which may be written

\[ g = -\frac{\mathbf{q} \cdot \nabla T}{T^2} + \frac{(-\mathbf{\sigma} : \nabla \mathbf{U})}{T} \]  (B.12)

where \( \mathbf{q} \) is some function of the temperature gradient for both conduction and convection, usually assumed to be directly proportional as follows:

\[ \mathbf{q} = -k \nabla T \]  (B.13)

where \( k \) is known as the coefficient of thermal conductivity.

Using (B.13), (B.12) becomes

\[ g = \frac{k(\nabla T)^2}{T^2} + \frac{(-\mathbf{\sigma} : \nabla \mathbf{U})}{T} \]  (B.14)

This final result shows that in any real system (i.e., one in which there is friction and temperature gradients), the irreversibilities are associated with the coupling of the temperature gradient and viscous terms as given on the right of (B.14).

Checking Equation (B.14) against experience. For Newtonian fluids \((-\mathbf{\sigma} : \nabla \mathbf{U})\) can be shown to be always positive [BS60], for most other fluids this term becomes intractable. Furthermore, since heat will always flow from a higher to a lower temperature, \( k \) must be positive (see Figure B.1). Thus in (B.14) both terms on the right are always positive which shows that the rate of entropy production is always positive.
\textbf{Figure B.1 \textit{Sign Convention}}

- \( q \) positive for negative temperature gradient
- Control volume
- Position coordinate increasing outwards
APPENDIX C

IDEAL ADIABATIC ANALYSIS

C.1 Introduction

The ideal adiabatic analysis (or simulation) of the Stirling cycle has been presented many times [F160.2, WK65, Be78, UK82, UB84]. The general assumptions of the ideal adiabatic analysis are identical to those of the Schmidt analysis with the exception that adiabatic working spaces are assumed. Some workers have suggested that the analysis is intractable and that solutions are only possible by explicit numerical integration of the system differential equations [F160.2, UK82]. A closed-form solution due to Rauch [Ra80] has been published. He, however, assumed that the temperatures in the working spaces vary as given by

\[
\left( \frac{T_1}{T_2} \right) = \left( \frac{p_0}{p_1} \right)^{(1-\gamma)/\gamma}.
\]

(C.1)

This, of course, is the adiabatic temperature equation for an enclosed volume. By applying this equation to the working spaces of a Stirling engine, Rauch neglects the substantial enthalpy flux carried by the gas in and out of the working spaces. Rauch has never indicated in the open literature the degree of error associated with this assumption.

Other workers who have analysed this cycle have always used numerical techniques, mostly time consuming explicit methods. Chen et al. [CG84] are the only group to have used an implicit method. Their technique
requires linearization but is much faster than the explicit methods.

In this appendix an approximate solution to the ideal adiabatic cycle is developed. The solution is iterative, however, a closed-form structure is retained in that the final equations describe global parameters rather than instantaneous parameters. Convergence is also rapid, the first iteration being approximately the Schmidt solution.

C.2 Analysis
The basic assumptions are:

1) The working gas in the cooler is at a constant temperature $T_k$.
2) The working gas in the heater is at a constant temperature $T_h$.
3) Regeneration is perfect.
4) There is no spatial pressure drop or leakage of the working gas.
5) The working gas is ideal.
6) Momentum of the working gas is neglected and thus by implication the kinetic energy of the working gas is also neglected.
7) A linear temperature profile is assumed along the regenerator.
8) The working spaces are assumed to be adiabatic and perfectly mixed.
9) Volume variations are sinusoidal.
10) The machine speed is constant and at cyclic steady state.

Following the isothermal analysis in Chapter 3, the total mass of gas in the engine is given by

$$m_t = p(v_c/ T_c + v_k/ T_k + v_r/ T_r + v_h/ T_h + v_e/ T_e)/R$$  \hspace{1cm} (C.2)
Since $T_c$ and $T_e$ are time dependent it is necessary to obtain two additional expressions.

![Diagram](3x2.png)

**Figure C.1** The Adiabatic Working Space

The energy equation for a general adiabatic working space is

$$\dot{H} - p \, \dot{V} = \dot{Q}$$  \hspace{1cm} (C.3)

From the definition of enthalpy:

$$\dot{H} = c_p \, T' \, w$$  \hspace{1cm} (C.4)

where $T'$ is the temperature of the gas crossing the control volume boundary.

From continuity:

$$w = \frac{d \, m}{dt}$$  \hspace{1cm} (C.5)
Thus (C.4) becomes:

\[ \dot{H} = c_p T' \, \text{d} m \]  \hspace{1cm} (C.6)

where

\[ T' \leftarrow T_{c,e} \text{ for } \text{d} m \leq 0 \]
\[ T' \leftarrow T_{k,h} \text{ for } \text{d} m > 0 \]

The internal energy is given by

\[ U = c_v T m \]

thus,

\[ \text{d} U = c_v (m \, \text{d} T + T \, \text{d} m) \]  \hspace{1cm} (C.7)

From the gas law:

\[ \text{d} m = \left[ (p/T) \text{d} V + (V/T) \text{d} p - (p V/T^2) \text{d} T \right] / R \]  \hspace{1cm} (C.8)

Substituting (C.8), (C.7) and (C.6) into (C.3) noting that \( c_p = \gamma R / (\gamma - 1) \) and \( c_p - c_v = R \) yields the following after simplification:

\[ \text{d} T = T(1 - T/T') \text{d} V / V + T[1 - T/(\gamma T')] \text{d} p / p \]  \hspace{1cm} (C.9)

When the flow is out of the working space, i.e., \( \text{d} m \leq 0 \) and \( T' \leftarrow T \), (C.9) becomes:
\[
d\frac{T}{T} = (1 - 1/\gamma)\frac{dP}{P} \quad \text{for } dm \geq 0 \quad (C.10)
\]

which is easily integrated to yield

\[
\ln T = (1 - 1/\gamma)\ln P + \ln C_1
\]
or

\[
T = C_1 P^{(1 - 1/\gamma)} \quad \text{for } dm \geq 0 \quad (C.11)
\]

For gas flow into the working space, i.e., \(dm > 0\) and \(T' \rightarrow \bar{T}\) \((C.9)\) becomes:

\[
d\frac{T}{T} = \left(1 - \frac{1}{\gamma} \right)\frac{d\nu}{\nu} + \left[1 - \frac{T}{(\gamma \bar{T})} \right]\frac{dP}{P} \quad \text{for } dm > 0 \quad (C.12)
\]

where \(\bar{T}\) is the relevant constant heat exchanger temperature.

\((C.12)\) is difficult to integrate directly and therefore the following simplification is made.

Assume that \(T = \langle T \rangle(1 + \delta T)\) where \(\delta T \ll 1\) \((C.13)\)

For practical engines \(\delta T\) is usually less than 0.1.

Substituting \((C.13)\) into the RHS of \((C.12)\) and dropping second order small terms:
\[ \frac{d\tau}{\tau} = (1 - <\tau>/\bar{\tau}) d\nu/\nu + [1 - <\tau>/(\gamma \bar{\tau})] dp/p \]  
\hspace{1cm} (C.14)

which may be integrated to give

\[ \tau = C_2 \nu (1 - \tau)^{p(1 - \tau/\gamma)} \]  
\hspace{1cm} for \( dm > 0 \)  
\hspace{1cm} (C.15)

where \( \tau = <\tau>/\bar{\tau} \) and \( C_2 \) is the constant of integration.

Equations (C.11) and (C.15) define the temperature variations in the working spaces.

Assume that pressure varies with the same form as given by the Schmidt analysis (Chapter 3), i.e,

\[ p = <p> [1 - 8p\cos(\Theta + \beta)] \]  
\hspace{1cm} (C.16)

where \( 8p \ll 1 \).

Using the binomial expansion, (C.11) may then be written

\[ \tau \approx C_1 [1 - 8p(1 - 1/\gamma)\cos(\Theta + \beta)] \]  
\hspace{1cm} for \( dm \leq 0 \)  
\hspace{1cm} (C.17)

where \( C_1 \) is now a new constant and higher order terms have been neglected.

Rewriting (C.17) for easier manipulation:

\[ \tau \approx B_1 - B_2 \cos(\Theta + \beta) \]  
\hspace{1cm} for \( dm \leq 0 \).  
\hspace{1cm} (C.18)
which is the final form of the working space temperature for the part of the cycle where \(d \tau \geq 0\).

(C.15) is treated a little differently. Rewriting (C.15) as the product of two functions

\[
T(p, \nu) = C_2 f(\nu) g(p) \quad \text{for } d \tau > 0
\]  

(C.19)

Both \(f(\nu)\) and \(g(p)\) are respectively non-linear functions of volume and pressure. An approximate method of linearizing (C.19) is to separately linearize \(f(\nu)\) and \(g(p)\) and then neglect second order small terms from their product.

Linearization of \(g(p)\) is straightforward and is given by

\[
g(p) \approx \langle p \rangle (1 - \tau / \gamma) [1 - \delta p (1 - \tau / \gamma) \cos(\theta + \beta)]
\]  

(C.20)

\(f(\nu)\) is more difficult to linearize since the variations of \(\nu\) are a large fraction of the mean value. In this case, Taylor’s expansion theorem is used.

Thus, expanding around the mean volume, i.e., \(\langle \nu \rangle = V_{C1} + V_{SW}/2\).

\[
f(\nu) \approx \langle \nu \rangle^{(1 - \tau)} [1 + (1 - \tau)(\nu / \langle \nu \rangle - 1) - (1/2)\tau(1 - \tau)(\nu / \langle \nu \rangle - 1)^2
\]

\[
+ (1/6)\tau(1 - \tau^2)(\nu / \langle \nu \rangle - 1)^3 + \ldots]
\]  

(C.21)
From (C.20) and (C.21), (C.19) now becomes

\[
\tau(p, \nu) \approx C_2 \left[ 1 + (1 - \tau)(\nu/\langle \nu \rangle - 1) - (1/2)\tau(1 - \tau)(\nu/\langle \nu \rangle - 1)^2 \\
+ (1/6)\tau(1 - \tau^2)(\nu/\langle \nu \rangle - 1)^3 + \ldots \right] - 8p(1 - \tau/\gamma)\cos(\theta + \beta) \quad \text{for } d.m > 0 \tag{C.22}
\]

where products of small quantities have been neglected and \( C_2 \) is now a new constant.

The following general volume variation is assumed:

\[
\nu = \nu_{cl} + (1/2) \nu_{sw} [1 + \cos(\theta + \alpha)]
\]
or

\[
\nu = \langle \nu \rangle + (1/2) \nu_{sw} \cos(\theta + \alpha). \tag{C.23}
\]

where for the compression space \( \alpha = 0 \) and for the expansion space \( \alpha = \alpha \).

Substituting for the volume variations (ie, (C.23) in (C.22)) and retaining only constants and the first harmonic:

\[
\tau \approx C_2 \left[ 1 + [(1/2)(1 - \tau)\nu + (1/96)\tau(1 - \tau^2)\nu^3]\cos(\theta + \alpha) \\
- 8p(1 - \tau/\gamma)\cos(\theta + \beta) \right] \quad \text{for } d.m > 0 \tag{C.24}
\]

where \( \nu = \nu_{sw} / \langle \nu \rangle \) and \( C_2 \) is again a new constant.
Contracting (C.24):\

\[ T \approx C_2 \left\{ A_0 + [A_1 \cos \alpha - A_2 \cos \beta] \cos \theta - [A_1 \sin \alpha - A_2 \sin \beta] \sin \theta \right\} \]

for \( d \tau > 0 \)

or

\[ T \approx B_3 + B_4 \cos(\theta + \zeta) \quad \text{for} \quad d \tau > 0 \quad \text(C.25) \]

where

\[ A_1 = C_2 \left[ (1/2)(1 - \tau) v + (1/96) \tau(1 - \tau^2) v^3 \right] \]

\[ A_2 = C_2 \delta p(1 - \tau / \gamma) \]

\[ B_3 = C_2 \left[ 1 - (1/16) \tau(1 - \tau)v^2 \right] \]

and \( B_4 \) is defined by the following trigonometric substitution:

![Figure C.2 Trigonometric Substitution for \( B_4 \) and \( \zeta \)]

For the working space temperatures to be continuous, the mean temperatures for each direction of flow must be equal (if equal half periods are assumed as is the case here), thus:
\( B_1 = B_3 \) \hspace{1cm} (C.26)

The temperatures in the working spaces for the entire cycle are required in the form

\[ T = \langle T \rangle (1 + \delta T) \] \hspace{1cm} (C.27)

where \( \delta T \) is the first harmonic fraction, i.e., \( \delta T = \Delta T / \langle T \rangle \).

From Fourier series, the temperatures may be written

\[ T = a_0 / 2 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \] \hspace{1cm} (C.28)

where

\[ a_n = \frac{1}{\pi} \int_{c}^{c+2\pi} T(\theta) \cos n\theta d\theta \]

\[ b_n = \frac{1}{\pi} \int_{c}^{c+2\pi} T(\theta) \sin n\theta d\theta \]

If \( dm \leq 0 \) for \( \theta_1 \leq \theta < \theta_1 + \pi \)

and \( dm > 0 \) for \( \theta_1 + \pi < \theta \leq \theta_1 + 2\pi \)

then from (C.28), (C.16) and (C.25) the mean value and first harmonic terms of the working space temperatures become
\[ a_0 = 2 \left[ B_1 + \left( B_2 / \pi \right) \sin(\theta_1 + \alpha) + \left( B_4 / \pi \right) \sin(\theta_1 + \zeta) \right] \]  \hspace{1cm} (C.29)

\[ a_1 = (1/2) \left( B_4 \cos \zeta - B_2 \cos \beta \right) \]  \hspace{1cm} (C.30)

\[ b_i = (1/2) \left( B_2 \sin \beta - B_4 \sin \zeta \right) \]  \hspace{1cm} (C.31)

Thus the final form of the temperatures in the working spaces may be written

\[ T = <T> - \Delta T \cos(\theta + \xi) \]  \hspace{1cm} (C.32)

where

\[ <T> = a_0 / 2 \]

and \( \Delta T \) and \( \xi \) are defined by the following trigonometric substitution:

\[ \Delta T \]

\[ \frac{1}{2} \left( a_2 \sin \beta - b_4 \sin \zeta \right) \]

\[ \frac{1}{2} \left( B_2 \cos \beta - B_4 \cos \zeta \right) \]

**Figure C.3** Trigonometric Substitution for \( \Delta T \) and \( \xi \)

In order to evaluate \( a_0 \) and hence \( <T> \), the value of \( \theta \) at flow reversal needs to be known, i.e. \( \theta_1 \). \( \theta_1 \) is obtained from the mass flow in the working space. Thus, substituting the pressure, temperature and volume variations
\[
\frac{dm}{d\theta} = -\left[ \frac{p}{(R\langle T\rangle)} \right] A_m \sin(\theta - \theta_1) \tag{C.33}
\]

where \( A_m \) and \( \theta_1 \) are obtained from the following trigonometric substitution:

\[
\left( \frac{V_{sw} \sin \alpha - \langle V \rangle \delta p \sin \beta + \langle V \rangle \delta T \sin \xi}{2} \right)
\]

\[
\left( \frac{V_{sw} \cos \alpha - \langle V \rangle \delta p \cos \beta + \langle V \rangle \delta T \cos \xi}{2} \right)
\]

**Figure C.4** Trigonometric Substitution for \( A_m \) and \( \theta_1 \).

The working spaces temperatures are now fully defined except for the original constants of integration which must be solved from appropriate boundary conditions.

From the energy equation (C.3), and integrating over a cycle:

\[
\frac{\delta H}{\delta p} - \frac{\delta p}{dV} = \frac{\delta dU}{dV}
\]

at cyclic steady state, the cyclic change in internal energy is zero, thus

\[
\frac{\delta H}{\delta p} = \frac{\delta p}{dV} \tag{C.34}
\]
which indicates that the net enthalpy flow into the working space is equal to the net $p\nu$ work done by the piston.

The enthalpy flow into the working space may be evaluated as follows:

\[
\langle H \rangle = c_p \left[ \int_{\theta_1}^{\pi + \theta_1} T \, dm + \int_{\pi}^{2\pi + \theta_1} T \, dm \right]
\]  
(C.35)

where $T$ is the working gas instantaneous temperature and $T$ is the associated constant heat exchanger temperature.

Substituting for $T$ for $dm \leq 0$ from (C.18) and for $dm$ from (C.33), (C.35) may be integrated to yield

\[
\langle H \rangle = -\left[ \frac{\gamma}{\gamma - 1} \right] \langle p \rangle \langle T \rangle A_m \left[ 2B_1 + B_2(\pi/2)\sin(\theta_1 + \beta) - 2T \right]
\]  
(C.36)

The $p\nu$ work is obtained from (C.16) and (C.23) giving:

\[
\int p \, d\nu = -\left( \frac{\pi}{2} \right) \langle p \rangle 8p V_{sw} \sin(\beta - \alpha)
\]  
(C.37)

where, again, $\alpha = 0$ for the compression space.

Equating (C.36) and (C.37), and noting the definitions of $B_1$ and $B_2$ from (C.17) and (C.18), the constant $C_1$ may be solved for:
\[ c_1 \left[ 1 + (\pi/4) \delta p (1 - 1/\gamma) \sin(\theta_1 + \beta) \right] \]
\[ = \bar{r} + (\pi/4) \delta p (1 - 1/\gamma) \left( \int \right) \nu_{sw} / A_m \sin(\beta - \alpha) \]  
(C.38)

Now \( B_1 = B_3 \) (from (C.26)) and \( C_1 = B_1 \) (again from (C.17) and (C.18)), thus from (C.25), the integration constant \( C_2 \) may be found:

\[ C_2 = C_1 / \left[ 1 - (1/16) \tau(1 + \tau) v^2 \right] \]  
(C.39)

This completes the necessary equations to solve the system. However, to proceed with the solution, the pressure given by (C.2) needs to be manipulated in order to obtain the form given by (C.16).

Substituting the temperature and volume variations given by (C.32) and (C.23) respectively into (C.2) and denoting compression and expansion space parameters by the subscripts \( c \) and \( e \) respectively, the pressure may be written

\[ p = \langle p \rangle \left[ 1 - (a_1 / S) \cos(\Theta + \beta) - (a_2 / S) \cos(2\Theta + \beta_2) \ldots + \text{(HOT)} \right] \]  
(C.40)

where

\[ S = A_{c_1} / \langle T_c \rangle + V_c / T_c + V_t / T_t + V_n / T_n + A_{e_1} / \langle T_e \rangle \]

\[ A_{c_1} = \langle V_c \rangle + (6 \tau / 4) \nu_{sw} c \cos \xi_c \]

\[ A_{e_1} = \langle V_e \rangle + (6 \tau / 4) \nu_{sw} c \cos(\alpha - \xi_e) \]
and the following trigonometrical substitutions have been necessary:

\[ a_1 = \frac{A_{c2}}{\langle T_c \rangle} \sin \phi_c + \frac{A_{e2}}{\langle T_e \rangle} \sin \phi_e \]

\[ = \frac{A_{c2}}{\langle T_c \rangle} \cos \phi_c + \frac{A_{e2}}{\langle T_e \rangle} \cos \phi_e \]

where \( A_{c2}, A_{e2}, \phi_c \) and \( \phi_e \) are defined by the following general substitution:

\[ a_2 = \frac{V_{sw}}{2} \sin \alpha + \langle \nu > B \sin \xi \]

\[ = \frac{V_{sw}}{2} \cos \alpha + \langle \nu > B \cos \xi \]

where the respective values for the compression and expansion spaces are used and \( \alpha = 0 \) for the compression space.

**Figure C.5** Trigonometrical Substitution for \( a_1, \beta, A_2 \) and \( \phi \)
The second harmonic term is obtained from:

\[
\begin{align*}
\Delta a_2 &= \left( \frac{A_{c3}}{\Delta T_C} \sin \xi_c + \frac{A_{e3}}{\Delta T_e} \sin(\alpha + \xi_e) \right) \\
\Delta \beta_2 &= \left( \frac{A_{c3}}{\Delta T_C} \cos \xi_c + \frac{A_{e3}}{\Delta T_e} \cos(\alpha + \xi_e) \right)
\end{align*}
\]

where

\[
A_{c3} = (8 T_C / 4) V_{swc}
\]

\[
A_{e3} = (8 T_e / 4) V_{swe}
\]

**Figure C.6** Trigonometrical Substitution for \( a_2 \) and \( \beta_2 \)

The second harmonic term is generally much smaller than the fundamental and does not contribute to the cyclic work, therefore it may be neglected.

Note also that

\[ \Delta p = a_1 / S \]

An iterative solution to the system proceeds as follows:

1) Assume initial values of \( \Delta T_C \) and \( \Delta T_e \) equal to zero. Thus \( \Delta T_C \) and \( \Delta T_e \) are also initially zero.

2) Solve for \( A_{c1} \) and \( A_{e1} \) (and \( A_{c3} \) and \( A_{e3} \), if second harmonic is required) (Equation (C.40) and Figure C.6).

3) Solve for \( A_{c2} \), \( A_{e2} \), \( \phi_c \) and \( \phi_e \) (Figure C.5).

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4) Solve for the fundamental in pressure, i.e., $a_1$ and $\beta$ (Figure C.5) and obtain $\delta p$. The first iteration yields the fundamental given by the Schmidt analysis.

5) Obtain flow reversal points and mass flow amplitudes, $\theta_{c1}$, $\theta_{e1}$ and $A_{mc}$, $A_{me}$ respectively (Figure C.4).

6) Solve for $C_{c1}$, $C_{e1}$ and $C_{c2}$, $C_{e2}$ from Equations (C.38) and (C.39).

7) Solve for $B_{c4}$, $B_{e4}$ and $\xi_C$, $\xi_e$ from Equation (C.25) and Figure C.2.

8) Solve for $B_{c2}$ and $B_{e2}$ from Equations (C.17) and (C.18).

9) Solve for $B_{c3}$, $B_{e3}$ and $B_{c1}$, $B_{e1}$ from Equations (C.25) and (C.26).

10) Find the mean temperatures $\langle T \rangle_C$ and $\langle T \rangle_e$ from Equations (C.32) and (C.29).

11) Find the temperature amplitudes $\Delta T_C$ and $\Delta T_e$ from Figure C.3.

12) Iterate on the mean temperatures $\langle T \rangle_C$ and $\langle T \rangle_e$ back to (1) until desired tolerance is obtained. Convergence of better than 1% on the mean temperatures is usually obtained in less than eight iterations.

13) Finally the cyclic work $W_C$ and $W_e$ is obtained from Equation (C.37). By inspection of a simple control volume containing either the cooler or heater, it follows that $Q_k = W_C$ and $Q_h = W_e$.

14) If the second harmonic in pressure is required, then this may now be evaluated from Figure C.6.

This procedure is included in the computer program in Appendix E.
A comparison of the iterative linear method with the more conventional explicit numerical integration technique [Be78, UBB4] is given in Table C.1 and Figures C.7 and C.8 for the SPIKE engine. The numerical integration technique obtains $T_c$ and $T_e$ by numerically integrating Equation (C.9) for each working space. In this method a problem arises in determining the total mass of gas ($m_t$) in the engine from the known conditions, i.e., temperature and pressure. Because of the phase displacement between pressure, temperature and volume, the mean pressure when the engine is at rest is not equal to the mean pressure when the engine is running. An approximate method is to obtain the mass of gas from the closed-form Schmidt result (Equation (3.29)) [UBB4]. A more accurate method is to adjust $m_t$ from cycle to cycle so that the calculated mean pressure (obtained by integrating the pressure over the cycle) eventually agrees with the real value. The iterative linear method does not require this additional calculation loop since the mean pressure is directly accounted for.

There is a further difference in the calculation methods in that the explicit integration method improves its accuracy as the number of time increments per cycle is increased. There is a practical limit to the number of time increments; ideally the number of time increments would be chosen so that the numerical solution deviates from the exact solution by an acceptable tolerance. Since the exact solution is not known; if the difference between successive number of time steps is suitably small, then it is assumed that the solution is acceptably accurate. Table C.1 lists numerical solutions for 20 and 90 time steps. It is interesting to note that
the numerical solution appears to approach the linear solution as the number of time steps increase.

<table>
<thead>
<tr>
<th>Calculated quantities</th>
<th>Numerical integration</th>
<th>Iterative linear</th>
</tr>
</thead>
<tbody>
<tr>
<td>( &lt;T_C&gt; ) [K]</td>
<td>311.710</td>
<td>307.442</td>
</tr>
<tr>
<td>( &lt;T_e&gt; ) [K]</td>
<td>746.636</td>
<td>751.666</td>
</tr>
<tr>
<td>( Q_k ) [J]</td>
<td>64.308</td>
<td>53.986</td>
</tr>
<tr>
<td>( Q_n ) [J]</td>
<td>112.841</td>
<td>117.651</td>
</tr>
<tr>
<td>( \mu ) [J]</td>
<td>67.107</td>
<td>66.559</td>
</tr>
<tr>
<td>Energy balance error [J]</td>
<td>18.574</td>
<td>2.894</td>
</tr>
<tr>
<td>Calculation time [secs]*</td>
<td>228.0</td>
<td>996.0</td>
</tr>
</tbody>
</table>

*All calculations done on a personal computer using an interpreted Basic language.

Convergence is obtained by calculating successive cycles until the difference in mean temperatures between cycles meets a defined tolerance. This may take ten or more cycles for the numerical integration method. The linear method, on the other hand, required only two cycles.

Figure C.7 indicates a slight phase displacement between the working spaces temperatures as calculated by the two methods. As the number of time steps increase the phase displacement diminishes but eventually approaches a finite value. The phase error could be partially due to round-off and truncation errors in the numerical integration technique.
**Figure C.7** Comparison of Working Space Temperatures between the Linear Analysis and Numerical Integration Method.

In Figure C.8 the cyclic pressure variations as calculated by both the Schmidt analysis and the numerical integration technique are compared to the linear analysis. All three methods can be seen to give very similar results for this particular case. Note, however, that the Schmidt pressure amplitude is slightly lower than the adiabatic analyses. This discrepancy tends to increase for higher compression ratios and can lead to significant error in the calculation of power and piston spring rate (for dynamic calculations in free-piston engines).

The numerical integration technique and the linear analysis in these examples were calculated to the same level of precision (i.e., to the same number of significant figures) and produce very similar results. The
important difference is that for comparable accuracy and precision, the
linear method is faster by two orders of magnitude. This advantage makes
it practical to do extensive analytical work on personal computers,
including optimization.

Figure C.8 Comparison of Cycle Pressures between the Linear Analysis,
Numerical Integration Method and the Schmidt Analysis
D.1 Introduction

Third order analyses make as few as possible simplifying assumptions and therefore, in theory, should most closely approach reality. The system of differential equations, which are complicated and unwieldy, are solved numerically by computer. Computation time required to solve the behaviour of a particular machine is much longer than other analytical methods, and therefore third order analyses are expensive when used indiscriminately. Also, the results of third order computation are machine specific and are not easily generalized.

Finkelstein presented the first effort that comfortably qualifies as a third order method [Fi75]. However, Finkelstein accounted for pressure drop effects in a limited form which has been shown to be inadequate [Be78]. He also ignored gas kinetic energy effects.

The first work to make a realistic attempt to account for gas momentum and kinetic energy effects was due to Urieli [Ur77]. This work was done at the University of the Witwatersrand under the supervision of Professor Railis and was presented at the 12th IECEC by Urieli et al. [Ur77]. Urieli applied his method to a hypothetical engine with constant flow area throughout the heat exchangers. The analysis was shown to be numerically well behaved and produced results which were qualitatively consistent when compared to experimental results of real engines. Urieli’s analysis
was not, however, directly extendable to engines with varying flow area. Furthermore, some heuristic arguments had been incorporated into the mathematics which had not been rigorously analysed.

Berchowitz, also while working under Professor Rallis, rederived the system equations from first principles in both the integral and differential forms [Be78]. In this way the structure of the integral equations, which are the basis for the numerical model, could be analysed in greater detail. In a critique of Urieli's work it was found that Urieli had incorrectly formulated the momentum equation and that discrepancies existed in the discretisation of the differential equations [Be78]. The numerical model was therefore reformulated to include the correct form of the momentum equation and extended to properly include the effects of flow area discontinuities. No intuitive steps were used to discretise the system differential equations. Other, smaller improvements were also made, such as accounting for the normal stress terms due to viscous effects in both the momentum and energy equations. The results of the model were tested against experiment with encouraging results [Be78, BR78]. Presentation of this work was made in Israel at the Joint Gas Turbine Congress (1979) and published in the Journal of Engineering for Power [BU80].

To date the basic model developed at the University of the Witwatersrand has never been improved. Better methods for integrating the system equations have, however, been employed by various workers thus improving accuracy and computation time. Recently, flow distribution (particularly in the regenerator and plenums) has been shown to have significant effects
on the performance of Stirling machines [Jo86, Ge85]. These effects can only be satisfactorily handled by at least two dimensional flow modelling. No simulation of Stirling machines has been published in which such effects have been included. Other effects that can be handled by two dimensional flow modelling include the distortion of the velocity and temperature profiles due to oscillatory flow. This would go far in accounting for the viscous and heat transfer enhancements that appear to occur in Stirling machines [SS86, HD83].

Since reference [BU80] is the most up to date report on the University of the Witwatersrand method, for completeness it is reproduced in the following pages.
A Numerical Model for Stirling Cycle Machines

The further development of a model to accurately simulate the performance and detailed behavior of Stirling cycle machines is described. The transport equation set (which describe the working gas) is derived in both the so-called integral and differential forms. Only the integral equation set is solved for the simulation. The differential equation set is used to investigate the structure of individual terms in the integral equation set. This procedure allows these terms to be more accurately understood and, hence, modeled. The energy equation includes kinetic energy and dissipation terms while the momentum equation includes the effects of working gas acceleration and viscous friction. Heat leakage and longitudinal conduction in the machine walls are accounted for and due regard is taken of the working gas instantaneous properties. The Reynolds analogy is used to calculate the local heat transfer coefficients.

The cell matrix is defined in terms of the matrix overall volume (being the void volume \( V_v \), plus the volume occupied by the matrix material \( V_m \)), porosity \( \psi \), temperature \( T_m \) and specific volume \( u_m \). Heat transfer by conduction between adjacent cell matrices \( Q_m \) takes place at nodes \( i \) through matrix effective cross sectional area \( A_m \). Heat transfer by conduction \( Q_{m+1} \) takes place between the matrix and containing wall through the effective area of contact \( A_{w+m} \). The

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**Fig. 1**: Cellular model [8]

**Fig. 2**: Generalized 7th elemental cell [8]

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containing wall associated with the \( i \)th cell is defined in terms of its volume \( V_{W, i} \), specific volume \( u_{W, i} \) and temperature \( T_{W, i} \). Heat conduction between adjacent walls \( Q_{W, i} \) occurs at node \( i \) through the wall cross-sectional area \( A_{W, i} \). Heat transfer with the external environment adjacent to the wall of the \( i \)th cell, \( Q_{W, i} \), takes place through the external wall area \( A_{W, i} \). For the regenerator cell it can generally be assumed that the regenerator is externally adiabatic (i.e., \( Q_{W, i} = 0 \)). In the special cases where there is no regenerator matrix i.e., that the wall acts as the regenerator or if the cell is part of the heater or cooler, the heat transfer to the working gas \( Q_{w} \) takes place between the working gas and the cell wall.

The boundary conditions for the series connected heat exchanger cells are the compression and expansion spaces respectively.

The cylinder wall temperature \( T_{W, c} \) for the compression space cannot be accurately determined analytically and depends inter alia upon the nature of the environment, the cylinder configuration, the mechanical friction heat generated by the piston seals and the heat transfer to the working fluid. It is usually assumed that the cylinder wall is adiabatic (for machines with separate heat exchangers). In this work it is further assumed that the gas momentum in the working spaces is negligible. Mechanical work is done on the external environment by virtue of a change in volume \( (pdV/dt) \).

The Transport Equation Set

The derivation of the transport equation set is directly pertinent to the simulation technique; for this reason it is included here.

The lumped parameter approach is used in the integral equations set to account for viscous effects (friction and dissipation) and heat transfer by means of friction factors and heat transfer coefficients. The integral equations are thus a pseudo three-dimensional system with only two independent variables: time \( (t) \) and axial position \( (x) \). These equations are then applied to the cellular model.

Consider a completely arbitrary control volume \( V \) within the flowing working gas (Fig. 4).

The control volume, which is located in an Eulerian framework is represented by its specific volume \( \nu \) (or density \( \rho \)), pressure \( P \), temperature \( T \) and momentum per unit volume \( g \). The momentum per unit volume \( g \) assumes the role of a mass flux density while crossing the control surface \( A \) transporting mass, energy and momentum into and out of the control volume. Heat \( Q \) crosses the control surface and mechanical work \( W \) is done by the control volume on the environment.

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**Nomenclature**

- \( A \) = free flow area
- \( A_m \) = effective area for conduction in matrix
- \( A_{W, i} \) = effective area for conduction in wall
- \( A_{W, i} \) = wetted area of matrix
- \( A_{W, i} \) = effective area of contact between wall and matrix
- \( A_{W, i} \) = effective area of contact between wall and environment
- \( B_m \) = heat capacity of matrix
- \( B_{W, i} \) = heat capacity of wall
- \( d \) = hydraulic diameter
- \( e \) = specific total energy
- \( E \) = rate of viscous dissipation
- \( F \) = force of fluid on solid surfaces: viscous forces
- \( F_i \) = Fanning friction factor
- \( F_r \) = Reynolds friction factor
- \( g \) = mass flux density of momentum per unit volume (note that \( g_{W, i} \) is velocity)
- \( h \) = mass flux density vector
- \( k \) = thermal conductivity
- \( k_{W, i} \) = specific internal energy
- \( L \) = internal energy
- \( m \) = specific volume of working gas
- \( u \) = specific volume of working gas
- \( v \) = specific volume of working gas
- \( w \) = specific volume of working gas
- \( n \) = node specific volume (working gas)
- \( u_{n} \) = specific volume of wall
- \( V \) = void volume
- \( V \) = control volume
- \( V_w \) = wall volume
- \( W \) = work
- \( x \) = axial position
- \( \gamma \) = ratio of specific heats (working gas)
- \( \Delta \) = small increment
- \( \psi \) = porosity
- \( \rho \) = density
- \( \mu \) = dynamic viscosity
- \( \sigma \) = stress tensor

**Subscripts**

- \( i \) = inlet conditions: location at cell \( i \)
- \( e \) = exit conditions
\[
\frac{d}{dt} \int_V \rho dV = \mathcal{S}_A \rho \mathbf{g} \cdot (-N)dA \\
\frac{d}{dt} \int_V \rho \mathbf{u} dV = -\int_V \rho \mathbf{g} \cdot N dA 
\]

Equation (1) is integrated for the integral form of the continuity equation:

\[
\frac{dm}{dt} = (gA)_i - (gA)_e 
\]

where \(i\) refers to the surface through which fluid flows into the cell and \(e\) refers to the surface through which fluid exits the cell. It is important to note that the integral result automatically assumes the area average value of the local flux parameter.

If the system is considered continuous in both time and space, it is possible to derive the differential continuity equation; applying Gauss' theorem to (2):

\[
\frac{d}{dt} \int_V \rho \mathbf{u} dV = -\int_V \mathbf{g} \cdot \mathbf{v} dV 
\]

By the Leibnitz rule for differentiating a volume integral:

\[
\int_V [\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \mathbf{g}] dV = 0 
\]

Since the control volume is entirely arbitrary and the integrand is continuous, the integrand must itself be zero:

\[
\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \mathbf{g} = 0 
\]

This is the differential equation of continuity for non-steady compressible flow.

2 Momentum (Equation of Motion). A momentum balance for the control volume is:

\[
\text{rate of change of momentum} = \text{net momentum flux} + \text{net forces} \\
\text{of momentum} = \text{through the control volume} + \text{acting on the fluid in the control volume} \\
\text{within the control volume} + \text{due to convection} + \text{due to pressure stresses} \\
\frac{d}{dt} \int_V \rho \mathbf{u} \mathbf{v} dV = \mathcal{S}_A \rho (\mathbf{v} \cdot \mathbf{g}) \cdot (-N)dA \\
\quad + \mathcal{S}_A \rho (-N)dA + \mathcal{S}_A \sigma (-N)dA 
\]

where:

\[
\mathbf{g} = \text{the velocity} \\
\sigma = \text{the viscous stress tensor} 
\]

In equation (7) only the surface forces are considered. Body forces such as gravitational, centripetal, magnetic and electrical have been neglected as irrelevant to the class of problems treated here.

Equation (7) is integrated to give the integral form of the momentum equation:

\[
\frac{d}{dt} (gV) = (g^2uA)_i - (g^2uA)_e + (pA)_i - (pA)_e - F 
\]

Note that \(F\) is the force of the fluid on the solid surfaces and is made up of the sum of all viscous and pressure forces experienced on these surfaces (viscous forces are empirically determined). The differential form of the momentum equation is derived in a manner similar to that used for the continuity equation, the result is:

\[
\frac{\partial \mathbf{g}}{\partial t} + [\mathbf{v} \cdot (\mathbf{g} \mathbf{v})] + \nabla p + [\nabla \cdot \sigma] = 0 
\]

Note that this equation has three components, one for each spatial direction.

3 The Mechanical Energy Equation. By forming the scalar product of the local velocity with equation (9) and using the differential continuity equation (6) it is possible to derive the following result [5]:

\[
\frac{d}{dt} (gV)^2/2 = -g\mathbf{u} \cdot \nabla (gV)^2/2 - u(\nabla \cdot \mathbf{g}) + u(\nabla \cdot [\sigma \cdot \mathbf{g}]) \\
-\nabla p (-\nabla \cdot \mathbf{g}) - u(-\sigma : \nabla \mathbf{g}) 
\]

Equation (10) is important for understanding the inter conversions of mechanical energy; the word statement is as follows:

rate of increase in specific kinetic energy = net input of specific kinetic energy by virtue of convection through control surface

rate of specific work done by pressure forces on control volume

rate of reversible conversion to specific internal energy due to compression

Bird, Stewart and Lightfoot [6] indicate that the term \((-\sigma : \nabla \mathbf{g})\) is always positive and is generally referred to as the dissipation function. It can thus be expected that all flow systems this term represents the degradation of mechanical energy into thermal energy and that therefore no real processes are reversible. If this term is ignored then all forms of energy in (10) would be freely interconvertible.

Furthermore, Bird, Stewart and Lightfoot point out that the \(p(\nabla \cdot \mathbf{g})\) term causes appreciable temperature effects for gas undergoing sudden expansion or compression. The \((\sigma \cdot \mathbf{g})\) term on the other hand causes measurable temperature changes in flow systems where the velocity gradients are large. Both these conditions are likely to be found in Stirling cycle machines.

Note that (10) is not a new equation: it is simply a different form of the momentum equation.

4 The Energy Equation. The word statement for this equation is as follows:

rate of increase in total energy (including kinetic, internal and potential) in the control volume = rate of total energy flux entering in control surface

+ rate of internal heat generation e.g., electrical and chemical + rate of energy input by conduction through the control surface

rate at which the fluid does work on its surroundings + rate of energy input by virtue of pressure and viscous stresses (normal and shear)

It is assumed that potential energy and internal heat generation by chemical or electrical means can be ignored as being irrelevant to the class of systems treated in this work. It should be noted, however, that internal heat absorption and release would be important in Stirling cycles operating with a dissociating gas [7].

Mathematically the verbal statement becomes:

\[
\frac{d}{dt} \int_V \rho \mathbf{e} dV = \mathcal{S}_A \rho \mathbf{e} (gV) \cdot (-N)dA - \mathcal{S}_A \rho (\mathbf{v} \cdot \mathbf{g}) \cdot N dA \\
\quad - \mathcal{S}_A [\sigma \cdot \mathbf{g}] \cdot N dA + \mathcal{S}_A \mathbf{q} \cdot (-N)dA 
\]

where \(\mathbf{e} = (gV)^2/2 + u \)

It is possible to derive an integral form of the energy equation directly from (11). However, a more convenient integral form can be derived from the simplified differential energy equation. Therefore the differential energy equation is derived first.

Using Gauss' theorem and the continuity equation (6), equation (11) can be manipulated to give [8]:

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\[
\frac{1}{\rho} \cdot (\frac{\partial}{\partial t} \cdot (\mathbf{v} \cdot \nabla) + \nabla \cdot (\rho \cdot u) + (\nabla \cdot (\mathbf{v} \cdot \nabla u) = 0
\]
which is the differential form of the energy equation.

Comparing equations (12) and (10) it is noted that the first four terms in the differential energy equation are repeated in the differential mechanical energy equation. Therefore subtracting (10) from (12):

\[
\frac{1}{\rho} \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{u} \right) = p(-\nabla \cdot \mathbf{v} + \nabla \cdot \mathbf{v} - \nabla q
\]
which is sometimes referred to as the equation of thermal energy. The word statement is as follows:

rate of gain of internal energy per unit volume = rate of reversible internal energy increase per unit volume by compression.

both locally and by convention

rate of irreversible internal energy + rate of internal energy input by increase per unit volume + conduction per volume by viscous dissipation

If the equations of mechanical and thermal energy are compared (10) and (13), the only terms that are common to both equations are \(p(-\nabla \cdot \mathbf{v})\) and \((-\nabla \cdot \mathbf{v} \cdot \nabla u)\). Thus these equations are coupled through these terms and they must therefore describe the interchange of mechanical and thermal energy. The term \(p(-\nabla \cdot \mathbf{v})\) can either be positive or negative, depending on whether the working gas is expanding or contracting, hence it represents a reversible mode of interchange. On the other hand, the term \((-\nabla \cdot \mathbf{v} \cdot \nabla u)\) is always positive as previously noted and therefore represents an irreversable degradation of mechanical energy into thermal energy.

Defining specific enthalpy:

\[
h = u + pv
\]
substituting (14) into (13) and after manipulation:

\[
\frac{\partial}{\partial t} \int_V pdV + \int_{\partial V} \mathbf{h} \cdot \mathbf{n} \cdot \mathbf{dA} = \int_V \mathbf{p} \cdot (\nabla \cdot \mathbf{v}) \cdot \mathbf{dA}
\]

Since pressures are assumed to be spatially constant over each cell and discontinuous at the nodes, it is possible to derive the integral form of equation (15) by integrating directly using Gauss' theorem and Liebnitz's rule:

\[
\frac{d}{dt} \int_V pdV + \int_{\partial V} \mathbf{h} \cdot \mathbf{n} \cdot \mathbf{dA} = \int_V \mathbf{p} \cdot (\nabla \cdot \mathbf{v}) \cdot \mathbf{dA}
\]

Evaluating:

\[
\int_V \frac{\partial}{\partial t} + (hA)_{\text{e}} = (hA)_{\text{i}} - (pA\mathbf{u})_{\text{e}} + (pA\mathbf{u})_{\text{i}} = -p[(A\mathbf{u})_{\text{e}} - (A\mathbf{u})_{\text{i}}]
\]

Note that the areas may be different for each group, for example the area for heat flux is generally different to the area for enthalpy flux.

No further manipulation can be performed on the dissipation work integral, this term is evaluated from empirical considerations, and is denoted by \(E\).

In accordance with the assumptions made by Uriei [2, 8], heat conduction along the gas is neglected. The only heat transfer accounted for is that due to convection, which is evaluated from empirical considerations. The heat flow terms are thus replaced by \(Q\).

Substituting for the perfect gas definitions of internal energy and enthalpy, (17) finally becomes:

\[
Q = \frac{R}{\gamma - 1} \frac{d}{dt} \left( \frac{\rho n T}{\gamma - 1} \right) - \frac{T}{\gamma - 1} \left[ \frac{\partial}{\partial t} \right] \left( \frac{\rho n T}{\gamma - 1} \right)
\]

This is the final form of the energy equation as used in this work.

**Empirical Factors**

There are three parameters which have to be evaluated empirically, they are:

(a) \(F\): The total force of the working gas on the solid surfaces. This includes pressure and viscous forces.

(b) \(E\): The dissipation of mechanical energy over the cell volume, also known as lost work.

(c) \(Q\): The heat transferred between the solid surfaces and the gas.

At the cellular level it can be shown [6] that by replacing the pressure term \((pA)_{\text{e}}\) \(\rightarrow \frac{\partial}{\partial x} (A\mathbf{u})\) in the momentum equation by \(A\frac{\partial}{\partial x}\), where \(A\) is the smallest area of flow, it becomes a reasonable approximation to assume that \(F\) is only the viscous force on the control volume. This simplification allows the momentum equation to be more easily modelled. Another potential difficulty is the fact that friction factors only account for the shear stresses. Fortunately, it is a simple matter to include the normal stresses which are evaluated from the differential momentum equation by making the relevant Newtonian assumptions. Writing the shear stresses in terms of the so-called Reynolds friction factor as suggested by Uriei [2, 8], and evaluating the normal stresses from the differential momentum equation, the final form of \(F\) is:

\[
F = \frac{1}{3} \frac{\rho}{\mu} \left[ \left( A \frac{\partial}{\partial x} (A\mathbf{u}) \right) + \left( A \frac{\partial}{\partial x} (A\mathbf{u}) \right) \right] + 2FR \frac{\partial}{\partial x} \left( A \frac{\partial}{\partial x} \mathbf{u} \right)
\]

where

\[
Fr \Delta F / Re
\]

\[
Re A |\partial u / \partial x|
\]

There is a certain degree of uncertainty associated with the Fanning friction factor \(F\) as applied to reversing flow. The Fanning friction factor is usually experimentally determined for steady, incompressible and isothermal flow. In this work the quasi-steady selection of friction factors coupled with the inclusion of the normal stresses help to limit the effects of the steady and incompressible requirements, whilst the general turbulent nature of the working gas flow in Stirling-type machines help to reduce the requirement for isothermality [5].

Dissipation can generally be divided into two main parts. Firstly, the viscous work done by standing eddies forming in the abrupt expansions and contractions, or in any other fittings that interrupt the smooth flow of the fluid; and secondly, the viscous work done in parallel flow. This distinction is purely empirical, since on an elemental level, both forms of dissipation are the result of the same mechanism, i.e., the irreversible work done against the local normal and shear stresses acting on the fluid element.

In Stirling machines the dissipation due to the standing eddies forming in the abrupt expansion or contraction to or from the working species is important and must be included in any analysis such as this. Generally in steady flow work, a pressure loss term is included to account for this dissipation. This term is either added to or subtracted from the pressure for expansion or contraction as the case may be. This approach was used by Uriei [2, 8] and Schock [9]. Unfortunately this approach is not valid for non-steady or reversing flow situations since it is not meaningful to add or subtract a loss term to or from the pressure at every differential time increment. A better approach is to relate the pressure loss term to the rate of dissipation since this is inherent in what it is derived from [6]. This is done by working "backwards" from the empirical result for \(\Delta p\) to evaluate the rate of dissipation which then features in the thermal energy equation as part of \(E\). From these arguments the following result can be derived [5].