Coherent Risk Measures and Arbitrage

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under the supervision of

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Declaration

I declare that this thesis is my own, unaided work. It is being submitted for the Degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other university.

Stuart F. Cullender	Date

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Preface

A basic requirement of risk management is the ability to quantify risk. In industry, the most popular tool to do this has been Value-at-Risk. Its simplicity, together with its easy interpretation as the amount of capital required to only have an $\alpha\%$ chance of ruin, made it attractive. As time wore on, it became clear that Value-at-Risk was lacking an important property. It did not respect risk diversification. This has caused much criticism by academics. In their seminal papers [6, 7], Artzner et al. conceived a set of axioms that a reasonable measure of risk should obey. This gave rise to the notion of a coherent risk measure. With a set of axioms in place, much rigorous analysis has been done on the properties of coherent risk measures. As the theory of coherent risk measures deepened, it became apparent that there was a non-trivial overlap with a number of other fields; no-arbitrage pricing theory, convex game theory, convex optimization, insurance risk pricing and utility theory, to name but a few.

In contrast to the theory of coherent risk measures, the problem of pricing and hedging derivatives has long been studied. In 1900, Bachelier [8] was among the first to price derivatives by taking an expectation of the payoff with an appropriate measure. It was only later, in 1973, that Black, Merton and Scholes [9, 45] connected the pricing of derivatives under an equivalent 'risk neutral' measure with the replication and dynamic hedging of derivatives by trading in the underlying asset. It is precisely this connection that made the celebrated Black-Scholes-Merton option pricing model robust enough to become an industrial standard. For the first time, the pricing of securities moved away from relying on the Law of Large Numbers and instead used the economically compelling principle of no-arbitrage. The argument is that the value of a security should be the cost of replicating it. If it were not, it would be possible for an agent to make a risk free profit by taking a position in the security and an opposite position in the replicating strategy.

These developments gave rise to the Fundamental Theorem of Asset Pricing:

Given a financial process S, there are no arbitrage opportunities in the market if and only if there exists an equivalent measure under which S is a martingale.

In this case, we may price a contingent claim on S by taking its expected payoff under this measure. This result was first considered by Ross [51] and later by Harrison, Kreps and Pliska in [29, 28, 43]. Since then, it has been vigorously studied and extended [12, 16, 17]. See [18, 55] for a thorough survey on this subject.

In a complete market model, the equivalent martingale measure is unique, and the value of a derivative is unambiguous. The Black-Scholes-Merton model is an example of this. However, when a model is extended to include market realities, such as transaction and liquidity costs, there are many equivalent martingale measures which preclude arbitrage. This produces an interval of no-arbitrage prices. The infimum and supremum of this price interval respectively represent the lower and upper no-arbitrage price bounds. In practice, these bounds tend to be quite wide, and the question of which price to choose arises. See [36, 37, 22] for the mathematical structure of valuation bounds in incomplete markets.

One approach is to adopt a utility function that specifies an agent's personal preference [30, 31, 41, 13, 25, 24, 40]. The use of a utility function allows the selection of a unique price, or at least tight price bounds, but suffers from being too subjective.

Another approach, taken by Černý and Hodges [63], is to strengthen the condition of no-arbitrage to exclude so called 'good deals'. This leads to good deal bounds that are tighter than no-arbitrage bounds. Jaschke and Küchler [35, 34] then made a vital connection between good deal bounds and coherent risk measures; modulo some technical conditions, they are in one-to-one correspondence. This allows good deal bounds to be specified in terms of an agent's appetite for risk. This approach fills a gap between the preference free no-arbitrage pricing on the one hand, and the utility based pricing on the other. The valuation bounds associated with a coherent risk measure are, to some extent, generic and independent of personal preference, yet tight enough to be used in practice. Jaschke and Küchler go on to prove an abstract version of the Fundamental Theorem of Asset Pricing in terms of a coherent risk measure, using mainly algebraic techniques. This result does not consider martingales and is phrased in terms of the existence of a pricing system being equivalent to the absence of good deals.

In this work, we will consider the Fundamental Theorem of Asset Pricing in the context of coherent risk measures.

We begin with a short survey of coherent risk measures, which is an extended version of [11] and serves as an introductory chapter. Here, our focus will be on exhibiting the shortcomings of Value-at-Risk and studying coherent alternatives for measuring risk. The core of our presentation is a characterization of coherent risk measures, due to Delbaen [15]. This fundamental result is ultimately where the theories of coherent risk measures and no-arbitrage pricing intersect. This characterization allows us to generate a plethora of coherent risk measures. In particular, we will study a popular coherent alternative to Value-at-Risk, called Expected Shortfall [1, 2, 61, 62]. We also consider the class of distortion risk measures, which arise from the Choquet Integral [19]. These risk measures have long been considered in the Actuarial Science literature [69, 67, 66, 64]. Under certain conditions, these risk measures are coherent, and many of the popular coherent risk measures may be represented in this context. This approach also leads to new coherent risk measures, such as the Wang Transform - a risk measure that 'goes beyond coherence' [66]. This risk measure is of particular interest as it may be used to recover the CAPM model, as well as the Black-Scholes-Merton option pricing model.

Next, we examine the Fundamental Theorem of Asset Pricing. We begin with the finite dimensional setting and present the finite discrete time version of this result by Harrison and Pliska [29]. In the infinite dimensional setting, it turns out that a continuous time version of this result does not exist. The condition of no-arbitrage needs to be complemented with a topological notion, which involves taking the closure of the set of super-replicable claims. We present a version of the Fundamental Theorem of Asset Pricing due to Kreps [43], and independently Yan [70], who introduced the stronger condition of no-free-lunch. We achieve this using an argument in the setting of Lindelöf spaces conceived by Rokhlin [50].

It is worth mentioning that a *finite discrete time* version of the Fundamental Theorem of Asset Pricing in terms of the no-arbitrage condition does indeed exist in the infinite dimensional setting, and was proved by Dalang, Morton and Willinger in [12].

We then proceed to the work of Jaschke and Küchler [34], where we present their unified framework. They showed that there is a one-to-one correspondence between the following economic objects:

- Coherent risk measures ρ.
- Cones A of acceptable risks, where $A = \{x : \rho(x) \leq 0\}$.
- Partial preferences $x \succeq y$, meaning that the cash stream x is at least as good as the cash stream y. This can be expressed as $x \succeq y \Leftrightarrow x - y \in A$.
- Valuation bounds $\overline{\pi}$ and $\underline{\pi}$ where $\rho(x) = \overline{\pi}(-x) = -\underline{\pi}(x)$.
- Sets K of admissible price systems given by $\pi \in K \Leftrightarrow \pi(x) \geq 0$ for all $x \succeq 0$.

Using an acceptance set (the cone of acceptable risks mentioned above) determined by a coherent risk measure, they introduce notions of good deals of the first and second kind. Subsequently, a version of the Fundamental Theorem of Asset Pricing characterising the condition of no good deals of the second kind is proved. This results in no good deal valuation bounds that are tighter than no-arbitrage valuation bounds. Since their approach focuses on using mainly algebraic techniques, they do not prove a corresponding result for the stronger condition of no good deals of the first kind.

Using an analogous approach to Kreps and Yan, we introduce the notion of no near-good deals of the first kind. We then prove the following generalisation of the Kreps-Yan Theorem, which is the centerpiece of this work.

Theorem *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X = L^p(\mathbb{P})$, endowed with the norm topology for $1 \leq p < \infty$ and the weak* topology $\sigma(L^\infty(\mathbb{P}), L^1(\mathbb{P}))$ for $p = \infty$.*

Let (M, π) be a market model in X induced by a financial process S and $M_0 = \pi^{-1}(0) \subset M$ be the linear subspace of marketed cashflows at price 0.

Suppose that ρ is a strictly expectation bounded coherent risk measure that is lower semi-continuous, and that $\rho_{\overline{A-M_0}}$ is the market aware risk measure where $A = A_\rho$. Then the following statements are true.

(a) There are no near-good deals of the first kind in the market if and only if there exists $\mathbb{Q} \in \mathcal{M}^e_A(S)$ with $\frac{d\mathbb{Q}}{d\mathbb{P}} \in X^* = L^q(\mathbb{Q})$, $p^{-1} + q^{-1} = 1$, under which S is a martingale.

(b) If there are no near-good deals of the first kind, then

$$\rho_{\overline{A-M}}(z) = \sup\left\{\mathbb{E}^{\mathbb{Q}}[-z] : \mathbb{Q} \in \mathcal{M}^e_A(S)\right\},$$

for all $z \in X$.

(c) *If there are no near-good deals of the first kind, then we have the near-good deal bounds*

$$\begin{aligned}\underline{\pi}_{A,M_0}(z) &= \inf\left\{\mathbb{E}^{\mathbb{Q}}[z] : \mathbb{Q} \in \mathcal{M}^e_A(S)\right\} \text{ and} \\ \overline{\pi}_{A,M_0}(z) &= \sup\left\{\mathbb{E}^{\mathbb{Q}}[z] : \mathbb{Q} \in \mathcal{M}^e_A(S)\right\},\end{aligned}$$

for all $z \in X$. Moreover, these bounds are at least as tight as the no-free-lunch bounds.

The above result is proved in the abstract setting of Lindelöf spaces and applies to popular risk measures, such as Expected Shortfall and the Wang Transform. Using similar techniques, we also recover a partial case of the Fundamental Theorem of Asset Pricing proved by Staum in [59], where he characterizes the slightly weaker condition of no near-arbitrage. Whilst the above result appears to be new, it does not produce tighter price bounds than the results of Staum.

The results in this work draw on elementary techniques from Functional Analysis, Convex Analysis and Duality Theory. To assist the reader who is not familiar with these subjects, we have included an appendix, where the required background and references may be found.

Stuart Cullender
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Chapter 1

A Survey of Coherent Risk Measures

1.1 Preliminaries

Throughout this survey, we will work with a general σ-additive probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The space of (classes of a.e. equal) measurable random variables $X : \Omega \to \mathbb{R}$ is denoted by $L^0(\Omega, \mathcal{F}, \mathbb{P})$, or $L^0(\mathbb{P})$ if there is no confusion possible.

If we equip $L^0(\mathbb{P})$ with the topology of convergence in probability, we reflect a natural mode of convergence therein; it is well known that this topology is not normable. Recall that a sequence of random variables $(X_n) \subset L^0(\mathbb{P})$ converges to X in probability if, for all $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \mathbb{P}[|X_n - X| \geq \varepsilon] = 0.$$

This is equivalent to the condition $\lim_{n\to\infty} \mathbb{E}[|X_n - X| \wedge \mathbf{1}] = 0$.

By $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ or $L^\infty(\mathbb{P})$, we mean the Banach space of all essentially bounded random variables. That is, all $X \in L^0(\mathbb{P})$ for which the norm

$$\|X\|_\infty := \inf\{K > 0 : \mathbb{P}[|X| > K] = 0\}$$

is finite. Notice how this norm does not depend on the underlying probability space.

The Banach spaces of p-integrable random variables are denoted by $L^p(\Omega, \mathcal{F}, \mathbb{P})$ or $L^p(\mathbb{P})$, where $1 \leq p < \infty$. These are the random variables $X \in L^0(\mathbb{P})$ for which the norm

$$\|X\|_p := \left(\int_\Omega |X|^p \,\mathrm{d}\mathbb{P} \right)^{1/p}$$

is finite. By Hölder's Inequality, it follows that

$$L^\infty(\mathbb{P}) \subset L^{p_2}(\mathbb{P}) \subset L^{p_1}(\mathbb{P}) \subset L^0(\mathbb{P})$$

for all $1 \leq p_1 < p_2 < \infty$.

For $1 < p < \infty$, let q be such that $p^{-1}+q^{-1} = 1$. Then the dual space (the space of all bounded linear functionals) of $L^p(\mathbb{P})$, denoted by $L^p(\mathbb{P})^*$, is isometrically isomorphic to $L^q(\mathbb{P})$. The isometry from $L^q(\mathbb{P})$ onto $L^p(\mathbb{P})^*$ is given by the mapping $g \mapsto f_g^*$ defined by

$$\langle f, f_g^* \rangle = \int_\Omega fg \,\mathrm{d}\mathbb{P}$$

for all $f \in L^p(\mathbb{P})$. Hölder's Inequality shows that this mapping is well defined and the Radon-Nikodým Theorem ensures that this mapping is surjective. In the case $p = 1$, the dual of $L^1(\mathbb{P})$ is $L^\infty(\mathbb{P})$.

The spaces $L^p(\mathbb{P})$ are reflexive for $1 < p < \infty$. That is, the canonical isometry $L^p(\mathbb{P}) \hookrightarrow L^p(\mathbb{P})^{**}$ is surjective so that $L^p(\mathbb{P}) = L^p(\mathbb{P})^{**}$. The spaces $L^1(\mathbb{P})$ and $L^\infty(\mathbb{P})$ are not reflexive. The dual of $L^\infty(\mathbb{P})$ is the Banach space $\mathbf{ba}(\Omega, \mathcal{F}, \mathbb{P})$ of all finitely additive measures μ of bounded variation on $(\Omega, \mathcal{F})$ such that $\mu(E) = 0$ when $\mathbb{P}(E) = 0$ for all $E \in \mathcal{F}$. Here, $\mathbf{ba}(\Omega, \mathcal{F}, \mathbb{P})$ is equipped with the variation norm defined by $\|\mu\| = |\mu|(\Omega) := \sup_{\mathcal{P}} \sum_{A \in \mathcal{P}} |\mu(A)|$, where the supremum is taken over all measurable, finite, pairwise-disjoint partitions $\mathcal{P}$ of Ω. We will use the notation $\mathbf{ba}(\mathbb{P})$ if there is no chance of confusion. Consequently, we have the canonical embedding $L^1(\mathbb{P}) \hookrightarrow \mathbf{ba}(\mathbb{P})$, which is not surjective. It can also be shown that $L^1(\mathbb{P})$ has no predual; i.e. it is not the dual of any normed space.

We shall, via the Radon-Nikodým Theorem, tacitly associate countably additive measures $\mathbb{Q}$ such that $\mathbb{Q} \ll \mathbb{P}$ with their corresponding densities $f := \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ in $L^1(\mathbb{P})$. In order for $\mathbb{Q}$ to be a probability measure, we must have $f \geq 0$ and $\mathbb{E}[f] = 1$. In the case where $\mathbb{P}[f > 0] = 1$, the induced measure $\mathbb{Q}$ is equivalent to $\mathbb{P}$.

We conclude this section with a characterization of convex cones $C \subset L^\infty(\mathbb{P})$ that are weak* closed. The Krein-Sḿulian Theorem says that for a Banach space X, a convex set $C \subset X^*$ is weak* closed if and only if $C \cap \lambda\, \mathrm{ball}(X^*)$ is weak* closed for all λ. When $X = L^\infty(\mathbb{P})$ and C is a cone, this result may be extended as follows (cf. [18, Proposition 5.2.4]).

Proposition 1.1.1 *Let C be a convex cone in $L^\infty(\mathbb{P})$. The following statements are equivalent:*

(a) C is weak closed.*

(b) $C \cap \mathrm{ball}(L^\infty(\mathbb{P}))$ is weak closed.*

(c) $C \cap \mathrm{ball}(L^\infty(\mathbb{P}))$ is $\|\cdot\|_p$-closed for every $0 < p < \infty$.

(d) $C \cap \mathrm{ball}(L^\infty(\mathbb{P}))$ is $\|\cdot\|_p$-closed for some $0 < p < \infty$.

(e) $C \cap \mathrm{ball}(L^\infty(\mathbb{P}))$ is closed with respect to the topology of convergence in probability.

In the literature, the above result is attributed to Grothendieck in [27, Part 4, Chapter 5, Exercise 1].

1.2 Introduction to Risk Measurement

1.2.1 The Risk Measurement Model

There are a myriad of risks a financial institution may face [68]. A financial institution's ability to accurately measure its market risk is central to determining capital adequacy requirements. *Market risk* is defined as the potential for unexpected change in a financial position due to fluctuations in market prices.

Market risk lends itself more naturally to quantification than other types of risk. For this reason, we will only concern ourselves with market risk in this work.

We consider a simple one-step model consisting of two time points; today and some point in the future. Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and denote the random variable of future profits and losses (P&L) by $X : \Omega \to \mathbb{R}$. We will sometimes refer to X as a *position*. For simplicity, it is assumed that all random profits and losses in the future have been discounted to today. This is equivalent to assuming that the risk free interest rate is zero.

Our aim is to associate with a given P&L distribution of a portfolio X, a number $\rho(X)$ that represents the risk of the position. This number can represent the 'value at risk' or 'capital requirement' or 'margin' required to hold the position X. We formalize this with a definition:

Definition 1.2.1 (Risk Measure) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{A} \subset L^0(\mathbb{P})$. Then $\rho : \mathcal{A} \to \mathbb{R} \cup \{\infty\}$ is called a *risk measure.*

The properties we impose on ρ determine the size of $\mathcal{A}$. Obviously, the more restrictions we place on a risk measure, the smaller its domain will become. In order for a risk measure to be sufficiently useful, we at least require that $L^\infty(\mathbb{P}) \subseteq \mathcal{A}$.

1.2.2 Quantiles

We recall the notion of a quantile and related elementary results.

Definition 1.2.2 (Quantiles) Let $X \in L^0(\mathbb{P})$ and $\alpha \in (0, 1)$.

(a) x is called an α-quantile of X if $\mathbb{P}[X \leq x] = \alpha$.

(b) The *lower α-quantile* of X is defined by

$$x_{(\alpha)} = q_\alpha(X) = \inf\{x \in \mathbb{R} : \mathbb{P}[X \leq x] \geq \alpha\}.$$

(c) The *upper α-quantile* of X is defined by

$$x^{(\alpha)} = q^\alpha(X) = \inf\{x \in \mathbb{R} : \mathbb{P}[X \leq x] > \alpha\}.$$

It follows from the definition that $x_{(\alpha)} \leq x^{(\alpha)}$. Observe that we may write

$$x_{(\alpha)} = \sup\{x \in \mathbb{R} : \mathbb{P}[X \leq x] < \alpha\}$$

and

$$x^{(\alpha)} = \sup\{x \in \mathbb{R} : \mathbb{P}[X \leq x] \leq \alpha\}.$$

We also have the following proposition, the proof of which is an easy exercise.

Proposition 1.2.3 *Let $X \in L^0(\mathbb{P})$ and $\alpha \in (0, 1)$ then*

(a) $q^\alpha(X) = -q_{1-\alpha}(-X)$ *and* $q_\alpha(X) = -q^{1-\alpha}(-X)$.

(b) $x_{(\alpha)} = x^{(\alpha)}$ *if and only if* $\mathbb{P}[X \le x] = \alpha$ *for at most one* x*, i.e. there is at most one* α*-quantile* x*.*

(c) In the case where there is no α*-quantile* x *associated with* α*, we have*

$$\mathbb{P}[X = x_{(\alpha)}] = \mathbb{P}[X = x^{(\alpha)}] > 0$$

and

$$\mathbb{P}[X \le x_{(\alpha)}] = \mathbb{P}[X \le x^{(\alpha)}] > \alpha.$$

(d) If $x_{(\alpha)} < x^{(\alpha)}$ *then*

$$\{x \in \mathbb{R} : \mathbb{P}[X \le x] = \alpha\} = \begin{cases} [x_{(\alpha)}, x^{(\alpha)}), & \mathbb{P}[X = x^{(\alpha)}] > 0; \\ [x_{(\alpha)}, x^{(\alpha)}], & \mathbb{P}[X = x^{(\alpha)}] = 0. \end{cases}$$

We are now prepared to define Value-at-Risk.

1.2.3 Value-at-Risk

One of the most popular risk measures used in industry is called Value-at-Risk (VaR). This risk measure seeks to answer the following question:

Given a profit and loss distribution of a portfolio X*, what is the minimum loss incurred in* $\alpha\%$ *of the worst cases?*

The number $\alpha \in (0, 1)$ is known as the *significance level* and is usually set to a small value (e.g. $\alpha = 0.05$ or $\alpha = 0.01$). Let us denote the answer to this question by $\mathrm{VaR}^{\alpha}(X)$. Then, if the capital requirement of the position X is set to $\mathrm{VaR}^{\alpha}(X)$, the probability of ruin is no greater than α. Thus, VaR is a risk measure that is only concerned with the frequency of a disaster, not with the extent of it. In view of this, VaR can be expressed in terms of a quantile.

The above shows that for $\alpha \in (0, 1)$, we may have many corresponding α-quantiles, or none at all. Thus, choosing a definition for VaR is not obvious. We follow [15] with the following definition. See Figure 1.2 for an illustration.

Definition 1.2.4 (Value-at-Risk) Let $X : \Omega \to \mathbb{R}$ denote the random variable of profits and losses of some portfolio. Then, for $\alpha \in (0, 1)$, we define *Value-at-Risk at level* α to be the quantity

$$\mathrm{VaR}^{\alpha}(X) = -q^{\alpha}(X) = -x^{(\alpha)}.$$

The position X is said to be *acceptable* if $\mathrm{VaR}^{\alpha}(X) \leq 0$.

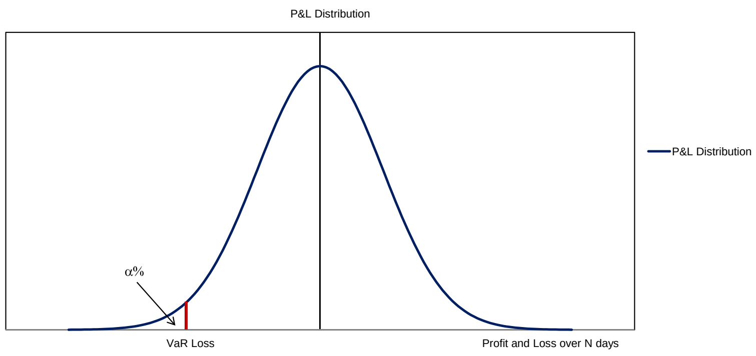

Figure 1.1: VaR *at a significance level of* α *on a normally distributed P&L distribution.*

The above definition can also be expressed as $\mathrm{VaR}^{\alpha}(X) = q_{1-\alpha}(-X)$. A positive VaR represents the extra capital required in order to reduce the probability of ruin to α. A negative VaR implies that capital may be withdrawn from the position or that more risk can be added to the position. We gather some important properties of VaR in the following proposition. The proof is straight forward.

Proposition 1.2.5 *Let* $X, Y \in L^0(\mathbb{P})$ *and* $\alpha \in (0,1)$. *Then* VaR *has the following properties:*

(a) $X \geq 0 \Rightarrow \mathrm{VaR}^{\alpha}(X) \leq 0$,

(b) $X \geq Y \Rightarrow \mathrm{VaR}^{\alpha}(X) \leq \mathrm{VaR}^{\alpha}(Y)$,

(c) $\mathrm{VaR}^{\alpha}(\lambda X) = \lambda \mathrm{VaR}^{\alpha}(X)$ *for all* $\lambda \geq 0$,

(d) $\mathrm{VaR}^{\alpha}(X + k) = \mathrm{VaR}^{\alpha}(X) - k$ *for all* $k \in \mathbb{R}$.

In particular, we have $\mathrm{VaR}^{\alpha}(X + \mathrm{VaR}^{\alpha}(X)) = 0$.

It is clear that the value of $\mathrm{VaR}^{\alpha}(X)$ depends only on the distribution of X and not the underlying probability space. This property is known as *law invariance*, which we will formalize later. In fact, VaR is law invariant in a very strong sense; it is not hard to think of different P&L distributions that produce the same VaR number (See Figure 1.1).

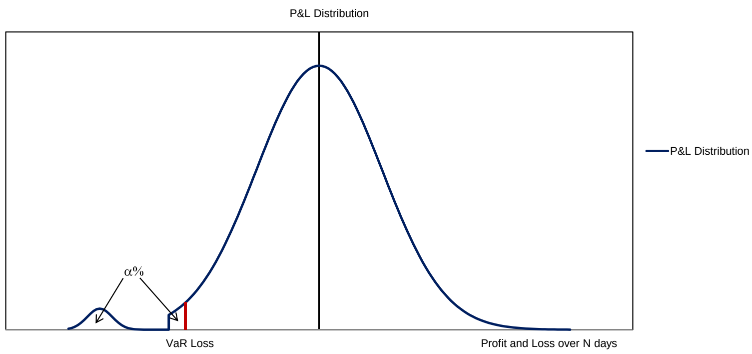

Figure 1.2: *An example of a different distribution with the same* VaR *as Figure 1.1.*

Since $\text{VaR}^{\alpha}(X)$ is defined for every $X \in L^0(\mathbb{P})$, one might suspect that such a high level of generality means that VaR could be missing an important property. Such suspicions would be well founded as it turns out that VaR is not sub-additive (i.e. VaR does not respect portfolio diversification). In other words, it is not always true that $\text{VaR}^{\alpha}(X+Y) \leq \text{VaR}^{\alpha}(X)+\text{VaR}^{\alpha}(Y)$. Consider the following simple example [68].

Example 1.2.6 Consider a portfolio consisting of a short out-the-money put and a short out-the-money call, written on the same asset. Suppose that each option has only a 4% chance of being in the money at maturity. Then the 95%VaR of each option will be zero. However, the combined portfolio has an 8% chance of being in-the-money, which means the 95%VaR is non-zero. Thus, the sum of individual risks of the options is smaller than the risk of the combined position.

Figure 1.3 shows the payoff diagram of the position in Example 1.2.6. Figure 1.4 shows a 7-day VaR surface simulation. Figure 1.5 shows the diversification benefit from combining the risks of the individual options, i.e. the difference between the sum of the individual risks and the risk of the combined position. Notice how the diversification benefit drops below zero.

It can be shown that VaR^{α} is sub-additive on risks with elliptical distributions provided that $0 < \alpha < 0.5$. For more examples of risks for which VaR fails to be sub-additive the reader may consult [6, 7, 61].

The lack of sub-additivity makes it difficult to decentralize risk management in financial institutions because the aggregation of risks of components of an institution does not provide an upper bound for the risk faced by the institution as a whole.

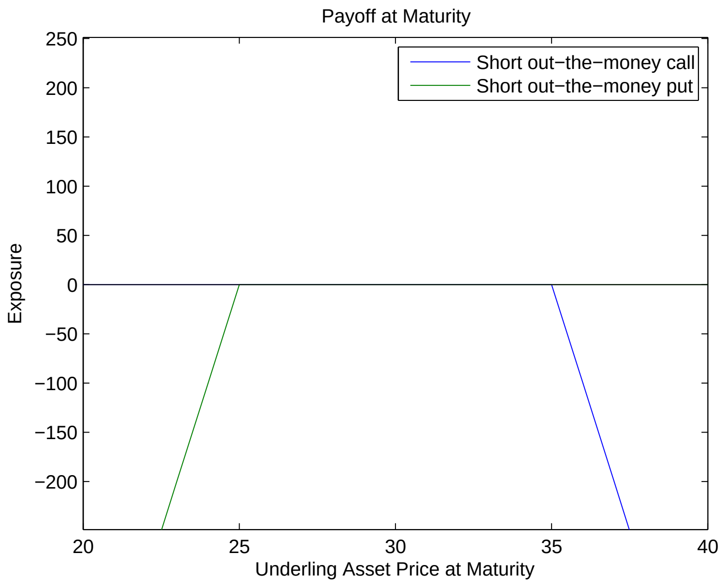

Figure 1.3: *The payoff of the portfolio from Example 1.2.6, containing a short call struck at \$35 and a short put struck at \$25. The notional size of each option is \$100.*

This problem may be compounded in the presence of a regulatory body that enforces capital adequacy requirements. Indeed, the institution may be tempted to create more subdivisions to artificially lower the perceived risk. For more on the virtues and shortcomings of VaR the reader can consult [23] and the references contained therein.

1.3 Coherent Risk Measures

1.3.1 Coherency Axioms

Value-at-Risk has been heavily criticized for its failure to consider the extent of a loss in the event of a disaster, as well as for its lack of sub-additivity [6, 7, 23]. In view of these shortcomings, Artzner et al. sought to axiomatize the desirable properties one would expect from a risk measure [6, 7]. This gave rise to a new class of risk measures known as 'Coherent Risk Measures'. We use the definition from [15].

Definition 1.3.1 (Coherent Risk Measure) *A mapping* $\rho : L^\infty(\mathbb{P}) \to \mathbb{R}$ *is called a* coherent risk measure *if the following properties hold:*

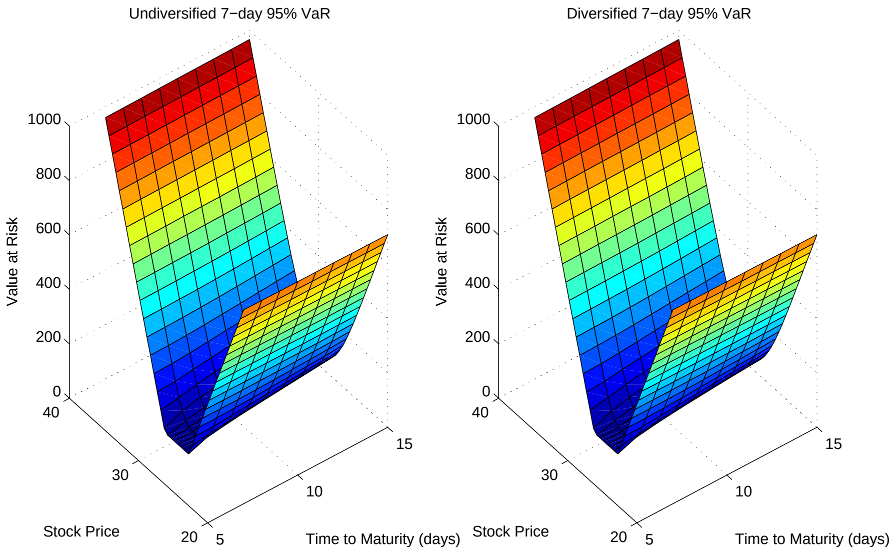

Figure 1.4: *The 7-day* VaR *surface of the portfolio in Example 1.2.6.*

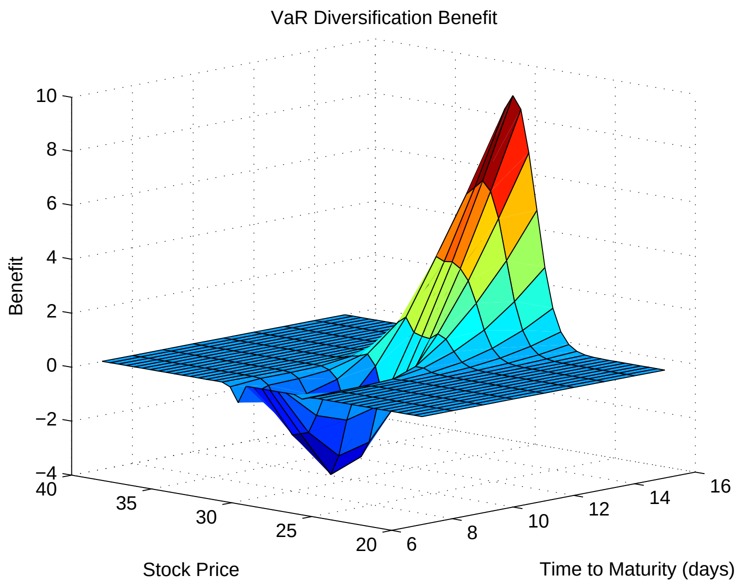

Figure 1.5: *The 7-day* VaR *diversification benefit of the portfolio in Example 1.2.6. Notice how the diversification benefit drops below zero.*

(a) Monotonicity: For all $X \in L^\infty(\mathbb{P})_+$ *we have* $\rho(X) \leq 0$.

(b) Sub-additivity: for all $X_1, X_2 \in L^\infty(\mathbb{P})$ *we have* $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$.

(c) Positive homogeneity: for all $X \in L^\infty(\mathbb{P})$ *we have* $\rho(\lambda X) = \lambda\rho(X)$ *for all* $\lambda > 0$.

(d) Translation invariance: for all $X \in L^\infty(\mathbb{P})$ *we have* $\rho(X + a) = \rho(X) - a$ *for all constant functions* a.

The above properties have important financial interpretations:

Monotonicity: If a position X is positive (i.e. $X \in L^\infty(\mathbb{P})_+$), then it means that the position cannot lose money. This constitutes an *acceptable position* and is represented by $\rho(X) \leq 0$. As with VaR , this means that we can withdraw capital from the position, or take on more risk.

Sub-Additivity: To repeat what has been mentioned above, we require that a coherent risk measure not punish its user for diversifying risk.

Positive Homogeneity: This is a natural requirement; the size of the risk of a position should scale with the size of the position.

Translation Invariance: Adding or removing a fixed amount of capital from a position alters risk of that position by the same amount. In particular, we have $\rho(X + \rho(X)) = \rho(X) - \rho(X) = 0$. Thus, the risk of a position represents the additional capital required to make that position acceptable.

According to the above definition, a coherent risk measure cannot take the value $\pm\infty$. Also, the domain of a coherent risk measure has been restricted to $L^\infty(\mathbb{P})$. The reason for this caution is due to the following negative result, proved in [15, Theorem 5.1].

Theorem 1.3.2 *If the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is atomless, then there is no real-valued coherent risk measure on* $L^0(\mathbb{P})$.

Proof. Suppose that there exists a coherent risk measure $\rho : L^0(\mathbb{P}) \to \mathbb{R}$. Define the translation invariant sub-modular function $\psi : L^0(\mathbb{P}) \to \mathbb{R}$ by $\psi(X) = \rho(-X)$ for each $X \in L^0(\mathbb{P})$. Then, ψ has the following obvious properties:

- $X \in L^0(\mathbb{P}), X \leq 0 \Rightarrow \psi(X) \leq 0$,
- $X_1, X_2 \in L^0(\mathbb{P}) \Rightarrow \psi(X_1 + X_2) \leq \psi(X_1) + \psi(X_2)$,
- $X \in L^0(\mathbb{P}) \Rightarrow \psi(\lambda X) = \lambda\psi(X)$ for all $\lambda > 0$,
- $X \in L^0(\mathbb{P}) \Rightarrow \psi(a + X) = \psi(X) + a$ for all constant functions a.

We will now show that ψ implies the existence of a non-zero positive linear functional on the vector lattice $L^0(\mathbb{P})$. This contradicts the fact that the only order bounded linear functional on $L^0(\mathbb{P})$ is the zero function, since all positive linear functionals are order bounded (see [72] for an example).

To this end, observe that $\psi(1) = 1$. Indeed, homogeneity implies that $\psi(0) = \psi(2 \cdot 0) = 2\psi(0)$ which gives $\psi(0) = 0$. Now, using the translation invariance, we obtain $\psi(1) = \psi(0 + 1) = \psi(0) + 1 = 1$ as required.

Consider the subspace $M = \{\alpha \cdot 1 : \alpha \in \mathbb{R}\} \subset L^0(\mathbb{P})$. Using translation invariance and homogeneity, it is easy to see that $\psi|_M$ is a linear functional on M. By the Hahn-Banach Theorem (in its most general form), there exists a linear functional $f : L^0(\mathbb{P}) \to \mathbb{R}$ that extends $\psi|_M$ with $f(X) \leq \psi(X)$ for all $X \in L^0(\mathbb{P})$. To see that f is positive, let $X \geq 0$. Then $-f(X) = f(-X) \leq \psi(-X) \leq 0$, from which $f(X) \geq 0$ follows. The fact that $f(1) = 1$ completes the proof. □

The above result allows for a distribution-free proof of the incoherency of VaR .

Corollary 1.3.3 VaR *is not a sub-additive risk measure. Consequently,* VaR *is not coherent.*

Proof. Let ρ be defined by $\rho(X) = \mathrm{VaR}^{\alpha}(X)$ for some $\alpha \in (0, 1)$. Then, as we have seen, ρ satisfies the properties of monotonicity, homogeneity and translation invariance. If ρ were sub-additive, then ρ would be a coherent risk measure with $\rho : L^0(\mathbb{P}) \to \mathbb{R}$. This contradicts the above result. □

Not to be discouraged by the above theorem, Delbaen [15] extended the notion of a coherent risk measure to all of $L^0(\mathbb{P})$ by allowing the risk measure to take on the value $\pm\infty$. The interpretation is as follows: If X is a very risky position in the sense that no amount of capital added to the position will make it acceptable, the risk assigned to this position would be ∞. It is absurd to assign the risk of the position

X a value of $-\infty$, because it would mean that an arbitrary amount of capital could be withdrawn from the position, and the position would remain acceptable. We do not exhibit Delbaen's extension to $L^0(\mathbb{P})$ here and continue in the setting of $L^\infty(\mathbb{P})$, safe in the knowledge that it can be extended if necessary.

1.3.2 Additional Properties for Risk Measures

Some additional properties for risk measures found in the literature [61, 15, 44, 67] are:

Definition 1.3.4 If ρ is a risk measure, then

(a) ρ is said to satisfy the *Fatou Property* if $\rho(X) \leq \liminf \rho(X_n)$ for every sequence of random variables (X_n) with $\sup_n \|X_n\|_\infty \leq 1$ that converges to a limit X in probability,

(b) ρ is *law invariant* if $\mathbb{P}(X \leq t) = \mathbb{P}(Y \leq t)$ for all $t \in \mathbb{R}$ implies $\rho(X) = \rho(Y)$ for any random variables X and Y,

(c) ρ is *co-monotonically additive* if for $Z \in L^0(\mathbb{P})$ and increasing functions f, g with $f \circ Z, g \circ Z \in L^0(\mathbb{P})$, we have $\rho(f \circ Z + g \circ Z) = \rho(f \circ Z) + \rho(g \circ Z)$.

We explain the interpretations of the above properties:

Fatou Property: It can be shown, in a similar way to the proof of Fatou's Lemma, that the Fatou Property is equivalent to $0 \leq X_n \leq 1, X_n \downarrow 0 \Rightarrow \rho(X_n) \uparrow 0$. Thus, the Fatou Property implies a type of continuity with respect to the partial ordering on $L^0(\mathbb{P})$. This property is required when working with σ-additive probability spaces. All the coherent risk measures we consider will have the Fatou Property.

Law Invariance: The risk associated with a position X depends only on the distribution of X. This property ensures that other factors, such as the structure of the underlying probability space, do not influence the risk associated with the position X.

Co-Monotonic Additivity: Two random variables X and Y are said to be *co-monotone* if there exist increasing functions f, g and a random variable Z such that $X = f \circ Z$ and $Y = g \circ Z$. Thus, if X and Y are co-monotone they share the same source of uncertainty. This can be viewed as a probability-free way of saying that X and Y are perfectly correlated, i.e. neither random variable provides a diversification benefit for the other. The property of co-monotonic additivity says that there is no diversification benefit to be gained from aggregating co-monotone risks.

As mentioned before, VaR is law invariant. It also turns out that VaR is co-monotonically additive. This follows easily from the identity

$$q^{\alpha}(u \circ X) = u \circ q^{\alpha}(X) \tag{1.3.1}$$

where $u : \mathbb{R} \to \mathbb{R}$ is an increasing function with no discontinuities in common with the distribution F_X of X. A corresponding statement holds for lower α-quantiles.

The proof of (1.3.1) is simple when both u and F_X are assumed to be one-to-one. Indeed, we have

$$F_{u(X)}(x) = \mathbb{P}[u(X) \le x] = \mathbb{P}[X \le u^{-1}(x)] = F_X(u^{-1}(x)) = F_X \circ u^{-1}(x).$$

Taking inverses on both sides, we arrive at (1.3.1). We omit the more complicated general case, which can be found in [19, Proposition 4.1].

1.3.3 Characterizations of Coherent Risk Measures

In this section, we present an important characterization of coherent risk measures with the Fatou Property, due to Delbaen [15]. This characterization will allow us to generate numerous examples of coherent risk measures.

Theorem 1.3.5 (Delbaen) *For a coherent risk measure $\rho : L^{\infty}(\mathbb{P}) \to \mathbb{R}$, the following statements are equivalent:*

(a) ρ satisfies the Fatou Property.

(b) There is a $\|\cdot\|_1$-closed, convex set $\mathcal{P}$ of probability measures, all of which are absolutely continuous with respect to $\mathbb{P}$, such that

$$\rho(X) = \sup_{\mathbb{Q} \in \mathcal{P}} \mathbb{E}^{\mathbb{Q}}[-X]$$

for all $X \in L^{\infty}(\mathbb{P})$.

(c) The convex cone $C_\rho := \{X \in L^\infty(\mathbb{P}) : \rho(X) \le 0\} \supset L^\infty(\mathbb{P})_+$ *is weak* closed and uniquely determines* ρ *via the relation* $\rho(X) = \inf\{\gamma \in \mathbb{R} : X + \gamma \in C_\rho\}$.

Proof. (a)⇒(c) Suppose that ρ satisfies the Fatou Property. By the positive homogeneity and sub-additivity of ρ, it follows that C_ρ is a (convex) cone. The monotonicity of ρ implies $L^\infty(\mathbb{P})_+ \subset C_\rho$.

To see that $\rho(X) = \inf\{\gamma \in \mathbb{R} : X+\gamma \in C_\rho\}$, observe that for all $\gamma \in \mathbb{R}$ with $X+\gamma \in C_\rho$, we have by translation invariance $\rho(X) - \gamma = \rho(X+\gamma) \le 0$. Thus $\rho(X) \le \gamma$. Moreover, since $\rho(X+\rho(X)) = 0$ it follows that $\rho(X) \in \{\gamma \in \mathbb{R} : X + \gamma \in C_\rho\}$, which proves the claim.

It remains to show that C_ρ is weak* closed. By Proposition 1.1.1, we need only show that $C_\rho \cap \mathrm{ball}(L^\infty(\mathbb{P}))$ is closed in probability. To this end, let $(X_n) \subset C_\rho \cap \mathrm{ball}(L^\infty(\mathbb{P}))$ be a sequence of random variables that converges in probability to X. The Fatou Property implies that $\rho(X) \le \liminf \rho(X_n) \le 0$. Thus, $X \in C_\rho \cap \mathrm{ball}(L^\infty(\mathbb{P}))$ so that C_ρ is weak* closed.

(b)⇒(a) For each $\mathbb{Q} \in \mathcal{P}$ we have, by Fatou's Lemma,

$$\mathbb{E}^{\mathbb{Q}}[-X] \le \liminf_{n\to\infty} \mathbb{E}^{\mathbb{Q}}[-X_n] \le \liminf_{n\to\infty} \rho(X_n)$$

for any sequence $(X_n) \subset \mathrm{ball}(L^\infty(\mathbb{P}))$ that converges to X in probability. Taking the supremum over $\mathcal{P}$ on the left hand side of the inequality shows that ρ has the Fatou Property.

(c)⇒(b) Consider the duality pair $(L^1(\mathbb{P}), L^\infty(\mathbb{P}), \langle\cdot,\cdot\rangle)$ where $\langle f, g\rangle := \mathbb{E}^{\mathbb{P}}[fg]$ for each $f \in L^1(\mathbb{P})$ and $g \in L^\infty(\mathbb{P})$. Then, the polar set $C_\rho^\circ \subset L^1(\mathbb{P})$ of the convex cone $C_\rho \subset L^\infty(\mathbb{P})$ is given by

$$C_\rho^\circ = \{f \in L^1(\mathbb{P}) : \mathbb{E}^{\mathbb{P}}[fX] \ge 0 \;\forall\; X \in C_\rho\}.$$

Since $L^\infty(\mathbb{P})_+ \subset C_\rho$, it follows that $C_\rho^\circ \subset L^1(\mathbb{P})_+$. Moreover, C_ρ° is a convex cone that is weakly closed (and thus norm closed). Define the set of measures

$$\begin{aligned}\mathcal{P} &= \{f \in C_\rho^\circ : \mathrm{d}\mathbb{Q} = f\mathrm{d}\mathbb{P} \text{ defines a probability measure}\}\\ &= \{f \in C_\rho^\circ : \mathbb{E}^{\mathbb{P}}[f] = 1\}.\end{aligned}$$

It is not difficult to check that $\mathcal{P}$ is convex and $\|\cdot\|_1$-closed. Also, we may write $C_\rho^\circ = \bigcup_{\lambda\ge 0} \lambda\mathcal{P}$. Indeed, since C_ρ° is a cone, we have the inclusion $\bigcup_{\lambda\ge0}\lambda\mathcal{P} \subset C_\rho^\circ$.

For the reverse inclusion, suppose there exists $f \in C_\rho^\circ \setminus \bigcup_{\lambda \geq 0} \lambda\mathcal{P}$. Then $\mathbb{E}^{\mathbb{P}}[f] \neq \lambda$ for all $\lambda \geq 0$. This is impossible because f is positive.

By the assumption, C_ρ is weak* closed. The Bi-Polar Theorem implies $C_\rho = C_\rho^{\circ\circ} = (\bigcup_{\lambda\geq 0} \lambda\mathcal{P})^\circ$. Thus,

$$\begin{aligned} C_\rho &= \left\{ X \in L^\infty(\mathbb{P}) : \mathbb{E}^{\mathbb{P}}[Xf] \geq 0 \ \forall\ f \in \bigcup_{\lambda\geq 0} \lambda\mathcal{P} \right\} \\ &= \{X \in L^\infty(\mathbb{P}) : \mathbb{E}^{\mathbb{P}}[Xf] \geq 0 \ \forall\ f \in \mathcal{P}\} \\ &= \{X \in L^\infty(\mathbb{P}) : \mathbb{E}^{\mathbb{Q}}[X] \geq 0 \ \forall\ \mathbb{Q} \in \mathcal{P}\}. \end{aligned}$$

Consequently, $\rho(X) \leq 0$ if and only if $\mathbb{E}^{\mathbb{Q}}[X] \geq 0$ for all $\mathbb{Q} \in \mathcal{P}$. This gives

$$\begin{aligned} \rho(X) &= \inf\{\gamma \in \mathbb{R} : X + \gamma \in C_\rho\} \\ &= \inf\{\gamma \in \mathbb{R} : \mathbb{E}^{\mathbb{Q}}[X + \gamma] \geq 0 \ \forall\ \mathbb{Q} \in \mathcal{P}\} \\ &= \inf\{\gamma \in \mathbb{R} : \mathbb{E}^{\mathbb{Q}}[-X] \leq \gamma \ \forall\ \mathbb{Q} \in \mathcal{P}\} \\ &= \sup\{\mathbb{E}^{\mathbb{Q}}[-X] : \mathbb{Q} \in \mathcal{P}\}, \end{aligned}$$

which concludes the proof. □

From the proof of the above theorem, it can be seen that there is a one-to-one correspondence between

(a) coherent risk measures ρ possessing the Fatou Property,

(b) closed, convex sets of probability measures $\mathcal{P} \subset L^1(\mathbb{P})$,

(c) weak* closed convex cones $C_\rho \subset L^\infty(\mathbb{P})$ such that $L^\infty(\mathbb{P})_+ \subset C_\rho$.

Remarks and Interpretations

- The set C_ρ is known as the *set of acceptable positions*. The quantity $\rho(X)$ represents the least (largest) amount of capital that has to be invested (withdrawn) in (from) the position X to achieve (maintain) an acceptable position.
- A coherent risk measure satisfying the Fatou property can be represented as a supremum of expectations taken over a collection of probabilities $\mathcal{P}$. One can interpret $\mathcal{P}$ as a range of scenarios, and the quantity $\rho(X)$ as the worst case scenario.

Extension to L^p-Spaces

Part of Theorem 1.3.5 can be extended from $L^\infty(\mathbb{P})$ to $L^p(\mathbb{P})$, where $1 \le p < \infty$. To achieve this, we need stronger condition than the Fatou Property on the coherent risk measure ρ.

Definition 1.3.6 Let $1 \le p < \infty$. If $\rho : L^p(\mathbb{P}) \to \mathbb{R}$ is a risk measure, then ρ is said to be *continuous* if there exists $K > 0$ such that $|\rho(X)| \le K\|X\|_p$ for all $X \in L^p(\mathbb{P})$.

The following result is due to Inoue [32].

Theorem 1.3.7 *Let $1 \le p < \infty$ and $p^{-1} + q^{-1} = 1$. For a risk measure $\rho : L^p(\mathbb{P}) \to \mathbb{R}$, the following statements are equivalent:*

(a) ρ is a continuous coherent risk measure.

(b) There exists a set $G \subset L^q(\mathbb{P})_+$ with $\mathbb{E}[g] = 1$ for each $g \in G$ such that

- $\sup_{g\in G} \|g\|_q < \infty$, *and*
- $\rho(X) = \sup_{g\in G} \mathbb{E}[-Xg]$ *for all $X \in L^p(\mathbb{P})$.*

Proof. (b)⇒(a) By Hölder's Inequality, we have

$$\rho(X) \le \sup\{\|g\|_q : g \in G\} \cdot \|X\|_p,$$

which implies that ρ is continuous. The axioms of coherency are also easily derived from the definition of ρ.

(a)⇒(b) Let $X \in L^\infty(\mathbb{P})$ and let X_n be a sequence in $L^\infty(\mathbb{P})$ that decreases to X. Then $\|X_n - X\|_p \to 0$ by the order continuity of $\|\cdot\|_p$. By the continuity of ρ we have $\rho(X_n) \to \rho(X)$, which implies that $\rho|_{L^\infty(\mathbb{P})}$ has the Fatou Property. By Theorem 1.3.5, there exists a set G of non-negative random variables g with $\mathbb{E}[g] = 1$ such that

$$\rho|_{L^\infty(\mathbb{P})}(X) = \sup_{g\in G} \mathbb{E}[-Xg],$$

for all $X \in L^\infty(\mathbb{P})$. Since ρ is continuous, it follows that

$$\rho|_{L^\infty(\mathbb{P})}(-X) = \sup_{g\in G} \mathbb{E}[Xg] \le K\|X\|_p.$$

Consequently,

$$\mathbb{E}[Xg] \leq K\|X\|_p$$

for all $X \in L^\infty(\mathbb{P})$ and $g \in G$. This implies that the functional $f_g \in L^p(\mathbb{P})^*$ defined on the dense subset $L^\infty(\mathbb{P})$ by the action $f_g(X) = \mathbb{E}[Xg]$ has norm $\|f_g\| \leq K$ for all $g \in G$. This shows that

$$\rho(X) = \sup_{g \in G} \mathbb{E}[-Xg]$$

is defined for all $X \in L^p(\mathbb{P})$ and that $\sup_{g \in G} \|g\|_q < \infty$. □

This result can be extended to include lower semi-continuous coherent risk measures, as is done in [46] for $L^1(\mathbb{P})$. We present a version for $L^p(\mathbb{P})$.

Definition 1.3.8 Let $1 \leq p \leq \infty$. If $\rho : L^p(\mathbb{P}) \to \mathbb{R}$ is a risk measure, then ρ is said to be *lower semi-continuous* (resp. *upper semi-continuous*) if for every $c \in \mathbb{R}$, the set

$$\{X \in L^p(\mathbb{P}) : \rho(X) \leq c\} \quad (\text{resp. } \{X \in L^p(\mathbb{P}) : \rho(X) \geq c\})$$

is closed with respect to the topology under consideration.

Theorem 1.3.9 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X = L^p(\mathbb{P})$ endowed with the norm topology for $1 \leq p < \infty$ and the weak* topology $\sigma(L^\infty(\mathbb{P}), L^1(\mathbb{P}))$ for $p = \infty$. For a risk measure $\rho : L^p(\mathbb{P}) \to \mathbb{R}$, the following statements are equivalent:*

(a) $\rho : X \to \mathbb{R}$ is a lower semi-continuous coherent risk measure.

(b) There exists a closed set $G \subset X_+^*$ with $\mathbb{E}[g] = 1$ for each $g \in G$ such that

- *$\rho(f) = \sup_{g \in G} \mathbb{E}[-fg]$ for all $f \in X$.*

(c) The convex cone $C_\rho := \{f \in X : \rho(f) \leq 0\} \supset X_+$ is closed and uniquely determines ρ via the relation $\rho(x) = \inf\{\gamma \in \mathbb{R} : x + \gamma \in C_\rho\}$.

Proof. (a)⇒(c) Follows directly from Theorem 1.3.5 and the lower semi-continuity of ρ.

(c)⇒(b) The proof is completely analogous to the argument used in Theorem 1.3.5.

(b)⇒(a) The fact that ρ is coherent follows easily from its definition in (b). Since the supremum of a collection of continuous functions is lower semi-continuous (cf. [4, Lemma 2.41]), we have (a). □

Observe that the Fatou Property is in fact lower-semi continuity with respect to the weak* topology on $L^\infty(\mathbb{P})$. This topology corresponds to the topology induced by bounded convergence in probability by Proposition 1.1.1.

1.3.4 Examples of Coherent Risk Measures

We are now in a position to generate some important examples of coherent risk measures using Theorem 1.3.5. To this end, fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and consider a coherent risk measure $\rho : L^\infty(\mathbb{P}) \to \mathbb{R}$ and a position $X \in L^\infty(\mathbb{P})$. For a set of measures $\mathcal{P} \subset L^1(\mathbb{P})$, we shall denote by $\overline{\text{co}}\,\mathcal{P}$ the closed convex hull with respect to $\|\cdot\|_1$.

Example 1.3.10 (Largest Coherent Risk Measure) Let

$$\begin{aligned}\mathcal{P} &= \overline{\text{co}}\,\{\mathbb{Q} : \mathbb{Q} \text{ a probability measure with } \mathbb{Q} \ll \mathbb{P}\} \\ &= \{f \in L^1(\mathbb{P})_+ : \mathbb{E}^{\mathbb{P}}[f] = 1\}.\end{aligned}$$

This produces the coherent risk measure

$$\rho(X) = \sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}^{\mathbb{Q}}[-X] = \text{ess-sup}\,(-X).$$

This risk measure gives the maximum loss of the position X. Here, X is acceptable if and only if X is non-negative. It is clear that using such a risk measure would stop all financial activities, as it is too conservative.

Example 1.3.11 (Average Loss) Let $\mathcal{P} = \{\mathbb{P}\}$. Then $\rho(X) = \sup_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}^{\mathbb{Q}}[-X] = \mathbb{E}^{\mathbb{P}}[-X]$. A position X is acceptable if and only if its average $\mathbb{E}^{\mathbb{P}}[X]$ is non-negative. This risk measure is too tolerant to use in practice.

Example 1.3.12 (Worst Conditional Expectation) Given a significance level $\alpha \in (0, 1)$, let

$$\mathcal{P}_{\text{WCE}} = \overline{\text{co}}\,\{\mathbb{P}[\,\cdot\,|A] : \mathbb{P}[A] > \alpha\}.$$

This produces the *Worst Conditional Expectation*

$$\mathrm{WCE}^{\alpha}(X) = \sup\{\mathbb{E}^{\mathbb{Q}}[-X] : \mathbb{Q} \in \mathcal{P}_{\mathrm{WCE}}\} = -\inf\{\mathbb{E}[X|A] : A \in \mathcal{F}, \mathbb{P}[A] > \alpha\}.$$

If the probability space is atomless then we can replace $\mathbb{P}[A] > \alpha$ with $\mathbb{P}[A] \geq \alpha$. One should be aware that the definition of Worst Conditional Expectation depends on the structure of the underlying probability space. The Radon-Nikodým derivative of $\mathbb{Q} = \mathbb{P}[\,\cdot\,|A]$ is of the form

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \frac{\mathbf{1}_A}{\mathbb{P}[A]}.$$

It follows that $\mathcal{P}_{\mathrm{WCE}}$ is $\|\cdot\|_\infty$-bounded by $1/\alpha$. Since $\mathcal{P}_{\mathrm{WCE}}$ is clearly uniformly integrable, we have that $\mathcal{P}_{\mathrm{WCE}}$ is weakly compact in $L^1(\mathbb{P})$ by the Dunford-Pettis Theorem.

Unfortunately, WCE is not of much practical use because it is difficult to compute. We shall remedy this situation with the next example.

Example 1.3.13 (Expected Shortfall) Given a significance level $\alpha \in (0,1)$, let

$$\mathcal{P}_{\mathrm{ES}} = \{f \in L^1(\mathbb{P})_+ : \|f\|_\infty \leq 1/\alpha,\, \mathbb{E}^{\mathbb{P}}[f] = 1\}.$$

Note that $\mathcal{P}_{\mathrm{ES}}$ is already closed and convex in $L^1(\mathbb{P})$. This produces the risk measure called *Expected Shortfall*, given by

$$\begin{aligned} \mathrm{ES}^{\alpha}(X) &= \sup\{\mathbb{E}^{\mathbb{Q}}[-X] : \mathbb{Q} \in \mathcal{P}_{\mathrm{ES}}\} \\ &= \frac{1}{\alpha}\Big(\mathbb{E}[-X \cdot \mathbf{1}_{\{X \leq x_{(\alpha)}\}}] - x_{(\alpha)}(\alpha - \mathbb{P}[X \leq x_{(\alpha)}])\Big). \end{aligned}$$

We shall prove the above formula later on. In the mean time, notice that $\mathcal{P}_{\mathrm{WCE}} \subset \mathcal{P}_{\mathrm{ES}}$. Consequently, $\mathrm{WCE}^{\alpha}(X) \leq \mathrm{ES}^{\alpha}(X)$ for all $X \in L^\infty(\mathbb{P})$. In the case where the underlying probability space is atomless, we get $\mathcal{P}_{\mathrm{WCE}} = \mathcal{P}_{\mathrm{ES}}$. Clearly, we then have $\mathrm{WCE}^{\alpha}(X) = \mathrm{ES}^{\alpha}(X)$ for all $X \in L^\infty(\mathbb{P})$.

By Theorem 1.3.5, ES is a coherent risk measure with the Fatou Property (as are all the other examples). We can thus avoid the technical proof of the coherency of ES given in [2, Proposition 3.1].

The above formula for Expected Shortfall is clearly law-invariant and easy to compute. ES can be viewed as the smallest law-invariant risk measure that dominates WCE. In the next chapter, we shall apply this risk measure to Example 1.2.6.

We remark that the so called 'Conditional Value-at-Risk' introduced in [62] is equivalent to Expected Shortfall. This is shown in [2, Corollary 4.3].

1.4 Expected Shortfall

1.4.1 Definition

In view of Example 1.3.13, we make the following definition:

Definition 1.4.1 Let $X \in L^\infty(\mathbb{P})$ denote the random variable of profits and losses of a portfolio. Then, for $\alpha \in (0,1)$, we define the *Expected Shortfall* of X to be

$$\mathrm{ES}_\alpha(X) = \frac{1}{\alpha}\left(\mathbb{E}[-X \cdot \mathbf{1}_{\{X \le x_{(\alpha)}\}}] - x_{(\alpha)}(\alpha - \mathbb{P}[X \le x_{(\alpha)}])\right).$$

As stated earlier in Example 1.3.13, Expected Shortfall is a coherent risk measure. To justify this, it remains to prove the following result.

Proposition 1.4.2 *For $X \in L^\infty(\mathbb{P})$ and $\alpha \in (0,1)$ we have*

$$\mathrm{ES}^\alpha(X) = \sup\{\mathbb{E}^{\mathbb{Q}}[-X] : \mathbb{Q} \in \mathcal{P}_{\mathrm{ES}}\},$$

where

$$\mathcal{P}_{\mathrm{ES}} = \{f \in L^1(\mathbb{P})_+ : \|f\|_\infty \le 1/\alpha,\, \mathbb{E}^{\mathbb{P}}[f] = 1\}.$$

Consequently, ES *is a coherent risk measure.*

Proof. First note that $1/\alpha > 1$ implies that $\mathcal{P}_{\mathrm{ES}}$ is non-trivial, i.e. non-empty and contains more than just the probability $\mathbb{P}$. Consider a measure $\mathbb{Q}_\alpha$ with a density $f_\alpha := \frac{\mathrm{d}\mathbb{Q}_\alpha}{\mathrm{d}\mathbb{P}}$ given by

$$\frac{\mathrm{d}\mathbb{Q}_\alpha}{\mathrm{d}\mathbb{P}} = \begin{cases} \frac{1}{\alpha}\mathbf{1}_{\{X \le x_{(\alpha)}\}}, & \mathbb{P}[X = x_{(\alpha)}] = 0; \\ \frac{1}{\alpha}\left[\mathbf{1}_{\{X \le x_{(\alpha)}\}} + \frac{\alpha - \mathbb{P}[X \le x_{(\alpha)}]}{\mathbb{P}[X = x_{(\alpha)}]}\mathbf{1}_{\{X = x_{(\alpha)}\}}\right], & \mathbb{P}[X = x_{(\alpha)}] > 0. \end{cases} \tag{1.4.2}$$

It is not difficult to check that $f_\alpha \ge 0$, $\|f_\alpha\|_\infty \le 1/\alpha$ and $\mathbb{E}^{\mathbb{P}}[f_\alpha] = 1$. Consequently, we have $\mathbb{Q}_\alpha \in \mathcal{P}_{\mathrm{ES}}$. Moreover,

$$\mathbb{E}^{\mathbb{Q}_\alpha}[-X] = \mathbb{E}^{\mathbb{P}}[-Xf_\alpha] = \frac{1}{\alpha}\left(\mathbb{E}[-X \cdot \mathbf{1}_{\{X \le x_{(\alpha)}\}}] - x_{(\alpha)}(\alpha - \mathbb{P}[X \le x_{(\alpha)}])\right).$$

To complete the proof, take an arbitrary $\mathbb{Q} \in \mathcal{P}_{\text{ES}}$ with density $f := \frac{d\mathbb{Q}}{d\mathbb{P}}$ and define $A = \{X \leq x_{(\alpha)}\}$. Then, bearing in mind that $0 \leq f \leq 1/\alpha$ and $\mathbb{E}^{\mathbb{P}}[f] = 1$, we deduce

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[-X] &= \int_\Omega (-X)\, d\mathbb{Q} = \int_A (-X) f \, d\mathbb{P} + \int_{A^c} (-X) f \, d\mathbb{P} \\
&= \frac{1}{\alpha} \int_A (-X)\, d\mathbb{P} + \int_A (-X) \left(f - \frac{1}{\alpha}\right) d\mathbb{P} + \int_{A^c} (-X) f \, d\mathbb{P} \\
&\leq \frac{1}{\alpha} \int_A (-X)\, d\mathbb{P} + \left(-x_{(\alpha)}\right) \int_A \left(f - \frac{1}{\alpha}\right) d\mathbb{P} + \left(-x_{(\alpha)}\right) \int_{A^c} f \, d\mathbb{P} \\
&= \frac{1}{\alpha}\mathbb{E}[-X \cdot \mathbf{1}_A] - x_{(\alpha)} \left(\mathbb{Q}(A) - \frac{1}{\alpha}\mathbb{P}[A] + \mathbb{Q}(A^c)\right) \\
&= \frac{1}{\alpha}\mathbb{E}[-X \cdot \mathbf{1}_A] - x_{(\alpha)} \left(1 - \frac{1}{\alpha}\mathbb{P}[A]\right) \\
&= \frac{1}{\alpha} \left(\mathbb{E}[-X \cdot \mathbf{1}_A] - x_{(\alpha)} \left(\alpha - \mathbb{P}[A]\right)\right) \\
&= \mathbb{E}^{\mathbb{Q}_\alpha}[-X].
\end{aligned}$$

This shows that $\text{ES}^{\alpha}(X) = \sup\{\mathbb{E}^{\mathbb{Q}}[-X] : \mathbb{Q} \in \mathcal{P}_{\text{ES}}\}$, as required. □

If the distribution of the position X is continuous at $x_{(\alpha)}$, we have $\mathbb{P}[X = x_{(\alpha)}] = 0$ and $\mathbb{P}[X \leq x_{(\alpha)}] = \alpha$. In this case, Expected Shortfall reduces to

$$\text{ES}^{\alpha}(X) = \mathbb{E}[-X | \{X \leq x_{(\alpha)}\}].$$

In other words, Expected Shortfall coincides with the so-called Tail Conditional Expectation (TCE). In general, TCE is not coherent. This is because quantiles are not continuous with respect to any reasonable topology.

Interpretation of Expected Shortfall

The following result [2, Proposition 4.1] provides a financial interpretation of Expected Shortfall.

Proposition 1.4.3 *Let $\alpha \in (0, 1)$ and $X \in L^\infty(\mathbb{P})$ be a random variable of profits and losses. If $X_1, X_2, \ldots$ denotes an independent sequence of random variables with the same distribution as X, then*

$$\text{ES}^{\alpha}(X) = \lim_{n\to\infty} \frac{-1}{\lfloor n\alpha \rfloor} \sum_{i=1}^{\lfloor n\alpha \rfloor} X_{i:n}$$

with probability 1. In fact, convergence holds in $\|\cdot\|_1$-norm as well. Here, the notation $X_{i:n}$ denotes the i-th smallest sample in the ranked sequence of n samples $X_1, X_2, \ldots, X_n$.

The above result says that Expected Shortfall can be estimated by the average of $\alpha\%$ of the most extreme losses drawn from X. So, in contrast to VaR, ES answers the following question:

Given a profit and loss distribution of a portfolio X, what is the average loss incurred in $\alpha\%$ of the worst cases?

Therefore, ES is a risk measure that is concerned with the extent of a loss, rather than just the probability of that loss happening. Intuitively, this should mean that Expected Shortfall is a more conservative risk measure than Value-at-Risk. This is confirmed by the following proposition.

Proposition 1.4.4 *For $X \in L^\infty(\mathbb{P})$ and $\alpha \in (0,1)$ we have*

$$\mathrm{ES}^{\alpha}(X) \geq \mathbb{E}[-X|\{X \leq x_{(\alpha)}\}] \geq \mathrm{VaR}^{\alpha}(X).$$

Proof. For the first inequality, observe

$$\begin{aligned}
\mathrm{ES}^{\alpha}(X) &= \frac{1}{\alpha}\Big(\mathbb{E}[-X\cdot \mathbf{1}_{\{X\leq x_{(\alpha)}\}}] - x_{(\alpha)}(\alpha - \mathbb{P}[X\leq x_{(\alpha)}])\Big)\\
&= \frac{1}{\alpha}\Big(\mathbb{E}[-X|\{X\leq x_{(\alpha)}\}]\mathbb{P}[X\leq x_{(\alpha)}] + x_{(\alpha)}(\mathbb{P}[X\leq x_{(\alpha)}]-\alpha)\Big)\\
&= \frac{1}{\alpha}\Big(\mathbb{E}[-X|\{X\leq x_{(\alpha)}\}](\mathbb{P}[X\leq x_{(\alpha)}]-\alpha) + x_{(\alpha)}(\mathbb{P}[X\leq x_{(\alpha)}]-\alpha)\\
&\qquad + \alpha\mathbb{E}[-X|\{X\leq x_{(\alpha)}\}]\Big)\\
&= \frac{1}{\alpha}\Big(\underbrace{(\mathbb{E}[-X|\{X\leq x_{(\alpha)}\}] - (-x_{(\alpha)}))}_{\geq 0}\underbrace{(\mathbb{P}[X\leq x_{(\alpha)}]-\alpha)}_{\geq 0}\\
&\qquad + \alpha\mathbb{E}[-X|\{X\leq x_{(\alpha)}\}]\Big)\\
&\geq \frac{1}{\alpha}\Big(\alpha\mathbb{E}[-X|\{X\leq x_{(\alpha)}\}]\Big)\\
&= \mathbb{E}[-X|\{X\leq x_{(\alpha)}\}].
\end{aligned}$$

For the second in equality, notice that

$$\mathbb{E}[-X|\{X\leq x_{(\alpha)}\}] \geq \mathbb{E}[-X|\{X\leq x^{(\alpha)}\}] \geq -x^{(\alpha)} = \mathrm{VaR}^{\alpha}(X).$$

□

We return to Example 1.2.6 with payoff at maturity given by Figure 1.3. Figure 1.6 shows a 7-day ES surface simulation. Figure 1.7 shows the diversification benefit from combining the risks of the individual options, i.e. the difference between the sum of the individual risks and the risk of the combined position. In contrast to the simulation done with VaR in Figures 1.4 and 1.5, notice how the diversification benefit does not drop below zero.

1.4.2 Properties of Expected Shortfall

We gather some important properties of Expected Shortfall in the following theorem. The proofs of these properties are adapted from [2, 61].

Theorem 1.4.5 *Let $X \in L^\infty(\mathbb{P})$ denote the random variable of profits and losses of some portfolio and $\alpha \in (0,1)$. Then* ES *has the following properties:*

(a) $\mathrm{ES}^{\alpha}(X) = -\frac{1}{\alpha}\int_0^\alpha x_{(u)}\,du = \frac{1}{\alpha}\int_{1-\alpha}^1 q^{(u)}(-X)\,du.$

(b) The mapping $\alpha \mapsto \mathrm{ES}^{\alpha}(X)$ *is continuous and decreasing.*

(c) ES *is a law-invariant coherent risk measure that satisfies the Fatou Property and is co-monotonically additive.*

(d) $$\mathrm{ES}^{\alpha}(X) = \sup\big\{\mathrm{WCE}^{\alpha}(X') : X' \in L^0(\Omega', \mathcal{F}', \mathbb{P}'),\ \mathbb{P}'[X' \le x] = \mathbb{P}[X \le x]\ \forall\, x \in \mathbb{R}\big\}.$$

Proof. (a) Let U be a uniformly distributed random variable on $(0,1)$. By the Inverse Transform Method, $Z := x_{(U)}$ has the same distribution as X. Using the fact that $u \mapsto x_{(u)}$ is non-decreasing, we arrive at the following inclusions:

$$\begin{aligned}\{U \le \alpha\} &\subset \{Z \le x_{(\alpha)}\}\\ \{U > \alpha\} \cap \{Z \le x_{(\alpha)}\} &\subset \{Z = x_{(\alpha)}\}.\end{aligned}$$

Consequently, we get

$$\begin{aligned}\int_0^\alpha x_{(u)}\mathrm{d}u &= \mathbb{E}[Z \cdot \mathbf{1}_{\{U \le \alpha\}}]\\ &= \mathbb{E}[Z \cdot \mathbf{1}_{\{Z \le x_{(\alpha)}\}}] - \mathbb{E}[Z \cdot \mathbf{1}_{\{U > \alpha\} \cap \{Z \le x_{(\alpha)}\}}]\\ &= \mathbb{E}[X \cdot \mathbf{1}_{\{X \le x_{(\alpha)}\}}] + x_{(\alpha)}(\alpha - \mathbb{P}[X \le x_{(\alpha)}]).\end{aligned}$$

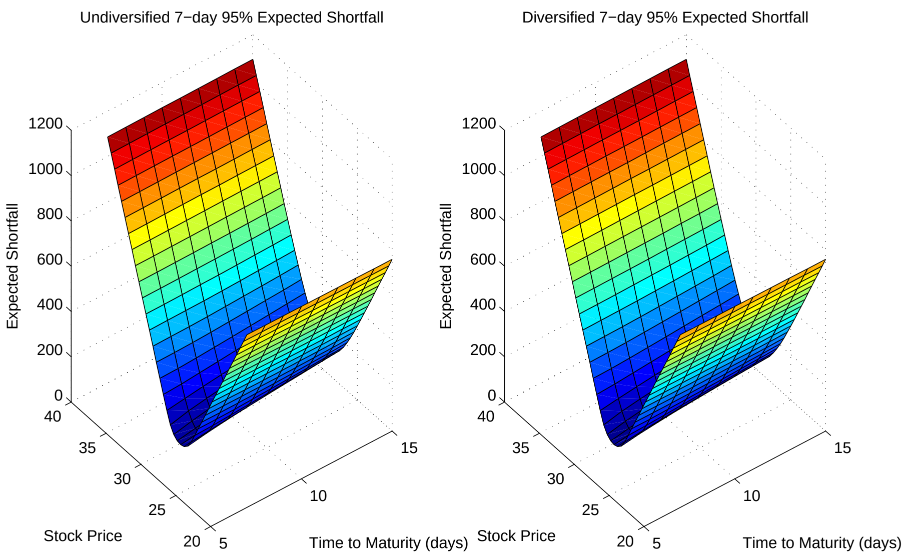

Figure 1.6: *The 7-day* ES *surface of the portfolio in Example 1.2.6.*

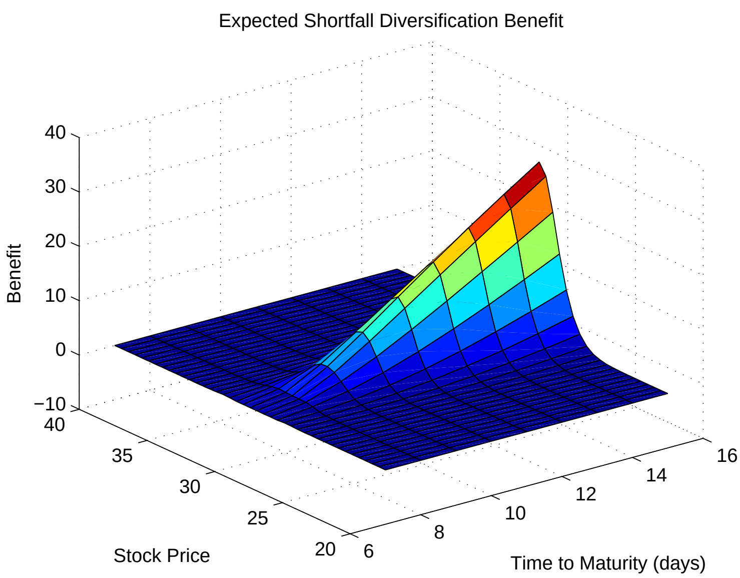

Figure 1.7: *The 7-day* ES *diversification benefit of the portfolio in Example 1.2.6. In contrast to* VaR *, notice how the diversification benefit does not drop below zero.*

Dividing through by $-\alpha$ proves the claim.

(b) The fact that the mapping $\alpha \mapsto \mathrm{ES}^{\alpha}(X)$ is continuous follows trivially from part (a). Let $\varepsilon > 0$ and let f_α be as in (1.4.2). It is then easy to check that

$$f_{\alpha+\varepsilon} - f_\alpha \begin{cases} \leq 0, & X < x_{(\alpha)}; \\ \geq 0, & X > x_{(\alpha)}. \end{cases}$$

Now let $A = \{X < x_{(\alpha)}\}$ and observe

$$\begin{aligned} \mathrm{ES}^{\alpha}(X) - \mathrm{ES}^{\alpha+\varepsilon}(X) &= \mathbb{E}^{\mathbb{P}}[-Xf_\alpha] - \mathbb{E}^{\mathbb{P}}[-Xf_{\alpha+\varepsilon}] \\ &= \mathbb{E}^{\mathbb{P}}[X(f_{\alpha+\varepsilon} - f_\alpha)] \\ &= \mathbb{E}^{\mathbb{P}}[\mathbf{1}_A \cdot (-X)\underbrace{(f_\alpha - f_{\alpha+\varepsilon})}_{\geq 0}] + \mathbb{E}^{\mathbb{P}}[\mathbf{1}_{A^c} \cdot X\underbrace{(f_{\alpha+\varepsilon} - f_\alpha)}_{\geq 0}] \\ &\geq \mathbb{E}^{\mathbb{P}}[\mathbf{1}_A \cdot (-x_{(\alpha)})(f_\alpha - f_{\alpha+\varepsilon})] + \mathbb{E}^{\mathbb{P}}[\mathbf{1}_{A^c} \cdot x_{(\alpha)}(f_{\alpha+\varepsilon} - f_\alpha)] \\ &= \mathbb{E}^{\mathbb{P}}[x_{(\alpha)}(f_{\alpha+\varepsilon} - f_\alpha)] \\ &= x_{(\alpha)}(1-1) \\ &= 0. \end{aligned}$$

This proves (b).

(c) After reading Example 1.3.13, the only thing left to prove is that ES is co-monotonically additive. But this fact follows easily from (1.3.1), the subsequent remarks, and part (a).

(d) This is also a trivial consequence of Example 1.3.13; we have $\mathrm{WCE}^{\alpha}(X) \leq \mathrm{ES}^{\alpha}(X)$ and $\mathrm{WCE}^{\alpha}(X) = \mathrm{ES}^{\alpha}(X)$ when the underlying probability space is non-atomic. Using the Inverse Transform Method, we can always find a random variable X' on an non-atomic probability space with the same distribution as X (note that X is not required to have a continuous distribution [53, Proposition 2.1]). □

In the literature, there are a variety of characterizations of coherent, law invariant, co-monotonically additive risk measures in terms of integrals of quantiles [61, 44, 67, 69]. Indeed, Theorem 1.4.5(a) is a special case of this. Kusuoka generalized with the following classical result [44, Theorem 4].

Theorem 1.4.6 (Kusuoka) *Let $\rho : L^\infty(\mathbb{P}) \to \mathbb{R}$. Then the following statements are equivalent:*

(a) *ρ is a law invariant coherent risk measure with the Fatou Property.*

(b) *There is a compact convex set $\mathcal{M}_0$ of probability measures on $[0,1]$ such that*

$$\rho(X) = \sup\left\{\int_0^1 \mathrm{ES}^{\alpha}(X)\, dm(\alpha) : m \in \mathcal{M}_0\right\}.$$

Moreover, ρ is co-monotonically additive if and only if the above supremum is attained.

In [39] it is shown that all law invariant risk measures already have the Fatou Property. Thus, the assumption of the Fatou property in the above theorem is superfluous and may be dropped.

1.4.3 The Relation with Value-at-Risk

In order to preserve the connection between the level of VaR and the probability of solvency, some would find it desirable to find a smallest coherent risk measure that dominates VaR [61]. In this case, the following theorem is a disappointment.

Theorem 1.4.7 *For each $X \in L^{\infty}(\mathbb{P})$ and $\alpha \in (0,1)$, we have that*

$$\mathrm{VaR}^{\alpha}(X) = \inf\left\{\rho(X) : \rho \geq \mathrm{VaR}^{\alpha},\ \rho \text{ coherent with the Fatou property}\right\}.$$

We omit the technical proof of this result, which can be found in [15, Theorem 6.8]. Since VaR is not coherent, this result shows that there is no smallest coherent risk measure that dominates VaR.

In order to find a smallest coherent risk measure, we must consider the smaller class of law invariant coherent risk measures that dominate VaR, where the underlying probability space is atomless. If we do this, the following result holds (cf. [15, Theorem 6.10]).

Theorem 1.4.8 *Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is an atomless probability space and $\alpha \in (0,1)$. Then, for any law-invariant coherent risk measure ρ, that satisfies the Fatou property and dominates VaR^{α}, we have $\rho \geq \mathrm{WCE}^{\alpha} = \mathrm{ES}^{\alpha}$.*

We have shown above that ES dominates VaR. The above result says that ES is, in some sense, the smallest coherent risk measure that dominates VaR. In practice, the difference between these two risk measures can be quite large. Figure 1.8 shows the difference between the 7-day ES surface and the 7-day VaR surface of the portfolio in Example 1.2.6. This makes it hard to see how the level of ES can be connected with the probability of default.

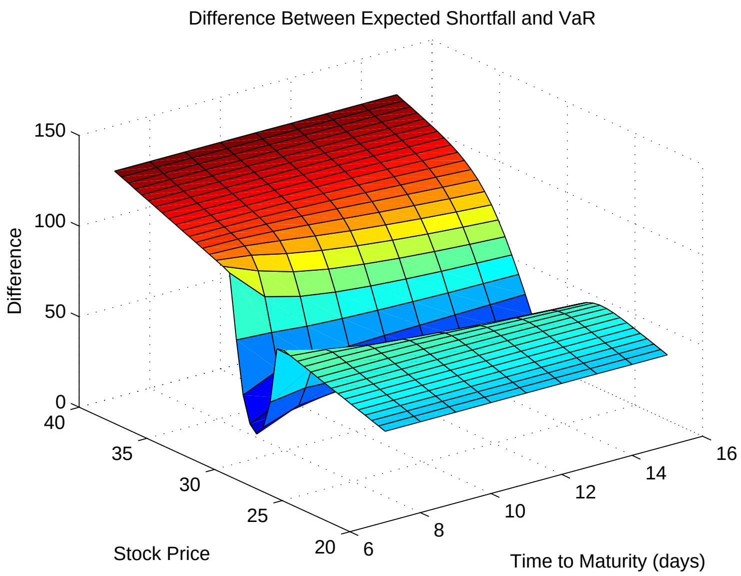

Figure 1.8: *The difference between the 7-day* ES *surface and the 7-day* VaR *surface of the portfolio in Example 1.2.6.*

1.5 Beyond Coherence

1.5.1 Distortion Measures and the Choquet Integral

We now generalize coherent risk measures further by considering the non-additive Choquet Integral. Consider a random variable X. If $X \geq 0$, we can write $X = \int_0^\infty \mathbf{1}_{\{X>u\}} \mathrm{d}u$. Similarly, if $X \leq 0$, we have $X = \int_{-\infty}^0 -\mathbf{1}_{\{X\leq u\}} \mathrm{d}u$. Consequently, for general X, we may write

$$X = \int_{-\infty}^0 -\mathbf{1}_{\{X\leq u\}} \mathrm{d}u + \int_0^\infty \mathbf{1}_{\{X>u\}} \mathrm{d}u.$$

Taking expectations on both sides gives

$$\begin{aligned}\mathbb{E}[X] &= \int_{-\infty}^{0} \mathbb{E}[-\mathbf{1}_{\{X\le u\}}]\,\mathrm{d}u + \int_{0}^{\infty} \mathbb{E}[\mathbf{1}_{\{X>u\}}]\,\mathrm{d}u \\ &= -\int_{-\infty}^{0} \mathbb{P}[X\le u]\,\mathrm{d}u + \int_{0}^{\infty} (1-\mathbb{P}[X\le u])\,\mathrm{d}u.\end{aligned}$$

This provides some insight into the following definitions.

Definition 1.5.1 (Distortion Probabilities) Let X be a random variable and $F_X(x) = \mathbb{P}(X \le x)$ be the distribution of X.

(a) If $g : [0, 1] \to [0, 1]$ is an increasing function with $g(0) = 0$ and $g(1) = 1$, then $F^*_X(x) = g(F_X(x))$ defines a *distorted probability distribution.*

(b) The function g is called a *distortion function.*

(c) The function g^* defined by $g^*(u) = 1 - g(1 - u)$ is called the *dual* distortion function.

Definition 1.5.2 (Choquet Integral) We define the *Choquet Integral*[1] with respect to the distortion function g and distorted probability $F^*_X(x) = g(F_X(x))$ to be

$$H_g[X] = -\int_{-\infty}^{0} g(F_X(x))\,\mathrm{d}x + \int_{0}^{\infty} [1 - g(F_X(x))]\,\mathrm{d}x.$$

If $X \ge 0$, we have

$$H_g[X] = \int_{0}^{\infty} [1 - g(F_X(x))]\,\mathrm{d}x.$$

The Choquet integral has long appeared in the insurance and actuarial literature [69, 65, 66, 64, 67]. It turns out that there is a significant overlap with the theory of risk measures [15, 66, 61].

Care must be taken when using the Choquet Integral because it is non-additive and asymmetrical. The theory of non-additive integration is treated in [19]. We collect some useful properties [67, Theorem 3]:

[1] In the literature, the Choquet integral is defined in terms of the survival function $S_X(x) = 1 - F_X(x)$. I.e. $H_h[X] = -\int_{-\infty}^{0}[1 - h(S_X(x))]\,\mathrm{d}x + \int_0^\infty [h(S_X(x))]\,\mathrm{d}x$. Since we are working with F_X, the distortion function g is dual to h. As a result, the properties listed for $H_g[\,\cdot\,]$ are symmetrical to those listed in the literature.

Proposition 1.5.3 *Let $X, Y \in L^\infty(\mathbb{P})$, where the underlying probability space is non-atomic. Then the Choquet Integral has the following properties:*

(a) $H_g[-X] = -H_{g^*}[X]$, *thus H_g asymmetrical,*

(b) $H_g[1] = 1$,

(c) $\mathbb{E}[X] \leq H_g[X]$ *for all X if and only if $g(u) \leq u$ for all $u \in [0,1]$,*

(d) $H_g[X] \leq \|X\|_\infty$,

(e) H_g is positively homogeneous,

(f) H_g is translation invariant,

(g) If g is convex, i.e. $g^{''} > 0$, then H_g is sub-additive and if g is concave, i.e. $g^{''} < 0$, then H_g is super-additive,

(h) H_g is law-invariant (and consequently has the Fatou Property [39]),

(i) $X \leq Y$ then $H_g[X] \leq H_g[Y]$,

(j) H_g is co-monotonically additive,

(k) $\lim_{d\to 0^+} H_g[(X-d)_+] = H_g[X]$ *and* $\lim_{d\to\infty} H_g[X \wedge d] = H_g[X]$.

If we are working with $L^\infty(\mathbb{P})$, we need consider only positive random variables. Indeed, by the translation invariance of the Choquet Integral, we can shift any $X \in L^\infty(\mathbb{P})$ by $\|X\|_\infty$ so that it is positive. We then subtract $\|X\|_\infty$ again from resulting calculation.

Definition 1.5.4 (Distortion Risk Measure) Let $X \in L^\infty(\mathbb{P})$ denote the random variable of profits and losses of some portfolio. Then, for a distortion function g, we define the *Distortion Risk Measure* of X to be

$$\rho_g(X) = H_g[-X] = -H_{g^*}[X].$$

Observe that g is convex if and only if g^* is concave. As a consequence of the above proposition we have the following result.

Theorem 1.5.5 *The Distortion Risk Measure ρ_g is a coherent risk measure provided that the distortion function g is convex (equivalently, the dual distortion function g^* is concave). Moreover, ρ_g enjoys all the properties listed in Proposition 1.5.3.*

1.5.2 Some Familiar Examples

Using the appropriate distortion function we can recover some familiar risk measures:

Example 1.5.6 (Value-at-Risk) With the distortion function

$$g(u) = \begin{cases} 1, & u > 1-\alpha; \\ 0, & u \leq 1-\alpha, \end{cases}$$

we obtain $\rho_g(X) = H_g[-X] = \mathrm{VaR}^{\alpha}(X)$. Notice that g is not convex or continuous. Consequently, ρ_g is not coherent.

Example 1.5.7 (Average Loss) If we define the distortion function to be $g(u) = u$, we obtain $\rho_g(X) = H_g[-X] = \mathbb{E}^{\mathbb{P}}[-X]$. Since g is convex, ρ_g is coherent.

Example 1.5.8 (Expected Shortfall) With the distortion function

$$g(u) = \begin{cases} \frac{\alpha+u-1}{\alpha}, & u > 1-\alpha; \\ 0, & u \leq 1-\alpha, \end{cases}$$

we obtain $\rho_g(X) = H_g[-X] = \mathrm{ES}_{\alpha}(X)$. Since g is convex, ρ_g is coherent.

The last two examples enjoy the properties listed in Proposition 1.5.3. Observe that the set $\{u \in (0,1) : g(u) = 0\}$ represents the portion of the distribution of X that is discarded when applying the Choquet integral.

Remark: We have seen that a convex distortion function g corresponds to a coherent risk measure ρ_g via the Choquet Integral. Since ρ_g has the Fatou Property, Theorem 1.3.5 implies that there is a $\|\cdot\|_1$-closed convex set of measures $\mathcal{P}_g$, all absolutely continuous to $\mathbb{P}$, so that $\rho_g(X) = \sup\{\mathbb{E}^{\mathbb{Q}}[-X] : \mathbb{Q} \in \mathcal{P}_g\}$. It would be interesting to characterize $\mathcal{P}_g$ in terms of g.

1.5.3 The Wang Transform

One of the criticisms leveled against Expected Shortfall is the fact that only the tail of the profit and loss distribution is considered. The information in the remaining portion of the distribution is discarded. Moreover, Expected Shortfall does not properly adjust for extreme low-frequency and high-severity losses because it does not take higher moments into account. In [66], a new distortion function is recommended that accounts for all this and thus 'goes beyond coherence'.

Definition 1.5.9 (Wang Transform) The *Wang Transform* is defined by the distortion function

$$g_\mu(u) = \Phi(\Phi^{-1}(u) - \mu).$$

We shall denote the corresponding Choquet integral applied to the random variable X by $H[X;\mu]$.

In the case of risk measures, it is useful to talk in terms of a significance level. For the *Wang Transform Risk Measure* we write

$$\mathrm{WT}^{\alpha}(X) = H[-X; -\Phi^{-1}(\alpha)] = -H[X; \Phi^{-1}(\alpha)]$$

for a given significance level $\alpha \in (0,1)$. Here, Φ denotes the standard normal cumulative distribution function.

If X is a standard normal random variable, then $H[X;\mu]$ has the action of setting the mean of X to μ, while leaving the standard deviation unchanged. We collect some properties of the the Wang Transform in the following Theorem (cf. [66, pp. 20–22]).

Theorem 1.5.10 (Wang Transform) The Wang Transform has the following useful properties:

(a) The first derivative of g_μ is

$$\frac{\mathrm{d}g_\mu(u)}{\mathrm{d}u} = \exp\left(\mu\Phi^{-1}(u) - \frac{\mu^2}{2}\right).$$

(b) The second derivative of g_μ is

$$\frac{\mathrm{d}^2 g_\mu(u)}{\mathrm{d}^2 u} = \frac{\mu\phi(\Phi^{-1}(u) - \mu)}{\phi(\Phi^{-1}(u))^2};$$

so that g_μ is convex for positive μ and concave for negative μ.

(c) The dual distortion operator of g_μ is

$$g^*_\mu(u) = 1 - g_\mu(1-\mu) = g_{-\mu}(u).$$

(d) $\text{ess-inf}\,(X) \le H[X;\mu] \le \text{ess-sup}\,(X)$.

(e) $H[X;\mu]$ is an increasing function of μ. Moreover, $H[X;\mu]$ approaches the above bounds when μ tends to $-\infty$ and ∞ respectively.

(f) $H[\,\cdot\,;\mu]$ is translation invariant.

(g) $H[\,\cdot\,;\mu]$ is positively homogeneous.

(h) For $\lambda < 0$ we have $H[\lambda X;\mu] = \lambda H[X;-\mu]$.

(i) $H[\,\cdot\,;\mu]$ is co-monotonically additive.

(j) $H[\,\cdot\,;\mu]$ is sub-additive when $\mu > 0$ and super-additive when $\mu < 0$. Correspondingly, WT is sub-additive when $0 < \alpha < \frac{1}{2}$ and super-additive when $\frac{1}{2} < \alpha < 1$.

(k) $H[X;\mu] > \mathbb{E}[X]$ when $\mu > 0$, $H[X;\mu] = \mathbb{E}[X]$ when $\mu = 0$ and $H[X;\mu] < \mathbb{E}[X]$ when $\mu < 0$.

(l) If $X \sim \mathcal{N}(\gamma,\sigma)$, then $g_\mu \circ F_X$ is a normal distribution given by $\mathcal{N}(\gamma+\mu\sigma,\sigma)$. Consequently, $H[X;\mu] = \mathbb{E}[X] + \mu\sigma[X]$. Note that the standard deviation is left unchanged.

(m) If $\log(X) \sim \mathcal{N}(\gamma,\sigma)$, then $g_\mu \circ F_X$ is a log-normal distribution which corresponds to a random variable whose logarithm is distributed $\mathcal{N}(\gamma+\mu\sigma,\sigma)$.

Figure 1.9 depicts the distortion function associated with the Wang Transform Risk Measure at a significance level of 5%. It is clearly convex, so that WT is coherent and enjoys the properties listed in Theorem 1.5.10.

Since the WT distortion function has the property that $g(u) \in \{0,1\}$ if and only if $u \in \{0,1\}$, the Wang Transform takes the entire distribution into account when measuring the risk. For normal and log-normals risks, the Wang Transform has the effect of moving the expected value to the α-th percentile of the original distribution.

Calculation of the Wang Transform

Calculation of the Choquet integral can prove to be cumbersome. When it comes to positive increasing functions of a standard normal random variable, the following result offers some assistance.

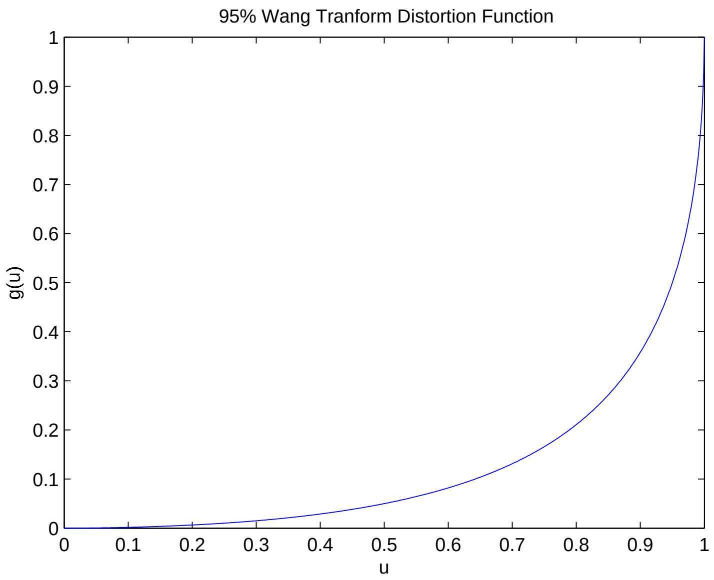

Figure 1.9: *The 95% Wang Transform distortion function.*

Proposition 1.5.11 *Let $k \in \mathbb{R}$ and $h : \mathbb{R} \to [0, \infty)$ be continuous on $\mathbb{R}$ and strictly increasing on the interval (k, ∞). Assume that $h((-\infty, k]) = \{0\}$ and $X = h(Z)$ where Z is a standard normal random variable. Then*

$$H[X; \mu] = \mathbb{E}[h(Z + \mu)].$$

Proof. Let $\varepsilon > 0$, then

$$\begin{aligned} H[X; \mu] &= \int_0^\infty [1 - g_\mu(F_X(x))] \, \mathrm{d}x \\ &= \int_\varepsilon^\infty [1 - g_\mu(F_X(x))] \, \mathrm{d}x + \int_0^\varepsilon [1 - g_\mu(F_X(x))] \, \mathrm{d}x \\ &= \int_\varepsilon^\infty [1 - g_\mu(\Phi(h^{-1}(x)))] \, \mathrm{d}x + \int_0^\varepsilon g_{-\mu}(\mathbb{P}[h(Z) > x]) \, \mathrm{d}x \\ &= \int_\varepsilon^\infty [1 - \Phi(h^{-1}(x) - \mu)] \, \mathrm{d}x + \int_0^\varepsilon g_{-\mu}(\mathbb{P}[h(Z) > x]) \, \mathrm{d}x \\ &= \int_\varepsilon^\infty [1 - \mathbb{P}[h(Z + \mu) \leq x]] \, \mathrm{d}x + \int_0^\varepsilon g_{-\mu}(\mathbb{P}[h(Z) > x]) \, \mathrm{d}x. \end{aligned}$$

To complete the proof, observe that

$$0 \leq \int_0^\varepsilon g_{-\mu}(\mathbb{P}[X > x]) \, \mathrm{d}x \leq \varepsilon$$

and let $\varepsilon \to 0$. □

When it comes to general distributions, a simple observation allows for an efficient Monte Carlo approximation. It is plain that $H_g[X] = \mathbb{E}[Y]$ where Y is drawn from the distorted distribution $g \circ F_X$. Thus,

$$\begin{aligned} H_g[X] &= \int_{-\infty}^{\infty} x \, \mathrm{d}(g \circ F_X)(x) \\ &= \int_{-\infty}^{\infty} x \, g'(F_X(x)) f_X(x) \, \mathrm{d}x \\ &= \int_0^1 F_X^{-1}(u) g'(u) \, \mathrm{d}u. \end{aligned}$$

Here, f_X denotes the probability density function of X. The above equation implies that the Choquet integral can be calculated as the mean of samples of X (generated using the Inverse Transform Method, say) multiplied by the derivative of g applied to the corresponding percentiles of X. In other words,

$$H_g[X] \sim \frac{1}{n} \sum_{i=0}^{N} x_i \cdot g'(F_X(x_i)),$$

where the x_i are samples drawn from the distribution of X. The derivative of g can be calculated using a finite difference method. In the case of the Wang Transform, we have an explicit formula for the derivative of g.

To illustrate this method, we return to Example 1.2.6 with payoff at maturity given by Figure 1.3. Figure 1.10 shows a 7-day WT surface simulation. Figure 1.11 shows the diversification benefit from combining the risks of the individual options.

Figure 1.12 depicts the difference between the Wang Transform surface and the corresponding Expected Shortfall and Value-at-Risk surfaces. Notice how the Wang Transform is dominated by Expected Shortfall. At first glance, this may seem to contradict Theorem 1.4.8. However, another glance at the adjacent picture shows that the Wang Transform does not dominate Value-at-Risk. This violates one of the conditions of the theorem.

1.5.4 The Relationship between the Wang Transform and CAPM

In this section, we recover the CAPM model using the Wang Transform. We make the crucial assumption that the prevailing price of an asset can be determined by applying $H[\,\cdot\,;\mu]$ to the discounted future asset price. This is much like assuming

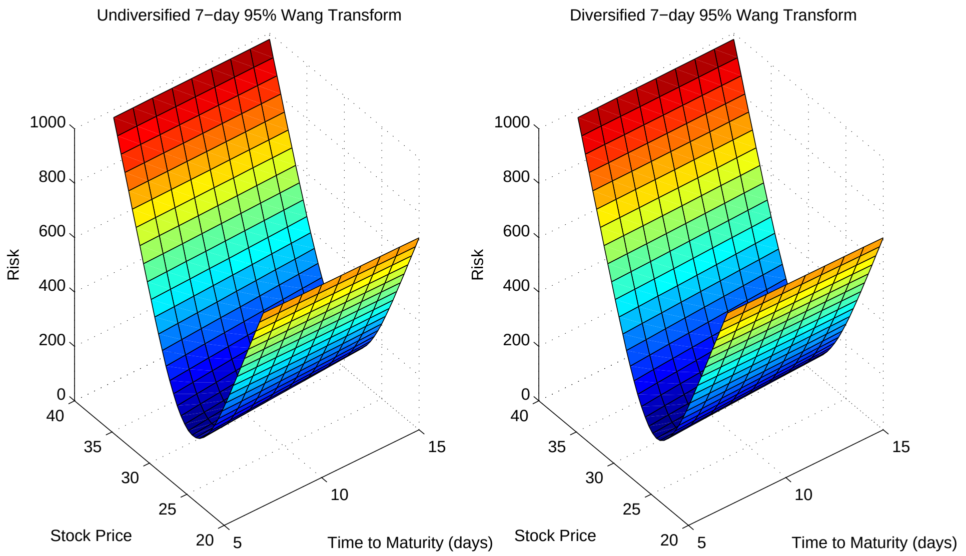

Figure 1.10: *The 7-day Wang Transform surface of the portfolio in Example 1.2.6.*

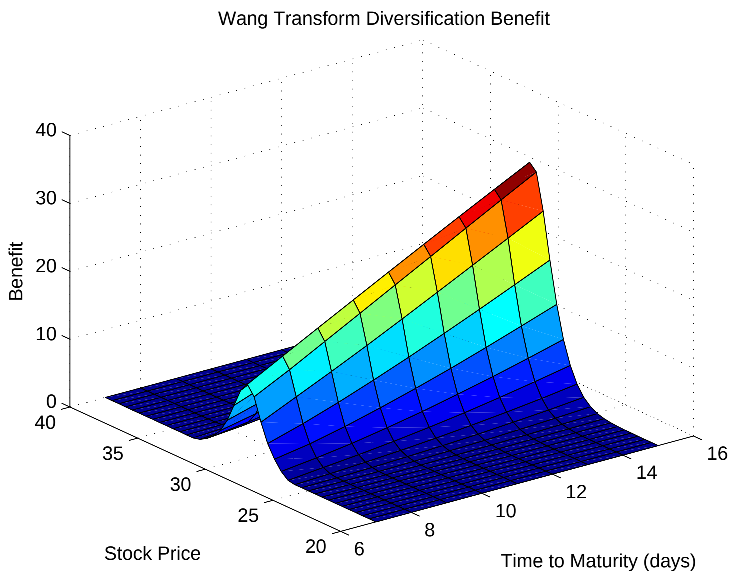

Figure 1.11: *The 7-day Wang Transform diversification benefit of the portfolio in Example 1.2.6. In contrast to* VaR*, notice how the diversification benefit does not drop below zero.*

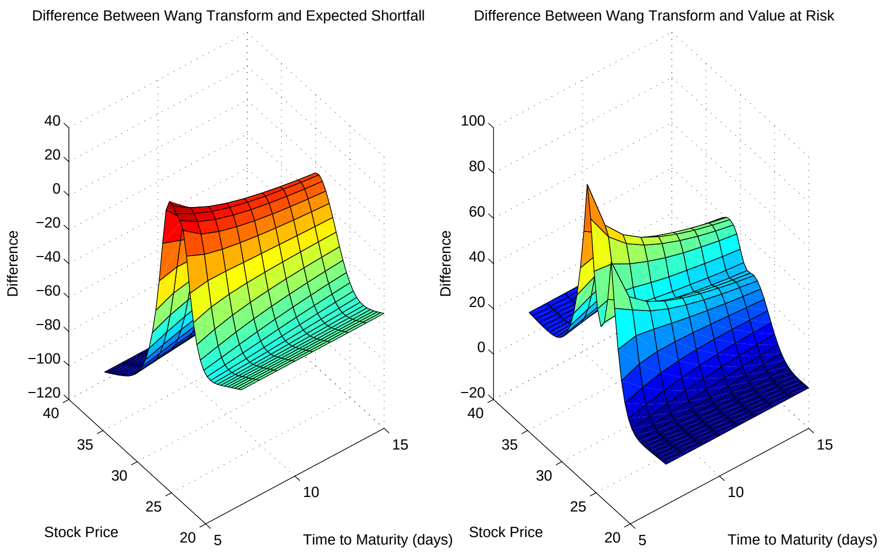

Figure 1.12: *The difference between the 7-day Wang Transform surface and the corresponding 7-day* ES *and* VaR *surfaces of the portfolio in Example 1.2.6.*

the asset price process is a martingale (or equivalently assuming the absence of arbitrage). Given this pricing assumption, we infer the value of μ from the market.

Let $A_i(0)$ denote the current price of asset i and $A(1)$ its asset price after one time step. Denote by $R_i = A_i(1)/A_i(0) - 1$ the simple return of asset i over that period. Assume that R_i is normally distributed with mean $\mathbb{E}[R_i]$ and standard deviation $\sigma[R_i]$. Assume that

$$A_i(0) = H\left[\frac{A_i(1)}{1+r_f}; -\mu_i\right] = H\left[A_i(0)\frac{1+R_i}{1+r_f}; -\mu_i\right],$$

where r_f denotes the deterministic risk free rate of return. It follows that

$$\begin{aligned} A_i(0)(1+r_f) &= A_i(0)(1 + A_i(0)H\left[R_i; -\mu_i\right]) \\ &\Rightarrow \quad r_f = H\left[R_i; -\mu_i\right] = \mathbb{E}[R_i] - \mu_i\sigma[R_i]. \end{aligned}$$

Consequently,

$$\mu_i = \frac{\mathbb{E}[R_i] - r_f}{\sigma[R_i]}.$$

For the market portfolio M, the risk adjusted rate of return must equal the risk free rate. This leads to

$$r_f = H\left[R_M; -\mu_M\right] = \mathbb{E}[R_M] - \mu_M\sigma[R_M]$$

so that

$$\mu_M = \frac{\mathbb{E}[R_M] - r_f}{\sigma[R_M]}.$$

This quantity is known as the *Sharpe ratio*.

The CAPM model asserts that

$$\mathbb{E}[R_i] = r_f + \beta_i[\mathbb{E}[R_M] - r_f]$$

where

$$\beta_i = \frac{\mathbb{C}\text{ov}\,[R_i, R_M]}{\sigma[R_M]^2}$$

is the beta of asset i.

This relationship can be rewritten as

$$\frac{\mathbb{E}[R_i] - r_f}{\sigma[R_i]} = \rho_{i,M}\frac{\mathbb{E}[R_M] - r_f}{\sigma[R_M]}$$

where $\rho_{i,M}$ is the correlation between R_i and R_M. $\rho_{i,M}$ is sometimes referred to as the *systematic risk* of A_i in relation to the market. Consequently, we have

$$\mu_i = \rho_{i,M} \cdot \mu_M$$

and

$$\mu_i \cdot \sigma[R_i] = \beta_i(\mu_M \cdot \sigma[R_M])$$

so that μ_i corresponds to the systematic risk and the beta of asset i.

1.5.5 Recovery of the Black-Scholes Formula

Suppose that the asset A_t follows the process

$$\mathrm{d}A_t = A_t\gamma\mathrm{d}t + A_t\sigma\mathrm{d}W_t$$

where W is a standard Brownian Motion, γ is the drift and σ is the volatility of A_t. The solution to this stochastic differential equation is given by

$$A_t = f_t(Z) := A_0 \exp\left(\left(\gamma - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}Z\right).$$

Note that f_t is a strictly increasing, continuous function on $\mathbb{R}$ taking values in $(0, \infty)$.

We will use the Wang Transform to recover the Black-Scholes formula for a European call option with strike K and maturity T. As before, we assume the absence of arbitrage, which means that

$$A_0 = H[\exp(-r_f T)A_T; -\mu] = \exp(-r_f T)H[A_T; -\mu],$$

where r_f is now the continuously compounded risk free rate of return. We may rewrite this as

$$A_0 = \exp(-r_f T)\mathbb{E}[B_T],$$

where B_T is drawn from the distorted distribution $F_{B_T} = g_{-\mu} \circ F_{A_T}$.

Since $\log\left(\frac{A_T}{A_0}\right) \sim \mathcal{N}\left(\left(\gamma - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$, it follows that

$$\log\left(\frac{B_T}{A_0}\right) \sim \mathcal{N}\left(\left(\gamma - \frac{\sigma^2}{2}\right)T - \mu\sigma\sqrt{T}, \sigma^2 T\right).$$

Thus,

$$\begin{aligned} A_0 &= \exp(-r_f T)A_0\mathbb{E}\left[\frac{B_T}{A_0}\right] \\ &= A_0 \exp\left(-r_f T + \left(\gamma - \frac{\sigma^2}{2}\right)T - \mu\sigma\sqrt{T} + \frac{\sigma^2 T}{2}\right) \\ &\Rightarrow 0 = (\gamma - r_f)T - \mu\sigma\sqrt{T} \\ &\Rightarrow \mu = \frac{(\gamma - r_f)\sqrt{T}}{\sigma}, \end{aligned}$$

so that μ is the market price of risk.

The payoff of the European call option is given by $C(A_T) = (A_T - K)_+$. Define $h(x) = C(f_T(x))$, then h is continuous, strictly increasing on the interval $(f_T^{-1}(K), \infty)$ and $h((-\infty, f_T^{-1}(K)]) = \{0\}$. With the help of Proposition 1.5.11, the price of the option may be calculated as:

$$\begin{aligned} & H[C(A_T); -\mu] \\ =\; & H[h(Z); -\mu] \\ =\; & \mathbb{E}[h(Z - \mu)] \\ =\; & \int_{-\infty}^{\infty} C\left[A_0 \exp\left(\left(\gamma - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}\left(x - \frac{(\gamma - r_f)\sqrt{T}}{\sigma}\right)\right)\right]\phi(x)\,\mathrm{d}x \\ =\; & \int_{-\infty}^{\infty} C\left[A_0 \exp\left(\left(r_f - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x\right)\right]\phi(x)\,\mathrm{d}x, \\ =\; & \int_{-\infty}^{\infty} \left[A_0 \exp\left(\left(r_f - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x\right) - K\right]_+ \phi(x)\,\mathrm{d}x, \end{aligned}$$

which is precisely the Black-Scholes formula for pricing call options. The same argument does not apply to put options.

Chapter 2

The Fundamental Theorem of Asset Pricing

2.1 Introduction

The Fundamental Theorem of Asset Pricing (FTAP) is a result that connects the pricing of derivatives via a replicating portfolio and the principle of no-arbitrage on the one hand, and pricing by taking expectations with respect to an equivalent risk neutral measure on the other. The result can be loosely formulated as follows:

Theorem 2.1.1 (The Fundamental Theorem of Asset Pricing) *For a model S of a financial market, the following statements are approximately equivalent:*

(a) S does not allow for arbitrage.

(b) There exists an equivalent probability measure $\mathbb{Q}$ on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ under which S is a martingale.

The word 'approximately' is used in the above theorem because these statements are not mathematically equivalent without additional definitions and assumptions. Many versions of this theorem exist in different settings and at different levels of generality. In this chapter we will showcase some of the earlier efforts at making this theorem precise. We first look at the finite dimensional case of Harrison and Pliska [29] and then the infinite dimensional case of Kreps [43].

The techniques developed here will add context to the next chapter where we will examine this theorem in the setting of coherent risk measures.

Our presentation follows the illuminating surveys [55] and [18, Chapter 2]. We refer the reader to these works for more detail.

2.2 The Finite Dimensional Setting

2.2.1 Preliminaries

Assume that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$ is a finite, filtered probability space. A *financial market model* is a stochastic process

$$\widetilde{S} = (\widetilde{S}_t)_{t=0}^T = (\widetilde{S}_t^{(0)}, \widetilde{S}_t^{(1)}, \ldots, \widetilde{S}_t^{(d)})_{t=0}^T,$$

taking values in $\mathbb{R}^{d+1}$. We assume that the zero co-ordinate satisfies $\widetilde{S}_t^{(0)} > 0$ for all $t = 0, \ldots, T$ and $\widetilde{S}_0^{(0)} = 1$. We will refer to $\widetilde{S}^0$ as the *numéraire asset* and it usually denotes a bank account.

A *trading strategy* $\widetilde{H} = (\widetilde{H}_t)_{t=0}^T = (\widetilde{H}_t^{(0)}, \widetilde{H}_t^{(1)}, \ldots, \widetilde{H}_t^{(d)})_{t=0}^T$ is a predictable (i.e. $\widetilde{H}_t$ is $\mathcal{F}_{t-1}$-measurable) process taking values in $\mathbb{R}^{d+1}$. Observe that between time t and $t-1$, the agent holds the quantity $\widetilde{H}_t^j$ of asset j and this quantity is determined at $t-1$. This explains the economic requirement of the predictability of $\widetilde{H}$.

We will say that the trading strategy $\widetilde{H}$ is *self financing* if we have

$$\langle \widetilde{H}_t, \widetilde{S}_t \rangle = \langle \widetilde{H}_{t+1}, \widetilde{S}_t \rangle$$

for all $t = 0, \ldots, T-1$. The quantity $\widetilde{V}_t = \langle \widetilde{H}_t, \widetilde{S}_t \rangle = \langle \widetilde{H}_{t+1}, \widetilde{S}_t \rangle$ is the value of the portfolio of assets described by $\widetilde{S}$ held in volumes described by $\widetilde{H}$. If $\widetilde{H}$ is self financing, it means that there is no in or out flow of money when altering the volumes of the assets in the portfolio. Adjustments to the portfolio are either funded by, or liquidated to, the numéraire asset.

It is easier to account in units of the numéraire asset. This is achieved by writing

$$S = (S_t)_{t=0}^T = (S_t^{(0)}, S_t^{(1)}, \ldots, S_t^{(d)})_{t=0}^T := \left(1, \frac{\widetilde{S}_t^{(1)}}{\widetilde{S}_t^{(0)}}, \ldots, \frac{\widetilde{S}_t^{(d)}}{\widetilde{S}_t^{(0)}}\right)_{t=0}^T.$$

S is known as the *discounted process*. Since co-ordinate zero is always equal to one, we may omit it from the above notation so that S takes its values in $\mathbb{R}^d$. I.e. we now write

$$S = (S_t^{(1)}, \dots, S_t^{(d)})_{t=0}^T.$$

Let H be the process obtained from $\widetilde{H}$ by discarding the first (numéraire) coordinate. In other words, H is the $\mathbb{R}^d$-valued process defined by

$$H = (H_t)_{t=0}^T = (H_t^{(1)}, \dots, H_t^{(d)})_{t=0}^T := (\widetilde{H}_t^{(1)}, \dots, \widetilde{H}_t^{(d)})_{t=0}^T.$$

Theorem 2.2.1 *For every predictable process $H = (H_t^{(1)}, \dots, H_t^{(d)})$ taking values in $\mathbb{R}^d$, there exists a unique self financing trading strategy $\widetilde{H} = (\widetilde{H}_t^{(0)}, \widetilde{H}_t^{(1)}, \dots, \widetilde{H}_t^{(d)})$ taking values in $\mathbb{R}^{d+1}$ such that $(\widetilde{H}_t^{(j)})_{t=1}^T = (H_t^{(j)})_{t=1}^T$ for $j = 1, \dots, d$ and $\widetilde{H}_1^{(0)} = 0$.*

The above result is easy to verify but economically important. It says that given any trading strategy $H = (H_t^{(1)}, \dots, H_t^{(d)})$ in d risky assets, we may always add an extra trading strategy $(\widetilde{H}_t^{(0)})$ in the numéraire asset such that the entire strategy becomes self financing. Moreover, if we normalize by requiring $\widetilde{H}_1^{(0)} = 0$, this trading strategy becomes unique.

The discounted portfolio value $V_t = \widetilde{V}_t / \widetilde{S}_t^{(0)}$ depends only on the $\mathbb{R}^d$-dimensional process H. Indeed,

$$\widetilde{V}_0 = V_0 = \langle \widetilde{H}_1, \widetilde{S}_0 \rangle = \langle H_1, S_0 \rangle,$$

using the convention $\widetilde{S}_0^{(0)} = 1$ and $\widetilde{H}_1^{(0)} = 0$. Moreover, since H is self financing, we have

$$\begin{aligned}
\Delta V_t &= V_t - V_{t-1} = \frac{\widetilde{V}_t}{\widetilde{S}_t^{(0)}} - \frac{\widetilde{V}_{t-1}}{\widetilde{S}_{t-1}^{(0)}} \\
&= \frac{\langle \widetilde{H}_t, \widetilde{S}_t \rangle}{\widetilde{S}_t^{(0)}} - \frac{\langle \widetilde{H}_{t-1}, \widetilde{S}_{t-1} \rangle}{\widetilde{S}_{t-1}^{(0)}} \\
&= \frac{\langle \widetilde{H}_t, \widetilde{S}_t \rangle}{\widetilde{S}_t^{(0)}} - \frac{\langle \widetilde{H}_t, \widetilde{S}_{t-1} \rangle}{\widetilde{S}_{t-1}^{(0)}} \\
&= \widetilde{H}_t^{(0)} + \langle H_t, S_t \rangle - (H_t^{(0)} + \langle H_t, S_{t-1} \rangle) \\
&= \langle H_t, \Delta S_t \rangle,
\end{aligned}$$

where $\Delta S_t := S_t - S_{t-1}$. Therefore, at maturity T, we may write

$$V_T = V_0 + \sum_{t=1}^T \langle H_t, \Delta S_t \rangle = V_0 + (H \cdot S)_T, \tag{2.2.1}$$

where $(H \cdot S)_T := \sum_{t=1}^{T} \langle H_t, \Delta S_t \rangle$ is the notation for a stochastic integral from the theory of stochastic integration. In this discrete setting, the stochastic integral takes the form of a Riemann sum. To know what the actual value of V_T at time T is, we need to make the calculation $\widetilde{V}_T = V_T \widetilde{S}_T^{(0)}$.

Given the above discussion, we will work with a discounted $\mathbb{R}^d$-valued financial process S and trading strategy H, safe in the knowledge that it may be uniquely transformed into a self-financing portfolio.

2.2.2 Attainable Claims and Martingale Measures

Definition 2.2.2 We call the subspace $K \subset L^0(\mathbb{P})$ defined by

$$K = \{(H \cdot S)_T : H \text{ a trading strategy in } \mathbb{R}^d\}$$

the *set of contingent claims attainable at price 0.*

The set K contains precisely those payoff functions at time T, depending on $\omega \in \Omega$, that an economic agent may replicate with zero initial investment, following some trading strategy H.

For $a \in \mathbb{R}$, the set $K_a := a + K$ is called the *set of contingent claims attainable at price a.* These are all the terminal portfolio values of the form (2.2.1). For convenience, K_0 is denoted by K.

Definition 2.2.3 We call the convex cone $C \subset L^\infty(\mathbb{P})$ defined by

$$C = \{g \in L^\infty(\mathbb{P}) : \exists\, f \in K \text{ such that } f \geq g\}$$

the set of contingent claims super-replicable at price 0.

The set C contains all the terminal payoffs that may be super-replicated at zero initial cost. In the event that the super-replication of $g \in C$ is strict, we may simply throw away money to arrive at g. This is known as *free disposal* and plays an indispensable role in the continuous version of the fundamental theorem of asset pricing later on. As before, we write $C_a := a + C$ for the set of all *contingent claims super-replicable at price $a \in \mathbb{R}$*. We are now in a position to formulate the notion of no-arbitrage mathematically.

Definition 2.2.4 (NO-ARBITRAGE) A financial market S satisfies the no-arbitrage condition (NA) if $K \cap L^0_+(\mathbb{P}) = \{0\}$.

Economically speaking, the no-arbitrage condition demands that any terminal payoff that precludes loss and is attainable at zero initial investment should not allow any chance of making money, no matter how small.

Proposition 2.2.5 *For the financial market model S and corresponding subspace K and cone C in $L^\infty(\mathbb{P})$ we have the following:*

(a) $C = K - L^\infty_+(\mathbb{P})$.

(b) The no-arbitrage condition $K \cap L^0_+(\mathbb{P}) = \{0\}$ is equivalent to $C \cap L^0_+(\mathbb{P}) = \{0\}$.

(c) If S satisfies the no-arbitrage condition, then $C \cap (-C) = K$.

Proof. (a) If $g \in K - L^\infty_+(\mathbb{P})$, then $g = f_1 - f_2$ where $f_1 \in K$ and $f_2 \in L^\infty_+(\mathbb{P})$. Consequently, $g \le f_1$ so that $g \in C$. Conversely, let $g \in C$. Then there exists $f_1 \in K$ such that $f_2 := f_1 - g \in L^\infty_+(\mathbb{P})$. Thus, $g = f_1 - f_2 \in K - L^\infty_+(\mathbb{P})$.

(b) Since $K \subset C$, we have that $C \cap L^0_+(\mathbb{P}) = \{0\}$ implies $K \cap L^0_+(\mathbb{P}) = \{0\}$. Conversely, assume $K \cap L^0_+(\mathbb{P}) = \{0\}$ and suppose that $0 \ne g \in C \cap L^0_+(\mathbb{P})$. Then there exists $f \in K$ such that $f \ge g$. Hence $0 \ne f \in K \cap L^0_+(\mathbb{P}) = \{0\}$, a contradiction.

(c) Clearly, $K \subset C \cap (-C)$. For the converse, let $g \in C \cap (-C)$. By part (a), we may write $g = f_1 - h_1$ with $f_1 \in K$ and $h_1 \in L^\infty_+(\mathbb{P})$. On the other hand, we may also write $g = f_2 + h_2$ with $f_2 \in K$ and $h_2 \in L^\infty_+(\mathbb{P})$. Consequently, $f_1 - f_2 = h_1 + h_2 \in L^\infty_+(\mathbb{P})$ so that $f_1 - f_2 \in K \cap L^\infty_+(\mathbb{P}) = \{0\}$. Plainly, $f_1 = f_2$ and $h_1 + h_2 = 0$. Since $h_1, h_2 \in L^\infty_+(\mathbb{P})$, we must have $h_1 = h_2 = 0$. It follows that $g = f_1 = f_2 \in K$, as required. □

Definition 2.2.6 (EQUIVALENT MARTINGALE MEASURE) Let $\mathbb{Q}$ be a probability measure on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=1}^T, \mathbb{P})$ and let S denote a financial market model on this space.

(a) S is called a martingale under $\mathbb{Q}$ if $\mathbb{E}^{\mathbb{Q}}[S_{t+1}|\mathcal{F}_t] = S_t$ for all $t = 0, \dots, T-1$.

(b) $\mathbb{Q}$ is called an equivalent martingale measure if $\mathbb{Q} \sim \mathbb{P}$ and S is a martingale under $\mathbb{Q}$.

(c) We denote by $\mathcal{M}^e(S)$ the set of all equivalent martingale measures with respect to S.

(d) We denote by $\mathcal{M}^a(S)$ the set of all probability measures $\mathbb{Q} \ll \mathbb{P}$ under which S is a martingale.

In the finite dimensional setting, $\mathbb{Q} \sim \mathbb{P}$ if and only if $\mathbb{Q}[\omega] > 0$ for each $\omega \in \Omega$. Also note that all probability measures $\mathbb{Q}$ automatically satisfy $\mathbb{Q} \ll \mathbb{P}$ in the finite dimensional setting. However, this is not the case when passing to infinite dimensions.

Lemma 2.2.7 *Let $\mathbb{Q}$ be a probability measure on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=1}^T, \mathbb{P})$. The following statements are equivalent:*

(a) $\mathbb{Q} \in \mathcal{M}^a(S)$.

(b) $\mathbb{E}^{\mathbb{Q}}[f] = 0$, *for all* $f \in K$.

(c) $\mathbb{E}^{\mathbb{Q}}[g] \leq 0$, *for all* $g \in C$.

Proof. (a)⇔(b) Given that (a) is true, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[f] &= \mathbb{E}^{\mathbb{Q}}\left[\sum_{t=1}^{T} \langle H_t, \Delta S_t \rangle\right] = \sum_{t=1}^{T} \mathbb{E}^{\mathbb{Q}}[\langle H_t, \Delta S_t \rangle] \\
&= \sum_{t=1}^{T}\sum_{j=1}^{d} \mathbb{E}^{\mathbb{Q}}[H_t^{(j)} \Delta S_t^{(j)}] = \sum_{t=1}^{T}\sum_{j=1}^{d} \mathbb{E}^{\mathbb{Q}}[H_t^{(j)} \mathbb{E}^{\mathbb{Q}}[\Delta S_t^{(j)} | \mathcal{F}_{t-1}]] \\
&= \sum_{t=1}^{T}\sum_{j=1}^{d} \mathbb{E}^{\mathbb{Q}}[H_t^{(j)} \underbrace{(\mathbb{E}^{\mathbb{Q}}[S_t^{(j)} | \mathcal{F}_{t-1}] - S_{t-1}^{(j)})}_{=0}] = 0,
\end{aligned}$$

for all $f \in K$. Conversely, consider trading strategies $(H_t)_{t=1}^T$ of the form $H_t = x\mathbf{1}_A \in \mathbb{R}^d$ for some $x \in \mathbb{R}^d$, $A \in \mathcal{F}_{t-1}$, $1 \leq t \leq T$, and $H_s = 0 \in \mathbb{R}^d$ for all $s \neq t$, $1 \leq s \leq T$. By assumption, for all such trading strategies, we have

$$0 = \mathbb{E}^{\mathbb{Q}}\left[\sum_{t=1}^{T} \langle H_t, \Delta S_t \rangle\right] = \mathbb{E}^{\mathbb{Q}}[\langle x\mathbf{1}_A, \Delta S_t \rangle] = \sum_{j=1}^{d} \mathbb{E}^{\mathbb{Q}}[x_j \mathbf{1}_A \cdot \Delta S_t^{(j)}].$$

By considering the unit vector basis for $\mathbb{R}^d$, we can deduce

$$\frac{\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_A(S_t - S_{t-1})]}{\mathbb{Q}[A]} = \mathbb{E}^{\mathbb{Q}}[(S_t - S_{t-1})|A] = 0 \in \mathbb{R}^d,$$

for all $A \in \mathcal{F}_{t-1}$ with $\mathbb{Q}[A] > 0$. Consequently,

$$\mathbb{E}^{\mathbb{Q}}[S_t - S_{t-1}|\mathcal{F}_{t-1}] = 0 \in \mathbb{R}^d.$$

This proves that S is a martingale.

(b)⇔(c) Suppose (b) is true and let $g \in C$. Then $g = f_1 - f_2$ with $f_1 \in K$ and $f_2 \in L_+^\infty(\mathbb{P})$. Consequently, $\mathbb{E}^{\mathbb{Q}}[g] = \mathbb{E}^{\mathbb{Q}}[f_1] - \mathbb{E}^{\mathbb{Q}}[f_2] \leq \mathbb{E}^{\mathbb{Q}}[f_1] = 0$. Conversely, suppose (c) is true. Since $K \subset C$, we have $\mathbb{E}^{\mathbb{Q}}[f] \leq 0$ for all $f \in K$. On the other hand, since K is a linear space, we have $\mathbb{E}^{\mathbb{Q}}[-f] \leq 0$ for all $f \in K$. Consequently, $\mathbb{E}^{\mathbb{Q}}[f] \geq 0$ so that $\mathbb{E}^{\mathbb{Q}}[f] = 0$. □

2.2.3 The Fundamental Theorem of Asset Pricing

After the above preparations we are able to prove the fundamental theorem of asset pricing in the finite dimensional setting, due to Harrison and Pliska [29].

In what follows, we consider the dual pair $(L^1(\mathbb{P}), L^\infty(\mathbb{P}), \langle\cdot,\cdot\rangle)$. The bilinear mapping $\langle\cdot,\cdot\rangle : L^1(\mathbb{P}) \times L^\infty(\mathbb{P}) \to \mathbb{R}$ is given by

$$\langle q, f\rangle = \int_\Omega f q \, \mathrm{d}\mathbb{P} = \sum_{i=1}^N (f_i q_i) p_i,$$

where $q = \sum_{i=1}^N q_i \mathbf{1}_{\{\omega_i\}} \in L^1(\mathbb{P})$, $f = \sum_{i=1}^N f_i \mathbf{1}_{\{\omega_i\}} \in L^\infty(\mathbb{P})$, $N = |\Omega|$ and $\mathbb{P}[\omega_i] = p_i$ for $1, \ldots, N$. This notation should not be confused with the scalar product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^d$. The correct operation will be clear from the context of its use.

Theorem 2.2.8 (FUNDAMENTAL THEOREM OF ASSET PRICING) *For a financial market S modeled on a finite filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$, the following statements are equivalent:*

(a) S satisfies the no-arbitrage condition.

(b) $\mathcal{M}^e(S) \neq \emptyset$.

Proof. (b)⇒(a) Let $\mathbb{Q} \in \mathcal{M}^e(S)$. By Lemma 2.2.7 we have $\mathbb{E}^{\mathbb{Q}}[g] \leq 0$ for all $g \in C$. Suppose $0 \neq g \in C \cap L_+^\infty(\mathbb{P})$, then $\mathbb{P} \sim \mathbb{Q}$ implies $\mathbb{E}^{\mathbb{Q}}[g] > 0$. A contradiction.

(a)⇒(b) By assumption, we have $K \cap L^\infty_+(\mathbb{P}) = \{0\}$. We would like to find a functional $q \in L^\infty(\mathbb{P})^* = L^1(\mathbb{P})$ that separates the closed linear space K from $L^\infty_+(\mathbb{P}) \setminus \{0\}$. To ensure the strict positivity of q, we consider the set

$$P := \left\{ \sum_{n=1}^{N} \alpha_n \mathbf{1}_{\{\omega_n\}} : \alpha_n \geq 0, \sum_{n=1}^{N} \alpha_n = 1, N = |\Omega| \right\},$$

instead of $L^\infty(\mathbb{P})_+ \setminus \{0\}$. P is a convex, compact subset of $L^\infty_+(\mathbb{P})$ which is disjoint from K by the no-arbitrage assumption. By the Hyperplane Separation Theorem, there exists $q = \sum_{i=1}^N q_i \mathbf{1}_{\{\omega_i\}} \in L^1(\mathbb{P})$ and $\alpha < \beta$ such that

$$\langle q, f \rangle \leq \alpha \quad \text{for all } f \in K$$

and

$$\langle q, h \rangle \geq \beta \quad \text{for all } h \in P.$$

Since K is a linear subspace, for any $f \in K$ we have $\langle q, f \rangle \leq \alpha$ and $-\langle q, f \rangle \leq \alpha$, which implies $0 \leq |\langle q, f \rangle| \leq \alpha$. Moreover, $\langle q, f \rangle \leq \frac{\alpha}{n}$ for all $n \in \mathbb{N}$. Hence, we may replace α with zero. Consequently, $\langle q, f \rangle = 0$ for all $f \in K$.

Denote the one-function by $\mathbf{1} = \sum_{i=1}^N \mathbf{1}_{\{\omega_i\}}$. Observe that $\langle q, h \rangle > 0$ for all $h \in P$ implies $q_i > 0$ for each $i = 1, \dots N$. This permits us to define

$$\mathbb{Q} = \frac{q}{\langle q, \mathbf{1} \rangle},$$

so that $\mathbb{E}^{\mathbb{Q}}[\mathbf{1}] = \langle \mathbb{Q}, \mathbf{1} \rangle = 1$. Thus, $\mathbb{Q}$ is a probability measure equivalent to $\mathbb{P}$ such that Lemma 2.2.7 (b) is true. It now follows that $\mathbb{Q} \in \mathcal{M}^e(S)$. □

Corollary 2.2.9 *Let S be a financial model satisfying the no-arbitrage condition on a finite filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$. For any $f \in K_a$, we have that the representation*

$$f = a + (H \cdot S)_T$$

is unique, where $a \in \mathbb{R}$ and H is some trading strategy. Moreover, for every $\mathbb{Q} \in \mathcal{M}^e(S)$, we have

$$\mathbb{E}^{\mathbb{Q}}[f] = a \quad \text{and} \quad \mathbb{E}^{\mathbb{Q}}[f|\mathcal{F}_t] = a + (H \cdot S)_t, \quad \text{for all } 0 \leq t \leq T.$$

Proof. For uniqueness, suppose that f has two representations

$$f = a_1 + (H^1 \cdot S)_T \quad \text{and} \quad f = a_2 + (H^2 \cdot S)_T,$$

with $a^1 > a^2$. By considering the trading strategy $H^2 - H^1$, we find an arbitrage opportunity $((H^2 - H^1) \cdot S)_T = a^1 - a^2 > 0$. This contradicts the assumption of no-arbitrage.

Now suppose that

$$f = a + (H^1 \cdot S)_T \quad \text{and} \quad f = a + (H^2 \cdot S)_T, \tag{2.2.2}$$

where the processes $H^1 \cdot S$ and $H^2 \cdot S$ are distinct. Then there exists t $(0 \le t \le T)$ such that $(H^1 \cdot S)_t \neq (H^2 \cdot S)_t$. We may then suppose that the event

$$A := \{\omega \in \Omega : (H^1 \cdot S)_t > (H^2 \cdot S)_t\} \in \mathcal{F}_t$$

is non-empty. Define the trading strategy $H = (H^2 - H^1)\mathbf{1}_A \cdot \mathbf{1}_{(t,T]}$. Economically, this strategy says we hold nothing until time t and, in the event A, then proceed with the strategy described by $H^2 - H^1$.

Using (2.2.2), we see that $(H \cdot S)_T = 0$ on $\Omega \setminus A$ and

$$\begin{aligned} (H \cdot S)_T &= (H^2 \cdot S)_{t+1}^T - (H^1 \cdot S)_{t+1}^T \\ &= (H^1 \cdot S)_t - (H^2 \cdot S)_t > 0 \end{aligned}$$

on A. This again contradicts the assumption of no-arbitrage.

The final part of the proof is completed by realizing that for any predictable process H and every $\mathbb{Q} \in \mathcal{M}^a(S)$, the process $H \cdot S$ is a martingale. Indeed,

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[(H \cdot S)_t | \mathcal{F}_{t-1}] &= \mathbb{E}^{\mathbb{Q}}\left[\sum_{s=1}^{t} \langle H_s, \Delta S_s \rangle \middle| \mathcal{F}_{t-1}\right] = \sum_{s=1}^{t} \mathbb{E}^{\mathbb{Q}}\left[\langle H_s, \Delta S_s \rangle | \mathcal{F}_{t-1}\right] \\ &= \sum_{s=1}^{t} \sum_{j=1}^{d} \mathbb{E}^{\mathbb{Q}}[H_s^{(j)} \Delta S_s^{(j)} | \mathcal{F}_{t-1}] \\ &= \sum_{s=1}^{t} \sum_{j=1}^{d} H_s^{(j)} (\underbrace{\mathbb{E}^{\mathbb{Q}}[S_s^{(j)} | \mathcal{F}_{t-1}] - S_{s-1}^{(j)}}_{0 \text{ when } s=t}) \\ &= \sum_{s=1}^{t-1} \sum_{j=1}^{d} H_s^{(j)} (S_s^{(j)} - S_{s-1}^{(j)}) \\ &= (H \cdot S)_{t-1}, \end{aligned}$$

as required. □

2.2.4 Pricing by No-Arbitrage

Proposition 2.2.10 *Let S be a financial model satisfying the no-arbitrage condition on a finite filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$. Then the polar cone C° of the cone C is equal to* $\operatorname{cone}\mathcal{M}^a(S)$. *Moreover, $\mathcal{M}^e(S)$ is dense in $\mathcal{M}^a(S)$. Hence, the following statements are equivalent:*

(a) $g \in C$,

(b) $\mathbb{E}^{\mathbb{Q}}[g] \leq 0$ *for all* $\mathbb{Q} \in \mathcal{M}^a(S)$,

(c) $\mathbb{E}^{\mathbb{Q}}[g] \leq 0$ *for all* $\mathbb{Q} \in \mathcal{M}^e(S)$.

Proof. Let

$$C^\circ = \{q \in L^1(\mathbb{P}) : \langle q, g \rangle \leq 0 \ \forall \ g \in C\}$$

be the polar cone of $C \subset L^\infty(\mathbb{P})$. Since $L^\infty_-(\mathbb{P}) \subset C$, it follows that $C^\circ \subset L^1_+(\mathbb{P})$. Let $0 \neq q \in C^\circ$. By Lemma 2.2.7, we have $\mathbb{Q} \in \mathcal{M}^a(S)$ where $\mathrm{d}\mathbb{Q} := (q/\langle q, \mathbf{1}\rangle)\,\mathrm{d}\mathbb{P}$. Consequently, $q \in \operatorname{cone}\mathcal{M}^a(S)$ so that $C^\circ \subset \operatorname{cone}\mathcal{M}^a(S)$. For the reverse inclusion, let $\mathbb{Q} \in \mathcal{M}^a(S)$ and observe that by Lemma 2.2.7 we have $\mathbb{E}^{\mathbb{Q}}[g] = \langle \mathbb{Q}, g \rangle \leq 0$ for all $g \in C$. This gives $\mathbb{Q} \in C^\circ$ and so $\mathcal{M}^a(S) \subset C^\circ$. Since C° is a cone, it follows that $\operatorname{cone}\mathcal{M}^a(S) \subset C^\circ$.

Note that, in our finite dimensional setting, C is closed. Indeed, by Proposition 2.2.5 we have $C = K - L^\infty_+(\mathbb{P})$, which is a finite dimensional algebraic sum of a linear space and a polyhedral cone.

By the Bi-Polar Theorem, we have $C = C^{\circ\circ} = (\operatorname{cone}\mathcal{M}^a(S))^\circ$. It follows that

$$\begin{aligned} g \in C \quad &\iff \quad \langle q, g \rangle \leq 0 \ \forall \ q \in C^\circ = \operatorname{cone}\mathcal{M}^a(S) \\ &\iff \quad \mathbb{E}^{\mathbb{Q}}[g] \leq 0 \ \forall \ \mathbb{Q} \in \mathcal{M}^a(S). \end{aligned}$$

This shows the equivalence of (a) and (b).

For the equivalence of (b) and (c), observe that the assumption of no-arbitrage implies that $\mathcal{M}^e(S) \neq \emptyset$. Choose $\mathbb{Q}^* \in \mathcal{M}^e(S)$ and for any $\mathbb{Q} \in \mathcal{M}^a(S)$, define the sequence $a_n := (1/n)\mathbb{Q}^* + (1 - 1/n)\mathbb{Q}$ for each $n \in \mathbb{N}$. Then $a_n \in \mathcal{M}^e(S)$ for each $n \in \mathbb{N}$ with $\lim_{n\to\infty} a_n = \mathbb{Q}$. This implies that $\mathcal{M}^e(S)$ is dense in $\mathcal{M}^a(S)$. The proof is now complete. □

A direct consequence of the above proposition is the following:

Proposition 2.2.11 *Let S be a financial model satisfying the no-arbitrage condition on a finite filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$. Then the following statements are equivalent:*

(a) $f \in K$,

(b) $\mathbb{E}^{\mathbb{Q}}[f] = 0$ *for all* $\mathbb{Q} \in \mathcal{M}^a(S)$,

(c) $\mathbb{E}^{\mathbb{Q}}[f] = 0$ *for all* $\mathbb{Q} \in \mathcal{M}^e(S)$.

Proof. By Proposition 2.2.5 (c) we have $f \in K$ if and only if $f \in C \cap (-C)$. Thus, the result follows directly from Proposition 2.2.10. □

Corollary 2.2.12 *Let S be a financial model satisfying the no-arbitrage condition on a finite filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$. If $f \in L^\infty(\mathbb{P})$ satisfies $\mathbb{E}^{\mathbb{Q}}[f] = a$ for all $\mathbb{Q} \in \mathcal{M}^e(S)$, then $f = a + (H \cdot S)_T$ for some trading strategy H.*

Proof. Since $\mathbb{E}^{\mathbb{Q}}[f - a] = 0$ for all $\mathbb{Q} \in \mathcal{M}^e(s)$, it follows from Proposition 2.2.11 that $f - a \in K$. This implies that $f \in K_a$. An application of Corollary 2.2.9 completes the proof. □

Corollary 2.2.13 (Complete Financial Markets) *Let S be a financial model satisfying the no-arbitrage condition on a finite filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$. Then the following statements are equivalent:*

(a) $\mathcal{M}^e(S)$ *consists of a single element* $\mathbb{Q}$.

(b) Each $f \in L^\infty(\mathbb{P})$ *may be represented as*

$$f = a + (H \cdot S)_T \quad \text{for some } a \in \mathbb{R} \text{ and } H \in \mathcal{H}.$$

In this case $a = \mathbb{E}^{\mathbb{Q}}[f]$*, the stochastic integral* $H \cdot S$ *is unique and we have* $\mathbb{E}^{\mathbb{Q}}[f|\mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[f] + (H \cdot S)_t$ *for* $t = 1, \dots, T$*.*

Proof. The result follows as a special case of Corollary 2.2.12. □

In the above result, the quantity $\mathbb{E}^{\mathbb{Q}}[f] = a$ is the unique no-arbitrage price of the contingent claim $f \in L^\infty(\mathbb{P})$. Notice that we can find such a unique price for every $f \in L^\infty(\mathbb{P})$. When a financial model S has this ideal property we refer to it as a *complete market*. We now turn our attention to the case where f may yield more than one arbitrage free price. In this case, we say the market is *incomplete*.

If a is an arbitrage free price for the claim $f \in L^\infty(\mathbb{P})$, we are able to enlarge the financial market S by introducing the financial instrument f without compromising the no-arbitrage condition. The instrument f is bought or sold at price a at time $t = 0$ and yields the random cashflow $f(\omega)$ at time $t = T$. The linear space $K^{f,a}$ generated by the set $K \cup \{f - a\}$ describes the enlarged set of attainable claims at price 0. The no-arbitrage condition for this enlarged market becomes $K^{f,a} \cap L^\infty_+(\mathbb{P}) = \{0\}$ which is satisfied if and only if a is indeed an arbitrage free price for f.

Theorem 2.2.14 (Pricing by No-Arbitrage) *Let* S *be a financial model satisfying the no-arbitrage condition on a finite filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbb{P})$*. For* $f \in L^\infty(\mathbb{P})$*, define the no-arbitrage bounds*

$$\begin{aligned} \overline{\pi}(f) &= \sup\{\mathbb{E}^{\mathbb{Q}}[f] : \mathbb{Q} \in \mathcal{M}^e(S)\} \quad \textit{and} \\ \underline{\pi}(f) &= \inf\{\mathbb{E}^{\mathbb{Q}}[f] : \mathbb{Q} \in \mathcal{M}^e(S)\}. \end{aligned}$$

Then either $\overline{\pi}(f) = \underline{\pi}(f)$*, in which case* f *is attainable at* $\pi(f) := \overline{\pi}(f) = \underline{\pi}(f)$*, or* $\underline{\pi}(f) < \overline{\pi}(f)$*, in which case*

$$(\underline{\pi}(f), \overline{\pi}(f)) = \{\mathbb{E}^{\mathbb{Q}}[f] : \mathbb{Q} \in M^e(S)\}$$

and a *is an arbitrage free price for* f *if and only if* $a \in (\underline{\pi}(f), \overline{\pi}(f))$*.*

Proof. The case $\overline{\pi}(f) = \underline{\pi}(f)$ follows from Corollary 2.2.12.

For the case $\underline{\pi}(f) < \overline{\pi}(f)$, first observe that $I := \{\mathbb{E}^{\mathbb{Q}}[f] : \mathbb{Q} \in \mathcal{M}^e(S)\}$ is a non-empty bounded interval in $\mathbb{R}$. Indeed, $I \subset [-\|f\|_\infty, \|f\|_\infty]$ shows that I is bounded. To see that I is an interval, observe that any convex combination of elements of

$\mathcal{M}^e(S)$ is again an element of $\mathcal{M}^e(S)$. With this in mind, let $x_1 < y < x_2$ with $x_1, x_2 \in I$ and let $\lambda \in (0,1)$ such that $\lambda x_1 + (1-\lambda)x_2 = y$. Then there exist $\mathbb{Q}_1, \mathbb{Q}_2 \in \mathcal{M}^e(S)$ with $\mathbb{E}^{\mathbb{Q}_1}[f] = x_1$ and $\mathbb{E}^{\mathbb{Q}_2}[f] = x_2$ whence $\mathbb{E}^{(\lambda\mathbb{Q}_1+(1-\lambda)\mathbb{Q}_2)}[f] = y$, implying $y \in I$.

Now suppose $a \in I$, then there exists $\mathbb{Q} \in \mathcal{M}^e(S)$ such that $\mathbb{E}^{\mathbb{Q}}[f - a] = 0$. By linearity, this implies that $\mathbb{E}^{\mathbb{Q}}[h] = 0$ for all $h \in K^{f,a}$. By Lemma 2.2.7, $\mathbb{Q} \in \mathcal{M}^e(S, f)$. Here, $\mathcal{M}^e(S, f)$ denotes the set of equivalent martingale measures with respect to the financial model S which is enlarged to include f. Thus, $K^{f,a} \cap L^\infty_+(\mathbb{P}) = \{0\}$ by Theorem 2.2.8.

Conversely, suppose that $K^{f,a} \cap L^\infty_+(\mathbb{P}) = \{0\}$. Theorem 2.2.8 and Lemma 2.2.7 imply that there exists $\mathbb{Q} \in \mathcal{M}^e(S, f)$ such that $\mathbb{E}^{\mathbb{Q}}[h] = 0$ for all $h \in K^{f,a}$. This, together with another application of Lemma 2.2.7, implies that $\mathbb{Q} \in \mathcal{M}^e(S)$ and $a = \mathbb{E}^{\mathbb{Q}}[f]$.

To conclude with the boundary case, assume $\overline{\pi}(f) \in I$. By definition, $\mathbb{E}^{\mathbb{Q}}[f - \overline{\pi}(f)] \leq 0$ for all $\mathbb{Q} \in \mathcal{M}^e(S)$ so that $f - \overline{\pi}(f) \in C$ by Proposition 2.2.10. Hence, there exists $h \in K$ such that $h - (f - \overline{\pi}(f)) \geq 0$. On the other hand, our assumption implies that there exists $\mathbb{Q}^* \in \mathcal{M}^e(S)$ such that $\mathbb{E}^{\mathbb{Q}^*}[f] = \overline{\pi}(f)$. Thus,

$$0 \leq \mathbb{E}^{\mathbb{Q}^*}[h - (f - \overline{\pi}(f))] = \mathbb{E}^{\mathbb{Q}^*}[h] - (\overline{\pi}(f) - \overline{\pi}(f)) = 0.$$

Consequently, $f - \overline{\pi}(f) = h \in K$, and an appeal to Proposition 2.2.11 produces $\mathbb{E}^{\mathbb{Q}}[f] = \overline{\pi}(f)$ for all $\mathbb{Q} \in \mathcal{M}^e(S)$. In other words, we have that I is a singleton - a contradiction. Using a similar argument applied to $-f$, we deduce that I must be an open interval. □

2.3 The Infinite Dimensional Setting

2.3.1 Introduction

Fix $1 \leq p \leq \infty$ and let q satisfy $p^{-1} + q^{-1} = 1$. Throughout, we consider the dual pair $(L^p(\mathbb{P}), L^q(\mathbb{P}), \langle \cdot, \cdot \rangle)$ where it is assumed that the underlying filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ is diffuse, and all processes are indexed by a continuous time interval $[0, T]$. The bilinear mapping $\langle \cdot, \cdot \rangle : L^p(\mathbb{P}) \times L^q(\mathbb{P}) \to \mathbb{R}$ is given by

$$\langle f, q \rangle = \int_\Omega f q \, \mathrm{d}\mathbb{P}$$

for all $f \in L^p(\mathbb{P})$ and $q \in L^q(\mathbb{P})$. Many of the definitions are analogous to the finite dimensional setting.

As before, we work with a discounted financial process S. For notational convenience, we will assume that S is one-dimensional. Generalizing to d assets is straight forward.

We consider the set of simple contingent claims attainable at price a defined by

$$M_a = \left\{ a + \sum_{i=1}^{n} H_i(S_{t_i} - S_{t_{i-1}}) \quad : \quad H \text{ bounded and predictable}, \right.$$
$$\left. 0 = t_0 < t_1 < \ldots < t_n = T \right\}.$$

Here, $(H_i)_{i=1}^n$ is a bounded process that is predictable in the sense that H_i is $\mathcal{F}_{t_{i-1}}$ measurable for $i = 1, \ldots, n$. Observe that M_0 is a linear space.

Define the vector space of *simple marketed claims*

$$M := \bigcup_{a \in \mathbb{R}} M_a \subset L^p(\mathbb{P}),$$

together with a pricing functional $\pi : M \to \mathbb{R}$. The elements of M are of the form

$$m = a + \sum_{i=1}^{n} H_i(S_{t_i} - S_{t_{i-1}})$$

and π is canonically defined as $\pi(m) = a$. Here, π plays the role of taking an expectation with respect to a martingale measure. We shall refer to the pair (M, π) as the *market model*.

In this setting, the assumption of no-arbitrage takes the following form:

Proposition 2.3.1 (No-Arbitrage) *Let (M, π) be a market model in $L^p(\mathbb{P})$. Then the no-arbitrage condition is satisfied if and only if for all $m \in M$ satisfying $m \geq 0$ and $\mathbb{P}[m > 0] > 0$, we have $\pi(m) > 0$.*

Proof. Suppose that $M_0 \cap L^p_+(\mathbb{P}) = \{0\}$ and let

$$m = a + \sum_{i=1}^{n} H_i(S_{t_i} - S_{t_{i-1}}) \in M$$

satisfy $m \geq 0$ and $\mathbb{P}[m > 0] > 0$. If $\pi(m) = a \leq 0$, then

$$\sum_{i=1}^{n} H_i(S_{t_i} - S_{t_{i-1}}) \geq -a \geq 0.$$

Thus,

$$0 \neq m \leq \sum_{i=1}^{n} H_i(S_{t_i} - S_{t_{i-1}}) \in M_0 \cap L^p_+(\mathbb{P}) = \{0\},$$

a contradiction.

Conversely, let $f \in M_0 \cap L^p_+(\mathbb{P})$. By definition, $\pi(f) = 0$ which leads to $\mathbb{P}[f > 0] = 0$ so that $f = 0$. □

The above proposition shows that arbitrage is excluded precisely when the functional $\pi : M \to \mathbb{R}$ is strictly positive.

The problem of constructing a martingale measure on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ now translates to finding a non-negative extension of π to all of $L^p(\mathbb{P})$. That is, to find $0 \leq \pi^* : L^p(\mathbb{P}) \to \mathbb{R}$ so that $\pi^*(m) = \pi(m)$ for all $m \in M$. Indeed, if such a π^* exists, then it is induced by a unique $q \in L^q(\mathbb{P})$ via the action

$$\pi^*(m) = \langle m, q \rangle = \int_\Omega mq \, \mathrm{d}\mathbb{P} = \mathbb{E}^{\mathbb{Q}}[m].$$

The non-negativity of π is equivalent to the non-negativity of q. By replacing q with $q/\mathbb{E}^{\mathbb{P}}[q]$, we may assume that $\pi^*(\mathbf{1}) = 1$ and we have found a probability measure $\mathbb{Q} \ll \mathbb{P}$ with Radon-Nikodým density $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = q$. Moreover, since M_0 contains all the simple integrals of the form

$$\sum_{i=1}^{n} H_i(S_{t_i} - S_{t_{i-1}})$$

where $0 = t_0 < t_1 < \ldots < t_n = T$, it follows that

$$\pi^* \left(\sum_{i=1}^{n} H_i(S_{t_i} - S_{t_{i-1}}) \right) = \mathbb{E}^{\mathbb{Q}} \left[\sum_{i=1}^{n} H_i(S_{t_i} - S_{t_{i-1}}) \right] = 0$$

for all such integrals. In a similar fashion to the proof of Lemma 2.2.7, we have that all finite subsequences $(S_{t_i})_{i=1}^n$ are martingales under $\mathbb{Q}$. Consequently, $(S_t)_{0 \leq t \leq T}$ is a continuous time martingale under $\mathbb{Q}$. In order for $\mathbb{Q} \sim \mathbb{P}$, we must have $\mathbb{P}[q > 0] = 1$, or in other words, π^* must be strictly positive. We summarise with the following proposition.

Proposition 2.3.2 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X = L^p(\mathbb{P})$ endowed with the norm topology for $1 \leq p < \infty$ and the weak* topology $\sigma(L^\infty(\mathbb{P}), L^1(\mathbb{P}))$ for $p = \infty$. Let (M, π) be a market model in X induced by the process S. The following statements are equivalent:*

(a) The market model (M, π) admits a strictly positive extension $\pi^* : X \to \mathbb{R}$.

(b) There exists a strictly positive $f \in X^*$ such that $f|_C \leq 0$, where $C := M_0 - X_+$.

(c) There exists a strictly positive $f \in X^*$ such that $f|_{M_0} = 0$.

(d) There exists a probability measure $\mathbb{Q} \sim \mathbb{P}$, with density function $f \in X^*$, under which S is a martingale.

Proof. (a)⇒(b) Let $f = \pi^*$, then $f|_{M_0} = 0$ and for all $x \in C$ we have $x = x_1 - x_2$ with $x_1 \in M_0$ and $x_2 \in X_+$. Thus, $f(x) = f(x_1) - f(x_2) = 0 - f(x_2) \leq 0$.

(b)⇒(c) Since $M_0 \subset C$, we have $f(x) \leq 0$ for all $x \in M_0$. Using the fact that M_0 is a linear space, we have $f(-x) = -f(x) \leq 0$. This implies $f|_{M_0} = 0$.

(c)⇒(d) By replacing f with $f/f(\mathbf{1})$ we have found a probability measure $\mathbb{Q} \sim \mathbb{P}$ with density $\frac{d\mathbb{Q}}{d\mathbb{P}} = f \in L^q(\mathbb{P})$ such that

$$f(x) = \langle x, f \rangle = \int_\Omega xf \, d\mathbb{P} = \mathbb{E}^{\mathbb{Q}}[x]$$

for all $x \in L^p(\mathbb{P})$. Since $\mathbb{E}^{\mathbb{Q}}[y] = 0$ for all $y \in M_0$, S is a martingale under $\mathbb{Q}$ by the above discussion.

(d)⇒(a) Define $\pi^*(x) = \mathbb{E}^{\mathbb{Q}}[x]$ for all $x \in X$. Since $\mathbb{Q} \sim \mathbb{P}$, π^* is strictly positive. By the fact that S is a martingale under $\mathbb{Q}$, we have $\pi^*|_{M_0} = 0$. Moreover, for $f \in M$, we have $f = a\mathbf{1} + m$, where $a \in \mathbb{R}$ and $m \in M_0$. Thus, $\pi^*(f) = a + 0 = a$ and we have $\pi^*|_M = \pi$. □

In view of Proposition 2.3.1, it is necessary that M does not contain any arbitrage opportunities in order for the market model (M, π) to admit a strictly positive extension.

The construction of a strictly positive $\pi^* : L^p(\mathbb{P}) \to \mathbb{R}$ is the difficult part in the proof of the Fundamental Theorem of Asset Pricing in infinite dimensions. It relies on the topological structure of the underlying space $L^p(\mathbb{P})$ as well as making additional assumptions on the closedness of $C = M_0 - L^p_+(\mathbb{P})$. This is the subject of the next section.

2.3.2 No-Free-Lunch

As with Theorem 2.2.8, we resort to the Hyperplane Separation Theorem to find the positive extension $\pi^* : L^p(\mathbb{P}) \to \mathbb{R}$ of the pricing functional $\pi : M \to \mathbb{R}$.

In the case of $L^\infty(\mathbb{P})$, Ross [51] proposed equipping a topology strong enough for the positive cone $L^\infty_+(\mathbb{P})$ to have non-empty interior. By Proposition 2.3.1, the linear space M_0 and the open set $\text{int}(L^\infty_+(\mathbb{P}))$ are disjoint precisely when the no-arbitrage condition is satisfied. An appeal to the Hyperplane Separation Theorem for open sets provides $\pi \in L^\infty(\mathbb{P})^*$ that is strictly positive on $\text{int}(L^\infty_+(\mathbb{P}))$ and non-positive on M_0.

Unfortunately, there are problems with this approach. In order for $\text{int}(L^\infty_+(\mathbb{P}))$ to be non-empty, either $L^\infty(\mathbb{P})$ must be finite dimensional or $L^\infty(\mathbb{P})$ must be equipped with the norm topology induced by $\| \cdot \|_\infty$. In both cases, there is no guarantee that π^* is strictly positive on $L^\infty_+(\mathbb{P}) \setminus \{0\}$ which means that there is no guarantee that the corresponding martingale measure $\mathbb{Q}$ is equivalent to $\mathbb{P}$. Moreover, in the latter case, we end up with a functional $\pi \in L^\infty(\mathbb{P})^*$ that may not be a member of $L^1(\mathbb{P})$. Recall that the norm dual of $L^\infty(\mathbb{P})$ is the space $\mathbf{ba}(\Omega, \mathcal{F}, \mathbb{P})$ of finitely additive measures with bounded variation, which is much larger than $L^1(\mathbb{P})$ in the infinite dimensional case. As such, we cannot induce a probability measure $\mathbb{Q} \ll \mathbb{P}$ under which the process S is a martingale because π^* may have a singular part.

In order for $\pi^* \in L^1(\mathbb{P})$, we have to work with the weak* topology on $L^\infty(\mathbb{P})$. However, under this topology, we have $\text{int}(L^\infty_+(\mathbb{P})) = \emptyset$. As examples show, it is then impossible to separate $L^\infty_+(\mathbb{P}) \setminus \{0\}$ from M_0 (cf. [21, Proposition 5.1.7]).

To circumvent this problem, it is necessary to employ a different version of the Hyperplane Separation Theorem, which says that it is always possible to strictly separate a closed convex set from a disjoint compact convex set by a continuous linear functional. To this end, Kreps considered the convex cone

$$C = \{g \in L^p(\mathbb{P}) : \exists\ f \in M_0 \text{ such that } f \geq g\}$$

of claims that are super replicable at price 0. In the same manner as the proof of Proposition 2.2.5, we have that $C = M_0 - L^p_+(\mathbb{P})$ and that the no-arbitrage property $M_0 \cap L^p_+(\mathbb{P}) = \{0\}$ is equivalent to $C \cap L^p_+(\mathbb{P}) = \{0\}$. In the infinite dimensional setting, C is not guaranteed to be weak* closed. To remedy this, Kreps formulates the following definition (cf. [43]), which is a strengthening of the no-arbitrage condition.

Definition 2.3.3 (No-Free-Lunch) Let (M, π) be a market model in $L^p(\mathbb{P})$, $1 \leq p \leq \infty$. We say that (M, π) satisfies the *no-free-lunch* condition if

$$\overline{C} \cap L^p_+(\mathbb{P}) = \{0\}.$$

Here, the closure of C is taken in the $\|\cdot\|_p$-topology for $1 \leq p < \infty$ and the $\sigma(L^\infty(\mathbb{P}), L^1(\mathbb{P}))$-topology for the case $p = \infty$.

Observe that, for $1 < p < \infty$, the norm, weak and weak* closures of the convex set C coincide.

If the no-arbitrage condition is violated, we can find $0 \neq g \in C \cap L^p_+(\mathbb{P})$. Thus, $0 \leq g = f - h$ where $f \in M_0$ and $h \geq 0$. Economically speaking, we were able to replicate the positive claim g with a zero-cost marketed claim f and by disposing of the positive cash flow h.

In the case of no-free-lunch being violated, we are not always able to replicate g, but instead are able to approximate g by elements of the form $g_\alpha = f_\alpha - h_\alpha \in M_0 - L^p_+(\mathbb{P})$. In other words, we are able to replicate g to some arbitrary precision using a zero-cost marketed claim and free disposal.

The no-free-lunch condition thus has a sensible economic interpretation and is crafted for the application of the Hyperplane Separation Theorem for closed convex sets. We are now in a position to prove the Kreps-Yan Theorem.

2.3.3 The Kreps-Yan Theorem

We present a proof of the Kreps-Yan Theorem by Rokhlin [50, Theorem 1.1]. Although this version is a partial case of [38, Theorem 3.1], its statement is simpler and the proof is cleaner.

Let $\langle X, Y \rangle$ be a pair of Banach spaces in separating duality. Suppose X is equipped with a locally convex topology τ which is compatible with the duality $\langle X, Y \rangle$ (in other words, preserves the continuity of the functionals induced by Y). Let $K \subset X$ denote a τ-closed pointed cone. An element $f \in Y$ is called *strictly positive* if $\langle x, f \rangle > 0$ for all $x \in K \setminus \{0\}$. An element $f \in Y$ is called *non-negative* if $\langle x, f \rangle \geq 0$ for all $x \in K$. We only consider cones K such that the set of strictly positive functionals is non-empty.

Definition 2.3.4 (Kreps-Yan Property) Let X be endowed with a locally convex topology τ compatible with the duality $\langle X, Y\rangle$ and $K \subset X$ be τ-closed pointed cone that admits a strictly positive functional.

(a) We say that the *Kreps-Yan Theorem is valid* for the ordered space (X, K) if for any τ-closed convex cone C with $-K \subset C$, the condition $C \cap K = \{0\}$ implies the existence of a strictly positive $f \in Y$ such that $f|_C \leq 0$.

(b) If this property holds for every τ-closed pointed cone $K \subset X$ that admits a strictly positive functional, then we say that (X, τ) has the *Kreps-Yan Property.*

Recall that a topological space (X, τ) is said to have the *Lindelöf Property* if every open cover of X has a countable subcover. We will refer to X as a *Lindelöf space* if $(X, \sigma(X, Y))$ has the Lindelöf Property.

Note that the space $(X, \sigma(X, Y))$ is a Lindelöf space if the Lindelöf Property can be verified for any topology τ compatible with the duality $\langle X, Y\rangle$. Indeed, for any open cover $\{U_\alpha\} \subset \sigma(X, Y) \subset \tau$ of X, there exists a countable subcover $\{U_{\alpha_i}\}_{i=1}^{\infty} \subset \tau$. But clearly $\{U_{\alpha_i}\}_{i=1}^{\infty} \subset \sigma(X, Y)$.

We now prove the Kreps-Yan Theorem in the abstract setting of Lindelöf spaces.

Theorem 2.3.5 *Let $(X, \sigma(X, Y))$ be a Lindelöf space. Then (X, τ) has the Kreps-Yan Property for any locally convex topology τ compatible with the duality $\langle X, Y\rangle$.*

Proof. Let $x \in K \setminus \{0\}$. Then $x \notin C$ and by the Hyperplane Separation Theorem, there exists $f_x \in Y$ such that

$$\langle y, f_x\rangle < \langle x, f_x\rangle$$

for all $y \in C$. Since C is a cone, we have that $\langle y, f_x\rangle \leq 0$ for all $y \in C$. Furthermore, $-K \subset C$ implies that $\langle x, f_x\rangle > 0$ and $\langle z, f_x\rangle \geq 0$ for all $z \in K$.

Consider the family of sets

$$A_x = \{y \in X : \langle y, f_x\rangle > 0\}$$

for all $x \in K \setminus \{0\}$ and let

$$A_0 = \{y \in X : |\langle y, \eta\rangle| < 1\},$$

where η is a strictly positive functional (whose existence is assumed). The sets A_x are open in the topology $\sigma(X,Y)$ and form an open cover of K. Moreover, the cone K is closed in $\sigma(X,Y)$ since all topologies compatible with the duality $\langle X,Y\rangle$ have the same collection of closed convex sets. An appeal to the Lindelöf Property implies the existence of the countable subcover $\cup_{i=0}^{\infty} A_{x_i} \supset K$, where $x_0 = 0$.

Let $\alpha_i = 1/(\|f_{x_i}\| 2^i)$, then $\sum_{i=1}^{\infty} \alpha_i f_{x_i}$ converges in the norm topology to some $f \in Y$. Clearly $f \leq 0$ on C. Moreover, f is strictly positive. Indeed, by the strict positivity of η, for any element $x \in K \setminus \{0\}$ there exists $\lambda > 0$ such that $\lambda x \neq A_0$. As a consequence, $\lambda x \in A_{x_k}$ for some $k \geq 1$ and

$$\langle \lambda x, f\rangle = \sum_{i=1}^{\infty} \alpha_i \langle \lambda x, f_{x_k}\rangle \geq \alpha_i \langle \lambda x, f_{x_k}\rangle > 0.$$

This completes the proof. □

We identify some spaces that have the Kreps-Yan Property.

Corollary 2.3.6 *The spaces $L^p(\mathbb{P})$, $1 \leq p < \infty$, have the Kreps-Yan Property for the norm topology and $L^\infty(\mathbb{P})$ has the Kreps-Yan Property for the weak* topology $\sigma(L^\infty(\mathbb{P}), L^1(\mathbb{P}))$.*

Proof. A topological space X is Lindelöf if it can be written as a countable union of compact subsets. Indeed, an open cover of X induces a finite subcover of each compact set. The union of these finite subcovers form a countable cover of X.

Therefore, by the Banach-Alaoglu Theorem, any dual space X^* is Lindelöf in the weak* topology $\sigma(X^*, X)$. Consequently, a reflexive space is Lindelöf in the weak topology $\sigma(X, X^*)$ due to the weak compactness of the unit ball. By the above theorem, reflexive spaces have the Kreps-Yan Property with respect to the norm topology (in view of the fact that the norm topology is a locally convex topology compatible with the duality $\langle X, X^*\rangle$) and dual spaces have the Kreps-Yan Property with respect to the weak* topology.

Lastly, recall that a Banach space X is called *weakly compactly generated* (WCG) if X contains a weakly compact subset whose span is dense in X. In [60] it is shown that all WCG spaces are Lindelöf with respect to the weak topology. Thus, all WCG spaces have the Kreps-Yan Property with respect to the norm topology.

The spaces $L^p(\mathbb{P})$ are reflexive for $1 < p < \infty$. Moreover, $L^1(\mathbb{P})$ is WCG (cf. [20]). This completes the proof. □

The Kreps-Yan Theorem falls out as a special case of the above result.

Theorem 2.3.7 (KREPS-YAN) *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X = L^p(\mathbb{P})$ endowed with the norm topology for $1 \leq p < \infty$ and the weak* topology $\sigma(L^\infty(\mathbb{P}), L^1(\mathbb{P}))$ for $p = \infty$.*

Let (M, π) be a market model in X induced by the process S. Then M satisfies the no-free-lunch condition if and only if there is an equivalent probability measure $\mathbb{Q}$ with $\frac{d\mathbb{Q}}{d\mathbb{P}} \in X^* = L^q(\mathbb{Q})$, $p^{-1} + q^{-1} = 1$, such that S is a $\mathbb{Q}$-martingale.

Proof. As before, let $C = \{g \in L^p(\mathbb{P}) : \exists\, f \in M_0 \text{ such that } f \geq g\} = M_0 - L^p_+(\mathbb{P})$, where $M_0 = \pi^{-1}(0)$. Suppose that there is an equivalent probability measure $\mathbb{Q}$ with $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \in X^* = L^q(\mathbb{P})$ such that S is a $\mathbb{Q}$-martingale. By Proposition 2.3.2, there exists a strictly positive functional $f \in X^*$ such that $f(g) \leq 0$ for all $g \in C$. By continuity, the inequality extends to $\overline{C}$. Now suppose $0 \neq g \in \overline{C} \cap L^p_+(\mathbb{P})$, then the strict positivity of f implies $f(g) > 0$. A contradiction.

For the converse, first observe that the set of strictly positive functionals with respect to $L^p(\mathbb{P})_+$ is non-empty. Indeed, the expectation functional is a member of this set. We may therefore apply Theorem 2.3.5. Suppose that M satisfies the no-free-lunch condition. By Corollary 2.3.6, X has the Kreps-Yan Property with respect to the above-mentioned topologies. As such, there exists a strictly positive functional $f \in X^*$ such that $\pi^*|_{\overline{C}} \leq 0$. An application of Proposition 2.3.2 completes the proof.
□

2.3.4 Further Developments

Dalang, Morton and Willinger apply the Kreps-Yan Theorem in $L^1(\mathbb{P})$ to prove a discrete time version of the Fundamental Theorem of Asset Pricing in terms of the no-arbitrage condition for a d-dimensional process S on a diffuse probability space [12].

Theorem 2.3.8 (DALANG-MORTON-WILLINGER) *Let $(\Omega, \mathcal{F}_T, \mathbb{P})$ be a probability space and let $S = (S_t)_{t=0}^T$ be an $\mathbb{R}^d$-valued stochastic process adapted to the discrete time filtration $(\mathcal{F})_{t=0}^T$. Then the no-arbitrage condition holds if and only if there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ so that:*

(a) $S_t \in L^1(\mathbb{P})$ *for all* $t = 0, \ldots, T$,

(b) S *is a* $\mathbb{Q}$*-martingale,*

(c) $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^\infty(\mathbb{P})$.

The proof of this theorem is difficult, and is as close as one can get to a general version of the Fundamental Theorem of Asset Pricing in terms of the no-arbitrage condition (cf. [16, §7]).

The Kreps-Yan Theorem was the first version of The Fundamental Theorem of Asset Pricing applicable to continuous time processes. The price of achieving this was to trade the no-arbitrage condition for the stronger no-free-lunch condition. The theorem also has other limitations. When applying the theorem for the case $1 \le p < \infty$, the martingale measure has density in $L^q(\mathbb{P})$ for $q > 1$. The q-th moment is not invariant under equivalent changes in measure. In other words, if we pass from the probability $\mathbb{P}$ to an equivalent probability $\mathbb{P}_1$, it does not follow from $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \in L^q(\mathbb{P})$ that $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}_1} \in L^q(\mathbb{P}_1)$. Only the spaces $L^0(\mathbb{P})$ and $L^\infty(\mathbb{P})$ remain unchanged under an equivalent change in probability. From a practical viewpoint, $L^\infty(\mathbb{P})$ is the most interesting case. However, the class of processes belonging to $L^\infty(\mathbb{P})$ is too restrictive for many applications. There is also the added complexity of dealing with the weak* topology.

One of the difficulties of working with the weak* topology is the interpretation of the no-free-lunch condition. Earlier, we mentioned that no-free-lunch means we are able to approximate an arbitrage opportunity g by elements of the form $g_\alpha = f_\alpha - h_\alpha \in M_0 - L^\infty_+(\mathbb{P})$. In general, the elements $\{f_\alpha - h_\alpha\}_{\alpha \in I}$ are indexed by an uncountable ordered set I. This is not very helpful in practical applications. The question arises as to whether we can replace $\{f_\alpha - h_\alpha\}_{\alpha \in I}$ with $(f_i - h_i)_{i=1}^\infty$. In the case of continuous processes, Delbaen [14] was able to provide a positive answer when the simple integrals in the market model (M, π) are indexed by stopping times instead of deterministic times. Another positive answer was provided by Schachermayer [54] in the case of discrete time processes with infinite time horizon.

When we are able to pass to the discrete sequence $(f_i - h_i)_{i=1}^\infty$ convergent in the weak* topology of $L^\infty(\mathbb{P})$, the Principle of Uniform Boundedness implies that

$$\sup_{i \in \mathbb{N}} \|f_i - h_i\|_\infty < \infty.$$

This means that the risk is bounded when approximating arbitrage opportunity g. This is known as *no-free-lunch with bounded risk* [54]. In the general case, this condition means that there is $M > 0$ such that $f_\alpha \geq -M$ $\mathbb{P}$-almost surely for all $\alpha \in I$.

Unfortunately, in the general setting of semi-martingales, this condition does not guarantee the existence of an equivalent martingale measure [16]. This fact suggested that the requirement of the market model (M, π) containing the simple integrals is not strong enough for a general theorem. An enrichment is needed. Indeed, the art of finding a Fundamental Theorem of Asset Pricing involves choosing the set M_0 of marketed cashflows at zero initial cost so that the $\|\cdot\|_\infty$-closure of the set $C = M_0 - L^\infty(\mathbb{P})$ is closed in the weak* topology of $L^\infty(\mathbb{P})$.

The admissibility criteria of the trading strategy H are a subtle issue. In the continuous time setting it is necessary to exclude trading pathologies, such as doubling strategies, in order to derive a viable theory. The criteria have to balance mathematical tractability with economic reality. A classical admissibility criterion is introduced in [28, 29] to exclude doubling strategies:

Definition 2.3.9 An S-integrable predictable process $H = (H_t)_{0 \leq t \leq T}$ is called *admissible* if there is a constant $M > 0$ such that

$$\int_0^t H_u \, \mathrm{d}S_u \geq -M$$

almost surely for all $t \in [0, T]$.

The interpretation of this is that the economic agent, trading according to the strategy H, has to adhere to a finite credit line. There are other variations of this condition in the literature; we refer the reader to [28, 71, 21, 55] for more information.

The approach of Delbaen and Schachermayer [16] was to define the set of marketed cashflows at zero as

$$M_0 = \left\{ \int_0^T H_t \, \mathrm{d}S_t : H \text{ admissable} \right\}$$

and then assume free disposal by defining

$$C = [M_0 - L^0_+(\mathbb{P})] \cap L^\infty(\mathbb{P}).$$

The difference in the approach of [16] to the classical Kreps-Yan Theorem is due to the following definition:

Definition 2.3.10 A locally bounded semi-martingale S satisfies the *no-free-lunch with vanishing risk* condition if

$$\overline{C} \cap L_{+}^{\infty}(\mathbb{P}) = \{0\},$$

where $\overline{C}$ denotes the $\|\cdot\|_{\infty}$-closure of C.

The process S fails the above condition if and only if there is $0 \neq g \in L_{+}^{\infty}(\mathbb{P})$ and a sequence (f_n) of the form

$$f_n = \int_0^T H_t^{(n)} \, \mathrm{d}S_t,$$

where the $H^{(n)}$ are admissible strategies, such that $f_n \geq g - \frac{1}{n}$. This is a weaker condition than no-free-lunch but stronger than no-arbitrage.

Economically, it means that the agent has to be willing to sacrifice at most $\frac{1}{n}$ when approximating the arbitrage opportunity g, which is easy to interpret and leads to the following general theorem [16].

Theorem 2.3.11 (Delbaen-Schachermayer) *Let S be a bounded (resp. locally bounded) real-valued semi-martingale. Then there is a probability measure $\mathbb{Q} \sim \mathbb{P}$ under which S is a martingale (resp. local martingale) if and only if S satisfies the no-free-lunch with vanishing risk condition.*

This result was extended to unbounded semi-martingales in [17], where the requirement of a martingale is weakened to sigma-martingale.

Theorem 2.3.12 (Delbaen-Schachermayer) *Let S be a (not necessarily bounded) $\mathbb{R}^d$-valued semi-martingale. Then there is a probability measure $\mathbb{Q} \sim \mathbb{P}$ under which S is a sigma-martingale if and only if S satisfies the no-free-lunch with vanishing risk condition.*

Chapter 3

Valuation Bounds and Risk Measures

3.1 Introduction

In this chapter, we present the work of Jaschke and Küchler [35, 34] (also see Staum [59]) on generalising the Fundamental Theorem of Asset Pricing in terms of coherent risk measures. We have seen earlier that the absence of arbitrage is equivalent to the existence of an equivalent martingale measure with which we can price contingent claims. If this measure is not unique (i.e. the market is not complete), we obtain the no-arbitrage bounds

$$\begin{aligned}\overline{\pi}(x) &= \sup\{\mathbb{E}^{\mathbb{Q}}[x] : \mathbb{Q} \in \mathcal{M}^e(S)\} \quad \text{and} \\ \underline{\pi}(x) &= \inf\{\mathbb{E}^{\mathbb{Q}}[x] : \mathbb{Q} \in \mathcal{M}^e(S)\}.\end{aligned}$$

In reality, these bounds may be quite wide which presents the problem of which price to choose. One approach is to select a unique price (or at least tighter price bounds) with the help of a utility function which characterizes an agent's preference [30, 31, 41, 13, 25, 24, 40]. The drawback to this approach is the tight coupling of contingent claim pricing and an agent's utility function, initial position and estimate of the real world probability measure. This can introduce significant model risk.

The approach of Jaschke and Küchler is to induce price bounds using a coherent risk measure. These turn out to be the same good deal bounds considered by Černý and Hodges in [63]. These bounds can be shown to be tighter than the no-arbitrage bounds but are still reasonably independent of personal preferences.

A remark on notation: Since this chapter involves many set theoretic arguments, we will use lowercase letters to denote random variables to avoid confusion with sets, which are denoted with uppercase letters.

3.2 The Space of Cash Streams

In what follows, let L denote a generic space of cash streams. We assume that it is possible to form a position on either side of a contract, which translates into L being a linear space. We also assume there exists a secure cash stream whose present value is 1. We denote this by $\mathbf{1} \in L$ and it is used as a reference or numéraire cash stream.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Some examples of the space L are:

(a) The space of stochastic cash streams on a finite time horizon $[0, T]$. Let L_{sm} denote the space of simple adapted processes

$$x(t,\omega) = \sum_{i=0}^{n} x_i \mathbf{1}_{E_i}(\omega) \mathbf{1}_{[\tau_i(\omega),T]}(t).$$

These are the cash streams which pay the amount x_i at a random time τ_i in the event E_i. The numéraire cash stream in this space is

$$\mathbf{1}(t,\omega) = \mathbf{1}_\Omega \mathbf{1}_{[0,T]}.$$

(b) The space of deterministic cash streams on a finite time horizon $[0, T]$. Let L_{dm} denote the space of piecewise constant functions

$$x(t) = \sum_{i=1}^{n} x_i \mathbf{1}_{[\tau_i,T]}(t).$$

These are the cash streams which pay the amount x_i at the deterministic time τ_i. The numéraire cash stream is given by

$$\mathbf{1}(t) = \mathbf{1}_{[0,T]}(t).$$

(c) The space of stochastic payments at one period. Let L_{so} denote the space of simple random variables

$$x(\omega) = \sum_{i=1}^{n} x_i \mathbf{1}_{E_i}(\omega).$$

The numéraire cash stream is given by

$$\mathbf{1}(\omega) = (1+r)x_\Omega(\omega),$$

where r is the risk free rate of return on the time interval $[0, T]$. The case where Ω is finite is treated in [7].

A *pricing system* $\pi : L \to \mathbb{R}$ assigns a fair value to a cash stream $x \in L$ before any kind of transaction costs. Since π represents frictionless pricing, it is naturally a linear function. We call a price system π *normalised* if $\pi(\mathbf{1}) = 1$. If π is normalised, then $\pi(x)$ amounts to the expected present value of the cash stream x.

Some examples of pricing systems are

(a) On the space L_{sm}, an important class of pricing systems are induced by pairs of numéraire processes N and probability measures $\mathbb{Q}$. These price systems are of the form $\pi_{N,\mathbb{Q}} : L_{\text{sm}} \to \mathbb{R}$ defined by

$$\pi_N(x) = \mathbb{E}^{\mathbb{Q}}\left[\int_0^T \frac{N_0}{N_t}\,\mathrm{d}x_t,\right]$$

where $N_t > 0$ is some process used for discounting - a money market account, for example.

(b) On the space L_{dm}, the cashflows are deterministic. Price systems on this space represent a term structure of interest rates since they apply corresponding discount factors $v(t)$ to the cashflows with maturity t. This can be expressed as

$$\pi_v(x) = \int_0^T v(t)\,\mathrm{d}x_t.$$

(c) On the space L_{so}, price systems are equivalent to taking expectations with respect to a probability measure $\mathbb{Q}$. I.e.

$$\pi_{\mathbb{Q}}(x) = \mathbb{E}^{\mathbb{Q}}[x] = \int_\Omega x(\omega)\,\mathrm{d}\mathbb{Q}(\omega).$$

3.3 A Unified Framework

A relation $\succeq$ is called a *pre-order* if it is both reflexive and transitive. A pre-order becomes a vector ordering if, in addition, the following two conditions hold:

- $x \succeq y \Leftrightarrow x - y \succeq 0$, and
- $x \succeq 0, \alpha > 0 \Rightarrow \alpha x \succeq 0$.

It is well known that there is a one-to-one correspondence between vector orderings $\succeq$ and cones A via the relation

$$x \succeq y \Leftrightarrow x - y \in A.$$

We also assume a natural vector ordering $\geq$ on L is given by $x \geq 0$ if every single payment of x is non-negative. The cone of non-negative cash streams is denoted by L_+.

Let $\succeq$ denote a preference relation, then $z \succeq 0$ means that z is preferable to the zero cash stream. For $x \in L$, define

$$\overline{\pi}(x) = \inf\{\alpha \in \mathbb{R} : \alpha \mathbf{1} \succeq x\}$$

and

$$\underline{\pi}(x) = \sup\{\alpha \in \mathbb{R} : \alpha \mathbf{1} \preceq x\}$$

as the upper and lower bound price of the cash stream x. The function ρ defined by

$$\rho(x) = \inf\{\alpha \in \mathbb{R} : \alpha \mathbf{1} + x \succeq 0\}$$

can be considered a risk measure. It denotes the smallest amount of a secure cash stream that needs to be added to x to make x preferable to zero. It is easy to see that $\rho(x) = \overline{\pi}(-x) = -\underline{\pi}(x)$ for all $x \in L$.

Modulo some technical conditions, there is a one-to-one correspondence between the following economic objects:

(a) Coherent risk measures ρ.

(b) Cones A of acceptable risks, where $A = \{x : \rho(x) \leq 0\}$.

(c) Partial preferences $x \succeq y$ meaning that x is at least as good as y. This can be expressed as $x \succeq y \Leftrightarrow x - y \in A$.

(d) Valuation bounds $\overline{\pi}$ and $\underline{\pi}$ where $\rho(x) = \overline{\pi}(-x) = -\underline{\pi}(x)$.

(e) Sets K of admissible price systems given by $\pi \in K \Leftrightarrow \pi(x) \geq 0$ for all $x \succeq 0$.

The relationship (a) and (b) was established for the case $L = L^\infty(\mathbb{P})$ in Theorem 1.3.5. We will prove the correspondence of the above in the more general setting of L. In the next definition, we formulate the conditions required for the economic objects $\succeq$, A, ρ and $(\underline{\pi}, \overline{\pi})$ to be equivalent.

Recall that a set A is called *absorbing* if for every $f \in L$ there exists $\alpha > 0$ such that $\alpha^{-1} f \in A$. The *radial interior* of A is the set

$$\{f \in A : A - f \text{ is absorbing}\}.$$

A set A is *radially open* if it coincides with its radial interior. A set A is called *radially closed* if its compliment is radially open.

Definition 3.3.1 (a) A vector ordering $\succeq$ on L is a *coherent partial preference* if:

(Cl) $\{x \in L : x \succeq 0\}$ is radially closed,

(M) $x \geq 0 \Rightarrow x \succeq 0$.

(b) A set $A \subset L$ is a *coherent acceptance set* if:

(C, PH) A is a cone,

(Cl) A is radially closed,

(T) $\mathbf{1} \in A$,

(M) $L_+ \subset A$.

(c) A function $\rho : L \to \mathbb{R}$ is a *coherent risk measure* if:

(C) ρ is convex,

(PH) ρ is positively homogeneous,

(T) ρ is translation invariant,

(M) ρ is monotone.

(d) The pair $(\underline{\pi}, \overline{\pi})$ are called *coherent valuation bounds* if $\overline{\pi}(-x) = -\underline{\pi}(x)$ for all $x \in L$ and $\rho := -\underline{\pi}$ is a coherent risk measure.

As a consequence, we have the identities $\underline{\pi}(x) = -\rho(x)$ and $\overline{\pi}(x) = \rho(-x)$ for all $x \in L$.

Observe that convexity and sub-additivity are equivalent under the assumption of positive homogeneity.

Theorem 3.3.2 *There is a one-to-one correspondence between the economic objects listed in Definition 3.3.1.*

Proof. (a)⇔(b) Let $A = \{x \in L : x \succeq 0\}$, then A is easily seen to be a coherent acceptance set. Conversely, for a coherent acceptance set A, the unique order relation $\succeq$ determined by $x \succeq y \Leftrightarrow x - y \in A$ satisfies the properties of (a) and is a coherent partial preference.

(b)⇔(c) For a coherent acceptance set A denote by

$$\rho_A(x) := \inf\{\alpha \in \mathbb{R} : \alpha\mathbf{1} + x \in A\}$$

the risk measure induced by A and for a coherent risk measure ρ denote by

$$A_\rho := \{x : \rho(x) \leq 0\}$$

the acceptance set induced by ρ.

It is easy to check the following equivalences:

- A $[A_\rho]$ is a cone if and only if ρ_A $[\rho]$ is convex and positively homogeneous.
- $\mathbf{1} \in A$ $[\mathbf{1} \in A_\rho]$ if and only if ρ_A $[\rho]$ is translation invariant.
- $L_+ \subset A$ $[L_+ \subset A_\rho]$ if and only if ρ_A $[\rho]$ is monotone.

What remains is to show that $A \mapsto \rho_A$ and $\rho \mapsto A_\rho$ are bijections and inverses of each other. First we show $\rho_{A_\rho} = \rho$. Indeed, by translation invariance

$$\begin{aligned} \rho_{A_\rho}(x) &= \inf\{\alpha \in \mathbb{R} : \alpha\mathbf{1} + x \in \{y \in L : \rho(y) \leq 0\}\} \\ &= \inf\{\alpha \in \mathbb{R} : \rho(x) \leq \alpha\} \\ &= \rho(x). \end{aligned}$$

Conversely, we show $A = A_{\rho_A}$. The inclusion $A \subset A_{\rho_A}$ is trivial. For the reverse inclusion, observe that A is a cone with $\mathbf{1} \in A$. Thus $x \in A$ and $\alpha \geq 0$ imply $x + \alpha\mathbf{1} \in A$. As such, the set

$$\{\alpha \in \mathbb{R} : \alpha\mathbf{1} + x \in A\}$$

is of the form $\emptyset$, $\mathbb{R}$, $[\rho, \infty)$ or (ρ, ∞) for some $\rho \in \mathbb{R}$. Since A is radially closed, the form (ρ, ∞) cannot occur. Indeed, $\rho\mathbf{1}+x \in L\setminus A$ implies that the set $L\setminus A - \{\rho\mathbf{1}+x\}$ is absorbing. Consequently, there exists $\gamma > 0$ such that

$$\gamma^{-1}\mathbf{1} \in L \setminus A - \{\rho\mathbf{1} + x\}$$

whence $(\gamma^{-1} + \rho)\mathbf{1} + x \in L \setminus A$. This contradicts the fact that

$$\rho = \inf\{\alpha \in \mathbb{R} : \alpha\mathbf{1} + x \in A\}$$

and so we must have $\rho\mathbf{1} + x \in A$. If $\rho \leq 0$, then $x = (\rho\mathbf{1} + x) - \rho\mathbf{1} \in A$. Adopting the convention $\inf\emptyset = \infty$ and $\inf\mathbb{R} = -\infty$ we have $\rho_A(x) \leq 0$ if and only if $x \in A$. It now follows that $A_{\rho_A} = \{x : \rho_A(x) \leq 0\} = A$.

(a)$\Rightarrow$(d) For $x \in L$, define

$$\overline{\pi}(x) = \inf\{\alpha \in \mathbb{R} : \alpha\mathbf{1} \succeq x\} \quad \text{and} \quad \underline{\pi}(x) = \sup\{\alpha \in \mathbb{R} : \alpha\mathbf{1} \preceq x\}.$$

Then, it is easily seen that $\overline{\pi}(-x) = -\underline{\pi}(x)$. Define

$$\rho(x) = -\underline{\pi}(x) = \inf\{\alpha \in \mathbb{R} : \alpha\mathbf{1} + x \succeq 0\}.$$

Then ρ is a coherent risk measure by virtue of the coherent partial preference $\succeq$.

(d)$\Rightarrow$(c) Trivial. □

Observe that $\rho_A(0) = 0$ if and only if $\mathbf{1} \in A$ and $-\mathbf{1} \notin A$. If A is a cone containing $\mathbf{1}$, then $\rho_A < \infty$ if and only if $\mathbf{1}$ is in the radial interior of A and $\rho_A > -\infty$ if and only if $-\mathbf{1}$ is in the radial interior of $L \setminus A$.

The monotonicity property of a convex measure ρ ensures that non-negative exposures are viewed as riskless ($x \geq 0 \Rightarrow \rho(x) \leq 0$). On the other hand, it is sensible to prevent non-positive positions from being acceptable. This is the aim of the following definition.

Definition 3.3.3 A coherent risk measure ρ is called *weakly relevant* if $x \leq 0$ and $x \neq 0$ imply $\rho(x) > 0$. We say that ρ is *strongly relevant* if $\rho(x) \leq 0$ and $\rho(-x) \leq 0$ imply $x = 0$.

The property of relevance ensures that the addition of a non-zero risk to an existing position will have a material impact on the risk of the portfolio (cf. [7]).

In terms of acceptance sets, weak relevance corresponds to the condition

$$A \cap (-L_+) = \{0\}.$$

Strong relevance corresponds to

$$A \cap (-A) = \{0\},$$

which is tantamount to the associated preference ordering being anti-symmetric.

As mentioned earlier, it is argued in [15] that for a coherent risk measure ρ, it does not make sense to allow $\rho(x) = -\infty$, as this would enable us to withdraw an arbitrary amount of capital from the position x without increasing its risk. This situation cannot occur if the corresponding (radially closed) acceptance set A_ρ is weakly relevant. Indeed, $-\mathbf{1}$ is contained in the radial interior of $L \setminus A_\rho$ when $A \cap (-L_+) = \{0\}$.

3.4 Dual Pricing Systems

Let $\langle L, \hat{L} \rangle$ be in separating duality. Note that $\hat{L}$ need not be the entire algebraic dual of L. Let

$$A^\circ := \{\pi \in \hat{L} : \pi(x) \geq 0 \;\forall\, x \in A\}$$

denote the polar cone in $\hat{L}$ of a cone $A \subset L$, and let

$$K^\circ := \{x \in L : \pi(x) \geq 0 \;\forall\, \pi \in K\}$$

denote the polar cone in L of a cone $K \subset \hat{L}$.

Definition 3.4.1 For an acceptance set $A \subset L$, we call the polar cone

$$K_A := A^\circ \subset \hat{L}$$

the associated set of *admissible price systems* and

$$D_A := \{\pi \in A^\circ : \pi(\mathbf{1}) = 1\}$$

the associated set of *normalised* admissible price systems.

We can now prove a generalization of Theorem 1.3.5.

Theorem 3.4.2 (Duality Theorem) *Let $A \subset L$ be a cone that contains $\mathbf{1}$, $K_A = A^\circ$ the associated set of admissible price systems, and $\rho = \rho_A$ its associated risk measure. Then the following statements are true:*

(a) $A = K_A^\circ$ if and only if A is $\sigma(L, \hat{L})$-closed.

(b) If A is $\sigma(L, \hat{L})$-closed then $D_A \neq \emptyset$ if and only if $-\mathbf{1} \notin A$.

(c) If A is $\sigma(L, \hat{L})$-closed then

$$\rho(x) = \sup_{\pi \in K_A} \pi(-x)/\pi(\mathbf{1}),$$

where we adopt the convention $0/0 = -\infty$.

(d) If A is $\sigma(L, \hat{L})$-closed then

$$\rho(x) = \sup_{\pi \in D_A} \pi(-x),$$

provided $\mathbf{1}$ is in the radial interior of A. We adopt the convention $\sup \emptyset = -\infty$.

Proof. (a) Assume A is a $\sigma(L, \hat{L})$-closed. Since A is a convex cone that contains 0, it follows from the Bi-Polar Theorem that

$$K_A^\circ = A^{\circ\circ} = \overline{\mathrm{co}}\,(A \cup \{0\}) = A.$$

Conversely, $A = K_A^\circ = A^{\circ\circ}$ implies A is $\sigma(L, \hat{L})$-closed.

(b) Observe that $A = K_A^\circ$ implies

$$x \in A \quad \Longleftrightarrow \quad \pi(x) \geq 0 \;\forall\; \pi \in K_A. \tag{3.4.1}$$

Suppose $D_A \neq \emptyset$. If $-\mathbf{1} \in A$ then $\pi(-\mathbf{1}) \geq 0$ for some $\pi \in D_A$, which implies $-1 \geq 0$. Thus, $-\mathbf{1} \notin A$. Conversely, if $-\mathbf{1} \notin A$, there exists $\pi \in K_A$ such that $\pi(-\mathbf{1}) < 0$. Thus, $\pi(\,\cdot\,)/\pi(\mathbf{1}) \in D_A$.

(c) By (3.4.1), we have

$$\begin{aligned}
\rho(x) &= \inf\{\alpha \in \mathbb{R} : \alpha\mathbf{1} + x \in A\} \\
&= \inf\{\alpha \in \mathbb{R} : \pi(\alpha\mathbf{1} + x) \geq 0 \;\forall\; \pi \in K_A\} \\
&= \inf\{\alpha \in \mathbb{R} : \alpha \geq \pi(-x)/\pi(\mathbf{1}) \;\forall\; \pi \in K_A\} \\
&= \inf\{\alpha \in \mathbb{R} : \alpha \geq \sup_{\pi \in K_A} \pi(-x)/\pi(\mathbf{1})\} \\
&= \sup_{\pi \in K_A} \pi(-x)/\pi(\mathbf{1})
\end{aligned}$$

for all $x \in L$.

(d) If $\mathbf{1}$ is in the radial interior of A, then $\pi(\mathbf{1}) > 0$ for all $0 \neq \pi \in K_A$. Indeed, pick $x \in L$ such that $\pi(x) > 0$. Since $A - \mathbf{1}$ is absorbing, there exists $\gamma > 0$ such that $-\gamma^{-1}x \in A - \{\mathbf{1}\}$. Thus, $\gamma\mathbf{1} - x \in A$. By (3.4.1), $\pi(\gamma\mathbf{1} - x) \geq 0 \Rightarrow \pi(\mathbf{1}) \geq \gamma^{-1}\pi(x) > 0$. The fact that

$$\rho(x) = \sup_{\pi \in D_A} \pi(-x)$$

now follows easily[1]. □

In view of the above result, we add to Definition 3.3.1.

Definition 3.4.3 (a) We call a coherent risk measure (and its equivalent representations) *closed* if its corresponding acceptance set is $\sigma(L, \hat{L})$-closed.

(b) We call $K_A \subset \hat{L}$ a *coherent set of admissible price systems* if

(C, PH) K_A is a cone,

(Cl) K_A is $\sigma(\hat{L}, L)$-closed,

(M) $x \in L_+ \Rightarrow \pi(x) \geq 0 \ \forall \ \pi \in K_A$.

By [34, Propostion 17], any closed acceptance set A is also radially closed.

Corollary 3.4.4 *There is a one-to-one correspondence between the closed versions of the economic objects listed in Definition 3.3.1 and the coherent sets of admissible price systems in Definition 3.4.3*

Proof. Given a $\sigma(L, \hat{L})$-closed coherent acceptance set $A \subset L$, define $K_A = A^\circ$. Then K_A is automatically a $\sigma(\hat{L}, L)$-closed convex cone. It is also easy to see that $x \in L_+ \Rightarrow \pi(x) \geq 0 \ \forall \ \pi \in K_A$.

Conversely, given a coherent set of admissible price systems $K \subset \hat{L}$, define $A_K = K^\circ$. Then A_K is a $\sigma(L, \hat{L})$-closed convex cone that contains L_+.

Lastly, the Bi-Polar Theorem asserts that $A = A_{K_A}$ for all closed coherent acceptance sets A and $K = K_{A_K}$ for all coherent sets of admissible price systems K. Thus, one-to-one correspondence is assured and the proof is complete. □

[1] In fact, D_A is a base for the cone K_A.

3.5 Good Deals

In this section we study *good deals*, which are a natural generalisation of arbitrage. A generalised version of the Fundamental Theorem of Asset Pricing will be derived.

Let $M \subset L$ be the set of all cash streams *available in the market at zero initial cost* and assume it is a cone.

As an illustration, consider the cash streams $\{z_i\}_{i=1}^n$. If it were possible to buy the cash stream z_i at the asked price $\overline{p}_i$, the net cash stream $\overline{c}_i = -\overline{p}_i\mathbf{1} + z_i$ is a member of M. Similarly, if it were possible to sell z_i at the bid price $\underline{p}_i$, then the net cash stream $\underline{c}_i = \underline{p}_i\mathbf{1} - z_i$ is also a member of M. Moreover, M contains the set

$$\text{cone}\left(\{\overline{c}_1, \ldots, \overline{c}_n\} \cup \{\underline{c}_1, \ldots, \underline{c}_n\}\right),$$

which represents the assumption of no trading constraints.

Conceptually, M plays the same role as the linear space K of attainable claims at price zero that was considered in the previous chapter. The assumption that M is a cone represents a market where transaction costs are taken into account. If M is linear, then the market can be viewed as frictionless.

Definition 3.5.1 (Good Deals) Fix a coherent acceptance set A and let M denote the cone of cash streams available in the market.

(a) We say that $0 \neq x \in M$ is a *good deal of the first kind* if $x \in A$.

(b) We say that $x \in M$ is a *good deal of the second kind* if there exists $\alpha > 0$ such that $x - \alpha\mathbf{1} \in A$.

A good deal of the first kind represents a strategy, with no initial cost, that achieves a terminal cashflow that is acceptable in terms of our measure of risk.

A good deal of the second kind is similar to a good deal of the first kind, except we are able to withdraw α units of $\mathbf{1}$ from the position without compromising its acceptability. Thus, a good deal of this kind allows the arbitrageur to determine whether the rewards of doing the deal outweigh the costs.

Proposition 3.5.2 *Let M denote the cone of cash streams available in the market. Fix a coherent risk measure ρ, let*

$$A = \{x \in L : \rho(x) \leq 0\}$$

and

$$C = \{x \in L : \rho(x) < 0\}.$$

Then the following statements are true:

(a) $M - A = \{z \in L : \exists\, x \in M \text{ such that } x \succeq z\}$.

(b) If ρ is strongly relevant, there are no good deals of the first kind if and only if $(M - A) \cap A = \{0\}$.

(c) There are no good deals of the second kind if and only if $(M - A) \cap C = \emptyset$.

Proof. (a) If $z \in M - A$, then $z = x - a$ where $x \in M$ and $a \in A$. Consequently, $x - z \in A$ so that $x \succeq z$. Conversely, for $z \in L$, if there exists $x \in M$ such that $x \succeq z$, we have $a := x - z \in A$. Thus, $z = x - a \in M - A$.

(b) Clearly, the absence of good deals of the first kind is equivalent to the condition $M \cap A = \{0\}$.

Since $M \subset M - A$, we have that $(M - A) \cap A = \{0\}$ implies $M \cap A = \{0\}$. Conversely, assume $M \cap A = \{0\}$ and suppose that $0 \neq z \in (M - A) \cap A$. Then there exists $x \in M$ such that $x \succeq z \succeq 0$. I.e. $x - z \in A$. The strong relevance of ρ implies that $-z \notin A$ and so $x \neq 0$. Hence, $0 \neq x \in M \cap A = \{0\}$, a contradiction.

(c) We first show that the absence of deals of the second kind is equivalent to the condition $M \cap C = \emptyset$. Indeed, if $x \in M \cap C$, we have that $\rho(x) = \beta < 0$. Hence, $0 = \rho(x) - \beta = \rho(x + \beta\mathbf{1})$. Letting $\alpha = -\beta > 0$, it follows that $x - \alpha\mathbf{1} \in A$ and so x is a good deal of the second kind. Conversely, if $x \in M$ is a good deal of the second kind, there exists $\alpha > 0$ so that $x - \alpha\mathbf{1} \in A$. Hence, $\rho(x - \alpha\mathbf{1}) = \rho(x) + \alpha \leq 0$. Consequently, $\rho(x) < 0$ which implies $x \in M \cap C$.

To conclude, observe that $M \subset M - A$ and so $(M - A) \cap C = \emptyset$ implies $M \cap C = \emptyset$. Conversely, assume $M \cap C = \emptyset$ and suppose that $z \in (M - A) \cap C$. Then $\rho(z) < 0$ and $z = x - a$ where $x \in M$ and $a \in A$. Consequently, $\rho(x) = \rho(z + a) \leq \rho(z) + \rho(a) < 0$. This contradicts $M \cap C = \emptyset$. □

If $A = L_+$ in the above proposition, the good deals of the first kind specialize to arbitrages.

Proposition 3.5.3 *Let A be a coherent acceptance set and M denote the cone of cash streams available in the market. Then the following statements hold:*

(a) There are no good deals of the second kind if and only if $\mathbf{1} \notin M - A$.

(b) If $-\mathbf{1} \notin A$, then every good deal of the second kind is also a good deal of the first kind.

Proof. (a) Let $x \in M$ and $a := x - \alpha\mathbf{1} \in A$ for some $\alpha > 0$. Since $M - A$ is a cone, it follows that $\mathbf{1} = \alpha^{-1}(x - a) \in M - A$. Conversely, suppose that $\mathbf{1} \in M - A$. Then, for any $\alpha > 0$, we may write $\alpha\mathbf{1} = x - a$ where $x \in M$ and $a \in A$. Consequently, $a = x - \alpha\mathbf{1} \in A$ so that x is a good deal of the second kind.

(b) As before, let $x \in M$ and $x - \alpha\mathbf{1} \in A$ for some $\alpha > 0$. Clearly, $-\alpha\mathbf{1} \notin A$ implies that $x \neq 0$. Since $\alpha\mathbf{1} \in A$, it follows that $x = (x - \alpha\mathbf{1}) + \alpha\mathbf{1} \in A$. Thus, x is a good deal of the first kind. □

We will focus on good deals of the second kind for the remainder of this chapter.

Definition 3.5.4 Let A be a coherent acceptance set and M denote the cone of cash streams available in the market, then we may define the market induced *good deal bounds*:

$$\begin{aligned} \overline{\pi}_{A,M}(z) &= \inf_{\alpha \in \mathbb{R}, x \in M} \{\alpha : x + \alpha\mathbf{1} - z \in A\}, \text{ and} \\ \underline{\pi}_{A,M}(z) &= \sup_{\alpha \in \mathbb{R}, x \in M} \{\alpha : x - \alpha\mathbf{1} + z \in A\}. \end{aligned}$$

The interval $[\underline{\pi}_{A,M}(z), \overline{\pi}_{A,M}(z)]$ is the interval of the prices for the cashflow z that exclude the possibility of good deals of the second kind.

As an illustration, suppose that an agent is willing to buy the cash stream z for a price $p > \overline{\pi}_{A,M}(z)$. Then there exists a market hedge x and price $\alpha < p$ such that

$$x + \alpha\mathbf{1} - z \in A.$$

Thus, we can sell z to the agent at price p, execute the strategy that generates x, and the resulting cash stream $y := x + p\mathbf{1} - z$ is a good deal of the second kind. Indeed, let $\gamma = p - \alpha > 0$, then $y \in M$ and $y - \gamma\mathbf{1} \in A$. A similar argument can be made for an agent willing to sell the cash stream z for a price less than $\underline{\pi}_{A,M}(z)$.

Proposition 3.5.5 *If A is a coherent acceptance set and M is a cone, then the good deal bounds $(\underline{\pi}_{A,M}, \overline{\pi}_{A,M})$ is a pair of coherent valuation bounds. Moreover,*

$$\rho(z) := -\underline{\pi}_{A,M}(z) = \rho_{A-M}(z) = \inf_{x \in M} \rho_A(x + z)$$

for all $z \in M$.

Proof. It follows easily from the definition that $\overline{\pi}_{A,M}(z) = -\underline{\pi}_{A,M}(-z)$. Also observe

$$\{\alpha : x + z - \alpha\mathbf{1} \in A, x \in M\} = \{\alpha : z - \alpha\mathbf{1} \in A - M\}.$$

Thus,

$$\begin{aligned}
\rho(z) &= -\underline{\pi}(z) \\
&= -\sup\{\alpha : x - \alpha\mathbf{1} + z \in A, x \in M\} \\
&= \inf\{-\alpha : x - \alpha\mathbf{1} + z \in A, x \in M\} \\
&= \inf\{\alpha : x + \alpha\mathbf{1} + z \in A, x \in M\} \\
&= \inf\{\alpha : z + \alpha\mathbf{1} \in A - M\} \\
&= \rho_{A-M}(z),
\end{aligned}$$

from which $\rho(z) = \inf_{x \in M} \rho_A(x + z)$ follows. Since $A - M$ is a cone that contains L_+, ρ is a coherent risk measure. If $A - M$ is radially closed, the relationship is one-to-one. □

We will refer to $\rho = \rho_{A-M}$ as the *market aware risk measure.*

The definition of $(\underline{\pi}_{A,M}, \overline{\pi}_{A,M})$ and $(\underline{\pi}, \overline{\pi})$ are similar, but their economic meaning is quite different. The raw valuation bounds $(\underline{\pi}_{A,M}, \overline{\pi}_{A,M})$ are independent of the market and based purely on an agent's preference. The bounds $(\underline{\pi}_{A,M}, \overline{\pi}_{A,M})$ price a security relative to the market and some trading strategy. These bounds can be viewed as a generalisation of the no-arbitrage price bounds seen earlier.

Definition 3.5.6 We call

$$\begin{aligned} K_{A,M} &:= A^\circ \cap (-M)^\circ \\ &= \Big\{\pi \in \hat{L} : \pi(x) \geq 0 \; \forall \; x \in A \text{ and } \pi(x) \leq 0 \; \forall \; x \in M\Big\} \\ &= \Big\{\pi \in \hat{L} : \pi(x) \geq 0 \; \forall \; x \in A \text{ and } \pi(x) \leq 0 \; \forall \; x \in M - A\Big\} \end{aligned}$$

the set of *consistent price systems* and

$$D_{A,M} = \{\pi \in K_{A,M} : \pi(\mathbf{1}) = 1\}$$

the set of *normalised* consistent price systems.

The set $D_{A,M}$ is analogous to the set of absolutely continuous martingale measures $\mathcal{M}^a(S)$ in the previous chapter.

As a simple consequence of Theorem 3.4.2, we obtain a generalised version of the fundamental theorem of asset pricing.

Theorem 3.5.7 (Jaschke-Küchler) *Let $A \subset L$ be a coherent acceptance set and $M \subset L$ be the cone of cashflows available in the market. If $A - M$ is $\sigma(L, \hat{L})$-closed, then the following statements are true for $z \in L$.*

(a) $A - M = K_{A,M}^\circ$.

(b) $D_{A,M} \neq \emptyset$ *if and only if there are no good deals of the second kind in the market.*

(c) $\rho_{A-M}(z) = \sup_{\pi \in K_{A,M}} \pi(-z)/\pi(\mathbf{1})$, *where we adopt the convention* $0/0 = -\infty$.

(d) $\rho_{A-M}(z) = \sup_{\pi \in D_{A,M}} \pi(-z)$, *provided* $\mathbf{1}$ *is in the radial interior of* $A - M$. *We use the convention* $\sup \emptyset = -\infty$.

Proof. (a) Observe that $K_{A,M} := A^\circ \cap (-M)^\circ = (A - M)^\circ$. Indeed, $\pi \in A^\circ \cap (-M)^\circ$ implies $\pi(a) \geq 0$ for $a \in A$ and $\pi(m) \leq 0$ for $m \in M$. Hence $\pi(a) - \pi(m) = \pi(a - m) \geq 0$ so that $\pi \in (A - M)^\circ$. Conversely, if $\pi \in (A - M)^\circ$, then $\pi(x) \geq 0$ for all elements of the form $x = 0 - m$ and $x = a - 0$ where $a \in A$ and $m \in M$. Thus, $\pi \in A^\circ \cap (-M)^\circ$.

The statement now follows from Theorem 3.4.2 (a) by replacing the acceptance set A with $A - M$.

(b) By Theorem 3.4.2 (b), $D_{A,M} \neq \emptyset$ if and only if $-\mathbf{1} \notin A - M$. By Proposition 3.5.3, $\mathbf{1} \notin M - A$ is equivalent to the absence of good deals of the second kind, from which the result follows.

(c), (d) Follows directly from Theorem 3.4.2. □

Corollary 3.5.8 *Let $A \subset L$ be a coherent acceptance set and $M \subset L$ be the cone of cashflows available in the market. If $A - M$ is $\sigma(L, \hat{L})$-closed, then $D_{A,M} \neq \emptyset$ if and only if $(M - A) \cap C = \emptyset$.*

Proof. The result follows from (b) in the above theorem and Proposition 3.5.2. □

Corollary 3.5.9 *Let $A \subset L$ be a coherent acceptance set and $M \subset L$ be the cone of cashflows available in the market. If $A - M$ is $\sigma(L, \hat{L})$-closed, then*

$$\begin{aligned} \underline{\pi}_{A,M}(z) &= \inf_{\pi \in D_{A,M}} \pi(z) \text{ and} \\ \overline{\pi}_{A,M}(z) &= \sup_{\pi \in D_{A,M}} \pi(z) \end{aligned}$$

for all $z \in L$, provided $\mathbf{1}$ is in the radial interior of $A - M$.

Proof. Follows directly from (d) in the above theorem and the identities implied by Proposition 3.5.5, i.e. $\underline{\pi}_{A,M}(z) = -\rho_{A-M}(z)$ and $\overline{\pi}_{A,M}(z) = \rho_{A-M}(-z)$ for all $z \in L$. □

As is evident from the above results, it is a requirement that $A - M$ be $\sigma(L, \hat{L})$-closed in order to achieve a fundamental theorem of asset pricing. This requires us to choose an appropriate M so as to complete $A - M$.

Chapter 4

A Generalised Kreps-Yan Theorem

4.1 Introduction

In [35, 34], Jaschke and Küchler prove a FTAP pertaining to good deals of the second kind only. It is argued that deals of this kind are practically relevant to the arbitrageur, because the value of doing the deal can be directly measured in units of the numéraire asset.

In this chapter, we will focus on the stronger condition of no good deals of the first kind, which is a natural extension of the no-arbitrage condition. As such, a continuous time, infinite dimensional version of the FTAP pertaining to these deals does not exist.

Following the approach of Kreps [43], we consider the condition of 'no near-good deals of the first kind', which is a natural extension of the no-free-lunch condition. This leads to a generalisation of the Kreps-Yan Theorem, where the notion of a free-lunch is extended to include (almost) super replicable contingent claims having acceptable risk specified by an agent's market aware coherent risk measure. Here, the super replication occurs according the agent's partial preference ordering induced by his/her risk measure. As a result, we obtain price bounds that are tighter than no-free-lunch price bounds.

In [59], Staum extends the work of Jaschke and Küchler. He formulates the notion of a near-arbitrage, which is analogous to a near-good deal of the first kind, but not equivalent. He goes on to prove a general version of the FTAP in terms of the absence of near-arbitrage. Using a different approach to that of Staum, we are able to prove a partial case of this result.

4.2 Preliminaries

Suppose that L is a topological vector space. In Theorem 3.4.2 (d), it is assumed that $\mathbf{1}$ is contained in the radial interior of the acceptance set A. This is a strong requirement. Indeed, an element e is in the radial interior of $A \subset L$ if and only if e is an *order unit* of L with respect to the ordering induced by A (cf. [56]), i.e. $I_e := \cup_{n\in\mathbb{N}}[-ne, ne] = L$, where we use the notation

$$[x, y] = \{z \in L : x \preceq z \preceq y\}.$$

If A is closed and the topology on L is completely metrizable (e.g. $L^0(\mathbb{P})$ equipped with the topology of convergence in probability), then e is in the interior of A if and only if e is an order unit of L.

It is well known that the positive cones $L^p_+(\mathbb{P})$, $1 \leq p < \infty$, have void interior with respect to the norm topology on $L^p(\mathbb{P})$. Also, the positive cone of $L^\infty(\mathbb{P})$ has void interior with respect to the weak* topology. In order to extend the above results to L^p-spaces, we use the notion of a quasi-interior point.

Definition 4.2.1 Let L be a locally convex topological vector space, ordered by the convex cone K. An element $e \in K$ is said to be a *quasi-interior point* of K if the set $\cup_{n\in\mathbb{N}}[-ne, ne]$ is dense in L. The set of all quasi-interior points of K will be denoted K_q.

One can check that $e \in K_q$ if and only if $\operatorname{span} K \cap (\{e\} - K)$ is dense in L. For a detailed account on quasi-interior points, the reader can consult [26] as well as [57, 56]. For convenience, we shall recall some basic results.

Proposition 4.2.2 *Let L be a locally convex topological vector space, ordered by the convex cone K. Then the following statements hold.*

(a) $\operatorname{int} K \subset K_q$.

(b) *If* $\text{int}K \neq \emptyset$ *then* $K_q = \text{int}K$.

(c) K_q *is a convex subset of* K.

(d) $K_q + K \subset K_q$.

(e) $K_q = \cup\{e + K : e \in K_q\}$.

(f) $K_q \cup \{0\}$ *is a convex cone.*

(g) $K_q \neq \emptyset$ *and* $L = K - K$ *imply* $L = K_q - K_q$.

(h) *If* $K_q \neq 0$ *then* $K \subset \overline{K_q}$. *If* K *is closed, then* $K = \overline{K_q}$.

(i) $(x + K)_q = x + K_q$ *for all* $x \in L$.

When K has non-empty interior, this interior is equal to K_q. When the interior of K is empty, the quasi-interior may be non-empty. Indeed, the L^p-spaces mentioned above are an example of this, since **1** is quasi-interior to the positive cone which has empty interior.

We turn our attention to strictly positive functionals.

Proposition 4.2.3 *Let* $\langle L, \hat{L}\rangle$ *be vector spaces in separating duality and suppose that* L *is ordered by the cone* $K \subset L$. *If* $f \in (K^\circ)_q$ *then* f *is strictly positive on* K.

Proof. Suppose $f \in (K^\circ)_q$ and choose $x \in K \setminus \{0\}$. If $\langle x, f\rangle = 0$, then $\langle y, f\rangle = 0$ for all $y \in I_f = \cup_{n\in\mathbb{N}}[-nf, nf] \subset \hat{L}$. Since I_f is dense in $\hat{L}$, it follows that $x = 0$, which is a contradiction. □

To prove the converse, we need to assume that K induces a lattice structure on the pair $\langle L, \hat{L}\rangle$ (cf. [5, Theorem 4.85] or [4, Theorem 8.54]).

We exhibit a condition on the acceptance set A that guarantees the existence of a strictly positive functional on A.

Definition 4.2.4 Let L be a vector space, ordered by the convex cone K. A convex subset $B \subset K$ is called a *base* for K if for each $0 \neq x \in K$, there exists a unique real number $\alpha > 0$ such that $\alpha^{-1}x \in B$.

A cone K admits a strictly positive functional if and only if it has a base. The discussion below proves this fact and is adapted from [3, Theorem 1.47].

In view of the above definition, define the function $\tau : K \to \mathbb{R}_+$ by $\tau(x) = \alpha$. Then τ is additive on K. To see this, let $x, y \in K$ and $b_1 = x/\tau(x)$ and $b_2 = y/\tau(y)$. Then $b_1, b_2 \in B$ and by the convexity of B, we have

$$b := \frac{\tau(x)b_1}{\tau(x) + \tau(y)} + \frac{\tau(y)b_2}{\tau(x) + \tau(y)} \in B.$$

Thus, $(\tau(x) + \tau(y))b = x + y$. By uniqueness, it follows that $\tau(x + y) = \tau(x) + \tau(y)$. An appeal to [3, Lemma 1.26] indicates that τ may be extended to a strictly positive linear function $f : K - K \to \mathbb{R}$ via the formula $f(x_1 - x_2) = f(x_1) - f(x_2)$, $x_1, x_2 \in K$.

Conversely, if there exists a strictly positive linear functional $f : K - K \to \mathbb{R}$, the set $B = K \cap f^{-1}(1)$ is a base for the cone K. To see this, let $\alpha = f(x)$ for each $x \in K$. Then $\alpha^{-1}x \in B$ and $\alpha > 0$ is unique. Indeed, suppose $\lambda > 0$ is such that $\lambda^{-1}x \in B$, then $f(\lambda^{-1}x) = 1 \Rightarrow f(x) = \lambda = \alpha$. In this case, we say that B is *defined* by f.

Note that if L is a locally convex topological vector space, f need not be continuous if the topology on L is not complete. We require that the strictly positive functional f be continuous. The next definition guarantees this.

Definition 4.2.5 Let L be a locally convex topological vector space, ordered by the convex cone K. Then K is said to be *well-based* if K has a base $B \subset K$ such that $0 \notin \overline{B}$.

The following result can be found in [47].

Proposition 4.2.6 *Let L be a locally convex topological vector space, ordered by the convex cone K. Then K admits a continuous strictly positive functional $f : L \to \mathbb{R}$ if and only if K is well-based. In this case, K is pointed and we may take the base B of K to be $B = K \cap f^{-1}(1)$.*

Proof. Assume K admits a continuous strictly positive functional $f : L \to \mathbb{R}$. Let $B = K \cap f^{-1}(1)$, then B is a convex base for K by the above discussion. Since f is continuous, we have that $0 \notin \overline{B}$.

Conversely, by the Hyperplane Separation Theorem, there exists a continuous functional f that strictly separates $\{0\}$ and $\overline{B}$. We may choose f so that $f(x) > 0$ for all $x \in \overline{B}$. For each $y \in K \setminus \{0\}$, we have $y = \lambda x$ where $\lambda > 0$ and $x \in B$ are uniquely determined. Hence, $f(y) = \lambda f(x) > 0$ so that f is strictly positive on K.

To complete the proof, observe that $K \cap (-K) \subset f^{-1}(0)$ and $K \cap f^{-1}(0) = \{0\}$. Thus, $K \cap (-K) \subset K \cap f^{-1}(0) = \{0\}$, which shows that K is pointed. □

In [33, Theorem 3.8.4], it is shown that if L is a normed space ordered by the cone K, then K is well-based by a bounded base B if and only if the dual cone K° in the norm dual L^* has non-empty interior.

For some interesting results concerning the geometry of cones, the interested reader can consult [47, 42].

4.3 Near-Good Deals of the First Kind

Definition 4.3.1 Let $\langle L, \hat{L} \rangle$ be in separating duality, $A \subset L$ a strongly relevant coherent acceptance set, and M a cone of marketed cash streams.

(a) We refer to $0 \neq z \in (\overline{M - A}) \cap A$ as a *near-good deal of the first kind.* Here, the closure of $M - A$ is taken in the $\sigma(L, \hat{L})$-topology.

(b) We say that there are no near-good deals of the first kind if $(\overline{M - A}) \cap A = \{0\}$.

It is obvious that every good deal of the first kind is also a near-good deal of the first kind.

In what follows, assume that $\langle L, \hat{L} \rangle$ are Banach spaces in separating duality. Note that $\hat{L}$ is not necessarily the continuous dual of L, for which we use the notation L^*.

Definition 4.3.2 Let $A \subset L$ be a closed coherent acceptance set.

(a) A functional $\pi \in \hat{L}$ is called *strictly positive* with respect to A if $\langle x, \pi \rangle > 0$ for all $x \in A \setminus \{0\}$.

(b) We use the notation

$$\widetilde{K}_A := \{\pi \in K_A : \pi \text{ is strictly positive with respect to } A\}$$

for the set of *strictly positive admissible price systems* and

$$\widetilde{D}_A := \{\pi \in \widetilde{K}_A : \pi(\mathbf{1}) = 1\}$$

for the set of *normalised* strictly positive admissible price systems.

(c) If $M \subset L$ is a cone of marketed cash streams, then define

$$\begin{aligned} \widetilde{K}_{A,M} &:= \widetilde{K}_A \cap (-M)^\circ \\ &= \Big\{\pi \in \hat{L} : \pi(x) > 0 \ \forall\, x \in A \setminus \{0\} \text{ and } \pi(x) \leq 0 \ \forall\, x \in M\Big\} \\ &= \Big\{\pi \in \hat{L} : \pi(x) > 0 \ \forall\, x \in A \setminus \{0\} \text{ and } \pi(x) \leq 0 \ \forall\, x \in M - A\Big\} \end{aligned}$$

to be the set of *strictly positive consistent price systems* and

$$\widetilde{D}_{A,M} := \{\pi \in \widetilde{K}_{A,M} : \pi(\mathbf{1}) = 1\}$$

to be the set of *normalised* strictly positive consistent price systems.

(d) Let $M \subset L$ be a cone of marketed cash streams. Then we define the market induced *no near-good deal valuation bounds* as:

$$\begin{aligned} \overline{\pi}_{A,M}(z) &= \rho_{\overline{A-M}}(-z) \text{ and} \\ \underline{\pi}_{A,M}(z) &= -\rho_{\overline{A-M}}(z), \end{aligned}$$

for all $z \in L$, where $\rho_{\overline{A-M}}$ denotes the coherent risk measure corresponding to the closed convex cone $\overline{A-M}$.

We now generalise the Kreps-Yan Theorem to the setting of coherent risk measures.

Theorem 4.3.3 *Suppose that $(L, \sigma(L, \hat{L}))$ is a Lindelöf space with $M \subset L$ a cone of marketed cash streams. Let $A \subset L$ be a closed and well-based coherent acceptance set, and let $\rho_{\overline{A-M}}$ be the market aware risk measure. Then the following statements are true:*

(a) There are no near-good deals of the first kind if and only if $\widetilde{D}_{A,M} \neq \emptyset$.

(b) There are no near-good deals of the first kind if and only if $\overline{A-M} = \widetilde{K}^\circ_{A,M}$.

(c) If there are no near-good deals of the first kind, then

$$\rho_{\overline{A-M}}(z) = \sup_{\pi \in \widetilde{K}_{A,M}} \pi(-z)/\pi(\mathbf{1}) = \sup_{\pi \in \widetilde{D}_{A,M}} \pi(-z).$$

(d) Using the convention $0/0 = -\infty$*, we have*

$$\rho_{\overline{A-M}}(z) = \sup_{\pi \in K_{A,M}} \pi(-z)/\pi(\mathbf{1}).$$

If $\mathbf{1}$ *is in the quasi-interior of* $\overline{A-M}$ *then*

$$\rho_{\overline{A-M}}(z) = \sup_{\pi \in D_{A,M}} \pi(-z),$$

where we use the convention $\sup \emptyset = -\infty$*.*

Proof. (a) Proposition 4.2.6 implies that A admits a strictly positive functional and is strongly relevant. Thus, if there are no near-good deals of the first kind, Theorem 2.3.5 implies the existence of a functional $\pi \in \widetilde{D}_{A,M}$. Conversely, if $\pi \in \widetilde{D}_{A,M}$, we have $A \cap \pi^{-1}(0) = \{0\}$. However, $(\overline{M-A}) \cap A \subset \pi^{-1}(0)$, which implies $(\overline{M-A}) \cap A = \{0\}$. Thus, there are no near-good deals of the first kind.

(b) By (a), we have that $\widetilde{K}_{A,M} \neq \emptyset$ and so we may choose $\pi^* \in \widetilde{K}_{A,M}$. Then, for any $\pi \in K_{A,M} = (\overline{A-M})^\circ$, define the sequence $a_n = (1/n)\pi^* + (1-1/n)\pi$. Clearly, $a_n \in \widetilde{K}_{A,M}$ for all $n \in \mathbb{N}$. Moreover, for any $x \in L$, we have

$$\begin{aligned} \lim_{n\to\infty} \langle x, a_n \rangle &= \lim_{n\to\infty} (1/n)\langle x, \pi^* \rangle + \lim_{n\to\infty} (1-1/n)\langle x, \pi \rangle \\ &= \langle x, \pi \rangle. \end{aligned}$$

This shows $\widetilde{K}_{A,M}$ is $\sigma(\hat{L}, L)$-dense in $K_{A,M}$. Since $\overline{A-M} = K^\circ_{A,M}$ by the Bi-Polar Theorem, we have

$$x \in \overline{A-M} \iff \langle x, \pi \rangle \geq 0 \text{ for all } \pi \in K_{A,M}. \tag{4.3.1}$$

Since $\widetilde{K}_{A,M}$ is dense in $K_{A,M}$, (4.3.1) is equivalent to

$$x \in \overline{A-M} \iff \langle x, \pi \rangle \geq 0 \text{ for all } \pi \in \widetilde{K}_{A,M} \tag{4.3.2}$$

by continuity. Consequently, $\overline{A-M} = \widetilde{K}^\circ_{A,M}$. Conversely, $\overline{A-M} = \widetilde{K}^\circ_{A,M}$ implies that $\widetilde{K}_{A,M} \neq \emptyset$, and consequently $\widetilde{D}_{A,M} \neq \emptyset$. By (a), there are no near-good deals of the first kind.

(c) Use the equivalence of (4.3.1) and (4.3.2) and the proof of Theorem 3.4.2 (c) with A replaced with $\overline{A-M}$. Normalising the elements of $\widetilde{K}_{A,M}$ yields the result.

(d) Since **1** is in the quasi-interior of $\overline{A-M}$, Proposition 4.2.3 implies that **1** is strictly positive on $K_{A,M}$. Proposition 4.2.6 implies that $D_{A,M}$ is a base for the cone $K_{A,M}$. The result now follows from Theorem 3.4.2 (c). □

Corollary 4.3.4 *Suppose that $(L, \sigma(L, \hat{L}))$ is a Lindelöf space with $M \subset L$ a cone of marketed cash streams. Let $A \subset L$ be a closed and well-based coherent acceptance set, and let $\rho_{\overline{A-M}}$ be the market aware risk measure. Then, the no near-good deal valuation bounds $(\underline{\pi}_{A,M}, \overline{\pi}_{A,M})$ may be calculated as*

$$\begin{aligned} \underline{\pi}_{A,M}(z) &= \inf_{\pi \in \widetilde{D}_{A,M}} \pi(z) \text{ and} \\ \overline{\pi}_{A,M}(z) &= \sup_{\pi \in \widetilde{D}_{A,M}} \pi(z) \end{aligned}$$

for all $z \in L$, provided there are no near-good deals of the first kind.

Proof. Follows directly from (d) in the above theorem and the identities $\underline{\pi}_{A,M}(z) = -\rho_{\overline{A-M}}(z)$ and $\overline{\pi}_{A,M}(z) = \rho_{\overline{A-M}}(-z)$ for all $z \in L$. □

Corollary 4.3.5 *Suppose that $(L, \sigma(L, \hat{L}))$ is an ordered Lindelöf space with $M \subset L$ a cone of marketed cash streams. Let $A \subset L$ be a closed and well-based coherent acceptance set. Then, the no near-good deal valuation bounds $(\underline{\pi}_{A,M}, \overline{\pi}_{A,M})$ are at least as tight as the no-free-lunch valuation bounds $(\underline{\pi}_{L_+,M}, \overline{\pi}_{L_+,M})$.*

Proof. The no-free-lunch valuation bounds $(\underline{\pi}_{L_+,M}, \overline{\pi}_{L_+,M})$ correspond to the acceptance set $\overline{L_+ - M}$. By the monotonicity of A, the set of strictly positive consistent price systems with respect to A, given by

$$\widetilde{K}_{A,M} = \left\{ \pi \in \hat{L} : \pi(x) > 0 \ \forall\, x \in A \setminus \{0\} \text{ and } \pi(x) \leq 0 \ \forall\, x \in M - A \right\},$$

is contained inside the set of strictly positive consistent price systems with respect to L_+, given by

$$\widetilde{K}_{L_+,M} = \left\{ \pi \in \hat{L} : \pi(x) > 0 \ \forall\, x \in L_+ \setminus \{0\} \text{ and } \pi(x) \leq 0 \ \forall\, x \in M - L_+ \right\}.$$

The containment is preserved for the corresponding normalised price systems. The result now follows easily from the above corollary. □

If the cone of marketed cashflows M is chosen so that the cone $A - M$ is $\sigma(L, \hat{L})$-closed, then near-good deals of the first kind may be replaced with good deals of the first kind in the above results.

4.4 Near-Arbitrage

In [59], Staum proves an abstract version of the FTAP (cf. [59, Theorem 6.2]) in terms of the absence of near-arbitrage. In this section, we recover a partial case of Staum's result.

Definition 4.4.1 Let L be an ordered vector space, $A \subset L$ be a coherent acceptance set and $M \subset L$ is a cone of marketed cash streams.

(a) We say there are no *near-arbitrages* in the market if $(\overline{M - A}) \cap L_+ = \{0\}$.

(b) In addition, we write

$$\widetilde{K}^{(\text{nna})}_{A,M} = \left\{\pi \in \hat{L} : \pi(x) > 0 \; \forall \; x \in L_+ \setminus \{0\} \text{ and } \pi(x) \leq 0 \; \forall \; x \in M - A\right\}$$

for the set of *strictly positive consistent price systems* and

$$\widetilde{D}^{(\text{nna})}_{A,M} := \{\pi \in \widetilde{K}_{A,M} : \pi(\mathbf{1}) = 1\}$$

for the set of *normalised* strictly positive consistent price systems.

The absence of near-arbitrage is a slightly weaker condition than the absence of near-good deals of the first kind and has a slightly more generic interpretation; a near-arbitrage is an (almost) super replicable claim that is non-negative. As with near-good deals of the first kind, super-replication takes place according to a preference ordering specified by an acceptance set. The main difference is that the claim has no downside, whereas with a near-good deal, the downside is acceptable in terms ones specified risk appetite. Of course, not everyone in the market uses the same risk measure.

We obtain the following result in an analogous manner to Theorem 4.3.3.

Theorem 4.4.2 (Staum) *Suppose that $(L, \sigma(L, \hat{L}))$ is an ordered Lindelöf space such that L_+ is well-based and closed. Let $M \subset L$ be a cone of marketed cash streams and let $A \subset L$ be a coherent acceptance set, with $\rho_{\overline{A-M}}$ the market aware risk measure. Then the following statements are true:*

(a) There are no near-arbitrages if and only if $\widetilde{D}^{(\text{nna})}_{A,M} \neq \emptyset$.

(b) *There are no near-arbitrages if and only if* $\overline{A-M} = (\widetilde{K}^{(\mathrm{nna})}_{A,M})^\circ$.

(c) *If there are no near-arbitrages, then*

$$\rho_{\overline{A-M}}(z) = \sup_{\pi \in \widetilde{K}^{(\mathrm{nna})}_{A,M}} \pi(-z)/\pi(\mathbf{1}) = \sup_{\pi \in \widetilde{D}^{(\mathrm{nna})}_{A,M}} \pi(-z).$$

Proof. (a) Proposition 4.2.6 implies that L_+ admits a strictly positive functional and is pointed. By the monotonicity of A, we have $-L_+ \subset \overline{M-A}$. Thus, if there are no near-arbitrages, Theorem 2.3.5 implies the existence of a functional $\pi \in \widetilde{D}^{(\mathrm{nna})}_{A,M}$. Conversely, if $\pi \in \widetilde{D}^{(\mathrm{nna})}_{A,M}$, we have $L_+ \cap \pi^{-1}(0) = \{0\}$. However, $(\overline{M-A}) \cap L_+ \subset \pi^{-1}(0)$, which implies $(\overline{M-A}) \cap L_+ = \{0\}$. Thus, there are no near-arbitrages.

(b) It follows easily from the monotonicity of A that $\widetilde{K}^{(\mathrm{nna})}_{A,M} \subset K_{A,M} = (\overline{A-M})^\circ$. In a similar fashion to Theorem 4.4.2 (b), we have that $\widetilde{K}^{(\mathrm{nna})}_{A,M}$ is dense in $K_{A,M}$ if and only if there are no near-arbitrages. The result follows from the Bi-Polar Theorem.

(c) Use (b) and the proof of Theorem 3.4.2 (c) with A replaced with $\overline{A-M}$. Normalising the elements of $\widetilde{K}^{(\mathrm{nna})}_{A,M}$ yields the result. □

Although the condition of no near-arbitrage is weaker than that of no near-good deals of the first kind, the resulting valuation bounds are identical.

Corollary 4.4.3 *Suppose that* $(L, \sigma(L, \hat{L}))$ *is an ordered Lindelöf space such that* L_+ *is well-based and closed. Let* $M \subset L$ *be a cone of marketed cash streams and let* $A \subset L$ *be a coherent acceptance set, with* $\rho_{\overline{A-M}}$ *the market aware risk measure. Then, the no near-arbitrage valuation bounds* $\left(\underline{\pi}^{(\mathrm{nna})}_{A,M}, \overline{\pi}^{(\mathrm{nna})}_{A,M}\right)$ *may be calculated as*

$$\begin{aligned} \underline{\pi}^{(\mathrm{nna})}_{A,M}(z) &= \inf_{\pi \in \widetilde{D}^{(\mathrm{nna})}_{A,M}} \pi(z) \text{ and} \\ \overline{\pi}^{(\mathrm{nna})}_{A,M}(z) &= \sup_{\pi \in \widetilde{D}^{(\mathrm{nna})}_{A,M}} \pi(z) \end{aligned}$$

for all $z \in L$*, provided there are no near-arbitrages. Moreover, if* A *is closed and well based, the no near-good deal valuation bounds* $(\underline{\pi}_{A,M}, \overline{\pi}_{A,M})$ *are identical to the no near-arbitrage valuation bounds.*

Proof. Follows directly from (c) in the above theorem and the identities

$$\underline{\pi}^{(\text{nna})}_{A,M}(z) = -\rho_{\overline{A-M}}(z)$$

and

$$\overline{\pi}^{(\text{nna})}_{A,M}(z) = \rho_{\overline{A-M}}(-z)$$

for all $z \in L$. The fact that these quantities correspond to the no near-good deal valuation bounds follows from the uniqueness of $\rho_{\overline{A-M}}$. □

In general, the condition $(\overline{M-A}) \cap H = \{0\}$, where H is a closed convex cone with $L_+ \subset H \subset A$, induces the same valuation bounds as the condition of no near-good deals of the first kind.

4.5 Application to L^p-Spaces

4.5.1 Expectation Boundedness

In order to apply Theorem 4.3.3 to a meaningful class of coherent risk measures, we need to strengthen the assumption of monotonicity. We follow Rockafeller et al. with the following definition [49, 48].

Definition 4.5.1 (Expectation Boundedness) We call a coherent risk measure $\rho : L^p(\mathbb{P}) \to \mathbb{R}$ *expectation bounded* if it satisfies

(EB) $\rho(x) \geq \mathbb{E}[-x]$ for all $x \in L^p(\mathbb{P})$,

and *strictly expectation bounded* if it satisfies

(SEB) $\rho(x) > \mathbb{E}[-x]$ for all non-constant $x \in L^p(\mathbb{P})$ and $\rho(x) = \mathbb{E}[-x]$ for all constant $x \in L^p(\mathbb{P})$.

Plainly, expectation boundedness implies monotonicity. Thus, when we consider expectation bounded coherent risk measures, we may replace the monotonicity axiom (M) in Definition 3.3.1 with axiom (EB) or (SEB).

It follows from Proposition 1.5.3 (c) that all distortion risk measures ρ_g are expectation bounded if and only if the distortion function has the property $g(u) \leq u$ for all $u \in [0, 1]$. Also, by Theorem 1.3.5, all coherent risk measures with the Fatou Property are expectation bounded.

In [67, Theorem 3], the following representation theorem is proved.

Theorem 4.5.2 *A pricing function $H : L^p(\mathbb{P}) \to \mathbb{R}$ has a Choquet representation*

$$H[X] = -\int_{-\infty}^{0} g(F_X(x))\, dx + \int_{0}^{\infty} [1 - g(F_X(x))]\, dx,$$

for a distortion function g if and only if H satisfies the following properties:

(a) $H[1] = 1$,

(b) H *is law-invariant,*

(c) $X \leq Y$ *then* $H[X] \leq H[Y]$,

(d) H_g *is co-monotonically additive,*

(e) $\lim_{d \to 0^+} H_g[(X - d)_+] = H_g[X]$ *and* $\lim_{d \to \infty} H_g[X \wedge d] = H_g[X]$.

In this case, all the properties listed in Proposition 1.5.3 hold.

Moreover, if the distortion function g satisfies $g(u) < u$ for all $u \in [0, 1]$, then the distortion measure ρ_g induced by H_g is strictly expectation bounded. Indeed, for non-constant $X \geq 0$, we have

$$\begin{aligned} g(u) < u \quad &\Rightarrow \quad g(F_X(x)) < F_X(x) \text{ for all } x \in [0, \infty) \\ &\Rightarrow \quad \int_0^\infty F_X(x) - g(F_X(x))\, \mathrm{d}x > 0 \\ &\Rightarrow \quad \int_0^\infty [1 - g(F_X(x))] - \int_0^\infty [1 - F_X(x)]\, \mathrm{d}x > 0 \\ &\Rightarrow \quad H_g[X] > \mathbb{E}[X]. \end{aligned}$$

For non-constant $X \leq 0$, we have

$$\begin{aligned} g(u) < u \quad &\Rightarrow \quad g(F_X(x)) < F_X(x) \text{ for all } x \in (-\infty, 0] \\ &\Rightarrow \quad -\int_{-\infty}^{0} g(F_X(x)) - F_X(x)\,\mathrm{d}x > 0 \\ &\Rightarrow \quad -\int_{-\infty}^{0} g(F_X(x)) + \int_{-\infty}^{0} F_X(x)\,\mathrm{d}x > 0 \\ &\Rightarrow \quad H_g[X] > \mathbb{E}[X]. \end{aligned}$$

Therefore, for general non-constant X, $\rho_g(X) = H_g[-X] > \mathbb{E}[-X]$ holds. The case of constant X is clear. As a consequence, Expected Shortfall is strictly expectation bounded, as well as the Wang Transform WT^{α} for $0 < \alpha < \frac{1}{2}$.

Strict expectation boundedness guarantees that the corresponding acceptance set is well-based.

Proposition 4.5.3 *An acceptance set $A \subset L^p(\mathbb{P})$ that corresponds to a strictly expectation bounded coherent risk measure ρ is well-based by the convex set $B = A \cap \{x \in L^p(\mathbb{P}) : \mathbb{E}[x] = 1\}$.*

Proof. Let $H_0 = \{x \in L^p(\mathbb{P}) : \langle x, \mathbf{1} \rangle = \mathbb{E}[x] > 0\}$. For non-constant $x \in A$, it follows from $\mathbb{E}[-x] < \rho(x) \leq 0$ that $x \in H_0$. For constant $0 \neq x \in A$, we have $\rho(x) = \mathbb{E}[-x] < 0$, which means $x \in H_0$. Therefore, $A \setminus \{0\} \subset H_0$. Thus, $\mathbf{1} \in A^\circ \subset L^q(\mathbb{P})$ $(p^{-1} + q^{-1} = 1)$ is a continuous functional which is strictly positive on the convex cone A. By Proposition 4.2.6, A is well-based by B. □

4.5.2 Good Deals and Martingales

We are now prepared to specialise to L^p-spaces.

Definition 4.5.4 Let $A \subset L^p(\mathbb{P})$, $1 \leq p \leq \infty$, be a strongly relevant coherent acceptance set and S be a financial process. Then we define

$$\mathcal{M}_A^e(S) = \left\{ \mathbb{Q} \in \mathcal{M}^e(S) : \mathbb{E}^{\mathbb{Q}}[x] > 0 \;\forall\; x \in A \setminus \{0\} \right\}$$

to be the set of equivalent martingale measures that are *strictly positive with respect to A*.

Note that the monotonicity of A implies that any probability measure $\mathbb{Q}$ with $\mathbb{E}^{\mathbb{Q}}[x] > 0 \ \forall \ x \in A \setminus \{0\}$ is equivalent to $\mathbb{P}$. The following result is analogous to Proposition 2.3.2. We present it again in the context of coherent acceptance sets for the sake of completeness.

Proposition 4.5.5 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X = L^p(\mathbb{P})$ endowed with the norm topology for $1 \leq p < \infty$ and the weak* topology $\sigma(L^\infty(\mathbb{P}), L^1(\mathbb{P}))$ for $p = \infty$.*

Let (M, π) be a market model in X induced by the process S, and $M_0 = \pi^{-1}(0) \subset M$ be the linear subspace of marketed cashflows at price 0. Suppose $A \subset X$ is a closed coherent acceptance set. Then the following statements are equivalent:

- *(a) The market model (M, π) admits an extension $\pi^* : X \to \mathbb{R}$ which is strictly positive on A.*
- *(b) There exists $f \in \widetilde{K}_{A,M_0}$.*
- *(c) There exists $f \in \widetilde{D}_{A,M_0}$.*
- *(d) There exists a probability measure $\mathbb{Q} \in \mathcal{M}^e_A(S)$ with $\frac{d\mathbb{Q}}{d\mathbb{P}} \in X^* = L^q(\mathbb{Q})$, $p^{-1} + q^{-1} = 1$.*
- *(e) There exists a $f \in X^*$ that is strictly positive on A such that $f(\mathbf{1}) = 1$ and $f|_{M_0} = 0$.*

Proof. (a)⇒(b) Let $f = \pi^*$, then $f|_{M_0} = 0$ and is strictly positive on A. For all $x \in C := M_0 - A$, we have $x = x_1 - x_2$ with $x_1 \in M_0$ and $x_2 \in A$. Thus, $f(x) = f(x_1) - f(x_2) = 0 - f(x_2) \leq 0$. Hence, $f \in \widetilde{K}_{A,M_0}$.

(b)⇒(c) Since $\mathbf{1} \in A$, replacing f with $f/f(\mathbf{1})$ gives (c).

(c)⇒(d) By the monotonicity of A, f is strictly positive on X_+. Thus, f corresponds to a probability measure $\mathbb{Q} \sim \mathbb{P}$ with density $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = f \in L^q(\mathbb{P})$ such that

$$f(x) = \langle x, f \rangle = \int_\Omega x f \, \mathrm{d}\mathbb{P} = \mathbb{E}^{\mathbb{Q}}[x]$$

for all $x \in L^p(\mathbb{P})$.

Since $M_0 \subset C$, we have $\mathbb{E}^{\mathbb{Q}}[x] \leq 0$ for all $x \in M_0$. Using the fact that M_0 is a linear space, we have $\mathbb{E}^{\mathbb{Q}}[-x] = -\mathbb{E}^{\mathbb{Q}}[x] \leq 0$. Therefore, $\mathbb{E}^{\mathbb{Q}}[x] = 0$ for all $x \in M_0$, whence

S is a martingale under $\mathbb{Q}$ by a similar argument to Lemma 2.2.7. We conclude that $\mathbb{Q} \in \mathcal{M}_A^e(S)$.

(d)⇒(e) Since S is a martingale under $\mathbb{Q}$, the restriction of the functional $f := \mathbb{E}^{\mathbb{Q}}[\,\cdot\,]$ to M_0 is 0. Thus, f satisfies the requirements of (e).

(e)⇒(a) For $x \in M$, we have $x = a\mathbf{1} + m$, where $a \in \mathbb{R}$ and $m \in M_0$. Thus, $f(x) = a + 0 = a$ and so we have $f|_M = \pi$ with f strictly positive on A. □

Theorem 4.3.3 can now be expressed in the following form.

Theorem 4.5.6 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X = L^p(\mathbb{P})$ endowed with the norm topology for $1 \leq p < \infty$ and the weak* topology $\sigma(L^\infty(\mathbb{P}), L^1(\mathbb{P}))$ for $p = \infty$.*

Let (M, π) be a market model in X induced by a financial process S and $M_0 = \pi^{-1}(0) \subset M$ be the linear subspace of marketed cashflows at price 0.

Suppose that ρ is a strictly expectation bounded coherent risk measure that is lower semi-continuous, and that $\rho_{\overline{A-M_0}}$ is the market aware risk measure where $A = A_\rho$. Then the following statements are true.

(a) There are no near-good deals of the first kind in the market if and only if there exists $\mathbb{Q} \in \mathcal{M}_A^e(S)$ with $\frac{d\mathbb{Q}}{d\mathbb{P}} \in X^* = L^q(\mathbb{Q})$, $p^{-1} + q^{-1} = 1$, under which S is a martingale.

(b) If there are no near-good deals of the first kind, then

$$\rho_{\overline{A-M}}(z) = \sup\left\{\mathbb{E}^{\mathbb{Q}}[-z] : \mathbb{Q} \in \mathcal{M}_A^e(S)\right\},$$

for all $z \in X$.

(c) If there are no near-good deals of the first kind, then we have the near-good deal bounds

$$\begin{aligned} \underline{\pi}_{A,M_0}(z) &= \inf\left\{\mathbb{E}^{\mathbb{Q}}[z] : \mathbb{Q} \in \mathcal{M}_A^e(S)\right\} \text{ and} \\ \overline{\pi}_{A,M_0}(z) &= \sup\left\{\mathbb{E}^{\mathbb{Q}}[z] : \mathbb{Q} \in \mathcal{M}_A^e(S)\right\} \end{aligned}$$

for all $z \in X$. Moreover, these bounds are at least as tight as the no-free-lunch bounds.

Proof. (a) The acceptance set A is closed by Theorem 1.3.9 and well-based by Proposition 4.5.3. By the proof of Corollary 2.3.6, X is a Lindelöf space. Theorem 4.3.3 (a) now asserts that there are no near-good deals of the first kind if and only if $\widetilde{D}_{A,M_0} \neq \emptyset$. The latter is equivalent to $\mathbb{Q} \in \mathcal{M}^e_A(S)$ with $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \in X^* = L^q(\mathbb{Q})$, $p^{-1} + q^{-1} = 1$, by Proposition 4.5.5.

(b) Follows from Theorem 4.3.3 (c) and the fact that $\widetilde{D}_{A,M_0} = \mathcal{M}^e_A(S)$.

(c) Follows from Corollary 4.3.4 and 4.3.5. □

4.5.3 Near-Arbitrage and Martingales

In the case of no near-arbitrage, we may drop the assumption of strict expectation boundedness and replace $\mathcal{M}^e_A(S)$ with a larger subset of $\mathcal{M}^e(S)$.

Definition 4.5.7 Let $A \subset L^p(\mathbb{P})$, $1 \leq p \leq \infty$, be a strongly relevant coherent acceptance set and S be a financial process. Then we define

$$\mathcal{M}^{(\mathrm{nna})}_A(S) = \Big\{\mathbb{Q} \in \mathcal{M}^e(S) : \mathbb{E}^{\mathbb{Q}}[x] \geq 0 \ \forall \ x \in A\Big\}$$

to be the set of equivalent martingale measures that are *positive with respect to A*.

Clearly, we have $\mathcal{M}^e_A(S) \subset \mathcal{M}^{(\mathrm{nna})}_A(S) \subset \mathcal{M}^e(S)$. With this definition, we obtain the following analogue of Proposition 4.5.5. We omit the proof, which is similar.

Proposition 4.5.8 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X = L^p(\mathbb{P})$ endowed with the norm topology for $1 \leq p < \infty$ and the weak* topology $\sigma(L^\infty(\mathbb{P}), L^1(\mathbb{P}))$ for $p = \infty$.*

Let (M, π) be a market model in X induced by the process S and $M_0 = \pi^{-1}(0) \subset M$ be the linear subspace of marketed cashflows at price 0. Suppose $A \subset X$ is a closed coherent acceptance set. Then the following statements are equivalent:

(a) The market model (M, π) admits a strictly positive extension $\pi^* : X \to \mathbb{R}$ which is non-negative on A.

(b) There exists $f \in \widetilde{K}^{(\mathrm{nna})}_{A,M_0}$.

(c) There exists $f \in \widetilde{D}^{(\mathrm{nna})}_{A,M_0}$.

(d) *There exists a probability measure* $\mathbb{Q} \in \mathcal{M}_A^{(\text{nna})}(S)$ *with* $\frac{d\mathbb{Q}}{d\mathbb{P}} \in X^* = L^q(\mathbb{Q})$, $p^{-1} + q^{-1} = 1$.

(e) *There exists a strictly positive* $f \in X^*$ *that is non-negative on* A*, such that* $f(\mathbf{1}) = 1$ *and* $f|_{M_0} = 0$.

Theorem 4.4.2 now specialises to the following.

Theorem 4.5.9 *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and* $X = L^p(\mathbb{P})$ *endowed with the norm topology for* $1 \leq p < \infty$ *and the weak* topology* $\sigma(L^\infty(\mathbb{P}), L^1(\mathbb{P}))$ *for* $p = \infty$.

Let (M, π) *be a market model in* X *induced by a financial process* S*, and* $M_0 = \pi^{-1}(0) \subset M$ *be the linear subspace of marketed cashflows at price* 0.

Suppose that ρ *is a coherent risk measure, and that* $\rho_{\overline{A-M_0}}$ *is the market aware risk measure where* $A = A_\rho$*. Then the following statements are true.*

(a) *There are no near-arbitrages in the market if and only if there exists* $\mathbb{Q} \in \mathcal{M}_A^{(\text{nna})}(S)$ *with* $\frac{d\mathbb{Q}}{d\mathbb{P}} \in X^* = L^q(\mathbb{Q})$, $p^{-1} + q^{-1} = 1$*, under which* S *is a martingale.*

(b) *If there are no near-arbitrages, then*

$$\rho_{\overline{A-M}}(z) = \sup\left\{\mathbb{E}^{\mathbb{Q}}[-z] : \mathbb{Q} \in \mathcal{M}_A^{(\text{nna})}(S)\right\},$$

for all $z \in X$.

(c) *If there are no near-arbitrages, then we have the near-arbitrage bounds*

$$\begin{aligned} \underline{\pi}_{A,M_0}^{(\text{nna})}(z) &= \inf\left\{\mathbb{E}^{\mathbb{Q}}[z] : \mathbb{Q} \in \mathcal{M}_A^{(\text{nna})}(S)\right\} \text{ and} \\ \overline{\pi}_{A,M_0}^{(\text{nna})}(z) &= \sup\left\{\mathbb{E}^{\mathbb{Q}}[z] : \mathbb{Q} \in \mathcal{M}_A^{(\text{nna})}(S)\right\} \end{aligned}$$

for all $z \in X$*. Moreover, these bounds are at least as tight as the no-free-lunch bounds.*

Proof. Since X is a Lindelöf space, Theorem 4.4.2 (a) asserts that there are no near-arbitrages if and only if $\widetilde{D}_{A,M_0}^{(\text{nna})} \neq \emptyset$. The latter is equivalent to $\mathbb{Q} \in \mathcal{M}_A^{(\text{nna})}(S)$ with $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \in X^* = L^q(\mathbb{Q})$, $p^{-1} + q^{-1} = 1$, by Proposition 4.5.8.

(b) Follows from Theorem 4.4.2 (c) and the fact that $\widetilde{D}^{(\mathrm{nna})}_{A,M_0} = \mathcal{M}^{(\mathrm{nna})}_{A}(S)$.

(c) Follows from Corollary 4.4.3. □

Conclusions and Directions

We have exhibited some of the overlap between the theory of coherent risk measures and the pricing and hedging of derivatives. The results of Jaschke and Küchler [34], which demonstrate a one-to-one correspondence between coherent risk measures and coherent valuation bounds, allow the Fundamental Theorem of Asset Pricing to be extended to price derivatives using good deal bounds, which are narrower than no-arbitrage bounds. These good deal bounds are specified by an acceptance set which incorporates one's preferences, risk appetite, and current inventory. The use of good deal bounds comes at the expense of making the stronger assumption of no good deals in the market. This assumption is safe in the sense that arbitrage opportunities are also excluded.

These price bounds can be used as bid/offer prices for over-the-counter derivatives, and serve as a methodology of pricing in incomplete markets. The pricing is consistent with the cost of super-replicating the claim using the partial preference ordering induced by the associated acceptance set.

The main result of this work is a generalisation of the Kreps-Yan Theorem [43, 70], where the condition of no-free-lunch has been replaced with the more general condition of no near-good deals of the first kind. This result, in some sense, compliments the work of Jaschke and Küchler, who prove a FTAP with regard to good deals of the second kind. In the context of L^p-spaces, this result also considers the underlying financial process that drives the cone of marketed cashflows M, and relates the condition of no near-good deals of the first kind to the existence of an equivalent martingale measure.

Since M only contains simple market strategies, this result also suffers from the problem of not being able to approximate good deals with countable sequences. A way to overcome this might be to adopt the 'vanishing risk' approach of Delbaen and Schachermayer [16] by making the following definition.

Definition 4.5.10 Let S be a (locally) bounded semi-martingale, $A \subset L^\infty(\mathbb{P})$ a strongly relevant coherent acceptance set, and

$$M_0 = \left\{ \int_0^T H_t \, \mathrm{d}S_t : H \text{ admissable} \right\}$$

the cone of marketed cash streams.

(a) We refer to $0 \neq z \in (\overline{M - A})^\infty \cap A$ as a *near-good deal of the first kind with vanishing risk*. Here, the closure of $M - A$ is taken in the $\| \cdot \|_\infty$-topology.

(b) We say that S satisfies the condition of no good deals of the first kind with vanishing risk (NGDVR) if $(\overline{M - A})^\infty \cap A = \{0\}$.

The crux is being able to prove the following analogue of [16, Theorem 4.2].

Theorem 4.5.11 *Let S be a (locally) bounded semi-martingale, $A \subset L^\infty(\mathbb{P})$ a strongly relevant coherent acceptance set, and M_0 the cone of marketed cash streams. If S satisfies (NGDVR), then*

(a) $C_0 := M_0 - A$ is Fatou closed in $L^0(\mathbb{P})$ and hence,

(b) $C = C_0 \cap L^\infty(\mathbb{P})$ is $\sigma(L^\infty(\mathbb{P}), L^1(\mathbb{P}))$-closed.

Here, a subset $D \subset L^0(\mathbb{P})$ is called *Fatou closed* if for every sequence $(f_n)_{n \geq 1}$ uniformly bounded below and such that $f_n \to f$ almost everywhere, we have $f \in D$. If D is a cone, then D is Fatou closed if for every sequence $(f_n)_{n \geq 1} \subset D$ with $f_n \geq 1$ and $f_n \to f$ almost surely, we have $f \in D$.

The proof of the above theorem is highly non-trivial in the case $A = L^\infty_+(\mathbb{P})$ ([16, Theorem 4.2]). The proof relies on a number of subtle convergence results, some of which rely on contradicting elaborate constructions. The author suspects that these results should still work for a general coherent acceptance set A, but careful checking is required. This goes beyond the scope of this work. Assuming this result is indeed true, one can employ Theorem 2.3.5 to construct an equivalent martingale measure in $L^1(\mathbb{P})$.

Another issue not addressed in this work is a procedure for constructing a coherent acceptance set A that represents ones beliefs and preferences as well as calculating the valuation bounds

$$\begin{aligned} \overline{\pi}(x) &= \sup\{\mathbb{E}^{\mathbb{Q}}[x] : \mathbb{Q} \in \mathcal{M}^e_A(S)\} \quad \text{and} \\ \underline{\pi}(x) &= \inf\{\mathbb{E}^{\mathbb{Q}}[x] : \mathbb{Q} \in \mathcal{M}^e_A(S)\}. \end{aligned}$$

See Staum [58] for more detail on this.

In closing, Staum mentions in [59] that the assumption of a market pricing function π obscures aspects of market modeling in his work. Theorem 4.5.9 avoids this problem by virtue of the fact that it is formulated in a more concrete setting; it relates the absence of near-arbitrage to an equivalent martingale measure on $L^p(\mathbb{P})$ (which Staum refers to as a strictly monotone consistent pricing kernel). Proving Theorem 4.5.11 would be a decent step forward in marrying the theories of coherent risk measures and no-arbitrage, pricing and hedging in a general setting.

Chapter A

Appendix

A.1 Functional Analysis

The reader is referred to [10, 52, 21, 3, 4, 56] for a comprehensive presentation of the material in this section.

A.1.1 Normed Spaces and Linear Operators

Definition A.1.1 Let X be a real vector space.

(a) A map $\|\cdot\| : X \to \mathbb{R}$ is called a *norm* if

  (i) $\|f\| > 0$ for all $f \in X$ and $\|f\| = 0$ if and only if $f = 0$,

  (ii) $\|\alpha f\| = |\alpha|\|f\|$ for all $f \in X$ and $\alpha \in \mathbb{R}$ and

  (iii) $\|f + g\| \leq \|f\| + \|g\|$ for all $f, g \in X$ (this is known as the triangle inequality).

(b) The pair $(X, \|\cdot\|)$ is called a *normed space.*

(c) If $(X, \|\cdot\|)$ is complete with respect to the norm, i.e. every norm Cauchy sequence has a limit in X, then $(X, \|\cdot\|)$ is called a *Banach space.*

(d) The set $\text{ball}(X) := \{x \in X : \|x\| \leq 1\}$ is called the *closed unit ball* in X.

Definition A.1.2 Let X and Y be vector spaces. We shall call a map $T : X \to Y$ a *linear operator* if we have $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for each $\alpha, \beta \in \mathbb{R}$, $x, y \in X$. Note that we sometimes denote $T(x)$ by Tx.

Definition A.1.3 Let X and Y denote normed spaces, and $T : X \to Y$ denote a linear operator.

(a) $T : X \to Y$ is called *bounded* if there exists a constant $C > 0$ such that $\|Tx\| \leq C\|x\|$ for all $x \in X$.

(b) $T : X \to Y$ is called *open* if $T(O)$ is open in Y for every open set $O \subset X$.

(c) $T : X \to Y$ is called an *isometry* if $\|Tx\| = \|x\|$ for all $x \in X$.

(d) $T : X \to Y$ is called an *isomorphism* if the exists a $K > 0$ such that $K^{-1}\|x\| \leq \|Tx\| \leq K\|x\|$ for all $x \in X$.

(e) $T : X \to Y$ is called a *metric surjection* if T is surjective and

$$\|y\| = \inf\{\|x\| : x \in X, \, Tx = y\}$$

for every $y \in Y$. Metric surjections are sometimes referred to as *quotient operators*.

(b) A linear operator $P : X \to X$ is called a *projection* if $P^2x = P(Px) = Px$ for all $x \in X$. P is called *contractive* if $\|Px\| \leq x$ for all $x \in X$.

It is easily shown that a linear operator is bounded if and only if it is continuous, therefore we will use these terms interchangeably.

Note that part (e) in the above definition is equivalent to $T : X \to Y$ mapping the open unit ball of X onto the open unit ball of Y. This implies that Y is isometrically isomorphic to the quotient space $X/\text{Ker}(T)$.

A.1.2 Dual Spaces

Definition A.1.4 Let X and Y be normed spaces.

(a) We define the normed space $\mathcal{L}(X, Y)$ by

$$\mathcal{L}(X, Y) := \{T \in L(X, Y) : \text{ T is bounded}\}$$

together with the *operator norm* $\|\cdot\|$ defined by

$$\|T\| = \sup\{\ \|Tx\| : \|x\| \leq 1\}$$

for all $T \in \mathcal{L}(X,Y)$. If $X = Y$ then we shall write $\mathcal{L}(X,X)$ as $\mathcal{L}(X)$.

(b) In the case where $Y = \mathbb{R}$, we shall write $\mathcal{L}(X,Y)$ as X^*. The elements of X^* are called *linear functionals* and X^* is called the *continuous dual* of X.

(c) We call $X^{**} = (X^*)^*$ the *continuous bidual* of X.

If X is a normed space and Y is a Banach space, then $\mathcal{L}(X,Y)$ is also a Banach space with respect to the operator norm. In particular, we have that X^* is a Banach space.

We note that a normed space X can be canonically embedded as a subspace of its bidual under the isometry $i_X : X \to X^{**}$ defined by $i_X(x)(x^*) = x^*(x)$ for all $x \in X$ and $x^* \in X^*$. We see this as an abstract containment where the normed structure is preserved and we denote this as $X \hookrightarrow X^{**}$. A normed space X is called *reflexive* if $X = X^{**}$. The elements of $X \hookrightarrow X^{**}$ are sometimes referred to as *induced linear functionals* on X^*. Since X^{**} is always a Banach space, the closure of X in X^{**} is complete, which shows every normed space has a completion.

A.1.3 Fundamental Results

Theorem A.1.5 *(a)* (Open Mapping Theorem) *A bounded linear surjection acting between Banach spaces is open.*

(b) (Closed Graph Theorem) *A linear operator between acting Banach spaces is bounded if and only if its graph is closed.*

(c) (Principle of Uniform Boundedness) *Let X and Y be Banach spaces and $S \subset \mathcal{L}(X,Y)$. If* $\sup\{\|Tx\| : T \in S\} < \infty$ *for all $x \in X$, then* $\sup\{\|T\| : T \in S\} < \infty$.

(d) (Hahn-Banach) *If f is a bounded linear functional on a subspace of a normed space, then f extends to the whole space with preservation of norm.*

Corollary A.1.6 (Hahn-Banach)

(a) If X is a normed linear space and $x \in X$, then there exists $x^* \in X^*$ of norm 1 such that $x^*(x) = \|x\|$.

(b) If X is a normed space, then for all $x \in X$ we have $\|x\| = \sup\{|x^*(x)| : \|x^*\| \leq 1, x^* \in X^*\}$.

(c) If X is a normed space and $x^*(x) = 0$ for all $x^* \in \mathrm{ball}(X^*)$, then $x = 0$; i.e. $\mathrm{ball}(X^*)$ separates the points in X.

Corollary A.1.7 (Hyperplane Separation Theorem) *Let X be a topological vector space and let A, B be convex non-empty subsets of X with $A \cap B = \emptyset$.*

(a) If A is open, there exists $f \in X^*$ and $t \in \mathbb{R}$ such that $f(a) < t \leq f(b)$ for all $a \in A$ and $b \in B$.

(b) If X is locally convex, A compact and B closed, then there exists $f \in X^*$ and $s, t \in \mathbb{R}$ such that $f(a) < t < s < f(b)$ for all $a \in A$ and $b \in B$.

A.1.4 Adjoint Operators

Definition A.1.8 Let X and Y be normed spaces.

(a) Let $T \in \mathcal{L}(X, Y)$. We define the *adjoint* $T^* : Y^* \to X^*$ by $(T^*y^*)(x) = y^*(Tx)$ for all $y^* \in Y^*$ and $x \in X$.

(b) For $T \in \mathcal{L}(X, Y)$, we call $T^{**} : X^{**} \to Y^{**}$ the *second adjoint* of T.

We collect some useful results involving adjoints.

Proposition A.1.9 *Let X and Y be normed spaces, then the following statements hold:*

(a) The mapping $T \mapsto T^*$ is an isometry of $\mathcal{L}(X, Y)$ into $\mathcal{L}(Y^*, X^*)$.

(b) The second adjoint $T^{**} : X^{**} \to Y^{**}$ is a unique continuous extension of $T : X \to Y$.

(c) $T : X \to Y$ is an isometry if and only if $T^* : Y^* \to X^*$ is a metric surjection.

(d) $T : X \to Y$ *is a metric surjection if and only if* $T^* : Y^* \to X^*$ *is an isometry.*

(e) If X *and* Y *are Banach spaces, then a bounded linear operator* $T : X \to Y$ *has closed range if and only if* $T^* : Y^* \to X^*$ *has closed range.*

A.1.5 Duality Pairs, Polar Sets, Convex Sets and Cones

Definition A.1.10 (a) Let X and Y be vector spaces and $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{R}$ be a bilinear (linear in each variable) map. Then $(X, Y, \langle \cdot, \cdot \rangle)$ is called a *dual pair* provided

- $\forall\, x \in X\ \exists\, y \in Y$ such that $\langle x, y \rangle \neq 0$, and
- $\forall\, y \in Y\ \exists\, x \in X$ such that $\langle x, y \rangle \neq 0$.

(a) Given a dual pair $(X, Y, \langle \cdot, \cdot \rangle)$,

(i) the *polar* of a set $A \subset X$ is defined by

$$A^\circ = \{y \in Y : \langle x, y \rangle \leq 1\ \forall\, x \in A\}.$$

(ii) Similarly, the *polar* of a set $B \subset Y$ is defined by

$$B^\circ = \{x \in X : \langle x, y \rangle \leq 1\ \forall\, y \in B\}.$$

Note that for any dual pair $(X, Y, \langle \cdot, \cdot \rangle)$, we have that $(X, Y, -\langle \cdot, \cdot \rangle)$ is also a dual pair. We may therefore, at our discretion, replace $\langle x, y \rangle \leq 1$ with $\langle x, y \rangle \geq -1$ in the above definition.

(c) For a vector space X, a set $C \subset X$ is called a *cone* if the following properties hold:

- $C + C \subset C$,
- $\lambda C \subset C$ for all $\lambda \geq 0$,

A cone C is called *pointed* or *proper* if $C \cap (-C) = \{0\}$.

(d) For a vector space X, a set $E \subset X$ is called *convex* if for every $x, y \in E$ we have that $tx + (1 - t)y \in E$ for all $t \in [0, 1]$.

(e) Let X be a topological vector space. For a set $D \subset X$, the closure of the smallest convex set containing D is denoted $\overline{\text{co}}\, D$. This is sometimes referred to as the *closed convex hull* of D.

Let $(X, Y, \langle \cdot, \cdot \rangle)$ be a dual pair. If $C \subset X$ is a cone, we have

$$C^\circ = \{y \in Y : \langle x, y\rangle \leq (\geq)\ 0\ \forall\ x \in C\}.$$

Similarly, if $D \subset Y$ is a cone, we have

$$D^\circ = \{x \in X : \langle x, y\rangle \leq (\geq)\ 0\ \forall\ y \in D\}.$$

Moreover, in the case that $A \subset X$ is a vector space, we have

$$A^\circ = A^\perp := \{y \in Y : \langle x, y\rangle = 0\ \forall\ x \in A\}.$$

Similarly, if $B \subset Y$ is a vector space, we have

$$B^\circ = B_\perp := \{x \in X : \langle x, y\rangle = 0\ \forall\ y \in B\}.$$

The vector space Y induces a locally convex topology $\sigma(X, Y)$ on the vector space X via point-wise convergence. That is, $x_\alpha \to x$ in the $\sigma(X, Y)$-topology if and only if $\langle x_\alpha, y\rangle \to \langle x, y\rangle$ for all $y \in Y$. Similarly, X induces a locally convex topology $\sigma(Y, X)$ on Y. If $f : X \to \mathbb{R}$ is linear, it can be shown that f is $\sigma(X, Y)$-continuous if and only if there exists $y_f \in Y$ such that $f(x) = \langle x, y_f\rangle$ for all $x \in X$. Thus, we have the identity $(X, \sigma(X, Y))^* = Y = \{\langle \cdot, y\rangle : X \to \mathbb{R} : y \in Y\}$.

In particular, for a normed space X, we have the dual pair $(X, X^*, \langle \cdot, \cdot \rangle)$, where $\langle x, x^*\rangle := x^*(x)$ for all $x \in X$ and $x^* \in X^*$. The topology $\sigma(X, X^*)$ on X called the *weak topology* on X and the topology $\sigma(X^*, X)$ on X^* is called the *weak* topology* on X^*.

Convex sets in Banach spaces have a useful property: they are norm closed if and only if they are weakly closed. Note that every cone and vector subspace is convex.

Proposition A.1.11 *Let $(X, Y, \langle \cdot, \cdot \rangle)$ be a dual pair, $\{A_i\}_{i \in I}$ be a family of sets in X and $A \subset X$. Then the following properties hold.*

- *(a) A° is convex and closed in the $\sigma(Y, X)$-topology, moreover $A^\circ = (\overline{\mathrm{co}}\, A)^\circ$.*
- *(b) $0 \in A^\circ$ and $A \subset A^{\circ\circ}$. Moreover, $A_1 \subset A_2 \Rightarrow A_2^\circ \subset A_1^\circ$.*
- *(c) $(\lambda A)^\circ = (1/\lambda) A^\circ$ for all $\lambda > 0$.*
- *(d) $(\bigcup_{i \in I} A_i)^\circ = \bigcap_{i \in I} A_i^\circ$.*

(e) $(\bigcap_{i\in I} A_i)^\circ \supset \overline{\text{co}} \bigcup_{i\in I} A_i^\circ$ *where the closure is taken in the* $\sigma(Y,X)$*-topology.*

(f) *For the dual pair* $(X, X^*, \langle\cdot,\cdot\rangle)$*, we have* $\text{ball}(X)^\circ = \text{ball}(X^*)$ *and* $\text{ball}(X^*)^\circ = \text{ball}(X)$*.*

A.1.6 The Bi-Polar, Krein-Sḿulian and Banach-Alaoglu Theorems

We list some fundamental results pertaining to weak topologies.

Theorem A.1.12 *(a)* (Bi-Polar Theorem) *Let* $(X, Y, \langle\cdot,\cdot\rangle)$ *be a dual pair. For a set* $A \subset X$ *we have* $A^{\circ\circ} = \overline{\text{co}}\,(A \cup \{0\})$*, where the closure is taken in the* $\sigma(X,Y)$*-topology.*

(b) (Krein-Sḿulian Theorem) *Let* X *be a Banach space and* $C \subset X^*$ *be convex. Then* C *is weak* closed if and only if* $C \cap (\lambda\,\text{ball}(X^*))$ *is weak* closed for all* $\lambda > 0$*.*

(c) (Banach-Alaoglu) *Let* X *be a Banach space, then the closed unit ball of* X^* *is* $\sigma(X^*, X)$ *compact.*

In particular, the Bi-Polar Theorem states that a convex set $C \subset X$ that contains zero is $\sigma(X,Y)$ closed if and only if $C = C^{\circ\circ}$.

If $C \subset X^*$ is a cone, where X is now a Banach space, the Krein-Sḿulian Theorem implies that C is weak* closed if and only if $C \cap \text{ball}(X^*)$ is weak* closed.

Lastly, the Banach-Alaoglu Theorem implies that every sequence in the closed unit ball of X^* has a weak* convergent subsequence.

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