Table 4.4 lists the flow dissipation results for easy reference. Note that the laminar and turbulent flow dissipations scale differently. In particular, the turbulent flow dissipation is directly proportional to the average gas density whereas the laminar flow dissipation is not a function of density in this first approximation.

The heat transfer per unit flow dissipation for turbulent flow may be obtained from Equations (4.7) and either of (4.51) and (4.52). This result in general form is as follows:

\[
\langle \phi \rangle/\langle \phi \rangle = \frac{16 k \Delta T_w g}{(\mu w^2 \nu_{sw}^2) h^2[(3/8)A_h^2]}
\]

Comparing this result to Equation (4.45), it can be seen from Table 4.3 that for equivalent geometries and operating conditions, the turbulent flow heat exchanger tends to have a better heat transfer per unit viscous dissipation than a laminar counterpart. For tubes, the heat exchanger geometry group (i.e. \((3/8)A_h^2\)), gives 0.231\(d^4\) which is an improvement over the laminar flow case by almost a factor of two. For rectangular passages the turbulent flow advantage drops off with increasing aspect ratio. At an aspect ratio of 8 the turbulent flow geometry group gives 24.03\(a^4\) which is an improvement of only 20% over the laminar flow case. This advantage is offset by the higher Reynolds numbers that necessarily occur in turbulent flow.

A useful measure of whether the flow is turbulent or not is given by the Reynolds number at half peak value. If the flow is turbulent by the time the
Reynolds number has reached its half peak value, then by the previous discussion (§4.2), the turbulence will persist for at least three-quarters of the total cycle. Thus defining the half peak Reynolds number as follows:

\[ \text{Re}_{hp} = \rho \langle u \rangle_{hp} \sigma_h / \mu \]

\[ = \omega_{hp} \sigma_h / (A \mu) \]

From the definition of \( \omega_{hp} \),

\[ \omega_{hp} = \omega_p / 2 \]

where \( \omega_p \) is the peak value of the mass flow in the heat exchanger of consideration.

Thus \( \text{Re}_{hp} \) becomes:

\[ \text{Re}_{hp} = \omega_p \sigma_h / (2A \mu) \quad (4.54) \]

Substituting for \( \omega_p \) from either of the approximate mass flow equations (Equations (3.67) and (3.71)) gives the general form of the Reynolds number in terms of engine parameters:

\[ \text{Re}_{hp} = \omega p \sigma_h \nu_{SW} / (4 \pi \rho g b A \mu) \quad (4.55) \]
where \( \langle T_g \rangle_b \) is the average gas bulk temperature. Usually, for this calculation, assuming \( \langle T_g \rangle_b \) equal to the local wall temperature does not introduce major errors.

The transition of laminar to turbulent flow occurs at different values of Reynolds number for different flow geometries. However, for most smooth geometries, transition begins at around 2300 and is complete at around 4000 to 10,000. To show the effect of the higher Reynolds numbers, (4.55) is rearranged to make the free-flow area the subject,

\[
A = \langle \mu \rangle / \rho \cdot c_h \cdot V_{sw} / (4Re_{hp} Re_{hp} \cdot R \cdot \langle T_g \rangle_b \mu),
\]

(4.56)

where for \( Re_{hp} > [Re_{hp}]_t \), i.e., greater than the transition half peak Reynolds number, the flow is assumed turbulent.

It can be seen that for a given engine geometry, the required \( A \) is larger for a smaller \( Re_{hp} \) which, from (4.53) would improve the heat transfer per unit viscous dissipation. Thus, in order to maximise the heat transfer per unit viscous dissipation, \( Re_{hp} \) should be kept low. This suggests that for maximum performance, a turbulent flow heat exchanger should be designed to operate at a point where the flow just begins to be dominantly turbulent.

Viscous losses associated with the standing eddies at the entrance and exit of the heat exchangers are also important, particularly in turbulent flow.
These entrance/exit losses are given by the following empirical result [Be76].

\[ \Phi_{h1} = 2K_p A U^3 \]  

(4.57)

where \( K \) is an empirical coefficient (also referred to as the head loss coefficient) dependent on the local Reynolds number, the flow geometry and direction of flow. Values for \( K \) are tabulated in standard texts on fluid flow (eg, [KL64]). Substituting for mass flow and density as before, the time averaged entrance/exit loss in general form becomes

\[ \langle \Phi_{h1} \rangle \approx K \langle p \rangle \left( \omega \nu_{sw} \right)^3 \]

\[ \frac{3\pi}{RT} \frac{R}{A^2} \]  

(4.58)

This result should always be included in the optimisation of the heat exchangers. Keeping the entrance/exit geometry well rounded and smooth will have a significant effect in reducing losses in these areas. Also, ensuring that the free-flow area of the heat exchanger plenums are at least equal to or greater than the free-flow area in the adjacent heat exchanger will reduce the entrance/exit losses.

To conclude this section, it is not altogether clear whether a laminar or turbulent flow heat exchanger is to be preferred. For a turbulent flow heat exchanger it appears that the Reynolds number should be kept low but high enough to ensure that most of the flow is turbulent. For the hypothetical case where the flow may be chosen to be either laminar or turbulent for a given geometry and Reynolds number, the turbulent flow heat exchanger
shows better heat transfer per unit viscous dissipation. However, for
rectangular flow passages of aspect ratio greater than 8, this advantage
starts dropping off. If one then includes the fact that the Reynolds numbers
have to be higher for turbulent flow, the advantage of the turbulent flow
heat exchanger is even slimmer. On the other hand, in order to improve the
specific size of an engine, the designer may choose higher frequencies and
higher pressures which may preclude a laminar flow heat exchanger unless
a large free flow area can be accommodated. In practical engines, it would
appear that both laminar and turbulent heat exchangers have been employed,
and indeed, in many cases the flow in strongly transitional. The laminar
flow heat exchangers are generally designed to operate at as high a
Reynolds number as possible while still maintaining laminar flow. For the
turbulent flow heat exchangers, the inverse holds true. Figure 4.7 shows
the heater head of a low pressure 1kW 60Hz engine (the hot end dome has
been omitted for clarity) in this machine the gas flow was laminar to
transitional hence the rectangular cross-section is the geometry of
preference.

4.5 Minimisation of Irreversibilities
As has been mentioned before, an optimum Stirling engine heat exchanger is
one in which the heat transfer has been maximised while minimising the
viscous dissipation. Strictly, such an optimisation should involve the
gine parameters in addition to the heat exchanger parameters. However,
there is an alternative approach which does allow some decoupling of the
engine proper and the heat exchangers. This approach involves minimising
the entropy generation due to the temperature difference and viscous
effects (a similar technique as been suggested by Bejan [Bej77] for the
The rate of entropy production per unit volume is given by the following result (see Appendix B for complete derivation):

$$g = - \frac{(q \cdot \nabla T)}{T^2} + \frac{(-\sigma : \nabla U)}{T}$$  \hspace{1cm} (4.59)

The first term on the right is due to temperature gradients and the second term is due to viscous dissipation.
Integrating (4.59) over the volume of interest gives the volumetric entropy generation in W/K, thus

\[ G = - \int_V (\mathbf{q} \cdot \nabla T) / T^2 \, dV + \int_V ( - \sigma : \nabla \mathbf{u}) / T \, dV \tag{4.60} \]

Now the dominant heat transfer occurs normal to the direction of flow and the major temperature gradient also occurs in this direction. Thus neglecting axial heat flow and axial temperature gradients, the first term on the right hand side is approximated by:

\[ - \int_V (\mathbf{q} \cdot \nabla T) / T^2 \, dV \approx - \frac{\partial}{\partial r} \left( \frac{\partial T}{\partial r} \right) \frac{V}{T_b^2} \]

Noting that

\[ \frac{\partial}{\partial r} = -k A_w \left( \frac{\partial T}{\partial r} \right) \]

and therefore

\[ \frac{\partial T}{\partial r} = - \frac{\partial}{\partial r} (k A_w) \]

the heat transfer contribution becomes

\[ - \int_V (\mathbf{q} \cdot \nabla T) / T^2 \, dV \approx \frac{\partial T}{\partial r}^2 \frac{V}{T_b^2} k A_w^2 \]

As would be expected, this quantity is always positive.

Referring to the second term on the right hand side of (4.60), the group

\[ \int_V ( - \sigma : \nabla \mathbf{u}) \, dV \]

is simply the viscous dissipation and in this application may be approximated by the time averaged value \( \nu \bar{z} \), viz, \( \Phi \). The general form of (4.60) in terms of heat exchanger parameters may now be given by
\[ G \approx \langle \hat{Q} \rangle^2 V / \langle k \langle T_b \rangle^2 A_w \rangle + \langle \Phi \rangle / \langle T_b \rangle \]  \hspace{1cm} (4.61)

A slightly more convenient form is as follows:

\[ G \approx \langle \hat{Q} \rangle^2 A_n / \langle n k L \langle T_b \rangle^2 \beta_n^2 \rangle + \langle \Phi \rangle / \langle T_b \rangle \]  \hspace{1cm} (4.62)

For well designed heat exchangers \( \langle T_b \rangle \approx T_w \). In other cases, the gas bulk temperature may be evaluated as follows:

\[ \langle T_b \rangle = T_w + \Delta T \]  \hspace{1cm} (4.63)

where \( \Delta T \) may be either positive or negative depending on the direction of heat flow. For a required \( \hat{Q} \), \( \Delta T \) may be evaluated from

\[ \Delta T = \langle \hat{Q} \rangle / (h A_w) \]  \hspace{1cm} (4.64)

Equations (4.61) and (4.62) are useful in the design of the cooler or heater. The regenerator, on the other hand, does have significant axial heat flow which would have to be accounted for in any optimisation. Similarly as before, but including the axial heat flow and assuming a linear axial temperature gradient, the entropy generation for the regenerator is as follows:
\[ G_r \approx \langle Q^2 \rangle V_f (k T_h T_k A_w^2) + Q_c (T_h + T_k)/(T_h T_k) + \langle \phi_f \rangle \ln(T_h/T_k)/(T_h - T_k) \]

(4.65)

where \( Q_c \) is the conduction down the regenerator through gas and solid.

Using Equations (4.61), (4.62) and (4.65) to minimise \( G \) will, in turn, minimise the total irreversibilities generated in the heat exchanger under consideration. For example, the heat exchanger length, \( L \), is a typical variable that might be optimised. Increasing the length reduces the \( \Delta T \) irreversibilities but increases the viscous irreversibilities. There is thus an optimum length.

Figure 4.8 illustrates the optimisation of length for a cooler and also shows the effect of assuming \( \langle T_g \rangle_b = T_w \). Figure 4.9 shows the effect of varying the number of parallel flow passages for the cooler example. A similar optimisation of regenerator porosity is given in Figure 4.10, here the length is held constant. In all these examples, parameters have been taken from the Sunpower SPIKE 1kW 60Hz free-piston engine (Appendix A).

Note that calculation of entropy production does not quantify the effect of a non-ideal heat exchanger on the ideal engine performance. Rather, it is a method of decoupling the design of the heat exchanger and therefore avoiding having to continuously iterate between engine parameters and the heat exchanger design.
**Figure 4.8** Entropy Generation at Constant Volume vs Length

This graph was generated holding volume constant. The parameters were as follows: $Q_k = 2546 \text{W}$, $K = 62.2 \times 10^{-6} \text{m}$, $V_{SWC} = 29 \times 10^{-5} \text{m}$, $T_k = 310 \text{K}$, $f = 60 \text{Hz}$, $n = 600$ and the gas type was helium.

**Figure 4.9** Entropy Generation at Constant Volume vs Length varying the Number of Parallel Paths
Figure 4.10 Entropy Generation at Constant Volume vs Porosity for various Wire Diameters
Data taken from Sunpower SPIKE machine (Appendix A.3)

4.6 Other Considerations
Thus far it has been assumed that the flow is always fully developed. Since Stirling engine heat exchangers often have small diameter to length ratios, the flow profile may be substantially different from that of developed flow. This could lead to significantly underestimated friction and heat transfer coefficients, particularly in laminar flow. In turbulent flow the flow generally develops far more rapidly, and the effect on the average friction and heat transfer coefficient is usually not important. In this work developing turbulent flow will not be considered.
For developing laminar flow, the following empirical equation for average friction factor by Kline and Shapiro [KS53] is used:

\[ f_{av} = 3.435 \sqrt{\frac{L}{d}} / Re \quad 10^{-5} < \frac{L}{d} / Re < 10^{-3} \quad (4.66) \]

This equation applies to tubes. For other geometries there will be different empirical equations. There are also analytical results (e.g., that of Langhaar [La42]), but these are not repeated here.

Developing flow heat transfer coefficients are dependent on the following general boundary conditions:

1. Constant wall temperature
2. Constant heat flux
3. Constant temperature difference

In Stirling engines the first boundary condition is the more appropriate. The following empirical result by Kays (given in [KK58]) for Pr=0.7 has been found to work well for tubes:

\[ Nu_{av} = 3.66 + 0.104[\frac{L}{d}/Pe]^{-1/2} [1 + 0.016[\frac{L}{d}/Pe]^{0.0}] \quad (4.67) \]

(for constant wall temperature and Pr = 0.7)

Figure 4.11 shows the effect of developing flow as given by (4.66) and (4.67) on the entropy generation. As can be seen, the minimum at zero length no longer occurs as in the fully developed case. Also the entropy
generation is generally higher, particularly for shorter lengths. The optimum length has been shifted only slightly.

Generally, the friction factors and heat transfer coefficients used in Stirling engine analysis have been those obtained for steady flow circumstances. In addition, the friction factors available in the open literature are further limited by being strictly applicable for isothermal incompressible flow only. The isothermal and incompressible assumptions are probably not too inaccurate when applied to either the cooler or heater. Generally, these heat exchangers are designed to have high NTU thus temperature differentials tend to be small compared to the mean bulk temperature of the gas. In addition, the viscous losses are held low, thus the flow is not usually subjected to rapid accelerations and therefore flow induced compressibility effects are not typically important. The steady flow assumption is more likely to introduce significant error, mainly due to the velocity profile being distorted by the interaction of the viscous and inertia forces and thus altering the shear at the wall. The bulk inertia of the gas is in most cases (but not all) likely to be small and is often neglected as in the preceding analysis. The temperature profile would also be affected by the modified velocity profile which in turn would alter the heat transfer between the wall and gas. The non-steady heat transfer effects, though understood heuristically, have never been included in Stirling engine analysis in the open literature. Steady flow heat transfer coefficients have always been used owing to a lack of suitable empirical data. The results as reported in the open literature have apparently been good.
Figure 4.11 Effect of Developing Flow on the Production of Irreversibility in a Cooler Heat Exchanger

Data taken from SPIKE engine (Appendix A.3)

Work that may be relevant to the question of oscillatory flow friction factors may be that presented by Trikha [Tr75]. Trikha analytically derived an approximate laminar flow friction factor for oscillatory flow in tubes as follows:

\[ f = f_0 + j4\omega(\rho\mu)[40/(\omega + 8000) + 8.1/(\omega + 200) + 1/(\omega + 26.4)] \]

where

\[ u = Ue^{j\omega t} \quad \text{(velocity)} \]

\[ a \text{ : radius of tube.} \]

\[ f_0 \text{ : steady flow friction factor.} \]

\(^1\)In the original paper by Trikha, the Darcy friction factor was used. Thus a factor 4 appeared in the second term on the right hand side.
The imaginary part of (4.68) modifies the fluid inertia whereas the real part is the viscous friction factor. Thus finding the real part:

\[ \tau = \tau_0 + (8/Re_p)\left[40/(1 + 64\cdot10^6/\omega^2) + 8.1/(1 + 4\cdot10^4/\omega^2) + 1/(1 + 700/\omega^2)\right] \]

(4.69)

For laminar flow:

\[ \tau_0 = 16/Re \text{ for tubes.} \]

Thus (4.69) becomes

\[ \tau/\tau_0 = 1 + (1/2)\left[40/(1 + 64\cdot10^6/\omega^2) + 8.1/(1 + 4\cdot10^4/\omega^2) + 1/(1 + 700/\omega^2)\right] \]

(4.70)

which gives the ratio of oscillatory friction factor to steady flow friction factor for laminar flow in tubes.

Trikha analysed the flow in an area well removed from the entrance or exit, thus he was able to neglect entrance/exit effects. It would therefore appear that entrance/exit effects should be included over and above the effects of flow oscillation. Trikha also pointed out that though his analysis applies strictly to laminar flow, there is evidence that the same results (i.e., Equation (4.68)) appear to predict the change in turbulent friction factors too. For turbulent flow in tubes

\[ \tau_0 = 0.079Re^{-1/4} \]

(4.71)
Thus in the case of turbulent flow, (4.69) becomes

\[
\frac{f}{f_0} = 1 + \frac{101.3}{(\text{Re}_p^{3/4})} \left[ \frac{40}{1 + 64 \cdot 10^6/\omega^2} + \frac{8.1}{1 + 4 \cdot 10^4/\omega^2} \right] + \frac{1}{1 + 700/\omega^2} \]  \tag{4.72}

If (4.72) is indeed valid for turbulent flow, then the hydraulic diameter should account for different flow section geometries. For laminar flow, on the other hand, the oscillatory friction factor may be quite different depending on the flow section geometry.

Equations (4.70) and (4.72) are shown plotted against peak Reynolds number in Figure 4.12. It is clear that flow oscillation introduces significantly higher viscous friction than in steady flow. For turbulent flow, the friction ratio tends to approach a constant value for higher Reynolds numbers. The oscillatory effect appears to be far more severe in laminar flow than in turbulent flow.

Since the friction factor in oscillatory flow is higher than for steady flow, the viscous losses would also be greater. Using Equations (4.70) and (4.72) as multipliers on the steady flow friction factor, the entropy generation for the SPIKE cooler of Figure 4.11 is recalculated in Figure 4.13.
**Figure 4.12** Oscillatory Friction Factor to Steady Flow Friction Factor vs Reynolds Number

**Figure 4.13** Entropy Production Including Developing Flow Effects and Oscillatory Friction Factor
Oscillatory flow effects on heat transfer have been studied by a few workers [HD83, NMB0, Ber69, KK58]. Unfortunately, there does not seem to be any general empirical or analytical result that may be immediately applicable to Stirling engine analysis. However, in circumstances similar to that encountered in Stirling engines, there appears to be strong evidence suggesting an increase in heat transfer over the steady flow correlations. In reference [KK58], work by F B West and A T Taylor is quoted in which heat transfer coefficients were increased by up to a factor of two over the steady flow values, the maximum increase occurring in turbulent flow. More recent work by Hwang and Dybbs [HD83] also show similar increases of the Nusselt number. Their maximum augmentation was measured at between two and three times that of the steady flow values. Hwang and Dybbs did obtain a correlation equation, but this equation was not general in geometry or frequency and is therefore not directly useful.

There is no question that the use of steady flow correlations for heat transfer will underestimate the capability of the heat exchangers. Though not satisfactory, the use of steady flow heat transfer correlations is generally not too critical whereas the use of steady flow friction factors could lead to a serious under prediction of flow losses. Figure 4.14 shows the effect on entropy production when arbitrarily increasing the heat transfer coefficient by 2.

Application of steady flow correlations to regenerator matrices is probably less likely to introduce significant errors. Owing to the low Reynolds numbers and large surface area to volume ratio, there is little fluid inertia
Figure 4.14 Entropy Generation with Nusselt Number Doubled to Account for Enhanced Heat Transfer in Oscillatory Flow

effect on the friction factor. In addition, the tortuous passages prevents the flow from relaxing over any appreciable distance thus developing flow effects are also insignificant. However, whereas it is generally admissible to ignore the implicit isothermal assumption for friction factors in the cooler or heater, this is not good practice to do so for the regenerator. The large axial temperature gradient causes appreciable density and velocity variations along the length of the regenerator which, in turn, affect the regenerator pressure drop.

There are many equations that have been proposed for describing pressure drop across porous columns. The equation used in this work was proposed by Ergun [Er52] and has been well reviewed in the literature [ME79]. More recently, work on regenerators specifically for use in Stirling engines has been presented by Miyabe et al [MT82]. Their work, however, relates
specifically to packed screen type regenerators and is not easily extended to the random wire regenerators used in the engines analysed in this work.

The Ergun equation is given by

\[
\left( \frac{p_0 - p_L}{G_0^2} \right) \frac{\mu}{\lambda} \left[ \psi^2 / (1 - \psi) \right] = 150(1 - \psi) / (G_p G_0 / \mu) + 1.75 \quad (4.73)
\]

where \( G_0 = \rho U_0 \), sometimes referred to as the mass flux density, where \( U_0 \) is the superficial velocity, i.e., the velocity the gas would have if the matrix were not there. \( G_p = 6/a_v \), where \( a_v \) is defined as the total particle surface area to the total particle volume.

From (4.73) the Fanning friction factor may be found (defined by Equation (4.35)):

\[
\lambda = 33.3/\text{Re} + 0.583 \quad (4.74)
\]

where the Reynolds number is defined as previously (see Equation (4.37)).

Since (4.74) is non-linear, in order to use it effectively in simple heat exchanger analysis it must first be linearised. As mentioned previously, the pressure amplitude is maybe 10% of the mean pressure and the regenerator pressure drop is smaller than that again. It is therefore reasonable to neglect pressure variation effects on the friction factor. Furthermore, at a local point in the regenerator, the temperature is also nearly constant. This
leaves only the velocity as the major contribution to the non-linearity of the friction factor.

Assuming sinusoidal velocity variations, the non-linear pressure drop may be integrated over a half cycle and be equated to the pressure drop given by the equivalent linear friction factor as follows:

\[ 2 \int_{\text{half cycle}} C_{Lr} \rho U^2 / (Re \, d_h) \, d\theta = 2 \int_{\text{half cycle}} (33.3 / Re + 0.583) \rho U^2 / d_h \, d\theta \]

taking density constant and letting \( U = U_p \sin \theta \), \( C_{Lr} \) may be solved for to give

\[ C_{Lr} = 33.3 + 0.583 \times (\pi / 4) Re_p \]

or

\[ C_{Lr} = 33.3 + 0.458 Re_p \] (4.75)

where \( Re_p \) is the peak Reynolds number.

Equation (4.75) is the linearised friction factor used for porous media in this work.

In order to determine the temperature effect, Equation (4.35) is again used:

\[ -dp/dx = 2 \rho \rho U^2 / d_h \]
or

\[-dp/dx = 2(\frac{C_L}{Re})\rho U^2/\alpha_h\]

Substituting for \(C_L\) from (4.75):

\[-dp/dx = 2[33.3/\text{Re} + 0.458(\text{Re}_p/\text{Re})]\rho U^2/\alpha_h\]  \hspace{1cm} (4.76)

Only the density and the velocity are functions of temperature. Since the instantaneous mass flow is only a function of time (instantaneous accumulation is small), the velocity may be written \(U = \frac{w}{(\rho A)}\).

Assuming a linear temperature gradient, (4.76) may be integrated over \(L\) to find the instantaneous pressure drop:

\[
|\Delta p| = [66.6\mu w/(A\alpha_h^2) + 0.916 w_p w/(A^2 \alpha_h)]R/p
\]

\[
\int_{L}[(\tau_h - \tau_k)/L + \tau_k]dx
\]

giving

\[
|\Delta p| = [66.6/\text{Re} + 0.916(\text{Re}_p/\text{Re})]w^2/(\bar{\rho}A^2 \alpha_h)\]  \hspace{1cm} (4.77)

where \(\bar{\rho}\) turns out to be the arithmetic mean density of the higher and lower temperatures. Thus in using steady isothermal friction factors to describe the pressure drop in porous media in simple heat exchanger analysis, it would appear that assuming the density to be the arithmetic
mean of the higher and lower temperatures is consistent with other basic assumptions in the analysis and should lead to better accuracy.

Heat transfer coefficients for porous media are generally defined as local values assumed constant over a cross-section of flow. One empirical correlation by Yoshida, Ramaswami and Hougen (quoted in [BS60]) is given as follows:

\[ \dot{h}_H = 0.91 \text{Re}^{-0.51} \theta \quad (\text{Re}_H < 50) \]  \hfill (4.78)

\[ \dot{h}_H = 0.61 \text{Re}^{-0.41} \theta \quad (\text{Re}_H > 50) \]  \hfill (4.79)

where the Colburn \( \dot{h}_H \) factor and Reynolds number are defined by

\[ \dot{h}_H = h_{\text{loc}}/(C_{pb} G_0 \text{Pr}^{2/3}) \]  \hfill (4.80)

\[ \text{Re}_H = G_0/(a u_f \theta) \]  \hfill (4.81)

In these equations \( a \) is the solid particle surface area per unit bed volume, \( C_{pb} \) is the bulk specific heat at constant pressure, \( G_0 = w/A_C \) is the superficial mass velocity where \( A_C \) is the canister free-flow area and \( f \) denotes properties evaluated at the film temperature. The quantity \( \theta \) is an empirical coefficient that depends on the particle shape, eg, for spheres \( \theta = 1.00 \) and for cylinders \( \theta = 0.91 \).
The actual heat transfer is usually written [BS60]:

\[ dQ = h_{1oc}(x) dA dx(T_w - T_b) \]  \hspace{1cm} (4.82)

For commonly used gases $Pr$ and $C_{pb}$ do not vary much and may be assumed constant. Also $\nu$ is spatially constant, thus from (4.80) the temperature dependence of $h_{1oc}$ is only related to $J_H$ which, if instantaneous mass accumulation is small, can be seen to be spatially constant from (4.81) and (4.78) or (4.79). It has thus been argued that for gases and for spatially constant mass flows the local heat transfer coefficient is also spatially constant. This suggests that the heat transfer is spatially uniform and therefore also the gas to matrix temperature differential. Equation (4.82) may thus be written

\[ \langle Q \rangle = h_{1oc} A_w (T_w - T_h) \] \hspace{1cm} (4.83)

where $A_w$ is the total wetted surface area.

The equations used here for porous media friction factors and heat transfer coefficients are generally more accurate at lower porosities. Since Stirling engines often require high porosity regenerators (in the region of 90%), it is suggested that the correlations given by (4.75), (4.78) and (4.79) be treated with caution. Accurate regenerator correlations are closely guarded proprietary secrets.
Table 4.4 Heat Exchanger Analysis Summary

**Heat Transfer in Cooler and Heater**

Laminar

\[ \dot{Q} = N u k A_w (T_w - T_{g_b})/\sigma_h \]

Turbulent

\[ \dot{Q} = \mu A T_w (T_w - T_{g_b})/(2\pi \mu A R T_w) \]

**Heat Transfer in Regenerator**

Ideal

\[ |Q| = \gamma (\gamma - 1)^{-1} \left( \frac{V_f^2 + V_{swc}^2 + V_{swe}^2 + 2(V_{swc} V_{swe} \cos \alpha)}{N k_{sc} \cos \beta - V_f V_{swe} \cos (\alpha - \beta)} \right)^{1/2} \]

where

\[ V_f = \frac{2(V_f/\gamma + k_c + k_{clc} + k_h + k_{cle}) + V_{swc} + V_{swe}}{S} \]

and \( S \) and \( \beta \) may be found in Table 3.2.

Actual

\[ |Q_{act}| = h A_w \Delta T r/(2\omega) \]

where

\[ r = \sqrt{2(1 - \cos \theta_w)} \]

**Regenerator Performance**

\[ \epsilon = N r/(\pi + N r) \]

Typically \( 77 < N < 156 \)

where

\[ N = \dot{n} A_w / [c_p(w_{kr})_{max}] \]
Table 4.4 Heat Exchanger Analysis Summary - Continued

Viscous Dissipation in Cooler and Heater

Laminar
\[ \langle \Phi \rangle \approx \mu L \frac{N \omega}{2 \pi n} \]

Turbulent
\[ \langle \Phi \rangle \approx \frac{f}{3n} \left( \frac{L}{d_h} \right) \frac{\rho}{\rho_g} \frac{(\omega \nu_{SW})^3}{A^2} \]

Refrigerator Viscous Dissipation
\[ \langle \Phi \rangle_r = 0.5\mu L \left[ \frac{1}{3} \frac{(\nu_{SW} \nu_{g} \nu_{t} \nu_{s})^3}{A^2} \right] \]

Head Loss
\[ \langle \Phi_\text{h} \rangle = \frac{1}{\pi} \left[ \frac{p}{RT} \right] (\omega \nu_{SW})^3 \]

Half Peak Reynolds Number
\[ \Re_{hp} = \omega \langle p \rangle \frac{d_h}{A} \frac{\nu_{SW}}{(AR \nu_{g} \nu_{b} A \mu)} \]

Entropy Generation

Cooler and Heater
\[ G \approx \langle \dot{Q}^2 \rangle / \left( n KL \langle \nu_{g} \nu_{b} \nu_{s} \nu_{n} \rangle \right) + \langle \Phi \rangle / \langle \nu_{g} \nu_{b} \rangle \]

Regenerator
\[ G_r \approx \langle \dot{Q}_r^2 \rangle (T_h - T_k) + \frac{\dot{Q}_r}{(T_h - T_k)} (T_h - T_k) + \frac{\Phi_r}{(T_h - T_k)} \ln \left( \frac{T_h}{T_k} / \frac{T_h}{T_k} \right) \]
5 LINEAR DYNAMICS OF FREE-PISTON ENGINES

5.1 Background

Free-piston engines use variations of working gas pressure to drive mechanically unconstrained reciprocating elements. Stirling cycle free-piston engines have the following important advantages over their crank driven counterparts:

1) There are negligibly small side loads on the moving parts which removes the need for liquid lubricants and allows the potential for long operating life.

2) The need for an external high pressure seal to the environment is removed since the machine is hermetically sealed.

3) All the out of balance forces are axial to the engine which allows simple vibration isolation by using a dynamic absorber. This has the advantages of making the engine extremely quiet and removing the damaging effects of engine vibration.

4) There are only two moving parts which improves reliability and reduces cost.

One typical configuration of free-piston Stirling engine driving a linear alternator is shown in Figure 5.1. Other applications are water pumps, and if the load is a second Stirling cycle operating as a heat pump, then domestic heat pumps, food freezers and natural gas liquefiers are also possible [Bea69, Be82, BR71, UP84, Co74].
Figure 5.1 Free-Piston Stirling Engine

The first mention of a Stirling cycle machine using freely moving components appears to be a British patent disclosure in 1876 [Po1876]. This machine was envisaged as a refrigerator (i.e., a reversed Stirling cycle) and the piston was therefore driven externally. The invention of the basic free-piston Stirling engine is generally attributed to W T Beale [Bea71], who in the early 1960s discovered that a rhombic drive machine could possibly work without its linkage. Independent inventions of similar types of engines were made by E H Cooke-Yarborough and C West at the Harwell
Laboratories of the UKAERE [Co67, Co70, We70, CF74]. G M Benson has also made important early contributions and has patented many novel free-piston configurations [Ben73, Ben77.1]. Others have since been working on various aspects and modifications of these original ideas [Ma75, Re79, GM77, Go79, GR79].

The first paper in the open literature to describe the workings of free-piston Stirling engines was that of W T Beale [Bea69] in 1969. In this paper Beale describes the simulation of free-piston engines by the use of the Continuous Systems Modelling Program (CSMP). Isothermal thermodynamics is used and good qualitative agreement between experiment and simulation is indicated. Later, Godin [Go78] presented a doctoral thesis in which a more complete account of the simulation of free-piston Stirling engines was given. Godin also assumed isothermal working spaces. Another effort at simulation was that of L Goldberg while at the University of the Witwatersrand [Go79]. Goldberg's thermodynamic analysis included the effects of adiabatic working spaces but did not account for gas momentum effects. Generally, he was able to show good agreement with experimental results. Other simulations which couple the thermodynamics and mechanical dynamics have been completed at various companies engaged in free-piston Stirling engine research [VR80, Gi80]. Generally the simulation method has shown good agreement with reality.

The simulation methods do, however, have some serious drawbacks in application to preliminary design of Stirling engines. For example, the simulation deals with a point design and does not automatically indicate how an engine might behave under other operating parameters. One might of
course vary the operating parameters and rerun the simulation. Doing this might indicate that the chosen design is unstable, that is, the engine might stop running or its amplitudes might continue to grow until the moving parts collide with the physical stops within the engine. The designer is now left with the question of what to change in order to obtain an acceptable operating characteristic. Since simulations are empirical exercises, the only recourse is to run many simulations while changing one parameter at a time until the desired result is obtained. It can be appreciated that this involves an inordinate amount of work, at least a number of simulations for each design change.

In general, simulation methods tend to be a slow and tedious process even on large main frame computers. Furthermore, simulation methods obscure the detailed mechanism of why a particular machine behaves as it does. For these reasons simulations are seldom used for preliminary design of free-piston machines.

Linear analyses are more commonly used for preliminary design and often for advanced design too. The first published linear analysis was that of J S Rauch [Ra75]. Rauch's analysis made the following major assumptions:

1) The component dynamics are not coupled to the thermodynamics of the cycle.
2) The working gas is assumed to act as a linear spring for purposes of analysing the engine dynamics.
3) The dissipative effects of the load are assumed to be linear dampers.

Of these assumptions only the first one is seriously limiting. His results