Having found the transformation laws of the curvature and field strength tensors we may define their covariant derivatives:

$$F_{ab}{}^{A}{}_{B;c} = F_{ab}{}^{A}{}_{B,c} + A_{c}{}^{A}{}_{D} F_{ab}{}^{D}{}_{B} - A_{c}{}^{D}{}_{B} F_{ab}{}^{A}{}_{D} \qquad (3.54)$$

$$F_{ab}{}^{k}{}_{;c} = F_{ab}{}^{k}{}_{,c} + A_{c}{}^{n} C_{n}{}^{k}{}_{m} F_{ab}{}^{m} \qquad (3.55)$$

It immediately follows they both obey the *Bianchi Identities*:

$$F_{ab}{}^{A}{}_{B;c} + F_{bc}{}^{A}{}_{B;a} + F_{ca}{}^{A}{}_{B;b} = 0 \qquad (3.56)$$

$$F_{ab}{}^{k}{}_{;c} + F_{bc}{}^{k}{}_{;a} + F_{ca}{}^{k}{}_{;b} = 0 \qquad (3.57)$$

Finally, we observe that the field strength tensor is the more fundamental since the curvature in any representation may be found from it once the generators of the representation are known. In particular, the curvature in the adjoint representation is

$$F_{ab}{}^{m}{}_{n} = F_{ab}{}^{k} C_{k}{}^{m}{}_{n} \qquad (3.58)$$

3.6 The Complete Internal System

We now have the identities in a fully covariant form and we may use them to ellucidate the nature of the theory. The usefulness of this approach stems from the fact that we do not have to know the detailed structure of the Lagrangian in order to make general statements about the theory itself. In fact we are in a position where we can compare different Lagrangians and the particular theories to which they give rise. This is especially important in the gauge theory of gravity which we shall deal with in later chapters.

The identity (3.48) stems from the global invariance of the theory and for this reason we will call it the *global invariance identity*. When the assumption of local invariance is made there arise a further two identities. Of these (3.43) is an immediate consequence of allowing the derivatives of the group parameters into the transformation laws. This is, of course, directly attributable to the assumption of local invariance and hence we shall call (3.43) the *local invariance identity*. Lastly, the identity (3.42) expresses a symmetry due essentially to the commutation of the partial derivatives of the parameters. As we have seen, in the process of writing the global identity in covariant form, (3.42) is directly responsible for the structure of the field strength tensor in the global identity. We shall therefore call (3.42) the *structure identity*.

Collecting these together we have:

The global identity:

$$\Pi_A T_k{}^A{}_m \psi^m + \Psi_A{}^a T_k{}^A{}_m \psi^m{}_{;a} + \frac{1}{2} \Upsilon^{ac}{}_n C_k{}^n{}_m F_{ac}{}^m = 0. \quad (3.59)$$

The local identity:

$$\Psi_A{}^a T_k{}^A{}_m \psi^m - \Pi^a{}_k = 0 \quad (3.60)$$

and, the structure identity:

$$\Upsilon^{ac}{}_k + \Upsilon^{ca}{}_k = 0 . \quad (3.61)$$

To these we must append the equations of motion:

$$\Psi_{A,a}{}^a - \Psi_A = 0 \quad (3.62)^a$$

$$\Upsilon^{ba}{}_{k,a} - \Upsilon^b{}_k = 0 . \quad (3.62)^b$$

Both of the concomitants associated with the field derivatives are tensors so we may define their covariant derivatives:

$$\Psi_{A;b}^{\ a} = \Psi_{A,b}^{\ a} - A_b^{\ k} T_{k\ A}^{\ m} \Psi_m^{\ a} \tag{3.63^a}$$

$$T^{ba}_{\ \ k;c} = T^{ba}_{\ \ k,c} - A_c^{\ n} C_{k\ n}^{\ m} T^{ba}_{\ \ m} . \tag{3.63^b}$$

Hence, using (3.25) and (3.39) the equations of motion may be written in covariant form:

$$\Psi_{A;a}^{\ a} - \Pi_A = 0 \tag{3.64}$$

$$T^{ba}_{\ \ k;a} - \Pi^b_{\ k} = 0 . \tag{3.65}$$

The three identities together with the equations of motion constitute our complete system.

3.7 The Current and its Conservation

It is customary to define the current as that which is coupled to the potential in the Lagrangian, i.e.

$$J^a_{\ k} = \frac{\partial L}{\partial A_a^{\ k}} = T^a_{\ k} . \tag{3.66}$$

We know, however, that $T^a_{\ k}$ is not a tensor so we will define instead

$$J^a_{\ k} \equiv \Pi^a_{\ k} \tag{3.67}$$

as the current of the system. From the local identity it follows immediately that

$$J^a_{\ k} = \Psi_A^{\ a} T_{k\ m}^{\ A} \psi^m . \tag{3.68}$$

We must emphasise that this current cannot yet be attributed solely to the matter since the Lagrangian is still extremely general and we have not specifically excluded couplings between the derivatives of the matter field and the potentials. For this reason the concomitant $\Psi_A^{\ a}$ may contain contributions from the potentials. We shall take this up again when we discuss minimal coupling. For the moment we must regard $J^a_{\ k}$ merely as the total gauge covariant current of the system.

By the equation of motion (3.65) we see that the current behaves like a source:

$$T^{ba}_{\ \ k;a} = J^b_{\ k} \quad . \tag{3.69}$$

Furthermore, using the equation of motion (3.64) together with the global identity we find:

$$J^a_{\ k;a} = -\tfrac{1}{2}T^{ac}_{\ \ n}\, C_{k\ m}^{\ n}\, F_{ac}^{\ \ m} \quad . \tag{3.70}$$

which shows that this current is not conserved without either a special choice of Lagrangian or some constraint being placed on the fields of the system. Another way to view the r.h.s. of (3.70) is to introduce the adjoint curvature (3.58):

$$J^a_{\ k;a} = +\tfrac{1}{2}\, T^{ac}_{\ \ n}\, F_{ac\ k}^{\ \ n} \tag{3.71}$$

and the conservation law now takes the form of an orthogonality condition between the concomitant $T^{ac}_{\ \ \cdot n}$ and the adjoint curvature. If these two are orthogonal then the current will be conserved.

3.8 Minimal Coupling

The problem of derivative couplings in the theory may easily be avoided by observing that the identities and the equations of motion are all linear in the Lagrangian so that it may be written as a sum of simpler parts each containing the derivative of only one field. The exact contents of these parts may be arrived at in a way we shall discuss next.

As we have seen the concomitants associated with field derivatives are tensors and so may vanish in a covariant manner. We interpret this to mean that the corresponding field derivatives may be omitted from the Lagrangian without destroying the symmetry (which is not true of the fields by themselves).

(a) The Matter Lagrangian

We omit the derivatives of the potential from the Lagrangian so that

$$T^{ac}{}_k = 0 \qquad (3.72)$$

and the field content of the Lagrangian reduces to:

$$L = L_m(\psi, \partial\psi, A) \cdot \qquad (3.73)$$

We shall call this the *matter* Lagrangian as is indicated by the subscript m. In view of (3.72) the identities which the matter Lagrangian must satisfy are (the overscript m refers to L_m):

$$\overset{m}{\Pi}_A T_k{}^A{}_B \psi^B + \overset{m}{\Psi}_A{}^a T_k{}^A{}_B \psi^B{}_{;a} = 0 \qquad (3.74)^a$$

$$\overset{m}{\Psi}_A{}^a T_k{}^A{}_B \psi^B - \overset{m}{\Pi}{}^a{}_k = 0 \qquad (3.74)^b$$

where

$$\overset{m}{\Pi}{}^a{}_k = \overset{m}{T}{}^a{}_k \quad . \qquad (3.75)$$

We are also now in a position to identify the *matter current*

$$\overset{m}{J}{}^a{}_k \equiv \overset{m}{\Pi}{}^a{}_k = \overset{m}{\Psi}_A{}^a T_k{}^A{}_B \psi^B . \qquad (3.76)$$

A short calculation shows that the Lagrangian

$$L_m' = L_m(\psi^A : \psi^A{}_{;a}) \qquad (3.77)$$

satisfies the identities provided that $L_m(\psi^A ; \psi^A_{,a})$ is globally invariant and the ordinary derivative is replaced by the covariant one.

(b) The Gauge Field Lagrangian

Since $\psi_A{}^a$ is a tensor we may omit the matter field derivatives so that it vanishes. Equation (3.24)^a then shows that ψ_A also becomes a tensor and hence we may omit the matter field altogether and so isolate the purely gauge part of the Lagrangian. The existence of such a Lagrangian shows that the gauge fields form a covariant dynamic system by themselves. Note that if, in a similar way, we had omitted both the gauge potential and its derivative in the construction of the matter Lagrangian the local identity (3.41) would demand that the matter be chargeless - the trivial case.

The Lagrangian is

$$L = L_g(A \; ; \partial A) \tag{3.78}$$

and t[he] identities it satisfies are:

$$\tfrac{1}{2} \overset{g}{T}{}^{ac}{}_n \, C_k{}^n{}_m \, F_{ac}{}^m = 0 \qquad (3.79)^a$$

$$\overset{g}{\Pi}{}^a{}_k = 0 \qquad \text{" b}$$

$$\overset{g}{T}{}^{ac}{}_n + \overset{g}{T}{}^{ca}{}_n = 0 \qquad \text{" c}$$

where

$$\overset{g}{\Pi}{}^a{}_k = \overset{g}{T}{}^a{}_k - \overset{g}{T}{}^{ac}{}_n \, C_m{}^n{}_k \, A_c{}^m \; . \tag{3.80}$$

(3.79)^a shows that the symmetry forces the $\overset{g}{T}{}^{ac}{}_n$ and the adjoint curvature to be orthogonal while (3.79)^b shows that the gauge fields do not carry a covariant charge.

(c) The Total Lagrangian

Taking advantage of the linearity of the identities we have the total Lagrangian

$$L = L_m + L_g \quad . \tag{3.81}$$

We will call a Lagrangian constructed in this way *minimally coupled* . We note that the derivatives of the various fields have been isolated in different terms of the Lagrangian.

The only field which the two parts of the Lagrangian have in common is the potential. We will therefore drop the cumbersome overscripts except on the concomitants associated with the potential.

We have

$$\Pi^a{}_k = \overset{m}{\Pi}{}^a{}_k + \overset{g}{\Pi}{}^a{}_k \tag{3.82}$$

and it follows from $(3.79)^b$ that the covariant current is carried solely by the matter

$$J^a{}_k = \overset{m}{\Pi}{}^a{}_k$$

$$= \Psi_A{}^a \, T_k{}^A{}_m \, \psi^m \quad . \tag{3.83}$$

The equations of motion read

$$\Psi^a{}_{A;a} - \Pi_A = 0 \qquad (3.84)^a$$

$$T^{ac}{}_{k;a} = -J^c{}_k \quad . \qquad (3.84)^b$$

$(3.74)^a$ and $(3.84)^a$ show that the current is covariantly conserved (which is consistent with (3.70) by $(3.79)^a$):

$$J^{a}{}_{k;a} = 0 \tag{3.85}$$

Note that if we had used the non-covariant definition of the current

$$j^{a}{}_{k} \equiv \Upsilon^{a}{}_{k} = \overset{m}{\Upsilon}{}^{a}{}_{k} + \overset{g}{\Upsilon}{}^{a}{}_{k} \tag{3.86}$$

then the local identities (3.79)^{b} and (3.74)^{b} would give

$$j^{a}{}_{k} = \Psi_{A}{}^{a}\, T_{k}{}^{A}{}_{B}\, \psi^{B} + \Upsilon^{ca}{}_{n}\, C_{k}{}^{n}{}_{m}\, A_{c}{}^{m} \tag{3.87}$$

and we find a non-covariant current contribution by the potentials. For this reason the potentials may be thought of as carrying charge, albeit non-covariant, in this general theory. This is unlike the case in electrodynamics in which the group is abelian so that its structure constants vanish and its potentials are automatically uncharged. If we use the equation of motion (3.67) we get

$$\Upsilon^{ba}{}_{k,a} = j^{b}{}_{k} \tag{3.88}$$

and, by the antisymmetry of $\Upsilon^{ba}{}_{k}$, we find that the non-covariant current is strictly conserved

$$j^{b}{}_{k,b} = 0 \quad . \tag{3.89}$$

This means, of course, that in any particular gauge the charge will be conserved but its non-covariant nature causes the amount to vary from one gauge to another.

3.9 A Particular Lagrangian

We already know that the Lagrangian

$$L_m = L_m(\psi^A : \psi^A{}_{;a})$$

will satisfy the matter invariance identities $(3.74)^{a,b}$. We have now to find a Lagrangian to satisfy the gauge field identities $(3.79)^{a,b,c}$. Clearly, there is no way to 'integrate' these uniquely since if L satisfies them then so will any function of L.

The simplest case is for an abelian group in which the structure constants vanish (the fields also lose their group indices k,m,n etc.) In this case the adjoint curvature vanishes identically and $(3.79)^a$ is satisfied. $(3.79)^b$ reduces to

$$T^a = 0 . \qquad (3.90)$$

Note that T^a is now a tensor (by (3.39)). $(3.79)^c$, the structure identity, shows that the derivatives should occur in the combination

$$F_{ac} = A_{a,c} - A_{c,a} \qquad (3.91)$$

and (3.90) can be satisfied by omitting the potentials themselves altogether. Thus L_g may be taken to depend only on F_{ac}. The simplest Lorentz scalar Lagrangian we can construct is

$$L_g = C\, F_{ac}\, F^{ac} \qquad (3.92)$$

where C is an arbitrary constant. We will take $C = \frac{1}{4}$.

It is natural to take the field strength tensor

$$F_{ac}{}^k = A_a{}^k{}_{,c} - A_c{}^k{}_{,a} - C_m{}^k{}_n\, A_a{}^m\, A_c{}^n \qquad (3.93)$$

as the generalization of (3.91). If the group is semi-simple then we may use the Killing-Cartan metric (1.12)

$$G_{mk} = C_{m\ n}^{\ p} C_{p\ k}^{\ n}$$

and take

$$L_g = \tfrac{1}{4} G_{mk} F_{ab}^{\ \ m} F_{cd}^{\ \ k} \eta^{ac} \eta^{bd}$$

$$= \tfrac{1}{4} F_{ab}^{\ \ k} F^{ab}_{\ \ k} \quad . \qquad (3.94)$$

By direct calculation we find

$$T^{ab}_{\ \ k} = F^{ab}_{\ \ k} \qquad (3.95)$$

and

$$T^{a}_{\ k} = - \quad C_{p\ k}^{\ m} F^{ea}_{\ \ m} A_e^{\ p} \quad . \qquad (3.96)$$

By definition of $F_{ab}^{\ \ k}$ the structure identity $(3.79)^c$ is satisfied. Substitution shows that the local identity $(3.79)^b$ is also satisfied. Finally,

$$T^{ab}_{\ \ m} C_{n\ p}^{\ m} F_{ef}^{\ \ n}$$

$$= \tfrac{1}{2} \eta^{ac} \eta^{bd} F_{cd}^{\ \ k} F_{ef}^{\ \ n} (G_{mk} C_{n\ p}^{\ m} + G_{mn} C_{k\ p}^{\ m})$$

by the symmetry of G_{mk} and the symmetry of

$$\eta^{ac} \eta^{bd} F_{cd}^{\ \ k} F_{ef}^{\ \ n}$$

in k and n. If we now use (1.12), the Jacobi identity and the antisymmetry of the structure constants in their lower indices, the term in parentheses vanishes identically. Hence $(3.79)^a$ is also satisfied. We have in fact shown that the field strength is orthogonal to the adjoint curvature.

The equations of motion in this case are:

$$\Psi_A{}^a{}_{;a} - \Pi_A = 0 \tag{3.97}$$

$$F^{ba}{}_{k;a} = J^b{}_k \tag{3.98}$$

where the current is

$$J^b{}_k = \Psi_A{}^b \, T_k{}^A{}_m \, \psi^m \tag{3.99}$$

and $$J^b{}_{k;b} = 0 \tag{3.100}$$

Written out in full the equ... of motion are

$$\Psi_A{}^a{}_{,a} - \Pi_A = \Psi_m{}^a \, T_k{}^m{}_A \, A_a{}^k \tag{3.101}$$

$$F^{ba}{}_{k,a} = \Psi_A{}^b \, T_k{}^A{}_m \, \psi^m + F^{ba}{}_m \, C_k{}^m{}_n \, A_a{}^n \tag{3.102}$$

which show how the fields source each other. Note that the last term of (3.102) is the non-covariant potential current which is still present even in the absence of matter showing that the potentials are self- sourced.

Finally, these equations must be suplemented by the Bianchi Identity:

$$F_{ab}{}^k{}_{;c} + F_{bc}{}^k{}_{;a} + F_{ca}{}^k{}_{;b} = 0 \, . \tag{3.103}$$

3.10 The Generalized Lorentz Condition

We note that tne gauge potentials are *massless* since L_g does not contain a mass term. In fact, the addition of a mass term of the form

$$M_{kn} \, A_a{}^k \, A_b{}^n \, \eta^{ab}$$

would destroy the gauge symmetry of the system because it is not gauge invariant. The masslessness of the potentials has the following fundamental significance. It follows from the general theory of Lorentz representations that the spin one representation of a massless field is reducible to a vector and a scalar (see, for example, Roman (1960), Ch.1) To eliminate this unwanted scalar component an additional condition must be imposed on the field which takes the form of a generalization of the Lorentz condition:

$$\partial^a A_a^{\ k} = 0 \tag{3.104}$$

and which cannot be deduced from the Lagrangian. We call (3.104) the *generalised Lorentz condition.* (3.104) is clearly not gauge covariant and its imposition severely restricts the possible gauge transformations which may be performed on the system, although not eliminating such transformations entirely. Contracting (3.104) with the generators we arrive at an equivalent condition on the gauge connections:

$$\partial^a A_{a\ B}^{\ A} = 0 \tag{3.105}$$

This can also hold in the transformed system if the transformation matrices are restricted by the condition:

$$A_{a\ D}^{\ C}\,\partial^a(D^A_{\ C}\,D^{-1D}_{\quad\ B}) - \partial^a(D^A_{\ C,a}\,D^{-1C}_{\quad\ B}) = 0 \tag{3.106}$$

which rises from the transformation law of the connections. Note that global transformations automatically satisfy this condition. Restricting ourselves to an infinitesimal transformation and using the structure relation we find that the infinitesimal parameters obey the linear wave equation:

$$\partial^a \partial_a \epsilon^m + A_a^{\ k} C_{k\ n}^{\ m} (\partial^a \epsilon^m) = 0 \tag{3.107}$$

In other words if we specify the parameters on some hypersurface then (3.107) shows how they must be propagated off this surface to the other points of the manifold in order that the transformed gauge potentials will also satisfy the Lorentz condition.

From a formal point of view the imposition of the Lorentz condition is no hinderance since it may always be imposed as a final formality once we have succeeded in constructing a local theory out of a global one .

APPENDIX 3A

The construction of the Invariance Identities of an Internal Group

To first order in the parameters we have

$$D^A{}_B = \delta^A{}_B + \epsilon^k T_k{}^A{}_B + O(2)$$

$$D^A{}_{B,a} = \epsilon^k{}_{,a} T_k{}^A{}_B + O(2)$$

from which we find that the only non-zero quantities are the following:

$$(D^A{}_B)_o = (D^{-1A}{}_B)_o = \delta^A{}_B \tag{3A.1}$$

$$\left|\frac{\partial D^A{}_B}{\partial \epsilon^k}\right|_o = -\left|\frac{\partial D^{-1A}{}_B}{\partial \epsilon^k}\right|_o = T_k{}^A{}_B \tag{3A.2}$$

$$\left|\frac{\partial D^A{}_{B,d}}{\partial \epsilon^k{}_{,b}}\right|_o = -\left|\frac{\partial D^{-1A}{}_{B,d}}{\partial \epsilon^k{}_{,b}}\right|_o = \delta^b{}_d T_k{}^A{}_B \tag{3A.3}$$

$$\left|\frac{\partial D^A{}_{B,dc}}{\partial \epsilon^k{}_{,ab}}\right|_o = \tfrac{1}{2}\{\delta^d{}_a \delta^c{}_b + \delta^c{}_a \delta^d{}_b\} T_k{}^A{}_B . \tag{3A.4}$$

Using these the transformations $(3.18)^{a,b,c,d}$ give:

$$\left|\frac{\partial \bar{\psi}^A}{\partial \epsilon^k}\right|_o = T_k{}^A{}_B \psi^B ; \quad \left|\frac{\partial \bar{\psi}^A{}_{,a}}{\partial \epsilon^k}\right|_o = T_k{}^A{}_B \psi^B{}_{,a} \tag{3A.5}$$

$$\left|\frac{\partial \bar{A}_a{}^A}{\partial \epsilon^k}\right|_o = T_k{}^A{}_C A_a{}^C{}_B - A_a{}^A{}_C T_k{}^C{}_B \tag{3A.6}$$

$$\left|\frac{\partial \bar{A}^{A}{}_{am,b}}{\partial \epsilon^{k}}\right|_{o} = A^{c}{}_{am,b}\, T_{k}{}^{A}{}_{c} - A^{A}{}_{ac,b}\, T_{k}{}^{c}{}_{m} \tag{3A.6}$$

$$\left|\frac{\partial \bar{\psi}^{A}{}_{,d}}{\partial \epsilon^{k}{}_{,b}}\right|_{o} = \delta^{b}{}_{d}\, T_{k}{}^{A}{}_{m}\, \psi^{m} \tag{3A.7}$$

$$\left|\frac{\partial \bar{A}^{A}{}_{am}}{\partial \epsilon^{k}{}_{,b}}\right|_{o} = -\,\delta^{b}{}_{a}\, T_{k}{}^{A}{}_{m} \tag{3A.8}$$

$$\left|\frac{\partial \bar{A}^{A}{}_{am,d}}{\partial \epsilon^{k}{}_{,b}}\right|_{o} = \delta^{b}{}_{d}\{T_{k}{}^{A}{}_{c}\, A^{c}{}_{am} - A^{A}{}_{ac}\, T_{k}{}^{c}{}_{m}\} \tag{3A.9}$$

$$\left|\frac{\partial \bar{A}^{A}{}_{dm,c}}{\partial \epsilon^{k}{}_{,ab}}\right|_{o} = \{\delta^{d}{}_{a}\delta^{c}{}_{b} + \delta^{c}{}_{a}\delta^{b}{}_{b}\}\, T_{k}{}^{A}{}_{m} \;. \tag{3A.10}$$

The identities follow easily on differentiating both sides of (3.17) with respect to the parameters and their derivatives and then setting all these to zero.

APPENDIX 3B

The Transformation Laws of the Concomitants

We differentiate both sides of (3.17) with respect to ψ^A and note that both $\bar{\psi}^A$ and $\bar{\psi}^A_{,a}$ depend on ψ^A to get

$$\frac{\partial \bar{L}}{\partial \bar{\psi}^B}\,\frac{\partial \bar{\psi}^B}{\partial \psi^A} + \frac{\partial \bar{L}}{\partial \bar{\psi}^B_{,a}}\,\frac{\partial \bar{\psi}^B_{,a}}{\partial \psi^A} = \frac{\partial L}{\partial \psi^A} \tag{3B.1}$$

From the transformations $(3.18)^{a,b}$ we get

$$\frac{\partial \bar{\psi}^B}{\partial \psi^A} = D^B_{\ A} \quad ; \quad \frac{\partial \bar{\psi}^B_{,a}}{\partial \psi^A} = D^B_{\ A,a} \tag{3B.2}$$

so that (3B.1) is

$$\bar{\Psi}_B\, D^B_{\ A} + \bar{\Psi}_B^{\ a}\, D^B_{\ A,a} = \Psi_A \ . \tag{3B.3}$$

Differentiate both sides of (3.17) with respect to $\psi^A_{,a}$

$$\frac{\partial \bar{L}}{\partial \bar{\psi}^B_{,b}}\,\frac{\partial \bar{\psi}^B_{,b}}{\partial \psi^A_{,a}} = \frac{\partial L}{\partial \psi^A_{,a}} \tag{3B.4}$$

and use $(3.18)^b$:

$$\frac{\partial \bar{\psi}^B_{,b}}{\partial \psi^A_{,a}} = \delta^a_{\ b}\, D^B_{\ A} \tag{3B.5}$$

so that (3B.4) becomes

$$\bar{\Psi}_B^{\ a}\, D^B_{\ A} = \Psi_A^{\ a} \ . \tag{3B.6}$$

We can invert (3B.3) and (3B.6) by invoking the inverse transformation (or by substituting (3B.6) into (3B.3)) to get finally:

$$\bar{\Psi}_A = D^{-1m}{}_A \Psi_m + D^{-1m}{}_{A,b} \Psi_m{}^b \qquad (3B.7)$$

$$\bar{\Psi}_A{}^a = D^{-1m}{}_A \Psi_m{}^a . \qquad (3B.8)$$

The transformation laws for $\Phi^a{}_A{}^m$ and $\Phi^{ab}{}_A{}^m$ follow in exactly the same way.

APPENDIX 3C

The Tensorial Concomitants

We see from the transformation laws that Ψ_A and $\Phi^{a}{}_{A}{}^{B}$ are not tensors. To construct equivalent tensors we use the transformation laws of the connections to eliminate the differentiated transformation matrices which occur in $(3.24)^{a,c}$:

$$D^{A}{}_{B,a} = D^{A}{}_{C}\, A_{a}{}^{C}{}_{B} - D^{C}{}_{B}\, \bar{A}_{a}{}^{A}{}_{C} .$$

Also $\quad D^{-1A}{}_{C}\, D^{C}{}_{B} = \delta^{A}{}_{B} \quad$ gives

$$D^{-1A}{}_{B,a} = D^{-1A}{}_{C}\, D^{-1D}{}_{B}\, D^{C}{}_{D,a} .$$

Substituting this into $(3.24)^{a}$ and collecting barred and unbarred quantities we find that

$$\Pi_{A} \equiv \Psi_{A} - \Psi_{B}{}^{b}\, A_{b}{}^{B}{}_{A}$$

transforms as a tensor. Similarly, when the process is repeated on equation $(3.24)^{c}$ we find that

$$\Pi^{a}{}_{A}{}^{B} \equiv \Phi^{a}{}_{A}{}^{B} - A_{b}{}^{C}{}_{A}\, \Phi^{ab}{}_{C}{}^{B} + A_{b}{}^{B}{}_{C}\, \Phi^{ab}{}_{A}{}^{C}$$

transforms as a tensor.

CHAPTER 4 The Physical and Geometrical Properties of the Gauge Potentials

4.1 Introduction

The gauge potentials have been introduced as Lorentz vectors and as such they are subject to the action of The Poincaré Group. We expect therefore that they should carry energy-momentum and angular momentum. In this chapter we make these properties explicit by calculating the canonical tensors in the manner of Chapter 2

Although it lies outside the scope of this work, we give a brief elementary treatment of the geometrical interpretation of the theory. It is based on the law of parallel transport derived from the covariant derivative and serves to shed some light on the nature of the gauge fields; particularly the curvature and the Bianchi Identity.

4.2 The Action of the Poincaré Group

The global Poincaré Group acts simultaneously with a local internal group so that the matter fields transform according to the direct product of the two. These representations therefore carry both internal indices A, B, ... and Lorentz indices α, β, ... and are symmetric under the interchange of the two

$$\psi^{A\alpha} = \psi^{\alpha A} \tag{4.1}$$

The Lagrangian is

$$L = L(\psi^{A\alpha} : \psi^{A\alpha}{}_{,a} : A_a{}^k : A_a{}^k{}_{,b}) \quad . \tag{4.2}$$

Since we are interested in the energy-momentum and angular momentum of the fields we will only consider the action of the global Poincare Group on these fields. This is given by

$$\bar{\psi}^{A\alpha} = D^\alpha{}_\beta \, \psi^{A\beta} \tag{4.3a}$$

$$\bar{\psi}^{A\alpha}{}_{,a} = \Lambda^{-1b}{}_a \, D^\alpha{}_\beta \, \psi^{A\beta}{}_{,b} \tag{4.3b}$$

$$\bar{A}_a{}^k = \Lambda^{-1b}{}_a \, A_b{}^k \tag{4.3c}$$

$$\bar{A}_a{}^k{}_{,c} = \Lambda^{-1b}{}_a \, \Lambda^{-1d}{}_c \, A_b{}^k{}_{,d} \tag{4.3d}$$

in which the parameters ϵ^{ab} $, \xi^a$ are constants.

We recall the equations of motion

$$\Psi_{A\alpha}{}^a{}_{;a} - \Pi_{A\alpha} = 0 \tag{4.4}$$

$$\Upsilon^{ab}{}_{k;b} - \Pi^a{}_k = 0 \quad . \tag{4.5}$$

4.3 Energy and Momentum

By the same arguments as given in Chapter 2 translational invariance demands that the Lagrangian satisfies the identity:

$$\partial_c L - \Psi_{A\alpha} \, \psi^{A\alpha}{}_{,c} - \Psi_{A\alpha}{}^a \, \psi^{A\alpha}{}_{,ac}$$
$$- \Upsilon^a{}_k \, A_a{}^k{}_{,c} - \Upsilon^{ab}{}_k \, A_a{}^k{}_{,bc} = 0 \quad . \tag{4.6}$$

This identity is not manifestly invariant under local internal transformations and we first bring it into this form. Use (3.25) and (3.39) to eliminate $\Psi_{A\alpha}$ and $\Upsilon^a{}_k$, then use the local internal identity (3.43) to eliminate the $\Pi_a{}^{k^k}$ so introduced. The term containing

$T^{ab}_{\ \ k}$ may be written:

$$- \tfrac{1}{2} T^{ab}_{\ \ k} F_{ab\ ,c}^{\ \ k}$$

$$= - T^{ab}_{\ \ k} F_{ab\ ;c}^{\ \ k} - \{\Pi_{A\alpha} T_{m\ \ m}^{\ A} \psi^{m\alpha} + \Psi_{A\alpha}^{\ \ a} T_{m\ \ m}^{\ A} \psi^{m\alpha}_{\ \ ;a}\} A_c^{\ m}$$

by definition of the covariant derivative of $F_{ab}^{\ \ k}$ and the internal global identity (3.59). Collecting terms the identity emerges as:

$$L_{;c} - \Pi_{A\alpha} \psi^{A\alpha}_{\ \ ;c} - \Psi_{A\alpha}^{\ \ a} \psi^{A\alpha}_{\ \ ;ac} - \tfrac{1}{2} T^{ab}_{\ \ k} F_{ab\ ;c}^{\ \ k} = 0 \quad (4.7)$$

now manifestly covariant.

In order to get a conservation law we introduce the commutator of two covariant derivatives, the Ricci Identity:

$$\psi^{A\alpha}_{\ \ ;ac} - \psi^{A\alpha}_{\ \ ;ca} = F_{ac}^{\ \ k} T_{k\ \ m}^{\ A} \psi^{m\alpha} . \quad (4.8)$$

Note that the Poincare index does not contribute to the covariant derivative because the group is acting globally. We also need the Bianchi Identity:

$$F_{ab\ ;c}^{\ \ k} + F_{bc\ ;a}^{\ \ k} + F_{ca\ ;b}^{\ \ k} = 0 . \quad (4.9)$$

If we now use (4.8) on the second last term of (4.7), (4.9) on the last term and the internal local identity (3.60), (4.7) becomes:

$$L_{;c} - \Pi_{A\alpha} \psi^{A\alpha}_{\ \ ;c} - \Psi_{A\alpha}^{\ \ a} \psi^{A\alpha}_{\ \ ;ca} - T^{ab}_{\ \ k} F_{cb\ ;a}^{\ \ k} = \Pi^{a}_{\ k} F_{ac}^{\ \ k} . \quad (4.10)$$

The equations of motion (4.4) and (4.5) now permit this to be written as

$$t^{a}_{\ c;a} = 0 \quad . \quad (4.11)$$

where

$$t^{a}{}_{c} \equiv L\,\delta^{a}{}_{c} - \Psi_{A\alpha}{}^{a}\,\psi^{A\alpha}{}_{;c} - T^{ab}{}_{k}\,F_{cb}{}^{k} \qquad (4.12)$$

which we will call the *total canonical energy-momentum tensor* of the system. Note that since $t^{a}{}_{c}$ is a set of internal group scalars it is, in fact, strictly conserved:

$$t^{a}{}_{c,a} = 0 \;. \qquad (4.13)$$

The above holds true for the general Lagrangian (4.2) but if we assume the minimally coupled Lagrangian

$$L = L_{m} + L_{g}$$

then we may get a detailed view of the interchange of energy and momentum between the fields. The Lagrangians L_m and L_g are separately translationally invariant,hence the identity (4.7) breaks into two parts:

(a) Matter

The invariance identity is

$$L_{m;c} - \Pi_{A\alpha}\,\psi^{A\alpha}{}_{;c} - \Psi_{A\alpha}{}^{a}\,\psi^{A\alpha}{}_{;ac} = 0 \qquad (4.14)$$

and, using the equation of motion,we find

$$\overset{m}{T}{}^{a}{}_{c,a} = J^{a}{}_{k}\,F_{ac}{}^{k} \qquad (4.15)$$

where

$$\overset{m}{T}{}^{a}{}_{c} \equiv L_{m}\,\delta^{a}{}_{c} - \Psi_{A\alpha}{}^{a}\,\psi^{A\alpha}{}_{;c} \qquad (4.16)$$

is the *gauge covariant canonical energy-momentum tensor* of matter.

(b) Gauge

The invariance identity is

$$L_{g;c} - \tfrac{1}{2} T^{ab}{}_{k} F_{ab}{}^{k}{}_{;c} = 0 \qquad (4.17)$$

together with which the equation of motion gives

$$\overset{g}{T}{}^{a}{}_{c,a} = - J^{a}{}_{k} F_{ac}{}^{k} \qquad (4.18)$$

where

$$\overset{g}{T}{}^{a}{}_{c} \equiv L_{g} \delta^{a}{}_{c} - T^{ba}{}_{k} F_{bc}{}^{k} \qquad (4.19)$$

is the *canonical energy-momentum tensor* of the gauge field.

Equation (4.15) shows that the energy-momentum of matter is not conserved but is scourced by the *generalized Lorentz force* vector

$$f_{c} \equiv J^{a}{}_{k} F_{ac}{}^{k} \qquad (4.20)$$

where $J^{a}{}_{k}$ is the covariant matter current given by (3.68). On the other hand the energy-momentum of the gauge field is not conserved either since (4.18) show. that the force exerts a 'back-reaction' on the gauge field. The total energy-momentum

$$t^{a}{}_{c} = \overset{m}{T}{}^{a}{}_{c} + \overset{g}{T}{}^{a}{}_{c} \qquad (4.21)$$

is still conserved.

Finally, if we take the special case of

$$L_{g} = \tfrac{1}{4} F_{ab}{}^{k} F^{ab}{}_{k} \qquad (4.22)$$

then

$$\overset{g}{T}{}^{a}{}_{c} = \tfrac{1}{4} F_{db}{}^{k} F^{db}{}_{k} \delta^{a}{}_{c} - F^{ba}{}_{k} F_{bc}{}^{k} \qquad (4.26)$$

which is symmetric

$$\overset{g}{T}{}^{ac} = \overset{g}{T}{}^{ca} \quad . \tag{4.27}$$

We will see in the next section that $\overset{g}{T}{}^{ac}$ must always be symmetric even if a more general Lagrangian than (4.22) is used.

4.4 Angular Momentum

Returning to the general Lagrangian (4.2), its global Lorentz invariance leads to the identity:

$$\Psi_{A\alpha} S_{de}{}^{\alpha}{}_{\beta} \psi^{A\beta} + \Psi_{A\alpha}{}^{a} \{S_{de}{}^{\alpha}{}_{\beta} \psi^{A\beta}{}_{,a} - S_{de}{}^{b}{}_{a} \psi^{A\alpha}{}_{,b}\}$$

$$- T^{a}{}_{k} S_{de}{}^{b}{}_{a} A_{b}{}^{k} - T^{ac}{}_{k} \{S_{de}{}^{b}{}_{a} A_{b}{}^{k}{}_{,c} + S_{de}{}^{b}{}_{c} A_{a}{}^{k}{}_{,b}\} = 0 \quad . \tag{4.28}$$

By a calculation similar to that of the last section we may bring this into manifestly covariant form:

$$\Pi_{A\alpha} S_{de}{}^{\alpha}{}_{\beta} \psi^{A\beta} + \Psi_{A\alpha}{}^{a} S_{de}{}^{\alpha}{}_{\beta} \psi^{A\beta}{}_{;a}$$

$$- \Psi_{A\alpha}{}^{a} S_{de}{}^{b}{}_{a} \psi^{A\alpha}{}_{;b} - T^{ac}{}_{k} S_{de}{}^{b}{}_{c} F_{ab}{}^{k} = 0 \; . \tag{4.29}$$

The equation of motion (4.4) allows this to be written as

$$(\Psi_{A\alpha}{}^{a} S_{de}{}^{\alpha}{}_{\beta} \psi^{A\beta})_{;a} - S_{de}{}^{b}{}_{a}(\Psi_{A\alpha}{}^{a} \psi^{A\alpha}{}_{;b} + T^{ac}{}_{k} F_{ab}{}^{k}) = 0 \tag{4.30}$$

which becomes, by definition of the total energy-momentum and the Lorentz generator,

$$(\Psi_{A\alpha}{}^{a} S_{de}{}^{\alpha}{}_{\beta} \psi^{A\beta})_{;a} - \tfrac{1}{2}(t_{de} - t_{de}) = 0 \; . \tag{4.31}$$

It follows that the *total canonical angular momentum* of the system

$$M^{a}{}_{de} \equiv \Psi_{A\alpha}{}^{a} S_{de}{}^{\alpha}{}_{\beta} \psi^{A\beta} - \tfrac{1}{2}(x_{d} t^{a}{}_{e} - x_{e} t^{a}{}_{d}) \tag{4.32}$$

is strictly conserved:

$$M^{a}{}_{de,a} = 0 \quad . \tag{4.33}$$

If we specialize to the minimally coupled Lagrangian

$$L = L_m + L_g$$

then the identity (4.29) breaks into two parts. We have also by (4.32) and (4.21)

$$M^{a}{}_{de} = \overset{m}{M}{}^{a}{}_{de} + \overset{g}{M}{}^{a}{}_{de} \tag{4.34}$$

where

$$\overset{m}{M}{}^{a}{}_{de} = \Psi_{A\alpha}{}^{a} S_{de}{}^{\alpha}{}_{\beta} \psi^{A\beta} - \tfrac{1}{2} (x_d \overset{m}{T}{}^{a}{}_{e} - x_e \overset{m}{T}{}^{a}{}_{d}) \tag{4.35}$$

and

$$\overset{g}{M}{}^{a}{}_{de} = - (x_d \overset{g}{T}{}^{a}{}_{e} - x_e \overset{g}{T}{}^{a}{}_{d}) \tag{4.36}$$

which are the angular momentum tensors of the matter and gauge fields respectively. Note that the gauge tensor does not contain an intrinsic spin term. This implies that the intrinsic spin of the gauge potentials have no dynamical role. In fact, it is not possible to give a gauge covariant definition of the intrinsic spin of the gauge potentials. This is, however, a somewhat involved topic which must rely on the quantum theory for a proper treatment. (See Jauch and Rohrlich (1955) for a discussion of the electrodynamic case.)

We have again two sets of conservation laws:

(a) Matter

The invariance identity is

$$\Pi_{A\alpha} S_{de}{}^{\alpha}{}_{\beta} \psi^{A\beta} + \Psi_{A\alpha}{}^{a} S_{de}{}^{\alpha}{}_{\beta} \psi^{A\beta}{}_{;a} - \Psi_{A\alpha} S_{de}{}^{b}{}_{a} \psi^{A\alpha}{}_{;b} = 0 \quad (4.37)$$

Using the equation of motion (4.4) this becomes

$$(\Psi_{A\alpha}{}^{a} S_{de}{}^{\alpha}{}_{\beta} \psi^{A\beta})_{;a} - S_{de}{}^{b}{}_{a} \overset{m}{T}{}^{a}{}_{b} = 0 . \quad (4.38)$$

If we take the covariant divergence of the matter angular momentum (4.35) and use (4.38) together with the conservation of matter energy-momentum (4.18) and the definition of the Lorentz force (4.20), we find the conservation equation of the angular momentum of matter:

$$\overset{m}{M}{}^{a}{}_{de,a} = \tfrac{1}{2}(x_e f_d - x_d f_e) . \quad (4.39)$$

We see that it is not conserved but is scourced by a 'torque' generated by the interaction. From (4.38) we see that this is also responsible for the antisymmetric part of the energy-momentum tensor.

(b) Gauge

The invariance identity is simply

$$T^{ac}{}_{k} S_{de}{}^{b}{}_{c} F_{ab}{}^{k} = 0 \quad (4.40)$$

which, by definition of the energy-momentum of the gauge field (4.19), may be written

$$S_{de}{}^{bc} \overset{g}{T}_{bc} = 0 \quad . \quad (4.41)$$

The generator is antisymmetric in b , c hence the gauge energy-momentum tensor must be symmetric in order for the theory to be globally Lorentz invariant.

Taking the divergence of (4.36) and using (4.18) we find

$$M^{g_a}_{de,a} = -\tfrac{1}{2}(x_e f_d - x_d f_e) \qquad (4.42)$$

showing that the angular momentum responds to a 'back-torque' in such a way that the total angular momentum remains conserved.

The above discussion shows clearly that the gauge and matter fields exchange energy, momentum and angular momentum with each other. This is, of course, what we would expect in a fully interacting theory. The system so far has contained only one matter field whose self-interaction is mediated by the gauge potentials. In general we encounter systems in which many diferent matter fields are present. A subset of these may be representations of the same group, for example they may be electrically charged, and hence subject to interactions with the same gauge field. These matter fields then interact indirectly with each other by way of the gauge fields and so are coupled togther.

In general we see that the gauging of a group leads directly to the existence of a force and we are lead to conclude that the forces of Nature have at the root of their existence the fact that the material fields which exist are representations of certain groups. It is presumably the task of the Experimentalist to find out which groups these are but we may also ask the very fundamental question as to why there appears to be, in Nature, more than one kind of force and many kinds of material fields. The answer to this lies in the domain of some Grand Unified Theory yet to be discovered which possibly depends only on a single group of which the groups currently found in Nature are non-trivial 'parts'. It is known, however, that it is not possible to unite an internal group and the Lorentz Group in any way other than the fairly trivial direct product of the two as we have done in this chapter (see O'Raifeartaigh (1965)).

4.5 The Geometric Interpretation

We conclude this chapter with an elementary discussion on the geometrical interpretation of the internal theory. We have formulated this theory in an abstract way and by using the covariant derivative so introduced to define the not.·n of parallel transport we can provide a direct geometrical interpretation.

4.5.1 Parallel Transport

The covar:ant derivative of the matter field is defined as

$$\psi^{A}{}_{;a} = \psi^{A}{}_{,a} + A_{a}{}^{A}{}_{B}\,\psi^{B} \,. \qquad (4.43)$$

If the field is such that its covariant derivative vanishes in some region of the manifold then we say thay the field is *covariantly constant* in that region. On the other hand, given the values of the components ψ^A at some point P and a curve C running from P to another point Q, we may *parallel transport* ψ^A from P to Q by holding the components covariantly constant along C. More precisely, the values of the transported components at Q on the given cu-ve C are given by the solution to the differential equation

$$\frac{\partial\psi^{A}}{\partial x^{a}} = - A_{a}{}^{A}{}_{B}\,\psi^{B} \qquad (4.44)$$

integrated along C with the initial values prescribed at P also on C. In integral form we write this as

$$\tilde{\psi}^{A}(Q) = \psi^{A}(P) - \int_{P}^{Q} A_{a}{}^{A}{}_{B}\,\psi^{B}\,dx^{a} \qquad (4.45)$$

It is clear that, in general, the transported components will depend on both the initial values and the curve and hence it is not possible to use the transport law to 'spread' the field out from the initial point over a finite region of the manifold in a unique way.

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4.5.2 The Curvature Tensor

The integrability condition for equation (4.44) is

$$\psi^{A}{}_{,ab} = \psi^{A}{}_{,ba} \; .$$

However, using (4.44) we find that

$$\psi^{A}{}_{,ab} - \psi^{A}{}_{,ba} = - F_{ab}{}^{A}{}_{B}\, \psi^{B} \tag{4.46}$$

and we conclude that such a field cannot exist in a region unless the curvature (field strength) vanishes in that region.

Alternatively, by noting that the gauge transformation law of the potentials is inhomogeneous,

$$\bar{A}_{a}{}^{A}{}_{B} = D^{A}{}_{C}\, A_{a}{}^{C}{}_{D}\, D^{-1D}{}_{B} - D^{A}{}_{C,a}\, D^{-1C}{}_{B} \tag{4.47}$$

we may enquire whether or not we can perform a transformation in which the transformed potentials vanish in some region. The group transformation matrices must then satisfy

$$D^{A}{}_{C,a} = D^{A}{}_{B}\, A_{a}{}^{B}{}_{C} \; . \tag{4.48}$$

The integrability condition for this equation again depends on the curvature

$$D^{A}{}_{B,ab} - D^{A}{}_{B,ba} = F_{ab}{}^{A}{}_{C}\, D^{C}{}_{B} \; . \tag{4.49}$$

Hence if the configuration of gauge potentials is such that the curvature tensor vanishes then we may simply transform these potentials away in which case the covariant derivatives reduce to ordinary derivatives. Of course, once the potentials have been transformed away, the system is restricted to global transformations only since local transformations will re-introduce them.

4.5.3 The Ricci and Bianchi Identities

If we expand the transport law (4.45) to second order and transport the field components around a closed square of side Δx^a , δx^b then we find that the change in these components is

$$\Delta\psi^A = \tilde{\psi}^A - \psi^A = F_{ab}{}^A{}_B \psi^B \Delta x^a \delta x^b \tag{4.50}$$

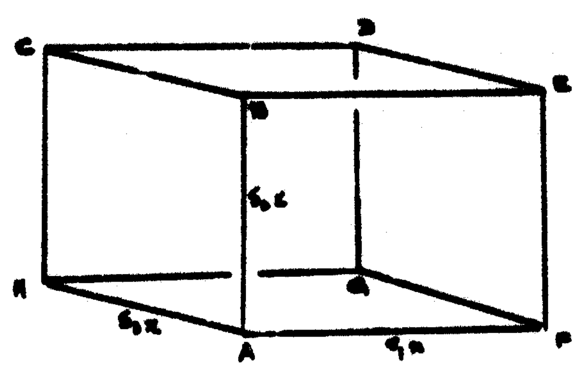
from which we conclude tnat the curvature provides a measure of the amount of 'distorsion' per unit area introduced by the gauge fields. This is essentially the content of the Ricci Identity.

In the same way we may transport ψ^A around the edges of a closed rectangular parallelepiped of sides $\delta_1 x^a$, $\delta_2 x^b$, $\delta_3 x^c$ in such a way that each edge is traversed exactly twice, once in each direction, as shown:

(A→B→C→D→E→B→A→F→G→H→A)

+ (A→H→C→B→A→F→E→D→G→F→A)

+ (A→B→E→F→A→H→G→D→C→H→A)



The opposed traversals of each edge ensures that all contributions cancel in-pairs and the net result of this transport is zero.

Explicitly this is (Feynman (1976))

$$(F_{ab}{}^{A}{}_{m;c} + F_{bc}{}^{A}{}_{m;a} + F_{ca}{}^{A}{}_{m;b})\ \psi^{m}\ \delta_1 x^a\ \delta_2 x^b\ \delta_3 x^c = 0 \qquad (4.51)$$

and since the δx's and the ψ^{m} are all independent we conclude that

$$F_{ab}{}^{A}{}_{m;c} + F_{bc}{}^{A}{}_{m;} + F_{ca}{}^{A}{}_{m;b} = 0 \qquad (4.52)$$

which is the Bianchi Identity. This simple argument shows that the Bianchi Identity imposes a fundamental geometric constraint which the curvature must satisfy. It is apparant that the identity represents a conservation condition and hence its use in the construction of conservation laws is clarified.

We have given the most rudimentary treatment of the geometrical aspects of gauge theory but there exists in the literature very sophisticated formalisms designed to exploit these aspects to the full. The most important of these are the Calculus of Forms (see, for example, von Westenholtz (1978)) and Fibre Bundle Theory (Trautman (1980)). Once again the reader is referred to an extensive literature.

PART III: Gravity as a Gauge Theory

CHAPTER 5 The Gauge Fields of the Poincaré Group

5.1 Introduction

In the previous chapters we have limited the theory to being globally invariant under the action of the Poincaré Group. We will now relax this restriction and show how gravity may be treated as a gauge theory in a manner conceptually similar to the internal theory we have discussed so far.

The Poincaré Group is the semi-direct product of two groups: the Translation and the Lorentz, a fact which complicates its implementation as a gauge group. Throughout the following we will find a remarkable symmetry between the objects associated with these two groups and, because of this duplication, there is greater freedom of choice in the Lagrangian. This is, however, the subject matter of later chapters.

The present chapter serves to introduce the fields with which we shall concern ourselves and the way in which the Poincaré Group is to be implemented. The Translation Group, in particular, differs somewhat in its action from the Lorentz Group. For this reason we will at first discuss the gauging of translations separately from that of Lorentz transformations. We will find that their gauging results in general co-ordinate transformations and that their potentials correspond to fields of orthonormal vectors called *tetrads*. Within this scheme the tetrads are unique only up to a Lorentz transformation on an 'internal' index and may be used to define representations of the Lorentz Group. We are then in a position to implement the full Poincaré Group which we do in Chapter 6.

Since we are dealing with general co-ordinate transformations we formulate the theory on a four-dimensional manifold on which is defined a non-singular, symmetric metric and an affine connection. We also suppose that the manifold has non-zero torsion curvature and it is a primary task of the theory to relate these quantities to the gauge fields of the group.

5.2 The Base Manifold

The theory is formulated on a four-dimensional differentiable manifold which we will refer to as the *Base Manifold*.

We will assign to it the co-ordinates

$$x^{\mu} , \quad \mu = 0,1,2,3$$

where the local co-ordinate indices will be denoted by Greek letters other than $\alpha , \beta , \gamma , \delta , \epsilon$ which are reserved for Lorentz representations.

We may define tensors locally on the manifold by their transformation properties under general co-ordinate transformations. For example, under the co-ordinate transformation:

$$x^{\mu} \rightarrow \bar{x}^{\mu} = \bar{x}^{\mu}(x^{\nu}) , \tag{5.1}$$

a second rank tensor transforms as

$$\bar{T}^{\mu}{}_{\nu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\lambda}} \frac{\partial x^{\rho}}{\partial \bar{x}^{\nu}} T^{\lambda}{}_{\rho} \tag{5.2}$$

and a scalar as

$$\bar{S} = S \tag{5.3}$$

We assume that the transformation (5.1) is such that the *Jacobian*

$$J \equiv \det\left(\frac{\partial x^{\nu}}{\partial x^{\mu}}\right) \tag{5.4}$$

is finite and non-zero.

The manifold carries a non-singular, symmetric *metric* :

$$g_{\mu\nu}(x) = g_{\nu\mu} \tag{5.5}$$

$$\det\,(g_{\mu\nu}) \neq 0 \tag{5.6}$$

and its inverse defined by

$$g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}{}_{\rho} \quad . \tag{5.7}$$

We will use this metric to raise and lower the manifold indices. As yet it has no particular signature but this will be determined later on.

We shall also suppose that the manifold is affinely connected in that it carries the *manifold connection*:

$$\Gamma_{\mu}{}^{\rho}{}_{\nu}$$

which may be used to define covariant derivatives of Base tensors, for example,

$$T^{\mu}{}_{\nu;\rho} \equiv T^{\mu}{}_{\nu,\rho} + \Gamma_{\rho}{}^{\mu}{}_{\lambda} T^{\lambda}{}_{\nu} - \Gamma_{\rho}{}^{\lambda}{}_{\nu} T^{\mu}{}_{\lambda} \quad . \tag{5.8}$$

To ensure that this is a tensorial quantity the connection must transform as:

$$\bar{\Gamma}_{\mu\ \nu}^{\ \lambda} = J_{\mu}^{\sigma} J_{\nu}^{\tau} K_{\rho}^{\lambda} \Gamma_{\tau\ \sigma}^{\ \rho} + K_{\rho}^{\lambda} J_{\mu\ \nu}^{\ \rho} \tag{5.9}$$

where we have used the notation:

$$J_{\mu}^{\sigma} = \frac{\partial x^{\sigma}}{\partial \bar{x}^{\mu}} \quad ; \quad K_{\rho}^{\lambda} = \frac{\partial \bar{x}^{\lambda}}{\partial x^{\rho}} \tag{5.10}$$

and

$$J_{\mu\ \nu}^{\rho} = \frac{\partial^2 x^{\rho}}{\partial \bar{x}^{\mu}\, \partial \bar{x}^{\nu}} \quad . \tag{5.11}$$

Note that

$$J_{\sigma}^{\rho} K_{\rho}^{\mu} = \delta_{\sigma}^{\mu} \quad . \tag{5.12}$$

From the covariant derivative of the metric,

$$g_{\mu\nu;\rho} = g_{\mu\nu,\rho} - \Gamma_{\rho\ \mu}^{\ \lambda} g_{\lambda\nu} - \Gamma_{\rho\ \nu}^{\ \lambda} g_{\mu\lambda} \ , \tag{5.13}$$

we deduce by an index permutation:

$$\Gamma_{\rho\ \nu}^{\ \sigma} = \left\{{}_{\rho}{}^{\sigma}{}_{\nu}\right\} - \left[{}_{\rho}{}^{\sigma}{}_{\nu}\right] - Q_{\nu\ \rho}^{\ \sigma} \tag{5.14}$$

where

$$\left\{{}_{\rho}{}^{\sigma}{}_{\nu}\right\} \equiv \tfrac{1}{2} g^{\sigma\mu} \left(g_{\mu\nu,\rho} + g_{\mu\rho,\nu} - g_{\nu\rho,\mu}\right) \tag{5.15}$$

is the *Christoffel symbol* and is not a tensor,

$$\left[{}_{\rho}{}^{\sigma}{}_{\nu}\right] \equiv \tfrac{1}{2} g^{\sigma\mu} \left(g_{\mu\nu;\rho} + g_{\mu\rho;\nu} - g_{\nu\rho;\mu}\right) \tag{5.16}$$

is the *metricity symbol* which is a tensor and

$$Q_{\nu\ \rho}^{\ \sigma} \equiv \tfrac{1}{2} \left(S_{\nu\ \rho}^{\ \sigma} + S_{\rho\nu}^{\ \ \sigma} + S_{\nu\rho}^{\ \ \sigma}\right) \tag{5.17}$$

is the *contorsion* defined in terms of

the *torsion*

$$S_{\nu\ \rho}^{\ \sigma} \equiv \Gamma_{\nu\ \rho}^{\ \sigma} - \Gamma_{\rho\ \nu}^{\ \sigma} \qquad (5.18)$$

both of which are tensorial. We will not immediately demand that the connection be metric, i.e. that $g_{\mu\nu;\rho}$ vanishes but we shall impose this condition later on.

Finally, the commutator of two covariant derivatives gives the Ricci Identity:

$$T^{\mu}_{\ ;\nu\rho} - T^{\mu}_{\ ;\rho\nu} = R_{\nu\rho\ \lambda}^{\ \ \mu} T^{\lambda} + S_{\nu\ \rho}^{\ \tau} T^{\mu}_{\ ;\tau} \qquad (5.19)$$

where

$$R_{\nu\rho\ \lambda}^{\ \ \mu} \equiv \Gamma_{\nu\ \lambda,\rho}^{\ \mu} - \Gamma_{\rho\ \lambda,\mu}^{\ \mu} - [\Gamma_{\nu\ \tau}^{\ \mu} \Gamma_{\rho\ \lambda}^{\ \tau} - \Gamma_{\rho\ \tau}^{\ \mu} \Gamma_{\nu\ \lambda}^{\ \tau}] \qquad (5.20)$$

is the *manifold curvature*. Together with the torsion it obeys the First and Second Bianchi Identities:

$$R_{\mu\lambda\ \sigma;\rho}^{\ \ \tau} + R_{\lambda\rho\ \sigma;\mu}^{\ \ \tau} + R_{\rho\mu\ \sigma;\lambda}^{\ \ \tau}$$

$$+ S_{\mu\ \lambda}^{\ \eta} R_{\eta\rho\ \sigma}^{\ \ \tau} + S_{\lambda\ \rho}^{\ \eta} R_{\eta\mu\ \sigma}^{\ \ \tau} + S_{\rho\ \mu}^{\ \eta} R_{\eta\lambda\ \sigma}^{\ \ \tau} = 0 \qquad (5.21)$$

$$S_{\mu\ \lambda;\rho}^{\ \tau} + S_{\lambda\ \rho;\mu}^{\ \tau} + S_{\rho\ \mu;\lambda}^{\ \tau}$$

$$+ S_{\mu\ \lambda}^{\ \eta} S_{\rho\ \eta}^{\ \tau} + S_{\lambda\ \rho}^{\ \eta} S_{\mu\ \eta}^{\ \tau} + S_{\rho\ \mu}^{\ \eta} S_{\lambda\ \eta}^{\ \tau}$$

$$+ R_{\mu\lambda\ \rho}^{\ \ \tau} + R_{\lambda\rho\ \mu}^{\ \ \tau} + R_{\rho\mu\ \lambda}^{\ \ \tau} = 0 \qquad (5.22)$$

both of which may be proved from the definitions.

5.3 Translational Invariance

5.3.1 General Co-ordinate Transformations as Local Gauge Translations

Consider a transformation of co-ordinates:

$$x^\mu \rightarrow \bar{x}^\mu$$

then, at any point P of the manifold, we may define the four quantities

$$\xi^\mu(P) \equiv \bar{x}^\mu(P) - x^\mu(P) \tag{5.23}$$

and the co-ordinate transformation may be viewed as the generalized translation

$$\bar{x}^\mu(P) = x^\mu(P) + \xi^\mu(P) \;. \tag{5.24}$$

The ξ^μ thus serve as parameters which keep track of the change in the numerical values of the co-ordinates in going from one co-ordinate system to another. Since the transformations are arbitrary but differentiable, the ξ^μ will vary from point to point on the manifold in a differentiable manner and we may define the *displacement field* $\xi^\mu(x^\nu)$ and its derivatives.

We have immediately, from (5.24) that

$$K^\mu_\nu = \delta^\mu_\nu + \xi^\mu{}_{,\nu} \tag{5.25}$$

and the corresponding J^μ_ν may be found by a series expansion in $\xi^\mu{}_{,\nu}$.

We may express the co-ordinate transformation as a gauge transformation with the parameters ξ^μ similar to that of the usual group transformations by observing that

$$\xi^\mu = \xi^\nu \delta^\mu_\nu = \xi^\nu \partial_\nu x^\mu$$

and

$$\frac{\partial^n x^\mu}{\partial x^{\nu_1} \ldots \partial x^{\nu_n}} = 0 \quad n > 1$$

hence, formally, we may write

$$\begin{aligned} \bar{x}^\mu &= x^\mu + \xi^\mu \\ &= \\ &= (1 + \xi^\nu \partial_\nu) \, x^\mu \\ &= \exp(\xi^\nu \partial_\nu) \, x^\mu \end{aligned}$$

So that these transformations may be viewed as gauge transformations with generators ∂_ν and parameters ξ^ν. Note that these are not infinitesimal transformations.

5.3.2 <u>The Action</u>

Introduce a scalar field $\varphi(x^\nu)$ on the manifold. It transforms as

$$\bar{\varphi} = \varphi \tag{5.26}$$

and its derivative as

$$\bar{\varphi}_{,\mu} = J_\mu^\nu \, \varphi_{,\nu} \, . \tag{5.27}$$

Define the action:

$$I = \int \mathcal{L}(\varphi : \varphi_{,\nu}) \, d^4x \tag{5.28}$$

where the Lagrangian $\mathcal{L}$ must transform as a scalar density:

$$\bar{\mathcal{L}} = J \, \mathcal{L} \tag{5.29}$$

to ensure that the action itself is an invariant scalar.

Instead of working with $\mathcal{L}$ directly we will assume that it may be factorized into the product of a suitable scalar density e (which we shall specify later) and a purely scalar Lagrangian L, i.e.

$$\mathcal{L} = e\, L \tag{5.30}$$

whose transformations are

$$\bar{L}(\bar{\varphi} : \bar{\varphi}_{,\nu}) = L(\varphi : \varphi_{,\nu}) \tag{5.31}$$

and

$$\bar{e} = J\, e\,. \tag{5.32}$$

5.3.3 General Co-ordinate Invariance - Introduction of the Tetrad

By not including the co-ordinates explicitly in the Lagrangian we have ensured that the theory is invariant under those transformations for which ξ^{μ} is constant (as in Chapter 2).

When the ξ^{μ} are not constants, the J^{μ}_{ν} in the transformation of the derivative introduces $\xi^{\mu}_{,\nu}$ into the transformations (see (5.27)). The r.h.s. of (5.31) is independent of all parameteric quantities so differentiating both sides with respect to $\xi^{\nu}_{,\rho}$ and setting all parametric quantities to zero we get:

$$\Phi \left(\frac{\partial \bar{\varphi}}{\partial \xi^{\nu}_{,\rho}}\right)_{0} + \Phi^{\mu} \left(\frac{\partial \bar{\varphi}_{,\mu}}{\partial \xi^{\nu}_{,\rho}}\right)_{0} = 0 \tag{5.33}$$

where the concomitants are defined to be

$$\Phi \equiv \frac{\partial L}{\partial \varphi} \quad \text{and} \quad \Phi^{\mu} \equiv \frac{\partial L}{\partial \varphi_{,\mu}}\,. \qquad (5.34\text{-}35)$$

From (5.26) $\bar{\varphi}$ is independent of $\xi^{\nu}{}_{,\rho}$ while (5.27) gives

$$\left(\frac{\partial\bar{\varphi}_{,\mu}}{\partial\xi^{\nu}{}_{,\rho}}\right)_{o} = \left(\frac{\partial J^{\sigma}_{\mu}}{\partial\xi^{\nu}{}_{,\rho}}\right)_{o} \varphi_{,\sigma}$$

$$= -\delta^{\rho}{}_{\mu}\,\varphi_{,\nu} \qquad (5.36)$$

where we have used (5.12) and (5.25) to get

$$\left(\frac{\partial J^{\sigma}_{\mu}}{\partial\xi^{\nu}{}_{,\rho}}\right)_{o} = -\,\delta^{\sigma}{}_{\nu}\,\delta^{\rho}{}_{\mu} \quad .$$

The identity (5.33) therefore reduces to

$$\Phi^{\cdot\rho}\varphi_{,\nu} = 0$$

or, since the derivative of φ is not zero everywhere

$$\Phi^{\rho} = 0 \qquad (5.37)$$

This may be satisfied by simply omitting the derivatives of φ from the Lagrangian. Once again, as in the internal theory, this is the trivial solution to the problem.

As before, if we want more interesting solutions we must replace the derivatives by covariant equivalents by introducing a set of vector fields as gauge potentials. We are dealing with a four-parameter group and so we need four vector fields which we take as the contrvariant vectors

$$e_{a}{}^{\mu}(x) \qquad a = 0,1,2,3$$

and we define the *translational covariant derivative* as

$$\varphi_{,a} \equiv e_{a}{}^{\mu}\varphi_{,\mu} \quad . \qquad (5.38)$$

Being contravariant vectors the potentials transform as

$$\bar{e}_a{}^{\mu} = K^{\mu}_{\nu}\ e_a{}^{\nu} \tag{5.39}$$

and the $\varphi_{,a}$ transform as a set of four Base scalars and not as the components of a single vector.

We must, of course, incorporate these potentials into a new Lagrangian

$$L(\varphi : \varphi_{,\mu} : e_a{}^{\nu}) \quad . \tag{5.40}$$

If we now differentiate the transformation law of this Lagrangian with respect to $\xi^{\nu}_{,\rho}$ we get the identity

$$\Sigma^{a}{}_{\nu}\, e_a{}^{\rho} - \Phi^{\rho}\varphi_{,\nu} = 0 \tag{5.41}$$

where

$$\Sigma^{a}{}_{\nu} \equiv \frac{\partial L}{\partial e_a{}^{\nu}} \tag{5.42}$$

and the appearence of the extra term avoids the conclusion (5.37).

We therefore have a viable theory in which the $e_a{}^{\mu}$ are a prescribed set of vector fields whose transformations are sufficient to ensure that the theory as a whole is gauge covariant. We may next include the derivatives of these potentials in the Lagrangian to develop a dynamic theory but it is far from being complete and will later emerge as a special case of a more general theory. Before continuing we will discuss some properties of these potentials.

(a) <u>Orthonormality</u>

If we take the determinant of both sides of (5.39) we get

$$\det(\bar{e}_a{}^{\mu}) = \det(K^{\mu}_{\nu}) \det(e_a{}^{\nu}) \tag{5.43}$$

then, since

$$\det(K^{\mu}_{\nu}) = \frac{1}{J} \neq 0$$

the property that

$$\det(e_a{}^{\mu}) \neq 0 \tag{5.44}$$

is gauge covariant. The efficacy of $e_a{}^{\mu}$ as a gauge potential lies purely in its transformation law (5.39) so that we may restrict the $e_a{}^{\mu}$ so that (5.44) is always satisfied. This being so we may define their inverses $e^a{}_{\mu}$ by

$$e_a{}^{\mu} e^a{}_{\nu} = \delta^{\mu}{}_{\nu} \tag{5.45}$$

Also, the transformation law (5.39) is scale invariant and we may normalize the potentials:

$$e_a{}^{\mu} e^b{}_{\mu} = \delta^b{}_a \tag{5.46}$$

The inverses must then transform as covariant vectors:

$$\bar{e}^b{}_{\mu} = J^{\nu}_{\mu} e^b{}_{\nu} \quad . \tag{5.47}$$

We conclude therefore that, at each point of the Base manifold, the $e_a{}^{\mu}$ forms an orthonormal set of four four-vectors which we will call a *tetrad*.

Finally, from (5.47) we find

$$\det(\bar{e}^b{}_{\mu}) = J \det(e^b{}_{\mu}) \tag{5.48}$$

indicating that $\det(e^b{}_{\mu})$ transforms as a scalar density. We can therefore take

$$e \equiv \det(e^b{}_\mu) \tag{5.49}$$

for the purpose of constructing a Lagrangian density in (5.30).

(b) The Metric

Introduce the numerical Minkowski matrix:

$$\{\eta^{ab}\} = \{\eta_{ab}\} = \text{diag}(-1,1,1,1) \tag{5.50}$$

and fix the manifold metric by taking

$$g_{\mu\nu} \equiv e^a{}_\mu \, e^b{}_\nu \, \eta_{ab} = g_{\nu\mu} \tag{5.51}$$

which is clearly a second rank Base tensor. Also

$$g^{\mu\nu} = e_a{}^\mu \, e_b{}^\nu \, \eta^{ab} \tag{5.52}$$

then, since $\eta^{ab}\,\eta_{ac} = \delta^a{}_c$ (5.53)

we have

$$g^{\mu\nu}\,g_{\nu\rho} = \delta^\mu{}_\rho \, . \tag{5.54}$$

By (5.51)

$$\begin{aligned} g &\equiv \det(g_{\mu\nu}) \\ &= \det(e^a{}_\mu) \det(e^b{}_\nu) \det(\eta_{ab}) \\ &= -e^2 \neq 0 \end{aligned} \tag{5.55}$$

so that $g^{\mu\nu}$ is non-singular and symmetric as is required of the manifold metric. Note that the particular form of the Minkowski matrix is extremely important since this imparts a Lorentzian structure to the manifold. In fact it is only once this form has been imposed that we may refer to the manifold as *space-time*. Based on this form the metric is not positive-definite but we may introduce the following (covariant) conventions:

A Base vector A^μ is said to be

$$\left.\begin{array}{l} \textit{Space-like} \\ \textit{Time-like} \\ \textit{Null} \end{array}\right\} \text{ if } g_{\mu\nu} A^\mu A^\nu \text{ is } \left\{\begin{array}{l} > 0 \\ < 0 \\ = 0 \end{array}\right. \tag{5.56}$$

We see immediately that $e_0{}^\mu$ is time-like while $e_1{}^\mu, e_2{}^\mu, e_3{}^\mu$ are all space-like.

Finally, since

$$\eta^{ac} g_{\mu\nu} e_c{}^\mu = e^a{}_\nu \tag{5.57}$$

we will use η^{ac} and $g_{\mu\nu}$ to raise and lower their respective indices.

(c) Tetrad Co-ordinates

The definition of the metric (5.51) is reminiscent of a co-ordinate transformation and we explore this by attempting to define a new set of Base co-ordinates y^a $a = 0,1,2,3$ by

$$\frac{\partial y^a}{\partial x^\mu} = e^a{}_\mu(x) \tag{5.58}$$

and taking the origin at the point P having the co-ordinates x_P^μ in the original system. If such a system could be found then

$$K^a_\mu = \frac{\partial y^a}{\partial x^\mu} = e^a{}_\mu$$

and the metric in the new system would be

$$\bar{g}^{ab} = K^a_\mu K^b_\nu g^{\mu\nu} = \eta^{ab}$$

everywhere. In other words we could transform the manifold to be Minkowskian everywhere.

However, the integrability condition for (5.58) is

$$e^a{}_{\mu,\nu} - e^a{}_{\nu,\mu} = 0 \qquad (5.59)$$

and it would be much too restrictive to impose such a condition on the potentials. On the other hand, if we restrict the transformation (5.58) to a small enough neighbourhood of a point of the manifold then we may use the tetrad at that point to define a *local co-ordinate system* in which we can express the Base tensors as 'local' tensors

$$\bar{A}_a(x) = e_a{}^\mu(x)\, A_\mu(x) \qquad (5.60)$$

The 'local' metric at each point being η^{ab}. Clearly, if (5.59) is not satisfied then we cannot patch together these local systems into a single co-ordinate system which covers the entire manifold.

It is also clear from (5.60) that these 'local' quantities are defined in a covariant way in that their components are reduced to sets of Base scalars invariant under general co-ordinate transformations.

5.4 Local Lorentz Transformations and Representations

Returning now to the general theory, we need to be able to define arbitrary Lorentz representations on the Manifold. In particular, we need to define spinorial representations since many elementary particles are found to be best described in this way. It is not surprising that the theory readily admits such representations since we have based its geometrical structure on the Minkowski matrix.

Consider an 'internal' transformation of the tetrad components which does not affect the Base co-ordinates and leaves the metric unchanged:

$$e'^{b}{}_{\nu} = \Lambda^{b}{}_{a}\, e^{a}{}_{\nu} \tag{5.61}$$

such that

$$e'^{a}{}_{\mu}\, e'^{b}{}_{\nu}\, \eta_{ab} = e^{a}{}_{\mu}\, e^{b}{}_{\nu}\, \eta_{ab} \tag{5.62}$$

The transformation matrix must therefore satisfy

$$\eta_{ab} = \Lambda^{c}{}_{a}\, \Lambda^{d}{}_{b}\, \eta_{cd} \tag{5.63}$$

which is a necessary and sufficient condition that the matrix $\Lambda^{a}{}_{b}$ be the self-representation of the Lorentz Group. What this means, of course, is that the tetrads as we have introduced them are unique only up to a Lorentz transformation. It is this non-uniqueness, however,which provides the opportunity to gauge the Lorentz Group.

From (5.61) it follows that the Latin indices are Lorentz indices and the local quantities defined by (5.60) are Lorentz vectors whose components are individually Base scalars.

On the other hand, Base tensors derived from local Lorentz tensors

$$A_{\mu} = e^{a}{}_{\mu}\, A_{a} \tag{5.64}$$

are invariant under these Lorentz transformations even if different transformations are applied at different points of the manifold. This is true, in particular, when the $\Lambda^a_{\ b}$ depend on parameters ϵ^{cd} which are differential functions of position:

$$e'^{b}_{\ \mu} = \Lambda^{b}_{\ a}(\epsilon^{cd}(x))\, e^{a}_{\ \mu}. \qquad (5.65)$$

Note that we must restrict the transformations to be proper (Chapter 1) to ensure that

$$e' = \det(\Lambda^{b}_{\ a})\, e$$

$$= + e \quad . \qquad (5.66)$$

We now have a tetrad at each point of the Base manifold with definite Lorentz transformation properties. We can therefore include matter fields in the form of arbitrary Lorentz representations by assigning to the tetrad at each point P a set of numbers ψ^{α} which are Base scalars under transformations of the manifold co-ordinates but which transform into each other under appropriate Lorentz transformations when the tetrad at P is transformed according to (5.65)

$$\bar{\psi}^{\alpha}(P) = D^{\alpha}_{\ \beta}(\epsilon^{cd}(P))\, \psi^{\beta}(P) \ . \qquad (5.67)$$

We will call the ψ^{α} *local spin-tensors* and demand that they be differentiable functions of the manifold co-ordinates.

Under a combined co-ordinate and local Lorentz transformation we have altogether

$$\bar{x}^{\mu} = x^{\mu} + \xi^{\mu} \qquad (5.68)^a$$

$$\bar{e}_{a}^{\ \mu} = X^{\mu}_{\nu}\, \Lambda^{-1b}_{\ \ a}\, e_{b}^{\ \nu} \qquad (5.68)^b$$

$$\bar{\psi}^{\alpha} = D^{\alpha}_{\ \beta}\, \psi^{\beta} \tag{5.68^c}$$

$$\bar{\psi}^{\alpha}_{\ ,\mu} = J^{\nu}_{\mu}\, (D^{\alpha}_{\ \beta}\, \psi^{\beta})_{,\nu} \quad . \tag{5.68^d}$$

Among the representations of the Lorentz Group there occur both tensor and spinor representations (Chapter 1). The former corresponds to local tensors and we may define Base equivalents for them by 'projection' using (5.64). For the local spinors no such Base equivalent can be defined (we may define Base equivalents for contracted pairs of spinors but not for individual ones). For this reason we will consider the local representation spin-tensors to be fundamental and the Base tensors to be derived. We remark that physically observable quantities are all Lorentz tensors and therefore have Base equivalents.

5.5 The Gauge Fields of the Poincaré Group

Since the Base tensors are all invariant under local Lorentz transformations we have, in a sense, introduced these transformations as 'internal' transformations quite separately from general co-ordinate transformations. This is, however, misleading since the tetrad components carry both types of index and are therefore susceptible to both transformations. We expect that the two will mix in some inseparable way as is indicated by the Poincaré Group composition law.

We observe from (5.68) that all quantities except the field derivatives are tensors:

$$\bar{\psi}^{\alpha}_{\ ,\mu} = J^{\nu}_{\mu}\, D^{\alpha}_{\ \beta}\, \psi^{\beta}_{\ ,\nu} + J^{\nu}_{\mu}\, D^{\alpha}_{\ \beta,\nu}\, \psi^{\beta} \tag{5.69}$$

and we need to define covariant equivalents. To do this we introduce a set of *Lorentz connections* which are elements of the Lie Algebra spanned by generators of the representation to which ψ^{α} belongs and, at the same time, Base manifold vectors:

$$W_{\mu}{}^{\alpha}{}_{\beta}(x) .$$

We define the *Poincaré covariant derivative*:

$$\psi^{\alpha}{}_{;a} \equiv e_{a}{}^{\mu} \{\psi^{\alpha}{}_{,\mu} + W_{\mu}{}^{\alpha}{}_{\beta} \psi^{\beta}\} . \qquad (5.70)$$

$\psi^{\alpha}{}_{;a}$ carries indices corresponding to two different representations of the Lorentz Group and we demand that it transforms as a tensor in the direct product of these two

$$\bar{\psi}^{\alpha}{}_{;a} = \Lambda^{-1b}{}_{a} D^{\alpha}{}_{\beta} \psi^{\beta}{}_{;b} . \qquad (5.71)$$

Using (5.70) and (5.68)^{c,d} this is

$$\{J^{\nu}_{\mu} D^{\alpha}{}_{\beta,\nu} \bar{e}_{a}{}^{\mu} + D^{\gamma}{}_{\beta} e_{a}{}^{\mu} \bar{W}_{\mu}{}^{\alpha}{}_{\gamma} - \Lambda^{-1b}{}_{a} D^{\alpha}{}_{\gamma} e_{b}{}^{\nu} W_{\nu}{}^{\gamma}{}_{\beta}\} \psi^{\beta}$$

$$+ \{J^{\nu}_{\mu} D^{\alpha}{}_{\beta} \bar{e}_{a}{}^{\mu} - \Lambda^{-1b}{}_{a} D^{\alpha}{}_{\beta} e_{b}{}^{\nu}\} \psi^{\beta}{}_{,\nu} = 0 . \qquad (5.72)$$

Since the matter field and its derivative are independent we get

$$\bar{e}_{a}{}^{\mu} = K^{\mu}_{\nu} \Lambda^{-1b}{}_{a} e_{b}{}^{\nu}$$

as we expect for the tetrad, and

$$\bar{W}_{\mu}{}^{\alpha}{}_{\beta} = J^{\nu}_{\mu} \{D^{\alpha}{}_{\gamma} W_{\nu}{}^{\gamma}{}_{\delta} D^{-1\delta}{}_{\beta} - D^{\alpha}{}_{\gamma,\nu} D^{-1\gamma}{}_{\beta}\} \qquad (5.73)$$

for the connection. Note that under pure co-ordinate transformations

$$D^{\alpha}{}_{\beta} = \delta^{\alpha}{}_{\beta} ; \quad D^{\alpha}{}_{\beta,\nu} = 0$$

the connection transforms as a set of Base vectors.

We may also specialize (5.73) to the vector representation of the Lorentz Group:

$$\overline{W}_{\mu}{}^{a}{}_{b} = J^{\nu}_{\mu} \{\Lambda^{a}{}_{c} W_{\nu}{}^{c}{}_{d} \Lambda^{-1d}{}_{b} - \Lambda^{a}{}_{c,\nu} \Lambda^{-1c}{}_{b}\} \cdot \quad (5.74)$$

5.6 Connections, Curvatures and Torsions

We have, finally, all the fields we will require and we conclude this chapter by relating the Base connection to the gauge fields.

By using the transformation laws of the connections, (5.73) and (5.9), we may define a covariant derivative of 'mixed' tensors; for example, suppose we have a tensor transforming as:

$$\overline{T}^{\alpha}{}_{\mu} = J^{\lambda}_{\mu} D^{\alpha}{}_{\beta} T^{\beta}{}_{\lambda} \quad (5.75)$$

and its ordinary derivative as:

$$\overline{T}^{\alpha}{}_{\mu,\rho} = J^{\nu}_{\rho} (J^{\lambda}_{\mu} D^{\alpha}{}_{\beta} T^{\beta}{}_{\lambda})_{,\nu} \quad . \quad (5.76)$$

Now,

$$J^{\nu}_{\rho} (J^{\lambda}_{\mu})_{,\nu} = \frac{\partial x^{\nu}}{\partial \overline{x}^{\rho}} \frac{\partial}{\partial x^{\nu}}\left(\frac{\partial x^{\lambda}}{\partial \overline{x}^{\mu}}\right) = \frac{\partial}{\partial \overline{x}^{\rho}}\left(\frac{\partial x^{\lambda}}{\partial \overline{x}^{\mu}}\right) = J^{\lambda}_{\rho\mu} \quad . \quad (5.77)$$

and , using (5.9) and (5.73) in the forms:

$$D^{\alpha}{}_{\beta,\nu} = D^{\alpha}{}_{\gamma} W_{\nu}{}^{\gamma}{}_{\beta} - K^{\mu}_{\nu} D^{\gamma}{}_{\beta} \overline{W}_{\mu}{}^{\alpha}{}_{\gamma} \quad (5.78)$$

$$J^{\lambda}_{\mu\rho} = J^{\lambda}_{\nu} \overline{\Gamma}_{\rho}{}^{\nu}{}_{\mu} - J^{\nu}_{\rho} J^{\sigma}_{\mu} \Gamma_{\nu}{}^{\lambda}{}_{\sigma} \quad (5.79)$$

as well as (5.77), we may eliminate the derivatives of the transformation matrices occuring in (5.76) to find that the quantity:

$$T^{\alpha}{}_{\mu;\rho} \equiv T^{\alpha}{}_{\mu,\rho} - \Gamma_{\rho}{}^{\sigma}{}_{\mu} T^{\alpha}{}_{\sigma} + W_{\rho}{}^{\alpha}{}_{\beta} T^{\beta}{}_{\mu} \tag{5.80}$$

transforms as a tensor. As is indicated we will define this to be the covariant derivative.

We have also the *Lorentz covariant derivative*

$$\psi^{\alpha}{}_{;\mu} = \psi^{\alpha}{}_{,\mu} + W_{\mu}{}^{\alpha}{}_{\beta} \psi^{\beta} . \tag{5.81}$$

As a particular instance of the derivative (5.80) we have that of the tetrad:

$$e^{a}{}_{\mu;\rho} = e^{a}{}_{\mu,\rho} - \Gamma_{\rho}{}^{\sigma}{}_{\mu} e^{a}{}_{\sigma} + W_{\rho}{}^{a}{}_{b} e^{b}{}_{\mu} . \tag{5.82}$$

So far the manifold connection is still arbitrary but we can fix it by making it such that the tetrad is covariantly constant:

$$e^{a}{}_{\mu;\rho} = 0 . \tag{5.83}$$

Then, inverting (5.82) we have

$$\Gamma_{\rho}{}^{\lambda}{}_{\mu} = W_{\rho}{}^{a}{}_{c} e^{c}{}_{\mu} e_{a}{}^{\lambda} + e_{a}{}^{\lambda} e^{a}{}_{\mu,\rho} . \tag{5.84}$$

This has the immediate interpretation that $\Gamma_{\rho}{}^{\lambda}{}_{\mu}$ is the Base manifold equivalent of the local quantity

$$W_{a}{}^{b}{}_{c} = e_{a}{}^{\mu} W_{\mu}{}^{b}{}_{c} \tag{5.85}$$

found by transforming from local co-ordinates to manifold co-ordinates in the manner of § 5.3.3. Equation (5.83) has the further consequence that, by definition of the metric,

$$g_{\mu\nu;\rho} = 0 \tag{5.86}$$

so that the connection Γ is a *metric connection*.

We next evaluate the commutators of various covariant derivatives to introduce the curvatures. The Lorentz derivative gives

$$\psi^{\alpha}{}_{;\tau\sigma} - \psi^{\alpha}{}_{;\sigma\tau} = G_{\tau\sigma}{}^{\alpha}{}_{\beta}\,\psi^{\beta} + V_{\tau}{}^{c}{}_{\sigma}\,\psi^{\alpha}{}_{;c} \tag{5.87}$$

where

$$G_{\tau\sigma}{}^{\alpha}{}_{\beta} \equiv W_{\tau}{}^{\alpha}{}_{\beta,\sigma} - W_{\sigma}{}^{\alpha}{}_{\beta,\tau} - [W_{\tau}{}^{\alpha}{}_{\gamma}\,W_{\sigma}{}^{\gamma}{}_{\beta} - W_{\sigma}{}^{\alpha}{}_{\gamma}\,W_{\tau}{}^{\gamma}{}_{\beta}] \tag{5.88}$$

is the *gauge curvature* and

$$V_{\tau}{}^{c}{}_{\sigma} \equiv W_{\tau}{}^{c}{}_{d}\,e^{d}{}_{\sigma} - W_{\sigma}{}^{c}{}_{d}\,e^{d}{}_{\tau} + e^{c}{}_{\sigma,\tau} - e^{c}{}_{\tau,\sigma} \tag{5.89}$$

is the *gauge torsion*. For the manifold covariant derivative we recall

$$A^{\mu}{}_{;\nu\rho} - A^{\mu}{}_{;\rho\nu} = R_{\nu\rho}{}^{\mu}{}_{\lambda}\,A^{\lambda} + S_{\rho}{}^{\sigma}{}_{\nu}\,A^{\mu}{}_{;\sigma} \tag{5.90}$$

where

$$R_{\nu\rho}{}^{\mu}{}_{\lambda} = \Gamma_{\nu}{}^{\mu}{}_{\lambda,\rho} - \Gamma_{\rho}{}^{\mu}{}_{\lambda,\nu} - [\Gamma_{\nu}{}^{\mu}{}_{\sigma}\,\Gamma_{\rho}{}^{\sigma}{}_{\lambda} - \Gamma_{\rho}{}^{\mu}{}_{\sigma}\,\Gamma_{\nu}{}^{\sigma}{}_{\lambda}] \tag{5.91}$$

is the manifold curvature, and

$$S_{\nu}{}^{\sigma}{}_{\rho} = \Gamma_{\nu}{}^{\sigma}{}_{\rho} - \Gamma_{\rho}{}^{\sigma}{}_{\nu} \tag{5.92}$$

is the manifold torsion. If we substitute (5.84) into (5.91) and (5.92) we get

$$S_{\nu}{}^{\sigma}{}_{\rho} = e_{a}{}^{\sigma}\,V_{\nu}{}^{a}{}_{\rho} \tag{5.93}$$

$$R_{\nu\rho}{}^{\mu}{}_{\lambda} = e^{c}{}_{\lambda}\,e_{a}{}^{\mu}\,G_{\nu\rho}{}^{a}{}_{c} \tag{5.94}$$

where now $G_{\nu\rho}{}^{a}{}_{c}$ has been specialised to the vector representation. We observe that if we had not imposed the condition (5.83) then (5.93) and (5.94) would have contained additional terms stemming from the arbitrariness of Γ and the gauge curvature would effectively have been decoupled from the manifold curvature. Because of (5.93)

We next evaluate the commutators of various covariant derivatives to introduce the curvatures. The Lorentz derivative gives

$$\psi^{\alpha}{}_{;\tau\sigma} - \psi^{\alpha}{}_{;\sigma\tau} = G_{\tau\sigma}{}^{\alpha}{}_{\beta}\,\psi^{\beta} + V_{\tau}{}^{c}{}_{\sigma}\,\psi^{\alpha}{}_{;c} \qquad (5.87)$$

where

$$G_{\tau\sigma}{}^{\alpha}{}_{\beta} \equiv W_{\tau}{}^{\alpha}{}_{\beta,\sigma} - W_{\sigma}{}^{\alpha}{}_{\beta,\tau} - [W_{\tau}{}^{\alpha}{}_{\gamma}\,W_{\sigma}{}^{\gamma}{}_{\beta} - W_{\sigma}{}^{\alpha}{}_{\gamma}\,W_{\tau}{}^{\gamma}{}_{\beta}] \qquad (5.88)$$

is the *gauge curvature* and

$$V_{\tau}{}^{c}{}_{\sigma} \equiv W_{\tau}{}^{c}{}_{d}\,e^{d}{}_{\sigma} - W_{\sigma}{}^{c}{}_{d}\,e^{d}{}_{\tau} + e^{c}{}_{\sigma,\tau} - e^{c}{}_{\tau,\sigma} \qquad (5.89)$$

is the *gauge torsion*. For the manifold covariant derivative we recall

$$A^{\mu}{}_{;\nu\rho} - A^{\mu}{}_{;\rho\nu} = R_{\nu\rho}{}^{\mu}{}_{\lambda}\,A^{\lambda} + S_{\rho}{}^{\sigma}{}_{\nu}\,A^{\mu}{}_{;\sigma} \qquad (5.90)$$

where

$$R_{\nu\rho}{}^{\mu}{}_{\lambda} = \Gamma_{\nu}{}^{\mu}{}_{\lambda,\rho} - \Gamma_{\rho}{}^{\mu}{}_{\lambda,\nu} - [\Gamma_{\nu}{}^{\mu}{}_{\sigma}\,\Gamma_{\rho}{}^{\sigma}{}_{\lambda} - \Gamma_{\rho}{}^{\mu}{}_{\sigma}\,\Gamma_{\nu}{}^{\sigma}{}_{\lambda}] \qquad (5.91)$$

is the manifold curvature, and

$$S_{\nu}{}^{\sigma}{}_{\rho} = \Gamma_{\nu}{}^{\sigma}{}_{\rho} - \Gamma_{\rho}{}^{\sigma}{}_{\nu} \qquad (5.92)$$

is the manifold torsion. If we substitute (5.84) into (5.91) and (5.92) we get

$$S_{\nu}{}^{\sigma}{}_{\rho} = e_{a}{}^{\sigma}\,V_{\nu}{}^{a}{}_{\rho} \qquad (5.93)$$

$$R_{\nu\rho}{}^{\mu}{}_{\lambda} = e^{c}{}_{\lambda}\,e_{a}{}^{\mu}\,G_{\nu\rho}{}^{a}{}_{c} \qquad (5.94)$$

where now $G_{\nu\rho}{}^{a}{}_{c}$ has been specialised to the vector representation. We observe that if we had not imposed the condition (5.83) then (5.93) and (5.94) would have contained additional terms stemming from the arbitrariness of Γ and the gauge curvature would effectively have been decoupled from the manifold curvature. Because of (5.93) and (5.94) we shall drop the distinction between the gauge curvature and torsion and the manifold curvature and torsion denoting them by R and S. Their tensorial nature follows directly from their definitions.

The Lorentz connection was introduced as an element of the Lie Algebra hence we may expand it in terms of the generators and in so doing introduce the *Lorentz potentials* $\omega_\mu{}^{ab}(x)$:

$$W_\mu{}^\alpha{}_\beta = \omega_\mu{}^{ab} S_{ab}{}^\alpha{}_\beta \tag{5.95}$$

where

$$\omega_\mu{}^{ab} = -\omega_\mu{}^{ba} . \tag{5.96}$$

By using the structure relation

$$S_{ab}{}^\alpha{}_\gamma S_{cd}{}^\gamma{}_\beta - S_{cd}{}^\alpha{}_\gamma S_{ab}{}^\gamma{}_\beta = C_{ab}{}^{ef}{}_{cd} S_{ef}{}^\alpha{}_\beta \tag{5.97}$$

it follows that $R_{\mu\nu}{}^\alpha{}_\beta$ is also an element of the Lie Algebra:

$$R_{\mu\nu}{}^\alpha{}_\beta = R_{\mu\nu}{}^{ab} S_{ab}{}^\alpha{}_\beta \tag{5.97}$$

where

$$R_{\mu\nu}{}^{ab} = -R_{\mu\nu}{}^{ba} \tag{5.98}$$

is the *gauge field strength tensor* given by

$$R_{\mu\nu}{}^{ab} = \omega_\mu{}^{ab}{}_{,\nu} - \omega_\nu{}^{ab}{}_{,\mu} - C_{cd}{}^{ab}{}_{ef}\, \omega_\mu{}^{cd} \omega_\nu{}^{ef} . \tag{5.99}$$

It follows that the field strength tensor is actually the curvature in the vector representation. Once it is known the curvature in any other representation may be found from (5.97).

Similarly, the connection in the vector representation is

$$W_\mu{}^a{}_b = \omega_\mu{}^{cd} S_{cd}{}^a{}_b = \omega_\mu{}^a{}_b \tag{5.100}$$

since we know the generator of the vector representation explicitly.

Hence we also know the transformation law of the potential explicitly (unlike the situation in the internal theory where the potential transformation was known only as an expansion in the parameters)

$$\bar{\omega}_{\mu}{}^{a}{}_{b} = J^{\nu}_{\mu} \{\Lambda^{a}{}_{c}\, \omega_{\nu}{}^{c}{}_{d}\, \Lambda^{-1d}{}_{b} - \Lambda^{a}{}_{c,\nu}\, \Lambda^{-1c}{}_{b}\} \ . \qquad (5.101)$$

The torsion is

$$S_{\nu}{}^{\sigma}{}_{\rho} = e_{a}{}^{\sigma} [\omega_{\nu}{}^{a}{}_{b}\, e^{b}{}_{\rho} + e^{a}{}_{\rho,\nu} - (\rho \leftrightarrow \nu)] \qquad (5.102)$$

and the Bianchi Identities are:

$$R_{\mu\lambda}{}^{ab}{}_{;\rho} + R_{\lambda\rho}{}^{ab}{}_{;\mu} + R_{\rho\mu}{}^{ab}{}_{;\lambda}$$
$$+ S_{\mu}{}^{\sigma}{}_{\lambda} R_{\sigma\rho}{}^{ab} + S_{\lambda}{}^{\delta}{}_{\rho} R_{\sigma\mu}{}^{ab} + S_{\rho}{}^{\sigma}{}_{\mu} R_{\sigma\lambda}{}^{ab} = 0 \qquad (1)$$

$$S_{\mu}{}^{a}{}_{\lambda;\rho} + S_{\lambda}{}^{a}{}_{\rho;\mu} + S_{\rho}{}^{a}{}_{\mu;\lambda}$$
$$- S_{\mu}{}^{\sigma}{}_{\lambda} S_{\rho}{}^{a}{}_{\sigma} - S_{\lambda}{}^{\sigma}{}_{\rho} S_{\mu}{}^{a}{}_{\sigma} - S_{\rho}{}^{\sigma}{}_{\mu} S_{\lambda}{}^{a}{}_{\sigma}$$
$$+ R_{\mu\lambda}{}^{a}{}_{\rho} + R_{\lambda\rho}{}^{a}{}_{\mu} + R_{\rho\mu}{}^{a}{}_{\lambda} = 0 \ . \qquad (2)$$

Finally a remark on the integrability of the gauge fields. For the tetrad this is (5.59) which we write in the form

$$S_{\mu}{}^{a}{}_{\nu} - \omega_{\mu}{}^{a}{}_{b}\, e^{b}{}_{\nu} + \omega_{\nu}{}^{a}{}_{b}\, e^{b}{}_{\mu} = 0 \qquad (5.103)$$

and for the Lorentz potential it is

$$R_{\mu\nu}{}^{a}{}_{b} = 0 \qquad (5.104)$$

Thus if the curvature vanishes then we transform the ω-field away but both the curvature and the torsion must vanish if we are to be able to integrate the tetrad field.

CHAPTER 6 Gravity

6.1 Introduction

In the last chapter we introduced all the fields and their transformations which we will require to construct a theory of gravity based on the Poincaré Group. In this chapter we proceed to the construction which is again based on the invariance identities satisfied by the functional derivatives of the Lagrangian and the equations of motion.

After deriving the identities and writing them in covariant form we use the equations of motion to deduce the conservation laws associated with the general Lagrangian. We find a pair of covariant tensors which behave as sources in the field equations but which are not conserved in the general case without further assumptions.

We next define a minimally coupled Lagrangian whose structure is determined by the identities. In this Lagrangian the field derivatives are separated into different terms and hence the contributions made by the individual fields may be distinguished. It is in this form that the standard theories of gravity are expressed and we investigate a number of particular cases in Chapter 7.

6.2 The Invariance Identities

The field content of the General Lagrangian is

$$L = L(\psi:\partial\psi:e:\partial e:\omega:\partial\omega) \tag{6.1}$$

and it transforms as

$$\bar{L} = L \tag{6.2}$$

under the field transformations

$$\bar{\psi}^{\alpha} = D^{\alpha}{}_{\beta}(\epsilon^{ab})\,\psi^{\beta} \tag{6.3^a}$$

$$\bar{\psi}^{\alpha}{}_{,\mu} = J^{\nu}_{\mu}\,(D^{\alpha}{}_{\beta}\psi^{\beta})_{,\nu} \tag{6.3^b}$$

$$\bar{e}^{a}{}_{\mu} = J^{\lambda}_{\mu}\,\Lambda^{a}{}_{b}\,e^{b}{}_{\lambda} \tag{6.3^c}$$

$$\bar{e}^{a}{}_{\mu,\rho} = J^{\nu}_{\rho}\,(J^{\lambda}_{\mu}\,\Lambda^{a}{}_{b}\,e^{b}{}_{\lambda})_{,\nu} \tag{6.3^d}$$

$$\bar{\omega}_{\mu}{}^{a}{}_{b} = J^{\lambda}_{\mu}\,\{\Lambda^{a}{}_{c}\,\omega_{\lambda}{}^{c}{}_{d}\,\Lambda^{-1d}{}_{b} - \Lambda^{a}{}_{c,\lambda}\,\Lambda^{-1c}{}_{b}\} \tag{6.3^e}$$

$$\bar{\omega}_{\mu}{}^{a}{}_{b,\rho} = J^{\nu}_{\rho}\,\{J^{\lambda}_{\mu}\,(\Lambda^{a}{}_{c}\,\omega_{\lambda}{}^{c}{}_{d}\,\Lambda^{-1d}{}_{b} - \Lambda^{a}{}_{c,\lambda}\,\Lambda^{-1c}{}_{b})\}_{,\nu} \tag{6.3^f}$$

where the parameters are twice diferentiable functions of the manifold co-ordinates. We deduce the identities by diferentiating both sides of (6.2) with respert to the parameters and their derivatives in turn and setting all these quantities to zero. In the transformations (6.3) the parameters occur together with their first and second derivatives and so we expect six sets of identities which are (Appendix CA):

Translations:

Global:

$$\partial_\mu L - \Psi_\alpha \psi^\alpha_{,\mu} - \Psi_\alpha^\rho \psi^\alpha_{,\rho\mu}$$
$$- \Sigma_a^\lambda e^a_{\lambda,\mu} - \Sigma_a^{\lambda\rho} e^a_{\lambda,\rho\mu}$$
$$- \Omega^\lambda_{ab} \omega_\lambda^{ab}{}_{,\mu} - \Omega^{\lambda\rho}{}_{ab} \omega_\lambda^{ab}{}_{,\rho\mu} = 0 \qquad (6.4)^a$$

Local:

$$\Psi_\alpha^\tau \psi^\alpha_{,\lambda} + \Sigma_a^\tau e^a_\lambda + \Omega^\tau_{ab} \omega_\lambda^{ab}$$
$$+ \Sigma_a^{\mu\tau} \{e^a_{\mu,\lambda} - e^a_{\lambda,\mu}\}$$
$$+ \Omega^{\mu\tau}{}_{ab} \{\omega_\mu^{ab}{}_{,\lambda} - \omega_\lambda^{ab}{}_{,\mu}\} = 0 \qquad (6.4)^b$$

Structure:

$$\Sigma_a^{\tau\sigma} + \Sigma_a^{\sigma\tau} = 0 \qquad (6.4)^c$$

Lorentz:

Global:

$$\Psi_\alpha S_{ab}{}^\alpha{}_\beta \psi^\beta + \Psi_\alpha^\mu S_{ab}{}^\alpha{}_\beta \psi^\beta_{,\mu}$$
$$+ \Sigma_c^\mu S_{ab}{}^c{}_d e^d_\mu + \Sigma_c^{\mu\rho} S_{ab}{}^c{}_d e^d_{\mu,\rho}$$
$$+ \Omega^\mu_{cd} C_{ab}{}^{cd}{}_{ef} \omega_\mu^{ef} + \Omega^{\mu\rho}{}_{cd} C_{ab}{}^{cd}{}_{ef} \omega_\mu^{ef}{}_{,\rho} = 0 \qquad (6.4)^d$$

Local:

$$\Psi_\alpha^\lambda S_{ab}{}^\alpha{}_\beta \psi^\beta + \Sigma_c^{\mu\lambda} S_{ab}{}^c{}_d e^d_\mu$$
$$- \Omega^\lambda_{ab} + \Omega^{\mu\lambda}{}_{cd} C_{ab}{}^{cd}{}_{ef} \omega_\mu^{ef} = 0 \qquad (6.4)^e$$

<u>Structure</u>:

$$\Omega^{\lambda\tau}{}_{ab} + \Omega^{\tau\lambda}{}_{ab} = 0 \quad . \qquad (6.4)^f$$

The concomitants are defined by:

$$\Psi_\beta \equiv \frac{\partial L}{\partial \psi^\beta} \; ; \; \Psi_\alpha{}^\rho \equiv \frac{\partial L}{\partial \psi^\alpha{}_{,\rho}} \qquad (6.5)^a$$

$$\Sigma_a{}^\lambda \equiv \frac{\partial L}{\partial e^a{}_\lambda} \; ; \; \Sigma_a{}^{\lambda\rho} \equiv \frac{\partial L}{\partial e^a{}_{\lambda,\rho}} \qquad (6.5)^b$$

$$\Omega^\lambda{}_{ab} \equiv \frac{\partial L}{\partial \omega_\lambda{}^{ab}} \; ; \; \Omega^{\lambda\rho}{}_{ab} \equiv \frac{\partial L}{\partial \omega_\lambda{}^{ab}{}_{,\rho}} \qquad (6.5)^c$$

We can find the transformation laws of these quantities by differentiating both sides of (6.2) with respect to the fields themselves and inverting the results (Appendix 6B). We get:

$$\bar{\Psi}_\beta = D^{-1\alpha}{}_\beta \Psi_\alpha + D^{-1\alpha}{}_{\beta,\rho} \Psi_\alpha{}^\rho \qquad (6.6)^a$$

$$\bar{\Psi}_\beta{}^\mu = K^\mu_\rho D^{-1\alpha}{}_\beta \Psi_\alpha{}^\rho \qquad (6.6)^b$$

$$\bar{\Sigma}_b{}^\mu = K^\mu_\rho (\Lambda^{-1a}{}_b \Sigma_a{}^\rho + \Lambda^{-1c}{}_{b,\lambda} \Sigma_c{}^{\rho\lambda}) \qquad (6.6)^c$$

$$\bar{\Sigma}_b{}^{\mu\nu} = K^\mu_\rho K^\nu_\lambda \Lambda^{-1c}{}_b \Sigma_c{}^{\rho\lambda} \qquad (6.6)^d$$

$$\bar{\Omega}^\mu{}_{ab} = K^\mu_\rho (\Lambda^{-1c}{}_a \Lambda^{-1d}{}_b \Omega^\rho{}_{cd} + (\Lambda^{-1c}{}_a \Lambda^{-1d}{}_b)_{,\sigma} \Omega^{\rho\sigma}{}_{cd}) \qquad (6.6)^e$$

$$\bar{\Omega}^{\mu\lambda}{}_{ab} = K^\mu_\nu K^\lambda_\tau \exp(-\epsilon^{ef} C_{ef}{}^{cd}{}_{ab}) \Omega^{\nu\tau}{}_{cd} \; . \qquad (6.6)^f$$

Note that the concomitants associated with the field derivatives are gauge tensors while the others are not. As in the case of the internal theory we can construct equivalent tensors for these (Appendix 6C):

$$\Phi_\alpha = \Psi_\alpha - \omega_\rho{}^{ab} S_{ab}{}^{\beta}{}_{\alpha} \Psi_\beta{}^{\rho} \qquad (6.7)^a$$

$$\Pi_a{}^{\rho} = \Xi_a{}^{\rho} - \omega_\mu{}^{cd} S_{cd}{}^{b}{}_{a} \Xi_b{}^{\rho\mu} \qquad (6.7)^b$$

$$\Sigma^{\mu}{}_{ab} = \Omega^{\mu}{}_{ab} - \omega_\nu{}^{cd} C_{cd}{}^{ef}{}_{ab} \Omega^{\mu\nu}{}_{ef} . \qquad (6.7)^c$$

6.3 The Identities in Covariant Form

As they stand the identities are not in a manifestly covariant form and it is our next task to bring them into this form.

$(6.4)^{c,f}$ are already covariant and $(6.4)^e$ is easily made so by using $(6.7)^c$:

$$\Sigma^{\lambda}{}_{ab} = \Psi_\alpha{}^{\lambda} S_{ab}{}^{\alpha}{}_{\beta} \psi^{\beta} + \Xi_c{}^{\mu\lambda} S_{ab}{}^{c}{}_{d} e^{d}{}_{\mu} . \qquad (6.8)$$

By using $(6.4)^e$ and $(6.7)^b$ to eliminate $\Omega^{\tau}{}_{ab}$ and $\Xi_a{}^{\rho}$ from $(6.4)^b$ we simultaneously introduce the covariant derivative of ψ^{β}, the torsion and the curvature. $(6.4)^b$ in covariant form is therefore:

$$\Psi_\alpha{}^{\tau} \psi^{\alpha}{}_{;\lambda} + \Pi_a{}^{\tau} e^{a}{}_{\lambda} + \Xi_a{}^{\mu\tau} S_{\lambda}{}^{a}{}_{\mu} + \Omega^{\mu\tau}{}_{ab} R_{\mu\lambda}{}^{ab} = 0 . \quad (6.9)$$

We next use $(6.7)^{a,b,c}$ to introduce Σ, Φ, Π into the global Lorentz identity $(6.4)^d$. This introduces three additional terms, one for each field, which contain contracted generators. We then use the structure relation (1.41) and the Jacobi Identity to commute these and in doing so break these terms into two parts; one part of each contributing to the introduction of the covariant derivative of the matter field, the curvature and the torsion, and the other parts, having a common factor ω, being collected together. The result is:

$$\Phi_\alpha S_{ab}{}^\alpha{}_\beta \psi^\beta + \Psi_\alpha{}^\rho S_{ab}{}^\alpha{}_\beta \psi^\beta{}_{;\rho} + \Pi_c{}^\rho S_{ab}{}^c{}_d e^d{}_\rho$$

$$+ \tfrac{1}{2} \Xi_c{}^{\mu\rho} S_{ab}{}^c{}_d S_\rho{}^d{}_\mu + \tfrac{1}{2} \Omega^{\mu\rho}{}_{cd} C_{ab}{}^{cd}{}_{ef} R_{\mu\rho}{}^{ef}$$

$$+ \omega_\rho^{cd} C_{ab}{}^{ef}{}_{cd} \{ \Sigma^\rho{}_{ef} - \Psi_\alpha{}^\rho S_{ef}{}^\alpha{}_\beta \psi^\beta + \Xi_g{}^{\mu\rho} S_{ef}{}^g{}_h e^h{}_\mu \} = 0 \quad (6.10)$$

in which the non-tensorial term vanishes because of the identity (6.8). In obtaining (6.10) we also had to make use of the structure identities $(6.4)^{c,f}$ to introduce the curvature and torsion. These and the local identities are therefore necessary to the local covariance of the global identity. This is to be expected, as we pointed out in the internal theory, since the local identities embody the assumption of local invariance. A similar situation exists when the global translational identity $(6.4)^a$ is written in a locally covariant form.

Using $(6.7)^{a,b,c}$ we introduce Σ, Φ, Π into $(6.4)^a$ and then eliminate Σ by using (6.8); next subtract (6.10)(without the non-tensorial term) contracted with ω to complete the covariant derivative of ψ and also subtract (6.9) contracted with Γ. Finally we collect terms to find the covariant derivatives of the curvature and the torsion and the term $\Pi_a{}^\lambda e^a{}_{\lambda;\mu}$ (the only one containing Π) which vanishes on account of the metricity assumption (5.83). We are left with

$$L_{;\mu} - \Phi_\alpha \psi^\alpha{}_{;\mu} - \Psi_\alpha{}^\rho \psi^\alpha{}_{;\rho\mu}$$

$$- \tfrac{1}{2} \Xi_a{}^{\lambda\rho} S_\rho{}^a{}_{\lambda;\mu} - \tfrac{1}{2} \Omega^{\lambda\rho}{}_{ab} R_{\lambda\rho}{}^{ab}{}_{;\mu} = 0 \qquad (6.11)$$

where we have used the fact that L is a scalar to replace its ordinary derivative by its covariant derivative. It is interesting to note that the covariance of this identity can only be established by using all the other identities including the global Lorentz identity.

For ease of reference we collect all the identities together:

$$L_{;\mu} - \Phi_\alpha \psi^\alpha_{\ ;\mu} - \Psi_\alpha^{\ \rho} \psi^\alpha_{\ ;\rho\mu}$$

$$- \tfrac{1}{2} \Sigma_a^{\ \lambda\rho} S_\rho^{\ a}{}_{\lambda;\mu} - \tfrac{1}{2} \Omega^{\lambda\rho}{}_{ab} R_{\lambda\rho}{}^{ab}{}_{;\mu} = 0 \qquad (6.12)^a$$

$$\Psi_\alpha^{\ \rho} \psi^\alpha_{\ ;\mu} + \Pi_a^{\ \rho} e^a_{\ \mu} + \Sigma_a^{\ \lambda\rho} S_\mu^{\ a}{}_\lambda + \Omega^{\lambda\rho}{}_{ab} R_{\lambda\mu}{}^{ab} = 0 \qquad (6.12)^b$$

$$\Sigma_a^{\ \rho\lambda} + \Sigma_a^{\ \lambda\rho} = 0 \qquad (6.12)^c$$

$$\Phi_\alpha S_{ab}{}^\alpha{}_\beta \psi^\beta + \Psi_\alpha^{\ \rho} S_{ab}{}^\alpha{}_\beta \psi^\beta_{\ ;\rho} + \Pi_c^{\ \rho} S_{ab}{}^c{}_d e^d_{\ \rho}$$

$$+ \tfrac{1}{2} \Sigma_a^{\ \mu\rho} S_{ab}{}^c{}_d S_\rho^{\ d}{}_\mu + \tfrac{1}{2} \Omega^{\mu\rho}{}_{cd} C_{ab}{}^{cd}{}_{ef} R_{\mu\rho}{}^{ef} = 0 \qquad (6.12)^d$$

$$\Psi_\alpha^{\ \rho} S_{ab}{}^\alpha{}_\beta \psi^\beta + \Sigma_c^{\ \lambda\rho} S_{ab}{}^c{}_d e^d_{\ \lambda} - \Sigma^\rho{}_{ab} = 0 \qquad (6.12)^e$$

$$\Omega^{\lambda\rho}{}_{ab} + \Omega^{\rho\lambda}{}_{ab} = 0 . \qquad (6.12)^f$$

From these we can clearly see the similarity between the roles played by the torsion and curvature; we may even go so far as to interpret the torsion as the curvature associated with the translation group.

6.4 The Equations of Motion

The action is

$$I = \int e \, L \, d^4x \qquad (6.13)$$

which gives rise to the equations of motion:

$$\partial_\mu \left[\frac{\partial(eL)}{\partial \psi^\alpha{}_{,\mu}} \right] - \frac{\partial(eL)}{\partial \psi^\alpha} = 0 \qquad (6.14)^a$$

$$\partial_\mu \left[\frac{\partial(eL)}{\partial e^a{}_{\rho,\mu}} \right] - \frac{\partial(eL)}{\partial e^a{}_\rho} = 0 \qquad (6.14)^b$$

$$\partial_\mu \left[\frac{\partial(eL)}{\partial \omega^{ab}_{\rho,\mu}} \right] - \frac{\partial(eL)}{\partial \omega^{ab}_\rho} = 0 \; . \qquad (6.14)^c$$

Introducing the concomitants and using the fact that

$$\frac{\partial e}{\partial e^a{}_\rho} = e\, e_a{}^\rho \qquad (6.15)$$

in $(6.14)^b$, we may write the equations of motion in the form:

$$\partial_\mu (e\, \Psi_\alpha{}^\mu) - e\, \Psi_\alpha = 0 \qquad (6.16)^a$$

$$\partial_\mu (e\, \Sigma_a{}^{\rho\mu}) - e\, \Sigma_a{}^\rho - e\, e_a{}^\rho L = 0 \qquad (6.16)^b$$

$$\partial_\mu (e\, \Omega^{\rho\mu}{}_{ab}) - e\, \Omega^\rho{}_{ab} = 0 \; . \qquad (6.16)^c$$

We may also write these in a manifestly covariant form: Since e is a scalar density its covariant derivative is

$$e_{;\mu} = e_{,\mu} - e\, \Gamma_\nu{}^\nu{}_\mu \; . \qquad (6.17)$$

Hence

$$(e\, \Psi_\alpha{}^\mu)_{;\mu} = (e\, \Psi_\alpha{}^\mu)_{,\mu} - e\, \omega_\mu{}^{ab} S_{ab}{}^\beta{}_\alpha \Psi_\beta{}^\mu \qquad (6.18)$$

and so, using $(6.7)^a$ to introduce Φ_α , (6.18) becomes

$$(e\,\Psi_\alpha{}^\mu)_{;\mu} - \Phi_\alpha = 0 \;. \qquad (6.19)$$

We have also

$$(e\,\Xi_a{}^{\rho\mu})_{;\mu}$$

$$= (e\,\Xi_a{}^{\rho\mu})_{,\mu} + e\,\Gamma_\mu{}^\rho{}_\nu\,\Xi_a{}^{\nu\mu} - e\,\omega_\mu{}^b{}_a\,\Xi_b{}^{\rho\mu} \;. \qquad (6.20)$$

By the structure identity $(6.12)^c$ we may write

$$\Gamma_\mu{}^\rho{}_\nu\Xi_a{}^{\nu\mu} = \tfrac{1}{2}\,S_\mu{}^\rho{}_\nu\,\Xi_a{}^{\nu\mu} \;. \qquad (6.21)$$

Then, using $(6.7)^b$ to introduce $\Pi_a{}^\rho$, (6.20) and (6.21), the equation of motion $(6.16)^b$ becomes

$$(e\,\Xi_a{}^{\rho\mu})_{;\mu} - \tfrac{1}{2}\,e\,S_\mu{}^\rho{}_\nu\,\Xi_a{}^{\nu\mu} = e\,e_a{}^\rho\,L + e\,\Pi_a{}^\rho \;. \qquad (6.22)$$

Finally, since $\Omega^{\rho\mu}{}_{ab}$ is an adjoint tensor its covariant derivative is

$$(e\,\Omega^{\rho\mu}{}_{ab})_{;\mu} = (e\,\Omega^{\rho\mu}{}_{ab})_{,\mu} + e\,\Gamma_\mu{}^\rho{}_\nu\,\Omega^{\nu\mu}{}_{ab}$$

$$- e\,\omega_\mu{}^{ef}\,C_{ef}{}^{cd}{}_{ab}\,\Omega^{\rho\mu}{}_{cd} \;. \qquad (6.23)$$

Using the structure identity $(6.12)^f$ as well as $(6.7)^c$ the equation of motion $(6.16)^c$ is

$$(e\,\Omega^{\nu\mu}{}_{ab})_{;\mu} - \tfrac{1}{2}\,e\,S_\mu{}^\rho{}_\nu\Omega^{\nu\mu}{}_{ab} = e\,\Sigma^\rho{}_{ab} \;. \qquad (6.24)$$

The equations of motion in covariant form are then (6.19), (6.22) and (6.24).

6.5 The Stress Tensors and their Conservation

It is customary to define a gravitational stress tensor by

$$t_a{}^{\mu} \equiv \frac{\partial(eL)}{\partial e^a{}_{\mu}} = e\, e_a{}^{\mu} L + e\, \boldsymbol{\Pi}_a{}^{\mu} . \qquad (6.25)$$

We know, however, that $\boldsymbol{\Pi}_a{}^{\mu}$ is not a tensor so we will call $t_a{}^{\mu}$ the *non-covariant stress density*. We will define instead

$$T_a{}^{\mu} \equiv e\, e_a{}^{\mu} L + e\, \Pi_a{}^{\mu} \qquad (6.26)$$

which is a tensor and which we will call the *total covariant stress tensor density*. We see that it is a source on the r.h.s. of the tetrad field equation which is now

$$(e\, \boldsymbol{\Pi}_a{}^{\rho\mu})_{;\mu} - \tfrac{1}{2} e\, S_{\nu}{}^{\rho}{}_{\mu}\, \boldsymbol{\Pi}_a{}^{\mu\nu} = T_a{}^{\rho} . \qquad (6.27)$$

Similarly the *non-covariant spin-stress density* is defined as

$$s^{\mu}{}_{ab} \equiv \frac{\partial(eL)}{\partial \omega_{\mu}{}^{ab}} = e\, \Omega^{\mu}{}_{ab} \qquad (6.28)$$

which, again, is not a tensor. Define instead

$$S^{\mu}{}_{ab} \equiv e\, \Sigma^{\mu}{}_{ab} \qquad (6.29)$$

as the *total covariant spin-stress tensor density*. The field equation (6.24) is then

$$(e\, \Omega^{\rho\mu}{}_{ab})_{;\mu} - \tfrac{1}{2} e\, S_{\mu}{}^{\rho}{}_{\nu}\, \Omega^{\nu\mu}{}_{ab} = S^{\rho}{}_{ab} . \qquad (6.30$$

The local identities $(6.12)^{b,e}$ give explicit expressions for these stresses. They are:

$$T_a^{\ \rho} = e\, e_a^{\ \lambda} \{ L\, \delta_\lambda^{\ \rho} - \Psi_\alpha^{\ \rho}\, \psi^\alpha_{\ ;\lambda} - \Sigma_b^{\ \mu\rho}\, S_{\lambda\ \mu}^{\ b} - \Omega^{\mu\rho}{}_{ac}\, R_{\mu\lambda}{}^{ac} \} \tag{6.31}$$

and

$$\mathcal{S}^\rho{}_{ab} = e\, \{ \Psi_\alpha^{\ \rho}\, S_{ab\ \beta}^{\ \ \alpha}\, \psi^\beta + \Sigma_c^{\ \mu\rho}\, S_{ab\ d}^{\ \ c}\, e^d_{\ \mu} \} . \tag{6.32}$$

Because these stresses are sources it is necessary to find their conservation laws. We could derive these directly from (6.31) and (6.32) in which case we would have to use the global identities, the Bianchi Identities and the equations of motion. Instead we will use the equations of motion directly; this is much shorter and produces the same results which stems from the fact that the equations of motion and the identities are not all entirely independent.

For the spin-stress we have, on taking the divergence of both sides of (6.30),

$$\mathcal{S}^\rho{}_{ab;\rho} = (e\, \Omega^{\rho\mu}{}_{ab})_{;\mu\rho} - \tfrac{1}{2} (e\, S_{\mu\ \nu}^{\ \rho}\, \Omega^{\nu\mu}{}_{ab})_{;\rho} . \tag{6.33}$$

By the Lorentz structure identity $(6.12)^f$ we may write

$$\begin{aligned}\Omega^{\rho\mu}{}_{ab;\mu\rho} &= \tfrac{1}{2} \{ \Omega^{\rho\mu}{}_{ab;\mu\rho} - \Omega^{\rho\mu}{}_{ab;\rho\mu} \} \\ &= R_{\mu\rho\ \tau}^{\ \ \ \rho}\, \Omega^{\tau\mu}{}_{ab} - \tfrac{1}{2} R_{\mu\rho}{}^{cd}\, C_{cd\ \ ab}^{\ \ ef}\, \Omega^{\rho\mu}{}_{ef} \\ &\quad + \tfrac{1}{2} S_{\mu\ \rho}^{\ \nu}\, \Omega^{\rho\mu}{}_{ab;\nu}\end{aligned} \tag{6.34}$$

where we have used the Ricci Identity on $\Omega^{\rho\mu}{}_{ab}$. We also have

$$e_{;\mu} = e_{,\mu} - e\, \Gamma_{\nu\ \mu}^{\ \nu}$$

which becomes, on using the definition of Γ given by (5.84),

$$e_{;\mu} = - S_{\nu}{}^{\nu}{}_{\mu}\, e \tag{6.35}$$

and hence

$$e_{;\mu\rho} = e\, S_{\lambda}{}^{\lambda}{}_{\rho}\, S_{\nu}{}^{\nu}{}_{\mu} - e\, S_{\nu}{}^{\nu}{}_{\mu;\rho} \quad . \tag{6.36}$$

Then, expanding out the last term of (6.33), using (6.34), (6.36) and the second Bianchi Identity, we find

$$\mathbf{S}^{\rho}{}_{ab;\rho} = \tfrac{1}{2}\, \Omega^{\mu\rho}{}_{cd}\, C_{ef}{}^{cd}{}_{ab}\, R_{\mu\rho}{}^{ef} \quad . \tag{6.37}$$

By an almost identical calculation we find the conservation law of the stress tensor. Taking the divergence of both sides of (6.27):

$$T^{\rho}{}_{a\,;\rho} = (e\, \Sigma_{a}{}^{\rho\mu})_{;\mu\rho} - \tfrac{1}{2}\,(e\, S_{\mu}{}^{\rho}{}_{\nu}\, \Sigma_{a}{}^{\nu\mu})_{;\rho} \quad . \tag{6.38}$$

The translational structure identity $(6.12)^{c}$ allows us to write

$$\Sigma_{a}{}^{\rho\mu}{}_{;\mu\rho} = R_{\mu\rho}{}^{\rho}{}_{\nu}\, \Sigma_{a}{}^{\nu\mu} - \tfrac{1}{2}\, R_{\mu\rho}{}^{b}{}_{a}\, \Sigma_{b}{}^{\rho\mu}$$

$$+ \tfrac{1}{2}\, S_{\mu}{}^{\nu}{}_{\rho}\, \Sigma_{a}{}^{\rho\mu}{}_{;\nu} \tag{6.39}$$

where we have again used the Ricci Identity. (6.36),(6.38) and the second Bianchi Identity now gives

$$T_{a}{}^{\rho}{}_{;\rho} = \tfrac{1}{2}\, \Sigma_{b}{}^{\mu\rho}\, R_{\mu\rho}{}^{b}{}_{a} \quad . \tag{6.40}$$

The similarity between these conservation laws is striking and they should be compared with that of the internal case (3.70). It is clear that neither stress will be coveriantly conserved without either a special choice of Lagrangian or some further restriction being placed on the gauge fields. As in the internal case we may interpret the conservation conditions as orthogonality conditions. We shall take this up in the next chapter when we consider some special cases.

6.6 Minimal Coupling

So far we have been dealing with the most general system which allows for coupling between derivatives and makes it impossible to distinguish the contributions made by the idividual fields to the stress tensors. For example, if there is a coupling between the derivative of the matter field and that of one of the potentials then this potential would make a contribution to the concomitant $\Psi_\alpha{}^\rho$ which would affect both the stress and the spin-stress of matter. We therefore, as in the internal case, separate the general Lagrangian into a sum of simpler ones each containing the derivatives of one field only.

We again observe that the concomitants associated with the field derivatives are tensors and interpret this to mean that the corresponding derivatives may be omitted from the Lagrangian without destroying the symmetry. In this way we can isolate the derivatives into separate terms and then use the linearity of the identities and the equations of motion to construct the complete Lagrangian as their sum.

a) The Matter Lagrangian

We omit the derivatives of both the potentials from the Lagrangian whose contents reduce to

$$L_m = L_m(\psi : \partial\psi : e : \omega) \quad . \tag{6.41}$$

It follows that

$$\overset{m}{\Omega}{}^{\rho\mu}{}_{ab} = 0; \quad \overset{m}{\Sigma}{}_a{}^{\rho\mu} = 0 \tag{6.42}$$

and hence

$$\overset{m}{\Sigma}{}_a{}^\rho = \overset{m}{\Pi}{}_a{}^\rho \tag{6.43a}$$

$$\overset{m}{\Sigma}{}^\rho{}_{ab} = \overset{m}{\Omega}{}^\rho{}_{ab} \tag{6.43b}$$

are both tensors by $(6.7)^{b,c}$. We will call L_m the *matter Lagrangian* and it must satisfy the identities:

$$L_{m;\mu} - \Phi_\alpha \psi^\alpha{}_{;\mu} - \Psi_\alpha{}^\rho \psi^\alpha{}_{;\rho\mu} = 0 \qquad (6.44)^a$$

$$\overset{m}{\Pi}_a{}^\rho = -\Psi_\alpha{}^\rho \psi^\alpha{}_{;\lambda} e_a{}^\lambda \qquad (6.44)^b$$

$$\Phi_\alpha S_{ab}{}^\alpha{}_\beta \psi^\beta + \Psi_\alpha{}^\rho S_{ab}{}^\alpha{}_\beta \psi^b{}_{;\rho} + \overset{m}{\pi}{}_c{}^\rho S_{ab}{}^c{}_d e^d{}_\rho = 0 \qquad (6.44)^c$$

$$\overset{m}{\Sigma}{}^\rho{}_{ab} = \Psi_\alpha{}^\rho S_{ab}{}^\alpha{}_\beta \psi \qquad (6.44)^d$$

We will omit the overscript m except where absolutely necessary.

b) <u>The Gauge Field Lagrangians</u>

We know that $\Psi_\alpha{}^\rho$ is a tensor and hence we may omit the derivative of the matter field from the general Lagrangian so that $\Psi_\alpha{}^\rho$ vanishes identically. The transformation equation $(6.6)^a$ then shows that Ψ_α also becomes tensorial and therefore we may omit the matter field from the Lagrangian entirely. We are then left with a purely gauge Lagrangian containing only the gauge fields and their derivatives. The existence of such a Lagrangian shows that the gauge fields by themselves form a closed dynamic system. The field content of this Lagrangian is

$$L_o = L_o(e : \partial e : \omega : \partial\omega) \quad . \qquad (6.45)$$

However, we can further divide the Lagrangian by omitting derivatives which avoids the possibility of having, for example, the curvature coupled directly to the torsion.

i) The Curvature Lagrangian

$\Sigma_a^{\ \mu\rho}$ is a tensor and so we omit the derivatives of the tetrad and take

$$L_\omega = L_\omega(e : \omega : \partial\omega) \tag{6.46}$$

then

$$\overset{\omega}{\Sigma}{}_a^{\ \mu\rho} = 0 \tag{6.47}$$

and

$$\overset{\omega}{\Sigma}{}_a^{\ \rho} = \overset{\omega}{\Pi}{}_a^{\ \rho} \tag{6.48^a}$$

$$\overset{\omega}{\Sigma}{}^{\rho}{}_{ab} = \overset{\omega}{\Omega}{}^{\rho}{}_{ab} - \omega_\nu^{\ cd}\, C_{cd}^{\ \ ef}{}_{ab}\, \Omega^{\rho\nu}{}_{ef} \, . \tag{6.48^b}$$

We will call L_ω the *curvature Lagrangian* and it must satisfy the identities:

$$L_{\omega;\mu} - \tfrac{1}{2}\, \Omega^{\lambda\rho}{}_{ab}\, R_{\lambda\rho}{}^{ab}{}_{;\mu} = 0 \tag{6.49^a}$$

$$\overset{\omega}{\Pi}{}_c^{\ \tau} = - e_c^{\ \lambda}\, \Omega^{\mu\tau}{}_{ab}\, R_{\mu\lambda}{}^{ab} \tag{6.49^b}$$

$$\overset{\omega}{\Pi}{}_c^{\ \mu}\, S_{ab\ d}^{\ \ c}\, e^d{}_\mu + \tfrac{1}{2}\, \Omega^{\mu\rho}{}_{cd}\, C_{ab}^{\ \ cd}{}_{ef}\, R_{\mu\rho}{}^{ef} = 0 \tag{6.49^c}$$

$$\overset{\omega}{\Sigma}{}^{\rho}{}_{ab} = 0 \tag{6.49^d}$$

$$\Omega^{\lambda\rho}{}_{ab} + \Omega^{\rho\lambda}{}_{ab} = 0 \quad . \tag{6.49^e}$$

ii) The Torsion Lagrangian

Finally, we omit the derivatives of the Lorentz potential ω :

$$L_e = L_e (e : \omega : \partial e) \tag{6.50}$$

then

$$\overset{e}{\Omega}{}^{\lambda\rho}{}_{ab} = 0 \tag{6.51}$$

and

$$\overset{e}{\Pi}{}_a{}^{\lambda} = \overset{e}{\Sigma}{}_a{}^{\lambda} - \omega_\rho{}^{cd} S_{cd}{}^b{}_a \Sigma_b{}^{\rho\lambda} \tag{6.52^a}$$

$$\overset{e}{\Sigma}{}^{\rho}{}_{ab} = \overset{e}{\Omega}{}^{\rho}{}_{ab} \quad . \tag{6.52^b}$$

We call L_e the *torsion Lagrangian* and it must satisfy:

$$L_{e;\mu} - \tfrac{1}{2} \Sigma_a{}^{\lambda\rho} S_\rho{}^a{}_{\lambda;\mu} = 0 \tag{6.53^a}$$

$$\overset{e}{\Pi}{}_c{}^{\tau} = - e_c{}^{\lambda} \Sigma_a{}^{\mu\tau} S_\lambda{}^a{}_\mu \tag{6.53^b}$$

$$\Sigma_a{}^{\mu\tau} + \Sigma_a{}^{\tau\mu} = 0 \tag{6.53^c}$$

$$\overset{e}{\Pi}{}_c{}^{\mu} S_{ab}{}^c{}_d e^d{}_\mu + \tfrac{1}{2} \Sigma_c{}^{\mu\rho} S_{ab}{}^c{}_d S_\rho{}^d{}_\mu = 0 \tag{6.53^d}$$

$$\overset{e}{\Sigma}{}^{\rho}{}_{ab} = \Sigma_c{}^{\mu\rho} S_{ab}{}^c{}_d e^d{}_\mu \quad . \tag{6.53^e}$$

In all of the above we have omitted all overscripts which are not absolutely necessary.

c) The Complete Lagrangian

Since the identities (6.12) are linear in the Lagrangian we can take the *complete minimally coupled Lagrangian* as

$$L = L_m + L_e + L_\omega \tag{6.54}$$

which will satisfy the identities (6.12) provided that the individual Lagrangians satisfy their respective identities.

We have also

$$\Pi_a{}^\rho = \overset{m}{\Pi}{}_a{}^\rho + \overset{e}{\Pi}{}_a{}^\rho + \overset{\omega}{\Pi}{}_a{}^\rho \tag{6.55}$$

from which it follows that

$$T_a{}^\rho = \overset{m}{T}{}_a{}^\rho + \overset{e}{T}{}_a{}^\rho + \overset{\omega}{T}{}_a{}^\rho \tag{6.57}$$

where, by the identities $(6.44)^b$, $(6.49)^b$ and $(6.53)^b$,

$$\overset{m}{T}{}_a{}^\rho = e\, e_a{}^\lambda \{L_m \delta_\lambda{}^\rho - \Psi_\alpha{}^\rho \psi^\alpha{}_{;\lambda}\} \tag{6.58}$$

is the *matter stress tensor density*,

$$\overset{e}{T}{}_a{}^\rho = e\, e_a{}^\lambda \{L_e \delta_\lambda{}^\rho - \Sigma_b{}^{\mu\rho} S_\lambda{}^b{}_\mu\} \tag{6.59}$$

is the *tetrad stress tensor density* and

$$\overset{\omega}{T}{}_a{}^\rho = e\, e_a{}^\lambda \{L_\omega \delta_\lambda{}^\rho - \Omega^{\mu\rho}{}_{ab} R_{\mu\lambda}{}^{ab}\} \tag{6.60}$$

is the *Lorentz stress tensor density*.

In the same way we have

$$\Sigma^\rho{}_{ab} = \overset{m}{\Sigma}{}^\rho{}_{ab} + \overset{e}{\Sigma}{}^\rho{}_{ab} + \overset{\omega}{\Sigma}{}^\rho{}_{ab} \tag{6.61}$$

hence

$$\mathbf{S}^{\rho}{}_{ab} = \overset{m}{\mathbf{S}}{}^{\rho}{}_{ab} + \overset{e}{\mathbf{S}}{}^{\rho}{}_{ab} + \overset{\omega}{\mathbf{S}}{}^{\rho}{}_{ab} \tag{6.62}$$

where, by the identities $(6.44)^d$, $(6.49)^d$ and $(6.54)^e$,

$$\overset{m}{\mathbf{S}}{}^{\rho}{}_{ab} = e\,\Psi_{\alpha}{}^{\rho}\, S_{ab}{}^{\alpha}{}_{\beta}\, \psi^{\beta} \tag{6.63}$$

is the *matter spin-stress density*,

$$\overset{e}{\mathbf{S}}{}^{\rho}{}_{ab} = e\,\mathfrak{X}_{c}{}^{\rho\mu}\, S_{ab}{}^{c}{}_{d}\, e^{d}{}_{\mu} \tag{6.64}$$

is the *tetrad spin-stress density* and

$$\overset{\omega}{\mathbf{S}}{}^{\rho}{}_{ab} = 0 \tag{6.65}$$

is the *Lorentz spin-stress density* which vanishes identically.

$\overset{m}{\mathbf{S}}{}^{\rho}{}_{ab}$ corresponds to what we previously called (Chap. 4) the intrinsic spin of matter. By analogy, we see that the tetrad also has a covariant spin while the Lorentz potential does not. Formally, the existence of a covariant tetrad spin is due to the fact that the tetrad is a gauge tensor unlike any of the other gauge potentials we have encountered. This has other consequences also in that in the internal theory, for example, when we specialized to the minimally coupled Lagrangian the identities demanded that the inhomogeneous term which is responsible for the non-conservation of the current vanished $(3.79)^a$. Here the tensorial nature of the tetrad leads to additional terms appearing in the equivalent identities $(6.49)^c$ and $(6.53)^d$ and hence the corresponding stresses are not automatically conserved when the Lagrangian is minimally coupled.

The vanishing of the Lorentz spin-stress implies that the Lorentz potential has no covariant intrinsic spin which corresponds, in the internal theory, to the gauge potentials having no covariant charge.

6.7 Detailed Conservation

The stress tensors are still coupled by the field equations since these equations are sourced by the total stresses.

From (6.58) we have

$$\overset{m}{T}_a{}^\rho{}_{;\rho} = e_a{}^\rho (e L_m)_{;\rho} - e_a{}^\lambda (e\, \Psi_\alpha{}^\rho\, \psi^\alpha{}_{;\lambda})_{;\rho} \,. \qquad (6.66)$$

Using the global identity $(6.44)^a$, the matter equation of motion (6.19), equation (6.17) and the Ricci Identity to commute the covariant derivatives of the matter field we arrive at

$$\overset{m}{T}_\lambda{}^\rho{}_{;\rho} = \overset{m}{T}_\sigma{}^\mu\, S_\lambda{}^\sigma{}_\mu + \overset{m}{S}{}^\mu{}_{ab}\, R_{\mu\lambda}{}^{ab} \,. \qquad (6.67)$$

In a similar way, except for the use of the first Bianchi Identity, (6.60) gives

$$\overset{\omega}{T}_\lambda{}^\rho{}_{;\rho} = \overset{\omega}{T}_\sigma{}^\mu\, S_\lambda{}^\sigma{}_\mu - \mathcal{S}^\mu{}_{ab}\, R_{\mu\lambda}{}^{ab} \qquad (6.68)$$

where $\mathcal{S}^\mu{}_{ab}$ is the total spin-stress. Because of (6.65) we can write this as:

$$\overset{\omega}{T}_\lambda{}^\rho{}_{;\rho} = \overset{\omega}{T}_\sigma{}^\mu\, S_\lambda{}^\sigma{}_\mu - (\overset{m}{S}{}^\mu{}_{ab} + \overset{e}{S}{}^\mu{}_{ab})\, R_{\mu\lambda}{}^{ab} \,. \qquad (6.69)$$

Finally, by a similar calculation proceeding from (6.59), or by the direct use of (6.40), we have

$$\overset{e}{T}_\lambda{}^\rho{}_{;\rho} = \tfrac{1}{2}\, \Sigma_b{}^{\mu\rho}\, R_{\mu\rho}{}^b{}_a\, e^a{}_\lambda - (\overset{m}{T}_\sigma{}^\mu + \overset{\omega}{T}_\sigma{}^\mu)\, S_\lambda{}^\sigma{}_\mu + \overset{e}{S}{}^\mu{}_{ab}\, R_{\mu\lambda}{}^{ab} \,. \qquad (6.70)$$

We see that the couplings, through which energy and momentum are exchanged between the fields, occur in essentially two ways: a stress-torsion coupling and a spin-curvature coupling both of which have the form of the generalized Lorentz force we found in the internal theory (Chap. 4). Note that, as is shown by the signs,