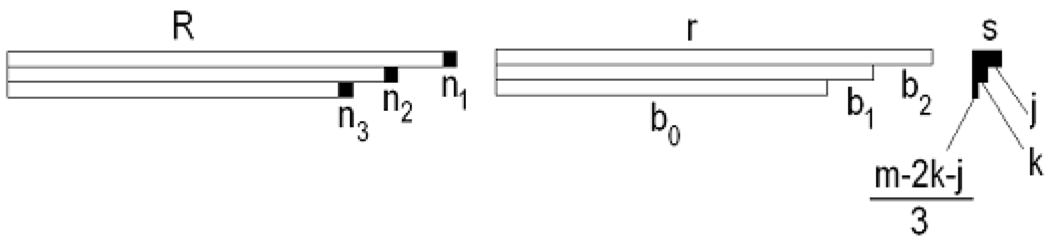


Figure 5.3: This figure summarizes how to translate between the original Young diagram labeling $O_{R,(r,s)}$ and the new $O(b_1,b_2,j,k,j^3,k^3,l^3)$ labeling. The boxes that must be removed from R to obtain r have been colored black. The number of boxes to be removed from the $i^{\rm th}$ row of R to obtain r is denoted n_i. We have $m = n_1+n_2+n_3$. The number of columns in r with 3 boxes is b_0, the number of columns with 2 boxes is b_1 and the number of columns with 1 box is b_2. The number of columns in s with 3 boxes is given by $\frac{m-j-2k}{3}$, the number of columns with two boxes is k and the number of columns with one box is j.

Although the general expression for the action of the dilatation operator can be computed using our methods, we have decided to focus on two special cases. For the first special case we choose $m = 3$ and $\Delta = (1, 1, 1)$ as this is the simplest possible case where we have a non-trivial multiplicity. The second special case greatly simplifies the action of the dilatation operator by considering $j^3 = O(1)$ while assuming the remaining quantum numbers (j, k, k^3, l^3 and m) are all order N.

5.2.2.1 $\Delta = (1, 1, 1)$ States of the $m = 3$ Sector

By applying the above results, it is straight forward to evaluate the action of the dilatation operator for the case that we have 3 Y fields and we set $\Delta = (1, 1, 1)$. There are four possible $U(3)$ states

$$|3,0,2,0,1\rangle \leftrightarrow \begin{bmatrix} 3 & & 0 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix} \qquad |0,0,0,0,0\rangle \leftrightarrow \begin{bmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix}$$

$$|1,1,1,0,1\rangle \leftrightarrow \begin{bmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix} \qquad |1,1,0,1,0\rangle \leftrightarrow \begin{bmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{bmatrix}$$

We see that the last two states belong to an inner multiplicity multiplet. This implies that there are a total of 6 symmetric group operators

$$P_1 = |3,0,2,0,1\rangle \langle 3,0,2,0,1| \qquad P_2 = |0,0,0,0,0\rangle \langle 0,0,0,0,0|$$

$$P_3^{(1,1)} = |1,1,1,0,1\rangle \langle 1,1,1,0,1| \qquad P_3^{(1,2)} = |1,1,1,0,1\rangle \langle 1,1,0,1,0|$$

$$P_3^{(2,1)} = |1,1,0,1,0\rangle \langle 1,1,1,0,1| \qquad P_3^{(2,2)} = |1,1,0,1,0\rangle \langle 1,1,0,1,0|$$

which define 6 restricted Schur polynomials. The corresponding normalized operators will be denoted $O_1(b_1, b_2)$, $O_2(b_1, b_2)$, $O_3(b_1, b_2)$, $O_4(b_1, b_2)$, $O_5(b_1, b_2)$ and $O_6(b_1, b_2)$. We will walk through the calculation of one of the terms of DO_2 and simply quote the complete result of DO_i. Consider the term where $R(r,s)jk = (r, \yng(1,1)) \to T(t,u)ml = (t, \yng(2))12$ and T is obtained from R by moving a box from the second row of R to the first row of R, resulting in $r_2 = t_2 + 1,\ r_1 = t_1 - 1$. In terms of our new labels $b_1, b_2, j, k, j^3, k^3, l^3$ we have $b_1 \to b_1 - 1$, $b_2 \to b_2 + 2$ and $|0,0,0,0,0\rangle \langle 0,0,0,0,0| \to |1,1,1,0,1\rangle \langle 1,1,0,1,0|$.

Notice that the multiplicity labels of $O_{T(t,u)ml}$ are swapped to lm in the trace appearing in $N_{R,(r,s)jk;T,(t,u)ml}$ of (5.3), so the traces we need to evaluate are

$$
\begin{aligned}
&\mathrm{Tr}_{V_p^{\otimes m}}\left(P_{R\to(r,s)jk}E^{(1)}_{aa}P_{T\to(t,u)lm}E^{(1)}_{bb}\right)+\mathrm{Tr}_{V_p^{\otimes m}}\left(P_{R\to(r,s)jk}E^{(1)}_{bb}P_{T\to(t,u)lm}E^{(1)}_{aa}\right)\\
&=\mathrm{Tr}_{V_p^{\otimes m}}\left(P_2E^{(1)}_{11}P_3^{(2,1)}E^{(1)}_{22}\right)+\mathrm{Tr}_{V_p^{\otimes m}}\left(P_2E^{(1)}_{22}P_3^{(2,1)}E^{(1)}_{11}\right)\\
&=\mathrm{Tr}_{V_p^{\otimes m}}\left(|\mathbf{0}\rangle\langle\mathbf{0}|E^{(1)}_{11}\,|1,1,0,1,0\rangle\,\langle 1,1,1,0,1|\,E^{(1)}_{22}\right)\\
&\qquad\qquad\qquad\qquad+\mathrm{Tr}_{V_p^{\otimes m}}\left(|\mathbf{0}\rangle\langle\mathbf{0}|E^{(1)}_{22}\,|1,1,0,1,0\rangle\,\langle 1,1,1,0,1|\,E^{(1)}_{11}\right).
\end{aligned}
$$

To evaluate the first trace, we act with $E^{(1)}_{11}$ on $|1,1,0,1,0\rangle$

$$
\begin{aligned}
E^{(1)}_{11}\,|1,1,0,1,0\rangle &= E^{(1)}_{11}\left|\left[\begin{smallmatrix}2&&1&&0\\&1&&1&\\&&1&&\end{smallmatrix}\right]\right\rangle\\
&= f_{\langle m_1s'_1\rangle}(1,1,0,1,0)\left|\left[\begin{smallmatrix}1&&1&&0\\&1&&1&\\&&1&&\end{smallmatrix}\right]\right\rangle\,,
\end{aligned}
$$

there is no state where s' is obtained from s by removing a box from the second row of s because $\left[\begin{smallmatrix}2&&0&&0\\&1&&1&\\&&1&&\end{smallmatrix}\right]$ is not a valid Gelfand-Tsetlin pattern. Taking the inner product with $\langle\mathbf{0}|=\left\langle\left[\begin{smallmatrix}1&&1&&1\\&1&&1&\\&&1&&\end{smallmatrix}\right]\right|$ we obtain (here $\circledast=1,1,0,1,0$)

$$
\mathrm{Tr}_{V_p^{\otimes m}}\left(\left|\left[\begin{smallmatrix}1&&1&&1\\&1&&1&\\&&1&&\end{smallmatrix}\right]\right\rangle f_{\langle m_1s'_1\rangle}(\circledast)f_{\langle m_1s'_3\rangle}(\mathbf{0})\left\langle\left[\begin{smallmatrix}1&&1&&0\\&1&&1&\\&&1&&\end{smallmatrix}\right]\middle|\left[\begin{smallmatrix}1&&1&&0\\&1&&1&\\&&1&&\end{smallmatrix}\right]\right\rangle\left\langle\left[\begin{smallmatrix}2&&1&&0\\&2&&0&\\&&1&&\end{smallmatrix}\right]\right|E^{(1)}_{22}\right).
$$

When $E^{(1)}_{22}$ acts on $\left\langle\left[\begin{smallmatrix}2&&1&&0\\&2&&0&\\&&1&&\end{smallmatrix}\right]\right|$ it yields (here $\circledcirc=1,1,1,0,1$)

$$
\left\langle\left[\begin{smallmatrix}2&&1&&0\\&2&&0&\\&&1&&\end{smallmatrix}\right]\right|E^{(1)}_{22}=f_{\langle m_2s'_1,m_{12}-1\rangle}(\circledcirc)\left\langle\left[\begin{smallmatrix}1&&1&&0\\&1&&0&\\&&1&&\end{smallmatrix}\right]\right|\ \oplus\ f_{\langle m_2s'_2,m_{12}-1\rangle}(\circledcirc)\left\langle\left[\begin{smallmatrix}2&&0&&0\\&1&&0&\\&&1&&\end{smallmatrix}\right]\right|\,,
$$

writing $\left|\left[\begin{smallmatrix}1&&1&&1\\&1&&1&\\&&1&&\end{smallmatrix}\right]\right\rangle$ as the sum $\sum_{s'}\sum_{M_{\mathbf{10}},M_{s'}}C^{M^i_s}_{M_{s'},M_{\mathbf{10}}}\,|M_{\mathbf{10}}\rangle\otimes|M_{s'},\beta\rangle$ and taking the trace over $V_p^{\otimes m}$ we get

$$
\mathrm{Tr}_{V_p^{\otimes m}}\left(|\mathbf{0}\rangle\langle\mathbf{0}|E^{(1)}_{11}\,|1,1,0,1,0\rangle\,\langle 1,1,1,0,1|\,E^{(1)}_{22}\right)
$$

$$
=d_{s'}f_{\langle m_2s'_3,m_{22}-1\rangle}(\mathbf{0})f_{\langle m_2s'_1,m_{12}-1\rangle}(\circledcirc)f_{\langle m_1s'_1\rangle}(\circledast)f_{\langle m_1s'_3\rangle}(\mathbf{0})=-\frac{1}{3\sqrt{3}}\,,
$$

where $s'=\begin{array}{|c|}\hline\\\hline\\\hline\end{array}$ and so $d_{s'}=1$. The second trace evaluates to zero, so all that is left is the coefficient $g^2_{\mathrm{YM}}\dfrac{c_{RR'}d_Tnm}{d_{R'}d_td_u(n+m)}\sqrt{\dfrac{f_T\,\mathrm{hooks}_T\,\mathrm{hooks}_r\,\mathrm{hooks}_s}{f_R\,\mathrm{hooks}_R\,\mathrm{hooks}_t\,\mathrm{hooks}_u}}\times d^2_{R^{m+1}}$

which reduces to $g^2_{\text{YM}}\dfrac{m}{d_u}\sqrt{\dfrac{\text{hooks}_s}{\text{hooks}_u}} = g^2_{\text{YM}}\dfrac{3}{\sqrt{2}}$ using the results of section 5.1.3. We have thus calculated that

$$DO_2(b_1,b_2) = g^2_{\text{YM}}\frac{1}{\sqrt{6}}O_4(b_1-1,b_2+2)\,.$$

Calculating all the other terms, we find that the action of the dilatation operator is given by

$$DO_i(b_1,b_2) = -g^2_{\text{YM}}\left(M^{(12)}_{ij}\Delta_{12}O_j(b_1,b_2) + M^{(13)}_{ij}\Delta_{13}O_j(b_1,b_2) + M^{(23)}_{ij}\Delta_{12}O_j(b_1,b_2)\right) \tag{5.11}$$

where

$$M^{(12)} = \begin{bmatrix} \frac{2}{3} & 0 & -\frac{2}{3\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 \\ 0 & \frac{2}{3} & 0 & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{2}{3\sqrt{2}} \\ -\frac{2}{3\sqrt{2}} & 0 & \frac{1}{3} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{2\sqrt{3}} & 1 & 0 & \frac{1}{2\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{2\sqrt{3}} & 0 & 1 & \frac{1}{2\sqrt{3}} \\ 0 & -\frac{2}{3\sqrt{2}} & 0 & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & \frac{1}{3} \end{bmatrix}$$

$$M^{(13)} = \begin{bmatrix} \frac{2}{3} & 0 & -\frac{2}{3\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{3\sqrt{2}} \\ -\frac{2}{3\sqrt{2}} & 0 & \frac{1}{3} & \frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & 1 & 0 & -\frac{1}{2\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & 0 & 1 & -\frac{1}{2\sqrt{3}} \\ 0 & -\frac{2}{3\sqrt{2}} & 0 & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & \frac{1}{3} \end{bmatrix}$$

$$M^{(23)} = \begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{3\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{2}{3} & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{3\sqrt{2}} \\ \frac{1}{3\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{5}{6} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{1}{2} & 0 & 0 & \frac{5}{6} \end{bmatrix} .$$

and

$$\Delta_{12} O(b_1, b_2, j, k, j^3, k^3, l^3) = -(2N + 2b_0 + 2b_1 + b_2) O(b_1, b_2, j, k, j^3, k^3, l^3)$$
$$+ \sqrt{(N + b_0 + b_1)(N + b_0 + b_1 + b_2)} \left(O(b_1 - 1, b_2 + 2, j, k, j^3, k^3, l^3) + O(b_1 + 1, b_2 - 2, j, k, j^3, k^3, l^3) \right) ,$$

$$\Delta_{13} O(b_1, b_2, j, k, j^3, k^3, l^3) = -(2N + 2b_0 + b_1 + b_2) O(b_1, b_2, j, k, j^3, k^3, l^3)$$
$$+ \sqrt{(N + b_0)(N + b_0 + b_1 + b_2)} \left(O(b_1 + 1, b_2 + 1, j, k, j^3, k^3, l^3) + O(b_1 - 1, b_2 - 1, j, k, j^3, k^3, l^3) \right) ,$$

$$\Delta_{23} O(b_1, b_2, j, k, j^3, k^3, l^3) = -(2N + 2b_0 + b_1) O(b_1, b_2, j, k, j^3, k^3, l^3)$$
$$+ \sqrt{(N + b_0)(N + b_0 + b_1)} \left(O(b_1 - 2, b_2 + 1, j, k, j^3, k^3, l^3) + O(b_1 + 2, b_2 - 1, j, k, j^3, k^3, l^3) \right) .$$

These $\Delta_{ij} O(b_1, b_2, j, k, j^3, k^3, l^3)$'s are the generalization to $p = 3$ rows of the linear combination $\Delta O(b_0, b_1, j, j^3)$ which featured in the dilatation operator of the two giant system. The combination Δ_{ij} is relevant for terms in the dilatation operator which allow a box to move between rows i and j. It will always be possible to express the action of the dilatation operator in terms of the Δ_{ij} combinations. To see how this comes about, first notice that the terms multiplying (as an example) $(N + b_0 + b_1 + b_2)$ come multiplied by

$$\langle G_1 | E_{11} | G_2 \rangle \langle G_2 | E_{11} | G_1 \rangle ,$$

the terms multiplying $\sqrt{(N + b_0 + b_1 + b_2)(N + b_0 + b_1)}$ come multiplied by

$$\langle G_1 | E_{11} | G_2 \rangle \langle G_2 | E_{22} | G_1 \rangle ,$$

and finally the terms multiplying $\sqrt{(N + b_0 + b_1 + b_2)(N + b_0)}\}$ come multiplied by

$$\langle G_1 | E_{11} | G_2 \rangle \langle G_2 | E_{33} | G_1 \rangle .$$

Using the identity $\mathbf{1} = E_{11} + E_{22} + E_{33}$ and $\langle G_1 | G_2 \rangle = 0$ (for the off diagonal terms in the dilatation operator G_1 and G_2 are by definition different states) we can write the first number above to be minus the sum of the second two. This and similar conditions which follow in the same way are precisely what we need to have a dependence only on the $\Delta_{ij} O(b_1, b_2, j, k, j^3, k^3, l^3)$'s. Note also that this argument generalizes trivially to $p > 3$ rows.

To obtain the action of D on the analogous sphere giant system where $\Delta = (1,1,1)$ we simply make substitutions of the form $N + b_0 \to N - b_0$, so all that changes in (5.11) is that the Δ_{ij}'s become

$$\Delta_{12} Q(b_1, b_2, j, k, j^3, k^3, l^3) = -(2N - 2b_0 - 2b_1 - b_2) Q(b_1, b_2, j, k, j^3, k^3, l^3)$$
$$+ \sqrt{(N - b_0 - b_1)(N - b_0 - b_1 - b_2)} \left(Q(b_1 - 1, b_2 + 2, j, k, j^3, k^3, l^3) + Q(b_1 + 1, b_2 - 2, j, k, j^3, k^3, l^3) \right) ,$$

$$\Delta_{13} Q(b_1, b_2, j, k, j^3, k^3, l^3) = -(2N - 2b_0 - b_1 - b_2) Q(b_1, b_2, j, k, j^3, k^3, l^3)$$
$$+ \sqrt{(N - b_0)(N - b_0 - b_1 - b_2)} \left(Q(b_1 + 1, b_2 + 1, j, k, j^3, k^3, l^3) + Q(b_1 - 1, b_2 - 1, j, k, j^3, k^3, l^3) \right) ,$$

$$\Delta_{23} Q(b_1, b_2, j, k, j^3, k^3, l^3) = -(2N - 2b_0 - b_1) Q(b_1, b_2, j, k, j^3, k^3, l^3)$$
$$+ \sqrt{(N - b_0)(N - b_0 - b_1)} \left(Q(b_1 - 2, b_2 + 1, j, k, j^3, k^3, l^3) + Q(b_1 + 2, b_2 - 1, j, k, j^3, k^3, l^3) \right) .$$

5.2.2.2 $j^3 = O(1)$ Sector:

We assume that the remaining quantum numbers (j, k, k^3, l^3 and m) are all order N. The Clebsch-Gordan coefficients simplify considerably in this limit, indeed the non-zero Clebsch-Gordan coefficients are

$$\langle j-1, k, j^3, k^3, l^3; m_1 | j, k, j^3, k^3, l^3 \rangle = \sqrt{\frac{j + k - k^3}{j + k}},$$

$$\langle j+1, k-1, j^3, k^3, l^3; m_1 | j, k, j^3, k^3, l^3 \rangle = \sqrt{\frac{k - k^3}{kj}},$$

$$\langle j, k+1, j^3, k^3 + 1, l^3; m_1 | j, k, j^3, k^3, l^3 \rangle = \sqrt{\frac{k^3}{j + k}},$$

$$\langle j-1, k, j^3, k^3 - 1, l^3 + 1; m_2 | j, k, j^3, k^3, l^3 \rangle = \sqrt{\frac{k^3 (k - k^3 - l^3)}{(j + k)(k - k^3)}},$$

$$\langle j+1, k-1, j^3, k^3, l^3; m_2 | j, k, j^3, k^3, l^3 \rangle = -\sqrt{\frac{k - k^3 - l^3}{k - k^3}},$$

$$\langle j,k+1,j^3,k^3,l^3+1;m_2|j,k,j^3,k^3,l^3\rangle = -\sqrt{\frac{(j+k-k^3)(k-k^3-l^3)}{(j+k)(k-k^3)}},$$

$$\langle j-1,k,j^3,k^3-1,l^3;m_3|j,k,j^3,k^3,l^3\rangle = -\sqrt{\frac{k^3l^3}{(j+k)(k-k^3)}},$$

$$\langle j+1,k-1,j^3,k^3,l^3-1;m_3|j,k,j^3,k^3,l^3\rangle = -\sqrt{\frac{l^3}{k-k^3}},$$

$$\langle j,k+1,j^3,k^3,l^3;m_3|j,k,j^3,k^3,l^3\rangle = \sqrt{\frac{(j+k-k^3)l^3}{(j+k)(k-k^3)}}.$$

Looking at the non-zero Clebsch-Gordan coefficients, the reason for the simplification of this limit is clear; notice that in the limit that we are considering the j^3 quantum number is fixed. This in turn implies that a single state from each inner multiplicity multiplet participates - a considerable simplification. Indeed, if j,k,m and the Δ weight $\Delta=(n_1,n_2,n_3)$ are given, then we know

$$k^3=\frac{m-3n_1-3j^3+2j+k}{3},\qquad l^3=\frac{m-3n_2+3j^3+k-j}{3}.$$

Thus, after specifying Δ and j^3 the k^3,l^3 labels are not needed. For this reason we can now simplify the notation for our operators to $O(b_1,b_2,j,k)$ for a given problem which is specified by j^3 and Δ[3]. The dilatation operator produces 45 terms when acting on $O(b_1,b_2,j,k)$, which can be grouped into 5 collections of 9 terms each

$$DO(b_1,b_2,j,k) = -g_{\rm YM}^2\Bigg[\frac{k^3(j+k-k^3)(k-k^3-l^3)}{3(j+k)^2(k-k^3)}\Delta^{(a)}\Delta_{12}O(b_1,b_2,j,k)$$
$$+\frac{l^3k^3(j+k-k^3)}{3(j+k)^2(k-k^3)}\Delta^{(a)}\Delta_{13}O(b_1,b_2,j,k)-\frac{l^3k^3(k-k^3-l^3)(j+k-k^3)}{3(j+k)^2(k-k^3)^2}\Delta^{(a)}\Delta_{23}O(b_1,b_2,j,k)$$
$$+\frac{l^3(k-k^3-l^3)(j+k-k^3)}{3(j+k)(k-k^3)^2}\Delta^{(b)}\Delta_{23}O(b_1,b_2,j,k)+\frac{k^3l^3(k-k^3-l^3)}{3(j+k)(k-k^3)^2}\Delta^{(c)}\Delta_{23}O(b_1,b_2,j,k)\Bigg] \tag{5.12}$$

where

[3] The symmetric group operators used to define the restricted Schur polynomials are $P=\sum|j,k,j^3,k^3,l^3\rangle\langle j,k,j^{3\prime},k^{3\prime},l^{3\prime}|$ where we could have $j^3\neq j^{3\prime}$, $k^3\neq k^{3\prime}$, $l^3\neq l^{3\prime}$. For simplicity we consider only the $j^3=j^{3\prime}$ case. It is a simple extension of our analysis to consider the general case.

$$\Delta^{(a)}O(b_1,b_2,j,k) = (2m+j-k)O(b_1,b_2,j,k)$$
$$-\sqrt{(m+2j+k)(m-j-2k)}\,(O(b_1,b_2,j-1,k-1)+O(b_1,b_2,j+1,k+1))$$

$$\Delta^{(b)}O(b_1,b_2,j,k) = (2m-2j-k)O(b_1,b_2,j,k)$$
$$-\sqrt{(m-j-2k)(m-j+k)}\,(O(b_1,b_2,j+1,k-2)+O(b_1,b_2,j-1,k+2))$$

$$\Delta^{(c)}O(b_1,b_2,j,k) = (2m+j+2k)O(b_1,b_2,j,k)$$
$$-\sqrt{(m+2j+k)(m-j+k)}\,(O(b_1,b_2,j-2,k+1)+O(b_1,b_2,j+2,k-1))\ .$$

The new $\Delta^{(a/b/c)}$ operators only change the j,k quantum numbers and so will be the same in the sphere giant case.

Chapter 6

Diagonalization of The Dilatation Operator

We will now diagonalize the action of the one loop dilatation operator on restricted Schur polynomials that we have obtained to determine the anomolous dimensions and corresponding eigenfunctions of the CFT duals of excited giant gravitons.

6.1 Two Giant Systems

The one loop dilatation operator when acting on two giant systems has already been diagonalized in [15]. We start with a quick review of this material because it is relevant for the multiple giant systems we consider next.

The action of the dilatation operator on restricted Schur polynomials of two long rows that we obtained in the previous chapter is

$$\begin{aligned} DO(b_0, b_1, j, j^3) \;&=\; g^2_{\mathrm{YM}}\Big[-h_1(j)\Delta O(b_0, b_1, j, j^3) + h_2(j)\Delta O(b_0, b_1, j+1, j^3) \\ &\quad + h_3(j)\Delta O(b_0, b_1, j-1, j^3)\Big] \;, \end{aligned}$$

where

$$h_1(j) = \frac{1}{2}\left(m - \frac{(m+2)(j^3)^2}{j(j+1)}\right) \;, \qquad h_2(j) = \sqrt{\frac{(m+2j+4)(m-2j)}{(2j+1)(2j+3)}}\,\frac{(j+j^3+1)(j-j^3+1)}{2(j+1)} \;,$$

$$h_3(j) = \sqrt{\frac{(m+2j+2)(m-2j+2)}{(2j+1)(2j-1)}}\,\frac{(j+j^3)(j-j^3)}{2j} \;=\; h_2(j-1)$$

and

$$\begin{aligned} \Delta O(b_0, b_1, j, j^3) \;&=\; \sqrt{(N+b_0)(N+b_0+b_1)}\big(O(b_0+1, b_1-2, j, j^3) + O(b_0-1, b_1+2, j, j^3)\big) \\ &\quad -(2N+2b_0+b_1)O(b_0, b_1, j, j^3) \;. \end{aligned}$$

To solve the eigenproblem

$$DO(p,n) = \kappa O(p,n) \ ,$$

where κ is the one loop anomalous dimension, we make the following ansatz for the operators of good scaling dimension[1]

$$O_{p,n} = \sum_{b_1} f(b_0,b_1)\, O_{p,j^3}(b_0,b_1) \ = \sum_{j,b_1} C_{p,j^3}(j)\, f(b_0,b_1)\, O_{j,j^3}(b_0,b_1) \ .$$

Plugging in the ansatz we get

$$\begin{aligned}\sum_{j,b_1}\Bigg\{ g^2_{\rm YM}\Bigg[& -h_1(j) C_{p,j^3}(j)\, f(b_0,b_1)\, \Delta O_{j,j^3}(b_0,b_1) \\ & + h_2(j) C_{p,j^3}(j)\, f(b_0,b_1)\, \Delta O_{j+1,j^3}(b_0,b_1) \\ & + h_3(j) C_{p,j^3}(j)\, f(b_0,b_1)\, \Delta O_{j-1,j^3}(b_0,b_1)\Bigg] \ = \ \kappa C_{p,j^3}(j)\, f(b_0,b_1)\, O_{j,j^3}(b_0,b_1)\Bigg\} \ .\end{aligned}$$

Equating the coefficient of $O_{j,j^3}(b_0,b_1)$ on both sides, we obtain the recursion relations

$$-\alpha_{p,j^3} C_{p,j^3}(j) = -h_1(j) C_{p,j^3}(j) + h_3(j) C_{p,j^3}(j-1) + h_2(j) C_{p,j^3}(j+1) \ , \quad (6.1)$$

and[2]

$$\begin{aligned}-\alpha_{p,j^3} g^2_{\rm YM}\Bigg[& \sqrt{(N+b_0)(N+b_0+b_1)}\big(f(b_0-1,b_1+2)+f(b_0+1,b_1-2)\big) \\ & -(2N+2b_0+b_1) f(b_0,b_1)\Bigg] = \kappa f(b_0,b_1) \ . \qquad (6.2)\end{aligned}$$

The first recursion relation is solved by

$$C_{p,j^3}(j) = (-1)^{\frac{m}{2}-p}\left(\frac{m}{2}\right)!\sqrt{\frac{(2j+1)}{\left(\frac{m}{2}-j\right)!\left(\frac{m}{2}+j+1\right)!}}\;{}_3F_2\left({{|j^3|-j,\, j+|j^3|+1,\, -p}\atop{|j^3|-\frac{m}{2},\, 1}}\Bigg|1\right) \ , \quad (6.3)$$

which is seen by substituting this solution into (6.1) and obtatining

$$-2p\,{}_3F_2\left({{j^3-j,\, j+1+j^3,\, -p}\atop{1,\, j^3-\frac{m}{2}}}\Bigg|1\right) \ = \ \frac{(j+j^3+1)(j-j^3+1)(m-2j)}{2(j+1)(2j+1)}\,{}_3F_2\left({{-1+j^3-j,\, j+2+j^3,\, -p}\atop{1,\, j^3-\frac{m}{2}}}\Bigg|1\right)$$

[1] $f(b_0,b_1)$ is not a function of b_0 and b_1 separately because $2b_0+b_1$ is fixed equal to the number of Z's.

[2] We have made replacements like $N+b_0+1 \to N+b_0$, which of course are valid in the large N limit.

$$-\left(\frac{m}{2}-\frac{(m+2)(j^3)^2}{2j(j+1)}\right){}_3F_2\left({j^3-j,j+1+j^3,-p \atop 1,j^3-\frac{m}{2}}\Big|1\right)+\frac{(j+j^3)(j-j^3)(m+2j+2)}{2j(2j+1)}{}_3F_2\left({1+j^3-j,j+j^3,-p \atop 1,j^3-\frac{m}{2}}\Big|1\right)$$

which is the recursion relation of the Hahn polynomials ${}_3F_2\left({j^3-j,j+1+j^3,-p \atop 1,j^3-\frac{m}{2}}\Big|1\right)$ (equation (1.5.3) in [48]). The second recursion relation (6.2) is solved by

$$f(b_0,b_1) = (-1)^n(\frac{1}{2})^{N+b_0+\frac{b_1}{2}}\sqrt{\binom{2N+2b_0+b_1}{N+b_0+b_1}\binom{2N+2b_0+b_1}{n}}{}_2F_1\left({-(N+b_0+b_1),-2n \atop -(2N+2b_0+b_1)}\Big|2\right), \tag{6.4}$$

which follows from the recursion relation of the Krawtchouk polynomials ${}_2F_1\left({-(N+b_0+b_1),-2n \atop -(2N+2b_0+b_1)}\Big|2\right)$ (equation (1.10.3) of [48])

$$\begin{aligned}-2n{}_2F_1\left({-(N+b_0+b_1),-2n \atop -(2N+2b_0+b_1)}\Big|2\right) &= \frac{1}{2}(N+b_0){}_2F_1\left({-(N+b_0+b_1+1),-2n \atop -(2N+2b_0+b_1)}\Big|2\right)\\ +\frac{1}{2}(N+b_0+b_1){}_2F_1\left({-(N+b_0+b_1-1),-2n \atop -(2N+2b_0+b_1)}\Big|2\right)&-\frac{1}{2}(2N+2b_0+b_1){}_2F_1\left({-(N+b_0+b_1),-2n \atop -(2N+2b_0+b_1)}\Big|2\right).\end{aligned} \tag{6.5}$$

The solutions (6.3) and (6.4) give the range of j and p to be $|j^3|\le j\le\frac{m}{2}$, $0\le p\le\frac{m}{2}-|j^3|$, and the associated eigenvalues are

$$-\alpha_{p,j^3}=-2p=0,-2,-4,...,-(m-2|j^3|)$$

and

$$\kappa=4n\alpha_{p,j^3}g_{\rm YM}^2=8png_{\rm YM}^2\qquad n=0,1,2,\dots .$$

Since our quantum numbers are very large, one might also consider examining the above recursion relations in a continuum limit where one would expect them to become differential equations. This is indeed the case[15]. Consider first (6.4). Introduce the continuous variable $\rho=\frac{b_1}{2\sqrt{N+b_0}}$ and replace $f(b_0,b_1)$ with $f(\rho)$. Now, expand

$$\sqrt{(N+b_0+b_1)(N+b_0)} = (N+b_0)\left(1+\frac{1}{2}\frac{b_1}{N+b_0}-\frac{1}{8}\frac{b_1^2}{(N+b_0)^2}+....\right)$$

and

$$f\left(\rho-\frac{1}{\sqrt{N+b_0}}\right)=f(\rho)-\frac{1}{\sqrt{N+b_0}}\frac{\partial f}{\partial\rho}+\frac{1}{2(N+b_0)}\frac{\partial^2 f}{\partial\rho^2}+...$$

These expansions are only valid if $b_1\ll N+b_0$, which is certainly not always the case. However, for eigenfunctions with all of their support in the small ρ region the continuum limit of the recursion relation will give accurate answers. The recursion relation becomes

$$\alpha_{p,j^3}g_{\rm YM}^2\left[-\frac{\partial^2}{\partial\rho^2}+\rho^2\right]f(\rho)=\kappa f(\rho) \tag{6.6}$$

which is a harmonic oscillator with frequency $2\alpha_{p,j^3}g_{\rm YM}^2$. We should only keep half of the oscillator states because b_1 and b_0 can never be negative for a valid Young diagram and so $\rho \geq 0$. Only wave functions that vanish at $\rho = 0$ are allowed solutions. Thus, the energy spacing of the half oscillator states is $4\alpha_{p,j^3}g_{\rm YM}^2$. Since each Z field contributes one unit of angular momentum to the giant, the limit of small ρ corresponds to giants that have similar radii. The strings stretching between these giants will have a smaller energy than those stretching between giants that have a larger separation. The description of the coefficients $f(b_0, b_1)$ obtained by solving (6.6) is accurate for any finite energy oscillator eigenstate.

Now consider (6.3) after setting $j^3 = 0$. We would like to take the limit $m \to \infty$, respecting our assumption that the number of Z's is much larger than the number of Y's and that b_1 is large enough that our simplification of Young's orthogonal representation is still valid. We can achieve this by considering a double scaling limit in which we take $m \to \infty$, $b_1 \to \infty$ keeping m/b_1 fixed and very small. In this limit

$$ {}_3F_2\left({}^{-j,j+1,-p}_{-\frac{m}{2},1};1\right) \to L_p\left(\frac{2j^2}{m}\right) $$

where $L_p(\cdot)$ is the Laguerre polynomial. Thus our coefficients

$$ C_p(j) \to (-1)^{\frac{m}{2}-p}\sqrt{\frac{2}{m}}\sqrt{2j+1}e^{-\frac{j^2}{m}}L_p\left(\frac{2j^2}{m}\right) \qquad 0 \leq j \leq \frac{m}{2}\,. $$

become the wave function of the s-wave sector of the 2d radial harmonic oscillator

$$ \frac{1}{2}\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+r^2\right]C_p(r) = (2p+1)C_p(r) $$

where $r = \sqrt{\frac{2}{m}}j$ ranges over $0 \leq r \leq \sqrt{\frac{m}{2}}$ and $C_p(r) = C_p(j)/\sqrt{r}$ in the continuum limit.

A few comments are in order. The solutions of the discrete recursion relations can be compared to the solution of the continuum differential equations. The agreement is perfect[15]. Although the solutions of our discrete recursion relations are in complete agreement with the solution of the corresponding differential equation obtained by taking a continuum limit, notice that the solution of the recursion relation does not make any additional assumptions (such as $b_1 \ll N + b_0$). Thus, although solving the differential equation is easier, the solution is not as general.

6.2 Multiple Giant Systems

Consider now the action of the dilatation operator when acting on three giant systems. We study the $m = 3$ example first.

6.2.1 $\Delta = (1, 1, 1)$ States of the $m = 3$ Sector

It is a simple matter to check that the matrices $M^{(12)}$, $M^{(13)}$ and $M^{(23)}$ appearing in (5.11) commute and hence can be simultaneously diagonalized. The result is the following 6 decoupled equations

$$\begin{aligned}
&DO_I(b_1,b_2) = -2g_{\rm YM}^2\Delta_{23}O_I(b_1,b_2)\,, && DO_{IV}(b_1,b_2) = 0\,,\\
&DO_{II}(b_1,b_2) = -2g_{\rm YM}^2\Delta_{12}O_{II}(b_1,b_2)\,, && DO_V(b_1,b_2) = -g_{\rm YM}^2(\Delta_{23}+\Delta_{12}+\Delta_{13})O_V(b_1,b_2)\,,\\
&DO_{III}(b_1,b_2) = -2g_{\rm YM}^2\Delta_{13}O_{III}(b_1,b_2)\,, && DO_{VI}(b_1,b_2) = -g_{\rm YM}^2(\Delta_{23}+\Delta_{12}+\Delta_{13})O_{VI}(b_1,b_2)\,.
\end{aligned} \tag{6.7}$$

Taking a continuum limit (Appendix D), assuming that $b_1, b_2 \ll N + b_0$ we find

$$\Delta_{23}O(b_1,b_2) \to \left(2\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2 O(x,y) - \frac{x^2}{4}O(x,y)$$

$$\Delta_{12}O(b_1,b_2) \to \left(\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}\right)^2 O(x,y) - \frac{y^2}{4}O(x,y)$$

$$\Delta_{13}O(b_1,b_2) \to \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 O(x,y) - \frac{(x+y)^2}{4}O(x,y)$$

where $x = b_1/\sqrt{N+b_0}$ and $y = b_2/\sqrt{N+b_0}$. These all correspond to oscillators with an energy level spacing of[3] 2. However, again because $b_1, b_2 > 0$ we keep only half the states and hence obtain oscillators with a level spacing of 4. The corresponding eigenvalues of the dilatation operator are $8ng_{\rm YM}^2$ with n an integer. This is remarkably consistent with what we found for the anomalous dimensions for the two giant system. Of course, a very important difference is that since these oscillators live in a two dimensional space, there will be an infinite discrete degeneracy in each level. Finally, it is also straight forward to show that

$$\Delta_{23} + \Delta_{12} + \Delta_{13} = 3\frac{\partial^2}{\partial x^{+2}} - \frac{3}{4}(x^+)^2 + 9\frac{\partial^2}{\partial x^{-2}} - \frac{1}{4}(x^-)^2$$

[3] For example, for the oscillator corresponding to Δ_{12} we have $H = \frac{1}{2}(aa^\dagger + a^\dagger a)$, $[a, a^\dagger] = 2$, $a = \frac{\partial}{\partial x} - 2\frac{\partial}{\partial y} + \frac{y}{2}$ and $a^\dagger = -\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y} + \frac{y}{2}$.

where

$$x^+ = \frac{x+y}{\sqrt{2}}, \qquad x^- = \frac{x-y}{\sqrt{2}} .$$

After rescaling the $x^- \to \sqrt{3}x^-$ we obtain a rotation invariant 2d harmonic oscillator with an energy level spacing of 3. Again because $b_1, b_2 > 0$ we keep only half the states and hence obtain oscillators with a level spacing of 6. The corresponding eigenvalues of the dilatation operator are $6ng^2_{\rm YM}$ with n an integer.

It is interesting to ask if we can diagonalize (6.7) directly without taking a continuum limit, since the resulting spectrum is not computed with the assumption $b_1, b_2 \sim \sqrt{N+b_0}$. Consider first the equation for $O_{II}(b_1, b_2)$. It is clear that Δ_{12} does not change the value of b_0. In addition, the dilatation operator does not change the number of Zs in our operator, so that $n_Z = 3b_0 + 2b_2 + b_1$ is fixed. This motivates the ansatz

$$O = \sum_{b_1} f(b_1, b_2) O_{II}(b_1, b_2)\Big|_{b_2 = n_Z - 3b_0 - 2b_1}$$

Requiring that $DO = 2g^2_{\rm YM}\alpha_n O$ we obtain the recursion relation

$$-(2N+2b_0+2b_1+b_2)f_n(b_1,b_2)+\sqrt{(N+b_0+b_1)(N+b_0+b_1+b_2+1)}f_n(b_1-1,b_2+2)$$
$$+\sqrt{(N+b_0+b_1+1)(N+b_0+b_1+b_2)}f_n(b_1+1,b_2-2) \;=\; 2g^2_{\rm YM}\alpha_n f_n(b_1,b_2)$$

where in the above equation $b_2 = n_Z - 3b_0 - 2b_1$. Using (6.5), it is a simple matter to verify that this recursion relation is solved by

$$f_n \;=\; (-1)^n \left(\frac{1}{2}\right)^{N+b_0+b_1+\frac{b_2}{2}} \sqrt{\binom{2N+2b_0+2b_1+b_2}{N+b_0+b_1+b_2}\binom{2N+2b_0+2b_1+b_2}{n}}\, {}_2F_1\left(\begin{matrix}-(N+b_0+b_1+b_2),-n\\-(2N+2b_0+2b_1+b_2)\end{matrix}\Big| 2\right)$$

$$2g^2_{\rm YM}\alpha_n = 4ng^2_{\rm YM}, \qquad n = 0,1,2,\cdots,\mathrm{int}\left(\frac{n_Z - 3b_0}{2}\right)$$

where n_Z is the number of Zs in the restricted Schur polynomial, b_0 is fixed, $b_2 = n_Z - 3b_0 - 2b_1$ and $\mathrm{int}(\cdot)$ is the integer part of the number in braces. Again, only half the states are retained because $b_1, b_2 > 0$ so that we finally obtain a spacing of $8ng^2_{\rm YM}$ - in perfect agreement with what we found above. Notice that we obtain a set of eigenfunctions for each value of b_0, so that at infinite N we have an infinite degeneracy at each level.

The equation for $O_{III}(b_1, b_2)$ can be solved in the same way. We find

$$f_n(b_0,b_1) = (-1)^n \left(\frac{1}{2}\right)^{N+b_0+\frac{b_1+b_2}{2}} \sqrt{\binom{2N+2b_0+b_1+b_2}{N+b_0+b_1+b_2}\binom{2N+2b_0+b_1+b_2}{n}}\, {}_2F_1\left(\begin{matrix}-(N+b_0+b_1+b_2),-n\\-(2N+2b_0+b_1+b_2)\end{matrix}\Big| 2\right)$$

$$n = 0, 1, ..., \min(J, n_Z - 2J)$$

where $J = b_0 + b_1$ is fixed, $b_2 = n_Z - 3b_0 - 2b_1$ and $\min(a, b)$ is the smallest of the two integers a and b. Only half the states are retained because $b_1, b_2 > 0$ and we again obtain a spacing of $8ng^2_{\rm YM}$. Notice that we obtain a set of eigenfunctions for each value of J, so that at infinite N we again have an infinite degeneracy at each level. For $O_I(b_1, b_2)$ we find

$$f_n(b_0, b_1) = (-1)^n \left(\frac{1}{2}\right)^{N+b_0+\frac{b_1}{2}} \sqrt{\binom{2N+2b_0+b_1}{N+b_0+b_1}\binom{2N+2b_0+b_1}{n}} {}_2F_1\left(\left.{-(N+b_0+b_1), -n \atop -(2N+2b_0+b_1)}\right| 2\right)$$

$$n = 0, 1, ..., \mathrm{int}\left(\frac{n_Z - J}{2}\right)$$

where $J = b_0 + b_1 + b_2$ is fixed and $b_2 = n_Z - 3b_0 - 2b_1$. Only half the states are retained because $b_1, b_2 > 0$ and we again obtain a spacing of $8ng^2_{\rm YM}$. Notice that we obtain a set of eigenfunctions for each value of J, so that at infinite N we again have an infinite degeneracy at each level. It would be interesting to solve the recursion relations arising from $O_V(b_1, b_2)$ and $O_{VI}(b_1, b_2)$. We will not do so here.

6.2.2 $j^3 = O(1)$

We now turn to the $j^3 = O(1)$ example. We have already studied the continuum limit of the operators Δ_{12}, Δ_{13}, and Δ_{23}. In addition to these three operators, we will also need the continuum limit of $\Delta^{(a)}$, $\Delta^{(b)}$ and $\Delta^{(c)}$. Taking $j, k \ll m$ and defining the continuum variables $w = k/\sqrt{m}$, $z = j/\sqrt{m}$ it is straight forward to obtain

$$\Delta^{(a)} O(j,k) \to \left(\frac{\partial}{\partial w} + \frac{\partial}{\partial z}\right)^2 - \frac{9}{4}(z+w)^2$$

$$\Delta^{(b)} O(j,k) \to \left(\frac{\partial}{\partial z} - 2\frac{\partial}{\partial w}\right)^2 - \frac{9}{4}w^2$$

$$\Delta^{(c)} O(j,k) \to \left(\frac{\partial}{\partial w} - 2\frac{\partial}{\partial z}\right)^2 - \frac{9}{4}z^2 \,.$$

These all correspond to oscillators with an energy level spacing of 3. Once again, because $j, k > 0$, only half the states are valid solutions implying a final level spacing of 6. Finally, we need to consider the continuum limit of the coefficients appearing in (5.12). Things simplify very nicely if we focus on those operators for which $\Delta = (n, n, n_3)$ and $n_3 \gg n$. In this case, we find

$$k^3 = l^3 = \frac{m}{3} - n$$

so that after taking the continuum limit (5.12) becomes

$$DO(w,x,y,z) = g_{\rm YM}^2 \frac{(k^3)^2}{3(j+k)^2}\left[9\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2 - \frac{(x-y)^2}{4}\right]\left[\left(\frac{\partial}{\partial w} + \frac{\partial}{\partial z}\right)^2 - 9\frac{(z+w)^2}{4}\right] O(w,x,y,z)$$

which is a direct product of harmonic oscillators! Although many interesting questions could be pursued at this point, we will not do so here.

6.2.3 A $p = 4$ Giant System

Finally, we have studied the action of the dilatation operator when acting on four giant systems. We will report the result for a four giant system with four impurities and $\Delta = (1, 1, 1, 1)$. There are a total of 24 operators that can be defined. The action of the dilatation operator when acting on these 24 operators can be written in terms of (only the labels of the Young diagram for the Zs is shown; the b_i are again the difference in the lengths of the rows)

$$\Delta_{12}O(b_1,b_2,b_3) = -(2N + 2b_0 + 2b_1 + 2b_2 + b_3)O(b_1,b_2,b_3) + \sqrt{(N+b_0+b_1+b_2)(N+b_0+b_1+b_2+b_3)}\,(O(b_1,b_2+1,b_3-2) + O(b_1,b_2-1,b_3+2))\ ,$$

$$\Delta_{13}O(b_1,b_2) = -(2N + 2b_0 + 2b_1 + b_2 + b_3)O(b_1,b_2,b_3) + \sqrt{(N+b_0+b_1)(N+b_0+b_1+b_2+b_3)}\,(O(b_1+1,b_2-1,b_3-1) + O(b_1-1,b_2+1,b_3+1))\ ,$$

$$\Delta_{14}O(b_1,b_2,b_3) = -(2N + 2b_0 + b_1 + b_2 + b_3)O(b_1,b_2,b_3) + \sqrt{(N+b_0)(N+b_0+b_1+b_2+b_3)}\,(O(b_1-1,b_2,b_3-1) + O(b_1+1,b_2,b_3+1))\ ,$$

$$\Delta_{23}O(b_1,b_2,b_3) = -(2N + 2b_0 + 2b_1 + b_2)O(b_1,b_2,b_3) + \sqrt{(N+b_0+b_1)(N+b_0+b_1+b_2)}\,(O(b_1+1,b_2-2,b_3+1) + O(b_1-1,b_2+2,b_3-1))\ ,$$

$$\Delta_{24}O(b_1,b_2,b_3) = -(2N + 2b_0 + b_1 + b_2)O(b_1,b_2,b_3) + \sqrt{(N+b_0)(N+b_0+b_1+b_2)}\,(O(b_1-1,b_2-1,b_3+1) + O(b_1+1,b_2+1,b_3-1))\ ,$$

$$\Delta_{34}O(b_1,b_2,b_3) = -(2N + 2b_0 + b_1)O(b_1,b_2,b_3) + \sqrt{(N+b_0)(N+b_0+b_1)}\,(O(b_1-2,b_2+1,b_3) + O(b_1+2,b_2-1,b_3))\ ,$$

After diagonalizing on the impurity labels we obtain the following decoupled problems: One BPS state

$$DO(b_1, b_2, b_3) = 0 \,, \tag{6.8}$$

six operators with two rows participating

$$DO(b_1, b_2, b_3) = -2g_{\rm YM}^2 \Delta_{ij} O(b_1, b_2, b_3), \quad (ij) = \{(12), (13), (14), (23), (24), (34)\}, \tag{6.9}$$

four doubly degenerate operators with three rows participating (so each equation appears twice) giving eight more operators

$$DO(b_1, b_2, b_3) = -g_{\rm YM}^2 (\Delta_{12} + \Delta_{13} + \Delta_{23}) O(b_1, b_2, b_3), \quad \text{plus 3 more}\,, \tag{6.10}$$

six operators of the type

$$DO(b_1, b_2, b_3) = -g_{\rm YM}^2 (\Delta_{12} + \Delta_{23} + \Delta_{34} + \Delta_{14}) O(b_1, b_2, b_3), \quad \text{plus 5 more}\,, \tag{6.11}$$

and finally three operators of the type

$$DO(b_1, b_2, b_3) = -2g_{\rm YM}^2 (\Delta_{12} + \Delta_{34}) O(b_1, b_2, b_3), \quad \text{plus 2 more}\,. \tag{6.12}$$

The equations (6.8), (6.9) and (6.10) can be solved with a very simple extension of what was done for the three giant system.

Chapter 7

Summary and Important Lessons

In this dissertation we have added to the technology developed in [5, 10, 11, 12, 13, 7, 9, 14, 15] for working with restricted Schur polynomials by describing a new version of Schur-Weyl duality that provides a powerful approach to the computation and manipulation of the symmetric group operators appearing in the restricted Schur polynomials. Using this new technology we have shown that it is straight forward to evaluate the action of the one loop dilatation operator on restricted Schur polynomials. We studied the spectrum of one loop anomalous dimensions on restricted Schur polynomials that have p long columns or rows. For $p = 3, 4$ we have obtained the spectrum explicitly in a number of examples, and have shown that it is identical to the spectrum of decoupled harmonic oscillators. This generalizes results obtained in [9, 14, 15]. The articles [9, 14, 15] provided very strong evidence that the one loop dilatation operator acting on restricted Schur polynomials with two long rows or columns is integrable. In this dissertation we have found evidence that the dilatation operator when acting on restricted Schur polynomials with p long rows or columns is an integrable system. To obtain this action we had to sum much more than just the planar diagrams so that *integrability in $\mathcal{N} = 4$ super Yang-Mills theory is not just a feature of the planar limit, but extends to other large N but non-planar limits.*

The operators we have studied are dual to giant gravitons in the $AdS_5 \times S^5$ background. These giant gravitons have a world volume whose spatial component is topologically an S^3. The excitations of the giant graviton will correspond to vibrational excitations of this S^3. At the quantum level, the energy in any particular vibrational mode will be quantized and consequently, the free theory of giant gravitons should be a collection of decoupled oscillators, which provides a rather natural interpretation of the oscillators we have found.

Giant gravitons are D-branes. Attaching open strings to a D-brane provides a concrete way to describe excitations. Are these open strings visible in our work? Recall that, since the giant graviton has a compact world volume, the Gauss Law implies that the total charge on the giant's world volume must vanish. When enumerating the possible stringy excitation states of a system of giant gravitons, only those states consistent with the Gauss Law should be retained. In [5], restricted Schur polynomials corresponding to giants with "string words" attached were constructed and, remarkably, the number of possible operators that could be defined in the gauge theory matches the number of stringy excitation states of the system of giant gravitons. In this study we have replaced open strings words with impurities Y, which does not modify the counting argument of [5]. Our results add something new and significant to this story: not only does the counting of states match with that expected from the Gauss Law, but, as we now explain, the structure of the action of the dilatation on restricted Schur polynomials itself is closely related to the Gauss Law. Consider the three giant system with $\Delta = (1, 1, 1)$. For this Δ we have three impurities and hence we consider open string configurations with 3 open strings participating. There are three rows in the Young diagrams, corresponding to three giant gravitons. Draw each giant graviton as a solid dot as shown in figure 7.1. The Gauss Law constraint then becomes the condition that there are an equal number of open strings coming to each particular dot as there are leaving the particular dot. We find six possible open string configurations consistent with the Gauss Law as shown in figure 7.1. Our results suggest that the action of the one loop dilatation operator is also coded into these diagrams. For each figure associate a factor of Δ_{ij} for a string stretching between dots i and j[1]. Since $\Delta_{ij} = \Delta_{ji}$, the last two figures shown translate into the same equation, but because the string orientations are different they do represent different states. A string starting and ending on the same dot does not contribute a Δ. Once the complete set of Δ_{ij} are read off the diagram, the action of the dilatation operator is given by summing them and multiplying by $-g_{\rm YM}^2$. Thus, the first diagram shown translates into

$$DO(b_1, b_2) = 0$$

The last two diagrams both give

$$DO(b_1, b_2) = -g_{\rm YM}^2(\Delta_{23} + \Delta_{12} + \Delta_{13})O(b_1, b_2)\,,$$

and the remaining three diagrams give

$$DO(b_1, b_2) = -2g_{\rm YM}^2\Delta_{12}O(b_1, b_2)\,, \qquad DO(b_1, b_2) = -2g_{\rm YM}^2\Delta_{13}O(b_1, b_2)\,,$$

[1] Recall that Δ_{ij} effectively moves boxes betweeen rows i and j.

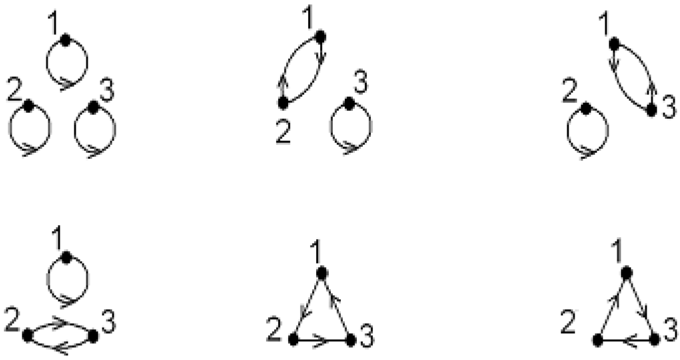


Figure 7.1: A schematic representation of the possible excitations of a three giant system that are consistent with the Gauss Law. Each giant graviton is represented by a labeled point. Lines represent open strings.

$$DO(b_1, b_2) = -2g_{\mathrm{YM}}^2 \Delta_{23} O(b_1, b_2) \, .$$

This is exactly the action we finally obtained in (6.7)! In Appendix E we have given a summary of another detailed computation we have performed: a three giant system with $\Delta = (3, 2, 1)$. The Gauss Law description is again perfect. This connection provides a remarkably simple and general way of describing the action of the one loop dilatation operator in the large N but non-planar limit. We learn that the action of the dilatation operator is given by summing a collection of operators Δ_{ij}, each appearing some integer n_{ij} number of times

$$DO(b_1, b_2) = -g_{\mathrm{YM}}^2 \sum_{ij} n_{ij} \Delta_{ij} \, O(b_1, b_2) \, . \tag{7.1}$$

[58] have proven that the one loop dilatation operator on restricted Schur polynomials in the displaced corners sector always diagonalizes on the impurity labels to this form[2], confirming the Gauss Law connection we have found.

In Appendix D we study the action (7.1) in a natural continuum limit and find it takes the form

$$-g_{\mathrm{YM}}^2 \sum_{ij} n_{ij} \Delta_{ij} \rightarrow g_{\mathrm{YM}}^2 \sum_{I} D_I \left[-\frac{\partial^2}{\partial x_I^2} + \frac{x_I^2}{4} \right] \, .$$

Thus, at one loop and in this continuum limit, the dilatation operator reduces to an infinite set of decoupled oscillators. The open string excitations of the

[2] This is done by identifying the space of excited giant gravitons with a double coset, and then using the Fourier transform on this coset.

p giant graviton system are, at low energy, described by a Yang-Mills theory with $U(p)$ gauge group. It seems natural to identify the $U(p)$ which played a central role in our new Schur-Weyl duality with this gauge group.

Although we have focused on the $SU(2)$ sector of the theory, it is not difficult to add another impurity flavor. Indeed, a remarkable and surprising result of [57] which studied the $p = 2$ case, is the fact that projectors from S_{n+m+q} to $S_n \times S_m \times S_q$ can be constructed by taking a direct product of two $SU(2)$ projectors.

The Gauss Law constraint is an exact statement about the worldvolume physics of giant gravitons. For this reason the connection we have found between the Gauss Law constraint and the action of the one loop dilatation operator should persist to higher loops. In an intense calculation, [59] have verified that the two loop dilatation operator does retain this Gauss Law connection. Clearly despite the enormous number of diagrams that need to be summed to construct this large N but non-planar limit, we are finding evidence that a simple integrable system emerges in the end!

Appendix A

Conformal Symmetry

Conformal transformations rescale the metric

$$g_{\alpha\beta}(x')\frac{\partial x'^{\alpha}}{\partial x^{\mu}}\frac{\partial x'^{\beta}}{\partial x^{\nu}} = \Omega^2(x) g_{\mu\nu}(x) \ . \tag{A.1}$$

For a scale factor $\Omega^2(x) = 1$, the condition above corresponds to the usual Poincaré group (translations, rotations and boosts). Consider now a rescaling

$$x'^{\mu} = \lambda x^{\mu} \ ,$$

if we choose $\Omega^2 = \lambda^2$, then the metric will be invariant under a rescaling! Consequently, conformal field theories cannot have any dimensionful parameters as these would impose a length scale on the theory.

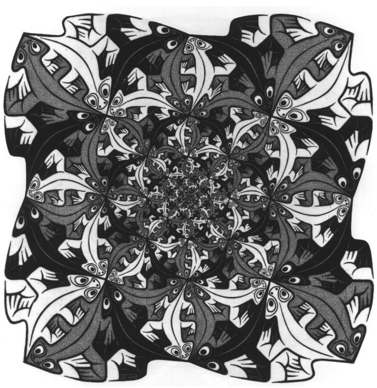


Figure A.1: A fractal pattern illustrating scale inavariance (M. C. Escher).

The only other transformations that will satisfy A.1 are the special conformal transformations

$$x'^{\mu} = \frac{x^{\mu} - b^{\mu}x^2}{1 - 2b\cdot x + b^2x^2} \ ,$$

with $\Omega(x) = (1 + 2b \cdot x + b^2 x^2)^{-2}$. By rewriting the above expression as

$$\frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} - b^\mu$$

we see that the special conformal transformations can be understood as an inversion of x^μ, followed by a translation b^μ, and followed again by an inversion.

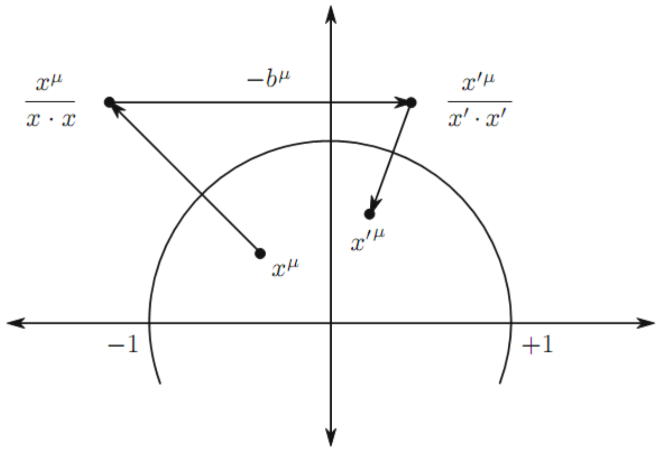


Figure A.2: Special conformal transformation in two dimensions [23].

Appendix B

Elementary Facts from $U(p)$ Representation Theory

In this appendix we collect the background $U(p)$ representation theory needed to understand our construction and diagonalization of the dilatation operator. There are many excellent references for this material. We have found [53, 54] useful. See also [55] for an extremely useful Clebsch-Gordan calculator.

B.1 The Lie Algebra $u(p)$

It is simpler to study the Lie algebra $u(p)$ instead of the group $U(p)$ itself. Most results obtained for representations of $u(p)$ carry over to $U(p)$. In particular, the Clebsch-Gordan coefficients (which play a central role in our construction) of their representations are identical. The structure of the $u(p)$ algebra is easily illustrated using a specific basis. Let E_{ij} with $1 \leq i, j \leq p$ be the matrix

$$(E_{ij})_{rs} = \delta_{ir}\delta_{js}\,,$$

so that it has only one non-zero matrix element. A convenient basis for the Lie algebra is generated by the matrices

$$iE_{kk}, \qquad 1 \leq k \leq p,$$

$$i(E_{k,k-1} + E_{k-1,k}), \qquad E_{k,k-1} - E_{k-1,k}, \qquad 1 < k \leq p\,.$$

$u(p)$ is spanned by real linear combinations of these matrices. The restriction of any irreducible representation of $GL(p, C)$ onto the subgroup $U(p)$ is also irreducible. Thus the carrier space of the irreducible representations of $U(p)$ share the same basis as the irreducible representations of $GL(p, C)$

and consequently, a labeling for $gl(p, C)$ irreducible representations is also a labeling for $u(p)$ irreducible representations.

B.2 Gelfand-Tsetlin Patterns

Gelfand and Tsetlin have introduced a powerful labeling for $u(p)$ irreducible representations and the basis states of their carrier spaces[60]. This labeling chooses basis states that are simultaneous eigenstates of all the matrices $J_z^{(l)}$, and further, explicit formulas are known for the matrix elements of the $J_\pm^{(l)}$ with respect to these basis states. An inequivalent irreducible representation for $GL(p, C)$ is uniquely given by specifying the sequence of p integers

$$\mathbf{m} = (m_{1p}, m_{2p}, \ldots, m_{pp}), \tag{B.1}$$

satisfying $m_{kp} \geq m_{k+1,p}$ for $1 \leq k \leq p-1$. Through out this dissertation we call this sequence the weight of the irreducible representation. The restriction of this irreducible representation onto the subgroup $GL(p-1, C)$ is reducible. It decomposes into a direct sum of $GL(p-1, C)$ irreducible representations with highest weights

$$\mathbf{m}' = (m_{1,p-1}, m_{2,p-1}, \ldots, m_{p-1,p-1}), \tag{B.2}$$

for which the "betweenness" conditions

$$m_{kp} \geq m_{k,p-1} \geq m_{k+1,p} \qquad \text{for} \qquad 1 \leq k \leq p-1$$

hold. The carrier spaces of the $GL(p, C)$ irreducible representations now give rise to (after restricting to the $GL(p-1, C)$ subgroup) $GL(p-1, C)$ irreducible representations. We can keep repeating this procedure until we get to $GL(1, C)$ which has one-dimensional carrier spaces. The Gelfand-Tsetlin labeling exploits this sequence of subgroups to label the basis states using what are called *Gelfand-Tsetlin patterns*. These are triangular arrangements of integers, denoted by M, with the structure

$$M = \begin{bmatrix} m_{1p} & & m_{2p} & \ldots & m_{p-1,p} & & m_{pp} \\ & m_{1,p-1} & & m_{2,p-1} & \cdots & m_{p-1,p-1} & \\ & & \ldots & \ldots & \ldots & & \\ & & m_{12} & & m_{22} & & \\ & & & m_{11} & & & \end{bmatrix}$$

The top row contains the weight that specifies the irreducible representation of the state and the entries of lower rows are subject to the betweenness condition. Thus, the lower rows give the sequence of irreducible representations our state belongs to as we pass through successive restrictions from $GL(p,C)$ to $GL(p-1,C)$ to ... to $GL(1,C)$. The dimension of an irreducible representation with weight $\mathbf{m}$ is equal to the number of valid Gelfand-Testlin patterns having $\mathbf{m}$ as their top row.

B.3 Σ and Δ Weights

We make extensive use of two weights in our construction: Σ weights and Δ weights. Define the row sum

$$\sigma_l(M) = \sum_{k=1}^{l} m_{k,l} \,.$$

The sequence of row sums defines the sigma weight

$$\Sigma(M) = (\sigma_p(M), \sigma_{p-1}(M), \cdots, \sigma_1(M)) \,.$$

The sigma weights do not provide a unique label for the states in the carrier space. Indeed, it is possible that $\Sigma(M) = \Sigma(M')$ but $M \neq M'$. The number of states $\vec{v}(M)$ in the carrier space that have the same Σ weight $\Sigma = \Sigma(M)$ is called the inner multiplicity $I(\Sigma)$ of the state. The inner multiplicity plays an important role in determining how many restricted Schur polynomials can be defined. The Δ weights are defined in terms of differences between row sums

$$\begin{aligned}\Delta(M) &= (\sigma_p(M) - \sigma_{p-1}(M), \sigma_{p-1}(M) - \sigma_{p-2}(M), \cdots, \sigma_1(M) - \sigma_0(M)) \\ &\equiv (\delta_p(M), \delta_{p-1}(M), \cdots \delta_1(M))\end{aligned}$$

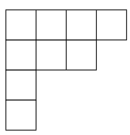
where $\sigma_0 \equiv 0$. We could also ask how many states in the carrier space have the same Δ, denoted $I(\Delta)$. It is clear that $I(\Delta) = I(\Sigma)$. The Δ weights play an important role in determining how the three Young diagram labels $R, (r,s)$ of the restricted Schur polynomials $\chi_{R,(r,s)jk}$ translate into a set of $U(p)$ labels. It tells us how boxes were removed from R to obtain r. Further, the multiplicity labels jk of the restricted Schur polynomial each run over the inner multiplicity.

B.4 Relation between Gelfand-Tsetlin Patterns and Young Diagrams

There is a one-to-one correspondence between Σ weights and Young diagrams, and between Gelfand-Tsetlin patterns and semi-standard Young tableaux. The language of semi-standard Young tableau is a key ingredient in understanding how the three Young diagram labels $R,(r,s)$ of the restricted Schur polynomials $\chi_{R,(r,s)jk}$ translate into the $U(p)$ language, so we will review this connection here. Recall that a Young diagram is an arrangement of boxes in rows and columns in a single, contiguous cluster of boxes such that the left borders of all rows are aligned and each row is not longer than the one above. The empty Young diagram consisting of no boxes is a valid Young diagram. For a $u(p)$ irreducible representation there are at most p rows. Every Young diagram uniquely labels a $u(p)$ irreducible representation. A (semi-standard) Young tableau is a Young diagram, with labeled boxes. The rules for labeling are that each box contains a single integer between 1 and p inclusive, the numbers in each row of boxes weakly increase from left to right (each number is equal to or larger than the one to its left) and the numbers in each column strictly increase from top to bottom (each number is strictly larger than the one above it). The basis states of a $u(p)$ representation identified by a given Young diagram D can be uniquely labeled by the set of all semi-standard Young tableaux. The dimension of a carrier space labeled by a Young diagram is equal to the number of valid Young tableaux with the same shape as the Young diagram. Each Gelfand-Tsetlin pattern M corresponds to a unique Young tableau. We will now explain how to construct the Young tableau given a Gelfand-Tsetlin pattern. Each step in the procedure is illustrated with a concrete example given by the following Gelfand-Tsetlin pattern

$$\begin{bmatrix} 4 & & 3 & & 1 & & 1 \\ & 3 & & 2 & & 1 & \\ & & 3 & & 2 & & \\ & & & 2 & & & \end{bmatrix}.$$

Start with an empty Young diagram (no labels). The first line of the Gelfand-Tsetlin pattern tells you the shape of the Young diagram - m_{in} is the number of boxes in row i. Thus, the information specifying the irreducible representation resides in the topmost row of the pattern. For the example we consider the Young diagram is



.

The last row of the Gelfand-Tsetlin pattern tells us which boxes are labeled with a 1. Imagine superposing the smaller Young diagram defined by the last row of the pattern onto the full Young diagram, so that the topmost and leftmost boxes of the two are identified. Label all boxes of this smaller Young diagram with a 1. For the example we consider

$$\begin{ytableau} 1 & 1 & {} & {} \\ {} & {} & {} \\ {} \\ {} \end{ytableau}\,.$$

The second last row of the pattern tells us which boxes are labeled with a 2. Again superpose the smaller Young diagram defined by the second last row of the pattern onto the full Young diagram and again identify the topmost and leftmost boxes of the two. Label all empty boxes of this smaller Young diagram with a 2. For the example we consider

$$\begin{ytableau} 1 & 1 & 2 & {} \\ 2 & 2 & {} \\ {} \\ {} \end{ytableau}\,.$$

Keep repeating this procedure until you have used the first row to identify the boxes labeled p. The result is a semi-standard Young tableau. The semi standard Young tableau for the example we consider is

$$\begin{ytableau} 1 & 1 & 2 & 4 \\ 2 & 2 & 4 \\ 3 \\ 4 \end{ytableau}\,.$$

The number of boxes containing the number l in tableau row k is given by $m_{kl} - m_{k,l-1}$ and we set $m_{kl} \equiv 0$ if $k > l$. The converse process of transcribing a semi-standard Young tableau to a Gelfand-Tsetlin pattern is now obvious. The components $\delta_l(M)$ of the Δ weight of a Gelfand-Tsetlin pattern M, is the number of boxes containing l in the tableau corresponding to M. Thus, the tableau corresponding to two patterns with the same Δ weight contain the same set of entries (i.e. the same number of l-boxes) but arranged in different ways. One interpretation for the inner multiplicity is that it simply counts the number of ways to arrange the relevant fixed set of entries in the tableau.

B.5 Clebsch-Gordon Coefficients

Let R and S be two irreducible unitary representations of the group $U(p)$. The tensor product of these representations decomposes into a direct sum of irreducible components

$$R \otimes S = \sum_T \oplus \nu(T) T\,. \tag{B.3}$$

In general a particular irreducible representation T can appear more than once in the product $R \otimes S$. The integer $\nu(T)$ indicates the multiplicity of T in this decomposition. For the applications we have in mind, we will need the direct product of an arbitrary representation with weight $\mathbf{m}_n$ with the defining representation which has weight $(1, \mathbf{0})$. In this case all multiplicities are equal to 1 and we need not worry about tracking multiplicities. Use the notation $\mathbf{m}_R$ to denote the weight of irreducible representation R and M_R to denote the Gelfand-Tsetlin pattern for a particular state in the carrier space of this irreducible representation. There are two natural bases for $R \otimes S$. The first is simply obtained by taking the direct product of the states spanning the carrier spaces of R and S. The states in this basis are labeled, using a bra/ket notation, as[1]

$$|\mathbf{m}_R, M_R; \mathbf{m}_S, M_S\rangle \, .$$

The second natural basis is given as a direct sum over the bases of the carrier spaces for the irreducible representations T appearing in the sum on the right hand side of (B.3). The states in this basis are labeled as[2]

$$|\mathbf{m}_T, M_T\rangle$$

where T runs over all irreducible representations appearing in the sum on the right hand side of (B.3). The Clebsch-Gordan coefficients supply the transformation matrix which takes us between the two bases. They are written as the overlap

$$\langle \mathbf{m}_R, M_R; \mathbf{m}_S, M_S | \mathbf{m}_T, M_T\rangle \, .$$

From now on we will drop the R, S, T labels which are actually redundant since the particular irreducible representations we consider are uniquely labeled by the weight which is recorded in the first row of the corresponding Gelfand-Tsetlin patterns. It is known that we can write the Clebsch-Gordan coefficients of $U(p)$ in terms of the Clebsch-Gordan coefficients of $U(p-1)$ as[3]

$$\langle \mathbf{m}_p, M; \mathbf{m}'_p, M' | \mathbf{m}''_p, M''\rangle = \left(\begin{array}{cc|c} \mathbf{m}_p & \mathbf{m}'_p & \mathbf{m}''_p \\ \mathbf{m}_{p-1} & \mathbf{m}'_{p-1} & \mathbf{m}''_{p-1} \end{array} \right) \langle \mathbf{m}_{p-1}, M_1; \mathbf{m}'_{p-1}, M'_1 | \mathbf{m}''_{p-1}, M''_1\rangle.$$

On the right hand side we have the Clebsch-Gordan coefficients of the group $U(p-1)$ and on the left hand side we have the Clebsch-Gordan coefficients

[1]When discussing and using the Clebsch-Gordan coefficients, we prefer to use a bra/ket notation. In our previous notation we could write this basis vector as $\vec{v}(M_R) \otimes \vec{v}(M_S)$.

[2]In general one would also need to include a multiplicity label among the labels for these states.

[3]Again, we are using the fact that for our applications multiple copies of the same representation are absent. In general one needs to worry about multiplicities.

of the group $U(p)$. The weights $\mathbf{m}_p, \mathbf{m}'_p, \mathbf{m}''_p$ label irreducible representations of $U(p)$, while weights $\mathbf{m}_{p-1}, \mathbf{m}'_{p-1}, \mathbf{m}''_{p-1}$ label irreducible representations of $U(p-1)$. The Gelfand-Tsetlin patterns M_1, M'_1 and M''_1 are obtained from M, M' and M'' respectively by removing the first row. Thus, the weights $\mathbf{m}_{p-1}, \mathbf{m}'_{p-1}, \mathbf{m}''_{p-1}$ correspond with the second rows in M, M' and M''. The coefficients $\left(\begin{array}{cc|c} \mathbf{m}_p & \mathbf{m}'_p & \mathbf{m}''_p \\ \mathbf{m}_{p-1} & \mathbf{m}'_{p-1} & \mathbf{m}''_{p-1} \end{array} \right)$ are called the scalar factors of the Clebsch-Gordan coefficients $\langle \mathbf{m}_p, M; \mathbf{m}'_p, M' | \mathbf{m}''_p, M'' \rangle$. Applying the above factorization to the chain of subgroups referenced by the Gelfand-Tsetlin pattern, we obtain

$$\begin{aligned} \langle \mathbf{m}_p, M; \mathbf{m}'_p, M' | \mathbf{m}''_p, M'' \rangle \quad &= \left(\begin{array}{cc|c} \mathbf{m}_p & \mathbf{m}'_p & \mathbf{m}''_p \\ \mathbf{m}_{p-1} & \mathbf{m}'_{p-1} & \mathbf{m}''_{p-1} \end{array} \right) \left(\begin{array}{cc|c} \mathbf{m}_{p-1} & \mathbf{m}'_{p-1} & \mathbf{m}''_{p-1} \\ \mathbf{m}_{p-2} & \mathbf{m}'_{p-2} & \mathbf{m}''_{p-2} \end{array} \right) \times \\ &\quad \times \left(\begin{array}{cc|c} \mathbf{m}_{p-2} & \mathbf{m}'_{p-2} & \mathbf{m}''_{p-2} \\ \mathbf{m}_{p-3} & \mathbf{m}'_{p-3} & \mathbf{m}''_{p-3} \end{array} \right) \cdots \end{aligned}$$

Thus, the Clebsch-Gordan coefficients can be written as a product of scalar factors.

There is a selection rule for the Clebsch-Gordan coefficients. The Clebsch-Gordan coefficients vanish unless

$$\sum_{i=1}^{j} m_{ij} + \sum_{i=1}^{j} m'_{ij} = \sum_{i=1}^{j} m''_{ij} \qquad j = 1, 2, ..., p \, .$$

The only Clebsch-Gordan coefficient that we will need for our applications come from taking the product of some general representation $\mathbf{m}_p$ with the fundamental representation. The weight of the fundamental representation is $(1, 0, ..., 0)$ with $p-1$ 0s appearing. The product we consider has been studied and the following result is known

$$\mathbf{m}_p \otimes (1, \mathbf{0}) = \sum_{i=1}^{m} \mathbf{m}_p^{+i} \, . \tag{B.4}$$

where $\mathbf{m}_p^{+i}$ is obtained from $\mathbf{m}_p$ by replacing m_{ip} by $m_{ip} + 1$. Of course, if this replacement does not lead to a valid Gelfand-Tsetlin pattern there is no corresponding representation. The term with the illegal pattern should be dropped from the right hand side of (B.4). From (B.4) we see that multiple copies of the same irreducible representation are absent on the right hand side. We have made use of this repeatedly in this subsection. These Clebsch-Gordan coefficients factor into products of scalar factors of the form

$$\left(\begin{array}{cc|c} \mathbf{m}_p & (1, \mathbf{0})_p & \mathbf{m}_p^{+i} \\ \mathbf{m}_{p-1} & (1, \mathbf{0})_{p-1} & \mathbf{m}_{p-1}^{+j} \end{array} \right) \qquad \text{or} \qquad \left(\begin{array}{cc|c} \mathbf{m}_p & (1, \mathbf{0})_p & \mathbf{m}_p^{+i} \\ \mathbf{m}_{p-1} & (0, \mathbf{0})_{p-1} & \mathbf{m}_{p-1} \end{array} \right) \, .$$

Explicit formulas for these scalar factors are known

$$\left(\begin{array}{cc|c} \mathbf{m}_p & (1,\mathbf{0})_p & \mathbf{m}_p^{+i} \\ \mathbf{m}_{p-1} & (1,\mathbf{0})_{p-1} & \mathbf{m}_{p-1}^{+j} \end{array}\right) = S(i,j)\left|\frac{\prod_{k\neq j}^{p-1}(l_{k,p-1}-l_{ip}-1)\prod_{k\neq i}^{p}(l_{kp}-l_{j,p-1})}{\prod_{k\neq i}^{p}(l_{kp}-l_{ip})\prod_{k\neq j}^{p-1}(l_{k,p-1}-l_{j,p-1}-1)}\right|^{\frac{1}{2}}$$

$$\left(\begin{array}{cc|c} \mathbf{m}_p & (1,\mathbf{0})_p & \mathbf{m}_p^{+i} \\ \mathbf{m}_{p-1} & (0,\mathbf{0})_{p-1} & \mathbf{m}_{p-1} \end{array}\right) = \left|\frac{\prod_{j=1}^{p-1}(l_{j,p-1}-l_{ip}-1)}{\prod_{j\neq i}^{p}(l_{jp}-l_{ip})}\right|^{\frac{1}{2}}$$

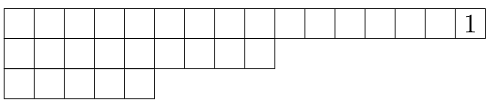
where $l_{sk} = m_{sk} - s$, $S(i,j) = 1$ if $i \le j$ and $S(i,j) = -1$ if $i > j$.

B.6 Explicit Association of labeled Young Diagrams and Gelfand-Tsetlin Patterns

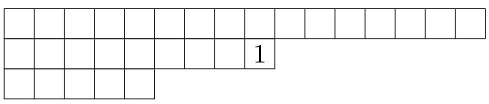
The association we spell out in this section is at the heart of our new Schur-Weyl duality and it demonstrates how we associate an action of $U(p)$ to a Young diagram with p rows or columns. First consider the case of a Young diagram with $O(1)$ rows and $O(N)$ columns. This situation is relevant for the description of *AdS* giant gravitons. We consider Young diagrams in which a certain number of boxes are labeled. To keep the argument general assume that the Young diagram has p rows. These labeled boxes are put into a one-to-one correspondence with p-dimensional vectors. If box i appears in the q^{th} row it is associated to a vector with components

$$\vec{v}(i)_k = \delta_{kq}\,.$$

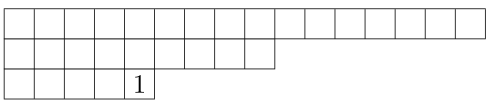
These states live in the carrier space of the fundamental representation of $U(p)$. In this subsection we would like to clearly spell out the Gelfand-Testlin pattern labeling of these vectors. We will spell out our conventions for $U(3)$. The generalization to any p is trivial. Our conventions are



$$\leftrightarrow \quad \begin{bmatrix} 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{bmatrix}$$



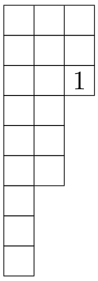
$$\leftrightarrow \quad \begin{bmatrix} 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{bmatrix}$$



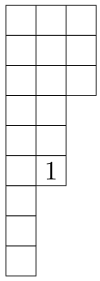
$$\leftrightarrow \quad \begin{bmatrix} 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{bmatrix}$$

The particular label (the 1 in this case) is irrelevant - its the row the label appears in that determines the pattern.

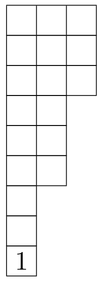
For the case of Young diagrams with $O(N)$ rows and $O(1)$ columns we have



$$\leftrightarrow \begin{bmatrix} 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{bmatrix},$$



$$\leftrightarrow \begin{bmatrix} 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{bmatrix},$$



$$\leftrightarrow \begin{bmatrix} 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{bmatrix}.$$

This situation is relevant for the description of sphere giant gravitons. Note that in addition to specifying the above correspondence between Gelfand-Tsetlin patterns and labeled Young diagrams, one also needs to assign the phases of the different states carefully. For a discussion see section 4.5.

B.7 Last Remarks

A box in row i and column j has a factor equal to $N - i + j$. To obtain the hook length associated to a given box, draw a line starting from the given box towards the bottom of the page until you exit the Young diagram, and another line starting from the same box towards the right until you again exit the diagram. These two lines form an elbow - the hook. The hook length for the given box is obtained by counting the number of boxes the elbow belonging to the box passes through. Here is a Young diagram with the hook lengths filled in

$$\begin{array}{|c|c|c|} \hline 5 & 3 & 1 \\ \hline 3 & 1 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array}$$

For Young diagram R we denote the product of the hook lengths by hooks_R.

Appendix C

Elementary Facts from S_n Representation Theory

The complete set of irreducible representations of S_n are uniquely labeled by Young diagrams with n boxes. From this Young diagram we can construct both a basis for the carrier space of the representation as well as the matrices representing the group elements. We will review these constructions in this Appendix. A useful reference for this material is [56].

C.1 Young-Yamonouchi Basis

A particularly convenient basis for the carrier space of an irreducible representation of the symmetric group is provided by the Young-Yamonouchi basis. The elements of this basis are labeled by numbered Young diagrams - a Young tableau. For a Young diagram with n boxes, each box in the tableau is labeled with a unique integer i with $1 \le i \le n$. In our conventions this numbering is done in such a way that if all boxes with labels less than k with $k < n$ are dropped, a valid Young diagram remains. As an example, if we consider the irreducible representation of S_4 corresponding to

$$\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline\end{array}$$

then the allowed labels are

$$\begin{array}{|c|c|}\hline 4 & 3 \\ \hline 2 & 1 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 4 & 2 \\ \hline 3 & 1 \\ \hline\end{array}\,.$$

Examples of labels that are not allowed include

$$\begin{array}{|c|c|}\hline 4 & 1 \\ \hline 3 & 2 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}\,.$$

For any given Young diagram the number of valid labels is equal to the dimension of the irreducible representation and each label corresponds to a vector in the basis for the carrier space. This basis is orthonormal so that, for example

$$\left\langle \begin{array}{|c|c|}\hline 4 & 3 \\ \hline 2 & 1 \\ \hline\end{array} \middle| \begin{array}{|c|c|}\hline 4 & 3 \\ \hline 2 & 1 \\ \hline\end{array} \right\rangle = 1, \qquad \left\langle \begin{array}{|c|c|}\hline 4 & 3 \\ \hline 2 & 1 \\ \hline\end{array} \middle| \begin{array}{|c|c|}\hline 4 & 2 \\ \hline 3 & 1 \\ \hline\end{array} \right\rangle = 0\,.$$

C.2 Young's Orthogonal Representation

A rule for constructing the matrices representing the elements of the symmetric group is easily given by specifying the action of the group elements on the Young-Yamonouchi basis. The rule is only stated for "adjacent permutations" which correspond to cycles of the form $(i, i+1)$. This is enough because these adjacent permutations generate the complete group. To state the rule it is helpful to associate to each box a factor[1]. The factor of a box in the i^{th} row and the j^{th} column is given by $K - i + j$. Here K is an arbitrary integer that will not appear in any final results. We will denote the factor of the box labeled l by c_l. Let $\hat{T}$ denote a Young tableau corresponding to Young diagram T and let $\hat{T}_{ij}$ denote exactly the same tableau, but with boxes i and j swapped. The rule for the action of the group elements on the basis vectors of the carrier space is

$$\Gamma_T\left((i, i+1)\right) \left|\hat{T}\right\rangle = \frac{1}{c_i - c_{i+1}} \left|\hat{T}\right\rangle + \sqrt{1 - \frac{1}{(c_i - c_{i+1})^2}} \left|\hat{T}_{i,i+1}\right\rangle\,.$$

C.3 Partially labeled Young Diagrams

Consider a Young diagram containing $n+m$ boxes so that it labels an irreducible representations of S_{n+m}. We will often consider "partially labeled" Young diagrams, which are obtained by labeling m boxes. The remaining n boxes are not labeled. We only consider labelings which have the property that if all boxes with labels $\leq i$ are dropped, the remaining boxes are still arranged in a legal Young diagram. We refer to this as a "sensible labeling". What is the interpretation of these partially labeled Young diagrams? To make the discussion concrete, we will develop the discussion using an explicit example. For the example we consider take $n = m = 3$ and use the

[1]This number is also commonly called the "weight" of the box. Here we will refer to it as the factor since we do not want to confuse it with the weight of the Gelfand-Tsetlin pattern.

following partially labeled Young diagram

$$\begin{array}{|c|c|c|}\hline \ & \ & 1 \\ \hline \ & 2 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array} \; . \tag{C.1}$$

If the labeling is completed, this partially labeled diagram will give rise to a number of Young tableau. For our present example two tableau are obtained

$$\begin{array}{|c|c|c|}\hline 6 & 5 & 1 \\ \hline 4 & 2 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 6 & 4 & 1 \\ \hline 5 & 2 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array} \; .$$

Each of these represents a vector in the carrier space of the S_6 irreducible representation labeled by the Young diagram $\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \ \\ \cline{1-2} \ \\ \cline{1-1}\end{array}$. Thus, a partially labeled Young diagram stands for a collection of states. Next, note that the subspace formed by this collection of states is invariant (you don't get transformed out of the subspace) under the action of the S_3 subgroup which acts on the boxes labeled 4,5 and 6. Thus, this subspace is a representation of S_3. In fact, it is easy to see that it is the irreducible representation labeled by $\begin{array}{|c|c|}\hline \ & \ \\ \hline \ \\ \cline{1-1}\end{array}$. This Young diagram can be obtained by dropping all the labeled boxes in (C.1). From this example we can now extract the general rule:

Key Idea: *A partially labeled Young diagram that has $n+m$ boxes, m of which are labeled, stands for a collection of states which furnish the basis for an irreducible representation of $S_n \times (S_1)^m$. The Young diagram that labels the representation of the S_n subgroup is given by dropping all labeled boxes.*

Finally, note that the only representations r that are subduced by R are those with Young diagrams that can be obtained by pulling boxes off R. This follows immediately from the well known subduction rule for the symmetric group which states that an irreducible representation of S_n labeled by Young diagram R with n boxes will subduce all possible representations R'_i of S_{n-1}, where R'_i is obtained by removing any box of R that can be removed such the we are left with a valid Young diagram after removal. Each such irreducible representation of the subgroup is subduced once.

C.4 Simplifying Young's Orthogonal Representation: The Displaced Corners Approximation

In this section we would like to consider a collection of partially labeled Young diagrams. A total of m boxes are labeled, with a unique integer i

$(1 \le i \le m)$ appearing in each box. The set of boxes to be removed are the same for every partially labeled Young diagram. The set of partially labeled Young diagrams we consider is given by including all possible ways in which the m boxes in the Young diagrams can sensibly be labeled. We can consider the action of the S_m subgroup which acts on the labeled boxes. This action will mix these partially labeled Young diagrams. We will consider Young diagrams with p rows built out of $O(N)$ boxes. For the generic operator we consider, the difference in the length between any two rows will be $O(N)$ which we refer to as the 'displaced corners approximation'. If we consider the case $m = \gamma N$ with $\gamma \sim O(N^0) \ll 1$, any two labeled boxes (i and j say) that are not in the same row will have factors that obey $|c_i - c_j| \sim O(N)$. Young's orthogonal representation is particularly useful because it simplifies dramatically in this situation. Indeed, if the boxes i and $i+1$ are in the same row, $i+1$ must sit in the next box to the left of i so that

$$\Gamma_R\left((i,i+1)\right)|\text{same row state}\rangle = |\text{same row state}\rangle\,. \tag{C.2}$$

The same state appears on both sides of this last equation. If i and $i+1$ are in different rows, then $c_i - c_{i+1}$ must itself be $O(N)$. In this case, at large N replace $\frac{1}{c_i - c_{i+1}} = O(b_1^{-1})$ by 0 and $\sqrt{1 - \frac{1}{(c_i - c_{i+1})^2}} = 1 - O(b_1^{-1})$ by 1 so that

$$\Gamma_R\left((i,i+1)\right)|\text{different row state}\rangle = |\text{swapped different row state}\rangle\,. \tag{C.3}$$

The notation in this last equation is indicating two things: i and $i+1$ are in different rows and the states on the two sides of the equation differ by swapping the i and $i+1$ labels. An example illustrating these rules is

$$\Gamma_R\left((1,2)\right)\left|\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline &&&&&&&&&3&2&1\\ \hline &&&&&\multicolumn{7}{c}{}\\ \cline{1-5}\end{array}\right\rangle = \left|\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline &&&&&&&&&3&2&1\\ \hline &&&&&\multicolumn{7}{c}{}\\ \cline{1-5}\end{array}\right\rangle$$

$$\Gamma_R\left((1,2)\right)\left|\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline &&&&&&&&&3&2\\ \hline &&&&1&\multicolumn{6}{c}{}\\ \cline{1-5}\end{array}\right\rangle = \left|\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline &&&&&&&&&3&1\\ \hline &&&&2&\multicolumn{6}{c}{}\\ \cline{1-5}\end{array}\right\rangle\,.$$

We will also consider Young diagrams with p columns built out of $O(N)$ boxes. For the generic operator we consider, the difference in the length between any two columns will be $O(N)$. Since we consider the case $m = \gamma N$ with $\gamma \sim O(N^0) \ll 1$, any two labeled boxes (i and j say) that are not in the same column will again have factors that obey $|c_i - c_j| \sim O(N)$. If the boxes i and $i+1$ are in the same column, $i+1$ must sit above i so that

$$\Gamma_R\left((i,i+1)\right)|\text{same column state}\rangle = -|\text{same column state}\rangle\,. \tag{C.4}$$

The same state appears on both sides of this last equation. If i and $i+1$ are in different columns, then $c_i - c_{i+1}$ must itself be $O(N)$. In this case, at large N again replace $\frac{1}{c_i - c_{i+1}} = O(b_1^{-1})$ by 0 and $\sqrt{1 - \frac{1}{(c_i - c_{i+1})^2}} = 1 - O(b_1^{-1})$ by 1 so that

$$\Gamma_R\left((i, i+1)\right) |\text{different column state}\rangle = |\text{swapped different column state}\rangle \tag{C.5}$$

The notation in this last equation is indicating two things: i and $i+1$ are in different columns and the states on the two sides of the equation differ by swapping the i and $i+1$ labels. An example illustrating these rules is:

$$\Gamma_R\left((1,2)\right) \left| \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline \ & 1 \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1} \ & \multicolumn{1}{c}{} \\ \cline{1-1} \ & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right\rangle = \left| \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline \ & 2 \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1} \ & \multicolumn{1}{c}{} \\ \cline{1-1} \ & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right\rangle \qquad \Gamma_R\left((1,2)\right) \left| \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1} \ & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right\rangle = - \left| \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1} \ & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right\rangle$$

Thus, the representations of the symmetric group simplify dramatically in this limit.

Appendix D

Continuum Limit

In this Appendix we will study the action of Δ_{ij} on a Young diagram with p rows. The row closest to the top of the page is row 1 and the row closest to the bottom of the page is row p. The number of boxes in row i minus the number of boxes in row $i+1$ is given by b_{p-i}. Δ_{ij} exchanges boxes between rows i and j; we always have $i \neq j$. If $|i-j| > 1$ we have

$$\Delta_{ij} O(b_0, ..., b_{p-1}) = -(2N + \sum_{k=0}^{p-j} b_k + \sum_{q=0}^{p-i} b_q) O(b_0, ..., b_{p-1})$$

$$+\sqrt{(N + \sum_{k=0}^{p-j} b_k)(N + \sum_{q=0}^{p-i} b_q)} \Big[O(b_0, ..., b_{p-j}-1, b_{p-j+1}+1, ..., b_{p-i}+1, b_{p-i+1}-1, ..., b_{p-1})$$

$$+O(b_0, ..., b_{p-j}+1, b_{p-j+1}-1, ..., b_{p-i}-1, b_{p-i+1}+1, ..., b_{p-1}) \Big] .$$

It proves convenient to introduce the variables

$$l_i = \sum_{k=1}^{p-i} b_k \qquad i = 1, 2, ..., p-1 .$$

Making the ansatz

$$O = \sum_{b_0, l_i, ..., l_{p-1}} f(b_0, l_1, ..., l_{p-1}) O(b_0, l_1, ..., l_{p-1})$$

for operators of a good scaling dimension, we find

$$\begin{aligned}\Delta_{ij}O &= \sum_{b_0,l_i,\dots,l_{p-1}} f(b_0,l_1,...,l_{p-1})\Delta_{ij}O(b_0,l_1,...,l_{p-1}) \\ &= \sum_{b_0,l_i,\dots,l_{p-1}} \tilde{\Delta}_{ij}f(b_0,l_1,...,l_{p-1})O(b_0,l_1,...,l_{p-1})\end{aligned}$$

where[1]

$$\begin{aligned}\tilde{\Delta}_{ij}f(b_0,l_1,...,l_{p-1}) &= -(2N+2b_0+l_i+l_j)f(b_0,l_1,...,l_{p-1}) \\ &\quad -\sqrt{(N+b_0+l_i)(N+b_0+l_j)}\Big[f(b_0,...,l_i-1,...,l_j+1,...,l_{p-1}) \\ &\quad + f(b_0,...,l_i+1,...,l_j-1,...,l_{p-1})\Big] .\end{aligned}$$

The continuum limit we consider takes $N+b_0\to\infty$ holding the variables

$$x_i=\frac{l_i}{\sqrt{N+b_0}}$$

fixed. Using the expansions

$$\begin{aligned}\sqrt{(N+b_0+l_i)(N+b_0+l_j)} &= N+b_0+\frac{l_i+l_j}{2}-\frac{(l_i-l_j)^2}{8(N+b_0)}+... \\ &= N+b_0+\frac{x_i+x_j}{2}\sqrt{N+b_0}-\frac{(x_i-l_x)^2}{8}+...\end{aligned}$$

and

$$f(b_0,...,l_i-1,...,l_j+1,...)\ \to\ f(b_0,...,x_i-\frac{1}{\sqrt{N+b_0}},...,x_j-\frac{1}{\sqrt{N+b_0}},...)$$

$$\begin{aligned}= f(b_0,...,l_i,...,l_j,...) &- \frac{1}{\sqrt{N+b_0}}\frac{\partial f}{\partial x_i}+\frac{1}{\sqrt{N+b_0}}\frac{\partial f}{\partial x_j}+\frac{1}{2(N+b_0)}\frac{\partial^2 f}{\partial x_i^2} \\ &+\frac{1}{2(N+b_0)}\frac{\partial^2 f}{\partial x_j^2}-\frac{1}{N+b_0}\frac{\partial^2 f}{\partial x_i\partial x_j}+...\end{aligned}$$

we find that in the continuum limit we have

[1] As the reader can easily check, this formula is also true when $|i-j|=1$ i.e. its completely general.

$$\tilde{\Delta}_{ij} f \;=\; \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)^2 f - \frac{(x_i - x_j)^2}{4} f \;=\; m_{ab}\left(\frac{\partial}{\partial x_a}\frac{\partial}{\partial x_b} - \frac{x_a x_b}{4}\right) f\,,$$

where

$$m_{ab} \;=\; \delta_{ai}\delta_{bi} + \delta_{aj}\delta_{bj} - \delta_{ai}\delta_{bj} - \delta_{aj}\delta_{bi}\,.$$

As proved in [58], the action of the dilatation operator in general is given by summing a collection of operators Δ_{ij}, each appearing some integer n_{ij} number of times

$$DO(b_1, b_2) = -g_{\rm YM}^2 \sum_{ij} n_{ij}\Delta_{ij}\, O(b_1, b_2)\,.$$

The result that we obtained above implies that in the continuum limit we have

$$\sum_{ij} n_{ij}\Delta_{ij} \;\to\; M_{ab}\left(\frac{\partial}{\partial x_a}\frac{\partial}{\partial x_b} - \frac{x_a x_b}{4}\right)\,,$$

where the explicite formula for M_{ab} depends on the n_{ij}. In terms of the orthogonal matrix V that diagonalizes M

$$V_{ik} M_{ij} V_{jl} = D_k \delta_{kl}$$

we define the new variable $y_k = V_{ik} x_i$. Written in terms of the new y variables we have

$$\sum_{ij} n_{ij}\Delta_{ij} \;\to\; \sum_a D_a \left(\frac{\partial^2}{\partial y_a^2} - \frac{y_a^2}{4}\right)\,,$$

which is (minus) the Hamiltonian of a set of decoupled oscillators. The D_a's, which are the eigenvalues of M, set the frequencies of the oscillators. For

$$\sum_{ij} n_{ij}\Delta_{ij} = 2\Delta_{12}\,,$$

we have

$$M = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}, \qquad D_1 = 0, \quad D_2 = 4\,.$$

For

$$\sum_{ij} n_{ij}\Delta_{ij} = \Delta_{12} + \Delta_{23} + \Delta_{13}\,,$$

we have

$$M = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \qquad D_1 = 0, \quad D_2 = 3 = D_3 \,.$$

These are perfectly consistent with the results given in chapter 6. One might wonder if the D_i are always integers. This is not the case. Indeed, for

$$\sum_{ij} n_{ij}\Delta_{ij} = \Delta_{12} + \Delta_{23} + \Delta_{34} + ... + \Delta_{1d} \,,$$

we have

$$M = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ . & . & . & . & \cdots & . & . \\ . & . & . & . & \cdots & . & . \\ -1 & 0 & 0 & 0 & \cdots & -1 & 2 \end{bmatrix} .$$

In this case it is rather simple to see that the eigenvalues are

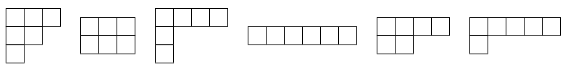
$$D_n = 2 - 2\cos\left(\frac{n\pi}{d}\right), \qquad n = 0, 1, ..., d \,.$$

These are not, in general, integer.

Appendix E

Gauss Law Example

In this Appendix we report the result of the computation of the action of the dilatation operator for restricted Schur polynomials with three rows and $\Delta = (3, 2, 1)$. There are a total of 60 states that can be obtained by removing 6 boxes as specified by the Δ weight. The 6 S_6 irreducible representations that can be suduced are



with the last two irreducible representations being suduced twice. Thus, there are a total of 12 operators that can be defined. After diagonalizing the action of the dilatation operator we find

$$DO = 0 \tag{E.1}$$

$$DO = -2g_{YM}^2 \Delta_{12} O \tag{E.2}$$

$$DO = -2g_{YM}^2 \Delta_{23} O \tag{E.3}$$

$$DO = -2g_{YM}^2 \Delta_{13} O \tag{E.4}$$

$$DO = -2g_{YM}^2 (\Delta_{12} + \Delta_{13}) O \tag{E.5}$$

$$DO = -2g_{YM}^2 (2\Delta_{12} + \Delta_{13}) O \tag{E.6}$$

$$DO = -2g_{YM}^2 (\Delta_{12} + \Delta_{23}) O \tag{E.7}$$

$$DO = -4g_{YM}^2 \Delta_{12} O \tag{E.8}$$

$$DO = -g_{YM}^2 (\Delta_{12} + \Delta_{13} + \Delta_{23}) O \tag{E.9}$$

$$DO = -g_{YM}^2 (\Delta_{13} + 3\Delta_{12} + \Delta_{23}) O \tag{E.10}$$

The last two equations each appear twice. The corresponding diagrams are shown in figure E.1.

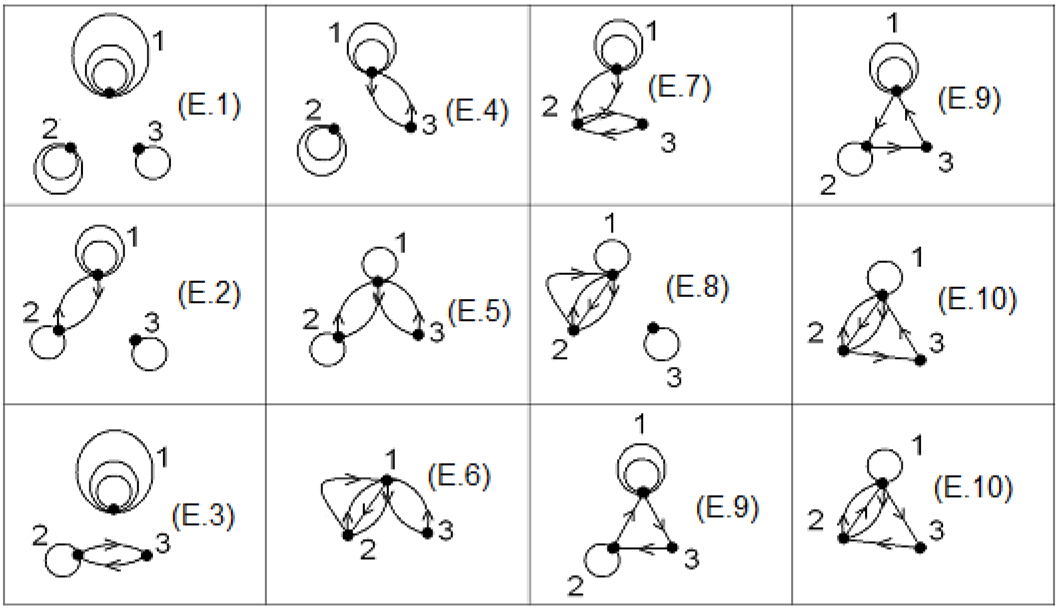


Figure E.1: The open string configurations consistent with the Gauss Law for a three giant system with Δ weight $\Delta = (3, 2, 1)$. The figure labels match the corresponding equation.

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