Figure 5.11 Piston Spring Rate versus Temperature [RB85]

With the isothermal approximations for $\alpha_p$ and $\alpha_r$ given by (5.53) and (5.54), the approximate stability criterion (5.34) becomes
\[
\frac{m_p}{m_d} \frac{\omega_p}{\omega_d} \left( \frac{\omega_p}{\omega_0 + \omega_p} \right) \left( \frac{\omega_p}{\omega_0 + \omega_p} \right) \left( 1 - \frac{T_k - A_R}{\tau_h} \right) \left( \frac{1}{A - A_R^2} \right) + \frac{m_p}{m_c} \frac{K_d}{K_p} \left( \frac{1}{A - A_R} \right)
\]

\[
AA_R \left[ \frac{(\Delta \omega/\omega_d)^2}{(1 + \sigma_d/\sigma_p)^2} \right]^{1/2} \left[ \frac{(\Delta \omega/\omega_p)^2}{(1 + \sigma_p/\sigma_d)^2} \right]^{1/2} > 1
\]

(5.57)

In practice, \(\Delta \omega/\omega_p\) and \(A_R/A\) are usually small, and therefore the stability criterion is approximately

\[
4\pi^2 \frac{m_p}{m_d} \frac{Q_dQ_p}{Q_p} \left( 1 - \frac{T_k - A_R}{\tau_h} \right) \left( \frac{A_R}{A} \right) + \frac{m_p}{m_c} \frac{K_d}{K_p} \left( \frac{1}{1 - A_R/A} \right)
\]

\[
\left[ \frac{(4\pi Q_d \Delta \omega/\omega_d)^2}{(1 + \sigma_d/\sigma_p)^2} \right]^{1/2} + \left[ \frac{(4\pi Q_p \Delta \omega/\omega_p)^2}{(1 + \sigma_p/\sigma_d)^2} \right]^{1/2} > 1
\]

(5.58)

It follows from (5.58) that the lowest \(T_h\) for onset of oscillation occurs when \(\Delta \omega = 0\), that is, when \(\omega_d = \omega_p\). For this case the start-up criterion becomes

\[
4\pi^2 \frac{m_p}{m_d} \frac{Q_dQ_p}{Q_p} \left( 1 - \frac{T_k - A_R}{\tau_h} \right) \left( \frac{A_R}{A} \right) + \frac{m_p}{m_c} \frac{K_d}{K_p} \left( \frac{1}{1 - A_R/A} \right) > 1
\]

(5.59)

At start-up sliding friction is generally dominant over viscous friction. Thus the viscous displacer-piston coupling that has been ignored in the
start-up criterion is not normally significant. \(Q_d\) and \(Q_p\) are therefore determined by sliding friction at start-up.

If sliding frictional forces \(F_{slD}\) and \(F_{slP}\) exist against displacer and piston motions respectively and are given by:

\[
F_{slD} = -|F_{slD}| \frac{\dot{x}_d}{|\dot{x}_d|}
\]

\[
F_{slP} = -|F_{slP}| \frac{\dot{x}_p}{|\dot{x}_p|}
\]

then at start-up, the quantities \(Q_d\) and \(Q_p\) are amplitude dependent according to:

\[
Q_d = m_d |x_d| w^2 / (8 |F_{slD}|)
\]

\[
Q_p = m_p |x_p| w^2 / (8 |F_{slP}|)
\]

and the start-up criterion for equal resonances of the piston and displacer becomes:

\[
\frac{n^2}{16} \frac{m_p (m_d m_d w^4)}{m_d} \frac{|x_d| |x_p|}{|F_{slD}| |F_{slP}|} \left(1 - \frac{r_k - A_R}{r_h} \frac{A}{A}ight) \left(\frac{A_R}{A} + \frac{m_p}{m_c} \frac{K_p}{1 - A_R/A} \frac{1}{1 - A_R/A} \right) > 1
\]
(5.64) shows that the effect of sliding friction is to impose a minimum product of amplitude, \(|X_d|X_P|\), as a condition for starting. Since (5.64) does not account for flow losses or external loading which become dominant over friction at larger amplitudes, the engine may well start if (5.64) is satisfied, but then fail to build up to design amplitudes.

An important conclusion that may be drawn from (5.59) and (5.64) is that a free-piston Stirling engine will self-start provided that the sliding friction is vanishingly small and that the small amplitude resonances of the displacer and piston are such that the stability criterion is satisfied. Practical considerations will often result in the resonances being detuned at rest. For example, the effect of gas spring centering ports, leakage between the gas spring and working gas, and drift of the moving parts under gravity. In these cases, and in cases where there is a small degree of sliding friction, the machine will require some input of energy to initiate oscillation.

Once the machine has started and reached cyclic steady state, the power follows from (5.41) and (5.54):

\[
P = \frac{\omega x p}{2} \sqrt{1 - (a/S)^2} \frac{\mathcal{A}}{\mathcal{F}} (1 - \frac{A_k}{A})(1 - \frac{\tau_k}{\tau_h} - \frac{A_k}{A}) |X_d|X_P|\sin\phi
\]  

(5.65)

In most cases \((a/S) \ll 1\), thus to good approximation, (5.65) reduces to

\[
P \approx \frac{\omega x p}{2} \sqrt{\frac{\mathcal{A}}{\mathcal{F}}} \left(1 - \frac{A_k}{A}\right) \left(1 - \frac{\tau_k}{\tau_h} - \frac{A_k}{A}\right) |X_d|X_P|\sin\phi
\]  

(5.66)
which translates to the isothermal result for work given by Equation (3.51) in Chapter 3.

Figure 5.12 shows the work as predicted by the linearized isothermal equation (Equation (5.66)) compared to experimental and computer simulation results. Also shown is the Schmidt solution (labeled non-linear). As the swept volumes become a larger fraction of the unswept volumes, the errors due to non-linearities become more significant.

\[
\frac{w_{SWC} + w_{SWE}}{w_{K} + w_{F} + w_{H} + w_{T}} = \frac{V_{SWC} + V_{SWE} T_{K}}{V_{K} + V_{F} T_{K} + V_{H} T_{K}}
\]

**Figure 5.12** Cyclic Work Predicted by the Linear Analysis
5.5 Calculation of Damping Coefficients

In §3.8 it was shown that the gas flow in the cooler is approximately 180° out of phase with the rate of change of the compression space volume. Similarly, it was shown that the gas flow in the heater is approximately in phase with the rate of change of the expansion space volume. Furthermore, it was shown that the mass flow is not strongly affected by the change in working gas pressures. Using this information, the gas flow velocities in the cooler and heater are given by:

\[
\vec{u}_k = -\frac{\dot{V}_c}{A_k} = -\frac{(A - A_R)}{A_k}(\dot{z}_d - \dot{z}_p)
\]  
\[
(5.67)
\]

\[
\vec{u}_h = \frac{\dot{V}_e}{A_h} = \frac{A}{A_h}\dot{z}_d
\]

(5.68)

For the regenerator, the velocity is evaluated from the mean mass flow through the regenerator, i.e.,

\[
\vec{u}_r = \frac{1}{2}\left(\frac{\dot{V}_e - \dot{V}_c}{\dot{V}_h}\right)\frac{\dot{F}_h - \dot{F}_k}{A_r \ln(\frac{\dot{F}_h}{\dot{F}_k})}
\]

\[= -\frac{1}{2}\left(\frac{A\dot{z}_d + (A - A_R)(\dot{z}_d - \dot{z}_p)}{\dot{V}_h}\right)\frac{\dot{F}_h - \dot{F}_k}{A_r \ln(\frac{\dot{F}_h}{\dot{F}_k})}
\]  
\[
(5.69)
\]

In heat exchangers where the flow is predominantly turbulent, the pressure drop is given by

\[
\Delta p = 2\rho(\frac{\dot{V}}{A_h + \kappa})\vec{u}|\vec{u}|
\]

(5.70)
where $\mathcal{K}$ is the head loss coefficient.

In turbulent flow it is reasonable to assume that $\mathcal{R}$ is approximately constant. As before, $\rho$ is assumed to vary only with local temperature and furthermore, as in Chapter 4, all the work done against the pressure drop is assumed to be dissipated.

Since (5.70) is non-linear, it is necessary to find an equivalent linear expression. The condition for equivalence is that the energy dissipated in the linear system must be equal to the energy dissipated in the non-linear system [An70]. Thus

$$2\rho (\mathcal{R}/\sigma + \mathcal{K}) \int_0^{\pi/2} (\ddot{U})^2 \vert \ddot{U} \vert \, dt = \int_0^{\pi/2} \ddot{U}^2 \, dt$$

which, assuming sinusoidal velocity variations, gives

$$c = (16/3)(\rho/\pi)(\mathcal{R}/\sigma + \mathcal{K}) \vert \ddot{U} \vert$$  \hspace{1cm} (5.71)

where $\ddot{U}$ (bold italic) denotes here the velocity magnitude (or amplitude).

Thus the linearized $\Delta p$ becomes:

$$\Delta p = c \ddot{U} = (16/3)(\rho/\pi)(\mathcal{R}/\sigma + \mathcal{K}) \vert \ddot{U} \vert \ddot{U}$$  \hspace{1cm} (5.72)

which from (5.67) through (5.69) is seen to be a linear function of piston and displacer velocities.

For the regenerator, the pressure drop is given by
\[ \Delta p_r = 2(\mu C_L L / \alpha_r^2) \bar{U}_r \]  
(Equation (4.39))  
(5.73)

where \( C_Lr \) is the flow coefficient and is given by (see §4.5)

\[ C_Lr = 33.3/\psi^2 + (0.458/\psi)\Re_p \]  
(5.74)

Generally, in a practical engine, the Reynolds number in the regenerator is low (around 10) and it is therefore possible to neglect the second term on the right of (5.74), thus,

\[ C_Lr \approx 33.3/\psi^2 \]  
(5.75)

which is constant with velocity. The regenerator pressure drop is thus approximately linear with velocity.

If the pressure calculated from the isothermal model is taken to represent the compression space pressure, then the pressure drop through the heat exchanger loop results in a force on the displacer given by

\[ \sum \Delta p = (A \Delta p)_k + (A \Delta p)_h + (A \Delta p)_r 
= -D_d \dot{x}_d - D_{dp} \dot{x}_p \]  
(5.76)

From (5.76) and the linearised pressure drops (ignoring, for the moment, gas spring hysteresis and other incidental dissipative losses), \( D_d \) and \( D_{dp} \) may be evaluated, giving:
\[ D_d = c_k (\lambda - \lambda_R) + c_h \lambda \]
\[ + \left( \mu C_p / \rho_h^2 \pi (A / \bar{T}_h + (A - A_R) / \bar{T}_k) (\bar{T}_h / \bar{T}_k - 1) / \ln(\bar{T}_h / \bar{T}_k) \right) \]
\[ (5.77) \]

\[ D_{dp} = - (\lambda - \lambda_R) \left[ c_k + \left( \mu C_p / \rho_h^2 \pi (\bar{T}_h / \bar{T}_k - 1) / \ln(\bar{T}_h / \bar{T}_k) \right) \right] \]
\[ (5.78) \]

where \( c_k \) and \( c_h \) are given by (5.71). The velocity amplitudes in \( c_k \) and \( c_h \) are obtained from (5.67) and (5.68) and are given by

\[ |\vec{U}_k| = \omega (X_d^2 - 2X_dX_p \cos \phi + X_p^2) / A_k \]
\[ (5.79) \]

\[ |\vec{U}_h| = \omega X_d^2 / A_h \]
\[ (5.80) \]

From (5.78) the displacer/piston viscous coupling \( D_{dp} \) is seen to be negative. This is due to the pressure drop force on the displacer increasing for positive piston displacement.

In turbulent flow heat exchangers, therefore, the damping is dependent on \( X_d, X_p \) and \( \phi \). Since damping increases with increasing displacer and piston amplitude, \( D_d \) decreases which, as shown in §5.3, Figure 5.6, tends to stabilise oscillations.

Furthermore, since the velocity amplitude in the cooler (Equation (5.79)) is
reduced for lower values of \( \phi \), the damping, and consequently the
dissipation is also reduced for lower values of \( \phi \). Ideal power is
maximised for \( \phi = 90^\circ \) (Equation (5.66)), thus, since dissipation results in
a direct reduction of the ideal power, it can be seen that optimum power
and/or efficiency will occur at \( \phi < 90^\circ \).

If the gas flow through the heat exchanger loop is entirely laminar, then
the gas flow friction is given by

\[
\tau = \frac{C_L}{Re} \quad (S4.4, \text{Equation (4.37)})
\]  \hspace{1cm} (5.81)

and the head loss is given by

\[
\Delta p_H = 2\mu C_{dH} \bar{U}.
\]  \hspace{1cm} (5.82)

Thus from (5.70)

\[
\Delta p = 2\mu (C_L/L/\sigma_h^2 + C_{dH}) \bar{U}
\]  \hspace{1cm} (5.83)

which gives the laminar flow values for \( c \) directly.

\[
c = 2\mu (C_L/L/\sigma_h^2 + C_{dH})
\]  \hspace{1cm} (5.84)

Thus the laminar flow damping coefficients do not vary with displacer and
piston motions and therefore \( \sigma_d \) is constant for a laminar flow engine.
This implies that laminar flow damping is not stabilizing as in the case of turbulent flow damping. The distinction between laminar and turbulent flow heat exchangers is therefore important to properly understand the dynamic behaviour of an engine. A laminar flow engine has no inherent mechanism for maintaining operation at the equilibrium point. It is therefore essential that they be controlled by a device that controls $Q_d$ or $Q_p$ or both. This does not suggest that a turbulent flow engine will not require a control device since in many instances the change of $Q$ is not sufficient for stability or, in other cases, more precise control may be required on the piston stroke. Discussion of engine control devices are beyond the scope of this thesis.

Note that gas spring hysteresis has not been included in the above damping equations. Typically, this should be small in a well designed engine, but should be included if the power dissipated in the gas spring (or bounce space) is near or greater than 10% of the total damping. Gas spring hysteresis is dealt with in detail in Chapter 6 and is also included in the following worked example.

5.6 An Application Example

Appendix A.3 gives some history and lists complete data on the Sunpower SPIKE engine. This is a recent free-piston/linear alternator combination which produces 1kW(e) at approximately 60Hz.

The SPIKE engine employs a gas spring to resonate the displacer at the appropriate frequency. It is therefore necessary to obtain the spring rate
for gas springs. In §6.3 it is shown that the pressure variations in a gas spring are close to those given by an adiabatic process. From Equation (6.8):

$$\frac{dp}{dV} \approx -\gamma \frac{p}{V}$$  \hspace{1cm} (5.85)

The linearized spring rate is therefore given by:

$$K = -\frac{d}{dx}\left(\frac{dp}{dx}\right) = \gamma \frac{p}{V} \frac{A^2}{V}$$  \hspace{1cm} (5.86)

Figure 5.13 shows the linear pressure and the non-linear pressure versus displacement. It is clear that large discrepancies can occur if linearity is assumed for highly non-linear springs.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{fig513.png}
\caption{Pressure versus displacement of gas springs}
\end{figure}
An alternative arrangement is to use two gas springs in a back to back arrangement as shown in Figure 5.14. In this configuration there is some balance between the non-linearities of the two gas springs.

![Diagram of Double Acting Gas Spring](image)

**Figure 5.14** Double Acting Gas Spring

Assuming sinusoidal displacement of the puck in Figure 5.15, the pressure variations in the two cavities would be

\[
p_1 = \langle p \rangle \left[ 1 + \left( AX / \langle V \rangle \right) \sin \theta \right]^{-\gamma}
\]

\[
p_2 = \langle p \rangle \left[ 1 - \left( AX / \langle V \rangle \right) \sin \theta \right]^{-\gamma}
\]

(5.87)

where \( X \) is the amplitude of the displacement.

If the ratio \( A / \langle V \rangle \) is similar for both cavities, then the total force on the puck is given by the sum of the pressure forces on both sides, as follows:
\[ F = A < p > \left\{ \left[ 1 - (A X / V) \sin \theta \right]^{-\gamma} - \left[ 1 + (A X / V) \sin \theta \right]^{-\gamma} \right\} \]

Since \((A X / V) < 1\), expanding by binomial series yields

\[ F = A < p > \left[ 2 \gamma (A X / V) \sin \theta + (1/3) \gamma (\gamma + 1)(\gamma + 2)(A X / V)^3 \sin^3 \theta \right. \]
\[ + \ldots \sin^6 \theta + \ldots \sin^7 \theta + \ldots \text{HOT} \] \hspace{1cm} (5.88)

From (5.88) it can be seen that there is a significant improvement in the linearity of the gas spring by utilizing the double acting arrangement. The mean pressure force acting on the puck of a double acting spring is shown in Figure 5.13 for comparison with the single acting spring.

Early on in the development of SPIKE engines, single acting gas springs were used, later on, a switch was made to double acting springs. Currently, all SPIKE engines use a double acting gas spring. The spring rate for the double acting gas spring is given by

\[ K \approx \gamma < p > \left[ (A^2 / V)_{\text{in}} + (A^2 / V)_{\text{out}} \right] \] \hspace{1cm} (5.89)

where "in" refers to the spring towards the hot end.

The bounce space is a simple single acting gas spring of low pressure ratio which contributes to the piston spring \(K_p\). A further advantage of the double acting gas spring is that the hysteresis
losses are generally much lower. The hysteresis loss in a gas spring is proportional to \((\Delta \nu/\nu)^2\) (see §6.3) which is typically much smaller for double acting gas springs. Gas spring losses are important to the dynamics of the engine since they constitute significant additional damping on the moving components. From §6.3 (Equation 6.30), the gas spring hysteresis loss is given by

\[
\langle \dot{\mathbf{w}} \rangle_{gs} = \left(\frac{k}{4}\right)\sqrt{\frac{\omega}{2\alpha}}(\gamma - 1)T_w A_w (\Delta \nu/\nu)^2
\]  

(5.90)

The energy dissipated in a linear damper is given by

\[
\dot{\mathbf{w}} = \frac{1}{2}D_s (\omega \chi)^2
\]  

(5.91)

where \(D_s\) is the gas spring damping coefficient and \(\chi\) is the amplitude of the damped motions.

Equating (5.90) and (5.91) and noting that \(\Delta \nu = \lambda \chi\)

\[
D_s = \left(\frac{k}{2}\right)\sqrt{\frac{\omega}{2\alpha}}(\gamma - 1)T_w A_w [\lambda/(\omega \chi)]^2
\]  

(5.92)

For a double acting spring, the total damping would be the sum of the damping for the "in" and "out" springs.

Thus for engines using gas springs, the damping coefficients \(D_a\) and \(D_p\) are increased by the added gas spring damping.
There is a similar hysteresis effect in the working gas too. However, this effect is believed to be small in the heat exchangers since the velocity and temperature fields swamp the pressure fluctuation effects. For the working spaces (i.e., the swept volume spaces) it is a different matter as velocity gradients are assumed to be much smaller here. Generally, hysteresis effects are taken to act only in the working spaces and other areas where the velocity gradients are small, e.g., plenum volumes (see paper by Lee et al. [L580]). It is assumed here that any area in the machine that has negligible pressure gradients compared to the pressure gradients in the heat exchangers is an area where velocity gradients are also negligible.

From §6.3, the lost work due to hysteresis effects is given by

$$\langle \dot{W} \rangle_g = \sqrt{\left( \frac{\omega}{32} \right)^2 A_w |p|^2 / (\gamma \langle p \rangle)}$$  \hspace{1cm} (5.93)

The effect of working gas hysteresis is to reduce the available work. The simplest way of including this effect on the dynamics of the engine is to modify the working gas pressure. Since the hysteresis effect is dissipative, the phase angle between the pressure and piston motions is reduced as is shown in Figure 5.15. This is the same effect that is noticed in the $pV$ diagram of a real gas spring with significant hysteresis loss (this point is also covered in §6). Since power is proportional to the pressure phase angle $\beta$, the new phase angle is obtained as follows
\[
\frac{\sin \beta'_a}{\sin \beta'} = \frac{P_a}{P}
\]  
(5.94)

where \( P \) is the \( pV \) power and the subscript refers to the actual value, i.e., the quantity with hysteresis accounted for.

![Diagram](image)

**Figure 5.15** Effect of Working Gas Hysteresis on Pressure Phase Angle.

From (5.94), \( \beta'_a \) may be found

\[
\beta'_a = \sin^{-1}\left[\left(\rho - \sum <\dot{w}>_g\right)\sin \beta'/P\right]
\]  
(5.95)

where the summation occurs over all working spaces and plenum volumes.
Leakage losses will also have a dissipating effect on the mechanical dynamics. In a properly operating SPIKE engine, the leakage effects are small and are therefore neglected here. However, in the design process it is important to account for internal leaks in a proper manner. This point will be detailed in S6 where the model is refined to include other incidental and major parasitic losses.

It is now possible to calculate the dynamic variables of a SPIKE type machine. The process is outlined as follows:

i) Assume a frequency, displacer/piston phase angle and stroke ratio

ii) Calculate resonances for displacer and piston

iii) Calculate the respective $Q$'s

iv) Calculate frequency, stroke ratio and piston/displacer phase angle

v) Iterate on stroke ratio by returning to (ii) until convergence is obtained

vi) Calculate power

If a full calculation with temperature drops and regenerator performance is required, then it will be necessary to assume initial gas temperature guesses in order to calculate heat transfer in each iteration. These initial guesses may be taken to be the respective wall temperatures. The temperature drops will vary for each iteration until convergence is obtained.

The following solutions were obtained using the computer programme in Appendix E with the working spaces selected as isothermal.
Figure 5.16 Linear Analysis Results of SPIKE Engine at 11bar absolute

(In the interests of clarity the pressure trace is displaced from its mean position)

Figure 5.16 was generated assuming perfect regeneration and no leakage or conduction losses. Temperature differentials in the heat exchangers have been included together with other dissipating influences such as gas hysteresis and viscous dissipation as these effects change the dynamics of the engine. Imperfect regeneration and conduction losses have second order effects on the dynamics but will, of course, change the system efficiency markedly. The general nature of parasitic losses is dealt with in §6.

For comparison, an experimental point of identical operating parameters is indicated in Figure 5.17. The photograph has been touched up with black
pen since the original picture did not reproduce well. Also, owing to a leaking gas spring, the displacer was running out slightly (towards the cold end). Table 5.1 compares the prediction of the linear analysis with the experimental results. The only additional empirical factor used in the linear analysis is a multiplier of 1.5 on the gas spring hysteresis. This is necessary on the third order simulation as well since the spring in this case is highly non-linear resulting in higher than predicted losses. Non-linearity of the gas spring also introduces higher order harmonics into the system in the form of non-sinusoidal displacer motion. The working gas pressure variation is therefore altered and in some circumstances may actually increase the cyclic work. Piston motions generally appear to retain their harmonic character.

Scale: same as Fig. 5.16

Figure 5.17 Oscilloscope Photograph of Displacer/Piston and Pressure/Piston Phase Plots of SPIKE Engine.

Note that the product of the piston area and pressure/piston phase area is the work per cycle, i.e., the $pV$ work.
Table 5.1 Comparison between Linear Analysis and Experiment

<table>
<thead>
<tr>
<th></th>
<th>Analytical</th>
<th>Experimental</th>
</tr>
</thead>
<tbody>
<tr>
<td>Power [W]</td>
<td>1.68E+03</td>
<td>1.57E+03</td>
</tr>
<tr>
<td>Ind. efficiency</td>
<td>0.39</td>
<td>0.26</td>
</tr>
<tr>
<td>Stroke ratio</td>
<td>0.96</td>
<td>0.72</td>
</tr>
<tr>
<td>Frequency [Hz]</td>
<td>59.3</td>
<td>60.0</td>
</tr>
<tr>
<td>Displacer phase [*]</td>
<td>54.2</td>
<td>65.7</td>
</tr>
<tr>
<td>Pressure phase [*]</td>
<td>-23.0</td>
<td>-20.6</td>
</tr>
<tr>
<td>Pressure ratio</td>
<td>0.12</td>
<td>0.13</td>
</tr>
</tbody>
</table>

Experimental errors on stroke, pressure amplitude and frequency are less than 1%. Phase angle errors vary with the size of the angle but are generally less than 5%.

Figures 5.18 and 5.19 are respectively the analytical and experimental results for the double acting spring system at 12bar absolute. From the nature of the phase plots in Figure 5.19 it is clear that the non-linearities have been greatly reduced. The early double acting spring designs suffered from serious leakage between the spring cavities. Leakage losses were therefore high and strongly affected the dynamic behaviour of the machine. It is therefore necessary to introduce an additional dissipative loss to the gas spring in order to obtain meaningful correlation. Accurately calculating the added loss would be difficult as it would involve the dynamics of seal rings and the specific machining and mechanical details of the engine. A simple approach is to assume that the amount of additional dissipative load is simply that which retains the same stroke ratio as the experimental runs (to within 5%). This approach is more relevant at this time and it has been used to generate Figure 5.18.
Correlation based on the equivalent stroke ratio assumption is indicated in Table 5.2.

Figure 5.18 Linear Analysis Results of SPIKE Engine at 12bar absolute
(Double Acting Spring)

Figure 5.19 Experimental Time Traces and Phase Plots of Double Acting Gas Spring SPIKE Engine
### Table 5.2 Comparison between Linear Analysis and Experiment (Double Acting Gas Spring)

<table>
<thead>
<tr>
<th></th>
<th>Analytical</th>
<th>Experimental</th>
</tr>
</thead>
<tbody>
<tr>
<td>Power [W]</td>
<td>1.45E+03</td>
<td>1.25E+03</td>
</tr>
<tr>
<td>Ind. efficiency</td>
<td>0.79</td>
<td>0.18-0.2</td>
</tr>
<tr>
<td>Stroke ratio</td>
<td>0.94</td>
<td>0.97</td>
</tr>
<tr>
<td>Frequency [Hz]</td>
<td>61.9</td>
<td>61.9</td>
</tr>
<tr>
<td>Displacer phase [°]</td>
<td>59.7</td>
<td>57.5</td>
</tr>
<tr>
<td>Pressure phase [°]</td>
<td>-17.0</td>
<td>-12.4</td>
</tr>
<tr>
<td>Pressure ratio</td>
<td>0.12</td>
<td>0.11</td>
</tr>
</tbody>
</table>

In the example of the SPIKE engine it can be seen that the linear analysis is fairly accurate in predicting dynamics and power even in the limited form in which it was applied. Efficiency was not well represented. The lack of correlation on efficiency will be shown in §6 to be mainly due to the exclusion of some parasitic losses and the assumption of perfect regeneration.

Another important attribute of the linear analysis is its speed. A simple sensitivity analysis is shown for the SPIKE engine in Figure 5.20. This sensitivity plot was generated interactively on a personal computer (Apple Macintosh 512K) in less than 5 minutes. Third order simulations running on typical large mainframes take much longer and cost a lot more to generate the same information.
Figure 5.20 Sensitivity Analysis Output for SPIKE Engine Generated by Programme in Appendix E
(Note that dependent variables are normalized)

5.7 Alternate Configurations
The linear analysis developed here has been applied to the most common free-piston configuration. It is obvious that the linear analysis may be applied to other engine arrangements with equal effect. Higher order engine arrangements have also been successfully analysed, eg, the Harwell thermo-mechanical generator [CF68, BW79 I, RB85] and Redlich's elegant analysis of the free-cylinder water pump [RB85]. Multiple degree of freedom duplex-Stirling heat pumps are also amenable to this approach.

An excellent monograph has recently been published by Walker and Senft on the general topic of free-piston engines [WS85]. In this book many
configurations of these devices are investigated and their relative merits explained. The interested reader is advised to pursue the subject of alternate arrangements there.

5.8 Conclusions
The following conclusions are drawn partly from the paper by Redlich and Berchowitz [RB85].

By the application of linear dynamics to free-piston Stirling engines the following has been shown:
1) Existence of a stability criterion which relates mechanical dynamics and thermodynamics and shows that there is a minimum hot end temperature for which oscillation may be expected.
2) That there is only one mode of oscillation for the engine treated here, i.e., only one pair of RHP roots.
3) Frequency is generally load dependent but the engine may be configured to operate at essentially constant frequency for wide changes in load.
4) Once the machine begins to oscillate, non-linearities must act to prevent a runaway condition.
5) The presence of sliding friction imposes a condition of a minimum amplitude product (|z_d||z_p|) for starting. In the absence of sliding friction, a properly configured engine will self-start at a minimum starting temperature.
6) Arranging \( \omega_d = \omega_p \) allows the engine to start at a lower hot end temperature.
7) The inclusion of viscous coupling between the displacer and piston will
increase the stroke ratio and reduce the displacer/piston phase angle.

6) Engines in which the flow remains totally or largely laminar will be less likely to operate stably without feedback control than engines in which the flow is predominantly turbulent.

9) And finally, simple isothermal thermodynamics coupled with viscous loss calculations appear to give good correlation in power and dynamics with experimental data taken from the SPIKE engine. However, the SPIKE engine is of particularly low compression ratio. Engines of higher compression ratio tend to have higher pressure swings than predicted by isothermal thermodynamics which then leads to errors in the calculation of the dynamics. In such cases more advanced thermodynamic analyses which account for the adiabatic effect in the working spaces would produce more accurate results.

Generally, the linear dynamics analysis has been shown to provide a direct means to identify engine behaviour over the complete operating parameter set and shows the influence of engine geometry on this behaviour. By virtue of being able to address the broad spectrum of operating conditions and design parameters, the linear analysis also offers a clear qualitative appreciation of these engine's idiosyncracies.

Developments to previous closed-form linear analyses include the accounting for displacer to piston viscous coupling and casing mass, both of which play significant roles in the dynamic behaviour of an engine.

Table 5.3 lists the pertinent equations derived in this section.
Table 5.3 Linear Dynamics Analysis Summary

Frequency of Operation

\[ \omega_0^2 = \omega_d \omega_d' + \omega_p \omega_p' - 2\pi (\omega_d \omega'_d/m_d)(\alpha_r/m_p) \omega_d \omega_p/(\omega_d \omega_p) \]

\[ \frac{\omega_d \omega_p}{\omega_d + \omega_p} \]

Borderline Equation

\[ \frac{\alpha_p'}{\alpha_r} = \frac{1}{4\pi^2} \left[ \frac{\omega_d \omega_p}{\omega_0} \left(1 - \frac{\alpha_p \alpha_r}{K_d K_p} \right) - \frac{\omega_d^2 + \omega_p^2 - \omega_0^2}{\omega_d \omega_p} \right]^{1/2} \]

Starting Criterion

\[ m_d m_p (\omega_0 + \omega_d)(\omega_0 + \omega_p) \left[ (\omega_0 - \omega_d)^2 + \frac{\omega_d^2}{4\pi^2 \omega_d^2} \right]^{1/2} \left[ (\omega_0 - \omega_p)^2 + \frac{\omega_p^2}{4\pi^2 \omega_p^2} \right]^{1/2} \]

\[ \left( \frac{\alpha_p}{D_d} \right)^2 + \omega_0^2 \right]^{1/2} \]

\[ \left| \alpha_r D_d \right| \]

Stroke Ratio

\[ \left| \frac{\alpha_d}{\alpha_r} \right| = \left( \frac{\alpha_p^2 + \omega_0^2 D_d^2}{\omega_d} \right)^{1/2} \left[ \left\{ \frac{\omega_0}{\omega_d} \right\}^2 + \left\{ \frac{\omega_0}{\omega_d} \frac{1}{2\pi \omega_d} \right\}^2 \right]^{-1/2} \]
Displacer to Piston Phase Angle

\[
\phi = \tan^{-1}\left[ \frac{-\omega_0 D_p \left( \frac{1}{\frac{D_p}{\omega_d}} \right) + \frac{\omega_0}{\omega_d} + \frac{1}{2\pi D_p}}{-\left( i - \frac{m_c}{\omega_d} \right) \frac{1}{\omega_d} - \frac{m_c}{\omega_d} \frac{D_p}{\alpha_p}} \frac{2\pi D_p}{\omega_d} \right]
\]

Pressure and Thermal Coupling Terms (Isothermal)

\[
\alpha_p = \frac{\langle p \rangle \sqrt{1 - (a/S)^2}}{S} A_R \frac{A}{A_R} (A - A_R) + \frac{m_p K_d}{m_c} \frac{\tilde{T}_k}{\tilde{T}_h}
\]

\[
\alpha_T = -\frac{\langle p \rangle \sqrt{1 - (a/S)^2}}{S} A \frac{A}{A_R} \left( 1 - \frac{\tilde{T}_k - A_R}{\tilde{T}_h} \right)
\]

where \( a \) and \( S \) are grouped terms from the Schmidt analysis (§3, Table 3.2)

Start-Up Criterion with Sliding Friction

\[
\frac{n^2}{16} \frac{m_p (m_d m_d^* \omega_0^2)}{m_d} \left\| K_d \right\| K_p \left( 1 - \frac{\tilde{T}_k - A_R}{\tilde{T}_h} \right)
\]

\[
\frac{\left( A_R + \frac{m_p}{m_c} \frac{K_d}{K_p} \frac{1}{1 - A_R/A} \right)}{1} > 1
\]

Power

\[
\rho = \frac{\omega_0^2}{2} \frac{\langle p \rangle \sqrt{1 - (a/S)^2}}{S} \frac{A}{A_R} \left( 1 - \frac{\tilde{T}_k - A_R}{\tilde{T}_h} \right) \left( 1 - \frac{\tilde{T}_k - A_R}{\tilde{T}_h} \right) \left\| K_d \right\| K_p \sin \phi
\]

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6 PARASITIC LOSSES

6.1 Background
There is a group of losses which is more or less unique to Stirling machines, not so much in the mechanisms of these losses but rather in their effect on the cycle. Typically, the technique that has found common practice is to calculate these losses separately and then evaluate their effect on the cycle. The analytical methods developed here are approximate and terms which contribute less than 10% of the overall effect are excluded. In some cases the general form of the problem is intractable and simplifying assumptions are necessary in order to obtain a solution. In such instances empirical experience is required to estimate the accuracy of the result.

This section discusses the more pertinent parasitic losses, i.e. those that may be 10% or more of the machine's power. There is no doubt that exclusions exist, some of which need yet to be properly identified. The results presented here have been, to some degree, verified experimentally and as such offer some confidence in their use. Figure 6.1 indicates diagrammatically the energy flows of most of the important parasitic losses.
Figure 6.1 Parasitic losses. For high pressure hydrogen engines it is reasonable to expect the maximum work fraction to be better than 0.5 with current technology. The range indicated is more typical for helium charged engines.

6.2 Conduction Losses

Significant conduction losses occur throughout Stirling engines, the most obvious being simple conduction along the casing walls from the hot end to the cold end. Other conduction paths are the displacer walls, the gas